Tree Builder Random Walk: recurrence, transience and ballisticity

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The Tree Builder Random Walk is a special random walk that evolves on trees whose size increases with time, randomly and depending upon the walker. After every \( s \) steps of the walker, a random number of vertices are added to the tree and attached to the current position of the walker. These processes share similarities with other important classes of markovian and non-markovian random walks presenting a large variety of behaviors according to parameters specifications. We show that for a large and most significant class of tree builder random walks, the process is either null recurrent or transient. If \( s \) is odd, the walker is ballistic, thus transient. If \( s \) is even, the walker’s behavior can be explained from local properties of the growing tree and it can be either null recurrent or it gets trapped on some limited part of the growing tree.

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1. Introduction

In this paper we consider a special random walk, which we call Tree Builder Random Walk (TBRW). It evolves on trees whose size increases randomly with time. Specifically, given \( s \in \mathbb{N} \) (a parameter of the model), after every \( s \) transitions of the walker a random number of vertices are added to the tree and attached to the current position of the random walk. As will become clear from our results, the TBRW has an intrinsic mathematical interest connected to other important classes of markovian and non-markovian random walks such as Random Walks in Random Environment [23, 24], Reinforced Random Walks [5, 9, 19, 20] and Excited Random Walks [17]. It is also important to mention that the study of random graphs [14, 16], and random graphs dynamics [8] has been a very active field of research motivated by the increasing applicability of network models to...
represent real life phenomena. Many interesting questions are related to the evolution of random walks on these growing/random networks [1, 6, 18, 21]. Despite the similarities with these models, the TBRW possesses the distinctive feature of having the evolution of the graph dependent upon the walker’s position, see [11] and references therein.

In order to present adequately our results and draw connections to previous works let us first introduce formally the model.

1.1. The model

Let $T$ be a tree and denote by $V(T)$ and $E(T)$ its vertex and edge sets, respectively. Let $\Omega$ be the collection of pairs $(T, x)$, where $T$ is a tree and $x \in V(T)$ is one of its vertices. Now fix a locally finite tree $T_0$, a positive integer $s$ and a sequence of non-negative integer random variables $\xi = \{\xi_n\}_{n \in \mathbb{N}}$. The TBRW is a stochastic processes $\{(T_n, X_n)\}_{n \geq 0}$ on $\Omega$ ($T_n$ denotes the tree at time $n$ and $X_n$ one of its vertices) defined inductively on almost every realization of $\xi$ according to the update rules below.

1. Obtain a locally finite tree $T_{n+1}$ from $T_n$ as follows:
   - if $n = 0 \text{ mod } s$, add $\xi_n$ new leaves to $X_n$,
   - if $n \neq 0 \text{ mod } s$, $T_{n+1} = T_n$.
2. Choose uniformly one edge in $\{\{X_n, y\} : \{X_n, y\} \in E(T_{n+1})\}$, i.e., an edge incident to $X_n$ in $T_{n+1}$, and set $X_{n+1}$ as the chosen neighbor of $X_n$.

We stress out the subscript $n+1$ of $T$ in (2); it means that we may add a new neighbor at (1) and choose it at (2). If $\xi$ is a sequence of independent random variables, the TBRW process is a Markov chain.

Note that $s$ and the sequence of random variables $\xi$ are parameters of the model. The first one allows the tree to grow only at times multiple of $s$, whereas the second controls the growth of the tree; for this reason we call the sequence $\xi$ environment process. We denote by $P_{T_0, x_0, s, \xi} (\cdot)$ the law of $\{(T_n, X_n)\}_{n \in \mathbb{N}}$ when $(T_0, X_0) = (T_0, x_0)$, and by $E_{T_0, x_0, s, \xi} (\cdot)$ the corresponding expectation.

For the sake of simplicity we will consider some nomenclature that will be useful. Given a realization of $T_n$, consider $x \in V(T_n)$, if $x$ has a single neighbor in $T_n$, we say that $x$ is a leaf of $T_n$ or a leaf of $z \in V(T_n)$ if $z$ is the single neighbor of $x$ in $T_n$. We also remark that we can allow $T_0$ to be a single vertex with a self-loop. This will be explained in the following sections.

The TBRW model generalizes a couple of models which have been recently studied; the NRRW (No Restart Random Walk) [11] and the BGRW (Bernoulli Growth Random Walk) [12]. In particular, TBRW reduces to NRRW assuming $\xi_n \equiv 1 \ \forall n$, whereas it reduces to BGRW assuming $s = 1$ and $\xi_n \sim \text{Ber}(p) \ \forall n$.

1.2. Environment conditions

The main goal of this paper is to provide conditions on the environment process $\xi = \{\xi_j\}_{j \in \mathbb{N}}$ under which we observe recurrence or transience of $\{X_n\}_{n \in \mathbb{N}}$. Since the process
ξ controls how the walk modifies its environment, we refer to any distributional condition on ξ as environment condition. In this section we list all conditions that will be used throughout the paper together with some brief discussion and examples. We reserve the letters \( P \) and \( E \) for the marginal distribution of \( ξ \) and the corresponding expectation.

Two basic hypothesis on the variables \( ξ_n, n \geq 1 \), are that they are independent or even i.i.d. In the first case we say that \( ξ \) is an independent environment and in the second that it is an i.i.d. environment. For the other ones we reserve special notation. The first condition, denoted by (UE) is the following one

\[
\inf_{n \in \mathbb{N}} P(\xi_n \geq 1) = κ > 0 .
\]

Tracing a parallel with the classical theory of random walk on random environment, the above condition is similar in spirit with the uniformly elliptic condition also denoted by (UE). In our case, whenever the walk can add a new leaf to its environment, it has bounded away from zero probability of adding at least one leaf. This fact will be crucial to prove ballisticity when \( s \) is odd, since we can use this property to “force routes of escape” as explained in Section 4.

The next condition imposes restrictions on the moments of the environment process. Given \( r > 0 \), we say that \( ξ \) satisfies condition (M)\( r \) if

\[
\sup_{n \in \mathbb{N}} E(\xi_n^r) \leq M < \infty .
\]

The moment conditions (M)\( r \) are required to control the growth of the graph, for instance to avoid the creation of traps for the random walk (see, Theorem 3.4).

The next two conditions are related to the asymptotic behavior of the environment process. For \( n \geq 1 \) define \( S_n := \sum_{j=1}^{n} \xi_j \). We say \( ξ \) satisfies assumption (S) if there exists a positive constant \( c > 0 \) and a function \( g : \mathbb{N} \setminus \{0\} \to \mathbb{R}^+ \) of non-summable inverse \( (\sum_{n=1}^{\infty} \frac{1}{g(n)} = \infty) \) such that

\[
P \left( \limsup_{n \to \infty} \frac{S_n}{g(n)} \leq c \right) = 1 .
\]

In essence condition (S) assures that the number of added leaves by time \( n \) is bounded from above by a function \( g(n) \), with \( g(n)/n \to 0 \) (as \( n \) grows), and thus the degree of a vertex cannot grow too fast.

We say the environment \( ξ \) satisfies condition (I) if there exist a constant \( c > 0 \) and a positive function \( f : \mathbb{N} \setminus \{0\} \to \mathbb{R}^+ \) of summable inverse \( (\sum_{n=1}^{\infty} \frac{1}{f(n)} < \infty) \), such that

\[
P \left( \liminf_{n \to \infty} \frac{S_n}{f(n)} \geq c \right) = 1 .
\]

Condition (I) requires that the number of added leaves by time \( n \) is bounded from below by a function \( f(n) \), with \( n/f(n) \to 0 \) (as \( n \) grows), and thus the degree of a vertex may grow very fast.
We end this section with some important examples of environment for which one may verify some of the above conditions. The first one is the particular case where the environment $\xi$ is i.i.d. with finite mean. It satisfies condition (UE) and (M). Moreover, the Strong Law of Large Numbers assures that condition (S) also holds. Even more generally, if the environment is an ergodic process, then (S) follows from the Ergodic Theorem.

On the other hand, for an independent environment such that $\xi_j$, $j \geq 1$, have heavy tails, then (I) holds. For instance, consider $\xi_j$ as independent random variables having power-law distributions such that $P(\xi_j \geq x) \geq \delta x^{-\alpha}$, $\forall j \geq 1$, for some $\delta > 0$ and $\alpha \in (0,1)$. For $\beta \in (\alpha,1)$ and all $i$ large enough we get

$$P\left(S_i \leq i^{\frac{1}{\beta}}\right) \leq P\left(\max\{\xi_1, \ldots, \xi_i\} \leq i^{\frac{1}{\beta}}\right) = \prod_{j=1}^i P(\xi_j \leq i^{\frac{1}{\beta}}) \leq \left(1 - \frac{\delta}{i^{\frac{1}{\beta}}}\right)^i \leq e^{-\delta i^{\gamma}},$$

where $\gamma = 1 - \frac{\alpha}{\beta}$. Since $\sum_{i \geq 1} e^{-\delta i^{\gamma}} < \infty$, by Borel-Cantelli lemma we have that $P\left(\lim \inf_{i \to \infty} S_i/i^{\frac{1}{\beta}} \geq 1\right) = 1$, i.e., condition (I) holds for $f(i) = i^{1/\beta}$ and $c = 1$.

### 1.3. Main results

In this section we present our results and trace a parallel with their possible counterparts in the more classical theory of random walk on random environments. Before we can state properly our main results we need to introduce some definitions. As said before we want to study recurrence/transience and related properties for a $(\xi, s)$-TBRW $\{(T_n, X_n)\}_{n \geq 0}$. Note that for independent environments $\xi$ the process $\{(T_n, X_n)\}_{n \geq 0}$ is markovian and under (UE) it is always transient in the usual sense, since $T_n$ increases. However, the process $\{X_n\}_{n \geq 0}$ is non-markovian and since $T_n$ increases, we need adequate definitions of recurrence and transience. With a slight abuse of terminology, when we say that the TBRW process is either recurrent or transient, we will always be referring ourselves to the corresponding process $\{X_n\}_{n \geq 0}$ and not to the pair $\{(T_n, X_n)\}_{n \geq 0}$.

Let $\{(T_n, X_n)\}_{n \geq 0}$ be a TBRW and $(T_0, x_0)$ its initial state. We say that $z \in T_0$ is recurrent if

$$P_{T_0, x_0, \xi}(X_n = z \text{ infinitely often}) = 1.$$ 

Since the graph structure is increasing, we also need to define recurrence for vertices that are added to the graph at some time. So given a fixed realization $(\bar{T}, x)$ of $(T_n, X_n)$ (i.e. $(\bar{T}, x)$, with $x \in V(\bar{T})$ is any possible pair attainable from $(T_0, x_0)$ at time $n$) and $z \in V(\bar{T})$, we say that $z$ is recurrent if

$$P_{\bar{T}, x, s, \theta_k(\xi)}(X_n = z \text{ infinitely often}) = 1,$$

where and $\theta_k$ as the forward time shift such that $\theta_k(\xi) = \{\xi_{j+k}\}_{j \in \mathbb{N}}$. In the above display there is some abuse of notation if the environment is not independent, indeed $P_{\bar{T}, x, s, \theta_k(\xi)}$
depends on the whole history of the process until time \( n \) and the display should be understood as the conditional probability

\[
P_{T_0,x_0,s,\xi}(X_n = z \text{ infinitely often } \mid T_n = \tilde{T}, X_n = x \in V(\tilde{T}), z \in V(\tilde{T})) = 1.
\]

The random walk in \((\xi,s)\)-TBRW is \textit{recurrent} if, for every \( n \) and every possible realization of \( T_n \), all \( z \in \cup_n V(T_n) \) are recurrent. If the TBRW is not recurrent, we say that it is \textit{transient}.

