Geometric Generated Family of Distributions: A Review

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Abstract. The present article represents a review of the geometric generated family of distributions. Based on this family of distribution, several distributions are proposed. The family can be proposed by using the compounding concept of zero truncated geometric distribution with any other model or family of distributions. Here, we provide a complete survey on this family of distributions and also listed the contributory related research work, their sub-models, hazard rates, and utilized real datasets. We also address 10 power series distributions, 60 distributions based on the geometric family of distribution. These numbers show the importance of the geometric family of distribution.

1 Introduction

In statistical literature, we suppose that possibly all the real phenomenons are generated by some lifetime models. If we know the model, we can completely specify our problem or phenomenon. For this purpose, several lifetime models have been developed. Geometric distribution is one of the famous model that also provide a family of distribution. By utilizing that family, several other lifetime models have been proposed and studied their properties by several authors.

The geometric generated family (GGF) of distribution can be constructed by using the concept of compounding. By compounding method, there are two different ways are available, one is by using power series distribution and other is by using zero truncated geometric distribution directly with other lifetime model. In this paper, we provides detailed study of both approaches for constructing GGF. The main idea of compounding is that lifetime of a system with $N$ (a discrete random variable) components and the failure time of $i^{th}$ component $T_i$ (say) follow some continuous lifetime distribution, then the maximum or minimum time of components of the system depending

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on the condition whether they are in parallel or series respectively. One of
the base paper that in-light in the field of compounding was proposed by

The main focus of the paper are two fold; first, we provide an updated
list of geometric generated family of distribution that will helpful for re-
searchers in the field of distribution theory. Secondly, it provide details of
real datasets used in different GGF models. There are lot of models available
in literature which are useful in a specific situation, but it may possible that
due to insufficient knowledge, one may use wrong model for conclusion to a
particular problem. Review papers help to tackle such situations and this
article can also be one of the contribution in this area; one may see Tahir
and Nadarajah (2015) and Tahir and Cordeiro (2016) for review on lifetime
models. Actually, this paper aims to discuss and promote to use of the GGF
models in various situations of real life.

1.1 Genesis for the geometric generated family of distribution
Let \( N \) be a random variable denoting the number of complementary risk
related to the occurrence of an event of interest. Further, we assume that \( N \)
has a zero truncated geometric distribution with probability mass function
given by

\[
P[N = n] = (1 - \theta)\theta^{n-1}; \quad n = 1, 2, \ldots ; 0 < \theta < 1.
\]

Now, let \( T_i (i = 1, 2, \cdots , N) \) denotes the time to failure of an event due
to \( i^{th} \) system, independent of \( N \), follow some distribution having prob-
ability density function (PDF) \( g(t) \) and cumulative distribution function
(CDF) \( G(t) \). Now, let the maximum time of system is denoted by \( X = \max(T_1, T_2, \cdots , T_N) \). Then the conditional distribution of \( X \) given \( N = n \)
is \( f(x|N = n) = ng(x)[G(x)]^{n-1} \) and the marginal distribution of \( X \) is,

\[
f(x) = (1 - \theta) \sum_{n=1}^{\infty} ng(x)[\theta G(x)]^{n-1} = \frac{(1 - \theta)g(x)}{[1 - \theta G(x)]^2} \quad \text{(1.1)}
\]

and the corresponding CDF is,

\[
F(x) = \frac{(1 - \theta)G(x)}{1 - \theta G(x)}; \quad x > 0, \quad 0 < \theta < 1. \quad \text{(1.2)}
\]

Now, if we do the same for \( X = \min(T_1, T_2, \cdots , T_N) \). Then the marginal
distribution of \( X \) is,

\[
f(x) = \frac{(1 - \theta)g(x)}{[1 - \theta(1 - G(x))]^2}; \quad x > 0, \quad 0 < \theta < 1. \quad \text{(1.3)}
\]
and the corresponding CDF is,
\[ F(x) = \frac{G(x)}{1 - \theta(1 - G(x))}; \quad x > 0, \quad 0 < \theta < 1. \] (1.4)

Then the random variable \( X \) is said to follow geometric generated family (GGF) of distribution if its CDF is given as in equation (1.2) or (1.4). These two equations can also be converted to each other by considering the parameter \( \theta = -\theta(1 - \theta)^{-1} \) in other. One point is also noticeable that, by using geometric series, both the equations (1.2) and (1.4) can be represented as an infinite linear combination of the baseline distribution, so it holds a number of mathematical properties as moments, percentiles, moment generating function, factorial moments, and others of its baseline distribution.

The rest of the paper is organised as follows; Section 2, deals with power series distribution method for generating geometric family of distribution. Section 3, deals with compounding technique for generating geometric family of distributions, this section is divided into two subsections, the first Subsection 3.1, gives the details of geometric family of distribution for parallel components or in case of maximum, while other Subsection 3.2, gives the details for series components or in case of minimum. The conclusion of the whole paper is summarized in Section 4.

2 Power series class of distribution

One of the important methods for constructing GGF is using power series distribution (PSD). The basic idea of PSD is that, let \( X \) be a random variable having some distribution function \( G(x) \). Further, let for given \( N; X_1, X_2, \ldots, X_N \) be independent and identical (iid) random variables having the same distribution function. Here, the number of iid random variables is suppose to be discrete random variable following zero truncated power series distribution, with PMF given as
\[ P(N = n) = \frac{a_n \theta^n}{C(\theta)}; \quad \theta > 0, n = 1, 2, \ldots \]
where \( a_n > 0 \), depends on \( n \) only and \( C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n \), such that \( C(\theta) \) is finite. Now, if we consider the distribution of maximum i.e. \( X_{(n)} = X = \max_{1 \leq i \leq N} X_i \) then the conditional distribution of \( X_{(n)} \mid N = n \) is \( G^n(x) \) and in this way the marginal distribution of \( X_{(n)} \) is
\[ F(x) = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} G^n(x) = \frac{C(\theta G(x))}{C(\theta)}. \] (2.1)
Similarly, if we consider the marginal distribution of minimum i.e. $X_{(1)} = \min_{1 \leq i \leq N} X_i$, then its CDF will be

$$F(x) = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} (1 - [1 - G(x)]^n) = 1 - \frac{C(\theta)(1 - G(x))}{C(\theta)}.$$  \hspace{1cm} (2.2)

Now, for the different baseline $G(x)$ and distributional nature of iid random variable (whether taken as minimum or maximum), different PSD can be generated. For particular choice of $a_n$ and $C(\theta)$, one can generate different PSD. For example, if we consider $a_n = 1$ and $C(\theta) = \theta(1 - \theta)^{-1}; \hspace{0.1cm} 0 < \theta < 1$, one can generate geometric power series distribution.

Here we have considered only those PSD which proposed GGF of distribution. The general form of the PSDs are summarized in a table. Some of the work related to GGFs of distribution are summarised below:

Chahkandi and Ganjali (2009) proposed exponential power series (EPS) class of distribution by considering the exponential model as a baseline by the method discussed in equation (2.2). After that, Morais and Barreto-Souza (2011) proposed Weibull power series (WPS) family of distribution by considering Weibull distribution in equation (2.2). These are two basic PSDs which are later generalized and extended by other authors.

Mahmoudi and Jafari (2012) generalized the exponential power series family of distribution and proposed generalized exponential power series (GEPS) class of distribution by considering $G(x) = (1 - e^{-\beta x})^\alpha$, as generalized exponential distribution in equations (2.1). They also proposed generalized exponential geometric distribution.

