On general notions of depth for regression

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Abstract

Depth notions in location have generated tremendous attention in the literature. In fact, data depth and its applications remain as one of the most active research topics in statistics over the last three decades. Most favored notions of depth in location include Tukey (1975) halfspace depth (HD), Liu (1990) simplicial depth, and projection depth (PD) (Stahel (1981) and Donoho (1982), Liu (1992), Zuo and Serfling (2000) (ZS00) and Zuo (2003)), among others.

Depth notions in regression have also been proposed sporadically, nevertheless. The regression depth (RD) of Rousseeuw and Hubert (1999) (RH99), the most famous, exemplifies a direct extension of Tukey HD to regression. Other notions include Carrizosa (1996) and the ones proposed in this article via modifying a functional in Marrona and Yohai (1993) (MY93). Is there any relationship between Carrizosa depth and the RD of RH99? Do these depth notions possess desirable properties? What are the desirable properties? Can existing notions serve well as depth notions in regression? These questions remain open.

The major objectives of the article include (i) revealing the connection between Carrizosa depth and RD of RH99; (ii) expanding location depth evaluating criteria in ZS00 for regression depth notions; (iii) examining the existing regression notions with respect to the gauges; and (iv) proposing the regression counterpart of the eminent location projection depth.


Key words and phrase: Depth, unfitness, linear regression, maximum depth regression estimating functionals, robustness.

Running title: Depth notions in regression.
1 Introduction

Notion of depth in location has attracted vast attention and has been increasingly pursued as a powerful tool for multi-dimensional nonparametric data analysis and inference.

Prevailing location depth notions include Tukey (1975) halfspace depth (HD) (popularized by Donoho and Gasko (1992)), Liu (1990) simplicial depth, the spatial depth (Vardi and Zhang (2000)), projection depth (PD) (Stahel (1981) and Donoho (1982), Liu (1992), Zuo and Serfling (2000) (ZS00), and Zuo (2003)), and zonoid depth (Koshevoy and Mosler (1997), Mosler (2002)). Applications of data depth in multivariate statistics include


(ii) multivariate exploratory data analysis, such as in geophysical, hydrological, and physio meteorological research (Liu, et al (1999), Chebana and Ouarda (2008, 2011));

(iii) outlier detection, such as in environmental studies (Dang and Serfling (2010), Serfling and Wang (2014), Wang and Serfling (2015, 2018), Febrero, et al (2007));


(v) multivariate risk measurement in financial engineering (Cascos and Molchanov (2007));

(vi) remote sensing and signal processing (Velasco-Forero and Angulo (2011, 2012)); robust linear programming (Mosler and Bazovkin (2014)), and

(vii) econometric and social studies (Caplin and Nalebuff (1988, 1991a, 1991b)),

among others. In fact, data depth and its applications remain as one of the most active research topics in statistics over the last three decades.


Mizera (2002) introduced a scheme as a calculus technique/tool to derive depth functions for different statistical models, and Mizera and Müller (2004) extended the tangent depth for location and regression to the location-scale setting. Data depth has also been employed as penalties in penalized regression (Majumdar and Chatterjee (2017)).
Depth notions in regression have been inevitably proposed, yet sporadically. Regression depth \((\text{RD}_{RH})\) by Rousseeuw and Hubert (1999) (RH99), the most famous, exemplifies a direct extension of Tukey location depth in regression. Others include pioneer Carrizosa depth (Carrizosa 1996) and the ones proposed here which are induced from Marrona and Yohai (1993) (MY93). Attention paid to regression depth has been disproportionally light, compared with its location counterpart. One of the reasons for this might be that there exist no clear fundamental principles to evaluate or measure proposed regression depth notions. Lack of the evaluation criteria not only prohibits any further advance and theoretical development of the depth notions in regression but also impedes their applications in practice. One major objective of the current article is to extend the set of criteria (or desired axiomatic properties) for depth notions in location in ZS00 to regression and to examine the existing regression notions of depth with respect to the proposed criteria.

There exist a variety of robust methods including M-estimate approach and ad hoc ones (see Rousseeuw and Leroy (1987) (RL87), Maronna, Martin, and Yohai (2006) (MMY06)) for estimating the parameters in a linear regression model. In this article, regression depth is utilized to introduce the median-type deepest estimating functionals for regression parameters, manifesting one of the prominent advantages of notions of depth. The functionals are the minimizers of the maximum of unfitness of regression parameters and recover in the empirical case the classical least squares, least absolute deviations, and other existing leading estimators. Under a general framework, depth notions induced from projection-pursuit approach include the \(\text{RD}_{RH}\) and projection regression depth (PRD) (induced from MY93) as special cases. The latter is an extension of the eminent PD in location to regression.

The rest of this article is organized as follows. Section 2 presents a general definition for notions of unfitness and depth in regression and puts forward four general approaches for introducing the notions of unfitness or depth and the maximum (deepest) depth functionals while examining three special examples. It is found that Carrizosa (1996) depth, \(D_C\), known of recovering the HD in location, is not identical but closed related to \(\text{RD}_{RH}\) in regression. Section 3 provides a rigorous definition of depth (or unfitness) notion in regression based on four axiomatic properties which then are employed for the evaluation of four types of special depth notions. Section 4 ends the article with brief concluding remarks. Section 5 (Appendix) collects some major proofs and derivations and auxiliary lemmas.

## 2 Definitions, approaches, and examples

### 2.1 Regression model

Consider a general linear regression model:

\[
y = \mathbf{x}' \mathbf{\beta} + e,
\]

where \(\mathbf{x}'\) denotes the transpose of a vector; random variable \(y\) and \(e\) are in \(\mathbb{R}^1\); and random vector \(\mathbf{x} = (x_1, \ldots, x_p)'\) and parameter vector \(\mathbf{\beta}\) are in \(\mathbb{R}^p\). Note that this general model includes the special case with an intercept term. For example, if \(\mathbf{\beta} = (\beta_1, \beta_2)'\) and \(x_1 = 1\), then one has \(y = \beta_1 + x_2 \beta_2 + e\), where \(\mathbf{x}_2 = (x_2, \ldots, x_p)' \in \mathbb{R}^{p - 1}\). Denote \(\mathbf{w} = (1, \mathbf{x}_2')'\), then
\[ y = \mathbf{w}'\beta + \epsilon. \] We use this model or (1) interchangeably, depending on the context. Denote by 
\[ F_{(y, x)} \] the joint probability distribution of \( y \) and \( x \) under the model (1).

In the following sections, we discuss the notions of unfitness or depth and general approaches to introduce regression depth and induced deepest estimating functionals.

### 2.2 Notions of unfitness and depth in regression

**Unfitness** of a candidate parameter \( \beta \): \( UF(\beta) \), is a function of the residual \( r(\beta) := (y - x'\beta) \). Namely, \( UF(\beta) = f(r(\beta)) \). Examples of \( f(x) \) include, \( x^2 \) and \( |x| \). Generally speaking, an even, monotonic in \( |x| \), and convex \( f(x) \) with its minimum value 0 at 0 will serve the purpose.

**Depth** of \( \beta \) then can be defined as a bounded reciprocal (reverse) function of \( \phi(F_R) \) (e.g. \( 1/(1 + x) \)), say on \([0, 1]\), where \( \phi \) is a functional on the distribution of \( R := UF(\beta) \). A typical example of \( \phi \) is the expectation or quantile functional, \( \phi(F_R) \) could also just be \( R \). Likewise, given its depth, one can define the unfitness of \( \beta \) to be a reciprocal function of the depth.

A minimizer \( \beta^* \) of unfitness function of \( \beta \) over all \( \beta \in \mathbb{R}^p \) can serve as a regression estimating functional for \( \beta \). Similarly, a maximizer of depth function plays the same role.

### 2.3 Four approaches for notions of unfitness and depth

#### 2.3.1 Classical objective function approach

Directly employing the scheme above, one can recover many classical regression estimators in the empirical distribution case (i.e. \( r_i(\beta) = y_i - x_i'\beta, \ i = 1, \cdots, n \)). Here the classical objective function in regression serves as the unfitness function, that is: \( UF(\beta) = f_{\text{Obj}}(r(\beta)) \).

Maximizing depth of \( \beta \) then is equivalent to the minimization of \( \phi(F_R) \) and then the minimizer denoted by \( \beta^* \) could serve as an estimator for \( \beta \in \mathbb{R}^p \). In the sequel, consider examples of \( \phi \): (i) the expectation functional \( \mu \), and (ii) quantile functional \( q_\tau \), \( \tau \in (0, 1) \).

**Example 2.1**

(I) If \( \phi = \mu \) and \( f(x) = x^2 \), then \( \beta^*(F_{(y, x)}) = \arg \min_{\beta \in \mathbb{R}^p} \int (t - s')^2 dF_{(y, x)}(t, s) \), which induces the least squares (LS) estimator when \( F_{(y, x)} \) is the empirical distribution.

(II) If \( \phi = \mu \) and \( f(x) = |x| \), then the approach above leads to the least absolute deviations (LAD) estimator;

(III) If \( \phi = \mu \) and \( f_r(x) = x(\tau - \mathbf{I}(x < 0)) \), \( \tau \in (0, 1) \), where \( \mathbf{I} \) is the indicator function, then the approach above results in the quantile regression estimator (Koenker and Bassett (1978)). When \( \tau = 1/2 \) it recovers the \( L_1 \) regression estimator, for related discussions on quantile regression, see Portnoy (2003, 2012);

(IV) If \( \phi = q_{0.5} \) and \( f(x) = x^2 \), then the approach above yields the least median squares (LMS) estimator (Rousseeuw (1984)); and

(V) If \( \phi = \mu \), coupled with an appropriately chosen function \( f(x) \), one can actually recover the M-estimators (Huber (1973)) (including the famous (a) Huber’s proposal 2 (Huber (1964)), (b) Hampel’s three-parts (Hampel (1974)), and (c) Tukey bisquare (Beaton and
Tukey, (1974)) ones, the L-estimators (Ruppert and Carroll (1980)), and the R-estimators (Koul (1970, 1971), Jureckova (1971), and Jaeckel (1972)).

2.3.2 Facility location approach

In addition to the general approach mentioned in the Section 2.3.1, there exist other approaches for the introduction of notions of depth or unfitness. The classical one is the facility location approach, prevailing in location analysis and operations research.

