The Hausdorff measure of the range and level sets of Gaussian random fields with sectorial local nondeterminism

CHEUK YIN LEE

Institut de Mathématiques, École Polytechnique Fédérale de Lausanne, Station 8, CH-1015 Lausanne, Switzerland. E-mail: cheuk.lee@epfl.ch

We determine the exact Hausdorff measure functions for the range and level sets of a class of Gaussian random fields satisfying sectorial local nondeterminism and other assumptions. We also establish a Chung-type law of the iterated logarithm. The results can be applied to the Brownian sheet, fractional Brownian sheets whose Hurst indices are the same in all directions, and systems of linear stochastic wave equations in one spatial dimension driven by space-time white noise or colored noise.

Keywords: Gaussian random fields; Hausdorff measure; local nondeterminism; Brownian sheet; harmonizable representation

1. Introduction and main results

Consider a centered, continuous, \( \mathbb{R}^d \)-valued Gaussian random field \( v = \{ v(x) : x \in \mathbb{R}^N \} \) with i.i.d. components, that is, \( v(x) = (v_1(x), \ldots, v_d(x)) \) and \( v_1, \ldots, v_d \) are i.i.d.

The class of Gaussian random fields \( v \) that we treat in this paper includes the Brownian sheet and other Gaussian random fields that share a similar structure with the Brownian sheet, such as the fractional Brownian sheets and the solution of a system of linear stochastic wave equations. One of the important properties that they have in common is that they all satisfy sectorial local nondeterminism (sectorial LND). The purpose of this paper is to establish a unified and general framework that incorporates these Gaussian random fields and allows us to study fine properties such as the Hausdorff measures of the associated random sets and sharp regularity properties for these random fields. In the main results of this paper, we identify the exact Hausdorff measure functions for the range and level sets of \( v \). We also prove a Chung-type law of the iterated logarithm (LIL). We use these results to outline subtle differences between Gaussian random fields with sectorial LND and those with stationary increments and strong LND.

It is well known that the fractional Brownian motion satisfies strong LND [29]. The exact Hausdorff measure function for the range of Lévy’s multiparameter Brownian motion was identified by Goldman [16] and the result was extended by Talagrand [34] to the case of the fractional Brownian motion. For Gaussian random fields with stationary increments and strong LND, the problem of finding the Hausdorff measure functions for the range and level sets was studied by Xiao [40, 41]. Baraka and Mountford [6] improved the result of Xiao [41] in the case of fractional Brownian motion and determined the exact value of the Hausdorff measure of the zero set. The Hausdorff measure function for the range of anisotropic Gaussian random fields was studied by Luan and Xiao [26].

On the other hand, the Brownian sheet does not satisfy strong LND (see [4]). Instead, it satisfies a weaker condition called sectorial LND [18, 19]. The Hausdorff measure functions for the range and graph of the Brownian sheet were determined by Ehm [13], and the level sets of the real-valued
Brownian sheet were dealt with by Zhou [44]. Moreover, the Brownian sheet is closely related to the additive Brownian motion, and the exact Hausdorff measure of the zero set of the latter was studied by Mountford and Nualart [28]. In [39], Wu and Xiao studied the local times of anisotropic Gaussian random fields with sectorial LND and obtained a partial result for the Hausdorff measure function for the level sets. We point out that the exponent of the logarithmic factor in the Hausdorff measure function for Gaussian fields with sectorial LND is typically not the same as those with stationary increments and strong LND, indicating subtle differences between these two classes of Gaussian fields.

Our framework is inspired by the structure of the Brownian sheet. Let us recall the local representation and behaviour of the Brownian sheet, which has been discussed in the literature, e.g. [17, 8]. Let \( B = \{ B(t_1,t_2) : (t_1,t_2) \in \mathbb{R}^2_+ \} \) be a Brownian sheet on \( \mathbb{R}^2_+ \). Fix \( s = (s_1,s_2) \in (0,\infty)^2 \) and let \( I = [s_1,s_1 + \rho] \times [s_2,s_2 + \rho] \) be a small interval. Then for all \( t = (t_1,t_2) \in I \), we can write

\[
B(t_1,t_2) = B(s_1,s_2) + \tilde{B}^1(t_1) + \tilde{B}^2(t_2) + R(t_1,t_2),
\]

where

\[
\tilde{B}^1(t_1) = B(t_1,s_2) - B(s_1,s_2), \quad \tilde{B}^2(t_2) = B(s_1,t_2) - B(s_1,s_2)
\]

and

\[
R(t_1,t_2) = B(t_1,t_2) - B(t_1,s_2) - B(s_1,t_2) + B(s_1,s_2).
\]

Note that \( \{ \tilde{B}^1(t_1) : t_1 \geq s_1 \} \) and \( \{ \tilde{B}^2(t_2) : t_2 \geq s_2 \} \) are two independent Brownian motions (up to scaling). Since \( \Var(R(t_1,t_2)) = (t_1-s_1)(t_2-s_2) \), the remainder process \( R \) is of smaller order compared to \( \tilde{B}^1 \) and \( \tilde{B}^2 \). Now consider the increments of \( B \) over the interval \( I \): for all \( t, t' \in I \),

\[
B(t) - B(t') = [\tilde{B}^1(t_1) - \tilde{B}^1(t'_1)] + [\tilde{B}^2(t_2) - \tilde{B}^2(t'_2)] + \text{remainder}. \tag{2}
\]

This suggests that the increments of \( B \) can be approximated by those of \( \tilde{B}^1 \) and \( \tilde{B}^2 \).

In this paper, we will consider the examples of fractional Brownian sheets and the solution of a linear stochastic wave equation (driven by a possibly colored noise), which share similar structures with the Brownian sheet. A technical difference for these examples of Gaussian fields is that the corresponding processes \( \tilde{B}^j \)'s may not have independent increments or Markov property unlike the Brownian sheet case, where the \( \tilde{B}^j \)'s are Brownian motions. Nonetheless, all of these Gaussian fields have certain harmonizable representation, which is a useful representation to create independence (see [11, 10]). These observations motivate us to introduce the processes \( \tilde{v}^j, j = 1, \ldots, N \), and the conditions for these processes in Assumption 1.2 below.

Let \( 0 < \alpha < 1 \) be a constant and \( T \) be a compact interval in \( \mathbb{R}^N \). Define \( \| X \|_{L^2} = (\mathbb{E} |X|^2)^{1/2} \) for a random vector \( X \), where \( | \cdot | \) denotes the Euclidean norm. We will consider the following assumptions for the Gaussian field \( v \). The first assumption specifies upper and lower bounds for the increments of \( v \) in \( L^2 \)-norm, and the property of sectorial local nondeterminism. This assumption appears in [42, 39].

**Assumption 1.1.** There exist positive finite constants \( c_0, c_1 \) such that the following properties hold:

(a) For all \( x, y \in T \),

\[
c_0^{-1} |x - y|^{\alpha} \leq \| v(x) - v(y) \|_{L^2} \leq c_0 |x - y|^{\alpha}.
\]

(b) (Sectorial local nondeterminism) For all integers \( n \geq 1 \), for all \( x, y^1, \ldots, y^n \in T \),

\[
\Var(v_1(x) | v_1(y^1), \ldots, v_1(y^n)) \geq c_1 \sum_{j=1}^{N} \min_{0 \leq i \leq n} |x_j - y^i_j|^{2\alpha},
\]

where \( x = (x_1, \ldots, x_N) \) and \( y^0 = 0 \).
For every \( s \in \mathbb{R}^N \) and \( r > 0 \), let \( I_r(s) \) denote the compact interval \( T \cap \prod_{j=1}^N [s_j - r, s_j + r] \). Let \( \mathcal{B}(\mathbb{R}_+) \) denote the \( \sigma \)-algebra of Borel sets in \( \mathbb{R}_+ = [0, \infty) \). The following assumption is a modification of Assumption 2.1 in [11] (see also [10]). Here, this assumption is adapted to the structure of the Brownian sheet as discussed above so that at the same time we are able to deal with other Gaussian random fields sharing a similar structure.

**Assumption 1.2.** For every \( s \in T \), there exists \( 0 < r_0 \leq 1 \), there exists, for each \( j = 1, \ldots, N \), an \( \mathbb{R}^d \)-valued Gaussian process \( \{ \tilde{\nu}^j(A, x_j) : A \in \mathcal{B}(\mathbb{R}_+), x_j \in [s_j - r_0, s_j + r_0] \} \), which has i.i.d. components and is a.s. continuous in \( x \) and there exists a constant \( c_2 \) depending on \( T \) but not on \( s \) such that the following properties hold:

(a) For each \( j \) and \( x, y \), \( A \mapsto \tilde{\nu}^j(A, x_j) \) is an \( \mathbb{R}^d \)-valued independently scattered Gaussian measure.

Moreover, whenever \( A \) and \( B \) are fixed, disjoint sets, the \( \sigma \)-algebra \( \sigma\{ \tilde{\nu}^1(A, \cdot), \ldots, \tilde{\nu}^N(A, \cdot) \} \) is independent of \( \sigma\{ \tilde{\nu}^1(B, \cdot), \ldots, \tilde{\nu}^N(B, \cdot) \} \).

(b) For all \( j \), for all \( 0 < r \leq r_0 \) and all \( x_j, y_j \in [s_j - r, s_j + r] \),

\[
\| \tilde{\nu}^j(\mathbb{R}_+, x_j) - \tilde{\nu}^j(\mathbb{R}_+, y_j) \|_{L^2} \leq c_2 |x_j - y_j|^\alpha.
\]

(c) Let \( \tilde{\nu}(A, x) = \sum_{j=1}^N \tilde{\nu}^j(A, x_j) \). The process \( \nu(\cdot) - \tilde{\nu}(A, \cdot) \) has i.i.d. components and there exist constants \( 0 \leq \gamma_1 < \alpha, \gamma_2 > 0 \) and \( a_0 \geq 0 \) such that for all \( 0 < r \leq r_0 \), for all \( x, y \in I_r(s) \), for all \( a_0 \leq a < b \leq \infty \),

\[
\| \nu(x) - \nu(y) - \tilde{\nu}([a, b), x) + \tilde{\nu}([a, b), y) \|_{L^2} 
\leq c_2 \left( a^{1-\alpha}|x-y| + r^{\gamma_1}b^{\gamma_1-\alpha} + r^{\gamma_2}|x-y|^\alpha \right).
\]

The next assumption originates from Assumption 2.4 in [11]. Part (a) states a uniform lower bound for the \( L^2 \)-norm of \( \nu \), and part (b) basically states that for any \( x \), one can find a reference point \( x' \) (which may depend on \( x \)) such that for all \( y, \bar{y} \) in a small neighbourhood of \( x \), the covariances in (3) are smoother than what one gets from the Cauchy–Schwarz inequality and the upper bound in Assumption 1.1(a) above.

**Assumption 1.3.** There exists constants \( 0 < \varepsilon_0 \leq 1, c_3 > 0, c > 0 \) and \( \delta \in (\alpha, 1) \) such that the following properties hold:

(a) For all \( x \in T^{(\varepsilon_0)} \), \( \| \nu(x) \|_{L^2} \geq c_3 \), where \( T^{(\varepsilon_0)} \) denotes the \( \varepsilon_0 \)-neighbourhood of \( T \) in the Euclidean norm.

(b) For every sufficiently small compact interval \( I \subset T \), and every \( 0 < \rho \leq \varepsilon_0 \), there exists a finite constant \( c_4 \) such that for any \( x \in I \), there exists \( x' \in I^{(\varepsilon_0)} \) such that for all \( y, \bar{y} \in I^{(\varepsilon_0)} \) with \( |x - y| \leq 2\rho \) and \( |x - \bar{y}| \leq 2\rho \), for all \( i = 1, \ldots, d \),

\[
|\mathbb{E}[(\nu_i(y) - \nu_i(\bar{y}))\nu_i(x')]| \leq c_4 |y - \bar{y}|^\delta.
\]

Let \( \mathcal{F}(T) \) be the collection of all compact intervals in \( T \), and let \( \lambda_N \) be the Lebesgue measure on \( \mathbb{R}^N \). For any subset \( F \) of \( \mathbb{R}^N \) or \( \mathbb{R}^d \), let \( \mathcal{H}_d(F) \) denote the Hausdorff measure of \( F \) with respect to the function \( \phi \), and for any interval \( J \subset \mathbb{R}^N \), let \( L(\cdot, J) \) denote the local time of \( \nu \) on \( J \) (see Section 2 for definitions). The range of \( \nu \) on \( J \) is the random set in \( \mathbb{R}^d \) defined by

\[
\nu(J) = \{ \nu(x) : x \in J \}.
\]
For any \( z \in \mathbb{R}^d \), the \( z \)-level set of \( v \) on \( J \) is the random set in \( \mathbb{R}^N \) defined by

\[
v^{-1}(z) \cap J = \{ x \in J : v(x) = z \}.
\]

The following are the main results of the paper, concerning the Hausdorff measure of the range and level sets of \( v \). Let \( T \) be a compact interval in \( \mathbb{R}^N \).

**Theorem 1.4.** Under Assumptions 1.1 and 1.2, if \( N < \alpha d \), then there exist positive finite constants \( C_1 \) and \( C_2 \) such that

\[
\mathbb{P}\left\{ C_1 \lambda_N(J) \leq \mathcal{H}_{\phi}(v(J)) \leq C_2 \lambda_N(J) \text{ for all } J \in \mathcal{I}(T) \right\} = 1,
\]

where \( \phi(r) = r^{N/\alpha} \left( \log \log(1/r) \right)^{N/\alpha} \).

We can compare Theorem 1.4 to the case of a fractional Brownian motion \( X \) from \( \mathbb{R}^N \) to \( \mathbb{R}^d \), for which we have \( 0 < \mathcal{H}_{\psi}(X(J)) < \infty \), where \( \psi(r) = r^{N/\alpha} \log \log(1/r) \); see Talagrand [34].