Mahmoudi and Shiran (2012) proposed exponentiated Weibull power series distribution, by using $G(x) = (1 - e^{-\beta x})^{\gamma}$, as exponentiated Weibull distribution by using the concept discussed in equations (2.1). They also proposed exponentiated Weibull geometric distribution. It reduces to Weibull PSD and generalization of all PSD discussed above.

Flores et al. (2013) proposed complementary exponential power series (CEPS) distribution by using PSD. The concept of CEPS is different from EPS as it utilizes the maximum concept instead of minimum, as given in equation (2.1).

Silva et al. (2013) proposed extended Weibull power series (EWPS) distribution by means of PSD by utilizing the similar concept as given in equation (2.2). They also proposed Pareto geometric distribution. Actually, EWPS generalizes the Weibull PSD and in this way extends the exponential PSD.

Silva and Cordeiro (2015) proposed Burr XII power series (B XII IPS) class of distribution by compounding the Burr type XII and power series distributions in equation (2.2). This family includes exponential and Weibull PSDs.
along with the Burr type XII distribution and its sub-models. They also proposed Burr XII geometric distribution.

Mahmoudi and Jafari (2017) proposed linear failure rate PSD (LFRPSD) by compounding linear failure rate distribution with power series distribution when the components are in series i.e., using the minimum concept as given in equation (2.2). They also proposed linear failure rate-geometric distribution. This PSD converted to the model proposed by Kuş (2007).

Elbatal et al. (2019) proposed generalized Burr XII PSD by compounding generalized Burr XII distribution with power series distribution by utilizing the concept as given in equation (2.1), which is generalization of Silva and Cordeiro (2015). Based on this concept, they proposed generalized Burr XII geometric distribution. They also proposed the above PSD for minimum case as given in equation (2.2).

Hassan and Assar (2019) proposed power function PSD by compounding power function distribution with power series distribution by utilizing concept as given in equation (2.1). The power function distribution is inverse of Pareto distribution. They showed that this PSD generates 12 different lifetime distributions and proposed power function geometric distribution.

The complete list of these PSDs are mentioned in Table 1, which shows the paper with the corresponding author(s). This table also discussed the general form of CDF of the related PSDs. Here the range of the parameters incorporated in CDFs are $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $x > k > 0$.

<table>
<thead>
<tr>
<th>No.</th>
<th>Proposed PSD</th>
<th>Author(s)</th>
<th>CDF $F(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Exponential</td>
<td>Chahkandi and Ganjali (2009)</td>
<td>$1 - \frac{C(\theta e^{-\beta x})}{C(\theta)\alpha}$</td>
</tr>
<tr>
<td>2</td>
<td>Weibull</td>
<td>Morais and Barreto-Souza (2011)</td>
<td>$1 - \frac{C(\theta e^{-\beta x^\alpha})}{C(\theta)\beta}$</td>
</tr>
<tr>
<td>3</td>
<td>Generalized exponential</td>
<td>Mahmoudi and Jafari (2012)</td>
<td>$\frac{C(\theta(1-e^{-\beta x^\alpha}))}{C(\theta)}$</td>
</tr>
<tr>
<td>4</td>
<td>Exponentiated exponential</td>
<td>Mahmoudi and Shiran (2012)</td>
<td>$\frac{C(\theta[1-e^{-\beta x^\alpha}])}{C(\theta)}$</td>
</tr>
<tr>
<td>5</td>
<td>Exponential</td>
<td>Flores et al. (2013)</td>
<td>$\frac{C(\theta(1-x^k))}{C(\theta)}$</td>
</tr>
<tr>
<td>6</td>
<td>Extended Weibull</td>
<td>Silva et al. (2013)</td>
<td>$1 - \frac{1}{C(\theta)}$</td>
</tr>
<tr>
<td>7</td>
<td>Burr XII</td>
<td>Silva and Cordeiro (2015)</td>
<td>$1 - \frac{C(1+x^\alpha)^{-\beta}}{C(\theta)}$</td>
</tr>
<tr>
<td>8</td>
<td>Linear failure rate</td>
<td>Mahmoudi and Jafari (2017)</td>
<td>$1 - \frac{C(\theta[e^{-\beta x^\alpha}])}{C(\theta)}$</td>
</tr>
<tr>
<td>9</td>
<td>Generalized Burr XII</td>
<td>Elbatal et al. (2019)</td>
<td>$\frac{C(\theta(1+(1+x^\alpha)^{-\beta}))}{C(\theta)}$</td>
</tr>
<tr>
<td>10</td>
<td>Power function</td>
<td>Hassan and Assar (2019)</td>
<td>$\frac{C(\theta x^\alpha e^{-\beta x})}{C(\theta)}$</td>
</tr>
</tbody>
</table>
3 Compound method

We follow two basic principles (the minimum and the maximum) used in series and parallel structure. The concept of generating a new lifetime model using compounding was started with Adamidis and Loukas (1998). This section is divided into two subsections according to condition whether the components of a system are in parallel or series, respectively. In these subsections, we will discuss the CDF of the GGF of distribution, the shape of its density, hazard rates, its sub-model(s), and utilized real dataset(s), respectively, whatever they discussed.

3.1 X is maximum

This condition occurs when the components of a system are in parallel. The method is already discussed in equation (1.2). In this subsection, we provide a summary when the maximum time of a system is considered.

Louzada et al. (2011) proposed exponential geometric (EG) distribution by compounding exponential distribution with geometric distribution. Later, Shahsanaei et al. (2012) proposed this distribution independently by using the concept of range distribution. Flores et al. (2013) proposed this model by means of PSD. The CDF of the model is

$$F(x) = \frac{(1 - \theta)(1 - e^{-\alpha x})}{1 - \theta(1 - e^{-\alpha x})}; \quad \alpha > 0.$$  

The shape of the PDF of the proposed model is either L-shaped or unimodal, and has increasing hazard rate (IHR). They considered two real datasets for model validation. The first dataset represents the lifetime of 23 ball bearings, shows the number of million revolutions before failure on an endurance test of deep-groove ball bearings (Lawless (2011)). Flores et al. (2013) also considered this dataset. The second dataset was reported by Perdona and Louzada (2006), shows serum-reversal time (in days) of 143 children contaminated with HIV by vertical transmission from the University Hospital of the Ribeiro Preto, School of Medicine.

Mahmoudi and Shiran (2012) proposed exponentiated Weibull geometric distribution by means of PSD. The CDF of the model is

$$F(x) = \frac{(1 - \theta)(1 - e^{-\alpha x^\beta})^\gamma}{1 - \theta(1 - e^{-\alpha x^\beta})^\gamma}; \quad \alpha, \beta, \gamma > 0.$$  

The shape of the PDF of the proposed model is either L-shaped or unimodal, and the hazard rates are increasing hazard rate (IHR), decreasing
hazard rate (DHR), bathtub (BT), and upside-down bathtub (UBT). The exponential-geometric, generalized exponential-geometric, Weibull-geometric, and exponentiated Rayleigh-geometric models are the special cases of the proposed model. In model validation, they had considered three real datasets. The first dataset shows the 67 fatigue life (rounded to the nearest thousand cycles) of Alloy T7987 that failed before having accumulated 300 thousand cycles of testing (Meeker and Escobar (2014)). The second dataset refers to 101 fatigue life of 6061-T6 aluminum coupons cut the direction of rolling and oscillated at 18 cycles per second with maximum stress per cycle 31,000 psi (Birnbaum and Saunders (1969)). The third dataset refers to fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed (Andrews and Herzberg (2012)).