Let $F_1 \in \mathbb{R}^2$ be a candidate for a facility location, and $P$ be the probability distribution of a random vector $X \in \mathbb{R}^2$ (or of consumers’ locations), and $d(F_1, X)$ (defined in Remarks 2.1 below) measures in some sense the closeness of $F_1$ to the distribution $P$ of $X$ (or the coverage of consumers). Let $F_2$ be a candidate for the facility location of another competitive company. Similarly, $d(F_2, X)$ measures the coverage of the consumers in the vicinity of the facility at $F_2$. With respect to the $F_1$, the maximum market share which can be captured by any other facility is $\sup_{F_2 \in \mathbb{R}^2} P(\omega : d(F_2, X(\omega)) < d(F_1, X(\omega)))$. Also, the $F_1$ should be chosen to maximize its market share. That is,

$$F_1^* = \arg \max_{F_1 \in \mathbb{R}^2} \left( 1 - \sup_{F_2 \in \mathbb{R}^2} P(\omega : d(F_2, X(\omega)) < d(F_1, X(\omega))) \right)$$

Carrizosa (1996) extended $\mathbb{R}^2$ above to $\mathbb{R}^p$ $(p \geq 2)$ and introduced a depth notion (normalized depth, see Definition 2.1 below). Let us use a generic term, Carrizosa depth, hereafter. The Carrizosa depth of $x$ w.r.t. $P$: $D_{C}(x; P)$ ($P$ and $F_X$ are used interchangeably), is defined as

$$D_{C}(x; P) = \inf_{y \in \mathbb{R}^p} P(\omega : d(x, X(\omega)) \leq d(y, X(\omega))).$$

Then $F_1^*$ in (2) is the maximum depth solution (functional) for the facility location problem.

Remarks 2.1

(I) Note that the distance measure $d$ above includes a class of possible choices. For example, $d$ could be an $L_p$ norm or weighted $L_p$ norm (see Zuo (2004)) $(p \geq 1)$.

(II) When $d(x, y) = \|x - y\|$, where “$\| \cdot \|$” stands for the Euclidean (or $L_2$) norm, (3) recovers the normalized depth $ND(x; P)$ of Carrizosa (1996) that is quoted below:

Definition 2.1 (Carrizosa 1996). The normalized depth of $ND(x; P)$ of a point $x \in \mathbb{R}^p$ in $P$ is defined as

$$ND(x; P) = \inf_{y \in \mathbb{R}^p} P(\{a : \|y - a\| \geq \|x - a\|\})$$

Henceforth we focus on $L_2$ norm for distance measure $d$, unless stated otherwise.
halfspace depth (HD)).

\[
\text{HD}(x; P) = \inf_H \{ P(H) : \text{H is a closed halfspace and } x \in H \}, \quad x \in \mathbb{R}^p. \tag{5}
\]

It turns out that \(D_C(x; P)\) can actually recover \(\text{HD}(x; P)\) as shown in Carrizosa (1996).

**Proposition 2.1** If \(d\) in (3) is the \(L_2\) norm, then \(D_C(x; P)\), equivalently \(\text{ND}(x; P)\) in (4), is identical to \(\text{HD}(x; P)\) in (5).

**Proof:** see the proof of Proposition 1 of Carrizosa (1996).

Although the depth \(D_C(x; P)\) above is introduced initially for the location problem, it can be extended for the regression problem, as done in Carrizosa (1996) with the \(L_1\) norm.

Indeed, given a probability measure \(P\) (or equivalently \(F_{(y, x)}\)) in \(\mathbb{R}^p\), one could define the \(D_C(\beta; P)\) of \(\beta = (\beta_1, \beta_2)' \in \mathbb{R}^p\) as follows: for \(|\beta_1| < \infty\)

\[
D_C(\beta; P) = \inf_{\alpha \in \mathbb{R}^p} P\left( d(y, (1, \alpha)' \beta) \leq d(y, (1, \alpha)' \beta) \right), \quad x \in \mathbb{R}^{p-1}, (p \geq 2) \tag{6}
\]

where \(\alpha = (\alpha_1, \alpha_2)'\) and \(\beta_2 \in \mathbb{R}^{p-1}\); if \(|\beta_1| \to \infty\), then define \(D_C(\beta; P) \to 0\). When \(d(x, y) = |x - y|\), (6) recovers the *depth in regression* in Carrizosa (1996). The latter seems to be the pioneer notion of depth in regression in the literature. Does it have anything to do with the \(R_{\text{RH}}\) of RH99? Let us first quote the original definition of RH99.

**Definition 2.2** (RH99). The regression depth of \(\beta\) is the minimum probability mass that needs to be passed when tilting \(\beta\) in any way until it is vertical.

Since \(D_C\) recovers HD in location and \(R_{\text{RH}}\) is an extension of HD in regression, naturally, one wonders whether \(D_C\) can recover \(R_{\text{RH}}\) in regression. The two are closely connected but not identical as revealed in Proposition 2.2 below.

The same idea of Carrizosa (1996) was proposed again in Adrover, Maronna, and Yohai (2002) (AMY02). AMY02 first flawlessly defined \(R_{\text{RH}}\) above to be

\[
R_{\text{RH}}(\beta, P) = \inf_{\lambda \neq 0} P\left( \frac{r(\beta)}{\lambda' x} < 0, \lambda' x \neq 0 \right), \tag{7}
\]

under assumptions (a) and (b) below, where, \(\lambda \in \mathbb{R}^p, r(\beta) = y - \beta' x\). They then proposed the depth:

\[
D(\beta, P) = \inf_{\gamma \in \mathbb{R}^p} P(|r(\beta)| \leq |r(\gamma)|). \tag{8}
\]

If the first coordinate \(x_1\) of \(x\) in (8) is 1 and \(d(x, y) = |x - y|\) in (6), then (8) and (6) coincides.

Under the assumptions (a) \(P(x'v = 0) = 0\) for all \(v \neq 0 \in \mathbb{R}^p\) and (b) \(P(r(\beta) = 0) = 0\) for all \(\beta \in \mathbb{R}^p\), AMY02 showed that (8) is equivalent to (7) (the last step of the proof is debatable though). (a) and (b) exclude *any* discrete distribution cases of \((y, x)\), nevertheless. The following result characterizes \(D_C(\beta; P)\) and reveals its connection with \(R_{\text{RH}}(\beta; P)\).

Write \(w = (1, x')'\) and \(r(\beta) = y - w' / \beta\). If \(d(x, y) = |x - y|\), then (6) is equivalent to

\[
D_C(\beta, P) = \inf_{\alpha \in \mathbb{R}^p} P(|r(\beta)| \leq |r(\alpha)|), \tag{9}
\]
For a given $\beta \in \mathbb{R}^p$ with $\|\beta\| < \infty$, denote by $H_\beta$ the unique hyperplane determined by $y = w'\beta$. Likewise, a given non vertical hyperplane $H$ uniquely identifies an $\alpha \in \mathbb{R}^p$ through $y = w'\alpha$. Define $S(\beta) := \{\alpha \in \mathbb{R}^p : H_\alpha \text{ intersects with } H_\beta\}$ for the given $\beta$.

**Proposition 2.2** If $d(x, y) = |x - y|$ in (6), then (i) $D_C(\beta; P) = P(r(\beta) = 0)$, and (ii) $\text{RD}_{RH}(\beta; P) = \inf_{\alpha \in S(\beta)} P(|r(\beta)| \leq |r(\alpha)|)$.

**Proof:** see the Appendix.

$D_C(\beta; P)$ in (9) is not identical to original $\text{RD}_{RH}$ of RH99, but is closely related to the latter. In fact, if the infimum in RHS of (9) performs over $S(\beta)$, then they are identical. This is another characterization of $\text{RD}_{RH}(\beta; P)$.

Based on the depth functional in (6), we can introduce the maximum regression depth estimating functional for $\beta$, which is defined, for $d(x, y) = |x - y|$, as

$$
\beta^*(P) = \arg\max_{\beta \in \mathbb{R}^p} D_C(\beta; P),
$$

$\beta^*(P)$ above is well defined. That is, the maximum on the RHS of (10) is attained at a bounded $\beta$. The latter is safeguarded by the result below under the assumption:

(A): $P(H_v) = 0$ for any vertical hyperplane $H_v$

**Proposition 2.3** Under (A), (i) $\lim_{\|\beta\| \to \infty} D_C(\beta; P) = 0$, and (ii) the maximum on the RHS of (10) exists and is attained at a bounded $\beta$.

**Proof:** see the Appendix.

### 2.3.3 Projection-pursuit approach

There is another approach based on the projection-pursuit (PP) scheme to induce the regression estimating functional for parameter $\beta$. One starts with a univariate regression estimating functional w.r.t. the univariate variable $u'x \in \mathbb{R}$ and $r(\beta)$ along each direction $u \in S^{p-1} := \{v, \|v\| = 1, v \in \mathbb{R}^p\}$ and calculates the UF$_u(\beta)$ (the unfitness along $u$) (see Section 2.2 for the definition of unfitness). Then one obtains UF(\beta), the supremum of UF$_u(\beta)$ over all $u \in S^{p-1}$. Finally, one minimizes UF(\beta) over all $\beta \in \mathbb{R}^p$ to obtain a regression estimating functional $\beta^*$ for $\beta$ via the min-max scheme.

**Remarks 2.4**

(I) The approach above actually can recover the maximum regression depth functional in RH99 and induce a maximum projection depth functional that is closely related to P1-estimate in MY93. We elaborate the two special cases in the following.

(II) A related PP approach was discussed in RL87 in the empirical case. To obtain the regression estimator $\beta^*$, it minimizes an objective dispersion function $s(r_1(\beta), \ldots, r_n(\beta))$, where $r_i(\beta) = y_i - x'_i\beta$, $s$ is just scale equivariant (not translation invariant), and $r_i(\beta)$ are regarded as a projection of the point $(y_i, x'_i)'$ onto $(1, -\beta)'$. By varying $s$, this approach covers a very large family of estimators (LS, LAD, LTS, LMS, and S, etc.).

**Example 2.2.** Regression depth and Maximum regression depth functional
Consider the linear model: \( y = \beta_1 + \mathbf{x}'\beta_2 + e \), where \( \mathbf{x}, \beta_2 \in \mathbb{R}^{p-1} \). Denote \( \mathbf{w} = (1, \mathbf{x}')' \), \( \beta = (\beta_1, \beta_2)' \). Then the model is: \( y = \mathbf{w}'\beta + e \). That is, \( \mathbf{w} \) here corresponds to \( \mathbf{x} \) in general model (1) and vice versa.

