Abstract

Three new mixed-effects regressions models using an extended Weibull distribution are defined for repeated measures, and their parameters are estimated by maximum likelihood. Monte Carlo simulations report the accuracy of the maximum likelihood estimators and the distribution of the quantile residuals in these regressions. The usefulness of the proposed regressions is illustrated in differential and integral class from the Exact Sciences Department at the University of São Paulo (Brazil) with the objective of showing a pedagogical alternative of learning diagnostic methodology as a game approach. The results indicate that the questions correctly answered by the students took less time to be solved than those incorrectly answered. In addition, the algebraic application and multiple representation questions has the lowest percentages of correct answers and, in general, the longest time to be solved. So, it is possible to note that the used game approach enables the identification of possible difficult points in a class and provides the teacher with the opportunity of search for different strategies to reduce these difficulties faced by differential and integral calculus students when entering higher education, which often result from basic education.

Keywords: Learning diagnostic methodology; Monte Carlo simulations; New Weibull-G family; Problems in teaching calculus; Repeated measures.

1 Introduction

A number of different regression models introduced recently consider that the elements of the response variable are statistically independent. For example, Prataviera et al. (2019), Vila et al. (2020), Prataviera et al. (2020), Vasconcelos et al. (2021a) and Vasconcelos et al. (2021b) reported some regressions for independent observations.

Sometimes, the observations form some groups and thus the response variables within the groups may not be mutually independent. This is the case, for example, of elements in certain clusters, people from the same family or animals from the same litter. It may also happen that for a single individual, the same event is observed several times (repeated measures or recurrent events) and then the response variable for the same element may present a correlation. The elements of the response
variable are not statistically independent and present a group structure in the cases mentioned. So, the traditional regressions cannot be used directly.

There are many experimental contexts with repeated measures. In this area, a study was conducted of the insertion of learning games in a class on differential and integral calculus in the Department of Exact Sciences at the University of São Paulo campus located in Piracicaba. With the isolation and social distancing caused by the SARS-CoV-2 pandemic (UNA-SUS, 2020), various learning activities have been conducted remotely as a strategy to guarantee continuity of education at all school levels. In compliance with the guidelines for schools during the pandemic published on April 28, 2020 by the Brazil’s National Education Council (CNE) in collaboration with the Ministry of Education (MEC), teachers and students in Brazil had to follow the restrictions imposed by the pandemic. So, various previously existing problems that permeate education have become more intense. This is certainly the case of the discipline differential and integral calculus. High failure rates in this subject have been experienced in many universities in Brazil and other countries. This has prompted many educators and researchers to seek responses and possible solutions regarding the problems of teaching this discipline (Rasmussen et al., 2014; da Fonseca et al., 2018).

In this sense, diagnostic tests can be used as an auxiliary tool in the teaching and learning process of calculus subjects. Macedo and Gregor (2020) reported that this tool enables an intervention that can benefit students and teachers, as it allows the identification of possible points of difficulties faced by students, directing teachers to adopt alternative methodology. So, it is possible to improve the learning of Differential and Integral Calculus.

Mathematical games can be adopted as diagnostic tests and one platform for it is the Kahoot! (https://kahoot.it/). Prieto et al. (2019) mentioned that the use of this digital platform increases the students’ motivation, and enhancing learning significantly. The authors highlighted that it is possible to apply several questions to students instantly, and they have immediate feedback with correct answers in a more playful way. Rodríguez-Fernández (2017), Prieto et al. (2019) and Wang and Tahir (2020) considered that Kahoot! was a game for enabling the inclusion of mobile devices and gamification tools in the classroom. In addition, it is possible to include images, music, videos, and a time for students to answer the questions.

Various studies have shown that software and computational games combined with online resolution of problems can enhance students’ learning of mathematical concepts (Fokides, 2018; Dele-Ajayi et al., 2019; Yuxuan, 2021). Therefore, one objective of this article is to demonstrate the effectiveness of inserting learning games in the teaching of differential and integral calculus as a diagnostic tool to identify difficulty points faced by students. In particular, the response time of students to solve questions that address issues inserted in the context of function, limit and derivative. Besides this, an effort is made to verify the influence on the response time when considering different types of questions centered on the concept, algebraic manipulations and multiple representations (diagram or graphics) of the differential and integral calculus. Each student answered to six questions giving rise to data with repeated measures. More details are discussed in Section 5. To analyze these data, we construct three mixed-effects regressions models from a new extended Weibull distribution.

It is frequent the occurrence of data with different behaviors (symmetrical, asymmetrical, bimodal, heavy tails, etc). Thus, it is convenient to consider parametric families of distributions that are flexible to capture a wide variety of behaviors, include as special cases classical distributions, and produce more robust estimates. Currently, proposing new distributions is a very important line of research in different areas. Thus, this work proposes to work with three generalizations of the Weibull distribution.

On the other hand, there are many generalizations of the Weibull distribution in the literature. For example, Lee et al. (2007) pioneered the beta Weibull (BW) distribution, and Cordeiro and Castro (2011) defined the Kumaraswamy Weibull (KW) distribution. The Weibull distribution is a special case of these two models. We work with the New Weibull-Weibull (NWW) distribution (Tahir et al., 2016) which presents flexibility, and it can be competitive to the BW and KW distributions to model data in several areas. These distributions have been applied in survival analysis, reliability, engineering, among other fields. One of the main issues in this work is to compare empirically these three models. Nadarajah et al. (2012) introduced a regression based on the KW distribution to cope with independence among the elements of the response variable. In a similar manner, Cordeiro et
al. (2013) proposed the BW regression.