**Remark 1.1.** Recurrence or transience for the \((\xi,s)\)-TBRW may depend on the choice of \( T_0 \), (even if \( T_0 \) is finite). Also, since the trees are connected, the TBRW is irreducible in the usual sense that every vertex is reachable from any given configuration with positive probability. So irreducibility has no role in the results.

Let \( \eta_z \) denote the first time the random walk visits vertex \( z \), i.e.,

\[
\eta_z := \inf \{ n \geq 1 \mid X_n = z \}.
\]  

(1)

Given \((T_0,x_0)\) and assuming the random walk in TBRW is recurrent, we say that this random walk is \textit{positive recurrent} if for any given realization of \((T_n,x_n)\) and \( z \in V(T_n) \)

\[
\mathbb{E}_{T_n,x_n,s,\theta,s,\xi}(\eta_z) < \infty.
\]

If the TBRW is recurrent but not positive recurrent, then we say that it is \textit{null recurrent}. This definition of positive recurrence is not directly comparable to the definition of positive recurrence for Markov chains. Since the tree is growing, there is no equilibrium and the mean return time to a vertex depends on the number of previous visits to that vertex. It might increase on each visit to that vertex and even diverge to infinity as the vertex keeps being visited.

As usual for trees, we will sometimes designate a particular vertex as the root of the tree, either because we simply want to fix a single vertex or because this vertex is special in some sense. We refer to this vertex simply as root.

We say that the TBRW is \textit{ballistic} if there exists a positive constant \( c \), such that

\[
\liminf_{n \to \infty} \frac{\text{dist}_{T_n}(X_n, \text{root})}{n} \geq c, \quad \mathbb{P}_{T_0,x_0,s,\xi} \text{ almost surely}.
\]

It is clear that every ballistic TBRW is transient.

We can now state the main results of this paper.

**Theorem 1.1** (Recurrence/Traps for \( s \) even). Consider a \((s,\xi)\)-TBRW process with \( s \) even. For every initial state \((T_0,x_0)\) with \( T_0 \) finite, there exist two regimes:

\( (i) \) (Recurrence is inherited) if \( \xi \) satisfies condition \((S)\), then the TBRW is recurrent.
(ii) (The dangerous environment) if $\xi$ is an independent environment satisfying condition (I), then there exists $n$ such that the walker gets trapped at time $n$, $\mathbb{P}_{T_0, x_0, s, \xi}$-almost surely, i.e.

$$\mathbb{P}_{T_0, x_0, s, \xi}(\text{there exists } x \in \bigcup_n V(T_n) \text{ and } k \text{ such that } X_{sn+k} = x \forall n) = 1.$$ 

For the specific case $s$ even and $\xi_j \equiv 1$, the recurrence of TBRW was proved in [11]; Theorem 1.1 generalizes the result to much more general environments $\xi$ and brings to light the possibility for the walker to get trapped in some environments, i.e., localization phenomenon occurs, with the walker being locked in the neighborhood of a (random) vertex. We point out that Reinforced random walks also presents the possibility to be either recurrent or to get trapped depending on the parameters of the model, see [5].

The name for regime (i) comes from the fact that under (S) and $s$ even the TBRW process propagates recurrence, i.e., it is enough to have a single recurrent vertex and all vertices which are eventually added to the tree inherit recurrence from their parents.

Although in Theorem 1.1 we do not need any further assumption on $T_0$ other than finiteness, we will always assume that $T_0$ has a self-loop at the root when $s$ is even. This indeed makes things more interesting, since it is the case where the height of the tree may increase. This will be carefully discussed in Sections 3 and 5 (see, Proposition 5.1).

Theorem 1.1 tells us that condition (S) guarantees the recurrence of every vertex. The next natural question regards the nature (null vs. positive) of this recurrence. As it turns out (S) alone does not guarantees neither null nor positive recurrence. Take, for instance, a sequence $\xi_n \sim \text{Ber}(n^{-2})$, then the Borel-Cantelli lemma assures that condition (S) is satisfied and the tree will almost surely be finite, which implies that all its vertices will be positive recurrent. The next proposition provides sufficient condition for null recurrence.

**Proposition 1.2** (Null recurrence for $s$ even). Consider a $(s, \xi)$ – TBRW process with $s$ even and independent environment $\xi$ satisfying conditions (S) and (UE). Then the TBRW is null recurrent.

We have another important result which is Theorem 3.4. We do not state it here to avoid excessive notation in this introduction. The theorem provides conditions on the parameters for i.i.d. environments that imply distinct local behaviors for the process. Specifically, either the exit time from a vertex through a non-leaf neighbor has infinite mean, which immediately implies null recurrence, or it has finite mean and the TBRW makes transitions between non leaves neighbors in finite mean times almost surely.

When $s$ is odd the TBRW process has a thoroughly different behavior.

**Theorem 1.3** (Ballisticity for $s$ odd). If $s$ is odd and $\xi$ is an independent environment satisfying (UE), then the TBRW is ballistic.

In [11], it is proved that the NRWW (TBRW with $\xi_j \equiv 1$, for every $j$) is transient for $s = 1$. There, it is also conjectured that the NRRW is transient for every $s$ odd. A first proof of ballisticity was given in [12] for the particular case with $s = 1$ and the
ξ_i.i.d. Bernoulli random variables. Our Theorem 1.3 proves the conjecture of [11] and generalizes the result in [12].

Let us draw a parallel with the classical theory of random walks on random environments. One of the main open problems in the theory of RWRE in $\mathbb{Z}^d$ is whether directional transience is equivalent to directional ballisticity for uniformly elliptic i.i.d. environments. The conjecture involves two conditions each of them playing specific roles: the uniform ellipticity, which is a local condition, and the directional transience, which is a global one. The former prevents the walk from getting trapped in substructures of the space, whereas the latter tells us how the walk explores the environment in the long run (the interested reader may read more about ellipticity and ballisticity in [22]). Interestingly, Theorem 1.3 reveals that for TBRW process with $s$ odd, uniform ellipticity is enough to obtain ballisticity. In other words, in this settings, a local condition is enough to drive the walk away from its initial position at linear speed.

Remark 1.2. We can define continuous versions of the TBRW by considering that the time between the $(n-1)$-th and $n$-th transitions are i.i.d. exponential random variables with parameter $\lambda_n > 0$, $n \geq 1$. It is standard to show that recurrence/transience of the continuous time version follows from the same property for the discrete time version. If $(\lambda_n)_{n \geq 1}$ is bounded away from $0$ and $\infty$, then the same holds for properties like ballisticity and null recurrence. It is also possible to have a null recurrent TBRW that generates a positive recurrent continuous time random walk which happens if $(\lambda_n)_{n \geq 1}$ diverges to infinity sufficiently fast. These derivative results do not bring novelty when compared to similar conclusions obtained for standard continuous time random walks.

The structure of the paper is the following: In Section 2 we prove that the time for the random walk to go from one vertex to other one sufficiently far apart has infinite mean (regardless of the parity of $s$). There we introduce an auxiliary process (called generalized loop-process) which will also play an important role in the proof of ballisticity. In Section 3 we prove Theorem 1.1 and other results for $s$ even, such as Theorem 3.4. Section 4 is devoted to the proof of ballisticity of TBRW for $s$ odd. In Section 5 we show that the height of the tree diverges to infinity as time goes to infinity (regardless of the parity of $s$) and we prove Proposition 1.2. Lastly, Section 6 finishes the paper with a brief discussion on possible extensions of our results.

2. Long hitting times

In this section we focus on understanding the hitting times of single vertices in the TBRW. More specifically, let us assume that the TBRW process starts on $(T_0, x_0)$ and that $z$ denotes a vertex at distance $\ell$ of $x_0$ in $T_0$. Recall the definition of the first hitting time $\eta_z$ from (1). In this section we study the distribution of $\eta_z$ and how it depends on $\ell$ in independent environments; the main results are Corollary 2.3 and Lemma 2.4.

In order to study $\eta_z$, we shall introduce a simpler process, called Generalized Loop Process. This simpler process is a generalization of the Loop process which was introduced...
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in [12] in the context of the BGRW, which is the TBRW for \( s = 1 \) and \( \xi \) i.i.d. sequence of Bernoulli’s random variables. Herein, building on the same ideas, we generalized such a process. Some of the results concerning the Generalized loop process are minor variations of the one presented in [12].

2.1. Generalized Loop Process

Roughly speaking, a generalized loop process on an initial graph \( G \) is a random walk such that at each time \( t = ms \) adds \( \xi_t \) loops to its position and then chooses uniformly one edge of its current position to walk on. Specifically, the number of vertices in the graph stays constant during the evolution of the process and it is equal to the number of vertices in the initial graph.

Although we can define the loop process over any graph, we will treat only the case in which it is defined over a specific graph called backbone. A finite graph \( B \) is a backbone of length \( \ell \) if \( B \) is a path of length \( \ell \) with a loop attached to its \((\ell + 1)\)-th vertex and possibly to remaining vertices, see Figure 1 below.

![Figure 1. A backbone of length \( \ell \)](image)

In this section, we denote by \( \deg_t(i) \) the number of edges attached to vertex \( i \) at time \( t \) counting loops only once. We refer to this quantity as degree of a vertex even though we do not count loops twice.

The process depends on an integer greater zero \( s \) and an independent environment process \( \xi = \{ \xi_n \}_{n \in \mathbb{N}} \), which we assume satisfies condition (UE). We denote the Generalized loop process by \( \{(B_t, X_{\text{loop}}^t)\}_{t \in \mathbb{N}} \) where \( B_t \) denotes the backbone at time \( t \) and \( X_{\text{loop}}^t \) is one of its \((\ell + 1)\) vertices. Similarly to the TBRW, the loop process is defined inductively according to the update rules presented below

1. Generate \( B_{t+1} \) from \( B_t \) as follows:
   - if \( t = 0 \mod s \) add \( \xi_t \) new loops to \( X_{\text{loop}}^t \),
   - if \( t \neq 0 \mod s \), \( B_{t+1} = B_t \).
2. Choose uniformly one edge attached to \( X_{\text{loop}}^t \) in \( B_{t+1} \). If the chosen edge is a loop, set \( X_{\text{loop}}^{t+1} = X_{\text{loop}}^t \). Otherwise, \( X_{\text{loop}}^{t+1} \) becomes the chosen \( X_{\text{loop}}^t \) neighbor.

Note the index \( t + 1 \) of \( B \) on the rule (2). This means we may add a loop at rule (1) and then select it at (2).
Given a backbone $B$ of length $\ell$ and $i \in \{0, 1, \cdots, \ell\}$ we denote by $\mathbb{P}_{B, i}$ the law of the loop process when $(B_0, X_0^{\text{loop}}) \equiv (B, i)$, and by $\mathbb{E}_{B, i}$ the corresponding expectation. We will also let $\eta_0^{\text{loop}}$ be the first time $X^{\text{loop}}$ visits 0, i.e.,

$$\eta_0^{\text{loop}} := \inf \left\{ t \geq 0 \mid X_t^{\text{loop}} = 0 \right\}.$$ 

The next result provides upper bounds for the cumulative distribution of $\eta_0^{\text{loop}}$.