Louzada et al. (2012) proposed long term exponential geometric distribution. The CDF of the distribution is

\[
F(x) = \frac{(1 - \theta)(1 - p)(1 - e^{-\alpha x})}{1 - \theta(1 - e^{-\alpha x})}; \quad \alpha > 0, 0 < p < 1.
\]

Actually, this model is based on the concept of long term survival on EG distribution. The long term survival concept (as the idea was given by Maller and Zhou (1996)), incorporate a probability \( p \) which represent that an individual belongs to the group out of risk individuals and the new survival function was given by the relation \( S(x) = p + (1 - p)S_0(x) \), where \( S_0(x) \) is old survival function. As the probability \( p \) goes to 0, it converted to the baseline model. The shape of the PDF of the proposed model is either L-shaped or uni-modal. It has IHR, DHR, and UBT type hazard rates. They had considered three real datasets for model validation. The first dataset shows 26 lifetimes of women with ovarian cancer (Maller and Zhou (1996)). The second dataset, shows 411 lifetimes of patients with glioma, which is a central nervous system tumor developed from glacial cells (http://portal.uni-freiburg.de/imbi/Royston-Sauerbrei-book) and the third dataset represent 507 times up to the default of clients credit score from a Brazilian bank.

Mahmoudi and Jafari (2012), Louzada et al. (2013) and Bidram et al. (2013b) proposed independently, generalized exponential-geometric distribution by compounding generalized exponential distribution with geometric distribution. The CDF of the model is

\[
F(x) = \frac{(1 - \theta)(1 - e^{-\alpha x})^\gamma}{1 - \theta(1 - e^{-\alpha x})^\gamma}; \quad \alpha > 0, \gamma > 0.
\]

The shape of the PDF of the proposed model is either L-shaped or uni-modal, according to the choice of the parameters, and incorporates IHR,
DHR, and BT type hazard rates. The generalized exponential and extended exponential geometric distributions are its sub-models. In model validation, Bidram et al. (2013b) considered two datasets. First one represents the 101 fatigue life of 6061-T6 aluminum coupons (Birnbaum and Saunders (1969)). This dataset was also taken by Mahmoudi and Jafari (2012). The second dataset represents 100 observations on breaking stress of carbon fibers measured in Gba (Nichols and Padgett (2006)). Whereas, Louzada et al. (2013) used three datasets. First dataset was serum-reversal time (Perdona and Louzada (2006)), second dataset shows 417 lifetimes (in hours) forty-watt, 110-volt internally frosted incandescent lamps (Davis (1952)). The last dataset refers to the survival times for laboratory mice, which were exposed to a fixed dose of radiation at the age of 5 to 6 weeks, reported by Hoel (1972). Mahmoudi and Jafari (2012) considered the second dataset shows the 67 fatigue life of Alloy T7987 data (Meeker and Escobar (2014)).

Tojeiro et al. (2014) proposed Weibull geometric (WG) distribution by means of compounding. The CDF of the model is

\[ F(x) = \frac{(1 - \theta)(1 - e^{-\alpha x^\beta})}{1 - \theta(1 - e^{-\alpha x^\beta})}; \quad \alpha > 0, \beta > 0. \]

The shape of the PDF of the proposed model is either L-shaped or unimodal, according to the choice of the parameters, and have IHR, DHR, and UBT type hazard rates. This model incorporates EG model as its sub-model. They considered three real datasets for model validations. The first dataset refers to 141 recovering addicts by the period 2000 to 2005, in treatment until relapse to drug use. The second dataset represents the 67 fatigue life of Alloy T7987 (Meeker and Escobar (2014)). And the last dataset was serum-reversal time data (Perdona and Louzada (2006)).

Afify et al. (2014) proposed transmuted Weibull geometric distribution by means of transmuting WG distribution. The CDF of the model is

\[ F(x) = (1 + \lambda) \left( \frac{(1 - \theta)(1 - e^{-\alpha x^\beta})}{1 - \theta(1 - e^{-\alpha x^\beta})} \right) - \lambda \left( \frac{(1 - \theta)(1 - e^{-\alpha x^\beta})}{1 - \theta(1 - e^{-\alpha x^\beta})} \right)^2; \quad \alpha > 0, \beta > 0, |\lambda| \leq 1. \]

The shape of the PDF of the proposed model is either L-shaped or unimodal having IHR, DHR, and UBT type hazard rates. There are eleven sub-models of the proposed one. They considered two real datasets to show the model applicability. The first dataset refers to survival in months of 20 acute myeloid leukemia patients (Pulsoni et al. (2004)). The second dataset refers to 100 breaking stress of carbon fibers (Nichols and Padgett (2006)).

Tahir and Cordeiro (2016) proposed general from of the CDF of four new family of distribution based on geometric distribution by method of
compounding namely; exponentiated generated geometric; exponentiated Kumaraswamy-geometric; McDonald-geometric and beta-geometric family of distributions. These family of distributions can easily obtainable from the equation (1.2) by just replacing baseline model $G(x)$ by $G^\alpha(x)$; $K^\alpha(x)$; $M(x)$ and $B(x)$ respectively. Here, $K(x)$ is CDF of Kumaraswamy family of distribution; $M(x)$ is CDF of McDonald family of distribution; $B(x)$ is CDF of beta family of distribution. There are scope for researchers to generate new lifetime models and studied their properties with real data applications.

Bidram and Nadarajah (2016) proposed exponentiated-exponential geometric distribution by rising an additional parameter in power form to exponential geometric distribution. The CDF of the distribution is

$$F(x) = \left(\frac{(1 - \theta)(1 - e^{-\alpha x})}{1 - \theta(1 - e^{-\alpha x})}\right)^\gamma; \quad \alpha, \gamma > 0.$$  

The shape of the PDF of the proposed model is either L-shaped or uni-modal and incorporates IHR, HDR, BT and UBT type of hazard rates. They considered a real dataset refers to the waiting times (in minutes) of 100 bank customers before service (Ghitany et al. (2008)), for model validation.

Saboor et al. (2016) proposed transmuted exponentiated Weibull geometric distribution by means of transmuting exponentiated Weibull geometric distribution. The CDF is given as

$$F(x) = \frac{(1 - \theta)(1 - e^{-\alpha x^\gamma})^\gamma}{1 - \theta(1 - e^{-\alpha x^\gamma})^\gamma}\left[1 + \lambda - \lambda \left(\frac{(1 - \theta)(1 - e^{-\alpha x^\gamma})^\gamma}{1 - \theta(1 - e^{-\alpha x^\gamma})^\gamma}\right)\right]; \quad \alpha, \beta, \gamma > 0, |\lambda| \leq 1.$$  

The proposed distribution is uni-modal, five parameter model having IHR, DHR, BT, and UBT type hazard rates. The exponentiated Weibull geometric distribution is a special case of this model. In model validation, they had used 100 observations of carbon fibers data (Nichols and Padgett (2006)).


$$F(x) = \frac{(1 - \theta)(1 - e^{-\alpha x - \frac{\beta}{2} x^2})^\gamma}{1 - \theta(1 - e^{-\alpha x - \frac{\beta}{2} x^2})^\gamma}; \quad \alpha, \beta, \gamma > 0.$$  

The proposed model is four parameter model and the shape of the PDF is either L-shaped or uni-modal. This model is very flexible in term of hazard rate as incorporating constant, IHR, DHR, BT and UBT hazard rates. The Linear failure rate geometric, Rayleigh geometric, generalized Rayleigh geometric, new generalized exponential-geometric and extended exponential-geometric distributions are sub-models of the proposed model. In model
validation, they had considered three real datasets. The first dataset represents the weights (in kg) of 110 students (http://www.statsci.org/data/oz/ms212.html). The second and third datasets show 102 and 101 fatigue life of 6061-T6 aluminium coupons cut the direction of rolling and oscillated at 18 cycles per second with maximum stress per cycle 26,000 psi and 31,000 psi respectively and reported by Birnbaum and Saunders (1969).