The main objectives of this paper is to extend these regressions to repeated measures by constructing three mixed-effects regressions models based on the NWW, BW and KW distributions, and also estimate their parameters highlighting an application in differential and integral calculus games.

The maximum likelihood method is adopted to estimate the parameters of these mixed-effects regressions models, and also carried out hypothesis tests based on the asymptotic distribution of the maximum likelihood estimators (MLEs). Further, to verify the model assumptions and detect outliers, some Monte Carlo simulations are done to identify the empirical distribution of the quantile residuals (qrs) for the three regressions. Confidence bands can be constructed from the generated envelope (Atkinson, 1985) as an additional measure of fit performance of these regressions.

The plan of the paper is as follows. Section 2 reviews the New Weibull-G (NW-G) family (Tahir et al., 2016) and the NWW distribution to model lifetime data from different areas. Section 3 defines the mixed-effects regressions models using the NWW, BW and KW distributions, and estimates their parameters. In Section 4, a Monte Carlo simulation study is conducted to examine the accuracy of the ML estimators. The importance of the regressions by means of a real application is addressed in Section 5. Finally, Section 6 offers some conclusions.

2 The NWW model

One of the main aims of modern distribution theory is to study new families to analyze real data in many fields, including engineering, reliability, management, medicine, actuarial science, medicine, biology, agriculture, and environmental sciences, among others. The addition of one or more parameter(s) using appropriate transformations in certain distribution functions has generated novel flexible families for modeling real data. The exponentiated-G (exp-G) (Gupta et al., 1998), Marshall-Olkin-G (Marshall and Olkin, 1997), gamma-G (Zografos and Balakrishnan, 2009), beta-G (Eugene et al., 1996), Kumaraswamy-G (Cordeiro and Castro, 2011) and, more recently, Weibull-G (Bourguignon et al., 2014) and NW-G (Tahir et al., 2016) are known classes of distributions, where the last four have two additional shape parameters.

The cumulative distribution function (cdf) of the NW-G family has the form

\[ F(y; a, b) = \exp \left\{ -a \left( -\log[G(y)] \right)^b \right\}, \quad y \in \mathbb{R}, \]  

(1)

where \( G(y) = G(y; \xi) \) is the parent cdf depending on a vector \( \xi \) omitted throughout the paper, and the positive parameters \( a \) and \( b \) are scale and shape, respectively.

Henceforth, \( Y \) refers to the cdf in Equation (1). Let \( W \) be a Weibull random variable with scale \( a \) and shape \( b \). The probability density function (pdf) of \( Y \) follows by differentiating \( P(Y \leq y) = P(W \geq -\log[G(y)]) \) as

\[ f(y; a, b) = \frac{abg(y)}{G(y)} \left\{ -\log[G(y)] \right\}^{b-1} \exp \left\{ -a \left( -\log[G(y)] \right)^b \right\}, \]  

(2)

where \( g(y) = dG(y)/dy \). The baseline \( G \) comes when \( a = b = 1 \).

Tahir et al. (2016) derived a linear representation for \( f(y; a, b) \) involving three infinite sums and two finite sums. Here, we provide a simpler representation for this pdf using Stirling polynomials with two infinite sums and one finite sum, and obtain some of its structural properties. The cdf and pdf of the Weibull distribution (with two parameters \( \lambda > 0 \) and \( c > 0 \)) are \( G_{\lambda, c}(y) = 1 - \exp[-(\lambda y)^c] \) and \( g_{\lambda, c}(y) = c\lambda^c y^{c-1} \exp[-(\lambda y)^c] \), respectively.

The NWW cdf follows by inserting \( G_{\lambda, c}(y) \) in Equation (1)

\[ F(y; \lambda, c, a, b) = \exp \left\{ -a \left[ -\log \left\{ 1 - \exp[-(\lambda y)^c] \right\} \right]^b \right\}, \quad y > 0. \]  

(3)
By inserting $G_{\lambda,c}(y)$ and $g_{\lambda,c}(y)$ in Equation (2) gives the NWW density function

$$f(y; \lambda, c, a, b) = \frac{abc \lambda^cy^{c-1} \exp[-(\lambda y)^c]}{1 - \exp[-(\lambda y)^c]} \left[ -\log \{1 - \exp[-(\lambda y)^c]\}\right]^{b-1} \times \exp \left\{ -a \left[-\log \{1 - \exp[-(\lambda y)^c]\}\right]^b \right\},$$

where all parameters are positive.

By inverting Equation (3), the quantile function (qf) of $Y$ can be expressed as

$$y = Q_Y(u) = Q_{\text{Weibull}} \left( \exp \left\{ - \left[ -a^{-1} \log(u) \right]^{1/b} \right\} \right), \quad u \in (0, 1),$$

where $Q_{\text{Weibull}}(u) = \lambda^{-1} \left[-\log(1-u)\right]^{1/c}$ is the Weibull qf.

Some forms of the NWW pdf and also histograms of one hundred thousand simulated observations of Equation 5 are reported in Appendix A.

The BW and KW distributions (with four positive parameters) and positive support are very popular models in applications from several areas such as survival analysis and reliability. Next, we present their pdfs in order to construct regressions with mixed-effects.

