SDEs with critical time dependent drifts: weak solutions

MICHAEL RÖCKNER\textsuperscript{1} and GUOHUAN ZHAO\textsuperscript{2,*}

\textsuperscript{1}Department of Mathematics, Bielefeld University, Germany, and Academy of Mathematics and Systems Science, Chinese Academy of Sciences (CAS), Beijing, P.R.China.
E-mail: roeckner@math.uni-bielefeld.de
\textsuperscript{2}Department of Mathematics, Bielefeld University, Germany. E-mail: *zhaoguohuan@gmail.com

For $d \geq 3$, we prove that time-inhomogeneous stochastic differential equations driven by additive noises with drifts in critical Lebesgue space $L^{p,q}([0,T]; L^p(\mathbb{R}^d))$, where $(p,q) \in (d, \infty] \times [2, \infty)$ and $d/p + 2/q = 1$, or $(p,q) = (d, \infty)$ and $\text{div} b \in L^{\infty}([0,T]; L^{d/2+\varepsilon}(\mathbb{R}^d))$, are weakly well-posed. The uniqueness in law is obtained by solving corresponding Kolmogorov backward equations in some second-order Sobolev spaces, which is analytically interesting in itself.

Keywords: Weak solutions; Ladyzhenskaya-Prodi-Serrin condition; Kolmogorov equations; De Giorgi’s method

1. Introduction

The main aim of this paper is to investigate the well-posedness of the following stochastic differential equation (SDE):

$$
\text{d}X_t = b(t, X_t) \text{d}t + \sqrt{2} \text{d}W_t, \quad X_0 = x \in \mathbb{R}^d,
$$

where $W$ is a $d$-dimensional standard Brownian motion and $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a vector field satisfying the following critical Ladyzhenskaya-Prodi-Serrin (LPS) type condition:

$$
b \in L^{p,q}_p(T) := L^q([0,T]; L^p(\mathbb{R}^d)) \quad \text{with} \quad p, q \in [2, \infty], \quad \frac{d}{p} + \frac{2}{q} = 1.
$$

The LPS condition, $b \in L^{p,q}$ with $d/p + 2/q < 1$, named after the authors who posed these conditions to prove global well-posedness of 3D Navier-Stokes equations and smoothness of solutions. For the regularity theory of Navier-Stokes equations, the endpoint case $(p,q) = (3, \infty)$, which triggered a lot of papers, is extremely difficult, and was finally solved by Escauriaza-Seregin-Šverák in [7] (see also [6] and [9]). As presented in [42] and [45], by letting $b$ in (1) be a solution to the Navier-Stokes equation (which is a divergence free vector field), the stochastic equation (1) can be related with the Navier-Stokes equation through Constantin and Iyer’s representation (see [5] and [40]). This deep connection between singular SDEs and Navier-Stokes equations is our main motivation to study (1) under the critical condition (2), especially for the borderline case $(p,q) = (d, \infty)$.

The study of classical strong solutions to SDEs in multidimensional spaces with singular drifts at least dates back to [36], where Veretennikov showed that (1) admits a unique strong solution, provided that $b$ is bounded measurable. Using Girsanov’s transformation and results from PDEs, Krylov-Röckner [22] obtained the existence and uniqueness of strong solutions to (1), when $b$ satisfies subcritical LPS type condition:

$$
b \in L_q^p(T) \quad \text{with} \quad p, q \in (2, \infty), \quad \frac{d}{p} + \frac{2}{q} < 1.
$$
After that, various works were devoted to generalize the well-posedness results and study the properties of solutions to SDEs with singular coefficients, among which we quote [8], [24], [25], [26], [30], [38], [39], [41].

However, for the critical regime (2), it has been a long-standing and challenging problem whether SDE (1) is well-posed or not in the strong sense under (2). In this regard, Beck-Flandoli-Gubinelli-Maurelli [2] showed that if $b$ satisfies (2) with $p > d$ or $p = d$ and $|b|_{L^d(\mathbb{T})}$ is sufficiently small, (1) has at least one strong solution starting from a diffusive random variable in a certain class. In [27], Nam proved that (1) admits a unique strong solution for each $x \in \mathbb{R}^d$ when the Lebesgue-type $L^q$ integrability in the time variable is replaced by a stronger Lorentz-type $L^{q,1}$ integrability condition:

$$b \in L^{q,1}([0,T];L^p(\mathbb{R}^d)) \text{ with } p, q \in (2, \infty), \quad \frac{d}{p} + \frac{2}{q} = 1$$

(see [10] for the definition of Lorentz spaces $L^{q,r}$). Very recently, Krylov made significant progress in his works [19], where the strong well-posedness is proved in the case that $b(t,x) = b(x) \in L^d(\mathbb{R}^d)$ with $d \geq 3$. His approach avoids Zvonkin type of changing variables (cf. [46]) and is based on his earlier work with Veretennikov [35] about the Wiener chaos expansion for strong solutions of (1), as well as some new estimates presented in [17] and [18]. Although the approaches in [19] may be extended to the time-dependent case, there are many technical challenges to overcome. In [31], the follow-up work of this paper, for the non-endpoint case, instead of utilizing the Yamada-Watanabe principle, we construct strong solutions directly using a compactness criteria for $L^2$ random fields in Wiener spaces and new estimates for some certain functionals of the solutions to (1). As a result, according to Cherny [4], the weak uniqueness for SDE (1) can play another crucial role in strong solvability. Surprisingly, as far as we know, there is no complete answer even for the weak well-posedness of (1) when $b$ only satisfies the critical LPS condition (2). The following primary finding of this paper provides an almost complete answer to this issue.

**Theorem 1.1.** Assume $d \geq 3$ and that $b$ satisfies one of the following two conditions:

(a) $b = b_0 + b_1$, where $b_1 \in \widetilde{L}^{p_1,q_1}(T)$ with $d/p_1 + 2/q_1 = 1$ and $p_1 \in (d, \infty]$, and $b_0 \in \tilde{L}^{d,\infty}(T)$ with $\|b_0\|_{\tilde{L}^{d,\infty}(T)} \leq \varepsilon$, for some constant $\varepsilon > 0$ only depending on $d,p_1,q_1$;

(b) $b \in \tilde{L}^{d,\infty}(T)$ and $\text{div}b \in \tilde{L}^{2,q_2}(T)$ with $q_2 \in (d/2, \infty)$.

Then there is a unique weak solution to (1) such that the following Krylov type estimate is valid:

$$\mathbb{E}\left(\int_0^T f(t,X_t)dt\right) \leq C \|f\|_{\tilde{L}^{p_3,q_3}(T)}, \quad \text{for any } p_3,q_3 \in (1, \infty) \text{ with } \frac{d}{p_3} + \frac{2}{q_3} < 2. \quad (4)$$

Here $\tilde{L}^{p,q}(T)$ ($\tilde{L}^{p,\infty}(T)$) is the localized space of $L^p(T)$ (weak $L^p(T)$) (see Section 2 for the precise definitions). $C$ is a constant, which does not depend on $f$.

**Remark 1.2.**

1. Our conditions in Theorem 1.1 cover the LPS condition, except for the endpoint case $(p,q) = (d, \infty)$, in which situation an additional assumption on the divergence of the drift is required.

2. For any $b \in C([0,T];L^d)$ and $\varepsilon > 0$, there exist two functions $b_0$ and $b_1$ such that $b = b_0 + b_1$ and $\|b_0\|_{L^d(T)} \leq \varepsilon$ and $b_1 \in \tilde{L}^{d,\infty}(T) \cap L^\infty([0,T] \times \mathbb{R}^d)$. Therefore, any functions in $C([0,T];L^d)$ meets condition (a) in Theorem 1.1 for arbitrary $\varepsilon > 0$.

3. As shown in [45, Theorem 5.1], for any $p \in (d/2, d)$, there exists a divergence free vector field $b \in L^p + L^\infty$ such that weak uniqueness of (1) fails. So our condition (b) is optimal when the drift is divergence free.
4. Our result can also be extended to general SDEs driven by multiplicative noises:

\[ \mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}^d. \]

It is sufficient to assume that \( \alpha := \frac{1}{2}\sigma\sigma^t \) is uniformly elliptic and uniformly continuous in \( x \) with respect to \( t \) if \( b \) satisfies assumption (a) in Theorem 1.1. When \( b \) satisfies condition (b), we need the additional condition that \( \partial_j \alpha^{ij} \) meets the same conditions as \( b \) in Theorem 1.1.

The following two intriguing examples are the main reason why we consider the weak Lebesgue spaces instead of the usual Lebesgue spaces here.

**Example.**

(i) \( b(x) = -\lambda x/|x|^2, \quad x \in \mathbb{R}^3 \) and \( \lambda > 0 \).

Obviously, \( b \notin L^3_{\text{loc}}(\mathbb{R}^3) \) but \( b \in L^{3,\infty}(\mathbb{R}^3) = \text{weak } L^3(\mathbb{R}^3) \) space. It was discussed in [2] and [44] that (1) has no weak solution if \( \lambda \) is large. However, when \( \lambda \) is sufficiently small, it was shown by Kinzebulatov-Semenov in [13] (see also [14]) that (1) has at least one weak solution. Our result shows the weak uniqueness holds for this example as well, as long as \( \lambda \) is small.

(ii) \( b(x) = \left( \frac{\lambda x_1 x_3}{(x_1^2 + x_2^2)|x|}, \frac{\lambda x_2 x_3}{(x_1^2 + x_2^2)|x|}, -\lambda \right), \quad x \in \mathbb{R}^3 \) and \( \lambda \in \mathbb{R} \).

Note that \( b \in L^{3,\infty}(\mathbb{R}^3) \setminus L^3_{\text{loc}}(\mathbb{R}^3) \) and \( \text{div} b \equiv 0 \). So in this case, our result implies that equation (1) has a unique weak solution for any \( \lambda \in \mathbb{R} \).

The preceding two examples also demonstrate that the drift’s integrability (or singularity) is not the only criterium for the well-posedness of (1). It is also influenced by the vector field’s structure.

To the best of the authors’ knowledge, there are few studies on the same subject prior to this paper, among them, we mention that Wei-Lv-Wu [37] studied the weak well-posedness of (1) when \( p, q \in [1, \infty) \), \( \frac{d}{p} + \frac{2}{q} = 1 \) and the quantity

\[ \sup_{t \in [0,T]} \left\| b(T-t) \right\|_{L^p(\mathbb{R}^d)}^{\frac{1}{2}} \]

is sufficiently small. Xia-Xie-Zhang-Zhao [38] proved the weak well-posedness of (1) when \( b \in C([0,T]; L^d(\mathbb{R}^d)) \). After this paper was announced, by refining the techniques in [13, 14], Kinzebulatov-Madou [12] also obtained that there exists a weak solution to (1) that is unique in an appropriate class when \( b \) satisfies

\[ \int_0^T \int_{\mathbb{R}^d} |(b\varphi)(t,x)|^2 \, dx \, dt \leq \delta \int_0^T \int_{\mathbb{R}^d} |\nabla \varphi(t,x)|^2 \, dx \, dt + \int_0^T g_\delta(t) \int_{\mathbb{R}^d} |\varphi(t,x)|^2 \, dx \, dt, \]

for all \( \varphi \in C_c^\infty([0,T] \times \mathbb{R}^d) \), some \( \delta \in (0, d^{-2}) \) and \( g_\delta \in L^1([0,T]) \). Such a condition is more general than the critical LPS condition with \( p > d \) (see Example 1.1 therein), but the main focus of this paper, namely, \( b \in L^d_{\text{loc}}(T) \), is much more delicate and cannot be addressed by the reasoning of [12] or [38].

We mention that in [44], Zhang and the second named author of this paper studied (1) when \( b \) does not satisfy the LPS condition and they proved that if \( b, \text{div} b \in L^q_{\text{loc}}(T) \) with \( p, q \in [2, \infty) \) and \( \frac{d}{p} + \frac{2}{q} \leq 2 \), then SDE (1) has at least one weak (martingale) solution. Later, the stochastic Lagrangian flow
associated to (1) was also considered by this paper’s second named author in [45]. We also point out that in a series of very recent works [16], [17], [20] and [21], when \( b \in L^{d+1}(\mathbb{R}^{d+1}) \), Krylov not only constructed a strong Markov process associated with (1), but also studied many properties of such process, such as Harnack’s inequality, higher summability of Green’s functions, and so on.