**Lemma 2.1.** Let $\xi$ be an independent environment satisfying condition (UE). Then there exist positive constants $c_1, c_2$ depending on $s$ and $\xi$ such that, for all integer $K \geq 1$ and for all $\beta \in (0, 1)$

$$\mathbb{P}_{B, \ell} \left( \eta_0^{\text{loop}} \leq e^{\beta K} \right) \leq \exp\{ -c_1 K \} + 1 - \left( \frac{1}{s} \right)^{c_2 K} + \exp\{ -(1 - \beta)K \}.$$

**Proof.** It will be useful to look to loop process only when it actually moves from its position. Thus, define inductively the following

$$\tau_0 \equiv 0, \quad \tau_k := \inf \left\{ t > \tau_{k-1} \mid X_t^{\text{loop}} \neq X_{\tau_{k-1}}^{\text{loop}} \right\}.$$ 

We point out that the stopping times $\tau_k$ are not necessarily finite almost surely. If $\tau_k = \infty$ it means that the process gets trapped. As it will be shown in Theorem 1.1 (Section 3) under condition (S) the times $\tau_k$ are finite almost surely for all $k \geq 1$. However, when this is not the case, we can condition on the event that $X^{\text{loop}}$ is able to reach 0 to estimate the probability in the statement, since on the complement of this event, we have $\eta_0^{\text{loop}} = \infty$.

We leave the details to the reader and, henceforth, we suppose that $\tau_k$ is finite almost surely for all $k \geq 1$. This allows us to define the process

$$Y_k := X_{\tau_k}^{\text{loop}}.$$ 

Note that by Strong Markov Property, $\{Y_k\}_{k}$ is a symmetric random walk on the segment $[0, \ell] \cap \mathbb{Z}$, with reflecting barriers. We also define another stopping time

$$\sigma := \inf \{ k > 0 \mid Y_k = 0 \},$$

and notice that $\eta_0^{\text{loop}} = \sigma$.

The idea of the proof is to show that the degree of vertex $\ell$ at time $\tau_\sigma$ is at least $e^K$ w.h.p. which, in turns, guarantees that $\sum_{j=0}^{\lfloor \tau_\sigma/s \rfloor} \xi_{s j}$, i.e., the number of leaves added up to time $\tau_\sigma$, is also at least $e^K$. Intuitively, having added an exponential number of leaves makes it harder for the walker to go back. In order to keep track of the degree of $\ell$, the following inequality (stochastic domination) will be useful

$$\text{deg}_{\tau_\sigma}(\ell) = 2 + \sum_{k=0}^{n-1} \mathbbm{1}\{Y_k = \ell\} \left( \sum_{m \in \{\tau_k, \tau_{k+1}\}, \ m \equiv 0 \mod s} \xi_m \right) \geq 2 + \sum_{k=0}^{n-1} \mathbbm{1}\{Y_k = \ell\} \text{Bin}(\lfloor \Delta \tau_k/s \rfloor, \kappa),$$
where $\Delta \tau_k := \tau_{k+1} - \tau_k$ and $\kappa$ is the uniform ellipticity constant given by condition (UE). Note that when $Y_k = \ell$, the amount of time $X^{\text{loop}}$ takes to leave $\ell$ is $\Delta \tau_k$, which in turns satisfies $\Delta \tau_k \geq \text{Geo}(1/\deg \tau_k(\ell))$. Since the degree is non-decreasing, for any $t \in [\tau_k, \tau_{k+1}]$, the probability of leaving $\ell$ given the past up to time $t - 1$ is at most $1/\deg \tau_k(\ell)$. Thus one may construct a coupling in such way that $\Delta \tau_k$ is greater than a geometric random variable with parameter $1/\deg \tau_k(\ell)$. Thus, we have

$$\deg \tau_k(\ell) \geq \sum_{k=0}^{n-1} \mathbb{1}\{Y_k = \ell\} \text{Bin} \left( \left\lfloor \text{Geo}\left(\frac{1}{\deg \tau_k(\ell)}\right)\mathbb{1}\{Y_k = \ell\}\right\rfloor, \kappa \right).$$

In order to avoid clutter, we simplify the notation defining $G_k := \text{Geo}(1/\deg \tau_k(\ell))$ and $d_k := \deg \tau_k(\ell)$. With this notation, we claim that

**Claim 2.1.** For all $\varepsilon \in (0, 1)$, there exists a constant $q = q(s, \kappa, \varepsilon) \in (0, 1)$ such that

$$\Pr_{B,\ell} \left( Y_k = \ell, \text{Bin} \left( \left\lfloor \frac{G_k}{s} \right\rfloor, \kappa \right) \geq \varepsilon \kappa \frac{d_k}{s} \left| \mathcal{F}_{\tau_k} \right. \right) \geq q \mathbb{1}\{Y_k = \ell\}, \quad \Pr_{B,\ell} \text{ - a.s.} .$$

**Proof of the claim:** By Chernoff bound, we have that

$$\Pr_{B,\ell} \left( Y_k = \ell, \text{Bin} \left( \left\lfloor \frac{G_k}{s} \right\rfloor, \kappa \right) \geq \varepsilon \kappa \frac{d_k}{s}, G_k \geq d_k \vee s \left| \mathcal{F}_{\tau_k}, G_k \right. \right)$$

$$\geq \left( 1 - e^{- \frac{(1-\varepsilon)^2}{2} \kappa \left( \frac{G_k}{s} \right)^2} \right) \mathbb{1}\{Y_k = \ell, G_k \geq d_k \vee s\}$$

$$\geq \left( 1 - e^{- \frac{(1-\varepsilon)^2}{2} \kappa} \right) \mathbb{1}\{Y_k = \ell, G_k \geq d_k \vee s\} .$$

Taking the conditional expectation with respect to $\mathcal{F}_{\tau_k}$ on the above inequality yields

$$\Pr_{B,\ell} \left( Y_k = \ell, \text{Bin} \left( \left\lfloor \frac{G_k}{s} \right\rfloor, \kappa \right) \geq \varepsilon \kappa \frac{d_k}{s} \left| \mathcal{F}_{\tau_k} \right. \right)$$

$$\geq \left( 1 - e^{- \frac{(1-\varepsilon)^2}{2} \kappa} \right) \Pr_{\ell}(Y_k = \ell, G_k \geq d_k \vee s | \mathcal{F}_{\tau_k})$$

$$\geq \left( 1 - e^{- \frac{(1-\varepsilon)^2}{2} \kappa} \right) \left( 1 - \frac{1}{d_k} \right)^{d_k \vee s} \mathbb{1}\{Y_k = \ell\} \geq \left( 1 - e^{- \frac{(1-\varepsilon)^2}{2} \kappa} \right) \left( 1 - \frac{1}{d_k} \right)^{d_k \vee s} \mathbb{1}\{Y_k = \ell\}$$

$$\geq \left( 1 - e^{- \frac{(1-\varepsilon)^2}{2} \kappa} \right) e^{- \frac{3}{2} s} \mathbb{1}\{Y_k = \ell\} ,$$

where in the last inequality we used that $(1 - \frac{1}{x})^x \geq e^{-\frac{3}{2}}, \forall x \geq 2$, and that $d_k$ is greater than 2 for all $k$.

By the claim, conditionally on the past, each time the process $\{Y_k\}_k$ visits $\ell$ it has a bounded away from zero probability of leaving $\ell$ with degree multiplied by $1 + \varepsilon \kappa/s$. \qed

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Thus, the degree of \( \ell \) must be at least exponential in the number of visits of \( \{Y_k\}_k \) to \( \ell \). Now we are left to control the number of visits to \( \ell \) by \( Y \). To do this, let \( N_\sigma(\ell) \) be the number of visits made by \( Y \) to \( \ell \) before time \( \sigma \). Recall that \( Y \) is a symmetric random walk on \( \{0,1,\ldots,\ell\} \), thus \( N_\sigma(\ell) \sim \text{Geo}(1/\ell) \). Moreover, the random variable \( W \) that counts how many times we have successfully multiplied the degree of \( \ell \) by \( 1 + \varepsilon \kappa/s \) may be written as follows

\[
W := \sum_{k=0}^{\sigma} \mathbb{1}\{Y_k = \ell\} \mathbb{1}\left\{ \binom{G_k}{s}, \kappa \geq \varepsilon \kappa \frac{d_k}{s} \right\},
\]

and dominates a random variable distributed as \( \text{Bin}(N_\sigma(\ell), q) \). Thus, for any \( K \geq 0 \)

\[
\mathbb{P}_{B,\ell} \left( \frac{K}{\log(1 + \varepsilon \kappa/s)} \leq \mathbb{P}_{B,\ell} \left( \binom{G_k}{s}, \kappa \geq \varepsilon \kappa \frac{d_k}{s} \right) \right) = \mathbb{P}_{B,\ell} \left( \binom{G_k}{s}, \kappa \geq \varepsilon \kappa \frac{d_k}{s} \right)
\]

\[
\leq \mathbb{P}_{B,\ell} \left( \binom{G_k}{s}, \kappa \geq \varepsilon \kappa \frac{d_k}{s} \right) \leq \frac{K}{q \log(1 + \varepsilon \kappa/s)} + \mathbb{P}_{B,\ell} \left( N_\sigma(\ell) \leq 2q^{-1} K \log(1 + \varepsilon \kappa/s) \right)
\]

\[
\leq \exp\{-c_1 K\} + 1 - \left(1 - \frac{1}{\ell}\right)^{c_2 K},
\]

with \( c_2 = \frac{2}{q \log(1 + \varepsilon \kappa/s)} \) and \( c_1 = \frac{1}{4q \log(1 + \varepsilon \kappa/s)} \) obtained using Chernoff bound. Finally, observe that if \( W \geq K / \log(1 + \varepsilon \kappa/s) \), then \( \text{deg}_{\tau_\sigma}(\ell) \geq 2K \). Thus, (2) yields

\[
\mathbb{P}_{B,\ell} \left( \text{deg}_{\tau_\sigma}(\ell) \leq 2e^K \right) \leq \exp\{-c_1 K\} + 1 - \left(1 - \frac{1}{\ell}\right)^{c_2 K}.
\]

Given \( K \), let \( \eta_* \) be the following stopping time: \( \eta_* := \inf \{ n \geq 0 : \text{deg}_{\eta_*}(\ell) \geq 2K \} \). By the Strong Markov Property applied to the loop process it follows that for any \( \beta \in (0,1) \)

\[
\mathbb{P}_{B,\ell} \left( \tau_\sigma \leq e^{\beta K}, \eta_* < \tau_\sigma \right) \leq \mathbb{E}_{B,\ell} \left( \mathbb{1}\{\eta_* < \tau_\sigma\} \mathbb{P}_{B,\eta_*} \left( \tau_\sigma \leq e^{\beta K} \right) \right).
\]

In the event \( \{\eta_* < \tau_\sigma\} \), the degree of vertex \( \ell \) in the backbone \( B_{\eta_*} \) is at least \( 2e^K \) and under such condition \( \tau_\sigma \) dominates a geometric random variable of parameter \( 1/(2e^K) \) which corresponds to the time needed for the walker to visit vertex \( \ell - 1 \). Thus, we obtain

\[
\mathbb{E}_{B,\ell} \left( \mathbb{1}\{\eta_* < \tau_\sigma\} \mathbb{P}_{B,\eta_*} \left( \tau_\sigma \leq e^{\beta K} \right) \right) \leq 1 - \left(1 - \frac{1}{2e^K}\right)^{e^{\beta K}} \leq \exp\{- (1 - \beta) K\}.
\]

Finally using (3), (4) and (5) we obtain that

\[
\mathbb{P}_{B,\ell} \left( \tau_\sigma \leq e^{\beta K} \right) \leq \mathbb{P}_{B,\ell} \left( \tau_\sigma \leq e^{\beta K}, \text{deg}_{\tau_\sigma}(\ell) \geq 2e^K \right) + \mathbb{P}_{B,\ell} \left( \text{deg}_{\tau_\sigma}(\ell) \leq 2e^K \right)
\]

\[
\leq \exp\{- (1 - \beta) K\} + \exp\{-c_1 K\} + 1 - \left(1 - \frac{1}{\ell}\right)^{c_2 K},
\]

proving the lemma. \( \square \)
The next proposition tells us that we may couple the TBRW and the Generalized Loop Process, GLP, in such way that $\eta_z$ is greater than $\eta_{\text{loop}}_0$ almost surely. The proposition is a mere generalization of Proposition 4.4 in [12] and its proof is in line with the one given therein and thence will be omitted. The reader may check it by just replacing $s = 1$ by any $s$ and taking into account that the environment process may be capable of adding more than one leaf at once.

