Modeling \( Z \)-valued time series based on new versions of the Skellam INGARCH model

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Abstract. Recently, there has been a growing interest in integer-valued time series models, including integer-valued autoregressive (INAR) models and integer-valued generalized autoregressive conditional heteroscedastic (INGARCH) models, but only a few of them can deal with data on the full set of integers, i.e., \( Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \). Although some attempts have been made to deal with \( Z \)-valued time series, these models do not provide enough flexibility in modeling some specific integers (e.g., 0, \( \pm 1 \)). A symmetric Skellam INGARCH(1,1) model was proposed in the literature, but it only considered zero-mean processes, which limits its application. We first extend the symmetric Skellam INGARCH model to an asymmetric version, which can deal with non-zero-mean processes. Then we propose a modified Skellam model which adopts a careful treatment on integers 0 and \( \pm 1 \) to satisfy a special feature of the data. Our models are easy-to-use and flexible. The maximum likelihood method is used to estimate unknown parameters and the log-likelihood ratio test statistic is provided for testing the asymmetric model against the modified one. Simulation studies are given to evaluate performances of the parametric estimation and log-likelihood ratio test. A real data example is also presented to demonstrate good performances of newly proposed models.

1 Introduction

Integer-valued time series are commonly encountered in many practical situations, such as the annual counts of major earthquakes in a particular region, the number of customers in a shopping mall, the price change (measured in cents) for a stock at each transaction time and so on. The observed data can be on \( \mathbb{N} \) or on \( \mathbb{Z} \). Many attempts have therefore been made to deal with integer-valued time series, but most of them are focused on time series of counts (i.e., \( \mathbb{N} \)-valued time series), see Weiß (2018) for a comprehensive summary on INAR models. As an alternative, the INGARCH model proposed...
by Ferland et al. (2006) and Fokianos et al. (2009) is also very popular, which is defined as follows

\[
\begin{align*}
X_t | \mathcal{F}_{t-1} & \sim \text{Poisson}(\lambda_t), \quad \forall t \in \mathbb{Z}, \\
\lambda_t & = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i} + \sum_{j=1}^{q} \beta_j \lambda_{t-j},
\end{align*}
\]

where \( \alpha_0 > 0, \alpha_i \geq 0, \beta_j \geq 0, \ i = 1, \ldots, p, \ j = 1, \ldots, q, \ p \geq 1, \ q \geq 0, \) and \( \mathcal{F}_{t-1} \) is the \( \sigma \)-field generated by \( \{X_{t-1}, X_{t-2}, \ldots\} \). Zhu (2011), Zhu (2012), Davis and Liu (2016), Silva and Barreto-Souza (2019) and Gonçalves et al. (2020) generalized the Poisson distribution assumption to the negative binomial, zero-inflated Poisson, exponential family, mixed Poisson and compound Poisson distributions, respectively. Neumann (2011) proved the absolute regularity for Poisson count process and Doukhan et al. (2012) showed the existence of moments. Christou and Fokianos (2014, 2015) studied the quasi maximum likelihood method based on negative binomial and non-linear mixed Poisson distributions. Tjøstheim (2016) and Fokianos (2016) gave excellent summaries about recent progress in this field.

Time series on \( \mathbb{Z} \) are also very frequent, for example, integer-valued time series with negative values, or we use the differencing operator to achieve stationarity in nonnegative integer-valued time series with a time trend or seasonality. For modeling \( \mathbb{Z} \)-valued time series data, Kim and Park (2008), Zhang et al. (2010), Chesneau and Kachour (2012) and Kachour and Truquet (2011) proposed some INAR models based on the signed thinning operator, while Kachour and Yao (2009) proposed an INAR model based on the rounding operator. In addition, Freeland (2010) and Andersson and Karlis (2014) introduced a new INAR model and studied the case where the innovation has Skellam (Poisson difference) distribution, which can allow negative values and negative correlations. Alzaid and Omair (2014) defined the Poisson difference INAR model based on the extended binomial thinning operator. Later, Nastic et al. (2016) and Barreto-Souza and Bourguignon (2015) studied INAR processes with symmetric discrete Laplace marginals and skew discrete Laplace marginals, respectively. Recently, Alomani et al. (2018) introduced an INGARCH(1,1) model based on the symmetric Skellam distribution, but they only considered the simplest case that the conditional expectation of process is equal to zero. So our first aim is to propose an INGARCH(1,1) model with the asymmetric Skellam distribution to capture volatility in non-zero-mean time series.

In some empirical data examples, however, above models in our current context cannot match the features for specific integers. This case corresponds to a zero-inflation phenomenon. Zero-inflation or excess zeros has been of interest in the literature. Karlis and Ntzoufras (2009) applied zero-inflated
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Poisson difference distribution to model football outcomes. Qi et al. (2019) introduced a stationary INAR(1) process with zero-and-one inflated Poisson innovations. But in some $Z$-valued time series models, $\pm 1$ counts also have special status as 0 counts. Chesneau and Kachour (2012) applied a signed INAR(1) model to 597 counts of the price changes $\{-2, -1, 0, 1, 2\}$ for an Australian firm, measured in cents at each transaction time during a day. Koopman et al. (2017) studied the discrete price changes of high-frequency financial data (the tick size accurate to $0.01$) at New York Stock Exchange with a careful treatment for small observations 0, $\pm 1$ by using a new dynamic modified Skellam model. They focused on modeling stochastic volatility based on the link function of signal process, which accommodates some salient features of intraday volatility. They also illuminated that their dynamic modified Skellam model performs better in terms of fit and forecasting power compared to other alternative modeling approaches. Nevertheless, there is no attempts to consider this modified Skellam distribution in the context of INGARCH models. So in this paper, we propose a modified Skellam INGARCH model to compensate for the over- or under-representation of specific integers (0, $\pm 1$), which can reasonably match the feature of the data.

The rest of the paper is organized as follows. In Section 2, we first briefly introduce the standard Skellam distributions (including symmetric and asymmetric forms) and their modified versions, then discuss some of their properties. In Section 3, we propose asymmetric and modified Skellam INGARCH(1,1) models. The maximum likelihood estimation procedure for unknown parameters of interest and the likelihood ratio test for testing the asymmetric model against the modified one are given in Section 4. Section 5 presents some simulation results and a real data example is given in Section 6. Section 7 concludes the whole paper with some comments for further work.

2 Skellam distribution

We start by presenting a short review of the standard Skellam distribution and the modified Skellam distribution, and then discuss some of their properties.

2.1 Standard Skellam distribution

The distribution of the difference of two independent Poisson random variables has been discussed by Skellam (1946). First, recall the definition of standard Skellam distribution. If $X \sim \text{Poisson}(\lambda_1)$, $Y \sim \text{Poisson}(\lambda_2)$, $X$ and
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Y are independent, then we say that \( Z = X - Y \) follows the Skellam distribution or the Poisson difference distribution with parameters \( \lambda_1 \) and \( \lambda_2 \), denoted as \( S(\lambda_1, \lambda_2) \). The probability mass function (pmf) of \( Z \) is given by

\[
P_S(Z = z) = f_S(z|\lambda_1, \lambda_2) = \exp\{-(\lambda_1 + \lambda_2)\} \left(\frac{\lambda_1}{\lambda_2}\right)^{z/2} I_{|z|}(2\sqrt{\lambda_1\lambda_2}),
\]

for all \( z \in \mathbb{Z} \), where \( I_r(x) \) is the modified Bessel function of order \( r \) and is defined by

\[
I_r(x) = \left(\frac{x}{2}\right)^r \sum_{k=0}^{\infty} \frac{(x^2/4)^k}{k!\Gamma(r+k+1)}.
\]

When \( \lambda_1 = \lambda_2 = \lambda \), (1) is simplified to symmetric Skellam (SS) distribution with pmf \( f_{SS}(z|\lambda) = e^{-2\lambda}I_{|z|}(2\lambda) \). When \( \lambda_1 \neq \lambda_2 \), we call (1) asymmetric Skellam (AS) distribution, denoted as \( AS(\lambda_1, \lambda_2) \). An interesting property is a “type” of symmetry given by \( f_S(z|\lambda_1, \lambda_2) = f_S(-z|\lambda_2, \lambda_1) \). The moment generating function (mgf) is

\[
M_S(t) = \mathbb{E}e^{tZ} = \exp\{-(\lambda_1 + \lambda_2) + \lambda_1 e^t + \lambda_2 e^{-t}\}.
\]

The expected value of the Skellam distribution is given by \( \mathbb{E}(Z) = \lambda_1 - \lambda_2 \), while the variance is \( \text{Var}(Z) = \lambda_1 + \lambda_2 \). In general, the odd cumulants are equal to \( \lambda_1 - \lambda_2 \), while the even cumulants are equal to \( \lambda_1 + \lambda_2 \). The skewness coefficient is \( (\lambda_1 - \lambda_2)/(\lambda_1 + \lambda_2)^{3/2} \), which means that the distribution is positive skewed if \( \lambda_1 > \lambda_2 \), negative skewed if \( \lambda_1 < \lambda_2 \) and symmetric for \( \lambda_1 = \lambda_2 \). The kurtosis coefficient is \( 3 + 1/(\lambda_1 + \lambda_2) \). Notice that as either \( \lambda_1 \) or \( \lambda_2 \) tends to infinity, the kurtosis coefficient tends to 3 and the skewness coefficient tends to zero in the context of constant difference \( \lambda_1 - \lambda_2 \).