Next, we will go through the methodology utilized in this paper. Take \( T > 0 \), \( f \in C_0^\infty(\mathbb{R}^{d+1}) \) and consider the equation

\[
\partial_t u - \Delta u - b \cdot \nabla u = f \quad \text{in} \quad (0, T) \times \mathbb{R}^d, \quad u(0) = 0.
\] (5)

The existence of weak solutions to (1) follows from a standard tightness argument and a global maximum principle for weak solutions to (5). For uniqueness, under the same conditions as in Theorem 1.1, we shall find a uniformly bounded and sufficiently regular solution \( u \) to the above parabolic equation so that a generalized Itô formula can be applied to any weak solution of (1) and the function \( u(T - t, x) \) to obtain

\[
u(0, X_T) - u(T, x) = - \int_0^T f(t, X_t) dt + \sqrt{2} \int_0^T \nabla u(t, X_t) \cdot dW_t.
\]

Then by taking expectations of both sides, we obtain

\[
\mathbb{E} \int_0^T f(t, X_t) dt = u(T, x),
\]

which is enough to guarantee the uniqueness of \( X \) in law. The solvability of equation (5) in certain second-order Sobolev spaces under condition (a) in Theorem 1.1 is relatively straightforward and will be illustrated in Theorem 3.1 using a perturbation argument combined with a parabolic type Sobolev inequality. Like the regularity theory for 3D Navier-Stokes equations, the endpoint case \( b \in L^{d+\delta}_\infty(T) \) (without the smallness condition on \( \|b\|_{L^{d+\delta}_\infty(T)} \)) is more difficult and we have no answer to the full borderline case without an additional assumption on \( b \). However, when \( \text{div} b \in L^{\frac{d+\delta}{\delta}}_\infty(T) \) (\( \delta > 0 \)), by means of De Giorgi’s method, we can show that any bounded weak solution of (5) is indeed Hölder continuous. After that, we apply another interpolation inequality of Nirenberg (7) involving Hölder seminorms to show that \( b \cdot \nabla u \in \mathbb{L}^{p_3}_{q_3}(T) \) with some \( p_3, q_3 \in (1, \infty) \) and \( d/p_3 + 2/q_3 < 2 \). This yields that the bounded weak solution \( u \) to (5) is indeed in \( \mathbb{L}^{2p_3}_{q_3}(T) \) (see Theorem 3.2), which is regular enough to apply the generalized Itô’s formula. The above mentioned analytic results seem also to be new, and are thus of independent interest.

2. Preliminary

In this section, we introduce some notations and present some lemmas, which will be frequently used in this paper.

For any real-valued function \( f \), we define \( f^+ := \max\{0, f\} \) and \( f^- := \max\{0, -f\} \). Let \( D \) be an open subset of \( \mathbb{R}^d \) and \( f : D \to \mathbb{R} \). Define

\[
[f]_{\alpha; D} := \sup_{x, y \in D, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}, \quad \|f\|_{C^\alpha(D)} := \|f\|_{L^\infty(D)} + [f]_{\alpha; D}.
\]
For any \( p \in [1, \infty] \), by \( L^{p, \infty}(D) \) we mean the weak \( L^p(D) \) space with finite quasi-norm given by
\[
\|f\|_{L^{p, \infty}(D)} := \sup_{\lambda > 0} \lambda \left| \{ x \in D : |f(x)| > \lambda \} \right|^{1/p}.
\]

For any \( r > 0 \), we define
\[
Q_r(t, x) = (t - r^2, t) \times B_r(x), \quad Q_r = Q_r(0, 0).
\]

Let \( I \) be an open interval in \( \mathbb{R} \) and \( Q = I \times D \). For any \( p, q \in [1, \infty] \), by \( L^p_q(Q) \) and \( L^{p, \infty}_q(Q) \) we mean the space of functions on \( Q \) with finite norm given by
\[
\|u\|_{L^p_q(Q)} := \|u(t, \cdot)\|_{L^p(I)} \quad \text{and} \quad \|u\|_{L^{p, \infty}_q(Q)} := \|u(t, \cdot)\|_{L^{p, \infty}(D)} \|L^q(I)\|.
\]

We write \( u \in V(Q) \) if
\[
\|u\|_{V(Q)}^2 := \|u\|_{L^2_\infty(Q)}^2 + \|\nabla u\|_{L^2_\infty(Q)}^2 < \infty,
\]
and \( u \in V^0(Q) \) if \( u \in V(Q) \) and for any \( t \in I \),
\[
\lim_{h \to 0} \|u(t+h, \cdot) - u(t, \cdot)\|_{L^2(D)} = 0.
\]

Given a constant \( T > 0 \), with a little abuse of notations, for each \( p, q \in [1, \infty] \), we set
\[
L^p_q(T) := L^q([0, T]; L^p(\mathbb{R}^d)) \quad \text{and} \quad L^p(T) := L^p_p(T).
\]

For \( p, q \in (1, \infty), s \in \mathbb{R} \), define
\[
\mathbb{L}^{s,p}_q(T) = L^q([0, T]; H^{s,p}(\mathbb{R}^d)),
\]
where \( H^{s,p} \) is the Bessel potential space. The usual energy space is defined as the following way:
\[
V(T) := \left\{ f \in L^2_\infty(T) \cap L^2([0, T]; H^1) : \|f\|_{V(T)} := \|f\|_{L^2_\infty(T)} + \|\nabla f\|_{L^2(T)} < \infty \right\}.
\]

In this paper, we will also use the localized versions of the above functional spaces. Throughout this paper we fix a cutoff function
\[
\chi \in C_c^\infty(\mathbb{R}^d; [0, 1]) \text{ with } \chi|_{B_1} = 1 \text{ and } \chi|_{B_2} = 0. \tag{6}
\]

For \( r > 0 \) and \( x \in \mathbb{R}^d \), let \( \chi^r(x) := \chi \left( \frac{x-y}{r} \right) \). For any \( p, q \in [1, \infty] \), define
\[
\tilde{L}^p_q(T) := \{ f \in L^q([0, T]; L^p_{loc}(\mathbb{R}^d)) : \|f\|_{\tilde{L}^p_q(T)} := \sup_{y \in \mathbb{R}^d} \|f \chi^r\|_{L^p_q(T)} < \infty \},
\]
and set \( L^p(T) := \tilde{L}^p(T) \). Similarly,
\[
\tilde{L}^{p, \infty}_q(T) := \{ f \in L^q([0, T]; L^{p, \infty}_{loc}(\mathbb{R}^d)) : \|f\|_{\tilde{L}^{p, \infty}_q(T)} := \sup_{y \in \mathbb{R}^d} \|f \chi^r\|_{L^{p, \infty}_q(T)} < \infty \}.
\]
The localized Bessel potential spaces and energy spaces are defined as following:

\[ \tilde{H}^{s,p}_q(T) := \{ f \in L^q([0,T]; H^{s,p}_{\text{loc}}) : \| f \|_{\tilde{H}^{s,p}_q(T)} := \sup_{y \in \mathbb{R}^d} \| f \chi^0_y \|_{H^{s,p}(T)}, \] 

\[ \tilde{V}(T) := \{ f \in L^2_{\text{loc}}(T) \cap \tilde{H}^{1,2}_q(T) : \| f \|_{\tilde{V}(T)} := \| f \|_{L^2(T)} + \| \nabla f \|_{L^2(T)} < \infty \}, \]

\[ \tilde{V}^0(T) := \{ f \in \tilde{V}(T) : \text{for any } y \in \mathbb{R}^d, t \mapsto f(t) \chi^0_y \text{ is continuous from } [0,T] \text{ to } L^2(\mathbb{R}^d) \}. \]

The following Nirenberg’s interpolation inequality involving Hölder seminorms and De Giorgi’s isoperimetric inequality are well-know (cf. [3, Lemma 1.4], [23, Theorem 2] or [29]).

**Lemma 2.1** (Nirenberg’s interpolation inequalities). Suppose \( d \geq 2 \), \( j, m \in \mathbb{N} \), \( 0 < j < m \), and \( 1 < p < q < \infty \) such that

\[ \frac{pm}{j} < q \leq \frac{p(m-1)}{j-1}, \quad \alpha = \frac{jq - mp}{q - p} \quad \text{and} \quad \theta = \frac{p}{q}. \]

Then

\[ \| \nabla^j u \|_q \leq C \| \nabla^m u \|_p^{\theta}, \quad \forall u \in H^{m,p} \cap C^\alpha. \quad (7) \]

**Lemma 2.2** (De Giorgi’s isoperimetric inequality). There exists a constant \( c_d > 0 \) depending only on \( d \) such that for any function \( u: \mathbb{R}^d \to \mathbb{R} \), set

\[ A_u := \{ u \geq 1/2 \} \cap B_1, \quad B_u := \{ u \leq 0 \} \cap B_1, \quad D_u := \{ u \in (0,1/2) \} \cap B_1, \]

then

\[ \| \nabla u^+ \|_2^2 \geq c_d \frac{|A_u|^2 |B_u|^{2 - \frac{2}{d}}}{|D_u|}. \]

The following conclusion is a variant of Theorem 1.1 in [15].

**Lemma 2.3.** Let \( p, q \in (1,\infty) \), \( \lambda > 0 \). For each \( u \in L^q(\mathbb{R}; H^2, p(R^d)) \cap H^1, q(\mathbb{R}; L^p(R^d)) \), it holds that

\[ \| \partial_t u \|_{L^p} + \lambda \| \nabla^2 u \|_{2,q} \leq C \| \partial_t u - \lambda \Delta u \|_{L^p}, \quad (8) \]

where \( C \) only depends on \( d, p, q \).

The following lemma about the \( L^q L^p \)-maximal regularity estimates will be used several times later.

**Lemma 2.4.** Let \( p, q \in (1,\infty) \), \( \alpha \in \mathbb{R} \). Assume that \( f \in \tilde{H}^{2+\alpha,p}_q(T) \), then the following heat equation admits a unique solution in \( \tilde{H}^{2+\alpha,p}_q(T) \):

\[ \partial_t u - \Delta u = f \text{ in } (0,T) \times \mathbb{R}^d, \quad u(0) = 0 \quad (9) \]
and

$$
\|\partial_t u\|_{\dot{H}^{\alpha,p}_q(T)} + \|u\|_{\dot{H}^{2+\alpha,p}_q(T)} \leq C \|f\|_{\dot{H}^{\alpha,p}_q(T)},
$$

(10)

where $C = C(d, p, q, T)$. Moreover, if $f \in \dot{H}^{\alpha,p}_q(T)$, then

$$
\|\partial_t u\|_{\dot{H}^{\alpha,p}_q(T)} + \|\nabla^2 u\|_{\dot{H}^{\alpha,p}_q(T)} \leq C_1 \|f\|_{\dot{H}^{\alpha,p}_q(T)},
$$

(11)

where $C_1 = C_1(d, p, q)$.

**Proof.** The estimate (11) is a consequence of Theorem 1.1 in [15]. For any $y \in \mathbb{R}^d$, by (9), we have

$$
\partial_t (u\chi_1^y) - \Delta (u\chi_1^y) = f\chi_1^y - 2\nabla u \cdot \nabla \chi_1^y - u \Delta \chi_1^y.
$$

Thus, due to [15, Theorem 1.2] and [44, Proposition 4.1], for each $\tau \in [0, T]$,

$$
\begin{align*}
\|\partial_t u\|_{\dot{H}^{\alpha,p}_q(\tau)} &+ \|u\|_{\dot{H}^{2+\alpha,p}_q(\tau)} \\
\leq C \sup_{y \in \mathbb{R}^d} \left(\|\partial_t (u\chi_1^y)\|_{\dot{H}^{\alpha,p}_q(\tau)} + \|u\chi_1^y\|_{\dot{H}^{2+\alpha,p}_q(\tau)}\right) \\
\leq C \sup_{y \in \mathbb{R}^d} \left(\|f\chi_1^y\|_{\dot{H}^{\alpha,p}_q(\tau)} + \|\nabla u \cdot \nabla \chi_1^y\|_{\dot{H}^{\alpha,p}_q(\tau)} + \|u\Delta \chi_1^y\|_{\dot{H}^{\alpha,p}_q(\tau)}\right) \\
\leq C \|f\|_{\dot{H}^{\alpha,p}_q(\tau)} + C \sup_{y \in \mathbb{R}^d} \left(\|\nabla (u\chi_2^y) \cdot \nabla \chi_1^y\|_{\dot{H}^{\alpha,p}_q(\tau)} + \|(u\chi_2^y)\Delta \chi_1^y\|_{\dot{H}^{\alpha,p}_q(\tau)}\right) \\
\leq C \left(\|f\|_{\dot{H}^{\alpha,p}_q(\tau)} + \|u\|_{\dot{H}^{\alpha+1,p}_q(\tau)}\right).
\end{align*}
$$

A basic interpolation inequality yields,

$$
\|\partial_t u\|_{\dot{H}^{\alpha,p}_q(\tau)} + \|u\|_{\dot{H}^{2+\alpha,p}_q(\tau)} \leq C_T \left(\|f\|_{\dot{H}^{\alpha,p}_q(\tau)} + \|u\|_{\dot{H}^{\alpha,p}_q(\tau)}\right), \quad \forall \tau \in [0, T].
$$

(12)

Since for each $\tau \in [0, T]$,

$$
\|u(\tau)\chi_1^y\|_{H^{\alpha,p}} \leq \int_0^\tau \|\partial_t u(r)\chi_1^y\|_{H^{\alpha,p}} \, dr \leq C_T \left(\int_0^\tau \|\partial_t u(r)\chi_1^y\|_{H^{\alpha,p}}^q \, dr\right)^{1/q},
$$

together with (12), we obtain

$$
\|u(\tau)\|_{H^{\alpha,p}} \leq C \|f\|_{\dot{H}^{\alpha,p}_q(T)} + C \left(\int_0^\tau \|u(r)\|_{H^{\alpha,p}}^q \, dr\right)^{1/q}.
$$

By Gronwall’s inequality, we obtain

$$
\sup_{\tau \in [0, T]} \|u(\tau)\|_{H^{\alpha,p}} \leq C \|f\|_{\dot{H}^{\alpha,p}_q(T)}
$$

which together with (12) yields the desired estimate. \qed
Next we attempt to prove a parabolic version of Sobolev inequality, which will play a crucial role in the proof of our main result. This goal can be achieved by using the Mixed Derivative Theorem, which goes back to the work of Sobolevskii’s (cf. [32]).