**Proposition 2.2** (Coupling TBRW and the GLP; Proposition 4.4 in [12]). Let $T_0$ be a rooted locally finite tree, $x_0$ one of its vertices different from the root and $z$ an ancestor of $x_0$ at distance at least 2 from $x_0$. Then, there exists a coupling of $\{(T_n, X_n)\}_{n \in \mathbb{N}}$ starting from $(T_0, x_0)$ and a generalized loop process $\{(B_n, X_{\text{loop}}^n)\}_{n \in \mathbb{N}}$ starting from $(B(T_0, z, x_0), x_0)$ such that $P(\eta_z \geq \eta_{\text{loop}}_0) = 1$, where $B(T_0, z, x_0)$ is the backbone obtained from $(T_0, x_0)$ by the following procedure: i) remove all vertices in $T_0$ at distance greater than 2 from the unique path connecting $x_0$ to $z$; ii) identify the remaining vertices at distance one from the path with their neighbors on the path and consequently, all remaining edges not belonging to the path turn into loops.

We combine the bound given by Lemma 2.1 with the above proposition to obtain an upper bound for the cumulative distribution of $\eta_z$ for a far enough $z$.

**Corollary 2.3.** Consider a TBRW started at $(T_0, x_0)$, with $T_0$ a rooted tree, $x_0$ a vertex different from the root (at distance at least $\ell$ from the root) and let $z$ be the ancestor of $x_0$ at distance $\ell$. Moreover, assume that $\xi$ satisfies condition (UE). Then, there exists a positive constant $C$ depending on $s$ and $\xi$ only, such that

$$P_{T_0, x_0, s, \xi}(\eta_z \leq e^{\sqrt{\ell}}) \leq \frac{C}{\sqrt{\ell}}.$$ 

**Proof.** By choosing $K = 2\sqrt{\ell}$ and $\beta = 1/2$ on Lemma 2.1 we obtain that under condition (UE) it holds

$$P_{B, \ell}(\eta_{\text{loop}}_0 \leq e^{\sqrt{\ell}}) \leq \frac{C}{\sqrt{\ell}},$$

for some positive $C$ depending on the environment process and $s$ only. Finally, using the coupling given by Proposition 2.2 the result follows. \qed

### 2.2. Infinite expectation for the hitting time of far away vertices

In this section, building on the previous (specifically on Lemma 2.1) we prove that the hitting time of sufficiently far away vertices has an infinite expectation. This immediately implies that the TBRW process is either transient or null-recurrent.
Lemma 2.4. Let $\xi$ be an independent environment satisfying condition (UE) and $\{(T_n, X_n)\}_{n \in \mathbb{N}}$ a $(\xi, s)$-TBRW. Then there exists $\ell_0 = \ell_0(s, \kappa)$ such that
\[ \mathbb{E}_{T_0, x, \xi}(\eta_z) = \infty, \]
for all vertices $x, z$ such that $\text{dist}_{T_0}(x, z) \geq \ell_0$.

Proof. The result will follow from our upper bound for the cumulative distribution of $\eta_z$ given in Corollary 2.3.

\[ \mathbb{E}_{T_0, x, s, \xi}(\eta_z) = \sum_{k=0}^{\infty} \mathbb{P}_{T_0, x, s, \xi}(\eta_z \geq k) \geq (e^{1/4} - 1) \sum_{m=0}^{\infty} \mathbb{P}_{T_0, x}(\eta_y \geq e^{m/3}) e^{(m-1)/3} \]
\[ \geq (e^{1/4} - 1) e^{-1/3} \sum_{m=0}^{\infty} e^{m/3} \left( 1 - \frac{1}{\ell} \right)^{c_2 m} e^{-c_1 m} - e^{-2/3m} \]
\[ = (e^{1/4} - 1) e^{-1/3} \left( \sum_{m=0}^{\infty} e^{m/3} \left( 1 - \frac{1}{\ell} \right)^{c_2 m} - \sum_{m=0}^{\infty} e^{m/3} e^{-c_1 m} - \sum_{m=0}^{\infty} e^{m/3} e^{-2m/3} \right), \]
where in the last inequality we used Lemma 2.1 and the coupling in Proposition 2.2. Clearly the last summation on the right-hand side converges and we can ignore this term. As regards the second summation, recalling from Lemma 2.1 that $c_1 = \frac{1}{1 + \log(1 + \varepsilon \kappa / s)}$ with $\varepsilon \in (0, 1)$ arbitrary, we can choose $\varepsilon$ sufficiently small (depending on $s$ and $\kappa$) such that $c_1 > 1/3$. This guarantees that the second summation also converges. In order to prove the claim we are left with showing that the first summation diverges. Using that $(1 - \frac{1}{\ell})^{\ell} \geq e^{-3/2}$ for $\ell \geq 2$ we obtain that
\[ \sum_{m=0}^{\infty} e^{m/3} \left( 1 - \frac{1}{\ell} \right)^{c_2 m} \geq \sum_{m=0}^{\infty} e^{m(\frac{3}{4} - \frac{3c_2}{2\ell})}. \]
Recall (from Lemma 2.1) that $c_2 = \frac{2}{q \log(1 + \varepsilon \kappa / s)}$; thus, given $s, \varepsilon, \kappa$ and $q$ it is possible to choose $\ell$ sufficiently large such that $1/3 - 3c_2/2\ell > 0$, which implies that the summation diverges.

3. The case $s$ even: Recurrence vs getting Trapped

In this section we study the TBRW when the step parameter is even for finite initial trees. We shall assume that the initial tree $T_0$ (finite) has a particular vertex $r$, called the root of the tree, which is the unique vertex with a self-loop.

As discussed in [11], the self-loop at the root plays a prominent role in TBRW when $s$ is even. Let us recall a few concepts to understand the impact of this local feature at the root. Let the level of a vertex be its distance (graph distance) from the root and define the level of the walker at time $n$ as $\text{dist}_{T_n}(X_n, \text{root})$. We say that the walker $X$ at time
$n$ is even (resp., odd) if $\text{dist}_{T_n}(X_n, \text{root}) + n$ is even (resp., odd). Note that, whenever the walker is even (resp., odd) new leaves can only be added to vertices with even (resp., odd) levels. In order for new leaves to be added to vertices whose levels have different parity, the walker must “change its parity”. As it turns out, the walker can change its parity only if it traverses the self-loop; indeed, this is the only case in which the distance from the root stays constant and the time increases by one (see, Figure 2). The change of parity of the walker imposes crucial constraints on the tree growth when $s$ is even.

- The tree can grow to deeper levels only if the walker $X$ changes its parity. Specifically, if we consider a leaf $i$ added at time $t = ms$, subsequent leaves can be added to $i$ only if the walker changes its parity after time $t$.
- If the walker $X$ does not change its parity, new leaves can only be added to a finite set of vertices (those whose levels have the same parity of the walker).

In most of the results presented in this chapter, we do not need the hypothesis of independence on the environment, and thus the Markov property. The reader should keep in mind that although the TBRW is not necessarily markovian, if we only observe the evolution of the walker $X$ on vertices with opposite parity as that of itself between consecutive uses of the self-loop, then this evolution is markovian.

Following [11], a few questions naturally arise: will the random walk change its parity an infinite number of times with probability one? What is the impact of the environment $\{\xi_j\}_{j \in \mathbb{N}}$ on the behavior of TBRW? Note that, if the random walk changes its parity a finite number of time with positive probability then with positive probability the tree will have a finite depth. A necessary condition to change parity an infinite number of times almost surely is that the walker visits the root an infinite number of times with probability one, i.e., that the root is recurrent. In this section we show that:

- If the environment $\{\xi_j\}_{j \in \mathbb{N}}$ satisfies assumption (S) (see, Section 1.2) then:
  i) the recurrence of the root is also a sufficient condition to assure the walker changes its parity infinitely often almost surely (Corollary 3.3).
ii) every vertex of the tree (also the root) is recurrent (Theorem 1.1).

- If the environment satisfies condition (I) (see, Section 1.2) then the walker gets trapped almost surely, i.e., will keep on bouncing from one (random) vertex to its neighbors and back forever (Theorem 1.1).

Let us mention that, for the specific case $(2^k,1)$-TBRW the recurrence is proved in [11] and that $(2^k,1)$-TBRW trivially satisfies condition (S).

Before proving the main theorem of this section we introduce some auxiliary results. The first one uses the fact that the random walk in TBRW is symmetric to conclude that if the walker visits a vertex $x$ an infinite number of times and traverse a specific edge incident to this vertex an infinite number of times, then it must traverse every edge incident to $x$ an infinite number of times.

**Lemma 3.1.** Consider the TBRW process $\{ (T_n, X_n) \}_{n \in \mathbb{N}}$ with initial condition $(T_0, x_0)$. If there exist $x, y \in V(T_0)$ such that

$$\sum_{n=1}^{\infty} \mathbb{1}\{X_n = x\} \mathbb{1}\{X_{n+1} = y\} = +\infty, \quad \mathbb{P}_{T_0, x_0, s, \xi} - a.s.,$$

then, for any $z$ such that $\{ x, z \} \in E(T_0)$ (z neighbor of $x$)

$$\sum_{n=1}^{\infty} \mathbb{1}\{X_n = x\} \mathbb{1}\{X_{n+1} = z\} = +\infty, \quad \mathbb{P}_{T_0, x_0, s, \xi} - a.s..$$

The proof of Lemma 3.1 is provided in the supplemental article [15].

The second auxiliary result states that, under condition (S) on the environment, the random walk does not get stuck bouncing from one vertex to its neighbors and back forever, whereas under condition (I), the walker has a positive probability to keep on bouncing back forever. Before stating the lemma, let us define

$$\tau_{exit} := \inf\{2n \in \mathbb{N} : X_{2n} \neq X_0\},$$

the first time the walker does not come back to the initial node after two steps.

**Lemma 3.2.** Consider an even $(s, \xi)$-TBRW process. Then, for every initial state $(T_0, x_0)$ with $T_0$ finite:

i) if $\xi$ satisfies condition (S), it holds that $\mathbb{P}_{T_0, x_0, s, \xi}(\tau_{exit}(x_0) < \infty) = 1$,

ii) if $\xi$ satisfies condition (I) and $x_0$ has at least a neighboring leaf in $T_0$ there exits a positive constant $C$, depending only on $s$ and on the distribution of $\xi$, such that

$$\mathbb{P}_{T_0, x_0, s, \xi}(\tau_{exit}(x_0) = \infty) > e^{-C(\deg_{T_0}(x_0) - \text{leaf}_{T_0}(x_0))} > 0,$$

where $\deg_{T_0}(x_0)$ denotes the degree of $x_0$ in $T_0$ and $\text{leaf}_{T_0}(x_0)$ the number of neighboring leaves.
Proof. Proof of item (i). Let us first assume that \( x_0 \) is different from the root. We shall prove the Lemma considering the “worst” possible scenario, i.e., the case in which \( T_0 \) is a star centered at \( x_0 \), whose degree is \( d \) and \( d - 1 \) neighbors of \( x_0 \) are leaves and one neighbor is the root with a self-loop. We shall assume that \( d \geq 2 \), i.e., \( x_0 \) has at least a neighboring leaf; the case where \( d = 1 \) is similar and easier. As it turns out, we shall prove that \( \tau_{exit}(x_0) < \infty \) almost surely, regardless of the value of \( d \), which “justify” why this choice of \( T_0 \) is the worst possible scenario. Note that for this choice of \( (T_0, x_0) \) the time \( \tau_{exit}(x_0) \) corresponds to the first time the walker traverses the self-loop. To show that \( \tau_{exit}(x_0) < \infty \) almost surely, it is enough to show that \( \tau_{exit}(x_0) < \infty \) a.s., where \( \tau_{exit} \) denotes the first time the walker visits the root. This is because every time the walker visits the root, it has a constant probability (equal to 1/2) to traverse the self-loop.