For the Skellam distribution, Fisz (1953) proved that as \( \lambda_1 \to \infty \) and \( \lambda_2 \to \infty \), we have

\[
\frac{Z - (\lambda_1 - \lambda_2)}{\sqrt(\lambda_1 + \lambda_2)} \to N(0,1) \text{ in distribution}.
\]

A refinement of this limit is dealt with in Fisz (1955). Tables given in Fisz (1953) suggests that for fairly small Poisson parameters the normal approximation is rather good. Strackee and van der Gon (1962) presented tables of the cumulative distribution function of the Skellam distribution to four decimal places for some combinations of values of the two parameters. Their tables also show the differences between the normal approximations. For more properties and applications of the Skellam distribution, see Karlis and Ntzoufras (2009), Alzaid and Omair (2010) and Freeland (2010).
2.2 Modified Skellam distribution

Following the idea of Koopman et al. (2017), we introduce the modified Skellam (MS) distribution to obtain a more flexible modeling framework. A random variable \( Z \) is said to follow an MS(\( \gamma, \lambda_1, \lambda_2 \)) distribution with pmf as

\[
P_{MS}(Z = z) = f_{MS}(z|\lambda_1, \lambda_2) = \begin{cases} 
P_S(Z = z), & z \in \mathbb{Z} \setminus \{0, \pm 1\}, \\ 
P_S(Z = z) - \frac{1}{2}\gamma\Delta, & z = -1 \text{ or } 1, \\ 
P_S(Z = z) + \gamma\Delta, & z = 0, \end{cases} \tag{2}
\]

where \( \Delta = P_S(Z = 0) - \min\{P_S(Z = -1), P_S(Z = 1)\} > 0 \) and \( P_S(Z = q) = f_S(q|\lambda_1, \lambda_2) \) for \( q \in \mathbb{Z} \) are given in (1). \( \gamma \) should lie in the range \((-P_S(Z = 0)/\Delta, 2\min\{P_S(Z = -1), P_S(Z = 1)\}/\Delta)\), which can be directly obtained from the last three equations in (2) since all probabilities need to be greater than 0. As Alzaid and Omair (2010) pointed out, the Skellam distribution is strongly unimodal, and from Figure 1 below or Figure 2 in Koopman et al. (2017), we can see that either AS or MS distribution has only one peak, i.e., the pmf shows different monotonicity on the left and right sides of the peak. Thus, no matter where the \( P_S(Z = 0) \) locates, we always obtain the result \( \Delta > 0 \). Therefore, the distribution is well defined.

In equation (2), the distance \( \Delta \) between the probabilities \( P_S(Z = 0) \) and \( \min\{P_S(Z = -1), P_S(Z = 1)\} \) is increased or decreased depending on the magnitude and sign of \( \gamma \). Intuitively, the sign of the coefficient \( \gamma \) determines whether we inflate or deflate the integer 0. Note that for \( \gamma = 0 \), we recover the standard Skellam distribution. In summary, the MS distribution transfers probability from \( Z = 0 \) to \( Z = 1 \) and \( Z = -1 \) for \( \gamma < 0 \) and transfers the other way around for \( \gamma > 0 \). So it can deal with both inflation and deflation phenomena.

In addition, it is easy to show that the mean and variance of \( Z \) with the pmf (2) are \( \mathbb{E}(Z) = \lambda_1 - \lambda_2 \) and \( \text{Var}(Z) = \lambda_1 + \lambda_2 - \gamma\Delta \). The mgf and \( k \)-order moment of \( Z \) are given by

\[
M_{MS}(t) = M_S(t) - \frac{1}{2}\gamma\Delta(e^{-t} + e^{t} - 2), \\
\mu_k^* = \mu_k - \gamma S_q\Delta, \quad k \in \mathbb{N}_+,
\]

respectively, where \( \mu_k \) and \( M_S(\cdot) \) are defined in Section 2.1 and \( S_q = 0 \) for \( q \) is odd number, \( S_q = 1 \) for \( q \) is even number. For ease of comparison, we present some examples of the above distributions by plotting pmf bars in Figure 1. The panels in Figure 1 reveals that different combinations of parameters in Skellam distribution and its modified version show different skewness and kurtosis.
3 Model formulation

In this section, we will establish and discuss INGARCH models based on AS and MS distributions.

3.1 Asymmetric Skellam INGARCH model

Let \{X_t\} be a sequence of integer-valued time series, we assume that the conditional distribution of \{X_t\} is specified by the AS distribution. To be specific, an AS-INGARCH(1,1) model (i.e., \(\lambda^2_t \neq \lambda^*_2\)) is defined as follows

\[X_t | \mathcal{F}_{t-1} \sim \text{AS}(\lambda^2_t, \lambda^*_2),\]
\[\lambda^2_t = \alpha_0 + \alpha_1 X^2_{t-1} + \beta_1 \lambda^2_{t-1}, \quad \lambda^*_2 = \alpha^*_0 + \alpha_1 X^2_{t-1} + \beta_1 \lambda^2_{t-1}, \quad (3)\]

where \(\alpha_0 > 0, \alpha^*_0 > 0, \alpha_1 \geq 0, \beta_1 \geq 0,\) and \(\mathcal{F}_{t-1}\) is the \(\sigma\)-field generated by \(\{X_{t-1}, X_{t-2}, \ldots\}\).

Remark 1 In the above two dynamic models in (3), we choose the same coefficients of the lagged values of the process (i.e., both dynamic models employ \(\alpha_1\) and \(\beta_1\)), which is not too restrictive. By adding these two dynamic equations, we have \(\lambda^2_t + \lambda^*_2 = (\alpha_0 + \alpha^*_0) + 2\alpha_1 X^2_{t-1} + \beta_1 (\lambda^2_{t-1} + \lambda^2_{t-1})\). Notice that this equation is similar to a GARCH(1,1) model with the conditional variance specified by its lagged conditional variance and lagged observations.
On the other hand, the conditional variance \((\lambda_t^2 + \lambda_t^4)\) contains two dynamics in (3), therefore our model has enough flexibility in modeling count process. If we do not fix \(\alpha_1\) and \(\beta_1\) for the processes \(\lambda_t^2\) and \(\lambda_t^4\), it would increase the instability of model (3) and lead to a larger volatility. As a result, either \(\lambda_t^2\) or \(\lambda_t^4\) tends to be fairly large, the values of the response \(X_t\) may explode to infinity in simulations.

**Remark 2** If \(\alpha_0 = \alpha_0^*\), then \(\lambda_t^2 = \lambda_t^4\) and model (3) reduces to an SS-INGARCH(1,1) model considered by Alomani et al. (2018). This model can be rewritten as

\[
X_t|\mathcal{F}_{t-1} \sim \text{SS}(\tilde{\lambda}_t^2),
\]

where \(\tilde{\lambda}_t^2 = \lambda_t^2 + \lambda_t^4 = 2\lambda_t^2\), \(\tilde{\alpha}_0 = \alpha_0 + \alpha_0^* = 2\alpha_0\) and \(\tilde{\alpha}_1 = 2\alpha_1\). If \(\beta_1 = 0\), then the AS-INGARCH model (3) reduces to AS-INARCH(1). Furthermore, if \(\alpha_0 = \alpha_0^*, \beta_1 = 0\), then the AS-INARCH(1) reduces to SS-INARCH(1) (see Alomani et al., 2018).

The conditional mean and conditional variance of \(\{X_t\}\) are given by

\[
\text{E}(X_t|\mathcal{F}_{t-1}) = \lambda_t^2 - \lambda_t^4,
\]

\[
\text{Var}(X_t|\mathcal{F}_{t-1}) = \lambda_t^2 + \lambda_t^4 > \text{E}(X_t|\mathcal{F}_{t-1}),
\]

respectively. For further properties, we give the following important results.

**Theorem 1** For a second-order stationary process \(\{X_t\}\) following (3), it is necessary to satisfy the condition \(2\alpha_1 + \beta_1 < 1\). Then, we have

1. The unconditional expectation of \(X_t\) is given by \(\mu := \text{E}(X_t) = \frac{\alpha_0 - \alpha_0^*}{1 - \beta_1}\);
2. The unconditional variance of \(X_t\) is \(\sigma^2 := \frac{\alpha_0 + \alpha_0^* + 2\alpha_1\mu^2}{1 - 2\alpha_1 - \beta_1}\), and the mixed moments fulfill

\[
\gamma_M(k) := \text{E}(X_tX_{t-k}) = \begin{cases} 
\sigma^2 + \mu^2, & k = 0, \\
(\alpha_0 - \alpha_0^*)\mu + \beta_1\gamma_M(k-1), & k \geq 1;
\end{cases}
\]

3. Furthermore, the autocovariances are given by

\[
\gamma_X(k) := \text{Cov}(X_t, X_{t-k}) = \begin{cases} 
\sigma^2, & k = 0, \\
\beta_1^k\sigma^2, & k \geq 1.
\end{cases}
\]

**Remark 3** The conditional mean and variance are changing as time varies and they are not constant. Take the conditional mean as an example, we know that \(\text{E}(X_t|\mathcal{F}_{t-1}) = \lambda_t^2 - \lambda_t^4 = (\alpha_0 - \alpha_0^*) + \beta_1(\lambda_{t-1}^2 - \lambda_{t-1}^4) = (\alpha_0 - \alpha_0^*) + \beta_1\text{E}(X_{t-1}|\mathcal{F}_{t-2})\), thus the conditional mean is not a constant. But under the second-order stationary condition, the unconditional mean and variance are supposed to be invariant so we can obtain the above formulas in Theorem 1.
Remark 4 By the result of Theorem 1, we can deduce that the autocorrelation \( \rho(k) = \frac{\gamma_X(k)}{\gamma_X(0)} = \beta_1^k \), like the autocorrelation of AR(1) model. In fact, it is reasonable due to the definition of model (3). Note that the conditional mean is driven by its lagged values, i.e., \( \lambda^2_t - \lambda^2_1 = (\alpha_0 - \alpha_0^*) + \beta_1(\lambda^2_{t-1} - \lambda^2_{1-1}) \), which does not depend on \( \alpha_1 \) but the conditional variance depends on \( \alpha_1 \) indeed.