Also, we remark that by Corollary 5.1 of [29], if \( N > \alpha d \) and \( T \) has interior points, then a.s. \( v(T) \) has positive Lebesgue measure in \( \mathbb{R}^d \). For the critical case \( N = \alpha d \), the problem of determining the exact Hausdorff measure function for \( v(T) \) is open; see Talagrand [35] for a partial result on the upper bound of the Hausdorff measure for the fractional Brownian motion.

**Theorem 1.5.** Under Assumptions 1.1, 1.2 and 1.3, if \( N > \alpha d \), then there exists a positive constant \( C \) such that for any fixed \( z \in \mathbb{R}^d \),

\[
\mathbb{P}\left\{ C \lambda_N(J) \leq \mathcal{H}_{\phi}(v^{-1}(z) \cap J) < \infty \text{ for all } J \in \mathcal{I}(T) \right\} = 1,
\]

where \( \phi(r) = r^{N-\alpha d} \left( \log \log(1/r) \right)^{\alpha d/N} \). Moreover, if \( N \leq \alpha d \), then \( v^{-1}(z) \cap T = \emptyset \) a.s.

Baraka and Mountford [6] proved that if \( N > \alpha d \) and \( X \) is a fractional Brownian motion from \( \mathbb{R}^N \) to \( \mathbb{R}^d \), then \( \mathcal{H}_{\psi}(X^{-1}(0) \cap J) = CL(0, J) \), where \( \psi(r) = r^{N-\alpha d} \left( \log \log(1/r) \right)^{\alpha d/N} \). We conjecture that Theorem 1.5 can be strengthened to \( \mathcal{H}_{\phi}(v^{-1}(0) \cap J) = CL(0, J) \) for some constant \( C \).

Another result of this paper is a Chung-type law of the iterated logarithm.

**Theorem 1.6.** Under Assumptions 1.1 and 1.2, for any fixed \( s \in T \), there exists a constant \( K \) that may depend on \( s \) such that

\[
\liminf_{r \to 0} \sup_{x \in I_r(s)} \frac{|v(x) - v(s)|}{r^{\alpha} \left( \log \log(1/r) \right)^{-\alpha}} = K \quad \text{a.s.}
\]

and \( K_0 \leq K \leq K_1 \), where \( K_0 \) and \( K_1 \) are positive finite constants depending on \( T \).

The paper is organized as follows. In Section 2, we review the notions of Hausdorff measure and local time, and some lemmas regarding Gaussian probability estimates. In Section 3, we derive Proposition 3.3, which is a precise probability estimate for the upper bound of the Chung-type LIL and is a key estimate for the proof of Theorems 1.4 and 1.5, and we also provide the proof of Theorem 1.6 at the end of that section. In Section 4, we prove some sojourn time estimates, which are needed in the proof of Theorem 1.4. Then we prove Theorems 1.4 and 1.5 in Sections 5 and 6 respectively. Finally,
in Section 7, we apply the results to the Brownian sheet, fractional Brownian sheets and systems of
linear stochastic wave equations.
Throughout the paper, unspecific constants are denoted by \( c, C, K \), etc., and their values may be
different in each appearance.

2. Preliminaries

Consider the class \( \Phi \) of nondecreasing, right-continuous functions \( \phi : (0, \delta_0) \to (0, \infty) \) for some \( \delta_0 > 0 \)
satisfying \( \phi(0+) = 0 \) and the condition: there exists \( c > 0 \) such that \( \phi(2r) \leq c \phi(r) \) for all \( 0 < r \leq \delta_0/2 \).

For \( \phi \in \Phi \) and \( F \subset \mathbb{R}^n \), the \( \phi \)-Hausdorff measure of \( F \) is defined by
\[
\mathcal{H}_\phi(F) = \lim_{\delta \to 0} \inf \left\{ \sum_{j=1}^{\infty} \phi(2r_j) : F \subset \bigcup_{j=1}^{\infty} B(x_j, r_j), 0 \leq r_j \leq \delta \text{ for all } j \right\},
\]
where \( B(x, r) \) denotes the open Euclidean ball centered at \( x \) with radius \( r \). It is known that \( \mathcal{H}_\phi \) is an
outer measure and every Borel set is \( \mathcal{H}_\phi \)-measurable. We say that \( \phi \) is an
exact Hausdorff measure function for \( F \) if \( 0 < \mathcal{H}_\phi(F) < \infty \).

When \( \phi(r) = r^\beta \), where \( \beta > 0 \), the \( \phi \)-Hausdorff measure is written as \( \mathcal{H}_\beta \) and is called the \( \beta \)-
dimensional Hausdorff measure. The Hausdorff dimension of \( F \) is defined as
\[
\dim_H F = \inf \{ \beta > 0 : \mathcal{H}_\beta(F) = 0 \}
= \sup \{ \beta > 0 : \mathcal{H}_\beta(F) = \infty \}.
\]
The equality between the \( \inf \) and the \( \sup \) is well known. We refer to [14, 30] for basic properties of
Hausdorff measure and Hausdorff dimension.

For any finite Borel measure \( \mu \) on \( \mathbb{R}^n \) and any function \( \phi \in \Phi \), the upper \( \phi \)-density of \( \mu \) at a point
\( x \in \mathbb{R}^n \) is defined as
\[
D_\mu^\phi(x) = \limsup_{r \to 0} \frac{\mu(B(x, r))}{\phi(r)}.
\]
The following lemma can be derived from the results in Rogers and Taylor [31] (see also Taylor and
Tricot [36, Theorem 2.1]) and will be used in the proof of Theorem 1.4.

**Lemma 2.1.** For any \( \phi \in \Phi \), there exists a positive finite constant \( C \) such that for any finite Borel measure \( \mu \) on \( \mathbb{R}^n \) and any Borel set \( F \subset \mathbb{R}^n \),
\[
\mu(F) \leq C \mathcal{H}_\phi(F) \sup_{x \in F} D_\mu^\phi(x).
\]

Let us recall the definition and basic properties of local times. We refer to [15] for more details. Let
\( T \) be a Borel set in \( \mathbb{R}^N \). The occupation measure of \( v \) on \( T \) is defined as
\[
\mu_T(A) = \lambda_N \{ x \in T : v(x) \in A \}, \quad A \in \mathcal{B}(\mathbb{R}^d),
\]
where \( \lambda_N \) denotes the Lebesgue measure on \( \mathbb{R}^N \). If \( \mu_T \) is absolutely continuous with respect to the
Lebesgue measure \( \lambda_d \) on \( \mathbb{R}^d \), we say that the local time of \( v \) exists on \( T \), and the local time is defined
as the Radon–Nikodym derivative
\[
L(z, T) = \frac{d\mu_T}{d\lambda_d}(z).
\]
It is clear that if the local time exists on $T$, then it also exists on any Borel subset of $T$.

Theorem 8.1 of [42] shows that under Assumption 1.1(a), $v$ has a local time $L(\cdot, T) \in L^2(\lambda_d \times \mathbb{P})$ if and only if $N > \alpha d$. Furthermore, by Theorem 8.2 of [42], if Assumption 1.1 is satisfied on a compact interval $T = \prod_{j=1}^N [t_j, t_j + h_j]$ and $N > \alpha d$, then $v$ has a jointly continuous local time in the sense that there exists a version of the local time, still denoted by $L(z, \cdot)$, such that a.s. $L(z, \prod_{j=1}^N [t_j, t_j + s_j])$ is jointly continuous in all variables $(z, s) \in \mathbb{R}^d \times \prod_{j=1}^N [0, h_j]$.

We will use a jointly continuous version of the local time whenever it exists. If $v$ is continuous and has a jointly continuous local time on $T$, then $L(z, \cdot)$ defines a Borel measure supported on the level set $v^{-1}(z) \cap T$. See [2, Theorem 8.6.1] or [15, p.12, Remark(c)].

Next, let us recall a small ball probability estimate for Gaussian processes that is due to Talagrand [34]. The following is a reformulation of this result and its proof can be found in Ledoux [21, p.257].

**Lemma 2.2.** Let $\{Z(t) : t \in S\}$ be a separable, real-valued, centered Gaussian process indexed by a bounded set $S$ with the canonical metric $d_Z(s, t) = (\mathbb{E}|Z(s) - Z(t)|^2)^{1/2}$. Let $N_e(S)$ denote the smallest number of $d_Z$-balls of radius $\varepsilon$ that are needed to cover $S$. If there is a decreasing function $\psi : (0, \varepsilon_1) \to (0, \infty)$ such that $N_e(S) \leq \psi(\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_1]$ and there are constants $a_2 \geq a_1 > 1$ such that

$$a_1 \psi(\varepsilon) \leq \psi(\varepsilon/2) \leq a_2 \psi(\varepsilon) \quad \text{for all } \varepsilon \in (0, \varepsilon_1],$$

(7) then for all $u \in (0, \varepsilon_1)$,

$$\mathbb{P} \left\{ \sup_{s, t \in S} |Z(s) - Z(t)| \leq u \right\} \geq \exp \left( -K \psi(u) \right),$$

where $K$ is a constant depending on $a_1$ and $a_2$ only.

The next lemma is an isoperimetric inequality for Gaussian processes; see [22, p.302].

**Lemma 2.3.** There is a universal constant $K$ such that the following statement holds. Let $S$ be a bounded set and $\{Z(s) : s \in S\}$ be a real-valued Gaussian process. Let $D = \sup \{d(s, t) : s, t \in S\}$ be the diameter of $S$ under the canonical metric $d_Z$. Then for all $h > 0$,

$$\mathbb{P} \left\{ \sup_{s, t \in S} |Z(s) - Z(t)| \geq K \left( h + \int_0^D \sqrt{\log N_e(S)} \, d\varepsilon \right) \right\} \leq \exp \left( - \frac{h^2}{D^2} \right).$$

(8)

The last lemma of this section provides a probability estimate for the uniform modulus of continuity for $v$. It will be needed later in the proofs of Theorems 1.4 and 1.5.

**Lemma 2.4.** Under Assumption 1.1(a), there exist constants $c$ and $C$ such that for all $s \in T$, for all $r > 0$ small, for all $L \geq C$,

$$\mathbb{P} \left\{ \sup_{x, y \in I_r(s)} |v(x) - v(y)| \geq L r^\alpha \sqrt{\log(1/r)} \right\} \leq r^{L^2/c^2}.$$ 

**Proof.** With the notations in Lemma 2.3, let $S = I_r(s)$ and $Z(x) = v(x)$. By Assumption 1.1(a), $d_Z(x, y) \leq c_0 |x - y|^{\alpha}$. It follows that the diameter of $S$ under $d_Z$ is $D \leq Cr^\alpha$ and $N_e(S) \leq C e^{-N/\alpha}$.
Note that for \(0 < x < x_0\) with \(x_0 > 0\) small, the inequality \(\int_0^x \sqrt{\log(1/\varepsilon)} \, d\varepsilon \leq C x \sqrt{\log(1/x)}\) holds and the function \(x \sqrt{\log(1/x)}\) is increasing. Then for some constant \(C\), for \(r > 0\) small, we have
\[
\int_0^D \sqrt{\log N_\varepsilon(S)} \, d\varepsilon \leq Cr^\alpha \sqrt{\log(1/r)}.
\]

Therefore, the result now follows from Lemma 2.3. \(\square\)

3. A Chung-type law of the iterated logarithm

The goal of this section is to derive the probability estimate in Proposition 3.3, which is a key ingredient of the proofs of Theorems 1.4 and 1.5, and to provide a proof for the Chung-type LIL of Theorem 1.6.

First, we need to establish two lemmas. Recall that \(I_r(s) = T \cap \prod_{j=1}^N [s_j-r, s_j+r]\).