As long as \( \tau_{exit} > sn \), we have that \( S_n := \sum_{j=0}^{n} \xi_j \) denotes the number of new leaves attached to \( x_0 \) up to time \( sn \). Note that, if the random walk steps towards a leaf (not the root), it will necessarily be at \( x_0 \) in the next step. The probability of choosing a leaf at time \( sn \) is \( 1 - \frac{1}{d+S_n} \). Therefore,

\[
\Pr_{T_0,x_0,s,\xi}(\tau_{exit} > sn) = E(\Pr_{T_0,x_0,s,\xi}(\tau_{exit} > sn|\xi)) = E\left(\prod_{j=0}^{n-1} \left(1 - \frac{1}{d+S_i}\right)^{s/2}\right) = E\left(\exp\left\{\frac{s}{2} \sum_{i=0}^{n-1} \log(1 - \frac{1}{d+S_i})\right\}\right),
\]

where \( E \) denotes the expectation with respect to the environment \( \xi \). Since \( \log(1-x) \leq -x \), for \( 0 \leq x < 1 \), we obtain that

\[
\lim_{n \to \infty} \Pr_{T_0,x_0,s,\xi}(\tau_{exit} > sn) \leq \lim_{n \to \infty} E\left(\exp\left\{-\frac{s}{2} \sum_{i=0}^{n-1} \frac{1}{d+S_i}\right\}\right) = E\left(\exp\left\{-\frac{s}{2} \lim_{n \to \infty} \sum_{i=0}^{n-1} \frac{1}{d+S_i}\right\}\right),
\]

where the last inequality follows from the Dominated Convergence Theorem. Moreover, using that we are under condition (S), it follows

\[
\limsup_{i \to \infty} \frac{S_i}{g(i)} \leq c, \quad P\text{-almost surely},
\]

which implies that there exists \( L(\omega) \), for almost every \( \omega \), such that \( S_i(\omega) < 2g(i)c \) for all \( i \geq L(\omega) \). Then

\[
\sum_{i=L(\omega)}^{n-1} \frac{1}{d+S_i(\omega)} \geq \sum_{i=L(\omega)}^{n-1} \frac{1}{d+2g(i)c} \to \infty \text{ as } n \to \infty,
\]

regardless the value of \( d \). Thus

\[
\Pr_{T_0,x_0,s,\xi}(\tau_{exit} = \infty) = \lim_{n \to \infty} \Pr_{T_0,x_0,s,\xi}(\tau_{exit} > sn) \leq E\left(\exp\left\{-\frac{s}{2} \lim_{n \to \infty} \sum_{i=0}^{n-1} \frac{1}{d+S_i}\right\}\right) = 0.
\]
Let us now consider the case $x_0 = \text{root}$. We shall prove the Lemma considering the “worst” possible scenario, which in this case corresponds to having $T_0$ a star centered at $x_0$ whose degree is $d$ (we can assume $d \geq 2$) and $d - 1$ neighbors of $x_0$ are leaves. Note that in this case $\tau_{\text{exit}}(x_0)$ corresponds to the first time the walker traverses the self-loop (at an odd time) and in the subsequent step visits a leaf. To show that $\tau_{\text{exit}}(x_0) < \infty$ almost surely, it is enough to show that $\bar{\tau}_{\text{exit}} < \infty$ a.s., where $\bar{\tau}_{\text{exit}}$ denotes the first time the walker uses the self-loop. This is because every time the walker traverses the self-loop at an odd time, it has a probability bigger than or equal to $\frac{d - 1}{d}$ to visit a leaf. Now the proof follows the same line of reasoning as above.

Proof of item $(ii)$. Given a vertex $x_0 \in V(T_0)$, we denote by $d = \deg_{T_0}(x_0)$ and by $\ell = \text{leaf}_{T_0}(x_0)$ its degree and the number of neighboring leaves in $T_0$, respectively. Note that, by hypothesis we have that $\ell \geq 1$, while for the tree structure we have $d - \ell \geq 1$. Define $\bar{\tau}_{\text{exit}}(x_0)$ as the first time the random walk visits a non-leaf vertex neighbor of $x_0$. Then

$$\mathbb{P}_{T_0, x_0, s, \xi} (\bar{\tau}_{\text{exit}}(x_0) > sn) \geq \mathbb{P}_{T_0, x_0, s, \xi} (\bar{\tau}_{\text{exit}}(x_0) > sn),$$

for every $s$ and $n$. Thus, to prove the claim it suffices to show that $\mathbb{P}_{T_0, x_0, s, \xi} (\bar{\tau}_{\text{exit}}(x_0) = \infty)$ is strictly positive. Notice that, as long as $\bar{\tau}_{\text{exit}}(x_0) > sn$, the probability of choosing a leaf at time $sn$ is equal to $1 - \frac{d - \ell}{d + S_i}$. Therefore, similarly to Equation (6), we have that

$$\mathbb{P}_{T_0, x_0, s, \xi} (\bar{\tau}_{\text{exit}}(x_0) > sn) = E \left( \exp \left\{ \frac{s}{2} \sum_{i=0}^{n-1} \log \left( 1 - \frac{d - \ell}{d + S_i} \right) \right\} \right),$$

where $E$ denotes the expectation with respect to the environment $\xi$. Recall that environment condition (I) assures that

$$P \left( \lim_{n \to \infty} \frac{S_n}{f(n)} \geq 2e \right) = 1. \tag{8}$$

Therefore, for every $0 < \varepsilon < 1$, we can find a measurable subset $\Omega_\varepsilon$ and $n_0 = n_0(\varepsilon)$ such that $P(\Omega_\varepsilon^c) > \varepsilon$ and $S_i \geq cf(i)$ on $\Omega_\varepsilon^c$, for all $i \geq n_0$. Hence for $n > n_0$ we get

$$E \left( \exp \left\{ \frac{s}{2} \sum_{i=0}^{n-1} \log \left( 1 - \frac{d - \ell}{d + S_i} \right) \right\} \right) \geq c^{\frac{n_0}{2}} \varepsilon e^{\frac{1}{d} \log(1 - \frac{d - \ell}{d + cf(i)})} \left\{ \exp \left\{ \frac{s}{2} \sum_{i=n_0}^{n-1} \log \left( 1 - \frac{d - \ell}{d + cf(i)} \right) \right\} \right\} \mathbb{1}_{\Omega_\varepsilon^c} \right).$$

Since $\log(1 - x) \geq -x - \frac{x^2}{1-x}$, for $0 \leq x < 1$, we have that

$$\sum_{i=n_0}^{n-1} \log \left( 1 - \frac{d - \ell}{d + cf(i)} \right) \geq - \sum_{i=n_0}^{n-1} \frac{d - \ell}{d + cf(i)} - \sum_{i=n_0}^{n-1} \frac{(d - \ell)^2}{(d + cf(i))(\ell + cf(i))}

$$

$$= - \sum_{i=n_0}^{n-1} \frac{d - \ell}{\ell + cf(i)} \geq - \sum_{i=n_0}^{n-1} \frac{d - \ell}{cf(i)}.$$
Moreover \( \log \left( 1 - \frac{d-\ell}{d} \right) = \log(\ell) - \log(d) \geq -(d-\ell) \). Using the hypothesis that \( \sum_{i=1}^{\infty} \frac{1}{i^{\ell}} < \infty \), there exists a constant \( \hat{C} > 0 \) such that

\[
P_{T_0,x_0,s} (\tau_{\text{exit}}(x_0) = \infty) = \lim_{n \to \infty} P_{T_0,x_0,s} (\tau_{\text{exit}}(x_0) > sn) \geq c e^{\frac{1}{2}n_0(d-\ell)} e^{-\frac{1}{2}(d-\ell)\hat{C}} > 0 .
\]

To finish the proof choose \( C = -\log c + \frac{1}{2}(n_0 + \hat{C}) \) and note that \( C \) is fixed, \( n_0 \) depends on \( \epsilon \) and on the distribution of \( \xi \), and \( \hat{C} \) depends only on the distribution of \( \xi \). We point out that \( n_0 \) should increase with \( \epsilon \) and we have not considered the optional choice for \( \epsilon \) here. \( \square \)

**Remark 3.1.** Note that in the above Lemma (part ii)), we implicitly use the fact that, at time 0 the walker is in \( x_0 \), and new leaf can be added to \( x_0 \) without the need to change parity. In particular, if the walker steps on a vertex with the “wrong” parity, it will not be able to add leaves to the vertex without changing its parity first, and in particular not before the corresponding \( \tau_{\text{exit}} \), thus the Lemma will not be true in this case! In the sequel we will need to use this Lemma together with the strong Markov property and therefore we must be sure the walker steps on a vertex with the right parity.

**Remark 3.2.** Lemma 3.2 also holds if the environment \( \xi \) is shifted by an almost surely finite random time. First consider (i). If \( \xi \) satisfies condition (S) then the environment shifted by an almost surely finite random time also satisfies condition (S). To see that, it is enough to show that condition (S) is preserved by any finite shift of the environment. For \( k \geq 0 \) let \( \theta_k \) be the forward time shift such that \( \theta_k(\xi) = \{ \xi_j \}_{j \in \mathbb{N}} \) and \( S_n \circ \theta_k = \sum_{j=k+1}^{n+k} \xi_j = S_{n+k} - S_k \). If \( \xi \) satisfies condition (S) with function \( g \) and constant \( c \), then for any \( k \geq 0 \), the environment \( \theta_k(\xi) \) satisfies condition (S) with function \( \tilde{g}(n) = g(n+k) \) and constant \( c \). Although \( k \) might be random, we still have an almost surely divergence in (7) and (i) holds. Now we consider (ii) which is even simpler. Suppose that \( \xi \) satisfies (8). Observe that \( f(n) \to \infty \). Then

\[
\frac{S_n \circ \theta_k}{f(n)} = \frac{f(k+n)}{f(n)} \frac{S_{n+k}}{f(k+n)} - \frac{S_k}{f(n)} \geq c ,
\]

for all sufficiently large \( n \) depending on the possibly random \( k \). This implies that the proof of (ii) is the same.

For the subsequent results we will need to define a random walk over an auxiliary graph that will play a crucial role in the proofs we are going to provide.

Let us denote by \( Y_n = X_{2n} \) the position of the walker after two steps, and define the sequence of stopping times: \( \phi_0 = 0 \) and for \( k \geq 1 \)

\[
\phi_k := \inf \{ n > \phi_{k-1} : Y_n \neq Y_{\phi_{k-1}} \} .
\]

Note that, under condition (S), Lemma 3.2 guarantees that \( \phi_k \) is almost surely finite, for every \( k \). Thus the process \( \{ Z_k \}_{k \in \mathbb{N}} \) defined as

\[
Z_k := Y_{\phi_k} ,
\]
Tree Builder Random Walk: recurrence, transience and ballisticity

is well defined. Interestingly, as long as $X$ does not traverse the self-loop, the process $Z$ is homogeneous. Specifically if $t_m$ denotes the $m$-th time $X$ crossed the self-loop, then $Z_k$ for $t_m \leq k \leq t_{m+1}$ is a symmetric random walk on a graph whose structure only depends on the TBRW up to time $t_m$ and remains fixed during $t_m$ and $t_{m+1}$ (see, Figure 3).