- **BW density function**

  It can be expressed as

  $$f(y) = c\lambda^c y^{c-1} \exp\{b(\lambda y)^c\} \{1 - \exp[-(\lambda y)^c]\}^{a-1},$$

  where $a$, $b$ and $c$ are shape parameters, $\lambda$ is a scale, and $B(a,b) = \Gamma(a+b)/[\Gamma(a)\Gamma(b)]$ is the beta function. General properties of the BW distribution can be found in Cordeiro et al. (2013).

- **KW density function**

  It has the form

  $$f(y) = abc \lambda^cy^{c-1} \exp\{- (\lambda y)^c\} \{1 - \exp[-(\lambda y)^c]\}^{a-1} \{1 - [1 - \exp\{- (\lambda y)^c\}]\}^{b-1},$$

  where $a$, $b$ and $c$ are shape parameters, and $\lambda$ is a scale. Nadarajah et al. (2012) obtained some of its properties.

In Appendix B, we provide some structural properties, estimation of the parameters, and simulations of the NWW model.

### 3 Three new mixed-effects regressions models

We propose three regressions with mixed-effects for repeated measures based on the NWW, BW and KW distributions. The sample is divided into $N$ groups multiplied by $n_i$ observations for the regressions with mixed-effects. Let $Y_{ij} = (Y_{i1}, \ldots, Y_{in_i})$ be the response variable for the $j$th observation in the $i$th group (for $j = 1, \ldots, n_i$ and $i = 1, \ldots, N$).

Consider that the response variable is related to fixed covariates $x_{ij} = (x_{ij1}, \ldots, x_{ijp})^T$, and that all individuals in the same group have common random effects (say $\gamma_i$). Assuming that the response from each individual has a common random effect, and that the random effects are unobserved random variables, the model for the parameter $\lambda_{ij}$ can be expressed as

$$\lambda_{ij} = \exp \left( x_{ij}^T \beta + \gamma_i \right),$$

where $\beta = (\beta_1, \ldots, \beta_p)^T$, and $\gamma_i$ denotes the random effects for the $i$th group. Each group from Equation (8) has a random effect $\gamma_i$ represented by independent and identically distributed random variables. The random effects are assumed independent of the response variable, $\gamma_i \sim N(0, \sigma^2_\gamma)$.
The mixed-effects BW regression model

\[ y_{ij} | \gamma_i \sim BW(a, b, c, \lambda_{ij}) \] with marginal pdf defined in Equation (6)

\[ f(y_{ij} | \gamma_i) = \frac{c \lambda_{ij}^c y_{ij}^{c-1} \exp\{-b(\lambda_{ij} y_{ij})^c\} \{1 - \exp\{-a(\lambda_{ij} y_{ij})^c\}\}^{a-1}}{B(a, b) y_{ij}^c} \]

3.1 Log-likelihood functions

The random variables \( Y_{ij} \) and \( \gamma_i \) are assumed independent. The \( i \)th group contribution to the marginal likelihood function can be expressed as

\[ \int \prod_{j=1}^{n_i} f(y_{ij} | \gamma_i) g(\gamma_i; \sigma_\gamma^2) \, d\gamma_i, \]  

where \( g(\gamma_i; \sigma_\gamma^2) \) is the normal density function of the random effects.

Further, if \( Y \) is a random variable with density function in Equation (9), the logarithm of the marginal likelihood function (10) has the form

\[ \ell_{NW-W}(\theta) = \sum_{i=1}^{N} \log \left\{ \prod_{j=1}^{n_i} \frac{a b c \lambda_{ij}^c y_{ij}^{c-1} \exp\{-b(\lambda_{ij} y_{ij})^c\} \{1 - \exp\{-a(\lambda_{ij} y_{ij})^c\}\}^{b-1}}{\sigma_b \sqrt{2\pi}} \right\} - \frac{\gamma_i^2}{2 \sigma_b^2} \, d\gamma_i \]

Analogously, the logarithms of the marginal likelihood functions for the mixed-effects BW regression model and mixed-effects KW regression mode are

\[ \ell_{KW}(\theta) = \sum_{i=1}^{N} \log \left\{ \prod_{j=1}^{n_i} \frac{a b c \lambda_{ij}^c y_{ij}^{c-1} \exp\{-b(\lambda_{ij} y_{ij})^c\} \{1 - \exp\{-a(\lambda_{ij} y_{ij})^c\}\}^{b-1}}{\sigma_b \sqrt{2\pi}} \right\} - \frac{\gamma_i^2}{2 \sigma_b^2} \, d\gamma_i \]
and

$$\ell_{KW}(\theta) = \sum_{i=1}^{N} \log \left\{ \int_{-\infty}^{+\infty} \prod_{j=1}^{n_i} a b c \lambda_i y_{ij}^{c-1} \exp\{-(\lambda_{ij} y_{ij})^c\}{1 - \exp[-(\lambda_{ij} y_{ij})^c]}^{a-1} \times \prod_{j=1}^{n_i} {1 - \exp[-(\lambda_{ij} y_{ij})^c]}^{b-1} \left[ \exp \left( \frac{-\gamma_i^2}{2\sigma_b^2} \right) \right] \frac{d\gamma_i}{\sigma_b \sqrt{2\pi}} \right\}$$

respectively.

The maximum likelihood estimates (MLEs) $\hat{\theta}$ of $\theta$ for the three regressions can be determined by maximizing the previous logarithms of the marginal likelihood functions. We use the \texttt{gamlss} (Stasinopoulos and Rigby, 2007) package in \texttt{R} software to calculate the estimate $\hat{\theta}$. Initial values for $\beta$, $\sigma_b^2$ and $c$ are taken from the fit of the Weibull regression model ($a = b = 1$).