**Lemma 3.1.** Under Assumption 1.2, there exist constants \(0 < K_0 < \infty\) and \(0 < \rho_0 < 1\) such that for all compact intervals \(I_r(s)\) in \(T\) with \(r \in (0, \rho_0)\), for all \(0 < \varepsilon < r^{\alpha}\) and all \(0 < a < b < \infty\),
\[
\mathbb{P} \left\{ \sup_{x \in I_r(s)} |\hat{\varphi}(a, b, x) - \hat{\varphi}(a, b, s)| \leq \varepsilon \right\} \geq \exp \left( -\frac{K_0 r}{\varepsilon^{1/\alpha}} \right).
\]

**Proof.** Since \(\hat{\varphi}\) has i.i.d. components, we only need to prove the lemma for \(d = 1\). Recall that \(\hat{\varphi}(a, b, x) = \sum_{j=1}^N \hat{\varphi}^j([a, b), x_j]\). Fix \(s \in T\). Take \(\rho_0 = r_0\) from Assumption 1.2. We first prove that for each \(j\),
\[
\mathbb{P} \left\{ \sup_{x_j \in [s_j-r, s_j+r]} |\hat{\varphi}^j([a, b), x_j) - \hat{\varphi}^j([a, b), s_j)| \leq \varepsilon/N \right\} \geq \exp \left( -\frac{K_0 r}{\varepsilon^{1/\alpha}} \right).
\]

We are going to use Lemma 2.2 to prove this inequality for fixed \(j\), \(0 < r < \rho_0\) and \(0 < a < b\). Take \(S = [s_j-r, s_j+r]\) and \(Z(x_j) = \hat{\varphi}^j([a, b), x_j)\). By Assumption 1.2(a), \(\hat{\varphi}^j([a, b), \cdot)\) and \(\hat{\varphi}^j([a, b), \cdot)\) are independent, thus
\[
\|\hat{\varphi}^j([a, b), x_j) - \hat{\varphi}^j([a, b), y_j)\|^2_{L^2} + \|\hat{\varphi}^j([a, b), x_j) - \hat{\varphi}^j([a, b), y_j)\|^2_{L^2} = \|\hat{\varphi}^j([a, b), x_j) - \hat{\varphi}^j([a, b), y_j)\|^2_{L^2}.
\]

Then Assumption 1.2(b) implies that for all \(x_j, y_j \in S\),
\[
d_Z(x_j, y_j) \leq \|\hat{\varphi}^j([a, b), x_j) - \hat{\varphi}^j([a, b), y_j)\|_{L^2} \leq c_2 |x_j - y_j|^\alpha.
\]

It follows that for all \(\varepsilon > 0\) small,
\[
N_\varepsilon(S) \leq \frac{r}{(c_2^{-1/\varepsilon})^{1/\alpha}} = C c_2, \alpha \left( \frac{r}{\varepsilon^{1/\alpha}} \right).
\]

Then we can take \(\psi(\varepsilon) = C c_2, \alpha (r/\varepsilon^{1/\alpha})\). This function satisfies (7) with constants \(a_1 = a_2 = 2^{1/\alpha}\) which are greater than 1. By Lemma 2.2, we can find a constant \(K\) depending on \(a_1, a_2, c_2, \alpha\) and \(N\) such that (10) is satisfied for all \(j\), for all \(0 < r < \rho_0\) and \(0 < \varepsilon < r^{\alpha}\).
Lemma 3.2. Under Assumptions 1.1 and 1.2, there exist positive finite constants $K$, $K'$ and $A_0$ such that for any $r > 0$ small, for any compact interval $I_r(s)$ in $T$, for any $0 < a < b < \infty$, we have
\[
\mathbb{P}\left\{ \sup_{x \in I_r(s)} |v(x) - v(s) - \tilde{v}([a, b], x) + \tilde{v}([a, b], s)| \geq h \right\} \leq \exp\left(-\frac{h^2}{K'A^2}\right)
\]
provided that $h \geq KA\sqrt{\log(Kr^\alpha/A)}$ and $A \leq A_0 r^\alpha$, where $A = a^{1-\alpha}r + r^{\gamma_1}b^{\gamma_1-\alpha} + r^{\gamma_2+\alpha}$, and $0 \leq \gamma_1 < \alpha$ and $\gamma_2 > 0$ are the constant in Assumption 1.2.

Proof. Since $v(\cdot) - \tilde{v}([a, b], \cdot)$ has i.i.d. components, we may assume that $d = 1$. With the notations in Lemma 2.3, set $S = I_r(s)$ and $Z(x) = v(x) - \tilde{v}([a, b], x)$. By Assumption 1.2(c), the canonical metric of $Z$ satisfies
\[
d_Z(x, y) \leq C \left( a^{1-\alpha} |x - y| + r^{\gamma_1} b^{\gamma_1-\alpha} + r^{\gamma_2} |x - y|^\alpha \right)
\]
for all $x, y \in S$. It follows that the diameter of $S$ in $d_Z$ satisfies
\[
D \leq C \left( a^{1-\alpha}r + r^{\gamma_1} b^{\gamma_1-\alpha} + r^{\gamma_2+\alpha} \right) = CA.
\]
Also, by Assumptions 1.1 and 1.2, for all $x, y \in S$,
\[
d_Z(x, y) \leq \|v(x) - v(y)\|_{L^2} + \sum_{j=1}^{N} \|\tilde{v}^j([a, b], x_j) - \tilde{v}^j([a, b], y_j)\|_{L^2} \leq C|x - y|^\alpha.
\]
This implies that
\[
N_\varepsilon(S) \leq C \left( \frac{r^N}{\varepsilon^{N/\alpha}} \right)
\]
and hence if $D \leq CA_0r^\alpha$ (which is the case if $A \leq A_0r^\alpha$), there is some large constant $K$ such that

$$\int_0^D \sqrt{\log N_\varepsilon(S)} \, d\varepsilon \leq K \int_0^D \sqrt{\log(Kr^\alpha/\varepsilon)} \, d\varepsilon.$$  

By the change of variable $\varepsilon = Kr^\alpha \exp(-u^2)$ and the elementary inequality $\int_x^\infty u^2 \exp(-u^2) \, du \leq Cx \exp(-x^2)$ for $x$ large, we deduce that

$$\int_0^D \sqrt{\log N_\varepsilon(S)} \, d\varepsilon \leq KD \sqrt{\log(Kr^\alpha/D)}.$$  

Since $D \leq CA$ and the function $f(x) = x \sqrt{\log(r^\alpha/x)}$ is increasing for $0 < x \ll r^\alpha$, the right-hand side is

$$\leq KA \sqrt{\log(Kr^\alpha/A)}$$

provided $A \leq A_0r^\alpha$, where $A_0$ is a sufficiently small constant. Therefore, the result now follows from Lemma 2.3.

\[\square\]

**Proposition 3.3.** Under Assumptions 1.1 and 1.2, there exist constants $0 < K_1 < \infty$ and $0 < \rho_0 \leq 1$ such that for any compact interval $I_\ell(s)$ in $T$, for any $0 < r_0 \leq \rho_0$.

$$\mathbb{P} \left\{ \exists r \in [r_0^2, r_0], \sup_{x \in I_\ell(s)} |v(x) - v(s)| \leq K_1r^\alpha \left( \log \log \frac{1}{r} \right)^{-\alpha} \right\} \geq 1 - \exp\left(-\left(\log \frac{1}{r_0}\right)^{1/2}\right).$$

**Remark 3.4.** This is a key estimate in the proofs of Theorems 1.4 and 1.5. Note that the exponent of the $\log \log \frac{1}{r}$ factor above is different from the one in Proposition 4.1 of [34] and the one in Proposition 2.3 of [11].

**Proof.** The proof follows the idea of Talagrand [34]. Let $r_0 > 0$. Fix $U > 1$. The value of $U$ will depend on $r_0$ and will be determined later. For $\ell \geq 1$, define $r_\ell = r_0U^{-2\ell}$ and $a_\ell = r_0^{-1}U^{2\ell-1}$. Let $\ell_0$ be the largest integer such that

$$\ell_0 \leq \frac{\log(1/r_0)}{2\log U}. \tag{11}$$

Then we have $r_0^2 \leq r_\ell \leq r_0$ for all $1 \leq \ell \leq \ell_0$. It suffices to prove that for some constant $K_1$,

$$\mathbb{P} \left\{ \exists 1 \leq \ell \leq \ell_0, \sup_{x \in I_\ell(s)} |v(x) - v(s)| \leq K_1r^\alpha \left( \log \log \frac{1}{r_\ell} \right)^{-\alpha} \right\} \geq 1 - \exp\left(-\left(\log \frac{1}{r_0}\right)^{1/2}\right). \tag{12}$$

By Lemma 3.1, if we take $K_1 = 2(4K_0)^{\alpha}$, then for all $1 \leq \ell \leq \ell_0$,

$$\mathbb{P} \left\{ \sup_{x \in I_\ell(s)} |\tilde{v}(a_{\ell}, a_{\ell+1}, x) - \tilde{v}(a_{\ell}, a_{\ell+1}, s)| \leq \frac{1}{2} K_1r^\alpha \left( \log \log \frac{1}{r_\ell} \right)^{-\alpha} \right\} \geq \exp\left(-\frac{1}{4} \log \log \frac{1}{r_\ell}\right) = \left(\log \frac{1}{r_\ell}\right)^{-1/4}. $$
By Assumption 1.2(a), the processes \( \hat{v}([a_\ell, a_{\ell+1}), \cdot) \), \( \ell = 1, \ldots, \ell_0 \), are independent, so

\[
\mathbb{P} \left\{ \exists 1 \leq \ell \leq \ell_0, \sup_{x \in I_{r_0}(s)} |\hat{v}(a_\ell, a_{\ell+1}, x) - \hat{v}(a_\ell, a_{\ell+1}, s)| \leq \frac{1}{2} K_1 r_0^\alpha \left( \log \log \frac{1}{r_0} \right)^{-\alpha} \right\}
\]

\[
= 1 - \prod_{\ell=1}^{\ell_0} \left( 1 - \mathbb{P} \left\{ \sup_{x \in I_{r_0}(s)} |\hat{v}(a_\ell, a_{\ell+1}, x) - \hat{v}(a_\ell, a_{\ell+1}, s)| \leq \frac{1}{2} K_1 r_0^\alpha \left( \log \log \frac{1}{r_0} \right)^{-\alpha} \right\} \right)
\]

\[
\geq 1 - \left( 1 - \left( \frac{1}{2} \right) \right)^{\ell_0}
\]

\[
\geq 1 - \exp \left( -\ell_0 \left( \frac{1}{2} \right)^{-1} \right).
\]

(13)

Let \( A_\ell = a_\ell^{1-\alpha} r_\ell^{1-\alpha} + a_{\ell+1}^{1-\gamma_1} r_\ell^{\gamma_1} + a_{\ell+1}^{1-\gamma_2} r_\ell^{\gamma_2} \), where \( 0 \leq \gamma_1 < \alpha \) and \( \gamma_2 > 0 \) are the constants given by Assumption 1.2(c). Then

\[
A_\ell r_\ell^{-\alpha} = (a_\ell r_\ell)^{1-\alpha} + (a_{\ell+1} r_\ell)^{-\alpha - \gamma_1} + r_\ell^{\gamma_2}.
\]

Note that \( a_\ell r_\ell = U^{-1} \) and \( a_{\ell+1} r_\ell = U \). Set \( \beta = \min\{1 - \alpha, \alpha - \gamma_1\} > 0 \). Then

\[
A_\ell r_\ell^{-\alpha} \leq 3 U^{-\beta}
\]

(14)

provided

\[
r_0^{\gamma_2} \leq U^{-\beta}.
\]

(15)

In particular, \( A_\ell r_\ell^{-\alpha} \leq A_0 \) if \( U \) is large enough. Set \( h_\ell = \frac{1}{2} K_1 r_\ell^\alpha \left( \log \log \frac{1}{r_\ell} \right)^{-\alpha} \). By Lemma 3.2, for some constants \( K \) and \( K' \), we have

\[
\mathbb{P} \left\{ \sup_{x \in I_{r_\ell}(s)} |v(x) - v(s) - \hat{v}(a_\ell, a_{\ell+1}, x) + \hat{v}(a_\ell, a_{\ell+1}, s)| \geq h_\ell \right\} \leq \exp \left( -\frac{h_\ell^2}{K'A_\ell^2} \right)
\]

provided \( h_\ell \geq K r_\ell^\alpha U^{-\beta} \sqrt{\log(KU)} \), that is, provided

\[
U_0^{\frac{\beta}{2}} \left( \log(KU) \right)^{-1/2} \geq \frac{2K}{K_1} \left( \log \log \frac{1}{r_0} \right)^{\alpha},
\]

(16)

which is possible if \( U \) is large enough (depending on \( r_0 \)). Then by (14), we get that

\[
\mathbb{P} \left\{ \sup_{x \in I_{r_\ell}(s)} |v(x) - v(s) - \hat{v}(a_\ell, a_{\ell+1}, x) + \hat{v}(a_\ell, a_{\ell+1}, s)| \geq \frac{1}{2} K_1 r_\ell^\alpha \left( \log \log \frac{1}{r_\ell} \right)^{-\alpha} \right\}
\]

\[
\leq \exp \left( -\frac{U_0^{2\beta}}{C(\log \log(1/r_0))^{2\alpha}} \right).
\]

(17)

Let

\[
F_\ell = \left\{ \sup_{x \in I_{r_\ell}(s)} |\hat{v}(a_\ell, a_{\ell+1}, x) - \hat{v}(a_\ell, a_{\ell+1}, s)| \leq \frac{1}{2} K_1 r_\ell^\alpha \left( \log \log \frac{1}{r_\ell} \right)^{-\alpha} \right\},
\]
The Hausdorff measure of the range and level sets of Gaussian random fields

\[
G_{\ell} = \left\{ \sup_{x \in I_{r_{\ell}}} |v(x) - v(s) - \tilde{v}([a_{\ell}, a_{\ell+1}], x) + \tilde{v}([a_{\ell}, a_{\ell+1}], s)| \geq \frac{1}{2} K_1 r_{\ell}^\alpha \left( \log \log \frac{1}{r_{\ell}} \right)^{-\alpha} \right\}.
\]

Then

\[
\mathbb{P} \left\{ \exists 1 \leq \ell \leq \ell_0, \sup_{x \in I_{r_{\ell}}} |v(x) - v(s)| \leq K_1 r_{\ell}^\alpha \left( \log \log \frac{1}{r_{\ell}} \right)^{-\alpha} \right\}
\]

\[
\geq \mathbb{P} \left( \bigcup_{\ell=1}^{\ell_0} (F_{\ell} \cap G_{\ell}^c) \right)
\]

\[
\geq \mathbb{P} \left( \bigcup_{\ell=1}^{\ell_0} F_{\ell} \right) \cap \left( \bigcap_{\ell=1}^{\ell_0} G_{\ell}^c \right)
\]

\[
\geq \mathbb{P} \left( \bigcup_{\ell=1}^{\ell_0} F_{\ell} \right) - \mathbb{P} \left( \bigcup_{\ell=1}^{\ell_0} G_{\ell} \right).
\]

By (13),

\[
\mathbb{P} \left( \bigcup_{\ell=1}^{\ell_0} F_{\ell} \right) \geq 1 - \exp \left( -\ell_0 \left( \log \frac{1}{r_0} \right)^{-1/4} \right),
\]

and by (17),

\[
\mathbb{P} \left( \bigcup_{\ell=1}^{\ell_0} G_{\ell} \right) \leq \ell_0 \exp \left( -\frac{U^{2\beta}}{C(\log \log (1/r_0))^{2\alpha}} \right).
\]