![Figure 3](image.png)

Figure 3. The process $Z$ and its graph corresponding to the situation depicted in Figure 2. As long as $Z$ (squared vertex) moves on the graph with black vertices, it will alter the gray graph structure. However, until the process $X$ crosses the self-loop (dotted line) $Z$ will not see the gray structure. As soon as $X$ crosses the self-loop the situation is interchanged.

We say that the walker gets trapped at time $n$ if $\tau_{\text{exit}}(X_n) = \infty$. Note that, by the definition of $\tau_{\text{exit}}$, if the walker gets trapped at time $n$ then it will also get trapped at time $n + 2k$ for every $k$.

**Remark 3.3.** If $X$ gets trapped at time $n$ with probability one, then all vertices at distance at least 2 from $X_n$ will be clearly transient, whereas the vertex $X_n$ will be recurrent. As a matter of fact, also all the vertices at distance equal to 1 from $X_n$ will be transient. Observe that if the walker gets trapped at time $n$ then it necessarily traverses a finite number of times the edge connecting $X_n$ to its parent in the tree; the symmetry of the random walk assures that the same must hold for every edge incident to $X_n$.

We can now prove Theorem 1.1 the main result of this section.

**Proof of Theorem 1.1.** Proof of item (i). To prove the first part, it is enough to show that the walker traverses the self-loop an infinite number of times almost surely. Indeed, whenever the latter happens, we know the root will be recurrent and, using Lemma 3.1, we can conclude that the walker traverses an infinite number of times any edge incident to the root. This will assure that the neighbors of the root will be recurrent. Knowing that these neighbors are recurrent and that the edges connecting them to the root are crossed an infinite number of times, by Lemma 3.1 we may conclude that all vertices at
distance 2 from the root are also recurrent. This argument can then be iterated along all vertices of the tree.

Given \( x, y \) vertices of the tree, let us define \( J_n(x, y) := \sum_{k=1}^{n-1} \mathbb{1}_{X_k = x} \mathbb{1}_{X_{k+1} = y} \). What we are after is to show that \( \lim_{n} J_n(\text{root}, \text{root}) = \infty \) a.s..

**Claim 3.1.** \( \lim_{n} J_n(\text{root}, \text{root}) = +\infty, \mathbb{P}_{T_0, x_0, s, \xi} \)-a.s.

The proof of Claim 3.1 is provided in the supplemental article [15].

**Proof of item (ii).** We need to show that \( \mathbb{P}_{T_0, x_0, s, \xi} (\exists n \geq 0 : \tau_{\text{exit}}(X_{2n}) = \infty) = 1 \). Due to the definition of \( \tau_{\text{exit}}(X_n) \) it is enough to show that

\[
\sum_{n=1}^{\infty} \mathbb{1}_{\tau_{\text{exit}}(X_{2n}) = \infty} = +\infty, \quad \mathbb{P}_{T_0, x_0, s, \xi} - \text{a.s.}
\]

In order to prove the above identity we argue in a similar way to the proof of item i). In this case, we show first that if the predictable process given the the Doob’s Decomposition Theorem converges to infinity, then the whole sum also goes to infinity. More formally, we prove the following claim first.

**Claim 3.2.**

\[
\sum_{k=1}^{\infty} \mathbb{E}_{T_0, x_0, s, \xi} \left( \mathbb{1}_{\tau_{\text{exit}}(X_{sk}) = \infty} \mid F_{sk} \right) = \infty, \quad \text{a.s.} \implies \sum_{k=1}^{\infty} \mathbb{1}_{\tau_{\text{exit}}(X_{sk}) = \infty} = \infty, \quad \text{a.s.}
\]

**Proof of the claim:** Let \( R_n = \sum_{k=1}^{n} \mathbb{1}_{\tau_{\text{exit}}(X_{sk}) = \infty} \), we can write \( R_n = M_n + A_n \), where,

\[
M_n := \sum_{k=1}^{n} \left( \mathbb{1}_{\tau_{\text{exit}}(X_{sk}) = \infty} - \mathbb{E}_{T_0, x_0, s, \xi} \left( \mathbb{1}_{\tau_{\text{exit}}(X_{sk}) = \infty} \mid F_{sk} \right) \right),
\]

is a bounded increments martingale and

\[
A_n := \sum_{k=1}^{n} \mathbb{E}_{T_0, x_0, s, \xi} \left( \mathbb{1}_{\tau_{\text{exit}}(X_{sk}) = \infty} \mid F_{sk} \right).
\]

Using Theorem 5.3.1 in [7], either \( \lim_n M_n \) exists and it is finite a.s., or \( \limsup_n M_n = \infty \) and \( \liminf_n M_n = -\infty \) a.s.. Using the assumption that \( \lim_n A_n = \infty \) a.s., in both cases we can conclude that \( \lim_n R_n = \infty \) a.s. \( \blacksquare \)

Using the above claim, to prove item ii) of Theorem 1.1, we have to show that

\[
\sum_{k=1}^{\infty} \mathbb{E}_{T_0, x_0, s, \xi} \left( \mathbb{1}_{\tau_{\text{exit}}(X_{sk}) = \infty} \mid F_{sk} \right) = \infty, \quad \mathbb{P}_{T_0, x_0, s, \xi} - \text{a.s.}
\]
The proof of (10) relies on Lemma 3.2 (point ii)). However, when using Lemma 3.2 together with the Markov Property (which holds given the independence assumption on the environment $\xi$) we need to account for how $\deg_{T_{sk}}(X_{sk}) - \text{leaf}_{T_{sk}}(X_{sk})$ grows with $k$. In order to do that, let us introduce a terminology: we say that a vertex is a \textit{quasi-star} if it has at least one neighboring leaf and only one non-leaf neighbor. Specifically, $X_n$ is a quasi-star if $\text{leaf}_{T_n}(X_{n}) \geq 1$ and $\deg_{T_n}(X_{n}) = \text{leaf}_{T_n}(X_{n}) + 1$. Then, we can write

$$
\mathbb{E}_{T_0,x_0,s,\xi} (\mathbb{1}\{\tau_{\text{exit}}(X_{sk}) = \infty\}|F_{sk}) = \\
\mathbb{E}_{T_0,x_0,s,\xi} (\mathbb{1}\{\tau_{\text{exit}}(X_{sk}) = \infty\}\mathbb{1}\{X_{sk} \text{ is a quasi-star}\}|F_{sk}) \\
+ \mathbb{E}_{T_0,x_0,s,\xi} (\mathbb{1}\{\tau_{\text{exit}}(X_{sk}) = \infty\}\mathbb{1}\{X_{sk} \text{ is not a quasi-star}\}|F_{sk})
$$

(11)

As far as (11) is concerned, applying the Markov Property, we obtain

$$
\mathbb{E}_{T_0,x_0,s,\xi} (\mathbb{1}\{X_{sk} \text{ is a quasi-star}\}\mathbb{1}\{\tau_{\text{exit}}(X_{sk}) = \infty\}|F_{sk}) = \\
\mathbb{1}\{X_{sk} \text{ is a quasi-star}\}\mathbb{E}_{T_{sk},X_{sk},s,\theta_{sk}}(\mathbb{1}\{\tau_{\text{exit}}(X_0) = \infty\})
$$

where $\theta_{sk}$ denotes the forward time shift $\theta_{sk}(\xi) = \{\xi_{j+sk}\}_{j \in \mathbb{N}}$, and in the last inequality we used that $X_{sk}$ is a quasi-star together with Lemma 3.2 (point ii). As regards (12), applying the Markov Property we have that

$$
\mathbb{E}_{T_0,x_0,s,\xi} (\mathbb{1}\{X_{sk} \text{ is not a quasi-star}\}\mathbb{1}\{\tau_{\text{exit}}(X_{sk}) = \infty\}|F_{sk}) = \\
\mathbb{1}\{X_{sk} \text{ is not a quasi-star}\}\mathbb{E}_{T_{sk},X_{sk},s,\theta_{sk}}(\mathbb{1}\{\tau_{\text{exit}}(X_0) = \infty\})
$$

If $X_{sk}$ is not a quasi-star, $X_{sk}$ may have no neighboring leaves in $T_{sk}$ and thus Lemma 3.2 (point ii) does not necessarily apply. To be able to apply Lemma 3.2, we multiply by $\mathbb{1}\{\xi_{sk} \geq 1\}$ and obtain that

$$
\mathbb{E}_{T_0,x_0,s,\xi} (\mathbb{1}\{X_{sk} \text{ is not a quasi-star}\}\mathbb{1}\{\tau_{\text{exit}}(X_{sk}) = \infty\}|F_{sk}) \\
\geq \mathbb{1}\{X_{sk} \text{ is not a quasi-star}\}\mathbb{1}\{\xi_{sk} \geq 1\}\mathbb{E}_{T_{sk},X_{sk},s,\theta_{sk}}(\mathbb{1}\{\tau_{\text{exit}}(X_0) = \infty\})
$$

$$
\geq \mathbb{1}\{X_{sk} \text{ is not a quasi-star}\}\mathbb{1}\{\xi_{sk} \geq 1\} \exp \{-C (\deg_{T_{sk}}(X_{sk}) - \text{leaf}_{T_{sk}}(X_{sk}))\}
$$

If $X_{sk}$ is not a quasi-star (but has at least a neighboring leaf), we do not have a straightforward upper bound for $\deg_{T_{sk}}(X_{sk}) - \text{leaf}_{T_{sk}}(X_{sk})$, uniformly in $sk$. However, if define

$$
c(T_{sk}) := \max_{x \in V(T_{sk})} \{\deg_{T_{sk}}(x) - \text{leaf}_{T_{sk}}(x)\}
$$

it holds that

$$
c(T_{sk}) \leq c(T_0) + \sum_{i=0}^{k} \mathbb{1}\{X_{ik} \text{ is a quasi-star}\}
$$

To see why the above inequality holds, note that when $\xi_{sk}$ new leaves are added to the tree, either $X_{sk}$ is a quasi-star (in $T_{sk+1}$) or not. If $X_{sk}$ is not a quasi-star (in $T_{sk+1}$) then the constant $c(T_{sk})$ does not change, since the degree of $X_{sk}$ in $T_{sk+1}$ has increased by
\( \xi_{sk} \) and all of them are leaves. If \( X_{sk} \) is a quasi-star (in \( T_{sk+1} \)) then the constant \( c(T_{sk}) \) may have increased by at most one unit (if the \( \xi_{sk} \) new vertices were added to a leaf).