Remark 5 Notice that \( 2\alpha_1 + \beta_1 < 1 \) is a necessary condition on the parameters of the model to ensure that \( \{X_t\}_{t \in \mathbb{Z}} \) is a second-order stationary process. If \( \alpha_0 = \alpha_0^* \) and \( \lambda^2_t = \lambda^2_1 \), as mentioned by Alomani et al. (2018), a sufficient and necessary condition for SS-INGARCH(1,1) process (4) to be stationary is \( 2\alpha_1 + \beta_1 < 1 \). Actually by adding the conditional variance equations in (3) under the symmetric case, we know that (4) is very similar to the original GARCH(1,1) process. Therefore, (4) can be rewritten as an infinite-dimensional ARCH(\( \infty \)) process. The basic idea for deducing the sufficient and necessary condition almost follows by the proof of Theorem 1 in Milhøj (1985) and some subsequent inductions of (4). See more details, please refer to the proof of Theorem 1 with \( p = 1, q = 1 \) case in Bollerslev (1986). However, when it comes to the non-zero mean case, our models (3.1) and (3.3) are different from the original GARCH(1,1) model proposed by Bollerslev (1986) and INGARCH(1,1) process introduced by Ferland et al. (2006).

In nonnegative time series, both mean and variance are positive, so overdispersion is often emphasized. In this context, we know for \( \mathbb{N} \)-valued data, the negative binomial distribution is an alternative of the Poisson distribution to deal with overdispersion phenomenon. But in \( \mathbb{Z} \)-valued time series, mean can be negative but variance is always positive, then in this case overdispersion always holds. As a referee pointed out, negative binomial distribution can also be considered to accommodate larger variance for \( \mathbb{Z} \)-valued data. In fact, the conditional variance of model (3) is always larger than its conditional mean, so our AS-INGARCH model can deal with overdispersion. To make it more clear, we conduct an experiment to compare the theoretical mean and variance with empirical mean and variance via simulations as suggested. Models A1 and A2 are specified in Section 5.1 and simulation results based on 500 times are shown in Table 1.

From the above results, we can find that for both cases, the empirical variance is fairly close to the theoretical variance. Moreover, the empirical variance is much larger than empirical mean, which further illustrates that our model is adequate for dealing with the overdispersion phenomenon.
3.2 Modified Skellam INGARCH model

Similar to Section 3.1, we consider the following model:

\[ X_t|\mathcal{F}_{t-1} \sim \text{MS}(\gamma, \lambda_t^2, \lambda_t^2), \]
\[ \lambda_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \lambda_{t-1}^2, \quad \lambda_t^2 = \alpha_0^* + \alpha_1 X_{t-1}^2 + \beta_1 \lambda_{t-1}^2, \quad (5) \]

where \( \gamma \in (\max_t\{-P_{0t}/\Delta_t\}, \min_t\{2\min_t(P_{-1t}, P_{1t})/\Delta_t\}) \), \( \Delta_t = P_{0t} - \min(P_{-1t}, P_{1t}) > 0 \), \( P_{qt} = P_{MS}(X_t = q), q \in \mathbb{Z}, \alpha_0 > 0, \alpha_0^* > 0, \alpha_1 \geq 0, \beta_1 \geq 0 \) and \( \mathcal{F}_{t-1} \) is the \( \sigma \)-field generated by \( \{X_{t-1}, X_{t-2}, \ldots\} \). The above model is denoted by MS-INGARCH(1,1). Note that for \( \gamma = 0 \), we recover the AS-INGARCH(1,1) model. The conditional mean and conditional variance of \( \{X_t\} \) are given by

\[ \mathbb{E}(X_t|\mathcal{F}_{t-1}) = \lambda_t^2 - \lambda_t^2, \quad \text{Var}(X_t|\mathcal{F}_{t-1}) = \lambda_t^2 + \lambda_t^2 - \gamma \Delta_t. \]

Similar to Theorem 1, we have the following results:

**Theorem 2** For a second-order stationary process \( \{X_t\} \) following MS-INGARCH(1,1) model (5), it is necessary to satisfy the conditions \( 2\alpha_1 + \beta_1 < 1 \) and \( \mathbb{E}(\Delta_t) = C \) for some constant \( C \in (0, 1) \). Then,

1. The unconditional expectation of \( X_t \) is given by \( \mu_* := \mathbb{E}(X_t) = \frac{\alpha_0 - \alpha_0^*}{1 - \beta_1}; \)

2. The unconditional variance of \( X_t \) is \( \sigma^2_* = \frac{\alpha_0 + \alpha_0^* + 2\alpha_1 \mu_*^2 - (1 - \beta_1) \gamma C}{1 - 2\alpha_1 - \beta_1} \), and the mixed moments fulfill

\[ \gamma^*_M(k) := \mathbb{E}(X_t X_{t-k}) = \begin{cases} 
\sigma^2_* + \mu_*^2, & k = 0, \\
(\alpha_0 - \alpha_0^*)\mu_* + \beta_1 \gamma^*_M(k-1), & k \geq 1;
\end{cases} \]

3. Furthermore, the autocovariances are given by

\[ \gamma^*_X(k) := \text{Cov}(X_t, X_{t-k}) = \begin{cases} 
\sigma^2_*, & k = 0, \\
\beta_1^* \sigma^2_*, & k \geq 1.
\end{cases} \]
Proof for the above theorem is deferred to the Appendix. By the way, when $\alpha_0 = \alpha_0^*$, then $\lambda_t^2 = \lambda_t^*$ and model (5) can be reduced to a symmetric MS-INGARCH(1,1) with one dynamic equation similar to the conditional variance equation (4). As a result, we can easily find that the necessary and sufficient for the second-order stationarity of this process is $2\alpha_1 + \beta_1 < 1$ and $\mathbb{E}(\Delta_t) = C$, which is similar to the discussion in Remark 5.

4 Estimation and Test

In this section, we study the conditional maximum likelihood (CML) estimation of the unknown parameters in both two types of INGARCH(1,1) models and derive the log-likelihood ratio test statistic for MS-INGARCH(1,1) model.

4.1 Estimation in AS-INGARCH model

Let the parameter vector $\phi = (\alpha_0, \alpha_0^*, \theta^\top)^\top$, where $\theta = (\alpha_1, \beta_1)^\top$. Write the true value of $\phi$ as $\phi^0$, suppose that we observe $\{X_t, t = 1, 2, \ldots, n\}$ from model (3). Conditional on $X_1 = x_1$, we know that the conditional likelihood function is

$$L(\phi|X_1, X_2, \ldots, X_n) = \prod_{t=2}^n \exp\{-\left(\lambda_t^2 + \lambda_t^*\right)\} \left(\frac{\lambda_t^2}{\lambda_t^*\lambda_t^{2*}}\right)^{X_t/2} \mathbb{I}_{|X_t|} \left(2\sqrt{\lambda_t^2\lambda_t^{2*}}\right),$$

then the log-likelihood function is given by

$$l(\phi) = \sum_{t=2}^n l_t(\phi) = \sum_{t=2}^n \left(-\left(\lambda_t^2 + \lambda_t^*\right) + \frac{X_t}{2} \left(\log \lambda_t^2 - \log \lambda_t^{2*}\right) + \log \mathbb{I}_{|X_t|} \left(2\sqrt{\lambda_t^2\lambda_t^{2*}}\right)\right).$$

The score function is defined by

$$S_n(\phi) = \frac{\partial l(\phi)}{\partial \phi} = \sum_{t=2}^n \frac{\partial l_t(\phi)}{\partial \phi}$$

and the CML estimate $\hat{\phi}$ is a solution to the equation $S_n(\hat{\phi}) = 0$. The Hessian matrix is given by

$$H_n(\phi) = -\sum_{t=2}^n \frac{\partial^2 l_t(\phi)}{\partial \phi \partial \phi^\top}. $$


Detailed expressions for partial derivatives in equations (6) and (7) are given in the Appendix. Asymptotic standard errors of CML estimates can be obtained from the matrix (see Ferland et al., 2006) \((\hat{U}_n\hat{V}_n^{-1}\hat{U}_n)^{-1}/n\), where

\[
\hat{V}_n = \frac{1}{n} \sum_{t=2}^{n} \frac{\partial l_t(\phi)}{\partial \phi} \frac{\partial l_t(\phi)}{\partial \phi^\top} \bigg|_{\phi = \hat{\phi}}, \quad \hat{U}_n = -\frac{1}{n} \sum_{t=2}^{n} \frac{\partial^2 l_t(\phi)}{\partial \phi \partial \phi^\top} \bigg|_{\phi = \hat{\phi}}.
\]