It follows that

\[
\mathbb{P} \left\{ \exists 1 \leq \ell \leq \ell_0, \sup_{x \in I_{r_{\ell}}} |v(x) - v(s)| \leq K_1 r_{\ell}^\alpha \left( \log \log \frac{1}{r_{\ell}} \right)^{-\alpha} \right\}
\]

\[
\geq 1 - \exp \left( -\ell_0 \left( \log \frac{1}{r_0} \right)^{-1/4} \right) - \ell_0 \exp \left( -\frac{U^{2\beta}}{C(\log \log (1/r_0))^{2\alpha}} \right).
\]

Therefore, the proof of (12) will be complete provided

\[
\exp \left( -\ell_0 \left( \log \frac{1}{r_0} \right)^{-1/4} \right) + \ell_0 \exp \left( -\frac{U^{2\beta}}{C(\log \log (1/r_0))^{2\alpha}} \right) \leq \exp \left( -\left( \log \frac{1}{r_0} \right)^{1/2} \right). \tag{18}
\]

Recall that conditions (15) and (16) are required for \( U \). Hence, we can take

\[
U = \left( \log \frac{1}{r_0} \right)^{\frac{1}{2\beta}}.
\]
Then for all \( r_0 \) small enough, by (11),
\[
\ell_0 > \frac{\beta}{2} \left( \log \frac{1}{r_0} \right) \left( \log \log \frac{1}{r_0} \right)^{-1} > 1.
\]
Therefore, the left-hand side of (18) is bounded above by
\[
\exp \left( -\left( \log \frac{1}{r_0} \right)^{3/4} \right) + \left( \log \frac{1}{r_0} \right) \exp \left( -\frac{\log \frac{1}{r_0}}{C r \log \log \frac{1}{r_0}^{2\alpha}} \right)
\]
\[
\leq \exp \left( -\left( \log \frac{1}{r_0} \right)^{1/2} \right)
\]
provided \( r_0 \) is small enough. This completes the proof of Proposition 3.3. \( \square \)

**Proof of Theorem 1.6.** We first prove that for each \( s \in T \), there exists a constant \( 0 \leq K \leq \infty \) such that
\[
\liminf_{r \to 0} \sup_{x \in I_r(s)} \frac{|v(x) - v(s)|}{r^\alpha (\log \log (1/r))^{-\alpha}} = K \quad \text{a.s.} \tag{19}
\]
We claim that the lim inf in (19) is equal to
\[
\liminf_{r \to 0} \sup_{x \in I_r(s)} \frac{|\tilde{v}(a_0, \infty, x) - \tilde{v}(a_0, \infty, s)|}{r^\alpha (\log \log (1/r))^{-\alpha}} \quad \text{a.s.} \tag{20}
\]
Indeed, by Assumption 1.2(c), there exists a constant \( C \) such that for \( r \) small, for all \( x, y \in I_r(s) \),
\[
|v(x) - v(y) - \tilde{v}(a_0, \infty, x) + \tilde{v}(a_0, \infty, y)|_{L^2} \leq C r^\gamma |x - y|^\alpha,
\]
where \( \gamma = \min\{1 - \alpha, \gamma_2\} > 0 \). Let \( S = I_r(s) \) and \( Z(x) = v(x) - \tilde{v}(a_0, \infty, x) \). Then the \( d_Z \)-diameter of \( S \) is \( D \leq C r^{\gamma + \alpha} \), and by the calculations in Lemma 3.2, we get that for \( r \) small,
\[
\int_0^D \sqrt{\log N_S(S)} d\varepsilon \leq K D \sqrt{\log (K r^{\alpha}) D} \leq K r^{\gamma + \alpha} \sqrt{\log (K/r)}.
\]
By taking \( r_n = 2^{-n} \) and using Lemma 2.3, we see that provided \( K \) is large,
\[
\mathbb{P} \left\{ \sup_{x \in I_{r_n}(s)} |v(x) - v(s) - \tilde{v}(a_0, \infty, x) + \tilde{v}(a_0, \infty, s)| \geq K r_n^{\gamma + \alpha} \sqrt{\log (1/r_n)} \right\} \leq \exp \left( -\frac{K^2 n}{C^2} \right)
\]
and hence by the Borel–Cantelli lemma,
\[
\limsup_{n \to \infty} \sup_{x \in I_{r_n}(s)} \frac{|v(x) - v(s) - \tilde{v}(a_0, \infty, x) + \tilde{v}(a_0, \infty, s)|}{r_n^{\gamma + \alpha} \sqrt{\log (1/r_n)}} \leq K \quad \text{a.s.}
\]
This implies that
\[
\lim_{r \to 0} \sup_{x \in I_r(s)} \frac{|v(x) - v(s) - \tilde{v}(a_0, \infty, x) + \tilde{v}(a_0, \infty, s)|}{r^\alpha (\log \log (1/r))^{-\alpha}} = 0 \quad \text{a.s.}
\]
From this, it follows that the lim inf in (19) is equal to the one in (20).

We continue with the proof of (19). For each $n \geq 0$, set $a_n = a_0 + n$. By Assumption 1.2(a),
\[ \hat{v}([a_0, \infty), x) = \sum_{n=0}^{\infty} \hat{v}([a_n, a_{n+1}), x) \quad \text{a.s.} \]
and the Gaussian fields $\hat{v}([a_n, a_{n+1}), \cdot)$, $n \geq 0$, are independent. Consider the $\sigma$-algebras $\mathcal{F}_n = \sigma(\hat{v}([a_m, a_{m+1}), \cdot) : m \geq n)$ and the $\sigma$-algebra $\mathcal{F}_\infty = \bigcap_{n=1}^{\infty} \mathcal{F}_n$ of all tail events. By Kolmogorov’s zero–one law, $\mathbb{P}(A) = 0$ or 1 for all $A \in \mathcal{F}_\infty$. This will imply (19) if we can show that the random variable in (20) is $\mathcal{F}_\infty$-measurable.

For each $n \geq 1$, we write $\hat{v}([a_0, \infty), x) = Y_n(x) + Z_n(x)$, where
\[ Y_n(x) = \sum_{i=0}^{n-1} \hat{v}([a_i, a_{i+1}), x) = \hat{v}([a_0, a_n), x) \quad \text{and} \quad Z_n(x) = \sum_{i=n}^{\infty} \hat{v}([a_i, a_{i+1}), x). \]

By Assumption 1.2(c), there is a constant $C$ (depending on $n$) such that for all $x, y \in I_r(s)$,
\[ \|Y_n(x) - Y_n(y)\|_{L^2} \leq \|v(x) - v(y) - \hat{v}([a_0, \infty), x) + \hat{v}([a_0, \infty), y)\|_{L^2} + \|v(x) - v(y) - \hat{v}([a_n, \infty), x) + \hat{v}([a_n, \infty), y)\|_{L^2} \leq Cr\gamma|x - y|^\alpha, \]
where $\gamma = \min\{1 - \alpha, \gamma_2\}$. Then, as in the first part of the proof, we use Lemma 2.3 to deduce that
\[ \lim_{r \to 0} \sup_{x \in I_r(s)} \frac{|Y_n(x) - Y_n(s)|}{r^\alpha (\log \log (1/r))^{-\alpha}} = 0 \quad \text{a.s.} \]

It follows the random variable in (20) is equal to
\[ \lim_{r \to 0} \sup_{x \in I_r(s)} \frac{|Z_n(x) - Z_n(s)|}{r^\alpha (\log \log (1/r))^{-\alpha}} = 0 \quad \text{a.s.} \]
Since $n \geq 1$ is arbitrary, this random variable is $\mathcal{F}_\infty$-measurable and therefore constant a.s. by Kolmogorov’s zero–one law. Combining with the claim in (20), we obtain (19) with $0 \leq K \leq \infty$.

Finally, we apply Proposition 3.3 with $r_0 = 2^{-n}$ and the Borel–Cantelli lemma to deduce the upper bound $K \leq K_1$, while the lower bound $K \geq K_0$ follows from Theorem 3.4 of [39] under Assumption 1.1.

\[ \square \]

4. Sojourn time estimates

In this section, we derive estimates for the sojourn times of $v$. For each $s \in T$ and $r > 0$, define
\[ \tau(s, r) = \int_T 1_{\{|v(x) - v(s)| \leq r\}} dx = \lambda_N \{ x \in T : |v(x) - v(s)| \leq r \}, \]
which is the sojourn time of $v$ in the ball centered at $v(s)$ with radius $r$. The following proposition provides upper bounds for the moments of $\tau(s, r)$.
Proposition 4.1. Let Assumption 1.1 hold. Then there exists a finite constant $K$ such that for all $n \geq 1$, for all $x^* \in T$, for all $r > 0$,

$$E[\tau(x^*, r)^n] \leq K^n(n!)^{\alpha n N/\alpha}.$$  

Proof. By Fubini’s theorem,

$$E[\tau(x^*, r)^n] = \int_T \prod_{1 \leq i \leq n} \mathbb{P}\left\{ \max_{1 \leq i \leq n} |v(x^i) - v(x^*)| \leq r \right\} dx^1 \ldots dx^n.$$  

For $m = 1, \ldots, n$, define the event $A_m$ by

$$A_m = \left\{ \max_{1 \leq i \leq m} |v(x^i) - v(x^*)| \leq r \right\}.$$  

Then

$$\mathbb{P}(A_m) = E \left[ \mathbf{1}_{A_{n-1}} \mathbb{P}\left\{ |v(x^n) - v(x^*)| \leq r \bigg| v(x^*), v(x^1), \ldots, v(x^{n-1}) \right\} \right].$$  

Since $v$ is Gaussian, so is the conditional distribution of $v(x^n)$ given $v(x^*), v(x^1), \ldots, v(x^{n-1})$ which, by sectorial LND [Assumption 1.1(b)], has conditional variance $\text{Var}(v(x^n)|v(x^*), v(x^1), \ldots, v(x^{n-1})) \geq c_1 \sum_{j=1}^N \min_z |x^n_j - x^j|^2 \alpha$, where the min is taken over $i \in \{*, 0, \ldots, n - 1\}$ and $x^0 = 0$. Then by Anderson’s inequality [3],

$$\mathbb{P}\left\{ |v(x^n) - v(x^*)| \leq r \bigg| v(x^*), v(x^1), \ldots, v(x^{n-1}) \right\} \leq K \min \left\{ 1, \frac{r^d}{\left( \sum_{j=1}^N \min_z |x^n_j - x^j|^{2 \alpha} \right)^{d/2}} \right\}.$$  

Note the elementary inequality $(\sum_{j=1}^N z_j)^\alpha \leq \sum_{j=1}^N z_j^{\alpha}$ for $0 < \alpha < 1$ and $z_j \geq 0$. Now, let us fix $x^*, x^1, \ldots, x^{n-1}$ and estimate the integral

$$\int_T \min \left\{ 1, \frac{r^d}{\left( \sum_{j=1}^N \min_z |x^n_j - x^j|^{2 \alpha} \right)^{d/2}} \right\} dx^n. \quad (21)$$  

We may assume that $x^*, x^1, \ldots, x^n$ are distinct, because the set of $(x^1, \ldots, x^n)$ for which they are not distinct has Lebesgue measure 0. Consider the following set consisting of $(n + 1)^N$ points:

$$Y = \prod_{j=1}^N \left\{ x^*_j, x^0_j, x^1_j, \ldots, x^{n-1}_j \right\}.$$  

We use these points to produce a partition of $T$ into subintervals $S$ such that for each subinterval $S$, there is one and only one $y \in Y$ such that

$$\min_i |x^n_i - x^j| = |x^n_j - y_j| \quad (22)$$  

for all $x^n \in S$ and all $j = 1, \ldots, N$. This can be done by first ordering, for each $j$, the coordinate points $x^*_j, x^0_j, \ldots, x^{n-1}_j$ with a permutation $\Pi_j$ on the set $\{*, 0, \ldots, n - 1\}$ such that

$$x^*_{j(\Pi_j(\ast))} < x^0_{j(\Pi_j(0))} < \cdots < x^*_{j(\Pi_j(n - 1))},$$
and then using the perpendicular bisectors of every two adjacent points to construct the partition. There are at most \((n + 1)^N = O(n^N)\) subintervals \(S\). Then the integral in (21) is equal to

\[
\sum_S \int_S \min \left\{ 1, \frac{r^d}{(\sum_{j=1}^N \min_i |x_j^n - x_i|^2)^{\alpha d/2}} \right\} \, dx^n.
\]

For each fixed subinterval \(S\), we use (22) and polar coordinates \(x^n - y = \rho \theta\) to get that

\[
\int_S \min \left\{ 1, \frac{r^d}{(\sum_{j=1}^N \min_i |x_j^n - y_j|^2)^{\alpha d/2}} \right\} \, dx^n \leq K \int_0^\infty \min \left\{ 1, r^d \rho^{-\alpha d} \right\} \rho^{N-1} \, d\rho
\]

\[
= K \int_0^{\rho_{1/\alpha}} \rho^{N-1} \, d\rho + K \int_{\rho_{1/\alpha}}^{\infty} r^d \rho^{N-\alpha d-1} \, d\rho
\]

\[
= K \rho_{N/\alpha}.
\]

It follows that

\[
\mathbb{E}[\tau(x^*, r^n)] = \int_{T^{n-1}} \mathbb{P}(A_{n-1}) dx^1 \cdots x^{n-1} \int_T \min \left\{ 1, \frac{r^d}{(\sum_{j=1}^N \min_i |x_j^n - x_i|^2)^{\alpha d/2}} \right\} \, dx^n
\]

\[
\leq Kn^{N/\alpha} \int_{T^{n-1}} \mathbb{P}(A_{n-1}) dx^1 \cdots dx^{n-1}
\]

\[
= Kn^{N/\alpha} \mathbb{E}[\tau(x^*, r^{n-1})].
\]

Therefore, the result can be deduced by induction. \(\square\)

From the moment estimates, we can derive the following asymptotic result for the sojourn times, which will be used in the proof of Theorem 1.4.