4 Simulations for three mixed-effects regressions models

We present a simulation study to assess the accuracy of the MLEs for all regressions described above. Regarding the amount of observations, we consider three scenarios: ($n_i = 6, N = 20$), ($n_i = 20, N = 20$), ($n_i = 60, N = 20$). The datasets are generated from the NWW, BW and KW distributions under the configurations:

- $c = 1$, $a = 0.8$ and $b = 0.7$; and the parameters for $\lambda_{ij}$ are: $\beta_0 = 0.6$, $\beta_1 = 0.2$ and $\beta_2 = 0.1$;
- $\sigma_b = 0.3$;
- The systematic component is $\lambda_{ij} = \exp (\beta_0 + \beta_1 x_{ij1} + \beta_2 x_{ij2} + \gamma_i)$.

The response variable, the random effects and the explanatory variable are obtained from

- $y_{ij} | \gamma_i \sim NWW(a, b, c, \lambda_{ij})$; $y_{ij} | \gamma_i \sim BW(a, b, c, \lambda_{ij})$; $y_{ij} | \gamma_i \sim KW(a, b, c, \lambda_{ij})$;
- $\gamma_i \sim \text{Normal}(0, \sigma_b^2)$; $x_{ij1} \sim \text{Uniform}(0, 0.5)$; $x_{ij2} \sim \text{Bernoulli}(0.5)$.

Figures 1, 2 and 3 from the simulations of the fitted mixed-effects NWW, BW and KW regressions models indicate that the square root of mean squared errors ($\sqrt{MSEs}$) decay to zero when the sample size increases.
Figure 1: Boxplots from $\sqrt{MSE}$ for the mixed-effects NWW regression model with $n_i = 6, n_i = 20, n_i = 40, n_i = 60$: (a) $\beta_0$; (b) $\beta_1$; (c) $\beta_2$; (d) $c$; (e) $a$; (f) $b$; (g) $\sigma_b$.

Figure 2: Boxplots from $\sqrt{MSE}$ for the mixed-effects BW regression model with $n_i = 6, n_i = 20, n_i = 40, n_i = 60$: (a) $\beta_0$; (b) $\beta_1$; (c) $\beta_2$; (d) $c$; (e) $a$; (f) $b$; (g) $\sigma_b$. 
Figure 3: Boxplots from $\sqrt{MSE}$ for the mixed-effects KW regression model with $n_i = 6$, $n_i = 20$, $n_i = 40$, $n_i = 60$: (a) $\beta_0$; (b) $\beta_1$; (c) $\beta_2$; (d) $c$; (e) $a$; (f) $b$; (g) $\sigma_b$.

An extra scenario for the mixed-effects NWW regression model is also reported when $c = 3$ (Figure 4). Note that $\sqrt{MSE}$ decreases when $n$ increases.

Figure 4: Boxplots from $\sqrt{MSE}$ for the mixed-effects NWW regression model with $n_i = 6$, $n_i = 20$, $n_i = 40$, $n_i = 60$ and $c = 3$: (a) $\beta_0$; (b) $\beta_1$; (c) $\beta_2$; (d) $c$; (e) $a$; (f) $b$; (g) $\sigma_b$. 
5 Application in differential and integral calculus classes

We present data analysis for four mixed-effects regressions models. We compare the performance of the mixed-effects NWW, BW, KW, Weibull and Normal regressions models to analyze the response time (in seconds) for solving problems in differential and integral calculus utilizing the Kahoot platform (Kahoot!) available at the link: https://kahoot.it/. Kahoot! enables the creation of various types of games, so it can be used as a methodological tool. We devise a mathematical game using Kahoot! in the form of a quiz as part of the calculus curriculum in the first semester of 2021 for students majoring in Agronomic Engineering and Forest Engineering at the University of São Paulo (campus located in Piracicaba city). The sample consists of 32 students of both major programs, who are assigned to answer six questions related to the subjects: functions, limits and derivatives. The maximum time to solve each problem is 90 seconds. The questions have multiple-choice answers and are focused on exploring the concepts, algebraic techniques and multiple representations (diagrams or graphics) of the differential and integral calculus, which are combined according to the contents covered. This ludic activity is applied at two moments during the semester, the first on May 10, 2021, containing questions only on types of functions, and the second on July 20, 2021, on limits and derivatives of functions. At the end of each of these two quizzes, it is possible to create a report of the ingenuity of each student regarding each question. And knowing the gender of each student, it is identified the influence of the covariables, type of question, and right/wrong answers to the questions and the time spent by each student to answer the questions.

The current variables are:

- $y_{ij}$: time taken by the student to respond (seconds) the question;
- $x_{ij1}$: correct/wrong answers to the question (with levels: 1-correct and 0-wrong);
- $x_{ij2}$: gender with two levels: (0=F, 1=M);
- $x_{ij3}$: type of question and contents with six levels: (A=Definition in the context of function, B=Applications of functions, C=Multiple representation of functions, D= Definition in the context of limits and derivatives, E=Applications of the limits of functions, F= Multiple representation of the limits of functions),

where $j = 1, \ldots, 6$ (observations by individual) and $i = 1, \ldots, 32$ (number of individuals). Thus, the total number of observations is 192. For more details on the experiment applied to students, see Appendix B.

The questions applied in games developed on the Kahoot! platform, and characteristics about the levels of $x_{ij3}$ are reported in the supplementary material. The image referring to Figure 13 was obtained from Stewart (2013).