Alizadeh et al. (2017) proposed four parameter exponentiated power Lindley-geometric (EPLG) distribution by compounding EPL distribution with geometric distribution. The proposed model is

$$F(x) = \frac{(1 - \theta) \left(1 - (1 + \frac{\alpha x^\beta}{1+\alpha})e^{-\alpha x^\beta}\right)^\gamma}{1 - \theta \left(1 - (1 + \frac{\alpha x^\beta}{1+\alpha})e^{-\alpha x^\beta}\right)^\gamma}; \quad \alpha, \beta, \gamma > 0.$$  

The PDF plot of the model is either L-shaped or uni-modal and incorporates IHR, DHR, BT, and UBT type hazard rates. Various lifetime distributions are sub-models of the proposed model as exponentiated power Lindley, new generalized Lindley, generalized Lindley, power Lindley, and Lindley geometric distributions. In model validation, they considered two datasets. The first one represents 63 strength of 1.5 cm glass fibers (Smith and Naylor (1987)), and the second dataset shows the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli reported by Bjerkedal (1960).

Elbatal et al. (2019) proposed generalized Burr XII geometric distribution based on PSD. The CDF of the model is

$$F(x) = \frac{(1 - \theta)[1 - (1 + x^c)^{-k}]^\alpha}{1 - \theta[1 - (1 + x^c)^{-k}]^\alpha}; \quad c, k, \alpha > 0.$$  

The PDF plot of the proposed model is uni-modal and has DHR and UBT type hazard rates. The Burr XII geometric distribution is sub-model of it. In model validation section, they had considered three real datasets. The first dataset shows 20 failure times of mechanical components (Murthy et al. (2004)). The second dataset represents 72 survival times of guinea pigs (Bjerkedal (1960)) and the third dataset refers to life of fatigue fracture of Kevlar 373/epoxy data Andrews and Herzberg (2012).

Hassan and Assar (2019) proposed power function geometric distribution by means of PSD. The CDF of the distribution is

$$F(x) = \frac{(1 - \theta)(\frac{x}{k})^\alpha}{1 - \theta(\frac{x}{k})^\alpha}; \quad \alpha > 0, x > k > 0.$$  

The PDF plot of the proposed model is either uni-modal or L-shaped having IHR, DHR and BT type hazard rates. They had considered two real datasets
to show applicability of the model. The first dataset shows the failure times of 50 items (Aarset (1987)) and the second dataset shows failure times of 30 devices (Meeker and Escobar (2014)).

### Table 2 List of papers on geometric family of distribution (max)

<table>
<thead>
<tr>
<th>No.</th>
<th>Proposed Model</th>
<th>Author(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Exponential geometric</td>
<td>Louzada et al. (2011)</td>
</tr>
<tr>
<td>2</td>
<td>Exponential geometric</td>
<td>Shahsanaei et al. (2012)</td>
</tr>
<tr>
<td>3</td>
<td>Exponentiated Weibull geometric</td>
<td>Mahmoudi and Shiran (2012)</td>
</tr>
<tr>
<td>4</td>
<td>Long term exponential geometric</td>
<td>Louzada et al. (2012)</td>
</tr>
<tr>
<td>5</td>
<td>Generalized exponential-geometric</td>
<td>Mahmoudi and Jafari (2012)</td>
</tr>
<tr>
<td>6</td>
<td>Generalized exponential-geometric</td>
<td>Louzada et al. (2013)</td>
</tr>
<tr>
<td>7</td>
<td>Generalized exponential geometric</td>
<td>Bidram et al. (2013b)</td>
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<td>Exponential geometric</td>
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<td>Weibull geometric</td>
<td>Tojeiro et al. (2014)</td>
</tr>
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<td>Transmuted Weibull geometric</td>
<td>Afify et al. (2014)</td>
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<td>Exponentiated-exponential geometric</td>
<td>Bidram and Nadarajah (2016)</td>
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<td>12</td>
<td>Exponentiated generated geometric family</td>
<td>Tahir and Cordeiro (2016)</td>
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<td>13</td>
<td>Exponentiated Kumaraswamy-geometric family</td>
<td>Tahir and Cordeiro (2016)</td>
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<td>McDonald-geometric family</td>
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<td>17</td>
<td>GLFR Geometric</td>
<td>Harandi and Alamatsaz (2017)</td>
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<tr>
<td>18</td>
<td>Exponentiated power Lindley geometric</td>
<td>Alizadeh et al. (2017)</td>
</tr>
<tr>
<td>19</td>
<td>Generalized Burr XII geometric</td>
<td>Elbatal et al. (2019)</td>
</tr>
<tr>
<td>20</td>
<td>Power function geometric</td>
<td>Hassan and Assar (2019)</td>
</tr>
</tbody>
</table>

The complete list of geometric generated family of distribution proposed by utilizing the maximum concept is mentioned in Table 2, which shows the paper with the corresponding author(s).

### 3.2 X is minimum

Marshall and Olkin (1997) proposed the famous Marshall-Olkin (M-O) family of distribution, which is similar to the form of the geometric generated family of distribution given in equation (1.4). But, the basic difference between both the family of distribution is the parameter $\theta$. In the GGF distribution, the parameter $\theta$ is a probability and in this way always lies between 0 to 1 i.e., $0 < \theta < 1$, whereas, the M-O family of distribution is based on survival concept with an additional parameter $\theta$, which is a real parameter, that can take any positive value i.e., $\theta > 0$. The M-O family also holds hazard rate order relationship to its baseline model, i.e. if $r(x)$ and $h(x)$ are the hazard rates of M-O model and its baseline model respectively. Then $\forall x > 0$, if $0 \leq \theta \leq 1 \Rightarrow r(x) \geq h(x)$ and if $\theta > 1 \Rightarrow r(x) \leq h(x)$. Mathematically, the GGF of distribution can be obtained from the M-O family of...
distribution, but conceptually, both are different.

Castellares and Lemonte (2016) showed the genesis of the M-O family of distribution and also proposed the geometric family of distribution for both when taken $X$ as maximum (see equation (1.2)) and as in minimum (see equation (1.4)). They also proposed Kumaraswamy-geometric distribution and named it as M-O Kumaraswamy distribution having the CDF

$$F(x) = \frac{1 - (1 - x^\beta)^\alpha}{1 - \theta(1 - x^\beta)^\alpha}; \quad \theta, \beta, \alpha > 0.$$ 

This model is very flexible as its PDF is uni-modal, BT type, increasing, decreasing, or constant (like Beta distribution) and has IHR, DHR, BT, and UBT type hazard rates. In the model validation section, they considered a real dataset of observed percentages of children living in households with per capita income less than $75.50 during 1991 in 5,509 Brazilian municipal districts. They were extracted from the Atlas of Brazil Human Development database available at http://www.pnud.org.br.

Adamidis and Loukas (1998) proposed exponential geometric (EG) distribution by compounding exponential distribution with geometric distribution. The CDF of the model is

$$F(x) = 1 - e^{-\lambda x^\theta}; \quad \lambda > 0.$$ 

(3.1)

The EG model has an L-shaped PDF and DHR model. The exponential distribution is a sub-model of it. To show the model validity, they had taken two real datasets. The first dataset represents the number of successive failures of the air conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes (Proschan (1963)) and the second dataset shows 109 observations on the period between successive coal-mining disasters (Cox (1966)).