When \( p = 2 \), we define \( F(\beta) = E(\mathbf{I}((y - \mathbf{w}'\beta)'\mathbf{v} \geq 0)) \), where \( F(\beta) \) stands for “fitness” of \( \beta \), \( \mathbf{I} \) for the indicator function, and \( \mathbf{v} = (-v_1, v_2) \), \( v_1 \in \mathbb{R} \), \( |v_2| = 1 \). When, \( v_2 = 1 \), it represents the total probability mass touched (covered) by tilting the line \( y = \beta_1 + \beta_2x \) counter-clockwise around the point \((v_1, \beta_1 + \beta_2v_1)\) to the vertical position (note that the point is the intersection point of the line with the vertical line \( x = v_1 \)). By considering the clockwise tilting \((v_2 = -1)\), it is seen that the closer to \( 1/2 \) the total mass is, the better (more balanced) the candidate parameter \( \beta \) is.

When \( p > 2 \), with the same \( F(\beta) \) as defined above, it can be shown that in the empirical case, minimizing \( F(\beta) \) over all \( v_1 \in \mathbb{R} \) and \( \mathbf{v}_2 \) with \( \|\mathbf{v}_2\| = 1 \) \((\mathbf{v} = (-v_1, \mathbf{v}_2')')\) leads essentially to the regression depth \( RD_{RH} \) of \( \beta \) in RH99 (See the derivations in the Appendix for the general case where it is shown that the approach here is equivalent to (12) below). That is,

\[
\inf_{v_1 \in \mathbb{R}, \|\mathbf{v}_2\|=1} E(\mathbf{I}((y - \mathbf{w}'\beta)'\mathbf{v}'\mathbf{w} \geq 0)) = RD_{RH}(\beta; P).
\tag{11}
\]

Equivalent definitions (or characterizations) of \( RD_{RH} \) in Definition 2.2 exist in the literature (see Remarks 5.1). The one given in Rousseeuw and Struyf (2004) (RS04) is

\[
RD_{RH}(\beta; P) = \inf_{D \in \mathcal{D}} \{ P((r(\beta) \geq 0) \cap D) + P((r(\beta) \leq 0) \cap D^c) \},
\tag{12}
\]

where \( \mathcal{D} \) is the set of all vertical closed halfspaces \( D \).

Now, we can define \( UF(\beta) \) as a simple reciprocal function of \( F(\beta) \) (e.g. \( f(x) = a(1-x)/x, \ a > 0 \)) such that it equals \( \infty \) if the latter equals zero and equals zero if the latter is 1. Maximizing \( UF(\beta) \) leads to the \( RD_{RH} \) of \( \beta \). Furthermore, minimizing the maximum of \( UF(\beta) \) over all \( \beta \in \mathbb{R}^p \) leads to the maximum regression depth functional \( \beta^* \). □

**Example 2.3. Projection regression depth and maximum depth functional**

Hereafter, assume that \( T \) is a univariate regression estimating functional which satisfies

(A1) regression, scale and affine equivariant, that is,

\[
T(F_{(y+xb, \ x)}) = T(F_{(y, \ x)}) + b, \ \forall \ b \in \mathbb{R};
\]

\[
T(F_{(sy, \ x)}) = sT(F_{(y, \ x)}), \ \forall \ s \in \mathbb{R}; \quad \text{and}
\]

\[
T(F_{(y, \ ax)}) = a^{-1}T(F_{(y, \ x)}), \ \forall \ a \in \mathbb{R} \text{ and } a \neq 0.
\]

respectively, where \( x, y \in \mathbb{R} \) are random variables.

(A2) \( \sup_{\|\mathbf{v}\|=1} |T(F_{(y, \ x')\mathbf{v}})| \leq \infty. \)

(A3) \( T(F_{(y-x'\beta, \ x')}) \) is quasi-convex and continuous in \( \beta \in \mathbb{R}^p \) for any fixed \( \mathbf{v} \in \mathbb{S}^{p-1} \).

Assume that \( S \) is a positive scale estimating functional such that \( S(F_{sz+b}) = |s|S(F_z) \) for random variable \( z \in \mathbb{R} \) and scalar \( b, s \in \mathbb{R} \); that is, \( S \) is scale equivariant and location invariant.
Remarks 2.5

(I) Note that, the $T$ above applies for the regression models that do not contain an intercept term (regression through the origin). The latter situation is required in certain applications (see page 62 in RL87) or is generally applicable by some simple treatments of original data (see Eisenhauer (2003)).

(II) Examples of $T(F(y, x')v)$ include mean and quantile functionals, among others. A particular example is $T(F(y, x')v) = \text{Med}_x v \neq 0 \{ y/x'v \}$, where Med stands for the median functional. Typical examples of $S$ include the variance functional and the median absolute deviations functional (MAD), etc.

(III) (A2) holds trivially if $T$ is a quantile-type functional (such as median functional) or mean-type functional if the moments of the underlying distribution exist. (A3) holds for those $T$ as long as integrands involved are quasi-convex and continuous.

Pairs of $T$ and $S$ induce a class of projection regression estimating functionals. Define

$$UF_v(\beta; F(y, x), T) := |T(F(y-x'\beta, x'v))|/S(F_y),$$

which represents unfitness of $\beta$ at $F(y, x)$ w.r.t. $T$ along the direction $v \in S^{p-1}$. Note that if $T$ is a Fisher consistent regression estimating functional, then $T(F(y-x'\beta_0, x'v)) = 0$ under the assumption $E(e|x) = 0$ in model (1) for some $\beta_0$ (the true parameter of the model) and $\forall v \in S^{p-1}; T$ could also be interpreted as a location estimating functional for the location of $y - x'\beta$, and the latter equals 0 for $\beta_0$ under the classical model assumption that 0 is some kind of center of the error distribution, and $x$ and $e$ are independent.

That is, overall one expects $|T|$ to be small and close to zero for a candidate $\beta$, independent of the choice of $v$ and $x'v$. The magnitude of $|T|$ measures the unfitness of $\beta$ along the $v$. Dividing here by $S(F_y)$ is simply to guarantee the scale invariance of $UF_v(\beta; F(y, x), T)$. Taking the supremum over all $v \in S^{p-1}$ and suppressing $T$, yields

$$UF(\beta; F(y, x)) = \sup_{\|v\|=1} UF_v(\beta; F(y, x), T),$$

the unfitness of $\beta$ at $F(y, x)$ w.r.t. $T$. Now applying the min-max scheme, we obtain the projection regression estimating functional

$$\beta^*(F(y, x)) = \arg\min_{\beta \in \mathbb{R}^p} UF(\beta; F(y, x)),$$

Remarks 2.6

(I) $UF(\beta; F(y, x))$ corresponds to the outlyingness $O(x, F_X)$, and $\beta^*(F(y, x))$ corresponds to the projection median functional $PM(F_X)$ in the location setting (see Zuo (2003)). In (13) (14) and (15), we have suppressed $S$ since it does not involve $v$ and is nominal (besides achieving the scale invariance). $T$ in (14) and (15) is also suppressed for convenience.

(II) A similar $\beta^*$ was first studied in MY93, where it was called P1-estimate (denote it by $TP_1$, see (16)). However, they are different. First, MY93 did not talk about the
“unfitness” (or “depth”). Second, the definition of \( \beta^* \) here is different from \( T_{P1} \) of MY93, the latter multiplies by \( S(F_{v'x}) \) instead of dividing by \( S(F_y) \) in (13). They instead defined the following
\[
A(\beta, v) = |T(F_{(y-\beta'x, v'x)})|S(F_{v'x}),
\]
where \( v, \beta \in \mathbb{R}^p \). Their P1-estimate is defined as
\[
T_{P1} = \arg \min_{\beta \in \mathbb{R}^p} \sup_{\|v\|=1} A(\beta, v). \tag{16}
\]
Later we will revisit \( T_{P1} \) and explain why we divide by \( S(F_y) \) in (13) instead of multiplying \( S(F_{y-x'}) \). Note that \( S(F_y) \) here could also be replaced by \( S(F_{y-x'}) \).

(III) The projection-pursuit idea here was first employed in a multivariate location setting by Stahel (1981) and Donoho (1982) independently.

**Projection regression depth (PRD)** One can also introduce the notion of projection depth in regression using the \( UF(\beta; F_{(y, x)}) \). For example, to make the depth between 0 and 1, define a projection regression depth (PRD) functional of \( F_{(y, x)} \) w.r.t. a pair \((T, S)\) as
\[
PRD(\beta; F_{(y, x)}) = (1 + UF(\beta; F_{(y, x)}))^{-1}. \tag{17}
\]
It is readily seen that (15) is also a maximum projection regression depth functional. For the specifical pair of \( T \) and \( S \) such as
\[
T(F_{(y-x'\beta, x'v)}) = \text{Med}_{x'v \neq 0} \left\{ \frac{y-x'\beta}{x'v} \right\}, \quad S(F_y) = \text{MAD}(F_y),
\]
we have
\[
UF(\beta; F_{(y, x)}) = \sup_{\|v\|=1} \left[ \text{Med}_{x'v \neq 0} \left\{ \frac{y-x'\beta}{x'v} \right\} \right] / \text{MAD}(F_y), \tag{18}
\]
and
\[
PRD(\beta; F_{(y, x)}) = \inf_{\|v\|=1, x'v \neq 0} \frac{\text{MAD}(F_y)}{\text{MAD}(F_y) + \text{Med}\left\{ \frac{y-x'\beta}{x'v} \right\}}. \tag{19}
\]

The empirical case of PRD above is closely related to the so-called “centrality” in Hubert, Rousseeuw, and Van Aelst (2001) (HRVA01). In the definition of the latter, all the terms of “MAD(\cdot)” on the RHS of (19) are divided by \( \text{Med}|x'v| \).

### 2.3.4 Other approaches

Besides the three approaches above, there are certainly other approaches (including ad hoc ones). Among them, Mizera (2000) is a famous one.

In extending the idea of \( RD_{RH} \) of RH99, Mizera (2002) (M02), with a decision-theoretic flavor and under the vector optimization framework (vector differential approach), introduced the notions of global, local and tangent depth rigorously. The former two are based on the so-called “critical” function. The latter (the tangent depth) is based on the vector differential
approach and includes local depth as a special case. The local depth in turn includes the global depth as its special case. They are identical under certain conditions.