Let $X$ be a Banach space and let $A : D(A) \to X$ be a closed, densely defined linear operator with dense range. Then $A$ is called sectorial, if

$$(0, \infty) \subseteq \rho(-A) \quad \text{and} \quad \left\| \lambda (\lambda + A)^{-1} \right\|_{\mathcal{L}(X)} \leq C, \quad \lambda > 0,$$

where $\rho(-A)$ is the resolvent set of $-A$. Set

$$\Sigma_\phi := \{ z \in \mathbb{C} \setminus \{0\} : |\arg z| < \phi \}.$$

We call $\phi_A := \inf \left\{ \phi \in [0, \pi) : \Sigma_{\pi - \phi} \subseteq \rho(-A), \quad \sup_{z \in \Sigma_{\pi - \phi}} \left\| z (z + A)^{-1} \right\|_{\mathcal{L}(X)} < \infty \right\}$

the spectral angle of $A$. For all $\theta \in (0, 1)$, the formulas

$$A^\theta x = \frac{\sin \theta \pi}{\pi} \int_0^\infty \lambda^{\theta - 1} (\lambda + A)^{-1} Ax \, d\lambda, \quad x \in D(A) \quad (13)$$

$$A^{-\theta} x = \frac{\sin \theta \pi}{\pi} \int_0^\infty \lambda^{-\theta} (\lambda + A)^{-1} x \, d\lambda, \quad x \in X \quad (14)$$

is valid (see [32]).

**Lemma 2.5 (Mixed Derivative Theorem).** Assume $A$ and $B$ are two sectorial operators in a Banach space $X$ with spectral angels $\phi_A$ and $\phi_B$, which are commutative and satisfy the parabolicity condition $\phi_A + \phi_B < \pi$. The coercivity estimate

$$\left\| Ax \right\|_X + \lambda \left\| Bx \right\|_X \leq M \left\| Ax + \lambda Bx \right\|_X, \quad \forall x \in D(A) \cap D(B), \lambda > 0$$

implies the estimate

$$\left\| A^{(1-\theta)} B^\theta x \right\|_X \leq C \left\| Ax + Bx \right\|_X, \quad \forall x \in D(A) \cap D(B), \theta \in [0, 1].$$

The above result suggests the following important parabolic type Sobolev inequality.

**Lemma 2.6.** Let $p, q \in (1, \infty)$, $r \in [p, \infty)$ and $s \in [q, \infty)$. Assume $\partial_t u \in L^p_q(T)$, $u \in L^{2,p}_q(T)$ and $u(0) = 0$. If $1 < d/p + 2/q = d/r + 2/s + 1$, then

$$\left\| \nabla u \right\|_{L^q_r(T)} \leq C_2 \left( \left\| \partial_t u \right\|_{L^p_q(T)} + \left\| \nabla^2 u \right\|_{L^p_q(T)} \right), \quad (15)$$

where $C_2$ only depends on $d, p, q, r, s$. 

Proof. Let $X = L^q(\mathbb{R}; L^p(\mathbb{R}^d))$, $A = \partial_t$ and $B = -\Delta$ in Lemma 2.5. It is well-known that

$$\phi_A = \frac{\pi}{2}, \quad \phi_B = 0.$$ 

Due to (8), for all $\lambda > 0$, we have

$$\|\partial_t u\|_{L^p_q} + \lambda \|\Delta u\|_{L^p_q} \leq C \|\partial_t u - \lambda \Delta u\|_{L^p_q},$$

where $C$ only depends on $d, p, q$. Thanks to Lemma 2.5, we get

$$\|\partial_t^{1-\theta} (-\Delta)^{\theta} u\|_{L^p_q} \leq C \|\partial_t u - \Delta u\|_{L^p_q} \leq C \left( \|\partial_t u\|_{L^p_q} + \|\nabla^2 u\|_{L^p_q} \right),$$  \hspace{1cm} (16)

for all $u \in H^{1,q}(\mathbb{R}, L^p) \cap L^q(\mathbb{R}, H^{2,p})$. By (14), one sees that

$$\partial_t^{1+\theta} f(t, x) = \frac{\sin(1-\theta)\pi}{\pi} \int_0^\infty \lambda^{-1+\theta} (\lambda + \partial_t)^{-1} f(t, x) \, d\lambda$$

$$= \frac{\sin(1-\theta)\pi}{\pi} \int_0^\infty \lambda^{-1+\theta} \int_{-\infty}^t e^{-\lambda(t-s)} f(s, x) \, ds \, d\lambda$$

$$= \frac{\Gamma(\theta) \sin(1-\theta)\pi}{\pi} \int_{-\infty}^t (t-s)^{-\theta} f(s, x) \, ds =: c_\theta (h_{\theta} * f)(t),$$  \hspace{1cm} (17)

where $h_{\theta}(t) := t^{-\theta} 1_{(0, \infty)}(t)$. Set $\theta = 1 + 1 - \frac{1}{p} = \frac{1}{2} + \frac{d}{2p} - \frac{d}{2p} \in \left[ \frac{1}{2}, 1 \right)$. Noting that $h_{\theta} \in L^{\frac{1}{\theta}, \infty}(\mathbb{R})$, by a refined version of Young’s inequality (cf. [1, Theorem 1.5]), we get

$$\|\partial_t^{1+\theta} f\|_{L^q(\mathbb{R})} \leq C \|h\|_{L^{1/q, \infty}(\mathbb{R})} \|f\|_{L^q(\mathbb{R})}, \quad \forall f \in L^q(\mathbb{R}),$$

which together with Sobolev’s inequality and the fact that $\frac{1}{p} = \frac{1}{p} - \frac{\theta-1/2}{d}$ yield

$$\|\nabla u\|_{L^q_x} \leq C \|\partial_t^{1-\theta} (-\Delta)^{\frac{1}{2}} u\|_{L^p_q} \leq C \|\partial_t^{1-\theta} (-\Delta)^{\theta} u\|_{L^p_q}.$$

Combing this and (16), we obtain

$$\|\nabla u\|_{L^q_x} \leq C \left( \|\partial_t u\|_{L^p_q} + \|\nabla^2 u\|_{L^p_q} \right),$$  \hspace{1cm} (18)

where $C$ only depends on $d, p, q, r, s$. Now if $u \in H^{2,p}_q(T)$, $\partial_t u \in L^p_q(T)$ and $u(0, x) = 0$, one can extend $u$ as

$$\bar{u}(t, x) := \begin{cases} 
 u(t, x) & \text{if } t \in [0, T] \\
 -3u(2T - t, x) + 4u \left( \frac{2T}{2} - \frac{t}{2}, x \right) & \text{if } t \in [T, 2T] \\
 4u \left( \frac{2T}{2} - \frac{t}{2}, x \right) & \text{if } t \in [2T, 3T] \\
 0 & \text{otherwise.}
\end{cases}$$

By the definition of $\bar{u}$, one sees that

$$100^{-1} \|\nabla^k \bar{u}\|_{L^p_q} \leq \|\nabla^k u\|_{L^p_q(T)} \leq \|\nabla^k \bar{u}\|_{L^p_q}$$
For any $q \geq 1$, we have

$$100^{-1}\|\partial_t u\|_{L^p_{q_1}} \leq \|\partial_t u\|_{L^2_{q_1}(T)} \leq \|\partial_t u\|_{L^p_{q_1}}.$$

Therefore, our desired result follows from (18).

### 3. Kolmogorov equation

Throughout this paper, $Q$ always means a domain in $\mathbb{R}^{d+1}$ and $T > 0$ is a time horizon. In this section, we study the unique solvability of the Kolmogorov equation (5) corresponding to (1) in some suitable $\mathbb{H}^{2,p_3}(T)$-space where $b$ satisfies the same assumptions as in Theorem 1.1.

#### 3.1. Case 1: $b$ satisfies condition (a) of Theorem 1.1

The main result in this subsection is

**Theorem 3.1.** Let $d \geq 3$, $b = b_0 + b_1$. Assume $b_1 \in \mathbb{L}^{p_1}_{q_1}(T)$ with $\frac{d}{p_1} + \frac{2}{q_1} = 1$ and $p_1 \in (d, \infty)$. Then for any $p_3 \in (1, d)$ and $q_3 \in (1, q_1)$ there is a constant $C = C(d, p_3, q_3) > 0$ such that for each $f \in \mathbb{L}^{p_3}_{q_3}(T)$, equation (5) has a unique solution $u \in \mathbb{H}^{2,p_3}(T)$, provided that $\|b_0\|_{L^{d,\infty}}(T) \leq \varepsilon$. Moreover,

$$\|\partial_t u\|_{\mathbb{H}^{2,p_3}(T)} + \|u\|_{\mathbb{H}^{2,p_3}(T)} \leq C\|f\|_{\mathbb{L}^{p_3}_{q_3}(T)},$$

where $C$ only depends on $d, p_1, q_1, \varepsilon, T$ and $b_1$.

**Proof.** To prove the desired result, it suffices to show (19) assuming that the solution already exists, since the method of continuity is applicable. We first establish the corresponding estimate in the usual space $\mathbb{H}^{2,p_3}(T)$, for any $p_3 \in (1, d)$, $q_3 \in (1, q_1)$ and some $\varepsilon = \varepsilon(d, p_3, q_3) > 0$.

Let $b_1^N := b_1 1_{\{|b_1| \leq N\}}$. Rewrite (5) as

$$\partial_t u - \Delta u = f + b_0 \cdot \nabla u + b_1^N \cdot \nabla u + (b_1 - b_1^N) \cdot \nabla u.$$ 

Thanks to [15, Theorem 1.2], for any $\tau \in [0, T]$,

$$\|\partial_t u\|_{\mathbb{H}^{2,p_3}_{q_3}(\tau)} + \|\nabla^2 u\|_{\mathbb{H}^{2,p_3}_{q_3}(\tau)} \leq C_3 \left( \|f\|_{\mathbb{L}^{p_3}_{q_3}(\tau)} + \|b_0 \cdot \nabla u\|_{\mathbb{L}^{p_3}_{q_3}(\tau)} + N \|\nabla u\|_{\mathbb{L}^{p_3}_{q_3}(\tau)} + \|(b_1 - b_1^N) \cdot \nabla u\|_{\mathbb{L}^{p_3}_{q_3}(\tau)} \right),$$

where $C_3 = C(d, p_3, q_3)$ does not depend on $t$. By [10, Exercise 1.4.19] and [34, Remark 5], we get

$$\|b_0 \cdot \nabla u\|_{\mathbb{L}^{p_3}_{q_3}(\tau)} \leq C \|b_0\|_{L^{d,\infty}} \|\nabla u\|_{\mathbb{L}^{p_3, (d-p_3)\cdot p_3}(\tau)} \leq C_4 \|b_0\|_{L^{d,\infty}} \|\nabla^2 u\|_{\mathbb{L}^{p_3}_{q_3}(\tau)},$$

where $L^{p,r}_q = L^q(\mathbb{R}_+; L^{p,r}(\mathbb{R}^d))$ (see [10] for the definition of $L^{p,r}(\mathbb{R}^d)$) and $C_4 = C_4(d, p_3)$. Set $1/r = 1/p_3 - 1/p_1$ and $1/s = 1/q_3 - 1/q_1$. By (15),

$$\|(b_1 - b_1^N) \cdot \nabla u\|_{\mathbb{L}^{p_3}_{q_3}(\tau)} \leq \|(b_1 - b_1^N)\|_{\mathbb{L}^{p_3}_{q_3}(T)} \|\nabla u\|_{\mathbb{L}^{p_3}_{q_3}(\tau)} \leq C_2 \|(b_1 - b_1^N)\|_{\mathbb{L}^{p_3}_{q_3}(T)} \left( \|\partial_t u\|_{\mathbb{L}^{p_3}_{q_3}(\tau)} + \|\nabla^2 u\|_{\mathbb{L}^{p_3}_{q_3}(\tau)} \right).$$
Let

\[ \varepsilon = \varepsilon(d, p_3, q_3) = (4C_3C_4)^{-1} > 0. \]

Noting that \( q_1 < \infty \), we can choose \( N \) sufficiently large so that \( \|b_1 - b_1^N\|_{L_{q_1}^p(T)} \leq (4C_2C_3)^{-1} \). By (20)-(22) and the choice of \( \varepsilon \) and \( N \), one sees that if \( \|b_0\| \leq \varepsilon \), then for each \( \tau \in [0, T] \),

\[
I(\tau) := \|\partial_t u\|_{L_{q_3}^{p_3}(\tau)}^{q_3} + \|\nabla^2 u\|_{L_{q_3}^{p_3}(\tau)}^{q_3} \leq C_5 \left( \|f\|_{L_{q_3}^{p_3}(\tau)}^{q_3} + N^{q_3} \|\nabla u\|_{L_{q_3}^{p_3}(\tau)}^{q_3} \right).
\]