Later, Adamidis et al. (2005) extend EG distribution and proposed extension of exponential geometric (EEG) distribution. The form of the distribution was the same, but the range of parameter $\theta$ was different. The former used the compounding method with geometric distribution with probability $\theta$, so the range is defined as $0 < \theta < 1$. But, the later used modified extreme value distribution as a conditional distribution concept; thus the parameter $\theta$ become a real parameter that can take any positive value, that is $\theta > 0$. The PDF of the EEG model is either L-shaped or uni-modal and has PDF IHR, DHR, and constant hazard rate. The EG model is a sub-model of it. They used two real datasets to show the model utility. The First dataset shows 100 observations of tensile fatigue characteristics of a polyester/viscose yarn, taken by Quesenberry and Kent (1982), and the second dataset
show the 107 failure times for right rear brakes on D9G-66A caterpillar tractors, reported by Barlow and Campo (1975).

Silva et al. (2010) generalized EG model by rising power and proposed exponentiated-exponential geometric distribution having CDF

$$F(x) = \left[\frac{1 - e^{-\alpha x}}{1 - \theta e^{-\alpha x}}\right]^{\beta}; \quad \alpha, \beta > 0.$$  

The PDF plot of the model is either L-shaped or uni-modal and have IHR, DHR and UBT type hazard rates and it converted to EG distribution. They used a real dataset for model validation, proposed by Birnbaum and Saunders (1969) shows 101 observations of fatigue life of aluminium coupons.

Gómez-Déniz (2010) proposed discrete generalized geometric distribution, based on the method given in equation (1.4). The PMF is given as

$$p(x) = \frac{(1 - \theta)\lambda^x (1 - \lambda)}{(1 - \theta \lambda^{x+1}) (1 - \theta \lambda^x)}; \quad \lambda > 0, \quad x = 0, 1, 2, \ldots$$

The proposed model has IHR and DHR type hazard rates and geometric distribution is its sub-model. They have considered four different real datasets. The first dataset shows the accidents to 647 women working on H E Shells for five weeks. The second dataset shows the number of carious teeth among the four deciduous molars in a sample of 100 children aged 10 and 11 years and reported by Krishna and Pundir (2009). The third dataset shows the results of ten shots fired from a rifle at each of 100 targets. The first and third datasets were reported by Consul and Jain (1973). The fourth dataset shows the fish catch data and reported by Kemp (2008).

Ortega et al. (2011) proposed generalized gamma geometric (GGG) distribution by the method of compounding the generalized gamma distribution with the geometric distribution. The CDF of the proposed model is given as

$$F(x) = \frac{\gamma \left[ k, \left( \frac{x}{\alpha} \right)^\beta \right]}{1 - \theta \left[ 1 - \gamma \left[ k, \left( \frac{x}{\alpha} \right)^\beta \right] \right]}; \quad \alpha, \beta, k > 0,$$  

where $\gamma \left[ k, x \right] = \int_0^x u^{k-1} e^{-u} du$ is incomplete gamma function and $\gamma \left[ k, x \right] = \frac{\gamma \left[ k, x \right]}{\Gamma(k)}$, the incomplete gamma function ratio. The PDF plot of the model is either L-shaped or uni-modal and has IHR, DHR, BT, and UBT type hazard rates. The generalized gamma, exponential geometric, and Weibull-geometric distributions are the sub-models of the proposed model.

Barreto-Souza et al. (2011) proposed Weibull geometric (WG) model by compounding Weibull and geometric distributions, having CDF

$$F(x) = \frac{1 - e^{-(\beta x)^\alpha}}{1 - \theta e^{-(\beta x)^\alpha}}; \quad \alpha, \beta > 0.$$
The PDF plot of the model is either L-shaped or uni-modal and has IHR, DHR, and UBT type hazard rates. Weibull, exponential, and EG models are sub-models of it. They had considered a real dataset represent 67 fatigue life Alloy T7987 data reported by Meeker and Escobar (2014).

Zakerzadeh and Mahmoudi (2012) proposed Lindley geometric distribution by compounding Lindley distribution with geometric distribution. The CDF of proposed model is

\[ F(x) = \frac{1 - \left(1 + \frac{\beta x}{\beta + 1}\right)e^{-\beta x}}{1 - \theta(1 + \frac{\beta x}{\beta + 1})e^{-\beta x}}; \quad \beta > 0. \]

The PDF of the model is either L-shaped or uni-modal and have IHR, DHR and BT type hazard rates. In model validation, they had considered two datasets. The first dataset shows 100 waiting times bank customers data (Ghitany et al. (2008)). The second dataset shows 34 vinyl chloride data obtained from clean up gradient monitoring wells in mg/L (Bhaumik et al. (2009)).

Bidram (2012) and Nassar and Nada (2012) independently proposed four-parameter beta exponential–geometric distribution from the logit of a beta random variable, extends the EG distribution. The CDF of the proposed model is

\[ F(x) = \frac{1}{B(\alpha, \beta)} \int_{0}^{G(x)} u^{\alpha-1}(1 - u)^{\beta-1} du; \quad \alpha, \beta > 0 \quad (3.4) \]

where the baseline CDF \( G(x) \) is exponential-geometric distribution given in equation (3.1) and \( B(\alpha, \beta) \) is beta function. The PDF plot of the model is either L-shaped or uni-modal with IHR, DHR and UBT type hazard rates. It reduces to EG and generalized exponential geometric distributions as its special cases. In real data application, Bidram (2012) considered a real dataset of 101 observation of fatigue life of aluminium coupons (Birnbaum and Saunders (1969)). And Nassar and Nada (2012) considered three real datasets. The first dataset represent 213 times of successive failures of the air-conditioning system, reported by Proschan (1963). The second data set shows the short-and long-term outcomes of constraint induced movement therapy after stroke, reported by Dahl et al. (2008) and the third dataset shows 23 ball bearing data reported by Lawless (2011).

Silva et al. (2013) proposed modified Weibull geometric distribution by means of PSD. The CDF is given as

\[ F(x) = \frac{1 - e^{-\alpha x^\beta e^{\lambda x}}}{1 - \theta e^{-\alpha x^\beta e^{\lambda x}}}; \quad \alpha, \beta, \lambda > 0. \]
The PDF of the model is either uni-modal or L-shaped and has IHR, DHR, BT, and UBT type hazard rates. The proposed model incorporates the Weibull-geometric model. In model validation, they had considered a real dataset of 128 soil fertility influence and the characterization of the biological fixation of \( N_2 \) for the Dimorphandra wilsonii rizz growth (Fonseca and Franca (2007)).

Alkarni (2013) proposed the geometric family of distribution with the concept of compounding any distribution with geometric distribution and proposed the same form as given in equation (1.4). And for illustration purpose, proposed the CDF and survival function of exponential geometric, Weibull geometric, Rayleigh geometric, and Pareto geometric distributions.

Wang (2013) proposed exponentiated Lindley geometric (ELG) distribution by compounding the exponentiated Lindley distribution with the geometric distribution. The CDF of the proposed model is

\[
F(x) = \left(1 - \frac{\beta+1+\beta x}{\beta+1} e^{-\beta x}\right)^{\alpha \theta + (1 - \theta)} \left(1 - \frac{\beta+1+\beta x}{\beta+1} e^{-\beta x}\right)^{\alpha}; \quad \alpha, \beta > 0.
\]

The PDF of the model is either uni-modal or L-shaped and has IHR, DHR, BT, and UBT type hazard rates. The ELG distribution contains Lindley geometric distribution as special cases. It can also be viewed as a mixture of exponentiated Lindley distribution. They illustrated the applicability of the model by using a real dataset of relief times of 20 patients receiving an analgesic given by Gross and Clark (1975).