Overall, we obtain that

\[
E_{T_0, x_0, s, \xi} \left( 1 \{ \tau_{\text{exit}}(X_{sk}) = \infty \} \middle| F_{sk} \right) \geq 1 \{ X_{sk} \text{ is a quasi-star} \} e^{-C} + \mathbb{1}\{ X_{sk} \text{ is not a quasi-star} \} \mathbb{1}\{ \xi_{sk} \geq 1 \} \exp \left\{ -C \left( c(T_0) + \sum_{i=0}^k \mathbb{1}\{ X_{ik} \text{ is a quasi-star} \} \right) \right\}.
\]

Note that, either \( \sum_{k=1}^{\infty} \mathbb{1}\{ X_{sk} \text{ is a quasi-star} \} = +\infty \) or \( \sum_{k=1}^{\infty} \mathbb{1}\{ X_{sk} \text{ is a quasi-star} \} = M < +\infty \). In the first case, clearly we obtain \( \sum_{k=1}^{\infty} E_{T_0, x_0, s, \xi} \left( 1 \{ \tau_{\text{exit}}(X_{sk}) = \infty \} \middle| F_{sk} \right) = +\infty \), whereas in the second case we obtain that

\[
\sum_{k=1}^{\infty} E_{T_0, x_0, s, \xi} \left( 1 \{ \tau_{\text{exit}}(X_{sk}) = \infty \} \middle| F_{sk} \right) \geq e^{-C c(T_0) + M} \sum_{k=1}^{\infty} \mathbb{1}\{ \xi_{sk} \geq 1 \} \mathbb{1}\{ X_{sk} \text{ is not a quasi-star} \} = e^{-C c(T_0) + M} \left( \sum_{k=1}^{\infty} \mathbb{1}\{ \xi_{sk} \geq 1 \} - M \right).
\]

The proof of (10) finally follows from noticing that condition (I) implies

\[
\sum_{k=1}^{\infty} \mathbb{1}\{ \xi_{sk} \geq 1 \} = +\infty, \quad P_{T_0, x_0, s, \xi} - \text{a.s.},
\]

since whenever \( S_n = \sum_{k=1}^{n} \xi_{sk} \) converges to a finite limit, condition (I) cannot hold.

One consequence of item (i) of Theorem 1.1 is the following corollary.

**Corollary 3.3.** Consider a \((s, \xi)\)-TBRW process with \( s \) even and environment \( \xi \) which satisfies condition (S). Then the walker changes its parity infinitely often almost surely.

### 3.1. Null recurrence driven by distinct local behaviors in i.i.d. environment

In this part, we consider the particular case where the environment process \( \xi \) is a sequence of i.i.d. random variables. The goal is to extract finer information about the exiting time \( \tau_{\text{exit}} \), specifically, regarding its expected value from different initial conditions.

By our discussion about conditions (S) and (I) at the beginning of this section and Proposition 1.2 it follows that under i.i.d. environments with non-zero finite mean the \((2k, \xi)\)-TBRW is null recurrent. However, we can identify two distinct regimes of the null recurrence for these processes. These two regimes are explained in terms of the random walk \( Z \) defined in Equation (9). Roughly speaking what we observe is that, under certain environment conditions, the random walk \( Z \) evolves in its graph with infinite mean time
transition and under other environment conditions, it indeed evolves with finite mean time transitions.

Below we state the main result of this section and explain how it is related with the two possible behaviors of the auxiliary random walk $Z$. In order to simplify the statement of the theorem, we use the following notation $P_{\ell+1, \text{root}, 2k, \xi}$ and $E_{\ell+1, \text{root}, 2k, \xi}$, to denote the TBRW process whose initial state $(T_0, x_0)$ is the root with $\ell$ leaves and a self-loop.

**Theorem 3.4.** Consider a $(2k, \xi) – TBRW$ process, with independent environment $\xi = \{\xi_j\}_{j \in \mathbb{N}}$ satisfying condition (M)$_1$. Then, TBRW is null recurrent. Furthermore:

(i) If $k < \mu := E(\xi_1)$, then $E_{\ell+1, \text{root}, 2k, \xi}(\tau_{\text{exit}}) = \infty$ for every $\ell \geq 1$;

(i*) If $\xi_1 \equiv k$ or $k = \mu$ and $\xi_1$ satisfies

\[
\limsup_{n \to \infty} \frac{\log(n)^2}{n} \log P(\xi > n/\log(n)) = -\infty,
\]

then $E_{\ell+1, \text{root}, 2k, \xi}(\tau_{\text{exit}}) = \infty$ for every $\ell \geq 1$;

(ii) If $k > \mu$ then there exists $\gamma > 0$ such that $E_{\ell+1, \text{root}, 2k, \xi}(\tau_{\text{exit}}) \geq \gamma \ell$ for every $\ell \geq 1$. Moreover, if there exists $\varepsilon > 0$ such that $\xi$ satisfies condition (M)$_{2+\varepsilon}$, then

\[
E_{\ell+1, \text{root}, 2k, \xi}(\tau_{\text{exit}}) = \Theta(\ell)
\]

The reader may notice the lack of a general initial state on the statement above, however, the general case is implicitly covered. Note that regardless of the initial state of the TBRW process items (i) and (i*) imply that when $X$ lands on a leaf at an even step by the Strong Markov Property the process $Z$ will take, in average, infinity time to move, although it will move almost surely since $X$ is recurrent. On the other hand, as we explained before the more non-leaves neighbors a vertex has the more likely is $X$ to leave it, thus in item (ii), under higher moment conditions, $X$ always moves in finite mean time.

The proof of Theorem 3.4 is provided in the supplemental article [15]. The proof of item (i*) is similar to that of item (i), but it requires a moderate deviations result for sums of i.i.d random variables. Condition (13) is a condition to guarantee this moderate deviations result. This sort of condition implies finite second moment and is implied by the existence of exponential moments. The interested reader may consult [10].

**Remark 3.4.** It is interesting to note that the local behavior described in Theorem 3.4 allows us to prove null recurrence, already established in Proposition 1.2 under much more general conditions on the environment. The proof of null recurrence through the local behaviour can be found in the supplemental material [15].
4. The Case $s$ odd: Ballisticity of the walker in TBRW

In this section we prove Theorem 1.3, which states ballistic behavior of TBRW for $s$ odd under condition (UE). To show ballisticity, i.e.,
\[
\liminf_{n \to \infty} \frac{\text{dist}_{T_n}(X_n, \text{root})}{n} \geq c, \quad \mathbb{P}_{T_0,x_0,s,\xi} \text{- almost surely ,}
\]
we rely on a general criterion which we extrapolate from the proof of ballisticity for BGRW given in [12]. We believe this general criterion may be of independent interest and may be applied to a wider class of similar processes.

4.1. General criterion for ballisticity

To provide some intuition about the general criterion for ballisticity in TBRW, let us begin building a bridge with the classical theory of Random Walk on Random Environments. In the latter context, the main mechanism behind ballisticity is the concept of \textit{regeneration time}, see [22]. Roughly speaking, it says the walk has regenerated if it does not return to a half-space after a certain time. This implies that from time to time, the walk is always exploring independent portions of the environment. The idea of regeneration also appears in the context of the TBRW when $s$ is odd. In essence, the regeneration is now due to two reasons combined:

1. The walk is capable of building long enough paths regardless the current tree;
2. Once the walk is at a tip of a path, it takes very long time to backtracking.

The two items above assure that the walker may “forget” the tree structure built up to a certain time and start afresh. Item (2) is related to the hitting times estimates proved in Section 2. Thus, item (2) holds in the presence of condition (UE) regardless the parity of $s$. However, as will be clearer in the sequel, item (1) requires $s$ to be odd. With $s$ odd, the walk may “push the tree forward” adding new leaves to the bottom of the tree. This feature gives the walk the ability of creating the escape routes it needs. This combined with (2) assures that the walker has a positive probability of never returning to some portions of the tree. The general criterion for ballisticity introduced in this section is, in essence, a quantitative version of (1) and (2).

For $r$ a positive integer, let us denote by $\Omega_r$ the subset of $\Omega$ formed by all pairs $(T,x)$ such that $T$ has height at least $r$ and $\text{dist}_{T}(x, \text{root}) \geq r$. Let $\eta_r$ be the first time $X$ hits the ancestor of its initial position at distance $r$ in the path connecting $X_0$ to the root of $T_0$, i.e.,
\[
\eta_r := \inf \{ n \geq 0 \mid \text{dist}_{T_n}(X_n, \text{root}) = \text{dist}_{T_0}(X_0, \text{root}) - r \}.
\]

We say that the process $\{(T_n,X_n)\}_{n \in \mathbb{N}}$ satisfies conditions (R) and (L) if: there exists $\alpha \in (0,1)$, $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $r$ such that
\[
\inf_{(T_0,x_0) \in \Omega} \mathbb{P}_{T_0,x_0,s,\xi} \left( \exists m \leq \exp\{r^\alpha\}, \text{dist}_{T_m}(X_m, \text{root}) \geq 2r \right) \geq 1 - \varepsilon/2 , \quad (R)
\]
Thus, if \( \{r \} \) are stopping times. First fix a positive integer \( r \). Let \( r \) denote the distance from the root by 2. We also set \( \Delta \) and by induction

\[
\sigma_k := \sigma_{k-1} \circ \theta_{\sigma_{k-1}}(\{(X_n)_{n \geq 0}\}),
\]

for \( k \geq 2 \), where \( \theta_k \) is the forward time shift by \( k \) units, i.e. \( \theta_k((X_n)_{n \geq 0}) = (X_{n+k})_{n \geq 0} \).

Clearly, these stopping times depend on \( r \), but we omit such a dependency to avoid clutter. From the definition it follows that, for all \( k \), \( \sigma_k \) is bounded from above by \( k \exp\{r^\alpha\} \).

We also set \( \Delta d_{k+1} = \text{dist}_{T_{\sigma_{k+1}}}(X_{\sigma_{k+1}}, \text{root}) - \text{dist}_{T_{\sigma_k}}(X_{\sigma_k}, \text{root}) \) and denote by \( \tilde{\mathcal{F}}_k \) the \( \sigma \)-field generated by the process \( \{(T_n, X_n)\}_{n \in \mathbb{N}} \) up to time \( \sigma_k \).

**Lemma 4.1.** Let \( \{(T_n, X_n)\}_{n \in \mathbb{N}} \) be a process satisfying conditions (R) and (L), \( T_0 \) a rooted locally finite tree, \( x_0 \) a vertex of \( T_0 \). There exists \( \varepsilon > 0 \) and \( r \) such that for all \( k \)

\[
\inf_{T_0, x_0} \mathbb{P}_{T_0, x_0} \left( \Delta d_{k+1} = r \right) \geq \frac{1}{2} (1 + \varepsilon).
\]

Thus, if \( \{S_k\}_{k \geq 0} \) denotes a \( \frac{1}{2}(1 + \varepsilon) \)-right biased simple random walk on \( \mathbb{Z} \), the process \( \{\text{dist}_{T_{\sigma_k}}(X_{\sigma_k}, \text{root})/r\}_{k \geq 0} \) and \( \{S_k\}_{k \geq 0} \) starting from \( \text{dist}_{T_0}(x_0, \text{root})/r \) can be coupled in such a way that

\[
\mathbb{P} \left( \text{dist}_{T_{\sigma_k}}(X_{\sigma_k}, \text{root}) \geq r S_k, \forall k \right) = 1.
\]

Once we have at our disposal (L) and (R), the proof of the above lemma is in line with Lemma 5.1 in [12]. A sketch of this proof is provided in the supplemental article [15].

We can now easily prove that a TBRW satisfying (L) and (R) is ballistic.
Proposition 4.2 (General criterion for Ballisticity). Let \( \{(T_n, X_n)\}_{n \in \mathbb{N}} \) be a process satisfying conditions (L) and (R), then there exists a positive constant \( c \), such that

\[
\liminf_{n \to \infty} \frac{\text{dist}_{T_n}(X_n, \text{root})}{n} \geq c, \quad \mathbb{P}_{T_0, x_0, s, \xi} \text{- almost surely,}
\]

for all initial conditions \((T_0, x_0)\).