### 4.2 Estimation in MS-INGARCH model

Let \(X_1, X_2, \cdots, X_n\) be observations from model (5). Then the parameter vector turns to be \(\phi^* = (\alpha_0, \alpha_0^*, \theta^\top, \gamma)^\top\), where \(\theta = (\alpha_1, \beta_1)^\top\). Consequently, we can easily obtain the conditional log-likelihood function as follows

\[
l^*(\phi^*) = \sum_{X_t=0} l_{1t}(\phi^*) + \sum_{X_t=1} l_{2t}(\phi^*) + \sum_{X_t=-1} l_{3t}(\phi^*) + \sum_{X_t\neq 0, \pm 1} l_{4t}(\phi^*), \quad (8)
\]

where

\[
l_{1t}(\phi^*) = \log \left\{ \exp\{-\left(\lambda_t^2 + \lambda_t^*s^2\right)\} I_0 \left( 2\sqrt{\lambda_t^2 \lambda_t^* s^2} + \gamma \Delta_t \right) \right\},
\]

\[
l_{2t}(\phi^*) = \log \left\{ \exp\{-\left(\lambda_t^2 + \lambda_t^* s^2\right)\} \left( \frac{\lambda_t^2}{\lambda_t^* s^2} \right)^{1/2} I_1 \left( 2\sqrt{\lambda_t^2 \lambda_t^* s^2} - \frac{1}{2} \gamma \Delta_t \right) \right\},
\]

\[
l_{3t}(\phi^*) = \log \left\{ \exp\{-\left(\lambda_t^2 + \lambda_t^* s^2\right)\} \left( \frac{\lambda_t^2}{\lambda_t^* s^2} \right)^{-1/2} I_1 \left( 2\sqrt{\lambda_t^2 \lambda_t^* s^2} - \frac{1}{2} \gamma \Delta_t \right) \right\},
\]

\[
l_{4t}(\phi^*) = \log \left\{ \exp\{-\left(\lambda_t^2 + \lambda_t^* s^2\right)\} \left( \frac{\lambda_t^2}{\lambda_t^* s^2} \right)^{X_t/2} I_{|X_t|} \left( 2\sqrt{\lambda_t^2 \lambda_t^* s^2} \right) \right\}
\]

\[
= -\left(\lambda_t^2 + \lambda_t^* s^2\right) + \frac{X_t}{2} \left( \log \lambda_t^2 - \log \lambda_t^* s^2 \right) + \log I_{|X_t|} \left( 2\sqrt{\lambda_t^2 \lambda_t^* s^2} \right).
\]

The first and second derivatives of (8) are also given in the Appendix and the CML estimate \(\hat{\phi}^*\) is a solution to \(\partial l(\phi^*)/\partial \phi^* = 0\). Notice that the second-order derivative with respect to the parameter \(\gamma\) of the above conditional log-likelihood function is zero, so we use \(\hat{V}_n^{-1}/n\) instead to calculate its standard error.

### 4.3 Testing MS-INGARCH model

To assess the superiority of MS-INGARCH(1,1) model over the AS-INGARCH(1,1) model, we need to test \(H_0 : \gamma = 0\) vs \(H_1 : \gamma \neq 0\). Now, we use the following log-likelihood ratio test (LRT) statistic:

\[
T_n = -2 \left[ l(\hat{\phi}) - l^*(\hat{\phi}^*) \right], \quad (9)
\]
where $\ell(\hat{\phi})$ is the log-likelihood function under the null hypothesis $H_0$ and $\ell^*(\hat{\phi}^*)$ is the log-likelihood function under the alternative hypothesis $H_1$. Since $\gamma$ in the alternative hypothesis $H_1$ is unrestricted, standard theory suggests that under the null hypothesis $H_0$, $T_n$ is asymptotically $\chi^2(1)$ distributed, as $n \to \infty$.

5 Simulation Studies

Here, we will evaluate the CML estimates for unknown parameters of interest by Monte Carlo simulations and conduct an experiment to discuss the performance of LRT.

5.1 Performances of the CML estimation

For estimation of the parameters, we first use the function \texttt{poissrnd} to generate random data in \texttt{Matlab}. Given the initial values randomly choosing from a uniform distribution whose domain is centered by the true value and perturbed with a little errors. We use the \texttt{Interior-point} algorithm in the constrained nonlinear optimization function \texttt{fmincon} to maximize the conditional log-likelihood function. Note that \texttt{Matlab} has a full coverage of all standard Bessel-related function and the function \texttt{besseli}(r,z) calculates the modified Bessel function of order $r$ in Section 2.1.

We consider the following configurations of the parameters:

- AS-INGARCH(1,1) models with $(\alpha_0, \alpha^*_0, \alpha_1, \beta_1)^\top$: 
  (A1) = (1, 0.5, 0.25, 0.1)$^\top$; (A2) = (0.5, 0.8, 0.2, 0.25)$^\top$;

- MS-INGARCH(1,1) models with $(\alpha_0, \alpha^*_0, \alpha_1, \beta_1, \gamma)^\top$:
  (B1) = (1, 1, 0.2, 0.15, 0.5)$^\top$; (B2) = (0.5, 0.3, 0.15, 0.3, -0.5)$^\top$.

In simulations, we choose sample size $n = 500$ and 1000 with $m = 500$ replications for each choice of parameters. The relative bias (RB) and mean square error (MSE) are calculated to evaluate the performance of the estimators according to the following formulas:

$$\text{RB} = \frac{1}{m} \sum_{j=1}^{m} \left( \frac{\hat{\varphi}_j - \varphi^0}{\varphi^0} \right), \quad \text{MSE} = \frac{1}{m} \sum_{j=1}^{m} (\hat{\varphi}_j - \varphi^0)^2,$$

where $\hat{\varphi}_j$ is the estimator of $\varphi^0$ in the $j$th replication. We also consider the consuming time in each simulation procedure.

The results are summarized in Table 2 with the last columns standing for the using time (in seconds). For $n = 500$ and 1000, all estimators generally show small values of RB and MSE. In all cases, the values of RB
Table 2 RBs and MSEs (within parentheses) for AS-INGARCH(1,1) and MS-INGARCH(1,1) models.

<table>
<thead>
<tr>
<th>Model</th>
<th>n</th>
<th>$\alpha_0$</th>
<th>$\alpha_0^*$</th>
<th>$\alpha_1$</th>
<th>$\beta_1$</th>
<th>$\gamma$</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>500</td>
<td>-0.0365</td>
<td>-0.0598</td>
<td>-0.0072</td>
<td>0.1790</td>
<td></td>
<td>182.6506</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0242)</td>
<td>(0.0125)</td>
<td>(0.0020)</td>
<td>(0.0075)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>-0.0093</td>
<td>-0.0202</td>
<td>0.0016</td>
<td>0.0350</td>
<td></td>
<td>344.3457</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0121)</td>
<td>(0.0072)</td>
<td>(0.0010)</td>
<td>(0.0031)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A2</td>
<td>500</td>
<td>-0.0609</td>
<td>-0.0459</td>
<td>-0.0155</td>
<td>0.0924</td>
<td></td>
<td>196.6288</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0174)</td>
<td>(0.0309)</td>
<td>(0.0019)</td>
<td>(0.0144)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>-0.0202</td>
<td>-0.084</td>
<td>-0.0140</td>
<td>0.0500</td>
<td></td>
<td>338.2725</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0090)</td>
<td>(0.0153)</td>
<td>(0.0009)</td>
<td>(0.0062)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B1</td>
<td>500</td>
<td>0.0083</td>
<td>0.0268</td>
<td>0.0035</td>
<td>-0.0467</td>
<td>0.1808</td>
<td>683.0255</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0431)</td>
<td>(0.0433)</td>
<td>(0.0019)</td>
<td>(0.0117)</td>
<td>(0.0773)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>0.0067</td>
<td>0.0261</td>
<td>-0.0125</td>
<td>-0.0307</td>
<td>0.1312</td>
<td>1316.7036</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0239)</td>
<td>(0.0243)</td>
<td>(0.0010)</td>
<td>(0.0067)</td>
<td>(0.0487)</td>
<td></td>
</tr>
<tr>
<td>B2</td>
<td>500</td>
<td>0.0244</td>
<td>0.0433</td>
<td>-0.0233</td>
<td>-0.0460</td>
<td>0.1158</td>
<td>695.1765</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0276)</td>
<td>(0.0141)</td>
<td>(0.0013)</td>
<td>(0.0407)</td>
<td>(0.0056)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>-0.0122</td>
<td>0.0160</td>
<td>-0.0007</td>
<td>0.0157</td>
<td>0.1054</td>
<td>1436.3719</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0093)</td>
<td>(0.0047)</td>
<td>(0.0004)</td>
<td>(0.0146)</td>
<td>(0.0056)</td>
<td></td>
</tr>
</tbody>
</table>

and MSE gradually decrease as the sample size increases. We also find that MS-INGARCH(1,1) models seem to use more time than AS-INGARCH(1,1) models, which may due to the complexity of the MS distribution. To check the asymptotic properties of the Monte Carlo estimates, Figures 2 and 3 show histograms and qq-plots for the sampling distribution of the standardized CML estimators based on models (A1) and (B1) with sample size $n = 1000$. These plots indicate our estimates are generally approximated to the normal distribution. Notice that the estimate for $\beta_1$ in model (B1) from Figure 3 is a little bit right-skewed, which is resulted from the constraints for the coefficient $\beta_1 > 0$ and $2\alpha_1 + \beta_1 < 1$.