(23)

Noting that

\[
\|u\|_{L_{q_3}^{p_3}(\tau)}^{q_3} = \int_0^\tau \|u(s, \cdot)\|_{L_{q_3}^{p_3}}^{q_3} ds = \int_0^\tau \left( \int_0^8 \|\partial_t u(r, \cdot)\|_{L_{q_3}^{p_3}}^{q_3} dr \right) ds.
\]

(24)

using an interpolation inequality, we obtain

\[
\|\nabla u\|_{L_{q_3}^{p_3}(\tau)}^{q_3} \leq \delta \|\nabla^2 u\|_{L_{q_3}^{p_3}(\tau)}^{q_3} + C_6 \|u\|_{L_{q_3}^{p_3}(\tau)}^{q_3}
\]

\[
\leq \delta I(\tau) + C_5 \int_0^\tau I(s) ds, \quad (\forall \delta > 0).
\]

(25)

Combing (23) and (25), we get

\[
I(\tau) \leq C_5 \delta N^{q_3} I(\tau) + C \|f\|_{L_{q_3}^{p_3}(\tau)}^{q_3} + C_6 N^{q_3} \int_0^\tau I(s) ds.
\]

Letting \( \delta = \delta(N) \) be small enough so that \( C_5 \delta N^{q_3} \leq 1/2 \), we get

\[
I(\tau) \leq C \|f\|_{L_{q_3}^{p_3}(\tau)}^{q_3} + C \int_0^\tau I(s) ds, \quad \forall \tau \in [0, T].
\]

Gronwall’s inequality yields \( I(T) \leq C \|f\|_{L_{q_3}^{p_3}(T)}^{q_3} \), which together with (24) implies

\[
\|\partial_t u\|_{L_{q_3}^{p_3}(T)} + \|u\|_{L_{q_3}^{p_3}(T)}^{q_3} \leq C \|f\|_{L_{q_3}^{p_3}(T)}^{q_3},
\]

(26)

where \( C = C(d, p_1, q_1, T, \varepsilon, b_1) \). Our desired estimate (19) is then obtained by (26) and an argument similar to the one in the proof for Lemma 2.4.

3.2. Case 2: \( b \in \mathbb{L}_{d,\infty}^d(T) \) and \( \text{div} b \in \mathbb{L}_{\infty}^{p_2}(T) \) with \( p_2 > d/2 \).

In this subsection, we will give an analogue of Theorem 3.1, where \( b \in \mathbb{L}_{\infty}^{d,\infty}(T) \) and \( \text{div} b \in \mathbb{L}_{\infty}^{p_2}(T) \). The main result is stated as follows:
Theorem 3.2. Let $d \geq 3$ and assume that $b \in \mathbb{L}^{d,\infty}_\infty(T)$ and $\text{div} b \in \mathbb{L}^{d,\infty}_\infty(T)$ for some $p_2 > d/2$. Then there are constants $p_3 \in (d/2, d)$ and $q_3 \in (2p_3/(2p_3 - d), \infty)$ such that for each $f \in \mathbb{L}^{d,\infty}_\infty(T)$, equation (5) has a unique solution $u \in \mathbb{L}^{p_3,q_3}(T)$. Moreover,

$$\|\partial_t u\|_{\mathbb{L}^{p_3,q_3}(T)} + \|u\|_{\mathbb{L}^{p_3,q_3}(T)} \leq C \|f\|_{\mathbb{L}^{d,\infty}_\infty(T)},$$

where $C$ only depends on $d, p_2, p_3, q_3, T \|b\|_{\mathbb{L}^{d,\infty}_\infty(T)}$ and $\|\text{div} b\|_{\mathbb{L}^{d,\infty}_\infty(T)}$.

Unlike the previous case, if $\|b\|_{\mathbb{L}^{d,\infty}_\infty(T)}$ is large, then $\|\partial_t u\|_{\mathbb{L}^{p_3,q_3}(T)}$ may not be controlled by $\|\partial_t u\|_{\mathbb{L}^{p_3,q_3}(T)} + \|u\|_{\mathbb{L}^{p_3,q_3}(T)}$, so the perturbation argument does not work any more. In order to overcome this difficulty, in this subsection, by means of De Giorgi’s method, we first show that any bounded weak solution of (5) is indeed Hölder continuous, provided that $b$ is in some Morrey type space and $\text{div} b \in \mathbb{L}^{d,\infty}_\infty(T)$. Then in the light of Nirenberg’s inequality (7), we show that $\nabla u$ is indeed in $\mathbb{L}^{r,q_3}_r(T)$ with some $r > d$, which implies $b \cdot \nabla u \in \mathbb{L}^{p_3,q_3}(T)$ with some $p_3 > d/2$ and $q_3 > 2p_3/(2p_3 - d)$. Our desired result then follows by Lemma 2.4.

We first give the precise definition of weak solutions to the equation

$$\partial_t u - \Delta u - b \cdot \nabla u = f \quad \text{in } Q = I \times D.$$  

(28)

Definition 3.3. Assume $b \in L^2_{\text{loc}}(Q)$. We say $u \in V_{\text{loc}}(Q)$ is a subsolution (supersolution) to (28) if for any $\varphi \in C^\infty_c(Q)$ with $\varphi \geq 0$,

$$\int_Q \left[ -u \partial_t \varphi + \nabla u \cdot \nabla \varphi - b \cdot \nabla u \varphi \right] \leq (\geq) \int_Q f \varphi.$$  

(29)

$u \in V_{\text{loc}}(Q)$ is a solution to (28) if $u$ and $-u$ are subsolutions to (28).

For any $p, q \in (1, \infty]$, here and below we define $p^*, q^* \in [2, \infty]$ by the relations

$$\frac{1}{p} + \frac{2}{p^*} = 1, \quad \frac{1}{q} + \frac{2}{q^*} = 1.$$  

(30)

The following two lemmas are crucial for proving Theorem 3.2, and their proofs are essentially contained in [44] and [45]. We provide sketches of their proofs in the Appendix for the reader’s convenience.

Lemma 3.4 (Energy inequality). Assume $0 < \rho < R \leq 1$, $k \geq 0$, $I \subseteq \mathbb{R}$ is an open interval, $Q = I \times B_R$ and $\eta$ is a cut off function in $x$, compactly supported in $B_R$, $\eta(x) \equiv 1$ in $B_\rho$, and $|\nabla \eta| \leq 2(\rho - \rho)$. Let $d \geq 2, p_i, q_i \in (1, \infty]$ satisfying $d/p_i + 2/q_i < 2, i = 2, 3$. Suppose that $b, \text{div} b \in L^2_{\text{loc}}(Q), f \in \mathbb{L}^{p_3,q_3}_{p_3,q_3}(Q)$ and $u \in V(Q)$ is a bounded weak subsolution to (28), then

$$(\int u_k^2 \eta^2)^{(t)} - (\int u_k^2 \eta^2)^{(s)} + \int_s^t \int \nabla (u_k \eta)^2$$

$$\leq C_6 \frac{((R - \rho)^2)}{(R - \rho)^2} \left( \|u_k\|^2_{L^2_{p_3,q_3}(A_s^k(k))} + \sum_{i=2}^3 \|u_k\|^2_{L^2_{p_i,q_i}(A_s^k(k))} \right) + C_6 \|f\|^2_{L^2_{p_3,q_3}(Q)} \|1_{A_s^k(k)}\|^2_{L^2_{q_3}(Q)},$$

(EI)

where $u_k = (u - k)^+ = \max\{0, u - k\}$, $A_s^k(k) = \{u > k\} \cap ([s, t] \times B_R)$ and $C_6$ only depends on $d, p_i, q_i, \|b\|_{L^{p_2}_{p_2}(Q)}$ and $\|\text{div} b\|_{L^{p_2}_{p_2}(Q)}$. 

Lemma 3.5. Let $d \geq 2$, $p_i, q_i \in (1, \infty)$ satisfying $d/p_i + 2/q_i < 2$, $i = 2, 3$. Suppose $b, \text{div} b \in \mathbb{L}^{p_i}_{q_i}(Q_1)$ and $u \in V(Q_1)$ is a bounded weak subsolution to (28) in $Q_1$. Then for any $f \in \mathbb{L}^{p_i}_{q_i}(Q_1)$,

$$
\|u^+\|_{L^\infty(Q_{1/2})} \leq C_7 \left( \|u^+\|_{L^2_{\beta}(Q_1)} + \sum_{i=2}^{3} \|u^+\|_{L^{p_i}_{q_i}(Q_1)}^* + \|f\|_{L^{p_i}_{q_i}(Q_1)} \right).
$$

( LM)

Here $C_7$ only depends on $d, p_i, q_i, \|b\|_{L^{p_i}_{q_i}(Q_1)}$ and $\|\text{div} b\|_{L^{p_i}_{q_i}(Q_1)}$.

In order to prove the Hölder estimate for the bounded weak solutions to (5), we also need some technical Lemmas. One of them is a parabolic version of De Giorgi’s Lemma. We set $Q'_1 := (-2, -1) \times B_1$. For any $u : Q_1 \cup Q'_1 \to \mathbb{R}$, define

$$
A_u := \{ f \geq 1/2 \} \cap Q_1, \quad B_u := \{ f \leq 0 \} \cap Q'_1, \quad D_u := \{ 0 < f < 1/2 \} \cap (Q_1 \cup Q'_1).
$$

Lemma 3.6. Let $p_i, q_i \in (1, \infty)$ satisfying $d/p_i + 2/q_i < 2$, $i = 2, 3$. Assume $b, \text{div} b \in \mathbb{L}^{p_i}_{q_i}(Q_2), \|f\|_{L^{p_i}_{q_i}(Q_2)} \leq 1$ and that $u$ is a weak subsolution to (28) in $Q_2$ with $u \leq 1$. Suppose that $\delta \in (0, 1)$ and that

$$
|A_u| \geq \delta \quad \text{and} \quad |B_u| \geq \delta.
$$

Then

$$
|D_u| = |\{ 0 < u < 1/2 \} \cap (Q_1 \cup Q'_1)| \geq \beta,
$$

where $\beta = \beta(d, p_i, q_i, \|b\|_{L^{p_i}_{q_i}(Q_2)}, \|\text{div} b\|_{L^{p_i}_{q_i}(Q_2)}, \delta)$ is a universal constant that does not depend on $u$.

Proof. By (EI), Hölder’s inequality and our assumption that $u \leq 1$, we have

$$
\begin{align*}
&\left( \int_{B_1} (u^+)^2(t) - \int_{B_1} (u^+)^2(s) \right) + \int_s^t \int_{B_1} |\nabla (u^+)|^2 \, dxdr \\
&\leq C \left( \|u^+\|_{L^2_{\beta}(Q_1)}^2 + \sum_{i=2}^{3} \|u^+\|_{L^{p_i}_{q_i}(Q_1)}^2 \right) \leq C|t - s|^\theta,
\end{align*}
$$

\begin{equation}
\tag{31}
\forall \theta = \frac{1}{2} \wedge (1 - \frac{1}{d^2}) \wedge (1 - \frac{1}{q_i}) > 0. \quad \text{Assume} \quad |D_u| < \beta, \quad \text{where} \quad \beta > 0 \quad \text{is a small number, which will be determined later.}
\end{equation}

Let

$$
a(t) = |\{ x \in B_1 : u^+(t, x) \geq 1/2 \}|, \\
b(t) = |\{ x \in B_1 : u^+(t, x) = 0 \}|, \\
d(t) = |\{ x \in B_1 : 0 < u^+(t, x) < 1/2 \}|.
$$

Set

$$
I_1 := \{ t \in (-2, 0) : d(t) \leq \sqrt{\beta} \} \quad \text{and} \quad I_2 := \left\{ t \in I_1 : b(t) > \frac{\delta}{100d^i} \text{ or } b(t) < \frac{\delta}{100d^i} \right\}.
$$
By the assumption and Chebyshev’s inequality, one sees that \(|I_1| \geq 2 - C\sqrt{\beta}\). Using (31) and Lemma 2.2, we have

\[
C \geq \int_{I_1} \int_{B_1} |\nabla u^+(t,x)|^2 \, dx \geq c_d \beta^{-\frac{1}{2}} \int_{I_1} a^2(t)b^2 \bar{\alpha}(t) \, dt.
\]

Thus,

\[
\int_{I_1} a^2(t)b^2 \bar{\alpha}(t) \, dt \leq C \sqrt{\beta} \to 0 \quad \text{as} \quad \beta \to 0.
\]

This together with the facts that

\[
\inf_{t \in I_1} [a(t) + b(t)] \geq |B_1| - \sqrt{\beta} \geq 1/d!
\]

and \(|I_1| \to 2 \) implies \(|I_2| \to 2 \) as \( \beta \to 0 \). Since the zero set of \( u^+ \) has mass \( \delta \) in \( Q_1' \), for small \( \beta \), there is some \( t_1 \in (-2, -1) \cap I_2 \) such that \( b(t_1) > |B_1| - \frac{\delta}{1000\sqrt{\beta}} \). Using the first term in the energy estimate (31), we see that for some universal small \( \tau > 0 \), there exists \( t_2 \in I_2 \) with \( t_2 \geq t_1 + \tau \) such that for all \( t \in [t_1, t_2] \cap I_2 \), \( b(t) > |B_1| - \frac{\delta}{10000\sqrt{\beta}} \). Iterating, we obtain that for all \( t \in [t_1, 0] \cap I_2 \), \( a(t) < \frac{\delta}{100000\sqrt{\beta}} \). That is a contradiction to \( |A_u| \geq \delta \) and completes the proof. \( \square \)

The next diminish of oscillation lemma is crucial for the Hölder estimate for the solutions to (5).