**Proposition 4.2.** Under Assumption 1.1, for all \(s \in T\),

\[
\limsup_{r \to 0} \frac{\tau(s, r)}{\phi(r)} \leq K \text{ a.s.,}
\]

where \(\phi(r) = r^{N/\alpha}(\log \log (1/r))^{N}\) and \(K\) is the constant in Proposition 4.1.

**Proof.** Consider a constant \(C > K\). By Fubini’s theorem and Jensen’s inequality,

\[
\mathbb{E}[\exp(C^{-1/N} r^{-1/\alpha} \tau(s, r)^{1/N})] = \sum_{n=0}^\infty \frac{1}{n!} C^{-n/N} r^{-n/\alpha} \mathbb{E}[\tau(s, r)^{n/N}]
\]

\[
\leq \sum_{n=0}^\infty \frac{1}{n!} C^{-n/N} r^{-n/\alpha} \mathbb{E}[\tau(s, r)^{n^{1/N}}].
\]

Then by Proposition 4.1,

\[
\mathbb{E}[\exp(C^{-1/N} r^{-1/\alpha} \tau(s, r)^{1/N})] \leq \sum_{n=0}^\infty (K/C)^{n/N} = A,
\]
where $A$ is finite since $C > K$. It follows that for any $\varepsilon > 0$,

$$
P \left\{ \tau(s, r) > (1 + \varepsilon)^N C r^{-N/\alpha} (\log \log(1/r))^N \right\}$$

$$= P \left\{ \exp(C^{-1/N} r^{-1/\alpha} \tau(s, r)^{1/N}) > (\log(1/r))^{1+\varepsilon} \right\} \leq \frac{A}{(\log(1/r))^{1+\varepsilon}}.
$$

Set $r_n = e^{-n/\log n}$. Then by the Borel–Cantelli lemma, we have

$$\limsup_{n \to \infty} \frac{\tau(s, r_n)}{\phi(r_n)} \leq (1 + \varepsilon)^N C \text{ a.s.}$$

From this, we can use a monotonicity argument to deduce that

$$\limsup_{r \to 0} \frac{\tau(s, r)}{\phi(r)} \leq (1 + \varepsilon)^N C \text{ a.s.}$$

Finally, we obtain the result by letting $\varepsilon \to 0$ and $C \downarrow K$ along rational sequences.

5. Proof of Theorem 1.4

It suffices to prove that for each compact interval $J$ in $T$ with rational vertices,

$$P \left\{ C_1 \lambda_N(J) \leq \mathcal{H}_\phi(v(J)) \leq C_2 \lambda_N(J) \right\} = 1. \quad (23)$$

Then (4) follows by taking sequences $J_n$ of compact intervals with rational vertices such that $J_n \uparrow J$ and by the continuity of $v$.

The proof of (23) is similar to the proof in [40]. We start with the proof of the lower bound. Let $J$ be any compact interval in $T$. Define a random Borel measure $\mu$ on $\mathbb{R}^N$ by

$$\mu(B) = \lambda_N \{ s \in J : v(s) \in B \}$$

for any Borel set $B$ in $\mathbb{R}^N$. Note that $\mu(\mathbb{R}^d) = \lambda_N(J)$ and $\mu$ is a finite measure. Consider the random set

$$F = \left\{ v(s) : s \in J \text{ and } \limsup_{r \to 0} \frac{\mu(B(v(s), r))}{\phi(r)} \leq K \right\},$$

where $K$ is the constant in Proposition 4.2. Then by Lemma 2.1, we have

$$\mu(F) \leq C \mathcal{H}_\phi(F) \sup_{s \in F} D_\mu^\phi(s) \leq CK \mathcal{H}_\phi(v(J)).$$

It remains to prove that $\mu(F) = \lambda_N(J)$ a.s. To see this, note that $\mu = \lambda_J \circ v^{-1}$, where $\lambda_J$ is the restriction of $\lambda_N$ on $J$. Then by the change of variable formula and Fubini’s theorem,

$$\mathbb{E}[\mu(F)] = \mathbb{E} \int_{\mathbb{R}^d} 1_F(z) \mu(dz) = \mathbb{E} \int_J 1_F(v(s)) \, ds$$

$$= \int_J P \left\{ \limsup_{r \to 0} \frac{\mu(B(v(s), r))}{\phi(r)} \leq K \right\} \, ds.$$
Then by Proposition 4.2, we have $E[\mu(F)] = \lambda_N(J)$. This implies that $\mu(F) = \lambda_N(J)$ a.s. and hence

$$\mathcal{H}_\phi(v(J)) \geq C^{-1} K^{-1} \lambda_N(J) \quad \text{a.s.}$$

We turn to the proof of the upper bound in (23). The idea is due to Talagrand [34]. For each $p \geq 1$, consider the random set

$$R_p = \left\{ s \in J : \exists r \in [2^{-2p}, 2^{-p}] \text{ such that } \sup_{x \in I_r(s)} |v(x) - v(s)| \leq K_1 r^\alpha \left( \log \log \frac{1}{r} \right)^{-\alpha} \right\}$$

and the event

$$\Omega_{p,1} = \left\{ \omega : \lambda_N(R_p) \geq \lambda_N(J) \left( 1 - \exp(-\sqrt{p}/4) \right) \right\}.$$

Note that

$$\Omega_{p,1}^c = \left\{ \omega : \lambda_N(J \setminus R_p) > \lambda_N(J) \exp(-\sqrt{p}/4) \right\}.$$

Then by Markov’s inequality,

$$P(\Omega_{p,1}^c) \leq \frac{E[\lambda_N(J \setminus R_p)]}{\lambda_N(J) \exp(-\sqrt{p}/4)}.$$ 

By Fubini’s theorem and Proposition 3.3, the numerator is equal to

$$E \int_J 1_{J \setminus R_p}(s) \, ds = \int_J P(s \notin R_p) \, ds \leq \lambda_N(J) \exp(-\sqrt{p}/2).$$

It follows that $\sum_{p=1}^{\infty} P(\Omega_{p,1}^c) \leq \sum_{p=1}^{\infty} \exp(-\sqrt{p}/4) < \infty$.

We will call an interval of the form $\prod_{j=1}^N [m_j 2^{-p}, (m_j + 1) 2^{-p})$, where $m_j \in \mathbb{Z}$, a dyadic cube of order $p$. Let $\mathcal{C}_p$ be the set of all dyadic cubes of order $p$ that intersect $J$, and for convenience, we replace each $C \in \mathcal{C}_p$ by the intersection $C \cap J$. Let $K_2$ be a constant and consider the event

$$\Omega_{p,2} = \left\{ \omega : \forall C \in \mathcal{C}_p \quad \sup_{x, y \in C} |v(x) - v(y)| \leq K_2 2^{-2p \alpha} \sqrt{p} \right\}.$$

Since the number of cubes in $\mathcal{C}_p$ is $\leq K_2 2^{2pN}$, it follows from Lemma 2.4 that $\sum_{p=1}^{\infty} P(\Omega_{p,2}^c) < \infty$ provided $K_2$ is large enough. Let $\Omega_p = \Omega_{p,1} \cap \Omega_{p,2}$. Then with probability 1, $\Omega_p$ occurs for all $p$ large.

Let $\omega \in \Omega_p$. We are going to construct a random cover for $v(J)$. We say that $C$ is a “good” dyadic cube of order $q$ if

$$\sup_{x, y \in C} |v(x) - v(y)| \leq 4K_1 2^{-q \alpha} \left( \log \log 2^q \right)^{-\alpha}. \quad (24)$$

Then for each $s \in R_p$ ($p$ large), there is at least one good dyadic cube of order $q$ ($p \leq q \leq 2p$) that contains $s$, and we can choose such a good dyadic cube $C$ of the smallest order. Indeed, if $s \in R_p$, then there is $r \in [2^{-q}, 2^{-q+1}]$ with $p + 1 \leq q \leq 2p$ such that

$$\sup_{x \in I_r(s)} |v(x) - v(s)| \leq K_1 r^\alpha \left( \log \log (1/r) \right)^{-\alpha}.$$
Then for the dyadic cube $C$ of order $q$ that contains $s$, we can verify that (24) is satisfied:

$$
\sup_{x,y \in C} |v(x) - v(y)| \leq \sup_{x \in C} |v(x) - v(s)| + \sup_{y \in C} |v(y) - v(s)|
\leq 2K_1 (2^{-q+1})^\alpha (\log \log 2^{q-1})^{-\alpha}
\leq 4K_1 2^{-q\alpha} (\log \log 2^q)^{-\alpha}.
$$

Therefore, we can obtain a family $\mathcal{H}_p^1$ of disjoint good dyadic cubes of order between $p$ and $2p$ that cover the set $R_p$.

Next, we let $\mathcal{H}_p^2$ be the family of dyadic cubes $C \in \mathcal{C}_{2p}$ of order $2p$ that are disjoint from the cubes of $\mathcal{H}_p^1$ and intersect $J$. We call these “bad” dyadic cubes. Then the cubes of $\mathcal{H}_p^2$ are contained in $J \setminus R_p$ and the family $\mathcal{H}_p := \mathcal{H}_p^1 \cup \mathcal{H}_p^2$ covers $J$.

For each dyadic cube $C$, let $s_C$ be the lower-left vertex of $C$ and

$$
r_C = \begin{cases}
4K_1 2^{-q\alpha} (\log \log 2^q)^{-\alpha} & \text{if } C \in \mathcal{H}_p^1 \text{ and } C \text{ is of order } q \\
K_2 2^{-2p\alpha} \sqrt{p} & \text{if } C \in \mathcal{H}_p^2.
\end{cases}
$$

Now, for $\omega \in \Omega_p$, define a family $\mathcal{F}_p$ of Euclidean balls in $\mathbb{R}^d$ by

$$
\mathcal{F}_p = \left\{ B(v(s_C), r_C) : C \in \mathcal{H}_p^1 \cup \mathcal{H}_p^2 \right\}.
$$

Note that for $C \in \mathcal{H}_p^1$, the image $v(C)$ is contained in the ball $B(v(s_C), r_C)$ because of (24), and this is also true for $C \in \mathcal{H}_p^2$ since $\omega \in \Omega_{p,2}$. Hence, $\mathcal{F}_p$ is a random cover for $v(J)$ as long as $\omega \in \Omega_p$.

Recall that $\phi(r) = r^{N/\alpha} (\log \log (1/r))^N$. If $C$ is a good dyadic cube in $\mathcal{H}_p^1$, then

$$
\phi(2r_C) \leq K [2^{-q\alpha} (\log \log 2^q)^{-\alpha}]^{N/\alpha} (\log \log 2^q)^N \leq K 2^{-qN}.
$$

Since the cubes of $\mathcal{H}_p^1$ are disjoint and contained in $J$, it follows that

$$
\sum_{C \in \mathcal{H}_p^1} \phi(2r_C) \leq K \sum_{C \in \mathcal{H}_p^1} \lambda_N(C) \leq K \lambda_N(J).
$$

If $C$ is a bad dyadic cube in $\mathcal{H}_p^2$, then

$$
\phi(2r_C) \leq K 2^{-2p\alpha} p^{N/(2\alpha)} (\log p)^N.
$$

Since $\omega \in \Omega_{p1}$, the number of cubes in $\mathcal{H}_p^2$ is at most

$$
2^{2pN} \lambda_N(J \setminus R_p) \leq 2^{2pN} \lambda_N(J) \exp(-\sqrt{p}/4).
$$

It follows that

$$
\sum_{C \in \mathcal{H}_p^2} \phi(2r_C) \leq K \lambda_N(J) \exp(-\sqrt{p}/4) p^{N/(2\alpha)} (\log p)^N.
$$
The Hausdorff measure of the range and level sets of Gaussian random fields

Therefore, with probability 1,
\[
\mathcal{H}_p(v(J)) \leq \liminf_{p \to \infty} \left[ 1_{\Omega_p} \sum_{C \in H^1_p \cup H^2_p} \phi(2r_C) \right]
\]
\[
\leq \liminf_{p \to \infty} \left[ K \lambda_N(J) \left( 1 + \exp(-\sqrt{p}/4) p^{N/(2\alpha)}(\log p)^N \right) \right]
\]
\[
= K \lambda_N(J).
\]

The proof of Theorem 1.4 is complete. \hfill \Box

6. Proof of Theorem 1.5

First, consider the case \( N > \alpha d \). For the lower bound in (5), it suffices to prove that there exists a positive finite constant \( C \) such that for each \( z \in \mathbb{R}^d \), for each compact interval \( J \) in \( T \) with rational vertices, we have
\[
P\left\{ CL(z, J) \leq \mathcal{H}_p(v^{-1}(z) \cap J) \right\} = 1.
\]
This is a consequence of Theorem 4.6 of [39] under Assumption 1.1. Since the local time \( L(z, J) \) of \( v \) is continuous in \( J \) [42, Theorem 8.2], we can then take, for any \( J \in \mathcal{I}(T) \), a sequence \( J_n \) of compact intervals with rational vertices such that \( J_n \uparrow J \) to deduce that
\[
P\left\{ CL(z, J) \leq \mathcal{H}_p(v^{-1}(z) \cap J) \text{ for all } J \in \mathcal{I}(T) \right\} = 1.
\]

To show that the Hausdorff measure above is finite simultaneously for all \( J \), we will prove that
\[
\mathbb{E}[\mathcal{H}_p(v^{-1}(z) \cap J)] \leq C \lambda_N(J)
\] (25)
for all \( z \in \mathbb{R}^d \) and all sufficiently small compact intervals \( J \) in \( T \). The proof of (25) below is similar to the proof of Theorem 4.2 in [41] and is based on the method of [35]. Let \( \varepsilon_0 \) be the constant given by Assumption 1.3. Fix a small constant \( 0 < \rho_0 \leq \varepsilon_0 \) whose value will be determined. Let \( 0 < \rho \leq \rho_0 \). We take \( J \) to be an interval \( I_\rho(x) = T \cap \prod_{j=1}^d [x_j - \rho, x_j + \rho] \) and let \( x' \in J(\varepsilon_\rho) \) be given by Assumption 1.3. We aim to prove that (25) holds for such an interval \( J \).