With mainly Euclidean norm (and/or $L_1$ norm) of $X - \theta$ (in location) and of $y - x'\beta$ (in regression) as the typical critical functions, M02 applied the notions of depth to location and (linear, nonlinear, and orthogonal) regression models and obtained specific depth functions in those models recovering mainly both the HD of Tukey (1975) in location and RD$_{RH}$ in linear regression under a single unified notion of depth (the tangent depth). It is not difficult to see that the critical function could be regarded as a form of unfitness measure of the underlying parameter (note that the words “unfitness”, “nonfit”, “critical function”, “objective function”, and “loss function” are interrelated in some sense. Different people have different preferences.) For the linear regression model, the critical function in M02 can be summarized as follows:

$$CF(\beta; F_{y, x}) = c_p \|y - x'\beta\|_p,$$

where $\| \cdot \|_p$ is the absolute value or squared value w.r.t. $p = 1$ or 2 respectively, and $c_p = 1/p$. The global depth of this leads to RD$_{RH}$ of RH99.

Based on the definitions of M02, one can introduce notions of depth in regression models with appropriate chosen critical functions. The key issue is how to construct “reasonable” or “optimal” critical functions besides the $L_1$ norm and the $L_2$ norm approaches given in M02. With the depth functions obtained via M02 approach, one can introduce the maximum (deepest) regression depth estimating functionals via the min-max scheme.

In addition to the approaches we have discussed so far, there are certainly other ones which introduce notions of unfitness or depth. Can all these notions really serve as depth notions in regression? Gauging or evaluating those notions naturally becomes an issue. Namely, all the unfitness or depth notions must satisfy some basic desired axiomatic properties or possess some desirable and intrinsic features and meet some criteria. What are the criteria?

In the following, we will propose and discuss four axiomatic properties that are deemed necessary for any notion of regression depth or unfitness, thereby providing a systematic basis for the selection and evaluation of a depth notion in regression.

3 Axiomatic properties for depth and unfitness

3.1 Four Axiomatic properties

Definition 3.1 (A depth notion in regression)

A non-negative functional $G$ defined on space $\mathbb{R}^p \times \mathcal{P} \rightarrow [0, \infty)$ is called a depth functional in regression, where $\mathcal{P}$ is the collection of distribution functions on $\mathbb{R}^{p+1}$, if it satisfies the following four properties:

(P1) Invariance (regression, scale, affine invariance) The functional $G$ is regression, scale
and affine invariant w.r.t. a given $F(y, x)$ iff, respectively,

$$
G(\beta + b; F_{(y + x'b, x)}) = G(\beta; F(y, x)), \forall b \in \mathbb{R}^p,
$$

$$
G(s\beta; F_{(sy, x)}) = G(\beta; F(y, x)), \forall s(\neq 0) \in \mathbb{R},
$$

$$
G(A^{-1}\beta; F_{(y, A'x)}) = G(\beta; F(y, x)), \forall \text{nonsingular } p \times p \text{ matrix } A.
$$

**P2** Maximal at center

The functional $G$ possesses its maximum over $\beta \in \mathbb{R}^p$ w.r.t. a given $F(y, x)$. That is, $\max_{\beta \in \mathbb{R}^p} G(\beta; F(y, x))$ exists. Furthermore, it is attained at $\beta^*$ if $\beta^*$ is the center of symmetry of $F(y, x)$ w.r.t. some notion of symmetry in regression.

**P3** Monotonicity relative to deepest point

With respect to a maximum depth point $\beta^*$ of the functional $G$, for any $\beta \in \mathbb{R}^p$ and $\lambda \in [0, 1]$,

$$
G(\lambda \beta^* + (1 - \lambda)\beta; F(y, x)) \geq G(\beta; F(y, x)).
$$

**P4** Vanishing at infinity

The functional $G$ is vanishing when $\|\beta\| \to \infty$. That is, $\lim_{\|\beta\|\to\infty} G(\beta; F(y, x)) = 0$.

Note that due to the reverse relationship, if the depth notion above changes to a unfitness notion, then the above four properties need obvious changes except the (P1). Maximum in (P2) becomes the minimum. (P3) changes maximum to minimum and reverses the direction of the inequality. (P4) becomes $\lim_{\|\beta\|\to\infty} UF(\beta; F(y, x)) = \infty$.

The four properties above were first investigated for the simplicial depth function in Liu (1990) and formulated for general depth functions in location in ZS00. They have been adopted and extended for depth notions in other settings, especially for the functional data in Nieto-Reyes and Battey (2016) (NRB16) from the topological validity point of view, for general functional data in Gijbels and Nagy (2017) (GN17), and for the relevance of halfspace depths in scatter, concentration and shape matrices in Paindaveine and Van Bever (2017+).

Sophisticated discussions on the adaptations and the replacements of the four properties and the appropriateness have been given in Dyckerhoff (2004) and Serfling (2006, 2019), and in NRB16 and GN17 for functional data. Here for the sake of consistency and simplicity, we keep focusing on the four core axiomatic properties and make some remarks below.

**Remarks 3.1**

(I) **(P1)** guarantees that the notion of depth in regression does not depend on the underlying coordinate system or measurement scale. This provides an advantage in the study of the depth induced functionals (estimators) by just dealing with an easily manageable special case (e.g. a spherically symmetric distribution) to cover a large class of cases (e.g. all elliptically symmetric distributions) without loss of generality (see e.g. VAR00).

(II) **(P2)** says that the maximum of $G$ always exists, and it is attained at the center of symmetry w.r.t. some notion of symmetry in regression, when there is such a center. This allows one to discuss the maximum regression depth estimating functional (or estimator in the empirical case). Note that the supremum of bounded $G$ always exists but not necessarily
for the maximum. If (P4) holds, one then can just focus on bounded \( \beta \), however, since \( G \) is not necessarily continuous in \( \beta \), the maximum of \( G \) is not guaranteed to exist. In the empirical distribution case, however, if there are only finitely many hyperplanes that need to be concerned for a given depth functional, then the maximum always exists.

(III) (P3) guarantees that \( G(\beta; \ F_{(y,x)}) \) is monotonically decreasing in \( \beta \) along any ray stemming from a deepest point. This is equivalent to the quasi-convexity of the depth functional under (P2), which further implies that the set of all \( \beta \) that has depth at least \( \alpha(\geq 0) \) is convex (which will be useful when studying the depth induced contours in the parameter space of \( \beta \in \mathbb{R}^p \)), and fewer ties in depth computations of \( \beta \) (in the strictly decreasing case) will be yielded.

(IV) (P4) dictates that when the hyperplane \( H_\beta \) determined by \( y = x'\beta \) becomes vertical, its depth should be vanishing. This makes sense since when the hyperplane is vertical, it can no longer serve as an estimating functional for a linear regression parameter. It is obviously no longer useful for the prediction of future responses as well. Note that \( ||\beta|| \to \infty \) could mean (i) \( |\beta_1| \to \infty \) and or (ii) \( ||\beta_2|| \to \infty \). (ii) just means the hyperplane \( H_\beta \) turns out to be vertical. When (i) happens, the intercept of the hyperplane \( H_\beta \) becomes unbounded, the hyperplane becomes useless and its depth logically should be vanishing.

3.2 Examining depth notions

Now that four axiomatic properties have been presented, a natural question is: do the regression depth functions induced from the four approaches in Sections 2.3.1 to 2.3.4 satisfy all the desired properties? That is, are they really notions of depth w.r.t. (P1)-(P4)? First, let us summarize the depth functionals from these sections.

The approach in Section 2.3.1 based on the classical objective functions induces a class of regression depth functionals, defined by

\[
D_{\text{Obj}}(\beta; \ F_{(y,x)}, \phi, f) = \left(1 + \phi(\text{R})\right)^{-1},
\]

\[\text{(21)}\]

where \( R = f(\beta(S(F_y))) \), \( \phi \) and \( f \) (objective function) are given in Section 2.2 or Example 2.1, and \( S(\cdot) \) is a scale functional that is translation invariant and scale equivalent; dividing by it achieves scale invariance of the depth functions; it is suppressed in \( D_{\text{Obj}} \).

Facility location approach in Section 2.3.2 induces \( D_C(\beta; F_{(y,x)}) \) (or \( D_C(\beta; P) \)) that is closely related to \( RD_{RH}(\beta; F_{(y,x)}) \) when the distance \( d \) in (6) is the \( L_1 \) norm. We will only consider \( D_C(\beta; F_{(y,x)}) \) with \( d(x, y) = |x - y| \) as the representative for this approach.

Typical depth functionals from Section 2.3.3 (the PP approach) are \( RD_{RH}(\beta; F_{(y,x)}) \) and \( PRD(\beta; F_{(y,x)}) \).

Mizera’s approach in Section 2.3.4 can recover Tukey HD in location and \( RD_{RH}(\beta; F_{(y,x)}) \) in regression, and its critical function (see (20)) could be regarded an objective function. Its general version of tangent depth in linear regression (on page 1694) essentially recovers \( RD_{RH}(\beta; F_{(y,x)}) \). No distinct depth function in linear regression from this approach will be discussed here.
Consequently, in the sequel we will investigate (i) $D_{Obj}(\beta; F_{(y,x)}, \phi, f)$, (ii) $D_C(\beta; F_{(y,x)})$, (iii) $RD_{RH}(\beta; F_{(y,x)})$, and (iv) $PRD(\beta; F_{(y,x)})$.

**Proposition 3.1** Regression depth functional (i), (ii), (iii), and (iv) satisfy (P1).

**Proof:** see the Appendix.

**Remarks 3.2**

(I) Without modifying the original function $A(\beta; v)$ of MY93 (see (II) of Remarks 2.6), the induced depth functional (iv), $PRD(\beta; F_{(y,x)})$, can never satisfy (P1).

(II) As a by-product of (P1), maximum regression depth functionals induced from regression depth notions in Proposition 3.1 are equivariant as declared in Corollary 3.1 below.

(III) Note that in (16), $T_{P1}(F_{s_y,x}) = s^2 T_{P1}(F_{y,x})$. That is, by definition below, $T_{P1}$ is not scale equivariant, contrary to popular belief in the literature.