**Lemma 3.7.** Let \( p_i, q_i \in (1, \infty) \) satisfying \( d/p_i + 2/q_i < 2 \), \( i = 2, 3 \). Assume \( b \in L^{p_2}_{Q_2} \) and \( \text{div} b \in L^{p_3}_{Q_2} \). Then there exist universal positive constants \( \mu < 1 \) and \( \varepsilon_0 > 0 \) only depending on \( d, p_i, q_i, \|b\|_{L^{p_2}_{Q_2}} \) and \( \|\text{div} b\|_{L^{p_3}_{Q_2}} \), such that for any \( f \in L^{p_3}_{Q_2} \) with \( \|f\|_{L^{p_3}_{Q_2}} \leq \varepsilon_0 \), and any weak subsolution \( u \) of (28) in \( Q_2 \) satisfying \( u \leq 1 \) and \(|\{u < 0\} \cap Q_1' | \geq |Q_1'|/2 \), the following estimate is valid:

\[
u \leq \mu \text{ in } Q_{1/2}.
\]

**Proof.** We consider \( u_k = 2^k \left(u - (1 - 2^{-k})\right) \), which fulfills for each \( k \geq 0 \),

\[
u_k \leq 1, \quad B_{u_k} := |\{u_k < 0\} \cap Q_1' | \geq |Q_1'|/2
\]

and

\[
\partial_t u_k - Lu_k = 2^k f =: f_k.
\]

Let \( \delta \in (0, 1) \) be a sufficiently small number such that \( 8C_7\delta \leq 1 \), where \( C_7 \) is the constant in (LM). Suppose \( \beta = \beta_\delta > 0 \) is the same constant as in Lemma 3.6 and set

\[
K := \lfloor 3|B_1|/\beta \rfloor + 1 \quad \text{and} \quad \varepsilon_0 := 2^{-K}\delta.
\]

By the definitions of \( K \) and \( \varepsilon_0 \), one can see that for each \( k \in \{1, 2, \cdots, K\} \),

\[
\|f_k\|_{L^{p_3}_{Q_2}} = 2^k ||f||_{L^{p_3}_{Q_2}} \leq 2^K \varepsilon_0 = \delta < 1.
\]  

(32)

We claim that

\[
\|u_+^K\|_{L^2_{Q_1}}^2 + \sum_{i=2}^3 \|u_+^K\|_{L^{p_i}_{Q_1}}^2 \leq 3\delta.
\]  

(33)
Assume (33) does not hold. Noting that \( u_k \) is decreasing, so

\[
\|u_k^+\|_{L^2(Q_1)}^2 + \sum_{i=2}^{3} \|u_k^+\|_{L^p_{q_i}(Q_1)}^2 > 3\delta, \quad \forall k \in \{1, 2, \cdots, K\}.
\]

By (34) and the fact that \( u_k \leq 1 \), we get

\[
\{u_k \geq 0\} \cap Q_1 \geq \frac{1}{3} \left( \|u_k^+\|_{L^2(Q_1)}^2 + \sum_{i=2}^{3} \|u_k^+\|_{L^p_{q_i}(Q_1)}^2 \right) > \delta.
\]

Thus,

\[
|A_{u_{k-1}}| := \{u_{k-1} \geq 1/2\} \cap Q_1 = |\{u_k \geq 0\} \cap Q_1| > \delta.
\]

Recalling that \( u_{k-1} \leq 1 \), \( |B_{u_{k-1}}| \geq |Q_1'|/2 \) and \( \|f_{k-1}\|_{L^p_{q_3}(Q_2)} \leq 1 \), by Lemma 3.6, we have

\[
|\{1 - 2^{-k+1} < u < 1 - 2^{-k}\} \cap (Q_1 \cup Q_1')| = |D_{u_{k-1}}| \geq \beta.
\]

Hence,

\[
2|B_1| \geq |\{0 < u < 1 - 2^{-K}\} \cap (Q_1 \cup Q_1')|
\geq \sum_{k=1}^{K} |\{1 - 2^{-k+1} < u < 1 - 2^{-k}\} \cap (Q_1 \cup Q_1')|
\geq K\beta = (3|B_1|/\beta + 1)\beta \geq 3|B_1|,
\]

which is a contradiction. So we complete the proof for (33). This together with (LM) yields

\[
\|u_K^+\|_{L^\infty(Q_{1/2})} \leq C_7 \left( \|u_K^+\|_{L^2(Q_1)}^2 + \sum_{i=2}^{3} \|u_K^+\|_{L^p_{q_i}(Q_1)}^2 + \|f_K\|_{L^p_{q_3}(Q_1)} \right)
\leq 4C_7\delta = 1/2,
\]

which implies

\[
\sup_{x \in Q_{1/2}} u \leq 1 - 2^{-K-1}.
\]

By letting \( \mu = 1 - 2^{-K-1} \), we complete our proof. \( \square \)

**Lemma 3.8.** Let \( p_i, q_i \in (1, \infty) \) satisfy \( d/p_i + 2/q_i < 2, \ i = 2, 3 \). Assume \( b, \text{div} b \in L^{p_3}_{q_3}(Q_2) \). Suppose \( u \) is a weak solution to (28) in \( Q_2 \) with \( f \in L^{p_3}_{q_3}(Q_2) \). Then

\[
\text{osc}_{Q_{1/2}} u \leq \mu \text{osc}_{Q_1} u + C\|f\|_{L^{p_3}_{q_3}(Q_2)},
\]

where \( \mu < 1 \) is the same constant as in Lemma 3.7 and \( C \) only depends on \( d, p_i, q_i, \|b\|_{L^{p_3}_{q_3}(Q_2)} \) and \( \|\text{div} b\|_{L^{p_3}_{q_3}(Q_2)} \).
\textbf{Proof.} Define

\[ u_\delta := \frac{u}{\delta + \|u^+\|_{L^\infty(Q_2)} + \varepsilon_0^{-1} \|f\|_{L^{p_3}_T(Q_2)}} \leq 1, \quad \delta > 0, \]

where \( \varepsilon_0 \) is the same constant as in Lemma 3.7. Then \( u_\delta \) satisfies

\[ \partial_t u_\delta - Lu_\delta = f_\delta := f(\delta + \|u^+\|_{L^\infty(Q_2)} + \varepsilon_0^{-1} \|f\|_{L^{p_3}_T(Q_2)})^{-1} \quad \text{in } Q_2 \]

and \( \|f_\delta\|_{L^{p_3}_T(Q_2)} \leq \varepsilon_0 \). By Lemma 3.7, we have \( u_\delta \leq \mu \) in \( Q_{1/2} \), so

\[ \|u^+\|_{L^\infty(Q_{1/2})} \leq \mu \liminf_{\delta \to 0} \left( \delta + \|u^+\|_{L^\infty(Q_2)} + \varepsilon_0^{-1} \|f\|_{L^{p_3}_T(Q_2)} \right) \]

\[ \leq \mu \|u^+\|_{L^\infty(Q_2)} + \mu \varepsilon_0^{-1} \|f\|_{L^{p_3}_T(Q_2)}. \]

Similarly, \( \|u^-\|_{L^\infty(Q_{1/2})} \leq \mu \|u^-\|_{L^\infty(Q_2)} + \mu \varepsilon_0^{-1} \|f\|_{L^{p_3}_T(Q_2)} \). So, we complete our proof. \( \square \)

Now we are at the point to show the Hölder regularity of the solutions to (5).

\textbf{Lemma 3.9.} Let \( p_i, q_i \in (1, \infty) \) satisfying \( d/p_i + 2/q_i < 2 \), \( i = 2, 3 \). Suppose that \( b, \text{div}b \in L^{p_3}_T(Q_2) \) and \( f \in L^{p_3}_T(Q_2) \). If there is a constant \( K_b \in (0, \infty) \) such that for all \( (t, x) \in [0, T] \times \mathbb{R}^d \) and \( r \in (0, 1) \),

\[ r^{1 - \frac{d}{p_2} - \frac{d}{q_2}} \|b\|_{L^{p_3}_T(Q_2)} \leq K_b, \]

then there are constants \( \alpha \in (0, 1) \) and \( C > 1 \) such that for any bounded weak solution \( u \in \overline{V}^0(T) \) to (5), it holds that

\[ \|u\|_{C^\alpha([0, T] \times \mathbb{R}^d)} \leq C \|f\|_{L^{p_3}_T(T)}, \]

where \( \alpha, C \) only depend on \( d, p_i, q_i, T, \|b\|_{L^{p_3}_T(T)} \) and \( K_b \).

\textbf{Proof.} For convenience, we extend \( u, b, f \) to be functions on \(( -\infty, T) \times \mathbb{R}^d \) by letting \( u(t, x) = b(t, x) = f(t, x) = 0 \), for all \( t \leq 0 \) and \( x \in \mathbb{R}^d \). By Definition 3.3, \( u \) is still a bounded weak solution to (28) on \(( -\infty, T) \times \mathbb{R}^d \). For any \( r \in (0, 1) \), \( (t_0, x_0) \in ( -\infty, T) \times \mathbb{R}^d \) \( \text{and } (t, x) \in Q_2 \), define \( u_r(t, x) := u(r^2t + t_0, rx + x_0) \), \( b_r(t, x) := rb(r^2t + t_0, rx + x_0) \), \( f_r(t, x) := r^2f(r^2t + t_0, rx + x_0) \). Then \( u_r \) satisfies

\[ \partial_t u_r - \Delta u_r - b_r \cdot \nabla u_r = f_r \quad \text{in } Q_2. \]

By our assumption, we have

\[ \|b_r\|_{L^{p_3}_T(Q_2)} = r^{1 - \frac{d}{p_2} - \frac{d}{q_2}} \|b\|_{L^{p_3}_T(Q_2)} \leq K_b, \]

\[ \|\text{div}b_r\|_{L^{p_3}_T(Q_2)} = r^{\alpha_2} \|\text{div}b\|_{L^{p_3}_T(Q_2)} \leq \|\text{div}b\|_{L^{p_3}_T(T)}, \]

\[ \|f_r\|_{L^{p_3}_T(Q_2)} = r^{\alpha_3} \|f\|_{L^{p_3}_T(Q_2)} \].
where $\kappa_i = 2 - d/p_i - 2/q_i > 0$, $i = 2, 3$. Using (35), we get

$$
\text{osc}_{Q_{2r}^+(t_0, x_0)} u \leq \mu \text{osc}_{Q_{2r}^+(t_0, x_0)} u + C r^{-1} \|f\|_{L_{p_2}^1(Q_{2r}(t_0, x_0))}, \quad \mu \in (0, 1).
$$

(37)

The desired estimate (36) follows by (37) and standard arguments (see [11, Lemma 3.4]).

\[\square\]

**Remark 3.10.** We should also point out that the Harnack inequality for Lipschitz continuous solutions to (28) with $f \equiv 0$ was also obtained in [28] by Moser iteration method.

To prove our desired result, we also need the following simple lemma.

**Lemma 3.11.** Let $1 < p < r < \infty$. Then there is a constant $C = C(d, p, r)$ such that

$$
\|f\|_{L_p} \leq C(d, p, r)\|f\|_{L_r^{\infty}}.
$$

(38)

**Proof.** Let $A$ be a Borel subset of $\mathbb{R}^d$ with finite Lebesgue measure. Set

$$
\mu_f(t) = |\{x \in A : |f(x)| > t\}|.
$$

Then,

$$
\int_A |f|^p = p \int_0^\infty t^{p-1} \mu_f(t) dt = p \int_0^\infty t^{p-1} |A| dt + p \|f\|_{L_{r, \infty}}^r(\lambda) \int_\lambda^\infty t^{p-r-1} dt
$$

$$
\leq \lambda^p |A| + p(r - p)^{-1} \|f\|_{L_{r, \infty}}^r(\lambda) \lambda^{p-r}.
$$

Letting $\lambda = (\frac{p}{r-p})^{1/r} \|f\|_{L_{r, \infty}}^r(\lambda) |A|^{-1/r}$, we obtain

$$
\|f\|_{L_p(A)} \leq 2^{1/p} \left(\frac{p}{r-p}\right)^{1/r} \|f\|_{L_{r, \infty}}^r(A) |A|^{1/p-1/r}.
$$

Thus,

$$
\|f\|_{L_p} \leq \sup_{y \in \mathbb{R}^d} \|f\|_{L_p(B_2(y))} \leq C(d, p, r)\|f\|_{L_r^{\infty}}.
$$

\[\square\]

Now we are in the position of proving Theorem 3.2.