Let us define
\[
v^2(y) = \mathbb{E}(v(y)|v(x')) \quad \text{and} \quad v^1(y) = v(y) - v^2(y).
\]
Then the Gaussian random fields \( v^1 \) and \( v^2 \) are independent. By Lemma 5.3 of [11], under Assumption 1.3, \( v^2 \) has a continuous version and there exists a finite constant \( K_3 \) and \( \alpha < \delta \leq 1 \) such that for all \( y, \bar{y} \in J \),
\[
|v^2(y) - v^2(\bar{y})| \leq K_3 |v(x')||y - \bar{y}|^\delta.
\] (26)

We define the random set \( R_p \) and the events \( \Omega_{p,1} \) and \( \Omega_{p,2} \) as in the proof of Theorem 1.4. In addition, we fix a constant \( \beta \) such that \( 0 < \beta < \delta - \alpha \) and consider the event
\[
\Omega_{p,3} = \{|v(x')| \leq 2p^{-\beta}\}.
\]
By (26), the Hölder constant of \( v^2 \) is not too large on \( \Omega_{p,3} \). Since \( v(x') \) is a Gaussian vector with independent components whose variances are bounded away from 0 (by Assumption 1.3(a)), we have 
\[
\sum_{p=1}^{\infty} \mathbb{P}(\Omega_{p,3}^c) < \infty.
\]
Let \( \Omega_p = \Omega_{p,1} \cap \Omega_{p,2} \cap \Omega_{p,3} \). Then with probability 1, \( \Omega_p \) occurs for all \( p \) large. Moreover, we consider the random set
\[
R_p' = \left\{ s \in J : \exists r \in [2^{-2p}, 2^{-p}] \text{ such that } \sup_{x \in I_r(s)} |v^1(x) - v^1(s)| \leq 2K_1 r^\alpha \left( \log \log \frac{1}{r} \right)^{-\alpha} \right\}.
\]
and the event
\[
\Omega_{p,4} = \left\{ \omega : \lambda_N(R_p') \geq \lambda_N(J) (1 - \exp(-\sqrt{p}/4)) \right\}.
\]
Since \( v = v^1 + v^2 \) and \( \alpha < \delta - \beta \), (26) implies that \( R_p \subset R_p' \) on \( \Omega_{p,3} \) for \( p \) large enough and hence
\[
\Omega_{p,1} \cap \Omega_{p,3} \subset \Omega_{p,4}. \quad (27)
\]
To prove (25), we let \( \omega \in \Omega_p \) and construct an efficient covering (depending on \( \omega \)) for the level set \( v^{-1}(z) \cap J \). Instead of (24), we now say that \( C \) is a “good” dyadic cube of order \( q \) if
\[
\sup_{y, \bar{y} \in C} |v^1(y) - v^1(\bar{y})| \leq 8K_1 2^{-q\alpha} (\log \log 2^q)^{-\alpha}. \quad (28)
\]
Then every \( s \in R_p' \) is contained in a good dyadic cube of order \( q \) with \( p \leq q \leq 2p \). This produces a family \( \mathcal{H}_p^1 \) of disjoint good dyadic cubes of order between \( p \) and \( 2p \) that cover the set \( R_p' \). Let \( \mathcal{H}_p^2 \) be the family of (bad) dyadic cubes of order \( 2p \) that are disjoint from the cubes of \( \mathcal{H}_p^1 \) and intersect \( J \). Let \( \mathcal{H}_p = \mathcal{H}_p^1 \cup \mathcal{H}_p^2 \). It follows that the cubes of \( \mathcal{H}_p^2 \) are contained in \( J \setminus R_p' \) and \( \mathcal{H}_p \) covers \( J \). Note that the families \( \mathcal{H}_p^1 \) and \( \mathcal{H}_p^2 \) depend on the process \( v^1 \) only. This property will be used later in the proof.

For each dyadic cube \( C \), define \( s_C \) as the lower-left vertex of \( C \) and
\[
r_C = \begin{cases} 
8K_1 2^{-q\alpha} (\log \log 2^q)^{-\alpha} & \text{if } C \in \mathcal{H}_p^1 \text{ and } C \text{ is of order } q \\
K_2 2^{-2p\alpha} \sqrt{p} & \text{if } C \in \mathcal{H}_p^2. 
\end{cases}
\]
Now, we construct the family \( \mathcal{G}_p = \mathcal{G}_p(\omega) \) of cubes by setting
\[
\mathcal{G}_p = \{ C \in \mathcal{H}_p : \omega \in \Omega_C \},
\]
where
\[
\Omega_C = \{ |v(s_C) - z| \leq 2r_C \}.
\]
We verify that \( \mathcal{G}_p \) covers \( v^{-1}(z) \cap J \) for \( \omega \in \Omega_p \) and \( p \) large. Let \( y \in v^{-1}(z) \cap J \). Then \( v(y) = z \) and \( y \in C \) for some dyadic cube \( C \in \mathcal{H}_p \) because \( \mathcal{H}_p \) covers \( J \). We need to show that \( C \) is a member of \( \mathcal{G}_p \). Consider the following two cases.

Case 1. If \( C \) is a good dyadic cube in \( \mathcal{H}_p^1 \) of order \( q \), where \( p \leq q \leq 2p \), then condition (28) is satisfied and thus
\[
|v^1(s_C) - v^1(y)| \leq r_C.
\]
Since \( \omega \in \Omega_{p,3} \), by (26),
\[
|v^2(s_C) - v^2(y)| \leq K2^{-q(\delta-\beta)},
\]
which is \( \leq 8K_12^{-qa_0}(\log \log 2^q)^{-\alpha} = r_C \) for \( p \) large because \( \alpha < \delta - \beta \). Since \( v = v^1 + v^2 \), it follows that
\[
|v(s_C) - z| = |v(s_C) - v(y)| \leq 2r_C.
\]
Hence \( \Omega_C \) occurs and \( C \in \mathcal{G}_p \).

Case 2. If \( C \) is a bad dyadic cube in \( \mathcal{H}^2_p \), then \( C \) of order \( 2p \). Since \( \omega \in \Omega_{p,2} \), we have
\[
|v(s_C) - z| = |v(s_C) - v(y)| \leq r_C
\]
and hence \( C \in \mathcal{G}_p \). Therefore, \( \mathcal{G}_p \) covers \( v^{-1}(z) \cap J \) for \( \omega \in \Omega_p \) and \( p \) large.

Consider the \( \sigma \)-algebra \( \Sigma_1 = \sigma\{v^1(y) : y \in J\} \). In order to estimate \( \mathbb{E}[\mathcal{M}_p(v^{-1}(z) \cap J)] \) using the random cover \( \mathcal{G}_p \), we need to first estimate the conditional probability \( \mathbb{P}(\Omega_C|\Sigma_1) \). Note that \( \text{Var}(v(y)|v(x')) \leq \mathbb{E}[(v(y) - v(x'))^2] \). Then by the conditional variance formula and Assumptions 1.1(a) and 1.3(a), we get that for all \( y \in J \),
\[
\text{Var}(v^2(y)) = \text{Var}(\mathbb{E}(v(y)|v(x')))
\]
\[
= \text{Var}(v(y)) - \mathbb{E}[\text{Var}(v(y)|v(x'))]
\]
\[
\geq c_3^2 - c_0^2|y - x'|^{2\alpha}.
\]
Since \( J \) has side length \( 2p \) and \( x' \in J(\rho p) \), with \( 0 < \rho \leq \rho_0 \), we can choose \( \rho_0 > 0 \) to be small enough so that the variance of \( v^2(y) \) is bounded below by a positive constant. Then \( \mathbb{P}(|v^2(y) - a| \leq r) \leq K r^d \) for all \( y \in J, a \in \mathbb{R}^d \) and \( r > 0 \), where \( K \) is a constant. Recall that \( \mathcal{H}^1_p \) and \( \mathcal{H}^2_p \) depend on \( v^1 \) only, so \( r_C \) is \( \Sigma_1 \)-measurable. Hence, by the independence of \( v^1 \) and \( v^2 \),
\[
\mathbb{P}(\Omega_C|\Sigma_1) \leq K r_C^d.
\] (29)

Now, we estimate \( \mathbb{E}[\mathcal{M}_p(v^{-1}(z) \cap J)] \) using the cover \( \mathcal{G}_p \). Recall that the cubes of \( \mathcal{H}^2_p \) are contained in \( J \setminus R'_p \), and from (27), we see that on the event \( \Omega_{p,1} \cap \Omega_{p,3} \), the set \( J \setminus R'_p \) has Lebesgue measure \( \leq \lambda_N(J) \exp(-\sqrt{p}/4) \). It follows that \( \Omega_p \) is contained in the event \( \tilde{\Omega}_p \) that the number of cubes in \( \mathcal{H}^2_p \) is at most \( \lambda_N(J) 2^{p \rho N} \exp(-\sqrt{p}/4) \). Then
\[
\mathbb{E}\left[ \mathbb{1}_{\Omega_p} \sum_{C \in \mathcal{G}_p} \varphi(\text{diam}(C)) \right] \leq \mathbb{E}\left[ \mathbb{1}_{\Omega_p} \sum_{C \in \mathcal{H}_p} \varphi(\text{diam}(C)) \mathbb{1}_{\Omega_C} \right].
\]
Since \( \tilde{\Omega}_p \) and \( \mathcal{H}_p \) are measurable with respect to \( \Sigma_1 \), we can take conditional expectation over \( \Sigma_1 \) and use (29) to get
\[
\mathbb{E}\left[ \mathbb{1}_{\tilde{\Omega}_p} \sum_{C \in \mathcal{H}_p} \varphi(\text{diam}(C)) \mathbb{P}(\Omega_C|\Sigma_1) \right] \leq K \mathbb{E}\left[ \mathbb{1}_{\tilde{\Omega}_p} \sum_{C \in \mathcal{H}_p} \varphi(\text{diam}(C)) r_C^d \right].
\]
Recall that \( \varphi(r) = r^{N - \alpha d}(\log \log (1/r)^{\alpha} \right)^d \). If \( C \in \mathcal{H}^1_p \) is a good dyadic cube of order \( q \), then
\[
\varphi(\text{diam}(C)) r_C^d \leq K 2^{-q(N - \alpha d)} (\log q)^{\alpha d} (2^{-q \alpha} (\log q)^{-\alpha})^d \leq K 2^{-q N}.
\]
It follows that
\[
\mathbb{E}\left[ \sum_{C \in \mathcal{H}_p^1} \varphi(\text{diam}(C)) r_C^d \right] \leq K \sum_{C \in \mathcal{H}_p^1} \lambda_N(C) \leq K \lambda_N(J).
\]
If $C \in H_p^2$ is a bad dyadic cube, then
\[ \varphi(\text{diam}(C))r_C^d \leq K2^{-2pN}(\log p)^{\alpha d}p^{d/2}. \]
Since the number of cubes in $H_p^2$ is \( \leq \lambda_N(J)2^{pN} \exp(-\sqrt{p}/4) \) on $\Omega_p$, it follows that
\[ \mathbb{E}\left[ \sum_{C \in H_p^2} \varphi(\text{diam}(C))r_C^d \right] \leq K\lambda_N(J)\exp(-\sqrt{p}/4)(\log p)^{\alpha d}p^{d/2}. \]
Therefore, by Fatou's lemma,
\[ \mathbb{E}\left[ \sum_{C \in G_p} \varphi(\text{diam}(C))r_C^d \right] \leq K\lambda_N(J)\exp(-\sqrt{p}/4)(\log p)^{\alpha d}p^{d/2}. \]

This completes the proof of (25) and establishes (5).

Consider the case $N \leq \alpha d$. Suppose, towards a contradiction, that the event $\Omega' := \{ v^{-1}(z) \cap J \neq \emptyset \}$ has positive probability. Then the event $\Omega' \cap \Omega^* = \bigcup_{n \geq 1} \bigcap_{p \geq n} (\Omega' \cap \Omega_p)$ also has positive probability. Consider the random variable
\[ X = \liminf_{p \to \infty} \mathbb{1}_{\Omega' \cap \Omega_p} \sum_{C \in G_p} \varphi(\text{diam}(C)). \]
By the calculations above, $\mathbb{E}(X) \leq K\lambda_N(J) < \infty$. On the other hand, for $p$ large, since $G_p$ covers $v^{-1}(z) \cap J$ on $\Omega_p$ and $v^{-1}(z) \cap J \neq \emptyset$ on $\Omega'$, we see that $G_p$ is nonempty on $\Omega' \cap \Omega_p$. Moreover, $N \leq \alpha d$ implies that $\varphi(r) \to \infty$ as $r \to 0$. It follows that $X = \infty$ on $\Omega' \cap \Omega^*$, which contradicts the fact that $\mathbb{E}(X) < \infty$. Therefore, we have $v^{-1}(z) \cap J \neq \emptyset$ a.s. for all sufficiently small compact intervals $J$ in $T$. This implies $v^{-1}(z) \cap T \neq \emptyset$ a.s. and completes the proof of Theorem 1.5.