**Corollary 3.1** The maximum regression depth functionals $\beta^*(F_{(y,x)})$ induced from (i), (ii), (iii) and (iv) are regression, scale, and affine equivariant. That is, respectively,

\[
\beta^*(F_{(y+\mathbf{x}'b, x)}) = \beta^*(F_{(y, x)}) + \mathbf{b}, \quad \forall \mathbf{b} \in \mathbb{R}^p;
\]

\[
\beta^*(F_{(s_y,x)}) = s \beta^*(F_{(y, x)}), \quad \forall \text{ scalar } s(\neq 0) \in \mathbb{R};
\]

\[
\beta^*(F_{(y, A'x)}) = A^{-1} \beta^*(F_{(y, x)}), \quad \forall \text{ nonsingular } A \in \mathbb{R}^{p \times p}.
\]

**Proof:** It is trivial.

If a maximum regression depth estimating functional $\beta^*(F_{(y,x)})$ is equivariant, then it is symmetric w.r.t. $(y, x)$ in the sense that $\beta^*(F_{(y,x)}) = \beta^*(F_{(-y,-x)})$. By virtue of the Corollary, one can assume (w.l.o.g.) that $\beta^*(F_{(y,x)})$ equals $0$.

For the joint distribution $F_{(y, x)}$ and the univariate regression estimating functional $T$ given in Example 2.3, $F_{(y, x)}$ is said to be $T$-symmetric about a $\beta_0$ iff for any $v \in \mathbb{R}^{p-1}$

\[
(C0) : \quad T(F_{(y-x'\beta_0, x'v)}) = 0,
\]

**Remarks 3.3:**

(I) $T$-symmetric $F_{(y, x)}$ includes a wide range of distributions. For example, if the univariate functional $T$ is the mean functional, then this becomes the classical assumption in regression when $\beta_0$ is the true parameter of the model: the conditional expectation of the error term $e$ (that is assumed to be independent of $x$) given $x$ is zero, i.e.

\[
(C1) : \quad T(F_{(y-x'\beta_0, x'v)}) = E(F_{(y-x'\beta_0, x'v)}|_{x=x_0}) = 0, \quad \forall x_0 \in \mathbb{R}^p,
\]

(II) When $T$ is the second most popular choice, the quantile functional, especially the median (Med) functional, the $T$-symmetric of $F_{(y, x)}$ about $\beta_0$ is closely related to a weaker
version (when \( \mathbf{v} = (1, 0, \cdots, 0) \)) of the so-called regression symmetry in RS04. Or precisely,

\[
(C2) : \quad T(F_{y-x'|\beta_0, x'_\mathbf{v}}) = \text{Med}(F_{y-x'|\beta_0 | x=x_0}) = 0, \quad \forall x_0 \in \mathbb{R}^p,
\]

For a thorough discussion of this type of symmetry, refer to RS04.

In the following, for \( D_{\text{Obj}}(\beta; F_{(y,x)}, \phi, f) \), we consider only the combinations \( \phi = \mu \), (a) \( f(x) = x^2 \) (the case (I) of Example 2.1) and (b) \( f(x) = |x| \) (the case (II) of Example 2.1).

**Proposition 3.2**  Regression depth function (i), (ii), (iii), and (iv) satisfy (P2) in the following sense.

(a) The maximum of regression depth (i) (i.e. \( D_{\text{Obj}}(\beta; F_{(y,x)}, \phi, f) \)) exists and is attained at \( \beta_0 \in \mathbb{R}^p \) if \( \phi = \mu, \ f(x) = x^2 \) and (C1) holds or if \( \phi = \mu, \ f(x) = |x| \) and (C2) holds.

(b) The maximum of regression depth (ii) (i.e. \( D_C(\beta; F_{(y,x)}) \)) exists if (A) holds and is attained at a bounded \( \beta_0 \in \mathbb{R}^p \).

(c) The maximum of regression depth (iii) (i.e. \( RD_{R_H}(\beta; F_{(y,x)}) \)) exists if (A) holds and is attained at \( \beta_0 \in \mathbb{R}^p \) if (C2) holds.

(d) The maximum of regression depth (iv) (i.e. \( PRD(\beta; F_{(y,x)}, T) \)) exists and is attained at \( \beta_0 \in \mathbb{R}^p \) if (C0) holds.

**Proof:** see the Appendix.

**Remarks 3.4**

(I) Part (a) of the Proposition could be extended to cover more cases. If functional \( \phi \) has the “monotonicity” property (\( \phi(F_{R1}) \leq \phi(F_{R2}) \) if \( R1 \leq R2 \)) and \( f(x) \) has the unique minimum value, then existence is guaranteed. When \( \phi \) is the expectation or quantile functional, then it has monotonicity, and if \( f(x) \) is even, monotonic in \( |x| \) and convex, then \( f(x) \) has a unique minimum value. This covers a large class of combinations of \( \phi \) and \( f \).

(II) Existence of maximum for \( D_C \) and \( RD_{R_H} \) in the Proposition is established under (A). The letter sufficient condition excludes the discrete distributions. In the empirical case, existence always holds true for both, nevertheless.

**Proposition 3.3**  Regression depth function (i), (iii), and (iv) satisfy (P3) in the following sense.

(a) The regression depth (i) (i.e. \( D_{\text{Obj}}(\beta; F_{(y,x)}, \phi, f) \)) monotonically decreases along any ray stemming from a deepest point if \( \phi \) has the monotonicity property (i.e. \( \phi(F_{R1}) \leq \phi(F_{R2}) \) if \( R1 \leq R2 \)), and \( f \) is quasi-convex and has a unique minimum.

(b) The regression depth (iii) (i.e. \( RD_{R_H}(\beta; F_{(y,x)}) \)) monotonically decreases along any ray stemming from a deepest point if (A) holds.

(c) The regression depth (iv) (i.e. \( PRD(\beta; F_{(y,x)}) \)) monotonically decreases along any ray stemming from a deepest point.

(d) The regression depth (ii) (i.e. \( D_C(\beta; P) \)) violates (P3) generally.
Proof: see the Appendix.

Remarks 3.5.

(I) When $\phi$ is the expectation or quantile functional in (a) of the Proposition, then it has the monotonicity property, and when $f$ is $x^2$ or $|x|$ or even the check function in (III) of Example 2.1, then it again meets all the requirements in (a) of the proposition.

(II) For $RD_{RH}$ to meet (P3) (or (P2)), we have to ask for (A) to hold. (P3) always holds for $PRD(\beta; F_{(y,x)})$ with $T$ in Example 2.3.

Proposition 3.4 Regression depth functional (i), (ii), (iii), and (iv) satisfy (P4) in the following sense.

(a) The regression depth (i): $D_{Obj}(\beta; F_{(y,w)}; \phi, f) \to 0$ when $\|\beta\| \to \infty$ and $\|\beta_2\| < \infty$ if $\phi(F_R) \to \infty$ as $|R| \to \infty$ and $f(x) \to \infty$ as $|x| \to \infty$.

(b) The regression depth (ii): $D_C(\beta; F_{(y,w)}) \to 0$ when $\|\beta\| \to \infty$ if (A) holds.

(c) The regression depth (iii): $RD_{RH}(\beta; P) \to 0$ when $\|\beta\| \to \infty$ if (A) holds.

(d) The regression depth (iv): $PRD(\beta; F_{(y,w)}; T) \to 0$ as $\|\beta\| \to \infty$ for $T$ in Example 2.3.

Proof: see the Appendix.

Remarks 3.6

(I) (a) is established under some assumptions on $\phi$ and $\beta$. If $\phi$ is the expectation or quantile functional and $f(x)$ is even, monotonic in $|x|$ and convex, then they satisfy the assumptions. (a) only treats one case of $\|\beta\| \to \infty$. This is, the intercept becomes unbounded while $\|\beta_2\| < \infty$ (as argued in (IV) of Remarks 3.1, in this case, the depth function ought to vanish). The other case of $\|\beta\| \to \infty$ remains untouched.

(II) (b) and (c) are established under the assumption (A). (d) holds for $PRD(\beta; F_{(y,x)}; T)$ with $T$ in Example 2.3 without any extra assumption. This $T$ could be the median or quantile functional or the weighted mean functional in WZ09.

4 Concluding remarks

This article extends four axiomatic properties (evaluation criteria) for location depth notions in ZS00 to depth notions in regression and discusses four general approaches for introducing notions of depth or unfitness in regression. The latter leads to four representative depth notions: (i) $D_{Obj}(\beta; F_{(y,x)}; \phi, f)$, (ii) $D_C(\beta; P)$, (ii) $RD_{RH}(\beta; P)$, and (iv) $PRD(\beta; F_{(y,x)})$.

It characterizes (ii) and reveals that this depth notion in regression is not identical yet closely related to the $RD_{RH}$ of RH99. The latter is contrary to a claim in the literature.

It further investigates the leading regression depth notions (i), (ii), (iii) and (iv) w.r.t. the evaluation criteria and shows that (a) $D_{Obj}(\beta; F_{(y,x)}; \phi, f)$ satisfy all the four properties under some conditions on $\phi$ and $f$, with (P4) proved under just one special case of $\|\beta\| \to \infty$; (b)
under (A), $D_C$ satisfy all (but P3) properties; (c) $\text{RD}_{RH}(\beta; P)$ satisfy all the four axiomatic properties if (A) holds; (d) $\text{PRD}(\beta; F_{(y,x)})$ satisfy all four properties.

Therefore, all but Carrizosa depth (ii) are real regression depth notions w.r.t. the four properties under those assumptions. Moreover, depth functions induced from PRD are representative extensions of eminent projection depth in location to regression.

As by-product of this article, two new characterizations of $\text{RD}_{RH}$ are obtained. One is in Proposition 2.2 and the other in Example 2.2. The latter one turns out to be extremely helpful in studying the asymptotics of the deepest regression estimator $\beta'_{\text{RD}_{RH}}$ (see Zuo (2019a)).

One of the primary advantages of the notions of depth is that it can be employed directly to define median-type deepest (or maximum depth) estimating functionals (estimators in the empirical distribution case) for parameters in regression or location models. The most outstanding feature of the univariate median is its exceptional robustness. Do the deepest regression estimating functionals induced from real regression depth notions here inherit this robustness property? Answers to this for most cases of (i) and for (iii) have been given in the literature (e.g. VAR00). Encouraging answers to (iv) have been established in Zuo (2018).

Besides (P1)-(P4), in evaluating and comparing the overall performance of various regression depth notions, one certainly has to further take into account the robustness and efficiency of their induced maximum depth estimators and their computability. Taking all these factors into consideration, preliminary results (see Zuo (2019b)) indicate that projection regression depth, just as its location counterpart, is favorable among leading competitors.