**Proof of Theorem 3.2.** Since $L_p \subseteq L_p^{0} (p > p')$, we can assume $p_2 \in (d/2, d)$. Letting $q_2 \in (1, \infty)$ such that $d/p_2 + 2/q_2 < 2$, by our assumptions on $b$, one sees that $b, \text{div} b \in L_{q_2}^{p_2}(T)$ and for any $r \in (0, 1)$, $t_0 \in [0, T]$ and $x_0 \in \mathbb{R}^d$,

$$
r^{1 - \frac{d}{p_2} - \frac{2}{q_2}} \left(\int_{t_0 - r^2}^{t_0} \|b(t, \cdot)\|_{L_{q_2}^{p_2}(B_r(x_0))}^q dt\right)^{1/q_2}
$$

\[\leq C r^{-\frac{2}{q_2}} \left(\int_{t_0 - r^2}^{t_0} \|b(t, \cdot)\|_{L_{q_2}^{p_2}(B_r(x_0))}^q dt\right)^{1/q_2} \leq C \|b\|_{L_{q_2}^{p_2}(B_r(x_0))}.
$$

(38)
Let \( p_3' \in (d/2, d), q_3' \in (1, \infty) \) be some constants satisfying \( d/p_3' + 2/q_3' < 2 \). Again by Lemma 3.11, \( f \in \mathcal{L}_{q_3}^3(T) \). Thanks to Lemma A.3 and Theorem 3.9, (5) admits a unique bounded weak solution \( u \), and there is a constant \( \alpha \in (0, 1) \) only depending on \( d, p_2, q_2, p_3', q_3' \), \( T, ||b||_{\mathcal{L}_{q_3}^3(T)} \) and \( ||\text{div} b||_{\mathcal{L}_{q_3}^3(T)} \) such that

\[
||u||_{C^\alpha([0,T] \times \mathbb{R}^d)} \leq C ||f||_{\mathcal{L}_{q_3}^3(T)} \leq C ||f||_{\mathcal{L}_{q_3}^\infty(T)}.
\] (39)

Next we fix

\[
s \in \left( 2 \sqrt{\frac{d(4 - 3\alpha)}{4 - 2\alpha}}, d \right), \quad q_3 > \frac{2s(2 - \alpha)}{2s(2 - \alpha) - d(4 - 3\alpha)}.
\] (40)

Rewrite (5) as \( \partial_t u - \Delta u = f + \text{div}(bu) - (\text{div} b) u \). It is easy to see that

\[
||f||_{\mathcal{L}_{q_3}^{1,s}(T)} \leq C ||f||_{\mathcal{L}_{q_3}^{d,\infty}(T)}, \quad ||\text{div}(bu)||_{\mathcal{L}_{q_3}^{1,s}(T)} \leq C ||b||_{\mathcal{L}_{q_3}^2(T)} ||u||_{L_{q_3}^\infty(T)} \leq C ||f||_{\mathcal{L}_{q_3}^{d,\infty}(T)}.
\] (39)

Noting that \( 1 < sd/(d + s) < d/2 < p_2 \) and using Sobolev embedding, we see that

\[
||((\text{div} b)u)||_{\mathcal{L}_{q_3}^{-1,s}(T)} \leq C ||((\text{div} b)u)||_{\mathcal{L}_{q_3}^{-2/(s+d)}}(T) \leq C ||f||_{\mathcal{L}_{q_3}^{d,\infty}(T)}.
\] (39)

By Lemma 2.4,

\[
||u||_{\mathcal{L}_{q_3}^{-1,s}(T)} \leq C ||f + \text{div}(bu) - (\text{div} b) u||_{\mathcal{L}_{q_3}^{1,s}(T)} \leq C ||f||_{\mathcal{L}_{q_3}^{d,\infty}(T)}.
\]

Using this and noting the fact that \( 1 < s/2 < d/2 \), we get

\[
||b \cdot \nabla u||_{\mathcal{L}_{q_3}^{s/2}(T)} \leq ||b||_{\mathcal{L}_{q_3}^{2,\infty}(T)} ||u||_{\mathcal{L}_{q_3}^{1,s}(T)} \leq C ||f||_{\mathcal{L}_{q_3}^{d,\infty}(T)}.
\]

Again by Lemma 2.4, we obtain

\[
||u||_{\mathcal{L}_{q_3}^{2,s/2}(T)} \leq C ||f + b \cdot \nabla u||_{\mathcal{L}_{q_3}^{s/2}(T)} \leq C ||f||_{\mathcal{L}_{q_3}^{d,\infty}(T)}.
\] (41)

In the light of Nirenberg’s inequality (7), we get

\[
||\nabla u||_{\mathcal{L}_{q_3}^{\theta}(T)} \leq C ||\nabla^2 u||_{\mathcal{L}_{q_3}^{s/2}(T)}^{\theta} ||u||_{C^\alpha([0,T] \times \mathbb{R}^d)}^{1 - \theta} \, (39),(41)
\]

where

\[
r = \frac{(2 - \alpha)s}{2 - 2\alpha}, \quad \theta = \frac{s}{2r} \in (0, 1).
\] (42)

Now letting

\[
\frac{1}{p_3'} = \frac{1}{r} + \frac{1}{s} = \frac{4 - 3\alpha}{s(2 - \alpha)},
\] (43)
by Hölder’s inequality,
\[ \|b \cdot \nabla u\|_{L^{p_3}_t(\mathbb{R}^d)} \leq C\|b\|_{L^{d,\infty}_t(\mathbb{R}^d)} \|\nabla u\|_{L^{p_3}_t(\mathbb{R}^d)} \leq C\|b\|_{L^{d,\infty}_t(\mathbb{R}^d)} \|f\|_{L^{d,\infty}_t(\mathbb{R}^d)} . \]

Using Lemma 2.4 again, we obtain
\[ \|\partial_t u\|_{L^{p_3}_t(\mathbb{R}^d)} + \|u\|_{L^{\infty,p_3}_t(\mathbb{R}^d)} \leq C \|f\|_{L^{d,\infty}_t(\mathbb{R}^d)} . \]

Combining (40), (42) and (43), we get
\[ \frac{d}{p_3} + \frac{2}{q_3} = \frac{d(4-3\alpha)}{s(2-\alpha)} + \frac{2}{q_3} < 2 . \]

So we complete our proof. \( \square \)

4. Proof of the main result

In this section, we present the proof of our main probabilistic results. Firstly, we give the precise definition of martingale solutions to (1).

**Definition 4.1.** For given \( x \in \mathbb{R}^d \), we call a probability measure \( \mathbb{P}_x \in \mathcal{P}(C([0,T];\mathbb{R}^d)) \) a martingale solution of SDE (1) with starting point \( x \) if

(i) \( \mathbb{P}_x(\omega_0 = x) = 1 \), and for each \( t \in [0,T] \),
\[ \mathbb{E}_x \int_0^t |b(s,\omega_s)|ds < \infty , \]
where \( \{\omega_t\}_{t \in [0,T]} \) is the canonical processes.

(ii) For all \( f \in C^2_c(\mathbb{R}^d) \),
\[ M_t^f (\omega) := f(\omega_t) - f(x) - \int_0^t (\Delta f + b \cdot \nabla f)(\omega_s)ds \]
is a \( \mathcal{B}_t \)-martingale under \( \mathbb{P}_x \), where \( \mathcal{B}_t := \sigma \{\omega_s : 0 \leq s \leq t\} \).

Let \( \rho \in C^\infty(\mathbb{R}^d) \) and \( \int_{\mathbb{R}^d} \rho = 1 \). Set \( \rho_n(x) := n^d\rho(nx) \) and \( b_n(t,x) = b(t,\cdot) \ast \rho_n(x) \). For each \( x \in \mathbb{R}^d \), we then consider the following modified SDE:
\[ dX^n_t(x) = b_n(t,X^n_t(x))dt + \sqrt{2}\,dW_t, \quad X^n_0 = x, \quad (44) \]

where \( W \) is a \( d \)-dimensional standard Brownian motion on some complete filtered probability space \( (\Omega,\mathcal{F},(\mathcal{F}_t)_{t \in [0,T]},\mathbb{P}) \). It is well known that there is a unique strong solution \( X^n_t(x) \) to the above SDE.

**Proof of Theorem 1.1.** Existence: Assume \( b \) satisfies condition (a) or (b) in Theorem 1.1 and \( p_3, q_3 \in (1,\infty) \) such that \( d/p_3 + 2/q_3 < 2 \). We first prove that there are constants \( \theta > 0 \) and \( C > 0 \) such that for any \( f \in C^\infty_c(\mathbb{R}^{d+1}) \) and \( 0 \leq t_0 < t_1 \leq T \),
\[ \sup_n \sup_{x \in \mathbb{R}^d} \mathbb{E} \int_{t_0}^{t_1} f(t, X^n_t(x))dt \leq C(t_1 - t_0)^\theta \|f\|_{L^{p_3}_t(\mathbb{R}^d)}. \]
Let $u_n$ be the smooth solution of the following backward PDE:
\[
\partial_t u_n + \Delta u_n + b_n \cdot \nabla u_n + f = 0, \quad u_n(t_1, \cdot) = 0.
\] (45)

By Itô’s formula we have
\[
u_n(t_1, X^n_{t_1}) = u_n(t_0, X^n_{t_0}) + \int_{t_0}^{t_1} (\partial_t u_n + \Delta u_n + b_n \cdot \nabla u_n)(t, X^n_t)dt + \sqrt{2} \int_{t_0}^{t_1} \nabla u_n(t, X^n_t)dW_t.
\]

Using (45) and taking expectation, we obtain
\[
\mathbb{E} \int_{t_0}^{t_1} f(t, X^n_t)dt = \mathbb{E}u_n(t_0, X^n_{t_0}) \leq \|u_n(t_0, \cdot)\|_{L^\infty}.
\]

Since $\frac{d}{p_3} + \frac{2}{q_3} < 2$, we can choose $q'_3 < q_3$ so that $\frac{d}{p_3} + \frac{2}{q'_3} < 2$. Thus, by Lemma A.3, we obtain
\[
\mathbb{E} \int_{t_0}^{t_1} f(t, X^n_t)dt \leq \|u_n(t_0, \cdot)\|_{L^\infty} \leq C\|f\mathbf{1}_{[t_0, t_1]}\|_{L^{p_3}} \leq C(t_1 - t_0)^{1 - \frac{q'_3}{q_3}}\|f\mathbf{1}_{[t_0, t_1]}\|_{L^{p_3}}. \tag{46}
\]

Now let $\tau \in [0, T]$ be any bounded stopping time. Note that
\[
X^n_{(\tau + \delta)\wedge T}(x) - X^n_{\tau}(x) = \int_{\tau}^{(\tau + \delta)\wedge T} b_n(t, X^n_t(x))dt + \sqrt{2}(W_{(\tau + \delta)\wedge T} - W_{\tau}), \quad \delta \in (0, 1).
\]

By (46) and Remark 1.2 in [44], we have
\[
\mathbb{E} \int_{\tau}^{(\tau + \delta)\wedge T} |b_n(t, X^n_t(x))|dt \leq C\delta^\theta \|b_n\|_{L^{p_3}(T)}.
\]

Thus,
\[
\mathbb{E} \sup_{0 \leq u \leq \delta} |X^n_{\tau + u}(x) - X^n_{\tau}(x)| \leq \mathbb{E} \int_{\tau}^{\tau + \delta} |b_n|(t, X^n_t(x))dt + \sqrt{2}\mathbb{E} \sup_{0 \leq u \leq \delta} |W_{\tau + \delta} - W_{\tau}|
\]
\[
\leq C\delta^\theta \|b_n\|_{L^{p_3}} + C\delta^{1/2} \leq C\delta^{\theta'},
\]

where $\theta', C > 0$ are independent of $n$. So by [43, Lemma 2.7], we obtain
\[
\sup_n \mathbb{E} \left( \sup_{t \in [0, T]; u \in [0, \delta]} |X^n_{(t+u)\wedge T}(x) - X^n_{t}(x)|^{1/2} \right) \leq C\delta^{\theta'}.
\]

From this, by Chebychev’s inequality, we derive that for any $\varepsilon > 0$,
\[
\lim_{\delta \to 0} \sup_n \mathbb{P} \left( \sup_{t \in [0, T]; u \in [0, \delta]} |X^n_{(t+u)\wedge T}(x) - X^n_{t}(x)| > \varepsilon \right) = 0.
\]

Hence, by [33, Theorem 1.3.2], $\mathbb{P}_x^n := \mathbb{P} \circ X^n(x)^{-1}$ is tight in $\mathcal{D}(C([0, T]; \mathbb{R}^d))$. Assume $\mathbb{P}_x$ is an accumulation point of $(\mathbb{P}_x^n)_{n \in \mathbb{N}}$, that is, for some subsequence $n_k$,
\[
\mathbb{P}_x^{n_k} \text{ weakly converges to } \mathbb{P}_x \text{ as } k \to \infty.
\]
Since (46) can be rewritten as
\[ SDEs \text{ with critical drifts} \]

\[ n \text{Note that for each } \]

\[ B \]

\[ f \]

\[ t \]

\[ - \text{measurable,} \]

\[ E \]

\[ \chi \]

\[ E_n \]

\[ f \]

\[ t \]

\[ n \]

\[ f \]

\[ M_{t_1}^f \]

\[ B_{t_0} \]

\[ M_{t_0}^f \]

\[ \mathbb{P}_x \text{-a.s.,} \]

\[ M_t^f := f(\omega_t) - f(\omega_0) - \int_0^t (\Delta + b \cdot \nabla) f(s, \omega_s)ds. \]