7. Examples

7.1. Brownian sheet and fractional Brownian sheets

Let $v = \{ v(x) : x \in \mathbb{R}^N_+ \}$ be a fractional Brownian sheet from $\mathbb{R}^N_+$ to $\mathbb{R}^d$ with Hurst indices $H_1 = \cdots = H_N = \alpha$, where $0 < \alpha < 1$, i.e., a centered, continuous Gaussian random field with i.i.d. components $v_1, \ldots, v_d$ and covariance
\[ \mathbb{E}[v_i(x)v_i(y)] = N \prod_{j=1}^N \frac{1}{2}(|x_j|^{2\alpha} + |y_j|^{2\alpha} - |x_j - y_j|^{2\alpha}). \]
When $\alpha = 1/2$, $v$ is a Brownian sheet from $\mathbb{R}^N_+$ to $\mathbb{R}^d$.

The uniform dimension of the level sets of the Brownian sheet was studied by Adler [1], and that of the images was studied by Mountford [27] and Khoshnevisan et al. [18]. The Hausdorff measures of
the range and graph of the Brownian sheet were studied by Ehm [13], and that of the level sets were studied by Zhou [44]. For fractional Brownian sheets, dimension results for the images and level sets can be found in [5] and [38].

It is known that for any compact interval $T$ in $(0, \infty)^N$, there exists a positive finite constant $c_0$ such that for all $x, y \in T$,

$$c_0^{-1} \sum_{j=1}^N |x_j - y_j|^{2\alpha} \leq \mathbb{E}(|v(x) - v(y)|^2) \leq c_0 \sum_{j=1}^N |x_j - y_j|^{2\alpha},$$  

(30)

and $v$ satisfies sectorial LND: there exists a positive constant $c_1$ such that for all integers $n \geq 1$, for all $x, y^1, \ldots, y^n \in T$,

$$\text{Var}(v_1(x)|v_1(y^1), \ldots, v_1(y^n)) \geq c_1 \sum_{j=1}^N \min_{0 \leq i \leq n} |x_j - y_j|^{2\alpha},$$  

(31)

where $y^0 = 0$. See [5, 38].

Let us recall the harmonizable representation for the fractional Brownian sheet [10, §5.1]. For any $x, y \in \mathbb{R}$, we have the identity

$$\frac{1}{2}(|x|^{2\alpha} + |y|^{2\alpha} - |x - y|^{2\alpha}) = c_\alpha \int_{\mathbb{R}} \left[ \frac{(1 - \cos x\xi)(1 - \cos y\xi)}{|\xi|^{2\alpha + 1}} + \frac{\sin x\xi \sin y\xi}{|\xi|^{2\alpha + 1}} \right] d\xi,$$

where $c_\alpha$ is a constant depending on $\alpha$. This implies that for all $x, y \in \mathbb{R}^N$,

$$\prod_{j=1}^N \frac{1}{2}(|x_j|^{2\alpha} + |y_j|^{2\alpha} - |x_j - y_j|^{2\alpha}) = c_\alpha \sum_{p \in \{0, 1\}^N} \int_{\mathbb{R}^N} \prod_{i=1}^N \frac{f_{p_i}(x_i\xi_i)f_{p_i}(y_i\xi_i)}{|\xi_i|^{2\alpha + 1}} d\xi,$$

(32)

where $f_0(x) = 1 - \cos x$ and $f_1(x) = \sin x$. Therefore, $v$ has the following representation:

$$v(x) \overset{d}{=} \sum_{p \in \{0, 1\}^N} \int_{\mathbb{R}^N} \prod_{i=1}^N \frac{f_{p_i}(x_i\xi_i)}{|\xi_i|^{\alpha + 1/2}} W_p(d\xi),$$

(33)

where $c = c_\alpha^{N/2}$ and $W_p(d\xi), p \in \{0, 1\}^N$, are i.i.d. $\mathbb{R}^d$-valued white noises on $\mathbb{R}^N$. In what follows, we assume that $v$ is defined by this representation, which allows us to introduce a set variable $A \in \mathcal{B}(\mathbb{R}_+^N)$ and define

$$v(A, x) = \sum_{p \in \{0, 1\}^N} \int_{|\xi|_\infty \in A} \prod_{i=1}^N \frac{f_{p_i}(x_i\xi_i)}{|\xi_i|^{\alpha + 1/2}} W_p(d\xi),$$

where $|\xi|_\infty = \max\{|\xi_j| : 1 \leq j \leq N\}$. For each $A$, $x \mapsto v(A, x)$ has a continuous version because of $\|v(A, x) - v(A, y)\|_{L^2} \leq \|v(x) - v(y)\|_{L^2}$ and (30), and we will use such a version.

Let $T$ be a compact interval in $(0, \infty)^N$. Fix $s \in T$. We are going to verify Assumption 1.2. By re-centering and re-scaling, we can consider the intervals $\prod_{j=1}^N [s_j, s_j + r]$ instead of $\prod_{j=1}^N [s_j - r, s_j + r]$. This will simplify notations.
Motivated by the representation (1) for the Brownian sheet, we define, for each \( j \), the Gaussian process \( \tilde{v}^j(A, x_j) := v(A, s_1, \ldots, s_{j-1}, x_j, s_{j+1}, \ldots, s_N) - v(A, s) \) by

\[
\tilde{v}^j(A, x_j) := c \sum_{p \in \{0, 1\}^N} \int_{|\xi| \in A} \frac{f_{p_j}(x_j \xi_j) - f_{p_j}(s_j \xi_j)}{|\xi_j|^{\alpha + 1/2}} \prod_{i \neq j} \frac{f_{p_i}(s_i \xi_i)}{|\xi_i|^{\alpha + 1/2}} W_p(d\xi).
\]

**Lemma 7.1.** Fix \( s \in T \). Take \( r_0 = 1 \), and let \( \tilde{v}(A, x) = \sum_{j=1}^N \tilde{v}^j(A, x_j) \). Then there exists a finite constant \( c \) depending on \( T \) but not on \( s \) such that the following statements hold.

(a) For each \( j \) and \( x_j \), \( A \rightarrow \tilde{v}^j(A, x_j) \) is an \( \mathbb{R}^d \)-valued independently scattered Gaussian measure.

The \( \sigma \)-algebra \( \sigma \{ \tilde{v}^1(A, \cdot), \ldots, \tilde{v}^N(A, \cdot) \} \) is independent of \( \sigma \{ \tilde{v}^1(B, \cdot), \ldots, \tilde{v}^N(B, \cdot) \} \) whenever \( A \) and \( B \) are fixed, disjoint sets.

(b) For all \( j \), for all \( x_j, y_j \in [s_j, s_j + r_0] \),

\[
\| \tilde{v}(\mathbb{R}^d, x_j) - \tilde{v}(\mathbb{R}^d, y_j) \|_{L^2} \leq c |x_j - y_j|^\alpha.
\]

(c) For all \( 0 < r \leq r_0 \), for all \( x, y \in \prod_{j=1}^N [s_j, s_j + r] \), for all \( 0 \leq a < b \leq \infty \),

\[
\| v(x) - v(y) - \tilde{v}([a, b), x) + \tilde{v}([a, b), y) \|_{L^2} \leq c \left( a^{1-\alpha} |x - y| + b^{-\alpha} + r^\alpha |x - y|^\alpha \right).
\]

Consequently, Assumption 1.2 is satisfied with \( \gamma_1 = 0 \), \( \gamma_2 = \alpha \) and \( a_0 = 0 \).

**Proof.** See Appendix A in the supplementary material [23].

It is known that when \( N > \alpha d \), the fractional Brownian sheet \( v \) has a jointly continuous local time \( L(z, J) \) on any closed interval \( J \) in \((0, \infty)^N\). See [43, 4].

**Theorem 7.2.** Let \( T \) be a compact interval in \((0, \infty)^N\).

(a) If \( N < \alpha d \), then there exist positive finite constants \( C_1 \) and \( C_2 \) such that

\[
P \left\{ \lambda_N(J) \leq \mathcal{H}_\phi(v(J)) \leq C_2 \lambda_N(J) \text{ for all } J \in \mathcal{J}(T) \right\} = 1,
\]

where \( \phi(r) = r^{N/\alpha} (\log \log(1/r))^N \).

(b) If \( N > \alpha d \), then there exists a positive finite constant \( C \) such that for any \( z \in \mathbb{R}^d \),

\[
P \left\{ C L(z, J) \leq \mathcal{H}_\varphi(v^{-1}(z) \cap J) < \infty \text{ for all } J \in \mathcal{J}(T) \right\} = 1,
\]

where \( \varphi(r) = r^{N-\alpha d} (\log \log(1/r))^{\alpha d} \). If \( N \leq \alpha d \), then \( v^{-1}(z) \cap T = \emptyset \) a.s.

**Proof.** By (30) and (31) above, \( v \) satisfies Assumption 1.1. By Lemma 7.1, it satisfies Assumption 1.2. Moreover, Lemma 5.2 of [10] shows that Assumption 1.3 holds. Therefore, the results follow from Theorems 1.4 and 1.5.
We remark that the exact Hausdorff measure function $\phi$ for the range of the Brownian sheet has been determined by Ehm [13]. For the real-valued Brownian sheet on $\mathbb{R}$, Zhou [44] proved that $CL(z, J) \leq \mathcal{H}_0(v^{-1}(z) \cap J) \leq C'L(z, J)$.

Theorem 7.3 below is a Chung-type law of the iterated logarithm for the fractional Brownian sheet, which is a direct consequence of Theorem 1.6. We point out that the exponent $-\alpha$ of the $\log \log$-factor is different compared to the fractional Brownian motion from $\mathbb{R}^N$ to $\mathbb{R}^d$, for which the exponent is $-\alpha/N$ (cf. [25]). Theorem 7.3 only holds for intervals away from the origin and the axes. In fact, Chung’s law at the origin takes a different form for the Brownian sheet [33] (see also [25]), and this is open for fractional Brownian sheets.

**Theorem 7.3.** Let $T$ be a compact interval in $(0, \infty)^N$. Then for any fixed $x_0 \in T$, there exists a constant $K$ depending on $x_0$ such that

$$\liminf_{r \to 0} \sup_{x: |x-x_0|_\infty \leq r} \frac{|v(x) - v(x_0)|}{r^\alpha (\log \log (1/r))^{-\alpha}} = K \quad a.s.$$ 

and $K_1 \leq K \leq K_2$ for some positive finite constants $K_1$ and $K_2$ that depend on $T$.

### 7.2. Systems of stochastic wave equations

Consider the following system of stochastic wave equations for $t \geq 0, x \in \mathbb{R}$:

$$\begin{cases}
\frac{\partial^2}{\partial t^2} U_j(t, x) = \frac{\partial^2}{\partial x^2} U_j(t, x) + \tilde{W}_j(t, x), & j = 1, \ldots, d, \\
U_j(0, x) = 0, & \frac{\partial}{\partial t} U_j(0, x) = 0.
\end{cases} \quad (35)$$

Here, $\tilde{W} = (\tilde{W}_1, \ldots, \tilde{W}_d)$ is an $\mathbb{R}^d$-valued Gaussian noise. We assume that $\tilde{W}_1, \ldots, \tilde{W}_d$ are i.i.d. and each $\tilde{W}_j$ is either

(i) white in time and colored in space with covariance:

$$E[\tilde{W}(t, x)\tilde{W}(s, y)] = \delta_0(t-s)|x-y|^{-\beta},$$

where $0 < \beta < 1$, or

(ii) a space-time white noise (set $\beta = 1$ in this case).

The noise $\tilde{W}_j$ is defined as a centered Gaussian process $\tilde{W}_j(\phi)$ indexed by compactly supported smooth functions $\phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R})$ such that for all $\phi_1, \phi_2 \in C_c^\infty(\mathbb{R}^d \times \mathbb{R})$,

$$E[\tilde{W}_j(\phi_1)\tilde{W}_j(\phi_2)] = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}} dy \int_{\mathbb{R}} dy' \phi_1(s, y)f(y, y')\phi_2(s, y'),$$

where $f(y, y') = |y-y'|^{-\beta}$ in case (i), and $f(y, y') = \delta_0(y-y')$ in case (ii). Following [9, 7], $\tilde{W}_j$ extends to a $\sigma$-finite $L^2$-valued measure $\tilde{W}_j(A)$, for bounded Borel sets $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$, such that

$$E[\tilde{W}_j(A)\tilde{W}_j(B)] = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}} dy \int_{\mathbb{R}} dy' \mathbf{1}_A(s, y)f(y, y')\mathbf{1}_B(s, y')$$

$$= \frac{C^{\beta}}{2\pi} \int_{\mathbb{R}_+} ds \int_{\mathbb{R}} d\xi \mathcal{F}\mathbf{1}_A(s, \cdot)(\xi)\mathcal{F}\mathbf{1}_B(s, \cdot)(\xi)|\xi|^\beta, \quad (36)$$
where $\mathcal{F}\phi(s, \cdot)$ is the Fourier transform of $y \mapsto \phi(s, y)$ defined by $\mathcal{F}\phi(s, \cdot)(\xi) = \int_{\mathbb{R}} e^{-i\eta \xi} \phi(s, y) dy$. The solution of (35) is given by

$$U(t, x) = \frac{1}{2} \int_0^t \int_{\mathbb{R}} 1_{\{\xi-y| \leq t-s\}} \bar{W}(ds, dy) = \frac{1}{2} \bar{W}(\Delta(t, x)), \quad (37)$$

where $\Delta(t, x) = \{(s, y) : 0 \leq s \leq t, |x-y| \leq t-s\}$, and $U = \{U(t, x) : t \geq 0, x \in \mathbb{R}\}$ is a Gaussian field with i.i.d. components $U_1, \ldots, U_d$. When $\bar{W}$ is a space-time white noise, it is known that

$$U(t, x) = \frac{1}{2} \bar{W} \left( \frac{t-x}{\sqrt{2}}, \frac{t+x}{\sqrt{2}} \right), \quad (38)$$

where $\bar{W}$ is the modified Brownian sheet defined by Walsh [37, Theorem 3.1].