Nadarajah et al. (2013) proposed geometric exponential Poisson (GEP) distribution. The CDF is given by

\[
F(x) = \frac{e^{-\beta e^{-\lambda x}} - e^{-\beta}}{1 - e^{-\beta} - (1 - \theta)(1 - e^{-\beta e^{-\lambda x}})}; \quad \beta, \lambda > 0.
\]

The PDF of the model is either uni-modal or L-shaped and have IHR, DHR and UBT type hazard rates. It incorporate exponential geometric and exponential Poisson distributions as its sub-models. They considered a real dataset of adult numbers of Tribolium confusum (Eugene et al. (2002)) to show the applicability of the model.

Bidram et al. (2013a) and Cordeiro et al. (2013) proposed independently; five-parameter beta Weibull geometric distribution from the logit of a beta random variable, extends the WG distribution. The CDF of the proposed model is given in equation (3.4), where the baseline CDF \( G(x) \) is given in equation (3.3). The PDF of the proposed model is either L-shaped or uni-modal with IHR, DHR, BT and UBT type hazard rates. This distribution

\[
F(x) = \frac{e^{-\beta e^{-\lambda x}} - e^{-\beta}}{1 - e^{-\beta} - (1 - \theta)(1 - e^{-\beta e^{-\lambda x}})}; \quad \beta, \lambda > 0.
\]
contains special sub-model as the Weibull geometric and exponential geometric distributions. The flexibility and potentiality of the distribution are examined by Bidram et al. (2013a) by using two real datasets. The first dataset is given by Birnbaum and Saunders (1969) represents 102 fatigue life of aluminium coupons data at 26,000 psi stress per cycle. The second dataset consists failures of the air-conditioning system analysed by Proschan (1963).

Ashour and Wahed (2014) proposed Kumaraswamy beta Weibull-geometric distribution by means of Kumaraswamy beta family of distribution with Weibull-geometric distribution as baseline. The CDF of the model is

\[ F(x) = K \int_0^{1-e^{-(\lambda x)}} u^{a-1}(1-u)^{b-1} e^{-dt} du; \quad \lambda, a, b, c, d > 0, \]

where \( K^{-1} = \int_0^1 u^{a-1}(1-u)^{b-1} e^{-dt} du \) is a normalizing constant. The PDF plot of the model is uni-modal and have IHR, DHR and BT type hazard rates. This model reduced to beta Weibull geometric, beta exponential geometric, generalized exponential geometric, Weibull geometric, exponential geometric, beta exponential distributions for particular choice of the parameter values. There are scope for researcher to explain its real data application.

Nadarajah et al. (2014) proposed four-parameter generalized linear failure rate-geometric distribution by compounding generalized linear failure rate distribution with geometric distribution. The CDF of the proposed model is

\[ F(x) = \frac{(1 - e^{-\alpha x - \beta x^2})^\gamma}{1 - \theta e^{-\alpha x - \beta x^2})^\gamma}; \quad \alpha, \beta, \gamma > 0. \]

The PDF of the proposed model is either L-shaped or uni-modal and have different type hazard rates as IHR, DHR, BT and UBT. The EG, WG, exponential distributions are the sub-models of it. A real dataset consists of remission times (in months) of a random sample of 128 bladder cancer patients, reported by Lee and Wang (2003) had considered showing the utility of the model.

Rezaei et al. (2014) proposed extended EG proportional hazard distribution by using the proportional hazard concept and extent Adamidis et al. (2005). The CDF of the model is

\[ F(x) = 1 - \left( \frac{\theta e^{-\alpha x}}{1 - \theta e^{-\alpha x}} \right)^{\exp(Y/\beta)}; \quad \alpha > 0, \]

where \( Y \), the model matrix; is a known as explanatory variable vector and \( \beta \) represent its coefficients.
Chung and Kang (2014) have proposed three parameter exponentiated Weibull-geometric (EWG) distribution by compounding exponentiated Weibull distribution with geometric distribution. The CDF of proposed model is

$$F(x) = \frac{(1 - e^{-x})^\beta}{1 - \theta(1 - (1 - e^{-x})^\beta)}; \quad \alpha, \beta > 0.$$ (3.5)

The PDF of the proposed model is either L-shaped or uni-modal with IHR, DHR and BT shaped hazard rate. This model also incorporates various sub-models like EG, WG, generalized exponential-geometric, exponentiated Rayleigh-geometric and Rayleigh-geometric models. In case of model validation, they used the two real datasets. The first data set shows the plasma concentrations of indomethacin following intravenous injection reported in Kwan et al. (1976). The second dataset shows of 128 observations on phosphorus concentration in the leaves, studied by Fonseca and Franca (2007).

Louzada et al. (2014) and later, Ristić and Kundu (2016) independently, proposed three parameter exponentiated exponential geometric distribution by compounding exponentiated exponential model with geometric distribution. The CDF of proposed model is

$$F(x) = \frac{(1 - e^{-\alpha x})^\gamma}{1 - \theta(1 - (1 - e^{-\alpha x})^\gamma)}; \quad \alpha, \gamma > 0.$$ 

The difference between both models is that the tilt parameter $\theta > 0$ for the model proposed by Ristić and Kundu (2016). The PDF of the model is either L-shaped or uni-modal and accommodates IHR, DHR and UBT type hazard rates. In model validation purpose, Louzada et al. (2014) considered three datasets. The first dataset refers to the survival times for 65 breast cancer patients treated over the period 1929–1938 (Boag (1949)), second dataset shows the survival times for patients with bile duct cancer, which took part in a study to determine whether a combination of a radiation treatment (R0Rx) and the drug 5-fluorouracil prolonged survival (Fleming et al. (1980)) and the third dataset shows the failures for the air conditioning system, taken by Proschan (1963). While, Ristić and Kundu (2016) considered a real dataset shows total cholesterol contents of 23 persons measured after 4th week and 20th week of the start of medicine, taken by Davis (2002).

Merovci and Elbatal (2014a) proposed transmuted Lindley geometric distribution by means of transmuting the Lindley–geometric distribution. The CDF of the proposed model is

$$F(x) = \frac{1 - (1 + \frac{\beta x}{\lambda + 1}) e^{-\beta x}}{1 - \theta \left(1 + \frac{\beta x}{\lambda + 1}\right) e^{-\beta x}} \left[1 + \lambda - \lambda \left(\frac{1 - (1 + \frac{\beta x}{\lambda + 1}) e^{-\beta x}}{1 - \theta \left(1 + \frac{\beta x}{\lambda + 1}\right) e^{-\beta x}}\right)\right]; \quad \beta > 0, |\lambda| \leq 1.$$


The PDF of the model is uni-modal and have IHR, DHR and UBT type hazard rates. It incorporate Lindley geometric distribution as its sub-model. In real life application, they had used 100 waiting times bank customers data taken by Ghitany et al. (2008).

Merovci and Elbatal (2014b) proposed transmuted Weibull geometric distribution by means of transmuting Weibull geometric distribution. The CDF is given as

\[
F(x) = \frac{1 - e^{-(\beta x)^\alpha}}{1 - \theta e^{-(\beta x)^\alpha}} \left[ 1 + \lambda - \lambda \left( \frac{1 - e^{-(\beta x)^\alpha}}{1 - \theta e^{-(\beta x)^\alpha}} \right) \right]; \quad \alpha, \beta > 0, |\lambda| \leq 1
\]

where \( \lambda \) is the transmuted parameter. The PDF of the model is either uni-modal or L-shaped and have IHR, DHR type hazard rates. Weibull geometric distribution is its sub-model. To show the applicability of the model, they had considered a real dataset representing 213 number of successive failures for the air conditioning system, reported by Proshan (1963).