By a standard monotone class argument, it is enough to show that for any \( \mathbb{P}_x \) is a martingale solution to (1), it suffices to prove that for any \( 0 \leq t_0 < t_1 \leq T \) and \( f \in C^2_c(\mathbb{R}^d) \),
\[ \mathbb{E}_x(M_{t_1}^f | B_{t_0}) = M_{t_0}^f, \quad \mathbb{P}_x \text{-a.s.,} \]

where
\[ M_t^f := f(\omega_t) - f(\omega_0) - \int_0^t (\Delta + b_n \cdot \nabla) f(s, \omega_s)ds, \quad t \in [0, T]. \]

We want to take weak limits, where the key point is to show
\[ \lim_{k \to \infty} \mathbb{E}_x^{n_k} \left( \int_0^t (b^{n_k} \cdot \nabla f)(s, \omega_s)ds \cdot G(\omega) \right) = \mathbb{E}_x \left( \int_0^t (b \cdot \nabla f)(s, \omega_s)ds \cdot G(\omega) \right). \]

Assume that \( \text{supp}(f) \subset B_R. \) By (46), we have
\[ \sup_{n \geq m} \mathbb{E}_x^n \left( \int_0^t ((b_m - b_n) \cdot \nabla f)(s, \omega_s)ds \cdot G(\omega) \right) \]
\[ \leq ||G||_\infty ||\nabla f||_\infty \sup_{n \geq m} \mathbb{E}_x^n \left( \int_0^t |(b_m - b_n)\chi_R^0|(s, \omega_s)ds \right) \]
\[ \leq C||G||_\infty ||\nabla f||_\infty \sup_{n \geq m} \|(b_m - b_n)\chi_R^0 1_{[0,t]}\|_{L^p_{R_3}} \to 0, \quad m \to \infty, \]

where the cutoff function \( \chi \) is defined by (6). Similarly, by (47),
\[ \mathbb{E}_x \left( \int_0^t ((b_m - b) \cdot \nabla f)(r, \omega_r)dr \cdot G(\omega) \right) \leq \|(b_m - b)\chi_R^0 1_{[0,t]}\|_{L^p_{R_3}} \to 0, \quad (m \to \infty). \]
On the other hand, for fixed \( m \in \mathbb{N} \),
\[
\omega \mapsto \int_0^t (b_m \cdot \nabla f)(r, \omega_r) dr \cdot G(\omega) \in C_b(C([0,T]; \mathbb{R}^d)),
\]
so we also have
\[
\lim_{k \to \infty} \mathbb{E}_x^{n_k} \left( \int_0^t (b_m \cdot \nabla f)(s, \omega_s) ds \cdot G(\omega) \right) = \mathbb{E}_x \left( \int_0^t (b_m \cdot \nabla f)(s, \omega_s) ds \cdot G(\omega) \right),
\]
which together with (49) and (50) implies (48).

**Uniqueness:** Let \( P_x^{(i)}, i = 1, 2 \) be two martingale solutions of SDE (1) and there is a constant \( C > 0 \) such that for all \( x \in \mathbb{R}^d \) and \( f \in \mathbb{L}_q^2(T) \),
\[
\mathbb{E}_x^{(i)} \left( \int_0^T f(t, \omega_t) dt \right) \leq C \| f \|_{\mathbb{L}^{p_3}_q(T)}, \quad \forall p_3, q_3 \in (1, \infty) \text{ with } \frac{d}{p_3} + \frac{2}{q_3} < 2. \tag{51}
\]
Let \((p_3, q_3)\) be the pair of constants in Theorems 3.1 and 3.2 with \( d/p_3 + 2/q_3 < 2 \), respectively. For any \( f \in C_c^\infty((0,T) \times \mathbb{R}^d) \), by Theorems 3.1 and 3.2, there is a unique solution \( u \in \mathbb{H}^{2,p_3}_q(T) \) with \( d/p_3 + 2/q_3 < 2 \) to the following backward equation:
\[
\partial_t u + Lu + f = 0, \quad u(T) = 0,
\]
where \( L := \Delta + b \cdot \nabla \). Let \( u_n(t, x) := u(t, \cdot) \ast \rho_n(x) \) be the mollifying approximation of \( u \). Then we have
\[
\partial_t u_n + Lu_n + g_n = 0, \quad u_n(T) = 0,
\]
where
\[
g_n = f \ast \rho_n + (Lu) \ast \rho_n - L(u \ast \rho_n).
\]
For \( R > 0 \), define
\[
\tau_R := \inf \{ t \geq 0 : |\omega_t| \geq R \}.
\]
By Itô’s formula, we have
\[
\mathbb{E}^{(i)} u_n(T \wedge \tau_R; \omega_T \wedge \tau_R) = u_n(0, x) - \mathbb{E}^{(i)} \left( \int_0^{T \wedge \tau_R} g_n(s, \omega_s) ds \right), \quad i = 1, 2. \tag{52}
\]
From the proofs for Theorems 3.1 and 3.2, one sees that
\[
\| (b \cdot \nabla u) - (b \cdot \nabla u) \ast \rho_n \|_{\mathbb{L}^{p_3}_q(T)} \to 0, \quad \| b \cdot \nabla u - b \cdot \nabla u \ast \rho_n \|_{\mathbb{L}^{p_3}_q(T)} \to 0.
\]
Using estimate (51), we have
\[
\lim_{n \to \infty} \mathbb{E}^{(i)} \left( \int_0^{T \wedge \tau_R} \left( (Lu) \ast \rho_n - L(u \ast \rho_n) \right)(s, \omega_s) ds \right) \leq C \lim_{n \to \infty} \| \chi_R \left( (b \cdot \nabla u) \ast \rho_n - b \cdot (\nabla u \ast \rho_n) \right) \|_{\mathbb{L}^{p_3}_q(T)}
\]

SDEs WITH CRITICAL DRIFTS

\[ \leq C \lim_{n \to \infty} \left\| \chi_R \left[ (b \cdot \nabla u) \ast \rho_n - b \cdot \nabla u \right] \right\|_{L^{p_3} \cap (\Theta)} + C \lim_{n \to \infty} \left\| \chi_R \left[ b \cdot \nabla u - b \cdot (\nabla u \ast \rho_n) \right] \right\|_{L^{p_3} \cap (\Theta)} = 0, \]

where the cutoff function \( \chi \) is defined by (6). Recalling that \( u \in \tilde{H}_{2,p_3} \), \( \partial_t u \in \tilde{L}^{p_3} \), and \( d/p_3 + 2/q_3 < 2 \), due to [22, Lemma 10.2] \( u \) is a bounded Hölder continuous function on \([0, T] \times \mathbb{R}^d\). Letting \( n \to \infty \) for both sides of (52) and by the dominated convergence theorem, we obtain

\[ E^{(i)} u(T \wedge \tau_R, \omega_{T \wedge \tau_R}) = u(0, x) - E^{(i)} \left( \int_0^{T \wedge \tau_R} f(s, \omega_s)ds \right), \quad i = 1, 2, \]

which, by letting \( R \to \infty \) and noting that \( u(T) = 0 \), yields

\[ u(0, x) = E^{(i)} \left( \int_0^T f(s, \omega_s)ds \right), \quad i = 1, 2. \]

This in particular implies the uniqueness of martingale solutions (see [33, Corollary 6.2.6]).

Appendix

In this section, we present sketches of proofs for Lemmas 3.4 and 3.5.

Proof of Lemma 3.4. As presented in the proof of [45, Lemma 3.2], for almost every \( s, t \in I \) with \( s < t \),

\[ \frac{1}{2} \left( \int_s^t u_k^2 \eta^2 \right) (t) - \frac{1}{2} \left( \int_s^t u_k^2 \eta^2 \right) (s) + \int_s^t \nabla u_k \cdot \nabla (u_k \eta^2) \]

\[ \leq - \int_s^t \int (u_k + k) b \cdot \nabla (u_k \eta) - \int_s^t \int \text{div}(u_k + k) u_k \eta + \int_s^t \int f u_k \eta^2. \]

Hölder’s inequality yields

\[ \int_s^t \int \nabla u_k \cdot \nabla (u_k \eta^2) = \int_s^t \int | \nabla u_k \eta |^2 + 2 \int_s^t \int (\nabla u_k \eta) \cdot (u_k \nabla \eta) \]

\[ \geq \frac{1}{2} \int_s^t \int | \nabla u_k \eta |^2 - \frac{C}{(R - \rho)^2} \left\| u_k \right\|_{L^2(\Theta)}^2. \]

(53)
Integration by parts and Hölder’s inequality yield

\[
- \int_s^t (u_k + k) b \cdot \nabla (u_k \eta^2) \\
= - \frac{1}{2} \int_s^t \eta^2 b \cdot \nabla (u_k^2) - 2 \int_s^t \eta^2 b \cdot \nabla u_k - k \int_s^t \eta^2 b \cdot \nabla \eta \\
= \left[ \int_s^t u_k^2 \eta b \cdot \nabla \eta + \frac{1}{2} \int_s^t \text{div} u_k^2 \eta^2 \right] - 2 \int_s^t u_k^2 \eta b \cdot \nabla \eta \\
+ \left[ 2k \int_s^t u_k \eta b \cdot \nabla \eta + k \int_s^t \text{div} u_k \eta^2 \right] - 2k \int_s^t u_k \eta b \cdot \nabla \eta \\
= - \int_s^t u_k^2 \eta b \cdot \nabla \eta + \frac{1}{2} \int_s^t \text{div} u_k^2 \eta^2 + k \int_s^t \text{div} u_k \eta^2.
\]

Therefore,

\[
- \int_s^t (u_k + k) b \cdot \nabla (u_k \eta^2) - \int_s^t \text{div} (u_k + k) u_k \eta^2 \\
= - \int_s^t u_k^2 \eta b \cdot \nabla \eta - \frac{1}{2} \int_s^t \text{div} u_k^2 \eta^2 \\
\leq \frac{2}{R - \rho} \int_s^t |b|_L^{p_q}(Q) \|u_k\|_{L^{p_q}(A^*_k)}^2 + (\text{div} \|u_k\|_{L^{p_q}(A^*_k)})^2 \\
\leq \frac{2}{(R - \rho)} \int_s^t |b|_L^{p_q}(Q) \|u_k\|_{L^{p_q}(A^*_k)}^2 \\
\leq \frac{1}{2} \|f\|_{L^{p_q}(Q)}^2 \|1_{A^*_k}\|_{L^{p_q}}^2 + \frac{1}{2} \|u_k\|_{L^{p_q}(A^*_k)}^2 \|1_{A^*_k}\|_{L^{p_q}}^2.
\]

By Hölder’s inequality,

\[
\int_s^t f u_k \eta^2 \leq \|f\|_{L^{p_q}(Q)} \|1_{A^*_k}\|_{L^{p_q}} \|u_k\|_{L^{p_q}(A^*_k)}^2 \\
\leq \frac{1}{2} \|f\|_{L^{p_q}(Q)}^2 \|1_{A^*_k}\|_{L^{p_q}}^2 + \frac{1}{2} \|u_k\|_{L^{p_q}(A^*_k)}^2.
\]

Combining (53)-(56) and using Hölder’s inequality, we obtain (EI).}

To prove Lemma 3.5, we need the following elementary lemma.