5 Appendix

Proof of Proposition 2.2:

PROOF OF PART (ii). Assume that $||\beta|| < \infty$, we need to show that

$$\text{RD}_{RH}(\beta; P) = \inf_{\alpha \in S(\beta)} P(|r(\beta)| \leq |r(\alpha)|).$$

(23)

Denote the angle between the hyperplane $H_\beta$ (determined by $y = w'\beta$) and the horizontal hyperplane plane $H_h$ (determined by $y = 0$) by $\theta_\beta$ (consider the acute one only, hereafter). That is, $\theta_\beta$ is the angle between the normal vector $(-\beta'_2, 1)'$ and the normal vector $(0', 1)'$ in the $(x', y)'$-space. Therefore, it is easy to see that $|\tan(\theta_\beta)| = ||\beta_2||$. For any $\alpha = (\alpha_1, \alpha'_2)' \in S(\beta)$ ($||\alpha|| < \infty$) define similarly (hereafter) $H_\alpha$ and $\theta_\alpha$.

First we show that the LHS of (23) is no less than its RHS. Tilting $\beta$ to a vertical position in Definition 2.2 means tilting $H_\beta$ along a hyperline $l_v(\beta)$ which is the intersection line of $H_\beta$ with some vertical hyperplane $H_v$. Let $P(l_v(\beta))$ be the minimum probability mass touched by tilting $H_\beta$ in the definition of $\text{RD}_{RH}$ to a vatical position along $l_v(\beta)$ in two ways. Then it is readily seen that

$$\text{RD}(\beta; P) = \inf_{l_v(\beta)} P(l_v(\beta)).$$

(24)
Let \( H_\gamma \) be the hyperplane with \( \theta_\gamma = \arctan (\|\alpha_2\| + \|\beta_2\|)/2 \) which contains the hyperline \( l_v(\beta) \). Then it is seen that \( H_\gamma \) is in-between \( H_\beta \) and \( H_\alpha \) (consider again the situation that the angle formed between \( H_\beta \) and \( H_\alpha \) is acute, w.l.o.g.). Furthermore, points on \( H_\gamma \) have the same vertical distances to \( H_\beta \) and \( H_\alpha \). That is, \( H_\gamma \) bisects the double wedge formed by \( H_\beta \) and \( H_\alpha \) (i.e. it bisects the vertical distance between the two hyperplanes).

Now it is not difficult to see that \( P (|r(\beta)| \leq |r(\alpha)|) \) equals the probability mass touched by tilting \( H_\gamma \) (towards \( H_\beta \) initially) along the hyperline \( l_v(\beta) \) to the vertical position. In order to reach the infimum over \( S(\beta) \), we need to seek \( \alpha \)'s such that the probability mass above becomes smaller.

Consider \( \alpha_m \in S(\beta) \) that approach \( \beta \) (or let \( \theta_{\alpha_m} \to \theta_\beta \)) while \( H_{\alpha_m} \) and \( H_\beta \) still intercept at \( l_v(\beta) \) (that is, tilting \( H_\alpha \) towards \( H_\beta \) along \( l_v(\beta) \) yields \( \alpha_m \)). As \( m \to \infty \), the probability mass contained in the interior of the double wedge formed between \( H_\beta \) and \( H_{\alpha_m} \) approaches zero and \( P (|r(\beta)| \leq |r(\alpha_m)|) \) decreases to the probability mass touched by tilting \( H_\beta \) to the vertical position along the hyperline \( l_v(\beta) \) in one of two ways (the other way is described below).

Consider \( \alpha_n \in S(\beta) \) that approach \( \beta \) (or let \( \theta_{\alpha_n} \to \theta_\beta \)) with \( H_{\alpha_n} \) being on the other side of \( H_\beta \) and still intercept at \( l_v(\beta) \) (i.e. if previously \( \theta_{\alpha_n} < \theta_\beta \), then \( \theta_{\alpha_n} > \theta_\beta \) now, vice versa). Using the same hyperline \( l_v(\beta) \) above, one can conclude similarly that \( P (|r(\beta)| \leq |r(\alpha_n)|) \) decreases to the probability mass touched by tilting \( H_\beta \) to the vertical position along the hyperline \( l_v(\beta) \) in the other way, as \( n \to \infty \).

The above results imply that \( \inf_{\alpha \in S(\beta)} P (|r(\beta)| \leq |r(\alpha)|) \leq P(l_v(\beta)) \). The arbitrariness of \( l_v(\beta) \) (which can be any hyperline that is the intersection line of \( H_\beta \) and any vertical hyperplane \( H_v \)), in conjunction with (24) implies that \( RD_{RH} (\beta; P) \geq \inf_{\alpha \in S(\beta)} P (|r(\beta)| \leq |r(\alpha)|) \).

Now we show that the LHS of (23) is no greater than its RHS. For a given \( \alpha \in S(\beta) \), \( H_\beta \) intersects \( H_\alpha \) at a hyperline, say \( l(\beta, \alpha) \). Replace \( l_v(\beta) \) with this line in the above proof, it is readily seen that for the given \( \beta, \alpha \in S(\beta) \), and \( l(\beta, \alpha) \), \( P (|r(\beta)| \leq |r(\alpha)|) \) equals the probability mass touched by tilting \( H_\gamma \) (towards \( H_\beta \) initially) along the hyperline \( l(\beta, \alpha) \) to the vertical position, which implies that \( P (|r(\beta)| \leq |r(\alpha)|) \geq P(l(\beta, \alpha)) \), where \( P(l(\beta, \alpha)) \) is again the minimum probability mass touched by tilting \( H_\beta \) along the hyperline \( l(\beta, \alpha) \) to the vertical position in two ways in the Definition of 2.2. Hence, the \( RD_{RH} (\beta; P) \leq \inf_{\alpha \in S(\beta)} P (|r(\beta)| \leq |r(\alpha)|) \) in light of (24). This completes the proof of (ii).

**Proof of part (i).** Consider only the \( \alpha \) that does not belong to \( S(\beta) \). Hence, \( H_\alpha \) is parallel to \( H_\beta \). Let \( H_\gamma \) be the hyperplane in the middle of the hyperstripe with \( H_\alpha \) and \( H_\beta \) as its two boundaries (i.e. \( \theta_\gamma = \arctan (||\alpha_2|| + ||\beta_2||)/2 \)). Then it is readily seen that \( P (|r(\beta)| \leq |r(\alpha)|) \) equals to the probability mass carried by the closed half of the hyperstripe with \( H_\gamma \) and \( H_\beta \) as its two boundaries.

Consider \( \alpha_n \notin S(\beta) \) that approach \( \beta \) (or let \( \alpha_n \to \beta_1 \)), it is readily seen that the probability mass contained in the interior of the half hyperstripe formed between \( H_\beta \) and \( H_{\gamma_n} \) approaches zero, and \( P (|r(\beta)| \leq |r(\alpha_n)|) \) decreases to \( P(H_\beta) = P(r(\beta) = 0) \) as \( n \to \infty \). Similarly to the proof of part (ii) above, it is readily shown that \( \inf_{\alpha \notin S(\beta)} P (|r(\beta)| \leq |r(\alpha)|) = P(r(\beta) = 0) \). On the other hand, by the proof of part (ii) above, it is readily seen that
\[
\inf_{\alpha \in S(\beta)} P (|r(\beta)| \leq |r(\alpha)| ) \geq P(r(\beta) = 0). \text{ This completes the proof of (i).} \]

**Proof of Proposition 2.3:**

(i) For any given \( \beta = (\beta_1, \beta_2)' \), let the angle between the hyperplane \( H_{\beta} \) (determined by \( y = w'\beta \)) and the horizontal hyperplane plane \( H_h \) (determined by \( y = 0 \)) be \( \theta \). That is, \( \theta \) is the angle between the normal vector \((-\beta_2', 1)'\) and the normal vector \((0', 1)'\). Therefore, it is easy to see that \( |\tan(\theta)| = |\beta_2| \). When \( ||\beta|| = (|\beta_1|^2 + |\beta_2|^2)^{1/2} \rightarrow \infty \), assume w.o.l.g. that \( |\beta_1| < \infty \) (otherwise \( DC(\beta; P) \rightarrow 0 \) by definition (12)), then \( |\tan(\theta)| \rightarrow \infty \), \( H_{\beta} \) turns to be vertical, which further implies by Proposition 2.2 that \( DC(\beta; P) \rightarrow 0 \) since the closed double wedge formed by \( H_{\beta} \) and its eventual vertical hyperplane \( H_v \) becomes smaller and smaller (in Lebesgue measure sense), and \( H_{\beta} \) approaches its eventual vertical hyperplane \( H_v \).

(ii) Part (i) implies that when \( ||\beta|| \) becomes unbounded, the RHS of (10) cannot reach its maximum value at such \( \beta \). For a fixed \( \alpha \), \( f(\beta; \alpha) = P(|r(\beta)| \leq |r(\alpha)|) \) is upper semicontinuous in \( \beta \), hence the infimum of upper semicontinuous functions \( DC(\beta; P) \) is also upper semicontinuous. The upper semicontinuity of \( DC(\beta; P) \) in \( \beta \) over a bounded set, in conjunction with the extreme value theorem, yields (ii).

**Proof of the statement in Example 2.3**

Let \( v = (-v_1, v_2)' \in \mathbb{R}^p \), \( v_1 \in \mathbb{R} \), \( v_2 \in \mathbb{R}^{p-1} \), and \( ||v_2|| = 1 \); \( r(\beta) = y - w'\beta \) and \( g(\beta, v) = r(\beta) \ast (v_2)'x - v_1) = r(\beta)w'v \). Here we wanted to show that

\[
RD_{RH}(\beta, P) = \inf_{||v_2|| = 1, v_1 \in \mathbb{R}} E (I(g(\beta, v) \geq 0))
\]

That is, the RHS above is equivalent to (12).

(i) Let us just focus on \( w'v \geq 0 \) (the case \( w'v \leq 0 \) can be treated similarly). Since \( w'v \geq 0 \) is equivalent to \( x'(v_2)' - v_1 \geq 0 \), the latter represents a closed halfplane \( H_x(v_1, v_2) \) in the \( x \)-hyperplane (horizontal hyperplane \( y = 0 \) in the \( (x', y)' \) space). From the \( (x', y)' \) space point of view, it represents a closed vertical halfspace \( D \) in \( (x', y)' \) space. The intersection of this \( D \) with \( x \)-hyperplane (or the vertical projection of \( D \) onto \( x \)-hyperplane) results in \( H_x(v_1, v_2) \).