5.2 Performances of the Log-likelihood ratio test

In this subsection, we will present a simulation study of the empirical size and power for LRT test statistic $T_n$ given by (9). From Section 4.3, we know that under the null hypothesis, the limit of LRT is $\chi^2(1)$ distribution. Let $c_\alpha$ be the $(1 - \alpha)$th quantile of $\chi^2(1)$, then at a nominal level $\alpha \in (0, 1)$, take $T_n > c_\alpha$ as the critical region based on our LRT. With this test procedure, we can easily obtain the corresponding empirical size and power.

For the log-likelihood ratio test, we conduct both INARCH and IN-GARCH cases for estimating the empirical size and power. We select the nominal significance level $\alpha = 0.05$, 0.1, the sample size 200, 500 and each experiment is based on 500 replications. For analyzing the empirical size,
Figure 2 From top to bottom: Histograms and qq-plots for the sampling distribution of the standardized CML estimators of model $A_1 = (1, 0.5, 0.25, 0.1)^T$. The smooth curve is the standard normal density function. The results are based on 1000 data points and 500 simulations.
Figure 3 From top to bottom: Histograms and qq-plots for the sampling distribution of the standardized CML estimators of model $B_1=(1, 1, 0.2, 0.15, 0.5)^\top$. The smooth curve is the standard normal density function. The results are based on 1000 data points and 500 simulations.
the following models are considered:

- AS-INARCH(1) models with \((\alpha_0, \alpha^*_0, \alpha_1)\)^T:
  
  \((C1) = (2, 1, 0.2)^T; \quad (C2) = (0.5, 1, 0.25)^T;\)

- AS-INGARCH(1,1) models with \((\alpha_0, \alpha^*_0, \alpha_1, \beta_1)\)^T:
  
  \((D1) = (1, 1, 0.2, 0.15)^T; \quad (D2) = (0.5, 0.3, 0.25, 0.1)^T.\)

To investigate the power of LRT under the alternative hypothesis, we work with the following models:

- MS-INARCH(1) models with \((\alpha_0, \alpha^*_0, \alpha_1, \gamma)\)^T:
  
  \((E1) = (2, 1, 0.2, 0.5)^T; \quad (E2) = (0.5, 1, 0.25, -0.5)^T;\)

- MS-INGARCH(1,1) models with \((\alpha_0, \alpha^*_0, \alpha_1, \beta_1, \gamma)\)^T:
  
  \((F1) = (1, 1, 0.2, 0.15, 0.3)^T; \quad (F2) = (0.5, 0.3, 0.25, 0.1, -0.5)^T.\)

Table 3: Empirical sizes and powers at the nominal levels 0.05 and 0.1 for LRT statistic \(T_n\).

<table>
<thead>
<tr>
<th></th>
<th>Level</th>
<th>0.05</th>
<th>0.1</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Size</td>
<td>C1</td>
<td>0.040</td>
<td>0.074</td>
<td>0.056</td>
<td>0.102</td>
</tr>
<tr>
<td></td>
<td>C2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>n = 200</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>n = 500</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>D1</td>
<td>0.048</td>
<td>0.096</td>
<td>0.050</td>
<td>0.104</td>
</tr>
<tr>
<td></td>
<td>D2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>E1</td>
<td>0.036</td>
<td>0.080</td>
<td>0.082</td>
<td>0.126</td>
</tr>
<tr>
<td></td>
<td>E2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>n = 200</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>n = 500</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>F1</td>
<td>0.332</td>
<td>0.484</td>
<td>0.496</td>
<td>0.678</td>
</tr>
<tr>
<td></td>
<td>F2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>n = 200</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>n = 500</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.906</td>
<td>0.908</td>
<td>0.968</td>
<td>0.986</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.948</td>
<td>0.948</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

The results are shown in Table 3. Empirical sizes for models (C1, C2, D1, D2) are close to the nominal size in both \(\alpha = 0.05\) and \(0.1\) cases. Note that the test statistics perform better when sample size increases from \(n = 200\) to \(n = 500\), so a large sample size is needed to obtain reasonable results. We also find that powers for INARCH models (E1, E2) are relatively small when \(n = 200\), but they will improve when \(n = 500\). As for INGARCH models, they all show high values when \(n = 200\) and \(n = 500\). It to some extent illustrates that the LRT for INGARCH model is more powerful than that for INARCH model, especially when the sample size is small.

In summary, our limited simulation study indicates that the LRT given in Section 4.3 may be quite useful in checking which model is better fitted between AS-INGARCH and MS-INGARCH specifications.
6 Real example

Here, we apply our proposed models to a real time series about the daily difference between the close and open prices of the Saudi Telecom in 2012, which has been investigated by Barreto-Souza and Bourguignon (2015). We consider the number of ticks by rescaling this time series as \((\text{close price} - \text{open price}) \times 10\), since the number of ticks can be treated as a Skellam-type distributed random variable which belongs to \(\mathbb{Z}\) and is our time series of interest here in this section. The data set are available online at the site http://www.tadawul.com.sa or can be obtained in Table 4 of Barreto-Souza and Bourguignon (2015) with the sample size 251. It is important to mention that the discrete nature of the price grid affects the empirical distribution of returns. It concentrates around the actual tick-sizes, is severely multi-modal, and consequently, is highly non-Gaussian, as Münnix et al. (2010) pointed out. Therefore, the ordinary GARCH modeling with error terms following Gaussian distribution may not be appropriate.

Table 4 Summary statistics for the number of ticks.

<table>
<thead>
<tr>
<th>No.</th>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>Minimum</th>
<th>Median</th>
<th>Maximum</th>
<th>(\rho(1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>251</td>
<td>0.5259</td>
<td>18.0663</td>
<td>0.9393</td>
<td>−18</td>
<td>0</td>
<td>22</td>
<td>0.1795</td>
</tr>
</tbody>
</table>

Statistics of the dataset are presented in Table 4. It is worth mentioning that with the efficient market hypothesis, the continuous price process should be a martingale to exclude arbitrage opportunity. As a result, an efficient financial market usually corresponds to the case of symmetric Skellam distribution. However in this paper, we mainly focus on investigating the integer-valued (discrete) process, which is different from the continuous case. Furthermore, due to the non-zero mean of the data, we are interested in applying this dataset to our proposed models, AS-INGARCH(1,1) and MS-INGARCH(1,1) to see whether they are well fitted.

Real data and their autocorrelation function (ACF) and partial autocorrelation function (PACF) are plotted in Figure 4. To compare the proposed models with existing ones, we consider TINAR(1) model (Freeland, 2010), SINARZ(1) model (Barreto-Souza and Bourguignon, 2015), SS-INARCH(1) and SS-INGARCH(1,1) models (Alomani et al., 2018) and our newly proposed integer-valued ARCH and GARCH models. For a goodness of fit, we use the root mean square error (RMS) and mean absolute error (MAE) to compare the above models, where the error is defined by \(X_t - \mathbb{E}(X_t|\mathcal{F}_{t-1})\) for \(t = 2,\ldots,n\) with \(X_t\) being the time series and \(n\) being the sample size. In addition, we employ Akaike information criterion (AIC) and Bayesian information criterion (BIC) to compare INARCH and INGARCH models. These
values and estimates of parameters with their standard errors are given in Table 5.

We calculate the estimated parameters by CML method for INARCH(1) and INGARCH(1,1) models, whereas by conditional least squares method for INAR(1)-type models since the likelihood functions for their models are cumbersome to work with. In fact, existing literature just considered this estimation method. From Table 5, we can find that both AS-INGARCH(1,1) and MS-INGARCH(1,1) models yield smaller values of RMS and MAE than INAR(1)-type models. Besides, notice that the estimates $\alpha_0$ and $\alpha_0^*$ based on AS and MS distributions show different values, which corresponds with the non-zero mean ($0.5259$) of our data. In terms of INARCH(1) models, three types of models show similar AIC values, but AS and MS have larger BIC values because of the increasing number of parameters. But for INGARCH(1,1) models, AS has the smallest values of AIC and BIC. Thus, we can know that AS-INGARCH(1,1) fits best among the INARCH(1) and INGARCH(1,1) models.

To make a full statistical data analysis, we also consider the Pearson residuals defined by $e_t = \frac{X_t - \mathbb{E}(X_t|\mathcal{F}_{t-1})}{\sqrt{\text{Var}(X_t|\mathcal{F}_{t-1})}}$ to further examine the adequacy fitting of our models. Under the correct model, the sequence $e_t$ should be a
Table 5 Estimated parameters with their standard errors (within parentheses) and values of some comparison criteria for number of ticks.