**Lemma A.2.** Suppose \(\{y_j\}_{j \in \mathbb{N}}\) is a nonnegative nondecreasing real sequence,

\[
y_{j+1} \leq NC^j y_j^{1+\varepsilon}
\]

with \(\varepsilon > 0\) and \(C > 1\). Assume

\[
y_0 \leq N^{-1/\varepsilon} C^{-1/\varepsilon^2}.
\]

Then \(y_j \to 0\) as \(j \to \infty\).
Proof of Lemma 3.5. (i) For any $k \in \mathbb{N}$, set
\[ t_k = -\frac{1}{2} (1 + 2^{-k}), \quad B_k' = B_{\frac{1}{2} (1 + 2^{-k})}, \quad Q_k' = (t_k, 0) \times B_k'. \]
The cut off functions $\eta_k$ is supported in $B_{k+1}'$ and equals to 1 in $B_k'$ such that $|\nabla^i \eta_k| \leq C 2^{|k|} (i = 0, 1, 2)$. Let $M > 0$, which will be determined later. Define
\[ M_k := M (2 - 2^{-k}), \quad u_k := (u - M_k)^+, \quad U_k := \|u_k\|^2_{L^\infty(Q_k')} + \sum_{i=2}^{3} \|u_k\|^2_{L^{p_i^*}(Q_k')} \]
and
\[ E_k := \sup_{t \in [t_k, 0]} \int (u_k \eta_k)^2 + \int_{t_k}^{0} \int |\nabla (u_k \eta_k)|^2. \]
For any $s, t$ satisfying $t_k \leq s \leq t_{k+1} \leq t \leq 0$, by Lemma 3.4, we have
\[ \int (u_{k+1} \eta_{k+1})^2 + \int_{s}^{t} \int |\nabla (u_{k+1} \eta_{k+1})|^2 \]
\[ \leq \int (u_{k+1} \eta_{k+1})^2(s) + C \|f\|^2_{L^{p_1^*}(Q_1)} \|1_{\{u_{k+1} > 0\}} \cap Q_{k} \|^2_{L^{p_1^*}(Q_1)} \]
\[ + C \|u_{k+1}\|^2_{L^2(Q_k')} + C \sum_{i=2}^{3} \|u_{k+1}\|^2_{L^{p_i^*}(Q_k')} \]
Using the range of $s, t$ and taking the mean value in $s$ between $t_{k+1}$ and $t_k$, we get
\[ \int (u_{k+1} \eta_{k+1})^2 + \int_{t_{k+1}}^{t} \int |\nabla (u_{k+1} \eta_{k+1})|^2 \]
\[ \leq 4 \cdot 2^k \int_{t_k}^{t_{k+1}} \int (u_{k+1} \eta_{k+1})^2 + C \|f\|^2_{L^{p_1^*}(Q_1)} \|1_{\{u_{k+1} > 0\}} \cap Q_{k} \|^2_{L^{p_1^*}(Q_1)} \]
\[ + C \|u_{k+1}\|^2_{L^2(Q_k')} + C \sum_{i=2}^{3} \|u_{k+1}\|^2_{L^{p_i^*}(Q_k')} \]
\[ \leq C \|u_{k+1}\|^2_{L^2(Q_k')} + C \sum_{i=2}^{3} \|u_{k+1}\|^2_{L^{p_i^*}(Q_k')} + C \|f\|^2_{L^{p_1^*}(Q_1)} \|1_{\{u_{k+1} > 0\}} \cap Q_{k} \|^2_{L^{p_1^*}(Q_1)}. \]
Choosing $M > C\|f\|_{L^p_{Q_1}}$, the above inequalities yield,

$$E_{k+1} \leq \sup_{t \in [t_{k+1}, 0]} \int (u_{k+1} \eta_{k+1})^2(t) + \int_{t_{k+1}}^0 \int \left| \nabla (u_{k+1} \eta_{k+1}) \right|^2 \leq Ck\|u_{k+1}\|_{L^2_{Q_k}}^2 + C 3 \sum_{i=2}^\infty \|u_{k+1}\|_{L^p_{Q_i}}^2 + C\|f\|_{L^p_{Q_k}}^2 \|1_{\{u_{k+1} > 0\} \cap Q_k}\|_{L^q_{Q_k}}^2 \tag{57}$$

$$\leq CkU_k + M^2\|1_{\{u_{k+1} > 0\} \cap Q_k}\|_{L^q_{Q_k}}^2.$$  

The quantity $E_{k+1}$ controls $u_{k+1} \eta_{k+1}$ in $L^2_\infty(Q'_{k+1})$, and thanks to Sobolev embedding, also in the space $L^2_\gamma(Q'_{k+1})$ for $\gamma = 2d/(d-2)$ if $d \geq 3$ and $L^2_\gamma(Q'_{k+1})$ for any $\gamma \in [2, \infty)$ if $d = 2$. So, by interpolation, $E_{k+1}$ controls the $L^r_\gamma(Q'_{k+1})$-norm of $u_{k+1} \eta_{k+1}$ for any

$$r, s \geq 2 \text{ with } \frac{d}{r} + \frac{2}{s} > \frac{d}{2}. \tag{58}$$

Noting that $d/p_i + 2/q_i < 2$ ($i = 2, 3$) and (30), we have

$$\frac{d}{p_i} + \frac{2}{q_i} > \frac{d}{2}, \text{ (i = 2, 3).}$$

By Hölder’s inequality and (57), one can see that there exists a constant $\varepsilon > 0$ such that

$$U_{k+1} = \|u_{k+1}\|_{L^2_{Q_{k+1}}} + \sum_{i=2}^\infty \|u_{k+1}\|_{L^p_{Q_i}}^2 \leq Ck\|u_{k+1}\|_{L^2_{Q_{k+1}}} + M^2\|1_{\{u_{k+1} > 0\} \cap Q_k}\|_{L^q_{Q_k}}^2 \leq CkU_k \|1_{\{u_{k+1} > 0\} \cap Q_k}\|_{L^q_{Q_k}}^2.$$ 

On the other hand,

$$M^2\|1_{\{u_{k+1} > 0\} \cap Q_k}\|_{L^{p'/2}_{Q_k}} \leq M^2(M2^{-k-1})^{-1}\|u_k\|_{L^{p'/2}_{Q_k}} \leq 2^{k+1}M\|u_k\|_{L^{p'/2}_{Q_k}} \|1_{\{u_{k+1} > 2^{-k-1}M\} \cap Q_k}\|_{L^{q'/2}_{Q_k}}^2 \leq 4 \cdot 4^k\|u_k\|_{L^{p'/2}_{Q_k}}^2 \leq CkU_k,$$

hence,

$$U_{k+1} \leq CkU_k \|1_{\{u_{k+1} > 0\} \cap Q_k}\|_{L^2_{Q_k}}^2 \leq CkU_k [2^{k+1}M^{-1}\|u_k\|_{L^2_{Q_k}}]^2 \leq M^{-2\varepsilon}C_k^\delta U_k^{1+\varepsilon}.$$
As showed in (54) and (56), we have

Choosing

\[ M := C_8 \| f \|_{L^{p_3}_{q_3}}(Q_1) + C_8^{1/(2 \varepsilon^2)} \left( \| u^+ \|_{L^2_2(Q_1)} + \sum_{i=2}^3 \| u^+ \|_{L^{p_i}_{q_i}}(Q_1) \right), \]

we have

\[ U_0 \leq \left( \| u^+ \|_{L^2_2(Q_1)} + \sum_{i=2}^3 \| u^+ \|_{L^{p_i}_{q_i}}(Q_1) \right)^2 \leq M^2 C_8^{-1/\varepsilon^2}. \]

By Lemma A.2,

\[ \|(u - 2M)^+\|_{L^2_2(Q_{1/2})} \leq \lim_{k \to \infty} U_k = 0. \]

By the definition of \( M \), we obtain

\[ \| u^+ \|_{L^\infty(Q_{1/2})} \leq 2M \leq C \left( \| u^+ \|_{L^2_2(Q_1)} + \sum_{i=2}^3 \| u^+ \|_{L^{p_i}_{q_i}}(Q_1) + \| f \|_{L^{p_3}_{q_3}}(Q_1) \right). \]

\[ \square \]

**Lemma A.3.** Let \( d \geq 3, p_i, q_i \in (1, \infty), i = 1, 2, 3 \) and \( f \in L^{p_3}_{q_3}(T) \) with \( d/p_3 + 2/q_3 < 2 \). Assume \( b \) satisfies one of the following two conditions:

(a) \( b = b_0 + b_1, \| b_0 \|_{L^\infty_{d,\infty}} \leq \varepsilon(d), \) for some \( \varepsilon(d) > 0 \) only depending on \( d \), and \( b_1 \in L^{p_1}_{q_1} \) with \( d/p_1 + 2/q_1 = 1 \) and \( p_1 \in (d, \infty) \),

(b) \( b, \text{div} b \in L^{p_2}_{q_2} \) with \( p_2, q_2 \in [2, \infty) \) and \( d/p_2 + 2/q_2 < 2 \).

Then equation (5) admits a unique weak solution \( u \in \bar{V}^0(T) \cap L^\infty(T) \). Moreover, the following estimate is valid:

\[ \| u \|_{\bar{V}(T)} + \| u \|_{L^\infty(T)} \leq C_9 \| f \|_{L^{p_3}_{q_3}(T)}, \]

where the constant \( C_9 \) only depends on \( d, p_1, q_1, p_3, q_3, \varepsilon, T \) and \( b_1 \) for the first case and \( d, p_2, q_2, p_3, q_3, T, \| b \|_{L^{p_2}_{q_2}} \) and \( \| \text{div} b \|_{L^{p_2}_{q_2}} \) for the second case.

**Proof.** Here we only give the proof for the first case, since the second case was essentially proved in [44] and [45].

For any \( x \in \mathbb{R}^d \), let \( \eta \in C^\infty_c(B_1) \) such that \( \eta \equiv 1 \) on \( B_2 \) and \( \eta_x := \eta(\cdot - x) \). As presented in the proof of [45, Lemma 3.2], for almost every \( t \in [0, T] \),

\[ \frac{1}{2} \left( \int_{B_2^2} u_k^2 \eta_x^2 \right)(t) + \int_0^t \int \nabla u_k \cdot \nabla (u_k \eta_x^2) \leq \int_0^t b \cdot \nabla u (u_k \eta_x^2) + \int_0^t f u_k \eta_x^2. \]  

As showed in (54) and (56), we have

\[ \int_0^t \int \nabla u_k \cdot \nabla (u_k \eta_x^2) \geq \frac{1}{2} \| \nabla u_k \eta_x \|_{L^2_2(t)}^2 - C \| u_k \|_{L^2_2(Q(t,x))}^2. \]
and
\[
\int_0^t \int f(u_k \eta_x^2) \leq \|f\|_{L^{p_3}_{q_3}(T)} \|1_{A(t,x;k)}\|_{L^{p_2}_{q_2}} \|u_k \eta_x\|_{L^{p_3}_{q_3}(Q(t,x))} \leq \frac{1}{15} \|u_k \eta_x\|_{V(t)}^2 + C \|f\|_{L^{p_3}_{q_3}(T)} \|1_{A(t,x;k)}\|_{L^{p_3}_{q_3}},
\]
where \(Q(t,x) = (0,t) \times B_1(x)\) and \(A(t,x;k) = \{u > k\} \cap Q(t,x)\). Let \(b_N^1 = (\neg N \lor b_1) \land N\). Then \(\delta_N := \|b - b_N^1\|_{L^{p_1}_{q_1}(T)} \to 0, (N \to \infty)\). Furthermore,
\[
\int_0^t \int b \cdot \nabla u (u_k \eta_x^2) = \int_0^t \int b_0 \cdot (\nabla u_k \eta_x) (u_k \eta_x) + \int_0^t \int (b - b_N^1) \cdot (\nabla u_k \eta_x) (u_k \eta_x) + \int_0^t \int b_N^1 \cdot (\nabla u_k \eta_x) (u_k \eta_x) =: I_1 + I_2 + I_3.
\]
For \(I_1\), by [10, Exercise 1.4.19] and [34, Remark 5], we have
\[
I_1 \leq \|b_0\|_{L^{p_1}_{q_1}(T)} \|\nabla u_k \eta_x\|_{L^2(\Omega)} \|u_k \eta_x\|_{L^2(\Omega)} \leq C_{11} \varepsilon \left( \|\nabla u_k \eta_x\|_{L^2(\Omega)} + \|u_k \nabla \eta_x\|_{L^1(\Omega)} \right).
\]
Here \(L^{p,q}\) is the Lorentz space (cf. [10]) and \(C_{11} = C_{11}(d)\). Choosing \(\varepsilon = \varepsilon(d) > 0\) small, then
\[
I_1 \leq \frac{1}{15} \|\nabla u_k \eta_x\|_{L^2(\Omega)} + C \|u_k\|_{V(t)}^2.
\]
For \(I_2\), we have
\[
I_2 \leq \|b - b_N^1\|_{L^{p_1}_{q_1}(T)} \|\nabla u_k \eta_x\|_{L^2(\Omega)} \|u_k \eta_x\|_{L^{p_1}_{q_1}(t)},
\]
where \(\frac{1}{p_1} = \frac{1}{2} - \frac{1}{p_1} \) and \(\frac{1}{q_1} = \frac{1}{2} - \frac{1}{q_1}\). Noting that \(d/p_1' + 2/q_1' = d/2\), by choosing \(N\) sufficiently large, we get
\[
I_2 \leq C \delta_N \left( \|\nabla u_k \eta_x\|_{L^2(\Omega)} \|u_k \eta_x\|_{V(t)} \right) \leq \frac{1}{15} \|\nabla u_k \eta_x\|_{L^2(\Omega)} + \frac{1}{15} \|u_k \eta_x\|_{V(t)}^2.
\]
For \(I_3\), by Hölder’s inequality, we also have
\[
I_3 \leq N \|\nabla u_k \eta_x\|_{L^2(\Omega)} \|u_k \eta_x\|_{L^2(\Omega)} \leq \frac{1}{15} \|\nabla u_k \eta_x\|_{L^2(\Omega)}^2 + C_N \|u_k\|_{L^2(\Omega)}^2.
\]
Combining (59)-(65), we obtain
\[
\|u_k \eta_x\|_{V(t)} \leq C \left( \|u_k\|_{L^2(\Omega)} + \|f\|_{L^{p_3}_{q_3}(T)} \|1_{A(t,x;k)}\|_{L^{p_3}_{q_3}} \right),
\]
where \(C\) only depends on \(d, p_1, q_1, \varepsilon\) and \(b_1\). Now our desired results can be obtained in the same way as in the proofs for Theorem 3.4 and Theorem 3.6 [45].
Acknowledgements

Research of Michael and Guohuan is supported by the German Research Foundation (DFG) through the Collaborative Research Centre (CRC) 1283 Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications.

The second named author is very grateful to Nicolai Krylov and Xicheng Zhang who encouraged him to persist in studying this problem, and also Moritz Kassmann for providing him an excellent environment to work at Bielefeld University.

References