Nadarajah et al. (2015) proposed exponentiated generalized geometric (EGG) family of distribution by considering generalized CDF in equation (1.4) and proposed two EGG distributions. The first one is exponentiated Weibull geometric (EWG) distribution by compounding the Weibull distribution with geometric distribution similar as given in equation (3.5) with an additional scale parameter. The second is exponentiated log-logistic geometric (ELLG) distribution by compounding log-logistic distribution with geometric distribution. Later, Mendoza et al. (2016) also proposed the ELLG model. The CDF of the proposed model is

\[
F(x) = \frac{(1 + (x)^{\gamma})^{-\beta}^{-\gamma}}{1 - \theta(1 - (1 + (x)^{\beta}^{-\gamma})^{-\gamma})}; \quad \alpha, \beta, \gamma > 0.
\]

The shape of the PDF is either uni-modal or L-shaped and have IHR, DHR, BT and UBT type hazard rates. They also proposed two other distributions by taking logarithmic of random variables follows EWG and ELLG distributions; named log-exponentiated Weibull geometric (LEWG) and log-exponentiated log-logistic geometric (LELLG) distribution. For the illustrative purpose, they used three types of real datasets; namely 744 environmental contamination data, 280 mortality death data on the age of death (in years) of retired women with temporary disabilities. The last is electrical insulation data, all are reported by Hirose (1993). Mendoza et al. (2016) considered two real datasets, first was 744 environmental contamination data and second dataset shows 60 accelerated voltage life test reported by Lawless (2011).
Wang and Elbatal (2015) proposed modified Weibull geometric distribution by compounding modified Weibull distribution with geometric distribution. The Model is

\[ F(x) = \frac{1 - e^{(\alpha x + \gamma x^\beta)}}{1 - \theta e^{(\alpha x + \gamma x^\beta)}}; \quad \alpha, \beta, \gamma > 0. \]

The shape of the PDF is either uni-modal or L-shaped and have DHR, UBT and down-UBT (DUBT) type hazard rates. This shows that it is a very flexible model, specially in reliability and survival analysis. It reduces to Weibull, exponential geometric, linear failure rate and modified Weibull distributions. They had considered a real dataset on remission time of 128 bladder cancer patients, reported by Lee and Wang (2003).

Silva and Cordeiro (2015) proposed Burr XII geometric distribution by utilizing the Burr XII PSD. The CDF of the proposed model is

\[ F(x) = \frac{1 - (1 + x^c)^{-k}}{1 - \theta (1 + x^c)^{-k}}; \quad c, k > 0. \]

The shape of the PDF is either uni-modal or L-shaped, and the hazard rates are DHR and UBT type. Burr XII, Weibull geometric, exponential geometric, Weibull and exponential distributions are sub-models of the proposed model. In the case of real illustration of the model, they considered two real datasets. The first dataset was taken by Blundell et al. (1998) and represent 1519 observations of budget share for fuel expenditure of British households. They were drawn from the 1980-1982 British Family Expenditure Surveys. The second dataset was taken by Murthy et al. (2004), represent the failure times of 20 mechanical components.

Elbatal et al. (2016) proposed five parameter additive Weibull geometric distribution by compounding additive Weibull distribution and geometric distribution. The CDF of proposed model is

\[ F(x) = \frac{1 - e^{-(\gamma x^\alpha + \beta x^\lambda)}}{1 - \theta e^{-(\gamma x^\alpha + \beta x^\lambda)}}; \quad \alpha, \beta, \lambda, \gamma > 0. \]

The PDF of the proposed model is either L-shaped or uni-modal and have different type hazard rates as IHR, DHR and BT type hazard rate. There are 10 models as the sub-models of the proposed one. They considered two real datasets to show the applicability of the proposed model. First dataset represents failure times are reported in Murthy et al. (2004) and second dataset represents 50 item failure data reported by Aarset (1987).

Bordbar and Nematollahi (2016) proposed a modified exponential geometric distribution by compounding the modified exponential distribution
with geometric distribution. The CDF of the model is
\[ F(x) = \frac{1 - e^{-\alpha x}}{1 - (1 - p)e^{-\alpha x} - p(1 - \theta)e^{-\alpha x}}; \quad \alpha > 0, \gamma > 0, p > 0. \]
The PDF of the model is L-shaped and have IHR and DHR hazard rates. The EG distribution is sub-model of it. In real data application, they had used a real data reported by Maguire et al. (1952) which gives 28 time intervals between two deadly accidents in the mines of the Division no. 5 of Great Britain National Cole Board in 1950.

Gitifar et al. (2016) proposed firstly linear failure rate-geometric distribution and later, Mahmoudi and Jafari (2017) also proposed the same distribution by utilizing the concept of PSD. The CDF of the model is
\[ F(x) = 1 - e^{-\lambda x - \frac{\beta}{x^2}}; \quad \lambda, \beta > 0. \]
The PDF plot of the model is either L-shaped or uni-modal have IHR, DHR, BT, and UBT shaped hazard rates. Exponential geometric and Rayleigh geometric are the sub-models of this model. They had considered two real datasets for model validation. The first dataset represents the lifetime (in days) of 40 patients suffering from leukaemia, from one of the Ministry of Health Hospitals in Saudi Arabia (Abouammoh et al. (1994)). The Second dataset shows the 63 glass fibers’ strengths data (Smith and Naylor (1987)).

Elbatal et al. (2017) proposed beta generalized inverse Weibull geometric (BGIWG) distribution. They firstly defined the GIWG distribution by compounding the GIW distribution with geometric distribution. The CDF of the GIWG model is
\[ F(x) = e^{-\gamma(\alpha x)^{-\beta}}; \quad \alpha, \gamma, \beta > 0. \]
The CDF of BGIWG model can be obtained by putting GIWG model as baseline CDF in beta family of distribution given as in equation (3.4). The PDF of BGIWG model is uni-modal and have IHR, DHR, BT and UBT shaped hazard rate. They had considered a real dataset of 20 relief times patients data, reported by Gross and Clark (1975).

Fattah et al. (2017) proposed exponentiated transmuted Weibull geometric distribution by means of compounding the transmuted Weibull model with geometric distribution. The CDF of the proposed model is
\[ F(x) = \frac{1 - e^{-\lambda x - \frac{\beta}{x^2}}}{1 - \theta e^{-\lambda x - \frac{\beta}{x^2}}}; \quad \lambda, \beta > 0. \]
The PDF of the model is uni-modal have IHR, DHR, BT and UBT type hazard rate. There are 22 sub-models of the proposed model. In real data illustration they used two real datasets; first is 100 breaking stress of carbon fibres data, given by Nichols and Padgett (2006) and second is 128 bladder cancer patients data given by Lee and Wang (2003).

Nofal et al. (2017) proposed a new family of distribution by utilizing exponentiated CDF in a transmuted family of distribution and named it a generalized transmuted family of distribution. The general form of the model is given as

\[ F(x) = G^{\alpha}(x)[(1 + \lambda) - \lambda G^{\beta}(x)]; \quad \alpha, \beta > 0, |\lambda| \leq 1 \]

where \( G(x) \) is baseline CDF. This family converted to exponentiated exponential and transmuted family of distribution. Based on the transformation, taking Weibull geometric distribution as a baseline model proposed generalized transmuted Weibull geometric distribution. The shape of the PDF is either uni-modal or L-shaped. In model validation, they had considered two real datasets. The first dataset shows 1206 nicotine data taken from http://pw1.netcom.com/rdavis2/smoke.html, and the second dataset shows 128 bladder cancer patients, taken by Lee and Wang (2003).