<table>
<thead>
<tr>
<th>Model</th>
<th>Estimates</th>
<th>RMS</th>
<th>MAE</th>
<th>AIC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>TINAR(1)</td>
<td>$\hat{\alpha} = 0.1795$</td>
<td>$\lambda = 7.3821$</td>
<td>4.1813</td>
<td>2.8282</td>
<td>-</td>
</tr>
<tr>
<td>SINARZ(1)</td>
<td>$\hat{\alpha} = 0.1795$</td>
<td>$\hat{\mu} = 0.5289$</td>
<td>4.1813</td>
<td>2.8282</td>
<td>-</td>
</tr>
<tr>
<td>SS</td>
<td>$\hat{\alpha}_0 = 10.3627$ (1.0552)</td>
<td>$\hat{\alpha}_1 = 0.4895$ (0.1633)</td>
<td>$\hat{\lambda} = 7.3821$</td>
<td>4.1813</td>
<td>2.8282</td>
</tr>
<tr>
<td>AS</td>
<td>$\hat{\alpha}_0 = 5.3392$ (0.566)</td>
<td>$\hat{\alpha}_1 = 0.5289$ (0.0792)</td>
<td>$\hat{\lambda} = 7.3821$</td>
<td>4.1813</td>
<td>2.8282</td>
</tr>
<tr>
<td>MS</td>
<td>$\hat{\alpha}_0 = 5.3525$ (0.3270)</td>
<td>$\hat{\alpha}_1 = 0.4969$ (0.4355)</td>
<td>$\hat{\lambda} = 7.3821$</td>
<td>4.1813</td>
<td>2.8282</td>
</tr>
<tr>
<td>INARCH(1)</td>
<td>$\hat{\alpha}_0 = 5.6584$ (2.1151)</td>
<td>$\hat{\alpha}_1 = 0.3334$ (0.3659)</td>
<td>$\hat{\beta}_1 = 0.3231$ (0.1745)</td>
<td>$\hat{\lambda} = 7.3821$</td>
<td>4.1813</td>
</tr>
<tr>
<td>AS</td>
<td>$\hat{\alpha}_0 = 1.3726$ (0.1186)</td>
<td>$\hat{\alpha}_1 = 0.2805$ (0.0111)</td>
<td>$\hat{\beta}_1 = 0.4294$ (0.0265)</td>
<td>$\hat{\lambda} = 7.3821$</td>
<td>4.1813</td>
</tr>
<tr>
<td>MS</td>
<td>$\hat{\alpha}_0 = 1.3866$ (0.2446)</td>
<td>$\hat{\alpha}_1 = 0.2805$ (0.0124)</td>
<td>$\hat{\beta}_1 = 0.4289$ (0.0124)</td>
<td>$\hat{\lambda} = 7.3821$</td>
<td>4.1813</td>
</tr>
</tbody>
</table>

Note: RMS, root mean square error; MAE, mean absolute error; AIC, Akaike information criterion; BIC, Bayesian information criterion.

![Figure 5](image-url) ACF (left) and cumulative periodogram (right) plots of the Pearson residuals based on fitted AS-INGARCH(1,1) model.
white noise sequence with constant variance. Therefore, we show ACF and cumulative periodogram plots (see Brockwell and Davis, 1991, Sec.10.2) for AS-INGARCH(1,1) model in Figure 5. From it, we can see that Pearson residuals seem to be uncorrelated and the standardized cumulative periodogram lies in the Kolmogorov–Smirnov bounds with level $\alpha = 0.05$, which further indicates the whiteness of the Pearson residuals.

As for testing MS-INGARCH(1,1) model, i.e., consider the null hypothesis $H_0 : \gamma = 0$ against the alternative hypothesis $H_1 : \gamma \neq 0$, we use the LRT statistic (see Section 4.3) and $T_n = 0.3640$. On comparing $\chi^2_{0.90}(1) = 2.7055$, we notice that LRT statistic is less than the nominal value, which implies that we cannot reject the $H_0$. This supports our conclusion that AS-INGARCH(1,1) model are more suitable to explain the data and fits better than other competitions.

7 Conclusion

The current paper considers appropriate and appealing models for $\mathbb{Z}$-valued time series data and proposes an asymmetric INGARCH model and a modified Skellam one, respectively. Both models can deal with non-zero-mean process, the latter can also modify the probabilities of 0 and $\pm 1$ to satisfy a special feature of the data. In other words, our models have better performances when $\mathbb{Z}$-valued time series data has non-zero mean and/or 0 and $\pm 1$ inflation features. For estimating parameters of our proposed models, the CML method has been developed. The numerical simulations show that the estimation results are reliable as long as the sample size is reasonably large. Moreover, a small simulation is provided to check the usefulness of LRT. We finally fit models to a real data example, which demonstrates that our proposed models are easy-to-use and flexible compared with existing models.

Theoretical properties including strict stationarity and ergodicity of new models are difficult to establish but it certainly is an important topic. After establishing these properties, we can discuss consistency and asymptotic normality of CML estimators.

As a referee pointed out, Liu and Luger (2015) recently proposed an unfolded GARCH model which can decompose returns into their signs and absolute values. They mainly considered the asymmetric distribution of the absolute values and showed how to find the model-implied centered moments. It is indeed of interest for us to extend this approach to INGARCH models as future work.
7.1 Appendix subsection

The proof of Theorem 1 is similar to that of Theorem 2, so we just give the proof of Theorem 2.

Proof of Theorem 2. As we discussed in Remark 4, if \( \{X_t\} \) is a second-order stationary, the unconditional variance will be unrelated with time \( t \). In light of the existence and the invariant property of the unconditional variance \( \sigma^2 \) (to be derived below), the process \( \{X_t\}_{t \in \mathbb{Z}} \) must satisfy necessarily the conditions \( 2\alpha_1 + \beta_1 < 1 \) and \( \mathbb{E}(\Delta_t) = C \in (0, 1) \).

(1) By the law of total expectation, we have

\[
\mu_* = \mathbb{E}(X_t) = \mathbb{E}(\mathbb{E}(X_t|\mathcal{F}_{t-1})) = \mathbb{E}(\lambda^2_t - \lambda_t^2) = (\alpha_0 - \alpha_0^*) + \beta_1\mathbb{E}(X_{t-1}),
\]

so \( \mu_* = (\alpha_0 - \alpha_0^*)/(1 - \beta_1) \).

(2) Let \( \sigma^2_* = \text{Var}(X_t) \), then \( \mathbb{E}(X_t^2) = \sigma^2_* + \mu^2_* \). According to the formula

\[
\text{Var}(X_t) = \mathbb{E}(\mathbb{E}(X_t|\mathcal{F}_{t-1})) + \mathbb{E}(\text{Var}(X_t|\mathcal{F}_{t-1}))
\]

and by iterating, we have

\[
\sigma^2_* = \beta_1^{2(t-1)}\text{Var}(\lambda^2_t - \lambda_t^2) + (\alpha_0 + \alpha_0^*)\sum_{k=1}^{t-1}\beta_1^{k-1}
\]

\[
+ 2\alpha_1\sum_{k=1}^{t-1}\beta_1^{k-1}(\sigma^2_* + \mu^2_*) + \beta_1^{t-1}\mathbb{E}(\lambda^2_t + \lambda_t^2) - \gamma\mathbb{E}(\Delta_t),
\]

combine with the precondition \( 0 \leq \beta_1 < 1, \mathbb{E}(\Delta_t) = C \in (0, 1) \) and as \( t \) goes to infinity, the unconditional variance turns to be

\[
\sigma^2_* = \frac{\alpha_0 + \alpha_0^* + 2\alpha_1\mu_* - (1 - \beta_1)\gamma C}{1 - 2\alpha_1 - \beta_1}.
\]

For \( k \geq 1, \)

\[
\gamma^*_M(k) = \mathbb{E}(X_tX_{t-k}) = \mathbb{E}(X_{t-k}\mathbb{E}(X_t|\mathcal{F}_{t-1}))
\]

\[
= (\alpha_0 - \alpha_0^*)\mu_* + \beta_1\mathbb{E}(X_{t-k}X_{t-1})
\]

\[
= (\alpha_0 - \alpha_0^*)\mu_* + \beta_1\gamma^*_M(k-1).
\]

For \( k = 0, \gamma^*_M(0) = \mathbb{E}(X_t^2) = \sigma^2_* + \mu^2_* \).

(3) For \( k \geq 1 \), let \( \mu_t = \mathbb{E}(X_t|\mathcal{F}_{t-1}) \), then

\[
\gamma^*_X(k) = \text{Cov}(X_t, X_{t-k}) = \text{Cov}(\mu_t, X_{t-k})
\]

\[
= \text{Cov}(\lambda^2_t - \lambda_t^2, X_{t-k}) = \text{Cov}(\beta_1(\lambda^2_{t-1} - \lambda_{t-1}^2), X_{t-k})
\]

\[
= \beta_1\gamma^*_X(k-1) = \ldots = \beta_1^k\gamma^*_X(0).
\]
Partial derivatives in Sections 4.1 and 4.2.