Table 1 reports the means and standard deviations (SDs) of the response times according to the type of question. In addition, the percentages of correct/wrong answers are also reported. The greatest average response times are associated to questions involving algebraic manipulations (questions B and E).

Note that the definition questions presented high percentages of correct answers. And the lowest percentages correspond to the multiple role representation questions. Figure 5 (a) shows the boxplot of the response time (seconds) according to the question type and contents classified as correct/wrong answers. It can be observed that the response time for students with questions answered correctly is less in comparison with students who answered wrong. On the other side, Figure 5(b) shows that, on average, the greatest times refer to application questions.
Table 1: Averages, SDs and percentages of correct/wrong answers to response times.

<table>
<thead>
<tr>
<th>Type of question and content</th>
<th>Average</th>
<th>SD</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct questions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Definition in the context of function (A)</td>
<td>12.95</td>
<td>4.15</td>
<td>65.62</td>
</tr>
<tr>
<td>Applications of functions (B)</td>
<td>24.37</td>
<td>15.09</td>
<td>62.50</td>
</tr>
<tr>
<td>Multiple representation of functions (C)</td>
<td>9.26</td>
<td>6.10</td>
<td>37.50</td>
</tr>
<tr>
<td>Definition in the context of limit and derivative (D)</td>
<td>8.01</td>
<td>4.86</td>
<td>87.50</td>
</tr>
<tr>
<td>Applications of the limits of functions (E)</td>
<td>41.62</td>
<td>22.50</td>
<td>56.25</td>
</tr>
<tr>
<td>Multiple representation of the limits of functions (F)</td>
<td>15.87</td>
<td>11.25</td>
<td>65.62</td>
</tr>
<tr>
<td>Wrong questions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Definition in the context of function (A)</td>
<td>32.48</td>
<td>18.21</td>
<td>34.38</td>
</tr>
<tr>
<td>Applications of functions (B)</td>
<td>33.52</td>
<td>20.16</td>
<td>37.50</td>
</tr>
<tr>
<td>Multiple representation of functions (C)</td>
<td>25.60</td>
<td>18.80</td>
<td>62.50</td>
</tr>
<tr>
<td>Definition in the context of limit and derivative (D)</td>
<td>17.50</td>
<td>6.06</td>
<td>12.50</td>
</tr>
<tr>
<td>Applications of the limits of functions (E)</td>
<td>53.32</td>
<td>30.66</td>
<td>43.75</td>
</tr>
<tr>
<td>Multiple representation of the limits of functions (F)</td>
<td>21.16</td>
<td>10.19</td>
<td>34.38</td>
</tr>
</tbody>
</table>

Figure 5: Boxplot (a) for \( x_{ij1} \) and Plot of observed profile (b) for \( x_{ij3} \).

The model for the parameter \( \lambda_{ij} \) is

\[
\lambda_{ij} = \exp \left( x_{ij}^T \beta + \gamma_i \right). \tag{12}
\]

The values of the Akaike and Bayesian information criteria (AIC and BIC respectively) in Table 2 reveal that the mixed-effects NWW regression model is the most adequate model among the four fitted regressions.

Table 2: AIC and BIC statistics.

<table>
<thead>
<tr>
<th>Model</th>
<th>NWW</th>
<th>KW</th>
<th>Weibull</th>
<th>BW</th>
<th>Normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIC</td>
<td>1457.9</td>
<td>1459.8</td>
<td>1460.2</td>
<td>1633.6</td>
<td>1611.148</td>
</tr>
<tr>
<td>BIC</td>
<td>1520.8</td>
<td>1522.2</td>
<td>1522.2</td>
<td>1685.4</td>
<td>1684.194</td>
</tr>
</tbody>
</table>
Figure 6(a) displays the index plot of the quantile residuals (qrs), and shows the adequacy of the mixed-effects NWW regression model. Figure 6(b) reports the plot of these residuals versus fitted values, and supports this regression. Also, the normal probability plot of the qrs with the simulated envelope (Atkinson, 1985) in Figure 6(c) leads to the same conclusion.

(a) (b) (c)

Figure 6: Residual analysis for the mixed-effects NWW regression model. (a) Plot of the qrs versus index. (b) Plot of the qrs versus fitted values. (c) Normal probability plot of the qrs with envelope.

Figure 7 shows the normal probability plot of the qrs with the simulated envelope. The mixed-effects Weibull and normal regression models are not adequate to explain these data because many points are outside the envelope.

(a) (b)

Figure 7: Normal probability plot of the qrs with envelope. (a) Mixed-effects Weibull regression model. (b) Mixed-effects Normal regression model.

Figure 8 reports the Q-Q plot of the mixed-effects NWW, Weibull and Normal regression models.
Figure 8: Q-Q plot. (a) Fitted mixed-effects NWW regression model. (b) Fitted mixed-effects KW regression model. (c) Fitted mixed-effects BW regression model. (d) Fitted mixed-effects Weibull regression model. (e) Fitted mixed-effects Normal regression model.

Table 3 reports the results from the fitted mixed-effects NWW regression model of the \( i \)th student on the response time. We provide some interpretations below. Note that some effects of the interaction between correct/wrong answers with the type of question and contents are significant, thus indicating that the student failing or succeeding in the question depends on the subject addressed. Note that the times spent by students who answered the application questions in the function context (B), the application questions in the context of function limit (E), and multiple representations of this function limit (F) differ significantly from the times spent by the students who correctly answered the multiple function representation questions (C). Figure 9 indicates that students who answered questions B, E and F correctly take longer to solve them than students who answer question C correctly. It is also noted that the students who do not get the answers right, in most cases, take longer to perform the solutions.