Bortolini et al. (2017) extend GGG model and proposed extended generalized gamma geometric distribution by method of rising power on survival function. The CDF of the proposed model is

\[ F(x) = 1 - \left( 1 - \frac{\gamma_1[k, (\frac{x}{\alpha})^\beta]}{1 - \theta[1 - \gamma_1[k, (\frac{x}{\alpha})^\beta]]} \right)^\lambda; \quad \alpha, \lambda, \beta, k > 0. \]

where the notations are same as given in equation 3.2. There are nearly 31 sub-models of the proposed model. The PDF of the proposed model is either L-shaped or uni-modal and has IHR, DHR, BT, and UBT type hazard rates. In the real data section, they considered an electronic survey of 147 permanence time (in years) in Japan of the Brazilian immigrants.

Rahmouni and Orabi (2018) proposed exponential generalized truncated geometric distribution, by compounding the exponential with generalized truncated geometric distribution. The CDF of the model is

\[ F(x) = \left( \frac{1 - e^{-\lambda x}}{1 - e^{-\lambda x}} \right)^\alpha; \quad \lambda > 0, \alpha = 1, 2, \ldots, N. \]

The PDF of the proposed model is either L-shaped or uni-modal and have IHR and DHR type hazard rates. The exponential geometric, extended exponential geometric distributions are sub-models of it. In real data section,
they had considered two real datasets. The first dataset shows 107 failure
times of right rear breaks, reported by Barlow and Campo (1975). The sec-
ond dataset shows 101 observations of the tensile fatigue characteristics of
a polyester/viscose yarn, reported by Picciotto and Hersh (1972).

The complete list of geometric generated family of distribution proposed
by utilizing the minimum concept is mentioned in Table 3, which shows the
paper with the corresponding author(s).

Table 4, on page 25, represent the list of all datasets utilized in correspond-
ing models. The column “Data Description”, shows the brief details of
datasets; “Reference”, shows the corresponding reference of the dataset.
The last column “Model”, shows the corresponding model that utilized the
datasets; the “max” shows that the model is defined for maximum criterion
as given in equation (1.2) and “min” shows that the model is defined for
minimum criterion as given in equation (1.4). This table shows that there
are 48 datasets used by 54 authors in 62 models.

<table>
<thead>
<tr>
<th>No.</th>
<th>Proposed Model</th>
<th>Author(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Kumaraswamy-geometric</td>
<td>Marshall and Olkin (1997)</td>
</tr>
<tr>
<td>2</td>
<td>Exponential geometric</td>
<td>Adamidis and Loukas (1998)</td>
</tr>
<tr>
<td>3</td>
<td>Extension of exponential geometric</td>
<td>Adamidis et al. (2005)</td>
</tr>
<tr>
<td>4</td>
<td>Exponentiated exponential geometric</td>
<td>Silva et al. (2010)</td>
</tr>
<tr>
<td>5</td>
<td>Discrete generalized geometric</td>
<td>Gómez-Déniz (2010)</td>
</tr>
<tr>
<td>6</td>
<td>Generalized gamma geometric</td>
<td>Ortega et al. (2011)</td>
</tr>
<tr>
<td>7</td>
<td>Weibull geometric</td>
<td>Barreto-Souza et al. (2011)</td>
</tr>
<tr>
<td>8</td>
<td>Lindley geometric</td>
<td>Zakerzadeh and Mahmoudi (2012)</td>
</tr>
<tr>
<td>9</td>
<td>Beta exponential geometric</td>
<td>Bidram (2012)</td>
</tr>
<tr>
<td>10</td>
<td>Beta exponential geometric</td>
<td>Nassar and Nada (2012)</td>
</tr>
<tr>
<td>11</td>
<td>Modified Weibull geometric</td>
<td>Silva et al. (2013)</td>
</tr>
<tr>
<td>12</td>
<td>Exponentiated Lindley geometric</td>
<td>Wang (2013)</td>
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<td>Geometric exponential Poisson</td>
<td>Nadarajah et al. (2013)</td>
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<td>Beta Weibull geometric</td>
<td>Bidram et al. (2013a)</td>
</tr>
<tr>
<td>15</td>
<td>Beta Weibull geometric</td>
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<td>Nadarajah et al. (2014)</td>
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<td>EEG proportional hazard</td>
<td>Rezaei et al. (2014)</td>
</tr>
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<td>18</td>
<td>Exponentiated Weibull geometric</td>
<td>Chung and Kang (2014)</td>
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<td>19</td>
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<td>Ashour and Wahed (2014)</td>
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</tr>
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<td>Transmuted Lindley geometric</td>
<td>Merovci and Elbatal (2014a)</td>
</tr>
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<td>Transmuted Weibull geometric</td>
<td>Merovci and Elbatal (2014b)</td>
</tr>
<tr>
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<td>Burr XII geometric</td>
<td>Silva and Cordeiro (2015)</td>
</tr>
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<td>25</td>
<td>Exponentiated generalized geometric</td>
<td>Nadarajah et al. (2015)</td>
</tr>
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<td>26</td>
<td>Exponentiated Weibull geometric</td>
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<td>Exponentiated Weibull geometric</td>
<td>Nadarajah et al. (2015)</td>
</tr>
<tr>
<td>28</td>
<td>Exponentiated log-logistic geometric</td>
<td>Nadarajah et al. (2015)</td>
</tr>
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</table>
4 Conclusion

In the present article, we have given a literature review on the geometric generated family of distribution and its associated distributions. There are several lifetime models available in the literature which are related to this family. Most of the researchers are not much aware of it. This article gives a complete list of the geometric generated family of distribution year by year.

In this context, we have listed 10 power series distributions by 10 different authors, 20 lifetime distributions by 16 authors based on the maximum concept in which exponentiated exponential geometric distribution and exponential geometric distribution have been proposed twice by various authors. Similarly, there are 40 distributions listed by 35 authors based on the minimum concept, in which Weibull geometric, exponentiated Weibull geometric, beta Weibull geometric, linear failure rate geometric, modified Weibull geometric, exponentiated exponential geometric distribution have been proposed two times by different authors. This means that there are a total of 60 models proposed by 51 authors based on a GGF family of distribution.

We have also discussed the shape of the density and hazard rates, their suitable sub-models, and the description of utilized real datasets for all the considered models. The utilized datasets have also been tabulated with their brief descriptions. There are 49 different datasets used for 60 models based on the maximum and minimum criterion of a geometric family of distribution.

Now, if we count all the listed lifetime distributions based on a geometric family of distribution, this number reaches to 70, which shows the significant role and importance of the geometric family of distribution. Therefore, this article can be a good contribution to the study of lifetime distributions.
Table 4: Table for dataset modelled by geometric family of distributions

<table>
<thead>
<tr>
<th>No.</th>
<th>Data Description</th>
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<th>Model</th>
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<tbody>
<tr>
<td>1</td>
<td>23 lifetime of ball bearings data</td>
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<tr>
<td>2</td>
<td>143 serum-reversal time data</td>
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<td>3</td>
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<td>Maller and Zhou (1996)</td>
<td>Exponential geometric (max)</td>
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<tr>
<td>4</td>
<td>411 lifetimes of glioma patients data</td>
<td><a href="http://portal.uni-freiburg.de/imbi/Royston-Sauerbrei-book">http://portal.uni-freiburg.de/imbi/Royston-Sauerbrei-book</a></td>
<td>Weibull geometric (max)</td>
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<tr>
<td>5</td>
<td>507 times clients credit score data</td>
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<td>Long term exponential geometric (max)</td>
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<td>6</td>
<td>101 fatigue life of 6061-T6 aluminium coupons data (with maximum stress per cycle 31,000 psi)</td>
<td>Binbaum and Saunders (1969)</td>
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<td>213 successive air conditioning failures data</td>
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<th>Model</th>
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<td>survival times for laboratory mice</td>
<td>Hoel (1972)</td>
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References