(ii) On the other hand, given a closed vertical halfspace \( D \) in \( (x', y)' \) space, it intercepts with the \( y = 0 \) hyperplane (or the \( x \)-hyperplane) at a closed halfplane \( H_x \) in \( x \)-hyperplane with its boundary a hyperline \( l_x \) in the \( x \)-hyperplane. Call the direction in the \( x \)-hyperplane that is perpendicular to the hyperline \( l_x \) and pointing into the halfplane \( H_x \) as \( v_2 \). Denote the distance from the origin to the point on \( v_2 \) and \( l_x \) as \( v_1 \); it then follows that \( D \) is equivalent to \( x'v_2 \geq v_1 \) in the \( (x', y)' \) space. That is, \( w'v \geq 0 \).

It is readily seen from (i) and (ii) above that the RHS of (25) is equivalent to (12). Also, it is straightforward to see that it is equivalent to (28) under the assumptions there. Furthermore, it can be shown that

\[
RD_{RH}(\beta, P) = \inf_{||v_2|| = 1, v_1 \in \mathbb{R}} E (I(g(\beta, v) \geq 0)) = \inf_{v \in \mathbb{R}^{p-1}} E (I(r(\beta) \ast (v'w) \geq 0)) \quad (26)
\]
Incidentally, it is seen that

\[
\inf_{\|v_2\|=1, v_1 \in \mathbb{R}} E \left( I \left( g(\beta, v) \geq 0 \right) \right) \\
= \inf_{\|v_2\|=1, v_1 \in \mathbb{R}} E \left( I \left( r(\beta) \ast (v'w) \geq 0 \right) \right) \\
= \inf_{\|v_2\|=1, v_1 \in \mathbb{R}} \min \left\{ E \left( I \left( r(\beta) \ast (v'w) \geq 0 \right) \right), E \left( I \left( r(\beta) \ast (-v)'w \geq 0 \right) \right) \right\} \\
= \inf_{\|v_2\|=1, v_1 \in \mathbb{R}} \min \left\{ E \left( I \left( r(\beta) \ast (v'w) \geq 0 \right) \right), E \left( I \left( r(\beta) \ast (v'w) \leq 0 \right) \right) \right\} \\
= \inf_{\|v_2\|=1, v_1 \in \mathbb{R}} \min \left\{ E \left( I \left( g(\beta, v) \geq 0 \right) \right), E \left( I \left( g(\beta, v) \leq 0 \right) \right) \right\}
\]

Note that the RHS of the last equality is the quantity used for the empirical regression depth calculation in RH99 (up to a constant factor \(n\)).

**Proof of Proposition 3.1:**

(i) For the \(D_{Obj}(\beta; F(\mathbf{y}, \mathbf{x})), \phi, f)\) in Section (2.3.1), notice the facts that

\[
(y + \mathbf{x}'b) - \mathbf{x}'(\beta + b) = y - \mathbf{x}'\beta, \\
s \ast y - x'(s \ast \beta) = s \ast (y - \mathbf{x}'\beta), s \neq 0 \\
y - \mathbf{x}'A(A^{-1}\beta) = y - \mathbf{x}'\beta.
\]

These, in conjunction with the scale equivalence of \(S\), yield the invariance of \(R = f(r(\beta)/S(F_y))\) and of the depth function. (P1) follows immediately for the \(D_{Obj}(\beta; F(\mathbf{y}, \mathbf{x})), \phi, f)\) in (21).

(ii) By (ii) of Proposition 2.3, \(D_C(\beta; P) = P(y - \mathbf{w}'\beta = 0)\), replacing the \(\mathbf{x}\) in (i) above verification by \(\mathbf{w}\), it is readily seen the (P1) follows immediately for \(D_C(\beta; P)\).

(iii) For \(RD_{RH}(\beta; P)\), in the empirical case, the fact that the latter satisfies it has already been declared in Section 2.1 of RH99. For the general population case, note that a characterization of \(RD_{RH}(\beta; P)\) is (see the proof above)

\[
RD_{RH}(\beta; P) = \inf_{\|v_2\|=1, v_1 \in \mathbb{R}} E(I(r(\beta) \ast (v_1, v_2)'w \geq 0)) = \inf_{v \in \mathbb{S}_{p-1}} E(I(r(\beta) \ast v'w \geq 0)),
\]

(27)

Similarly to the proof in (i), (P1) follows immediately for \(RD_{RH}(\beta; P)\).

(iii) For \(PRD(\beta; F(\mathbf{y}, \mathbf{x}))\) in (17). (P1) follows straightforwardly from (13), (14), and (17), coupled with (A1) and (A4).

**Remarks 5.1**

(I) (27) is one of representations of the \(RD_{RH}\). Many other characterizations exist. For example, one is given in RS04 displayed in (12) and another one given in VAR00 is:

\[
RD_{RH}(\beta; P) = \inf_{u \in \mathbb{R}_{p-1}, v \in \mathbb{R}} \left\{ P \left( r(\beta) > 0 \cap \mathbf{x}'u < v \right) + P \left( r(\beta) < 0 \cap \mathbf{x}'u > v \right) \right\}. 
\]

(28)

They assumed that \(P(\mathbf{x}'u = v) = 0\) (and implicitly assumed that (A0): \(P(r(\beta) = 0) = 0\)).

(II) Another representation of the \(RD_{RH}\) given in AMY02 is displayed in (7), which is slightly more general than (28) but again also implicitly made the assumptions above. The
latter implies that these representations are valid only for regression lines or hyperplanes that do not contain any probability mass. The empirical version of (7) was also given on page 158 of Maronna, Martin, and Yohai (2006) (MMY06).

(III) Empirical versions of the regression depth of RH99 and its relationship to the location (halfspace) depth were also extensively investigated in Mizera (2002) (page 1689-1690).

(IV) Another empirical version (which actually is slightly different from \(RD_{RH}\)) was given in Bai and He (1999):

\[RD_{RH}(\beta, Z_n) = \inf_{|u|=1, v \in \mathbb{R}} \min \left\{ \sum_{i=1}^{n} I(r_i(\beta)(u'x_i - v) > 0), \sum_{i=1}^{n} I(r_i(\beta)(u'x_i - v) < 0) \right\},\]

where \(y_i = \beta_0 + x_i'\beta_1 + e_i, \beta' = (\beta_0, \beta_1') \in \mathbb{R}^p, x_i \in \mathbb{R}^{p-1}, r_i(\beta) = y_i - (1, x_i')\beta, \) and \(Z_n = \{(x_i, y_i), i = 1, \cdots, n\}.\) They again implicitly assumed that (A0) hold and \(P(\mathbf{x}'u = v) = 0.\)

\[\square\]

Proof of Proposition 3.2:

(a) In light of (21), the existence of maximizer of \(D_{Obj}\) is equivalent to the existence of the minimizer of \(\phi(F_R),\) where \(R = f(r(\beta)/S(F_y)).\) The latter holds true by virtue of the property ("monotonicity") of the given functional \(\phi\) (i.e. \(\phi(F_{R1}) \leq \phi(F_{R2})\) if \(R1 \leq R2\)) and the unique minimizer 0 of the given \(f\) with the minimum value 0. Under (C1) or (C2), it is readily seen that the maximizer is \(\beta_0,\) in the respective cases.

(b) For \(D_c(\beta; P),\) \(P2\) follows directly from Proposition 2.3.

(c) Following the proof of Proposition 2.3, it can show similarly that under the given condition \(P(H_v) = 0, RD_{RH}(\beta; P) \to 0\) when \(|\beta| \to \infty\) for \(\beta = (\beta_1, \beta_2')'\) with bounded \(\beta_1.\) For \(|\beta_1| \to \infty\) case, we have to adopt the slightly modified definition for \(RD_{RH},\) as done in (6) for \(D_C.\) That is, \(RD_{RH}(\beta; P) \to 0\) when \(|\beta_1| \to \infty.\) Following the arguments given in (ii) of Proposition 2.3, we see that the maximum of \(RD_{RH}(\beta; P)\) exists and is attained at a bounded \(\beta^*\) (note that \(RD_{RH}(\beta; P)\) is upper semi-continuous). Now, if (C2) holds, i.e. \(F_{(y,x)}\) is regression symmetric about the \(\beta_0,\) then by Theorem 3 of RS04

\[RD_{RH}(\beta_0) = \frac{1}{2} + \frac{1}{2}P(y - w'\beta_0 = 0),\]

which is the maximum possible depth value for all \(\beta \in \mathbb{R}^p\) in this case.

(d) We have to show that (a) the depth value can not be maximized when the norm of \(\beta \in \mathbb{R}^p\) becomes unbounded and (b) within the set of bounded \(\beta \in \mathbb{R}^p,\) there exits a \(\beta_0\) which can attain the maximum depth value. For (a), by Lemma 5.1, one immediately sees that \(PRD(\beta) \to 0\) as \(|\beta| \to \infty.\) For (b), first, the continuity of \(PRD(F_{(y,x)}; \beta)\) in \(\beta\) follows directly from the (A3) and the property of supremum; second, by the extreme value theorem, the existence of a bounded maximizer \(\beta_0\) is guaranteed. When \(F_{(y,x)}\) is \(T\)-symmetric about \(\beta_0 \in \mathbb{R}^p,\) (iii) of Lemma 5.1 yields the desired result. \[\square\]
A function \( f \) from \( \mathbb{R}^d \to \mathbb{R} \) is *quasi-concave* if \( f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}, \forall \lambda \in [0,1] \) and \( x, y \in \mathbb{R}^d \) \((d \geq 1)\). For the distribution \( F_X \) of any random vector \( X \), denote its empirical version by \( F^n_X \).

**Lemma 5.1** The projection regression depth \( \text{PRD}(\beta; F_{(y,x)}) \) in (17) is

(i) affine invariant, quasi-concave and continuous in \( \beta \),

(ii) vanishing when \( \|\beta\| \to \infty \),

(iii) maximized at the center \( \beta_0 \) of T-symmetric \( F_{(y,x)} \),

(iv) continuous in \( F_{(y,x)} \) in the sense that \( \text{PRD}(\beta; F^n_{(y,x)}) \xrightarrow{\text{m}} \text{PRD}(\beta; F_{(y,x)}) \) in the same mode as \( F^n_{(y,x)} \xrightarrow{\text{m}} F_{(y,x)} \) when \( n \to \infty \), provided that

(a) \( T(F^n_{(y-x';\beta,x'\nu)}) \xrightarrow{\text{m}} T(F_{(y-x';\beta,x'\nu)}) \) uniformly in \( \nu \in S^{p-1} \) and \( S(F^n_{y}) \xrightarrow{\text{m}} S(F_{y}) \),

(b) \( \sup_{\|\nu\|=1} |T(F_{(y-x';\beta,x'\nu)})| < M_T \), and \( \inf_n S(F^n_{y}) > M_S > 0 \),

where convergence mode “m” could be in \( o_P(1), o(1) \) a.s., or in \( O_P(n^{-1/2}) \).