Note also that the students who get the definition questions right for both contents (A and D) and the multiple representations of function questions (C) do not show significant differences in terms of the spent time. These questions are solved in shorter times (Figure 9). Questions A and D have the highest percentages of correct answers, possibly indicating they have mastered the idea.
of functions, limits and derivatives of functions, but they make more mistakes on the questions in application and interpretation of graphs, thus indicating possible difficulties in resolving the issues.

The questions of multiple function representations and the application questions presented the lowest percentages of correct answers and, in general, the longest times (Figure 9). These results can indicate that the longer response time is related to lack of previous mastery of the basic concepts underpinning the differential and integral calculus. Macêdo and Gregor (2020) stated that the difficulties in mastering basic operations and fundamental mathematical contents can negatively influence the performance of new undergraduate students, since mastery of those aspects is fundamental to perform well in the subject of differential and integral calculus. Agustin and Agustin (2009) stressed that while students may understand the concepts of differential and integral calculus, their lack of mastery of the required algebraic manipulations has a negative influence on the development of the potential of students in this discipline. Other studies also have found this situation (Ferrer et al., 2017; Vandenbussche et al., 2018). Therefore, if these limitations are not corrected in the initial semester of exact sciences programs, this will hamper the learning of students throughout the undergraduate period.

We consider the transformation $g(c) = \log(c)$, where $g(\cdot)$ is a known one-to-one continuous and twice differentiable function.

Table 3: Findings from the fitted mixed-effects NWW regression model

<table>
<thead>
<tr>
<th>Effects</th>
<th>Parameter</th>
<th>MLE</th>
<th>SE</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>$\beta_0$</td>
<td>-3.443</td>
<td>0.106</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>Gender (M)</td>
<td>$\beta_1$</td>
<td>-0.125</td>
<td>0.087</td>
<td>0.152</td>
</tr>
<tr>
<td>correct/wrong answers to the question (Correct)</td>
<td>$\beta_2$</td>
<td>0.996</td>
<td>0.173</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>Type of question (A)</td>
<td>$\beta_{32}$</td>
<td>-0.235</td>
<td>0.203</td>
<td>0.248</td>
</tr>
<tr>
<td>Type of question (B)</td>
<td>$\beta_{33}$</td>
<td>-0.255</td>
<td>0.181</td>
<td>0.160</td>
</tr>
<tr>
<td>Type of question (D)</td>
<td>$\beta_{34}$</td>
<td>0.366</td>
<td>0.532</td>
<td>0.492</td>
</tr>
<tr>
<td>Type of question (E)</td>
<td>$\beta_{35}$</td>
<td>-0.687</td>
<td>0.196</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>Type of question (F)</td>
<td>$\beta_{36}$</td>
<td>0.176</td>
<td>0.263</td>
<td>0.503</td>
</tr>
<tr>
<td>Correct/wrong answers (Correct) $\times$ Type of question (A)</td>
<td>$\beta_{42}$</td>
<td>-0.147</td>
<td>0.334</td>
<td>0.659</td>
</tr>
<tr>
<td>Correct/wrong answers (Correct) $\times$ Type of question (B)</td>
<td>$\beta_{43}$</td>
<td>-0.685</td>
<td>0.264</td>
<td>0.010</td>
</tr>
<tr>
<td>Correct/wrong answers (Correct) $\times$ Type of question (D)</td>
<td>$\beta_{44}$</td>
<td>-0.196</td>
<td>0.558</td>
<td>0.725</td>
</tr>
<tr>
<td>Correct/wrong answers (Correct) $\times$ Type of question (E)</td>
<td>$\beta_{45}$</td>
<td>-0.819</td>
<td>0.280</td>
<td>0.003</td>
</tr>
<tr>
<td>Correct/wrong answers (Correct) $\times$ Type of question (F)</td>
<td>$\beta_{46}$</td>
<td>-0.680</td>
<td>0.317</td>
<td>0.033</td>
</tr>
<tr>
<td>$\log(c)$</td>
<td>$a$</td>
<td>0.406</td>
<td>0.053</td>
<td>0.925</td>
</tr>
<tr>
<td></td>
<td>$b$</td>
<td>1.451</td>
<td>0.077</td>
<td>0.100</td>
</tr>
</tbody>
</table>
We reviewed the New Weibull-Weibull (NWW) distribution and derived a novel linear representation for its density function. Based on this distribution, and the Beta Weibull (BW) and Kumaraswamy Weibull (KW), we proposed three mixed-effects regressions models to repeated measures using the \texttt{gamlss} package (Stasinopoulos and Rigby, 2007). The new regressions are important extensions to some known regressions. Further, the mixed-effects NWW regression model served as a good alternative for the analysis of repeated measures, i.e., when there is no independence between observations, and it can be more flexible than other regressions. The estimation was conducted by the maximum likelihood method. Some simulations indicated that the empirical distribution of the quantile residuals for the three mixed-effects regressions models were approximately normal distributed. In this sense, we fitted the proposed regressions to the first term student data of Agro-nomic Engineering and Forest Engineering courses. The answered to the questions correctly took less time to solve than those who answered incorrectly. In addition, we noticed that students have more difficulties in algebraic questions and the interpretation of graphs. However, they did a better job understanding limit and differential concepts. The results indicated that the probing test applied using Kahoot! was efficient to identify possible points of difficulty faced by students, thus allowing the teacher to search for different teaching strategies that can facilitate the learning of differential and integral calculus.