The score function is given by \((6)\) with the first derivatives

\[
\frac{\partial l_t(\phi)}{\partial \alpha_0} = \left( \frac{X_t}{2\lambda_t} - 1 + \frac{|X_t|}{2\lambda_t^2} \frac{I_{[X_t]|+1}}{I_{[X_t]}} \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{1/2} \right) \frac{\partial \lambda_t^2}{\partial \alpha_0},
\]

\[
\frac{\partial l_t(\phi)}{\partial \alpha_0^*} = \left( -\frac{X_t}{2\lambda_t^2} - 1 + \frac{|X_t|}{2\lambda_t^2} \frac{I_{[X_t]|+1}}{I_{[X_t]}} \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{1/2} \right) \frac{\partial \lambda_t^2}{\partial \alpha_0^*},
\]

\[
\frac{\partial l_t(\phi)}{\partial \theta} = \left( \frac{X_t}{2\lambda_t^2} - 1 + \frac{|X_t|}{2\lambda_t^2} \frac{I_{[X_t]|+1}}{I_{[X_t]}} \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{1/2} \right) \frac{\partial \lambda_t^2}{\partial \theta},
\]

\[
- \left( \frac{X_t}{2\lambda_t^2} + 1 - \frac{|X_t|}{2\lambda_t^2} \frac{I_{[X_t]|+1}}{I_{[X_t]}} \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{1/2} \right) \frac{\partial \lambda_t^2}{\partial \theta},
\]

\[
\frac{\partial \lambda_t^2}{\partial \alpha_0} = 1 + \beta_1 \frac{\partial \lambda_t^2}{\partial \alpha_1}, \quad \frac{\partial \lambda_t^2}{\partial \alpha_1} = \frac{\partial \lambda_t^2}{\partial \alpha_0},
\]

\[
\frac{\partial \lambda_t^2}{\partial \alpha_0^*} = 1 + \beta_1 \frac{\partial \lambda_t^2}{\partial \alpha_0^*}, \quad \frac{\partial \lambda_t^2}{\partial \alpha_0^*} = \frac{\partial \lambda_t^2}{\partial \alpha_0},
\]

where \(I_r\) is short for \(I_r \left( 2\sqrt{\lambda_t^2 \lambda_t^2} \right)\).

The Hessian matrix is given by \((7)\) with the second derivatives as

\[
\frac{\partial^2 l_t(\phi)}{\partial \alpha_0^2} = \left( \frac{X_t + |X_t|}{2\lambda_t^4} + \frac{I_{[X_t]|+2}}{I_{[X_t]}} \left( \frac{\lambda_t^2}{\lambda_t^2} \right) - \frac{I_{[X_t]|+1}^2}{I_{[X_t]^2}} \left( \frac{\lambda_t^2}{\lambda_t^2} \right) \right) \frac{\partial \lambda_t^2}{\partial \alpha_0},
\]

\[
\frac{\partial^2 l_t(\phi)}{\partial \alpha_0^*^2} = \left( \frac{X_t - |X_t|}{2\lambda_t^4} + \frac{I_{[X_t]}}{I_{[X_t]}} \left( \frac{\lambda_t^2}{\lambda_t^2} \right) - \frac{I_{[X_t]|+1}^2}{I_{[X_t]^2}} \left( \frac{\lambda_t^2}{\lambda_t^2} \right) \right) \frac{\partial \lambda_t^2}{\partial \alpha_0^*},
\]

\[
\frac{\partial^2 l_t(\phi)}{\partial \alpha_0 \partial \theta^T} = \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{1/2} \frac{I_{[X_t]|+1}}{I_{[X_t]}} + \frac{I_{[X_t]|+2}}{I_{[X_t]}} \left( \frac{\lambda_t^2}{\lambda_t^2} \right) - \frac{I_{[X_t]|+1}^2}{I_{[X_t]^2}} \left( \frac{\lambda_t^2}{\lambda_t^2} \right) \frac{\partial \lambda_t^2}{\partial \alpha_0 \partial \theta^T},
\]

\[
\frac{\partial^2 l_t(\phi)}{\partial \alpha_0^* \partial \theta^T} = \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{1/2} \frac{I_{[X_t]|+1}}{I_{[X_t]}} + \frac{I_{[X_t]|+2}}{I_{[X_t]}} \left( \frac{\lambda_t^2}{\lambda_t^2} \right) - \frac{I_{[X_t]|+1}^2}{I_{[X_t]^2}} \left( \frac{\lambda_t^2}{\lambda_t^2} \right) \frac{\partial \lambda_t^2}{\partial \alpha_0^* \partial \theta^T},
\]
where

\[ \frac{\partial^2 I_t (\phi)}{\partial \theta \partial \theta^\top} = \left( -\frac{X_t + |X_t|}{2\lambda^2_t} + \frac{I_{[X_t]} + |X_t + 1|}{I_{[X_t]}} \left( \frac{\lambda^2_t}{\lambda^2_t} \right) - \frac{I_{[X_t] + 1}}{I_{[X_t]}} \left( \frac{\lambda^2_t}{\lambda^2_t} \right) \right) \frac{\partial \lambda_t^2}{\partial \theta} \frac{\partial \lambda_t^2}{\partial \theta^\top} + \left( \frac{X_t - |X_t|}{2\lambda^2_t} + \frac{I_{[X_t]} + |X_t + 1|}{I_{[X_t]}} \left( \frac{\lambda^2_t}{\lambda^2_t} \right) \right) \frac{\partial \lambda_t^2}{\partial \theta^\top} \frac{\partial \lambda_t^2}{\partial \theta} + \left( \frac{X_t}{2\lambda^2_t} - 1 + \frac{I_{[X_t] + 1}}{I_{[X_t]}} \left( \frac{\lambda^2_t}{\lambda^2_t} \right)^{1/2} \right) \frac{\partial^2 \lambda_t^2}{\partial \theta \partial \theta^\top}, \]

The first derivatives of (8) are obtained as follows

\[ \partial I^*(\phi^*) = \left( \sum_{X_t = 0} K_1 + \sum_{X_t = 1} K_2 + \sum_{X_t = -1} K_3 \right) \frac{\partial \lambda_t^2}{\partial \alpha_0} + \sum_{X_t \neq 0, \pm 1} \frac{\partial I_t (\phi)}{\partial \alpha_0}, \]

\[ \partial I^*(\phi^*) = \left( \sum_{X_t = 0} K_4 + \sum_{X_t = 1} K_5 + \sum_{X_t = -1} K_6 \right) \frac{\partial \lambda_t^2}{\partial \alpha^*_0} + \sum_{X_t \neq 0, \pm 1} \frac{\partial I_t (\phi)}{\partial \alpha^*_0}, \]

\[ \partial I^*(\phi^*) = \left( \sum_{X_t = 0} K_1 + \sum_{X_t = 1} K_2 + \sum_{X_t = -1} K_3 \right) \frac{\partial \lambda_t^2}{\partial \theta} + \sum_{X_t \neq 0, \pm 1} \frac{\partial I_t (\phi)}{\partial \theta}, \]

\[ \partial I^*(\phi^*) = \left( \sum_{X_t = 0} K_4 + \sum_{X_t = 1} K_5 + \sum_{X_t = -1} K_6 \right) \frac{\partial \lambda_t^2}{\partial \theta} + \sum_{X_t \neq 0, \pm 1} \frac{\partial I_t (\phi)}{\partial \theta}, \]

\[ \frac{\partial I^*(\phi^*)}{\partial \gamma} = \sum_{X_t = 0} \frac{\Delta_t}{e^{2\lambda_t^2(\phi^*)}} + \sum_{X_t = 1} \frac{-\Delta_t}{2e^{2\lambda_t^2(\phi^*)}} + \sum_{X_t = -1} \frac{-\Delta_t}{2e^{2\lambda_t^2(\phi^*)}}, \]
where

\[ K_1 = \frac{e^{-\left(\lambda_t^2 + \lambda_t^2\right)}}{e^{l_{1t}(\phi^*}) \left[ \frac{\lambda_t^2}{\lambda_t^2} I_1 - I_0 \right], \]

\[ K_2 = \frac{e^{-\left(\lambda_t^2 + \lambda_t^2\right)}}{e^{l_{2t}(\phi^*}) \left[ \left( \frac{1}{\lambda_t^2 \lambda_t^2} \right)^{\frac{1}{2}} - \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{\frac{1}{2}} \right) I_1 + I_2 \right], \]

\[ K_3 = \frac{e^{-\left(\lambda_t^2 + \lambda_t^2\right)}}{e^{l_{3t}(\phi^*}) \left[ \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{\frac{1}{2}} I_2 - \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{\frac{1}{2}} I_1 \right], \]

\[ K_4 = \frac{e^{-\left(\lambda_t^2 + \lambda_t^2\right)}}{e^{l_{4t}(\phi^*}) \left[ \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{\frac{1}{2}} I_1 - I_0 \right], \]

\[ K_5 = \frac{e^{-\left(\lambda_t^2 + \lambda_t^2\right)}}{e^{l_{5t}(\phi^*}) \left[ \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{\frac{1}{2}} I_2 - \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{\frac{1}{2}} I_1 \right], \]

\[ K_6 = \frac{e^{-\left(\lambda_t^2 + \lambda_t^2\right)}}{e^{l_{6t}(\phi^*}) \left[ \left( \frac{1}{\lambda_t^2 \lambda_t^2} \right)^{\frac{1}{2}} - \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{\frac{1}{2}} \right) I_1 + I_2 \right], \]