**Proof of Lemma 5.1**

(i) This is a straightforward verification, by (A1), (A3), and (A4) and (13), (14), and (17).

(ii) In light of (A1), it is readily seen that \( T(F_{(y-x';\beta,x'\nu)}) = T(F_{(y,x')}) - \|\beta\| \) for \( \beta \neq 0 \) and \( \nu = \beta/\|\beta\| \). This, in conjunction with (A3), (13), (14), and (17), yields \( \text{PRD}(\beta; F_{(y,x)}) \to 0 \) as \( \|\beta\| \to \infty \).

(iii) In virtue of the definition of T-symmetric about \( \beta_0 \) in (22), (13), (14), and (17), one sees that \( \text{PRD}(\beta_0; F_{(y,x)}) \) attains its maximum possible value 1.

(iv) Write \( G \) for \( \text{PRD} \), and in light of (17), (14), and (13), a simple derivation leads to

\[
|G(\beta; F^n_{(y,x)}) - G(\beta; F_{(y,x)})| \leq \sup_{\|\nu\|=1} |UF_{\nu}(\beta; F^n_{(y,x)}; T) - UF_{\nu}(\beta; F_{(y,x)}, T)| \\
\leq \sup_{\|\nu\|=1} \frac{D(T_n)S(F_y)}{S(F^n_y)S(F_y)} |T(F_{(y-x';\beta,x'\nu)})| \\
\leq \frac{1}{M_S} \sup_{\|\nu\|=1} D(T_n) + \frac{M_T}{M_S S(F_y)} D(S_n)
\]

by the given (b), where \( D(T_n) = |T(F^n_{(y-x';\beta,x'\nu)}) - T(F_{(y-x';\beta,x'\nu)})| \) and \( D(S_n) = |S(F^n_{y}) - S(F_{y})| \). This, in conjunction with the given (a), leads immediately to (v). \( \blacksquare \)

**Remarks 5.2**

(i) The assumption (a) in (v) of the Lemma holds for classical regression estimating functionals \( T \) such as Med functional or trimmed and winsorized mean functionals (Wu and Zuo (2009) (WZ09)) and for \( S \) such as MAD or trimmed and winsorized standard deviations.
functionals (Wu and Zuo (2008)). Uniformity in $v \in S^{p-1}$ can usually be established via the II.4. and II.5. of Pollard (1984).

II.4. and II.5. The assumption (b) in (v) of the Lemma holds true for $T$ and $S$ above, as long as $S(F_y) > \delta > 0$ and $|T(F_{y-x';x'}(\beta, x'))| < M_v < \infty$ for any $v \in S^{p-1}$.

**Proof of Proposition 3.3:**

It is readily seen that to prove (P3), it suffices to show that (1) there exists a deepest point $\beta_0$ of $G(\beta; P)$, and (2) the regression depth function $G(\beta; P)$ is quasi-concave in $\beta$.

(a) For regression depth (i) (i.e. $D_{obj}(\beta; F_{y-x';x'; \phi}; f)$), (1) follows from the given condition and (I) of Remarks 3.5. For (2), in light of (21), we only need to show that $\phi(F_{r})$ is quasi-convex in $\beta$ with $R = f(r(y - x')/S(F_y))$. The latter follows from the quasi-convexity of $f$ and the “monotonicity” of $\phi$.

(b) For regression depth (iii) (i.e. $R_{D_{RH}}$), (1) follows from (ii) of Proposition 3.2 and (2) follows directly from (iii) of Lemma 5.2.

(c) For the projection regression depth functional, (1) follows from the (i) and (ii) of Lemma 5.1. For (2), in virtue of definition (17), it is seen that it suffices to show that $\text{UF}(\beta_1; F_{y-x';x'}; P)$ is quasi-convex in $\beta$. By (A3), coupled with (13) and (14), it is readily seen that for any $\beta = \lambda \beta_1 + (1 - \lambda) \beta_2, \lambda \in [0, 1]$

$$\text{UF}(\beta; F_{y-x';x'}; P) \leq \max \{\text{UF}(\beta_1; F_{y-x';x'}; P); \text{UF}(\beta_2; F_{y-x';x'}; P)\},$$

This completes the proof of (P3) for PRD$(\beta; F_{y-x';x'}; P)$.

(d) Let $\lambda \in (0, 1)$ is fixed. For given $\beta_1$ and $\beta_2 (\neq \beta_1)$ in $R^p$, assume that $P(r(\beta_i) = 0) = 1/2, i = 1, 2$. Then we have that $D_C(\lambda \beta_1 + (1 - \lambda) \beta_2; P) = 0$ and $D_C(\beta_i; P) = 1/2$ by Proposition 2.2. This implies that

$$D_C(\lambda \beta_1 + (1 - \lambda) \beta_2; P) < \min\{D_C(\beta_1; P); D_C(\beta_2; P)\} = D_C(\beta_2; P),$$

That $D_C(\beta; P)$ attains it maximum value at $\beta_1$ yields the desired result.

**Lemma 5.2.** $R_{D_{RH}}(\beta; P)$ of RH99 is

(i) upper semicontinuous in $\beta$, and continuous in $\beta$ if the density of $P$ exists and
discontinuous in $\beta$ generally;

(ii) continuous in $P$ in the sense that $R_{D_{RH}}(\beta; Q_n)$ converges to $R_{D_{RH}}(\beta; P)$ in the
same mode (in distribution, in probability, with probability one) as $Q_n$ converges to $P$;
If $Q_n$ is the empirical version $P_n$ of $P$, then $R_{D_{RH}}(\beta; P_n)$ converges to $R_{D_{RH}}(\beta; P)$
almost surely and uniformly in $\beta \in R^p$.

(iii) quasi-concave in $\beta \in R^p$.

**Proof of Lemma 5.2:**
(i) For fixed \( v \), \( f(\beta; v) = P(r(\beta) \ast v'w) \) in (27) is upper semicontinuous in \( \beta \), hence the infimum of upper semicontinuous functions \( RD_{RH}(\beta; P) \) is also upper semicontinuous. If a density of \( P \) exists, then \( f(\beta; v) \) is continuous in \( \beta \) and so is the infimum of continuous functions \( RD_{RH}(\beta; P) \). We focus on the discontinuity part. Suppose that the distribution of \( (y, x) \) has its entire probability mass on the hyperplane determined by \( y = w'\beta_0 \) for some \( \beta_0 \in \mathbb{R}^p \), and any hyperline contains zero probability mass; then \( RD_{RH}(\beta_0; P) = 1 \), and \( RD_{RH}(\beta; P) = 0 \) for any \( \beta(\neq \beta_0) \in \mathbb{R}^p \). Thus, when \( \beta \) approaches \( \beta_0 \), \( RD_{RH}(\beta; P) \) can never approach \( RD_{RH}(\beta_0; P) \).

(ii) First part follows directly from the characterization of \( RD_{RH}(\beta; P) \) given in (27) and the continuity of the infimum function; the second part follows from standard empirical process theory, such as Pollard (1984).

(iii) Let \( \beta_1, \beta_2 \in \mathbb{R}^p \) and \( \lambda \in [0, 1] \), and \( \beta := \lambda \beta_1 + (1 - \lambda) \beta_2 \). Let \( H_\beta \) be the hyperplane determined by \( y = w'\beta \), and \( a = \min\{w'\beta_1, w'\beta_2\} \), \( b = \max\{w'\beta_1, w'\beta_2\} \). Denote by \( W(H_{\beta_1}, H_{\beta_2}) = \{(x', y) : x \in \mathbb{R}^{p-1}, y \in [a, b]\} \) the closed double wedge formed by two hyperplanes \( H_{\beta_1} \) and \( H_{\beta_2} \) (assume w.l.o.g. that \( H_{\beta_1} \) is not parallel to \( H_{\beta_2} \)).

By Definition 2.2, \( RD_{RH}(\beta; P) \) is the minimum probability mass that needs to pass when \( H_\beta \) is tilted into a vertical position. Notice that the position of \( H_\beta \) is in-between that of \( H_{\beta_1} \) and \( H_{\beta_2} \), it is readily seen that

\[
RD_{RH}(\lambda \beta_1 + (1 - \lambda) \beta_2; P) \geq \min \{RD_{RH}(\beta_1; P), RD_{RH}(\beta_2; P)\} \\
+ \min \{P(W(H_\beta, H_{\beta_1})), P(W(H_\beta, H_{\beta_2}))\} \\
\geq \min \{RD_{RH}(\beta_1; P), RD_{RH}(\beta_2; P)\}
\]

This completes the proof of part (iii).

Proof of Proposition 3.4:

(a) By virtue of its definition (21), it suffices to show that \( \phi(F_R) \rightarrow \infty \) when \( ||\beta|| \rightarrow \infty \) with \( R = f(r(\beta)/S(F_y)) \). The latter, in light of given conditions, follows if we can show that \( |r(\beta)| \rightarrow \infty \) when \( |\beta_1| \rightarrow \infty \). Note that \( S(F_y) \) is a fixed positive number and \( |r(\beta)| \geq |\beta_1| - |y - \beta_2'x| \geq |\beta_1| - |\beta_2'x| - |y| \geq |\beta_1| - |y| - \|\beta_2\||x| \rightarrow \infty \) with probability one. This implies that \( |R| \rightarrow \infty \) by virtue of the given condition on \( f \), which in turn implies that \( \phi(F_R) \rightarrow \infty \).

(b) \( D_C(\beta, P) \) satisfies \( \textbf{P4} \) follows directly from the Proposition 2.3.

(c) \( RD_{RH}(\beta; P) \) satisfies \( \textbf{P4} \) has been proved in (c) of the proof of Proposition 3.2.

(d) This part was given in (ii) of Lemma 5.1.

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References


[86] Zuo, Y. (2010), “Is $t$ procedure $\bar{x} \pm t_\alpha (n - 1)s/\sqrt{n}$ optimal?”, The American Statistician, 64(2), 170-173.