Acknowledgment
This work was supported by CAPES and CNPq, Brazil.

Appendix A: Implementation and plots of the NWW density function for some scenarios

The implementation of the NWW distribution in \texttt{gamlss} (Stasinopoulos and Rigby, 2007) package in R can be obtained from the GitHub link:

\url{https://github.com/juliocezarvasconcelos/New-Weibull-Weibull-distribution}

Some shapes of Equation (4) in Figure 10 reveal that the NWW density function is very flexible.
Figure 10: Plots of the NWW density function. (a) $c = 10$, $a = 0.8$, $b = 0.7$. (b) $\lambda = 1.5$, $a = 0.8$, $b = 0.7$. (c) $\lambda = 1.5$, $c = 5$, $b = 0.25$. (d) $\lambda = 1.5$, $c = 5$, $a = 0.5$.

Histograms of one hundred thousand simulated observations and density function plots of the NWW distribution in two scenarios are displayed in Figure 11, thus revealing that the simulated data are consistent with this distribution.

Figure 11: Histograms and plots of the NWW density function.

Appendix B: Structural properties, estimation and simulations for the NWW model

We determine expansions for some mathematical quantities of the NW-G family which can be implemented on computational platforms for analytical and numerical calculations. This representation has Stirling polynomials and only three sums, and it is simpler than that one with four sums given by Tahir et al. (2016). The estimation of the parameters of the NWW distribution by maximum likelihood and some simulations are also reported.

Linear representation
First, we derive a novel linear representation for the NW-G family density. The power series holds (Ward, 1934)

\[- \log(1 - t) \] = ∞ \sum_{i=0}^{\infty} \rho_i(p) t^i, \quad |t| < 1,

(13)

where

\[ \rho_0(p) = 1 \quad \text{and} \quad \rho_i(p) = p \psi_{i-1}(i + p - 1), \quad i \geq 1, \]

and \[ \psi_0(p) = 1/2, \psi_1(p) = (2+3p)/24, \psi_2(p) = (p + p^2) / 48 \]

and \[ \psi_3(p) = (-8 - 10p + \frac{15p^2 + 15p^3}{5760}, \]

etc, are the Stirling polynomials.

By using Equation (13) in Equation (1) gives

\[ F(y; a, b) = \exp \left\{ -a \sum_{i=0}^{\infty} \rho_i(b) [1 - G(y)]^{i+b} \right\}. \]

By expanding the binomial term, we have

\[ F(y; a, b) = \exp \left[ \sum_{j=0}^{\infty} \delta_j(a, b) G(y)^j \right], \]

where \[ \delta_j(a, b) = (-1)^{j+1} a \sum_{i=0}^{\infty} \rho_i(b) \binom{i+b}{j} \] (for \( j \geq 0 \)).

The formula for the exponential of a power series (Henrice, 1974) gives

\[ F(y; a, b) = \sum_{j=0}^{\infty} \omega_j(a, b) G(y)^j, \]

(14)

where \[ \omega_0(a, b) = \exp[\delta_0(a, b)] \] and (for \( j \geq 1 \))

\[ \omega_j(a, b) = \frac{1}{j} \sum_{m=1}^{j} m \delta_m(a, b) \omega_{j-m}(a, b). \]

From now on, let \( Z_c \sim \text{exp-G}(c) \) represent the exponentiated-G (exp-G) random variable with power parameter \( c > 0 \), \( \text{cdf} \ H_c(y) = G(y)^c \), and \( \text{pdf} \ \pi_c(y) = c G(y)^{c-1} g(y) \). Many exponentiated distributions were addressed by Tahir and Nadarajah (2015).

By differentiating Equation (14) and using the concept of the exp-G distribution, we obtain

\[ f(y; a, b) = \sum_{j=0}^{\infty} \omega_{j+1}(a, b) \pi_{j+1}(y). \]

(15)

Equation (15) represents a novel linear representation for the NW-G family density function. Thus, some properties of the NW-G distribution can be determined from those of the exp-G distribution.

**Moments and generating function**

The \( n \)th ordinary moment of \( Y \) follows from Equation (15) as

\[ \mu'_n = E(Y^n) = \sum_{j=0}^{\infty} \omega_{j+1}(a, b) E(Z_{j+1}^n) = \sum_{j=0}^{\infty} (j + 1) \omega_{j+1}(a, b) \tau_j^{(n)}, \]

where \[ \tau_j^{(n)} = \int_0^1 Q_G(u)^n u' du. \]

For the exp-Weibull distribution, Mudholkare et al. (1995) derived a finite sum for \( E(Z_{j+1}) \).
The $n$th incomplete moment of $Y$, say $m_n(y) = E(Y \mid Y < y)$, can be expressed as

$$m_n(y) = \sum_{j=0}^{\infty} \omega_{j+1}(a, b) \sum_{j=0}^{\infty} (j + 1) \omega_{j+1}(a, b) \int_0^{G(y)} Q_G(u)^n u^j du.$$ 

All moments of $Y$ can be found numerically.