By Proposition 4.1 of [12], for any compact interval $T \subset (0, \infty) \times \mathbb{R}$, there exists a positive finite constant $c_0$ such that for all $(t, x), (s, y) \in T$,

$$c_0^{-1}(|t-s| + |x-y|)^{2-\beta} \leq \mathbb{E}(|U(t, x) - U(s, y)|^2) \leq c_0 (|t-s| + |x-y|)^{2-\beta}. \quad (39)$$

It follows that $U$ has a version that is locally Hölder continuous in $(t, x)$ of any exponent $< (2-\beta)/2$. By Proposition 2.1 of [24], $U$ satisfies sectorial LND: there exist constants $c_1 > 0$ and $\delta_0 > 0$ such that for all $n \geq 1$, for all $(t, x), (t^1, x^1), \ldots, (t^n, x^n) \in T$ with $|t-t^1| + |x-x^1| \leq \delta_0$ for all $i$,

$$\text{Var}(U_1(t, x)|U_1(t^1, x^1), \ldots, U_1(t^n, x^n)) \geq c_1 \left( \min_{1 \leq i \leq n} |(t+x) - (t^i+x^i)|^{2-\beta} + \min_{1 \leq i \leq n} |(t-x) - (t^i-x^i)|^{2-\beta} \right). \quad (40)$$

By Proposition 9.2 of [11], the solution $U$ of (35) has the same law as the Gaussian random field $V = \{V(t, x) : t \geq 0, x \in \mathbb{R}\}$ defined by

$$V(t, x) = C_0 \text{Re} \int_{\mathbb{R}} \int_{\mathbb{R}} F(t, x, \tau, \xi)|\xi|^{(\beta-1)/2} W(d\tau, d\xi), \quad (41)$$

where $C_0$ is a constant,

$$F(t, x, \tau, \xi) = e^{-i\tau \xi} \left( \frac{e^{-i\tau \xi} - e^{i|\xi|}}{\tau + |\xi|} - \frac{e^{-i\tau \xi} - e^{-i|\xi|}}{\tau - |\xi|} \right)$$

and $W$ is a $\mathbb{C}^d$-valued space-time white noise, that is, $\text{Re} W$ and $\text{Im} W$ are independent $\mathbb{R}^d$-valued space-time white noises with i.i.d. components. Here, $\text{Re}$ and $\text{Im}$ stand for real part and imaginary part respectively. We refer to (41) as the harmonizable representation for the solution $U(t, x)$ of the stochastic wave equation.

In view of (38) and (40), it is natural to change coordinates by a rotation of $45^\circ$:

$$(\eta, \theta) = \left( \frac{t-x}{\sqrt{2}}, \frac{t+x}{\sqrt{2}} \right), \quad \text{or} \quad (t, x) = \left( \frac{\eta + \theta}{\sqrt{2}}, \frac{-\eta + \theta}{\sqrt{2}} \right).$$

With the $(\eta, \theta)$ coordinate system, we write $f(\eta, \theta, \tau, \xi) = F\left( \frac{\eta + \theta}{\sqrt{2}}, \frac{-\eta + \theta}{\sqrt{2}}, \tau, \xi \right)$ and

$$v(\eta, \theta) = C_0 \text{Re} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\eta, \theta, \tau, \xi)|\xi|^{(\beta-1)/2} W(d\tau, d\xi). \quad (42)$$
It follows that, as processes on \{ (\eta, \theta) : \eta + \theta \geq 0 \},
\[ v(\eta, \theta) \overset{d}{=} U \left( \frac{\eta + \theta}{\sqrt{2}}, \frac{-\eta + \theta}{\sqrt{2}} \right). \] (43)

For any \( A \in \mathcal{B}(\mathbb{R}_+) \), define
\[ v(A, \eta, \theta) = C_0 \Re \int_{|\tau| \vee |\xi| \in A} f(\eta, \theta, \tau, \xi) |\xi|^{(\beta - 1)/2} W(d\tau, d\xi), \]
where \( |\tau| \vee |\xi| = \max\{ |\tau|, |\xi| \} \). For each \( A \), by \( \| v(A, \eta, \theta) - v(A, \eta', \theta') \|_{L^2} \leq \| v(\eta, \theta) - v(\eta', \theta') \|_{L^2} \)
and (39), we can choose a version of \( v(A, \cdot) \) such that \( (\eta, \theta) \mapsto v(A, \eta, \theta) \) is continuous.

Let \( T \) be a compact rectangle in \{ (\eta, \theta) : \eta + \theta > 0 \}. Suppose \((\eta_0, \theta_0) \in T\) is fixed. Then for any \( \eta \geq \eta_0 \) and \( \theta \geq \theta_0 \), define
\[
\begin{align*}
\tilde{v}^1(A, \eta) &= v(A, \eta, \theta_0) - v(A, \eta_0, \theta_0), \\
\tilde{v}^2(A, \theta) &= v(A, \eta_0, \theta) - v(A, \eta_0, \theta_0).
\end{align*}
\]
Set \( \tilde{v}(A, \eta, \theta) = \tilde{v}^1(A, \eta) + \tilde{v}^2(A, \theta) \).

**Lemma 7.4.** Let \( T \) be a compact rectangle in \{ (\eta, \theta) : \eta + \theta > 0 \} and let \((\eta_0, \theta_0) \in T\) be fixed. Take \( r_0 = 1 \) and \( \alpha = (2 - \beta)/2 \). Then, for some constants \( c \) and \( a_0 > 1 \), the following statements hold.

(a) For each \( \eta \in [\eta_0, \eta_0 + r_0] \) and \( \theta \in [\theta_0, \theta_0 + r_0] \), \( A \mapsto \tilde{v}^1(A, \eta) \) and \( A \mapsto \tilde{v}^2(A, \theta) \) are independently scattered Gaussian measures. Also, the \( \sigma \)-algebra \( \sigma(\{ \tilde{v}^1(A, \cdot), \tilde{v}^2(A, \cdot) \}) \) is independent of \( \sigma(\{ \tilde{v}^1(B, \cdot), \tilde{v}^2(B, \cdot) \}) \) whenever \( A \) and \( B \) are fixed, disjoint sets.

(b) For all \( \eta_1, \eta_2 \in [\eta_0, \eta_0 + r_0] \),
\[ \| \tilde{v}^1(\mathbb{R}_+, \eta_1) - \tilde{v}^1(\mathbb{R}_+, \eta_2) \|_{L^2} \leq c |\eta_1 - \eta_2|^\alpha. \]
For all \( \theta_1, \theta_2 \in [\theta_0, \theta_0 + r_0] \),
\[ \| \tilde{v}^2(\mathbb{R}_+, \theta_1) - \tilde{v}^2(\mathbb{R}_+, \theta_2) \|_{L^2} \leq c |\theta_1 - \theta_2|^\alpha. \]

(c) Take \( \gamma_1 = 0 \) if \( \beta = 1 \); take \( \gamma_1 \) such that \( \frac{1 - \beta}{2} < \gamma_1 < \frac{1}{2} \) if \( 0 < \beta < 1 \). Take \( \gamma_2 = 1/2 \). Then for all \( \eta \in [\eta_0, \eta_0 + r] \times [\theta_0, \theta_0 + r] \) and \( a_0 \leq a < b \leq \infty \),
\[
\begin{align*}
&\| v(\eta_1, \theta_1) - v(\eta_2, \theta_2) - \tilde{v}([a, b), \eta_1, \theta_1) + \tilde{v}([a, b), \eta_2, \theta_2) \|_{L^2} \\
&\phantom{=} \leq c \left[ a^{1 - \alpha} (|\eta_1 - \eta_2| + |\theta_1 - \theta_2|) + r \gamma_1 b^{\gamma_1 - \alpha} + r^{1/2} (|\eta_1 - \eta_2| + |\theta_1 - \theta_2|)^\alpha \right].
\end{align*}
\]

**Proof.** See Appendix B in the supplementary material [23].

In [11], it is shown that the stochastic wave equation in spatial dimension \( k \geq 1 \) with \( \beta \geq 1 \) satisfies Assumption 1.3 with \( \delta = 2 - \beta \). We verify this assumption for \( k = 1 \) and \( 0 < \beta \leq 1 \).

**Lemma 7.5.** Let \( T \) be a compact rectangle in \( D := \{ (\eta, \theta) : \eta + \theta > 0 \} \). Let \( 0 < \varepsilon_0 \leq 1 \) be such that the closure of \( T^{(\varepsilon_0)} \) is contained in \( D \). Take \( c = 1 \) and \( \delta = 1 \).

(a) There exists a positive constant \( c_3 \) such that \( \| v(\eta, \theta) \|_{L^2} \geq c_3 \) for all \( (\eta, \theta) \in T^{(\varepsilon_0)} \).
(b) Let \( I \subset T \) be a compact rectangle and \( 0 < \rho \leq \varepsilon_0 \). For \( (\eta_0, \theta_0) \in I \), take \( (\eta'_0, \theta'_0) = (\eta_0, \theta_0) \).

Then there is a constant \( c_4 \) such that for all \( (\eta_1, \theta_1), (\eta_2, \theta_2) \in I^4(\rho) \) with \( |\eta_1 - \eta| + |\theta_1 - \theta| \leq 2\rho \)
and \( |\eta_2 - \eta| + |\theta_2 - \theta| \leq 2\rho \), for all \( i = 1, \ldots, d \),
\[
|\mathbb{E}[v_i(\eta_1, \theta_1) - v_i(\eta_2, \theta_2)]v_i(\eta'_0, \theta'_0)]| \leq c_4 (|\eta_1 - \eta_2| + |\theta_1 - \theta_2|).
\]

**Proof.** See Appendix C in the supplementary material [23].

By (39) and (40), \( v(\eta, \theta) \) satisfies Assumption 1.1. It follows from the results of [42, 39] that if \( d < 4/(2 - \beta) \), then \( v(\eta, \theta) \) [and hence the solution \( U(t, x) \) of (35)] has a jointly continuous local time
\( L(z, J) \) on any compact rectangle \( J \) away from \( t = 0 \).

We obtain the following result for the range and level sets of \( v(\eta, \theta) \) in joint variables \( (\eta, \theta) \).

**Theorem 7.6.** Suppose that \( T \) is a compact rectangle in \( \{ (\eta, \theta) \in \mathbb{R}^2 : \eta + \theta > 0 \} \). Let \( \phi(r) = r^{4/(2 - \beta)}(\log \log(1/r))^{2} \) and \( \varphi(r) = r^{2 - d(2 - \beta)/2} (\log \log(1/r))^{d(2 - \beta)/2} \).

(a) If \( d > 4/(2 - \beta) \), then there exist positive finite constants \( C_1 \) and \( C_2 \) such that
\[
\mathbb{P}\left\{ C_1 \lambda(J) \leq \mathcal{H}_\phi(v(J)) \leq C_2 \lambda(J) \text{ for all } J \in \mathcal{F}(T) \right\} = 1.
\]

In particular, \( \dim_H v(T) = 4/(2 - \beta) \) a.s.

(b) If \( d < 4/(2 - \beta) \), then there exists a positive finite constant \( C \) such that for any \( z \in \mathbb{R}^d \),
\[
\mathbb{P}\left\{ \mathcal{C}L(z, J) \leq \mathcal{H}_\varphi(v^{-1}(z) \cap J) < \infty \text{ for all } J \in \mathcal{F}(T) \right\} = 1.
\]

In particular, \( \dim_H [v^{-1}(z) \cap T] = 2 - d(2 - \beta)/2 \) a.s. on the event \( \{L(z, T) > 0\} \).

If \( d \geq 4/(2 - \beta) \), then \( v^{-1}(z) \cap T = \emptyset \) a.s.

**Proof.** Note that it suffices to prove the theorem for all sufficiently small rectangles in \( T \) because we can cover \( T \) with finitely many small rectangles. Then, without loss of generality, we can assume that the sides of \( T \) have length \( \leq \delta_0/\sqrt{2} \), so that \( v \) satisfies (40) on \( T \). Together with (39), we see that \( v \) satisfies Assumption 1.1. Moreover, Lemmas 7.4 and 7.5 above show that \( v \) satisfies Assumptions 1.2 and 1.3 respectively. Hence, the results follow from Theorems 1.4 and 1.5.

Finally, by Theorem 1.6, we get a Chung-type LIL for the solution of (35).

**Theorem 7.7.** Let \( T \) be a compact rectangle in \((0, \infty) \times \mathbb{R}\). Then for any fixed \((t_0, x_0) \in T\), there exists a constant \( K \) depending on \((t_0, x_0)\) such that
\[
\liminf_{r \to 0} \sup_{(t, x) \in [t - t_0, x - x_0] \subset T} \frac{|U(t, x) - U(t_0, x_0)|}{r^{(2 - \beta)/2} (\log \log(1/r))^{-(2 - \beta)/2}} = K \quad \text{a.s.}
\]

and \( K_1 \leq K \leq K_2 \), where \( K_1 \) and \( K_2 \) are positive finite constants that depend on \( T \).

**Acknowledgements**

The author wishes to thank Thomas Mountford and Yimin Xiao for stimulating discussions and helpful comments that have led to improvements in the paper.
Supplementary Material

Supplement to “The Hausdorff measure of the range and level sets of Gaussian random fields with sectorial local nondeterminism”
The supplement contains the proofs of Lemmas 7.1, 7.4 and 7.5.

References


[23] C.Y. Lee, Supplement to “The Hausdorff measure of the range and level sets of Gaussian random fields with sectorial local nondeterminism”.