The second derivatives of (8) are expressed as

\[
\frac{\partial^2 l^*}{\partial \alpha_0^2} = \left( \sum_{X_t=0} L_1 + \sum_{X_t=1} L_2 + \sum_{X_t=-1} L_3 \right) \left( \frac{\partial \lambda_t^2}{\partial \alpha_0} \right)^2 + \sum_{X_t \neq 0, \pm 1} \frac{\partial^2 l_t(\phi)}{\partial \alpha_0^2} \\
- \left( \sum_{X_t=0} K_1 \frac{\partial l_{1t}(\phi^*)}{\partial \alpha_0} + \sum_{X_t=1} K_2 \frac{\partial l_{2t}(\phi^*)}{\partial \alpha_0} + \sum_{X_t=-1} K_3 \frac{\partial l_{3t}(\phi^*)}{\partial \alpha_0} \right) \frac{\partial \lambda_t^2}{\alpha_0},
\]

\[
\frac{\partial^2 l^*(\phi^*)}{\partial \alpha_0^2} = \left( \sum_{X_t=0} N_1 + \sum_{X_t=1} N_2 + \sum_{X_t=-1} N_3 \right) \left( \frac{\partial \lambda_t^2}{\partial \alpha_0} \right)^2 + \sum_{X_t \neq 0, \pm 1} \frac{\partial^2 l_t(\phi)}{\partial \alpha_0^2} \\
- \left( \sum_{X_t=0} K_4 \frac{\partial l_{1t}(\phi^*)}{\partial \alpha_0} + \sum_{X_t=1} K_5 \frac{\partial l_{2t}(\phi^*)}{\partial \alpha_0} + \sum_{X_t=-1} K_6 \frac{\partial l_{3t}(\phi^*)}{\partial \alpha_0} \right) \frac{\partial \lambda_t^2}{\alpha_0},
\]

\[
\frac{\partial^2 l^*(\phi^*)}{\partial \alpha_0 \partial \alpha_0} = \left( \sum_{X_t=0} M_1 + \sum_{X_t=1} M_2 + \sum_{X_t=-1} M_3 \right) \left( \frac{\partial \lambda_t^2}{\partial \alpha_0} \right) \left( \frac{\partial \lambda_t^2}{\partial \alpha_0} \right) + \sum_{X_t \neq 0, \pm 1} \frac{\partial^2 l_t(\phi)}{\partial \alpha_0 \partial \alpha_0} \\
- \left( \sum_{X_t=0} K_1 \frac{\partial l_{1t}(\phi^*)}{\partial \alpha_0} + \sum_{X_t=1} K_2 \frac{\partial l_{2t}(\phi^*)}{\partial \alpha_0} + \sum_{X_t=-1} K_3 \frac{\partial l_{3t}(\phi^*)}{\partial \alpha_0} \right) \frac{\partial \lambda_t^2}{\partial \alpha_0},
\]
\[
\frac{\partial^2 l^*(\phi^* \theta^*)}{\partial \alpha_0 \partial \theta^\top} = \left( \sum_{X_t=0} L_1 + \sum_{X_t=1} L_2 + \sum_{X_t=-1} L_3 \right) \left( \frac{\partial \lambda_t^2}{\partial \alpha_0} \right) \left( \frac{\partial \lambda_t^2}{\partial \theta^\top} \right) + \left( \sum_{X_t=0} M_1 + \sum_{X_t=1} M_2 + \sum_{X_t=-1} M_3 \right) \left( \frac{\partial \lambda_t^2}{\partial \alpha_0} \right) \left( \frac{\partial \lambda_t^2}{\partial \theta^\top} \right) + \left( \sum_{X_t=0} K_1 + \sum_{X_t=1} K_2 + \sum_{X_t=-1} K_3 \right) \frac{\partial^2 (\lambda_t^2)}{\partial \alpha_0 \partial \theta^\top} + \sum_{X_t \neq 0, \pm 1} \frac{\partial^2 l_t(\phi)}{\partial \alpha_0 \partial \theta^\top},
\]

\[
\frac{\partial^2 l^*(\phi^* \theta^*)}{\partial \theta \partial \theta^\top} = \left( \sum_{X_t=0} N_1 + \sum_{X_t=1} N_2 + \sum_{X_t=-1} N_3 \right) \left( \frac{\partial \lambda_t^2}{\partial \theta} \right) \left( \frac{\partial \lambda_t^2}{\partial \theta^\top} \right) + \sum_{X_t \neq 0, \pm 1} \frac{\partial^2 l_t(\phi)}{\partial \theta \partial \theta^\top},
\]

\[
\frac{\partial^2 l^*(\phi^* \theta^*)}{\partial \theta \partial \theta^\top} = \left( \sum_{X_t=0} L_1 + \sum_{X_t=1} L_2 + \sum_{X_t=-1} L_3 \right) \left( \frac{\partial \lambda_t^2}{\partial \theta} \right) \left( \frac{\partial \lambda_t^2}{\partial \theta^\top} \right) + \sum_{X_t \neq 0, \pm 1} \frac{\partial^2 l_t(\phi)}{\partial \theta \partial \theta^\top},
\]

\[
2 \left( \sum_{X_t=0} M_1 + \sum_{X_t=1} M_2 + \sum_{X_t=-1} M_3 \right) \left( \frac{\partial \lambda_t^2}{\partial \theta} \right) \left( \frac{\partial \lambda_t^2}{\partial \theta^\top} \right) + \left( \sum_{X_t=0} N_1 + \sum_{X_t=1} N_2 + \sum_{X_t=-1} N_3 \right) \left( \frac{\partial \lambda_t^2}{\partial \theta} \right) \left( \frac{\partial \lambda_t^2}{\partial \theta^\top} \right) + \left( \sum_{X_t=0} K_1 + \sum_{X_t=1} K_2 + \sum_{X_t=-1} K_3 \right) \frac{\partial^2 (\lambda_t^2)}{\partial \theta \partial \theta^\top} + \left( \sum_{X_t=0} K_4 + \sum_{X_t=1} K_5 + \sum_{X_t=-1} K_6 \right) \frac{\partial^2 (\lambda_t^2)}{\partial \theta \partial \theta^\top},
\]

where

\[
L_1 = \frac{e^{-(\lambda_t^2 + \lambda_t^2)}}{e^{d_{11}^2(\phi^*)}} \left[ \left( \frac{\lambda_t^2}{\lambda_t^2} \right) I_2 - 2 \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{\frac{1}{2}} I_1 + I_0 \right],
\]

\[
L_2 = \frac{e^{-(\lambda_t^2 + \lambda_t^2)}}{e^{d_{21}^2(\phi^*)}} \left\{ \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{\frac{1}{2}} I_3 + 2 \left( \frac{1}{\lambda_t^2} - 1 \right) I_2 + \left[ \left( \frac{\lambda_t^2}{\lambda_t^2} \right)^{\frac{1}{2}} - \frac{2}{\sqrt{\lambda_t^2 \lambda_t^2}} \right] I_1 \right\},
\]
\[ L_3 = \frac{e^{-(\lambda_t^2 + \lambda_t^2)}}{e^{e^{(\phi^*)}}} \left\{ \frac{\lambda_t^2}{\lambda_t^4} \right\} I_3 + \left[ \frac{1}{2\lambda_t^2} - \left( \frac{\lambda_t^2}{\lambda_t^4} \right)^\frac{1}{2} \right] I_2 + I_1 \],

\[ M_1 = \frac{e^{-(\lambda_t^2 + \lambda_t^2)}}{e^{e^{(\phi^*)}}} \left\{ I_2 + \left[ \frac{1}{\sqrt{\lambda_t^2 \lambda_t^4}} - \left( \frac{\lambda_t^2}{\lambda_t^4} \right)^\frac{1}{2} \right] I_1 + I_0 \right\}, \]

\[ M_2 = \frac{e^{-(\lambda_t^2 + \lambda_t^2)}}{e^{e^{(\phi^*)}}} \left\{ \frac{\lambda_t^2}{\lambda_t^4} \right\} I_3 + 2 \left( \frac{1}{\lambda_t^2} - 1 \right) I_2 + \left[ \left( \frac{\lambda_t^2}{\lambda_t^4} \right)^\frac{1}{2} - \frac{1}{\sqrt{\lambda_t^2 \lambda_t^4}} \right] I_1 \],

\[ M_3 = \frac{e^{-(\lambda_t^2 + \lambda_t^2)}}{e^{e^{(\phi^*)}}} \left\{ I_3 + \frac{3}{2\sqrt{\lambda_t^2 \lambda_t^4}} - \left( \frac{\lambda_t^2}{\lambda_t^4} \right)^\frac{1}{2} - 1 \right\} I_2 + \left[ \left( \frac{\lambda_t^2}{\lambda_t^4} \right)^\frac{1}{2} - \frac{1}{\lambda_t^2 \lambda_t^2} \right] I_1 \].

\[ N_1 = \frac{e^{-(\lambda_t^2 + \lambda_t^2)}}{e^{e^{(\phi^*)}}} \left\{ \frac{\lambda_t^2}{\lambda_t^4} \right\} I_2 - 2 \left( \frac{\lambda_t^2}{\lambda_t^4} \right)^\frac{1}{2} I_1 + I_0 \],

\[ N_2 = \frac{e^{-(\lambda_t^2 + \lambda_t^2)}}{e^{e^{(\phi^*)}}} \left\{ \frac{\lambda_t^2}{\lambda_t^4} \right\} I_3 + \left[ \frac{1}{2\lambda_t^2} - \left( \frac{\lambda_t^2}{\lambda_t^4} \right)^\frac{1}{2} \right] I_2 + I_1 \],

\[ N_3 = \frac{e^{-(\lambda_t^2 + \lambda_t^2)}}{e^{e^{(\phi^*)}}} \left\{ \frac{\lambda_t^2}{\lambda_t^4} \right\} I_3 + 2 \left( \frac{1}{\lambda_t^2} - 1 \right) I_2 + \left[ \left( \frac{\lambda_t^2}{\lambda_t^4} \right)^\frac{1}{2} - \frac{2}{\sqrt{\lambda_t^2 \lambda_t^4}} \right] I_1 \].

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