The generating function (gf) $M(t) = E(e^{tX})$ of $Y$ follows from Equation (15)

$$M(t) = \sum_{j=0}^{\infty} \omega_{j+1}(a, b) M_j+1(t) = \sum_{j=0}^{\infty} (j + 1) \omega_{j+1}(a, b) \zeta_{j+1}(t),$$

where $M_j+1(t)$ is the gf of $Z_{j+1}$ and

$$\zeta_{j+1}(t) = \int_0^1 \exp[tQ_G(u)] u^j du.$$

### Estimation and simulations

The log-likelihood function for $\theta = (a, b, \lambda, c)\top$ from a random sample of size $n$ from the NWW$(a, b, \lambda, c)$ distribution has the form

$$\ell(\theta) = n \log(ab^c \lambda^c) + (c - 1) \sum_{i=1}^{n} \log(y_i) - \sum_{i=1}^{n} (\lambda y_i)^c - \sum_{i=1}^{n} \log[1 - \exp[-(\lambda y_i)^c]] +$$

$$(b - 1) \sum_{i=1}^{n} \log[-\log[1 - \exp[-(\lambda y_i)^c]]] - a \sum_{i=1}^{n} \log[-\log[1 - \exp[-(\lambda y_i)^c]]] +$$

$$b \sum_{i=1}^{n} \log[-\log[1 - \exp[-(\lambda y_i)^c]]].$$

We maximize Equation (17) in the gamlss package (Stasinopoulos and Rigby, 2007) available in the R programming language (R Core Team, 2021). The parameters are estimated by a backfitting procedure, alternating between the linear terms and the random effects. This results in the standard errors and $p$-values of the linear parameters being conditional on the final fitted random effects. The standard errors (and hence the corresponding $p$-values) for the linear parameters are obtained using the square root of the diagonal elements of the inverse of the (conditional) observed information matrix. One thousand Monte Carlo simulations are carried out in gamlss to examine the consistency of the MLEs for the NWW distribution.

A random sample is drawn from the NWW$(\lambda, c, a, b)$ model for sample size $n = 50, 100, 200$ and $300$, and the MLEs are calculated in each of these replications. We obtain NWW observations by generating $u \sim \text{U}(0, 1)$ from Equation (5) for fixed $\lambda = 1, c = 3, a = 0.7$ and $b = 0.6$ (based on Figure 11(a) in Appendix A). The average estimates (AEs), biases, and means squared errors (MSEs) from these simulations in Table 4 reveal that the estimates are accurate and consistent since their biases and MSEs converge to zero when $n$ increases.

### Table 4: Simulation results from the NWW distribution.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$n = 50$</th>
<th>$n = 100$</th>
<th>$n = 200$</th>
<th>$n = 300$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.885 -0.115 0.048</td>
<td>0.942 -0.058 0.032</td>
<td>0.942 -0.058 0.032</td>
<td></td>
</tr>
<tr>
<td>$c$</td>
<td>4.047 1.047 2.969</td>
<td>3.430 0.430 0.909</td>
<td>3.430 0.430 0.909</td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td>0.510 -0.190 0.097</td>
<td>0.603 -0.097 0.057</td>
<td>0.603 -0.097 0.057</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>0.719 0.119 0.050</td>
<td>0.667 0.067 0.022</td>
<td>0.667 0.067 0.022</td>
<td></td>
</tr>
<tr>
<td>$\lambda$</td>
<td>1.001 0.001 0.017</td>
<td>1.025 0.025 0.015</td>
<td>1.025 0.025 0.015</td>
<td></td>
</tr>
<tr>
<td>$c$</td>
<td>3.051 0.051 0.252</td>
<td>2.937 -0.063 0.153</td>
<td>2.937 -0.063 0.153</td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td>0.691 -0.009 0.029</td>
<td>0.722 0.022 0.024</td>
<td>0.722 0.022 0.024</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>0.626 0.026 0.008</td>
<td>0.613 0.013 0.005</td>
<td>0.613 0.013 0.005</td>
<td></td>
</tr>
</tbody>
</table>
Appendix C: Some final remarks regarding Section 5

- **Definition in the context of function**
  The graph of an even function is symmetric about the $y$ axis.

  (a) True  (b) False

- **Applications of functions**
  Determine the value of $m$ so that the affine function $f(x) = (m - 1)x + 2$ is increasing.

  (a) $\{S \in \mathbb{R}| m < 1\}$  (b) $\{S \in \mathbb{R}| m > 1\}$
  (c) $\{S = m \in \mathbb{R}| m = 1\}$  (d) $\{S = m \in \mathbb{R}| m \neq 1\}$

- **Multiple representations of functions**
  The diagram shown below is characteristic of a function:

  (a) Injection  (b) Surjection
  (c) Bijection  (d) None of the alternatives

  ![Figure 12: Diagram of a surjective function considered in the Kahoot! game.](image)

- **Definition in the context of limits and derivatives**
  The derivatives are interpreted as slopes and rates of change.

  (a) True  (b) False

- **Applications of the limits of functions**
  Compute:

  $$\lim_{x \to 0} \frac{\sin(x)}{x}$$

  (a) 1  (b) $e$
  (c) 0  (d) $\infty$

- **Multiple representation of the limits of functions**
  For the function $f(x)$, whose graph is shown below, the value of $\lim_{x \to 1} f(x)$ is:

  (a) $e$  (b) 2
  (c) 1  (d) It does not exist

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The game created on the Kahoot! platform allows to include different resources (image or music), which makes it possible to create a more fun and interactive virtual environment. Students used mobile devices (smartphones, tablets or notebooks) to access the game. The questions and answers were viewed by each student as illustrated in Figure 14.

References