**Proof.** By Strong Law of Large Numbers, given a \( \frac{1}{2}(1+\varepsilon) \)-biased random walk \( \{S_k\}_{k \in \mathbb{N}} \) and Lemma 4.1 we already have that for any initial condition \((T_0, x_0)\),

\[
\liminf_{k \to \infty} \frac{\text{dist}_{T_{\sigma_k}}(X_{\sigma_k}, \text{root})}{k} \geq \varepsilon r , \quad \mathbb{P}_{T_0, x_0, s, \xi} \text{- almost surely.}
\]

To pass from the subsequence to the whole sequence is a standard argument. The key point is to observe that by definition of the stopping times, it follows that

\[
|\text{dist}_{T_{\sigma_{k+1}}}(X_{\sigma_{k+1}}, \text{root}) - \text{dist}_{T_{\sigma_k}}(X_{\sigma_k}, \text{root})| \leq r \quad \text{and} \quad |\sigma_{k+1} - \sigma_k| \leq \exp\{r^\alpha\},
\]

hold almost surely, for every \( k \). The details can be checked in Proposition 5.3 in [12]. \( \square \)

4.2. Proof of Theorem 1.3

In light of Proposition 4.2, in order to prove Theorem 1.3, it is enough to show that the TBRW process with \( s \) odd and \( \xi \) is an independent environment satisfying condition (UE) fulfills conditions (L) and (R).

We begin recalling Corollary 2.3, which states that for \( s \) odd and under condition (UE) there exists a positive \( C \) depending on \( s \) and \( \xi \) only such that

\[
\mathbb{P}_{T_0, x_0, s, \xi} \left( \eta_\varepsilon \leq e^{\sqrt{T}} \right) \leq \frac{C}{\sqrt{\ell}}.
\]

By setting \( \ell = r^{2\alpha} \) with \( \alpha \in (0,1) \) and choosing \( r \) sufficiently large the above upper bound implies condition (L), since the above bound is uniform for \((T_0, x_0) \in \Omega_r\).

Thus, in order to prove Theorem 1.3 we are left to prove that for \( s \) odd and under (UE) the TBRW satisfies (R). However, instead of showing it directly, we will show that an auxiliary condition (R) \( \mathcal{M} \) is satisfied, which implies (R) for some values of \( \mathcal{M} \).

Let us define the condition (R) \( \mathcal{M} \): we say the TBRW satisfies (R) \( \mathcal{M} \) if there exists \( n_0 = n_0(s, M, \xi) \in \mathbb{N} \), depending only on \( s, M \) and \( \xi \) such that, for all \( n \geq n_0 \), all finite trees \( T_0 \) and all \( x_0, y \in T_0 \),

\[
\inf_{(T_0, x_0) \in \Omega} \mathbb{P}_{T_0, x_0, s, \xi} \left( \exists m \leq n : \text{dist}_{T_m}(X_m, y) \geq \log^M n \right) \geq 1 - e^{-n^{1/4}}. \quad (R_\mathcal{M})
\]

Note that (R) \( \mathcal{M} \), for \( M > 1 \), implies (R): let \( n = \exp\{r^\alpha\} \), choose \( \alpha \) such that \( \alpha M > 1 \), and choose a large enough \( r \). Thus, to prove Theorem 1.3, it remains to show that the TBRW satisfies (R) \( \mathcal{M} \) for some \( M > 1 \). For the sake of clarity and organization, we will divide the latter proof into subsections, each one corresponding to a step of the proof.
4.2.1. General idea of the proof

The proof that TBRW satisfies condition (R) is similar to the proof for the BGRW (which is the TBRW for \( s = 1 \) and \( \xi \) an i.i.d. sequence of Bernoulli’s random variables) treated in [12]. For this reason, we will trace a parallel between the latter and the general case investigated here, pointing out and proving the main modification needed in order to extend the proof to any \( s \) odd and general environment process \( \xi \) satisfying (UE).

The general idea is to bootstrap (R)\( _M \), i.e., we show that the condition is satisfied for small values of \( M \) and then use it to show that (R)\( _{M+1/2} \) is satisfied as well. Once we have proven (R)\( _M \), we combine it with (L), which says that \( X \) is unlikely to decrease its distance from the root by a certain amount. In essence we show that the process is likely to behave as follows: if (R)\( _M \) holds, in \( n \) steps we are likely to see \( X \) at distance \( \log M^2 n \) away from the root; by (L) it is unlikely that in \( n \) steps the walker backtrack half of this distance. Thus, instead of backtracking half the distance, the walker increases its distance by another \( \log M^2 n \) and this argument allows us to pass from (R)\( _M \) to (R)\( _{M+1/2} \).

4.2.2. Small distance: Proving (R)\( _{1/2} \)

In the particular case of BGRW, at each step the walker has probability at least \( p/2 \) of increasing its distance by one: if it is on a leaf, it adds a new leaf with probability \( p \) (since \( s = 1 \) it has a chance of adding a new leaf at each step) and then jumps to it with probability \( 1/2 \), and this is the worst scenario. Thus, if \( M \) is small, in \( n \) steps we are likely to see the walker taking \( \log M^2 n \) steps down in a row. For general \( s \) odd we do not have this feature, so we overcome this by looking the process only at times multiple of \( s \). The following lemma formalizes this argument.

**Lemma 4.3.** Consider a TBRW with \( s \) odd and independent environment process satisfying condition (UE). Then, it satisfies (R)\( _{1/2} \).

The proof of the above lemma relies on the following technical result.

**Lemma 4.4** (Lemma 3.6 of [12]). Suppose \( (I_j)_{j \in \mathbb{N}\setminus\{0\}} \) are indicator random variables. Assume \( \mu \) is such that \( \mathbb{P}(I_1 = 1) \geq \mu \) and \( \mathbb{P}(I_j = 1 \mid I_1, \ldots, I_{j-1}) \geq \mu, \forall j > 1 \). Then for any \( k, m \in \mathbb{N}\setminus\{0\} \)

\[
\mathbb{P}\left( \text{at least } k \text{ consecutive } 1\text{'s in the sequence } (I_j)_{j=1}^m \right) \geq 1 - (1 - \mu^k)^{\lfloor m/k \rfloor}.
\]

**Proof of Lemma 4.3.** Define, for each \( 1 \leq k \leq \lfloor n/s \rfloor \),

\[
I_k := \mathbb{1}\{\text{dist}_{T,s}(X_{sk}, y) \geq \text{dist}_{T_s(k-1)}(X_{s(k-1)}, y) + 1\},
\]

and observe that by the Markov property:

\[
\mathbb{P}_{T_0,x_0,s,\xi}(I_k = 1 \mid I_1, \ldots, I_{k-1}) \geq \inf_{(T,x) \in \Omega} \mathbb{P}_{T,x,s,\xi}(I_1 = 1) \geq \frac{\kappa}{2} \left( \frac{\mu^k}{2} \right)^{\lfloor \frac{m}{k} \rfloor} \geq \kappa 2^{-s},
\]
where the second inequality is justified by the following observation: whenever $X$ is not on a leaf, it has probability at least $1/2$ of jumping down. Thus, for our bound we may consider the worst case possible which is $X_0$ is a leaf. Since our process satisfies condition (UE), with probability at least $\kappa$ we add at least one leaf to $X_0$. Then, with probability at least $1/2$ we jump to one of the new leaves. Repeating this bouncing back argument on the leaves, paying at least $1/2$ to jump to a leaf and letting them push the walker back, we have that after $s$ steps $X_s$ is on a leaf of $x_0$ with probability at least $\kappa 2^{-\lfloor (s+1)/2 \rfloor}$. Setting $k = \log^2 n$, $m = n$ and $\mu = \kappa 2^{-s}$ in Lemma 4.4, proves the result.

4.2.3. Small growth in distance

Now, we will show the key step to derive $(R)_{M+1/2}$ from $(R)_M$. This relation relies on the following crucial lemma.

**Lemma 4.5** (Small growth in distance). Consider a TBRW with $s$ odd and independent environment process satisfying condition (UE). Also assume $(R)_M$ is satisfied for some $M \geq 1/2$. Then there exists $n_1(s, \xi, M) \in \mathbb{N}$ such that, for $n \geq n_1$, the following property holds:

$$\mathbb{P}_{T_0, x_0, s, \xi} \left( \exists t \leq n : \text{dist}_{T_t}(X_t, y) = \text{dist}_{T_0}(x_0, y) + 1 \right) \geq 1 - 2 \frac{(\log \log n)^2}{\log M n},$$

for all finite tree $T_0$ and $x_0, y \in T_0$ with $\text{dist}_{T_0}(x_0, y) \geq \log M n$.

The above lemma basically says that, when $(R)_M$ is satisfied and $x_0$ is “far” from $y$, then it is likely that the distance between the walker and $y$ will increase by at least one unit by time $n$. This probability is large enough that we are likely to see many such increases in a small time window.

The proof of Lemma 4.5 for the particular case BGRW is done in [12] and relies on condition $(R)_M$, which may be see as a global condition since it gives information on how the walker $X$ is exploring/building the tree, and also on a local feature of the TBRW in this particular case: at each step $X$ has probability at least $p/2$ (where $p$ is the parameter of the model) of increasing its distance from the root by one unit.

This local feature is important because if the walker hits the bottom of the tree many times then it is likely that after one of these hits it adds a new leaf and jump to it. However, in the case $s > 1$ this local feature is lost since the walk may hit the bottom with the wrong parity (at times not multiple of $s$) and then it goes back with probability one. Fortunately, a local correction is possible at the cost of a fixed probability depending on $s$. This is the core of our next result and will be a key step for proving Lemma 4.5.

**Lemma 4.6** (Correcting the parity). Consider a TBRW with $s$ odd and independent environment process satisfying condition (UE), then

$$\forall t \in \mathbb{N}, \inf_{T, x_0} \mathbb{P}_{T, x_0, s, \xi} \left( \exists m \leq 2s, \text{dist}_{T_{t+m}}(X_{t+m}, y) = \text{dist}_{T}(x_0, y) + 1 \mid X_t = x_0 \right) \geq \frac{\kappa}{2^{s+1}}.$$


Observe that for \( s = 1 \) the lemma follows immediately, since the walker has probability at least \( \kappa \) of attaching a new leaf on \( x_0 \) and probability at least \( 1/2 \) of jumping to it.

**Proof of Lemma 4.6.** We split the proof into cases.

Given \( T(x_0) \geq 2 \). We know that every time the walker visits \( x_0 \) it has probability at least \( \frac{d - 1}{d} \geq \frac{1}{2} \geq \frac{\kappa}{2r} \) to jump to a neighbor \( x' \) of \( x_0 \) with \( \text{dist}(x', y) = \text{dist}(x_0, y) + 1 \).

**Case \( t = 0 \mod s \).** The walker has probability at least \( \kappa/2 \) to attach at least one new leaf to \( x_0 \) and to jump to one of these new neighbors \( x' \) of \( x_0 \) with \( \text{dist}(x', y) = \text{dist}(x_0, y) + 1 \).

Case \( T(x_0) = 1 \) and \( t = ls + r \), with \( 0 < r < s \).

We know that \( x_0 \) has a unique neighbor \( x_1 \) (belonging to the path connecting \( x_0 \) and \( y \)) and it must be the case that \( \deg_T(x_1) \geq 2 \). We then consider two sub-cases: \( s - r \) is even and \( s - r \) is odd.

- **\( s - r \) is even:** After visiting \( x_0 \) the walker necessarily will visit \( x_1 \). Then, with probability at least \( 1/2 \) the walker will visits one of the neighbors of \( x_1 \) (recall that \( \deg_T(x_1) \geq 2 \)) which are not in the path connecting \( x_1 \) to \( y \) (\( x_0 \) is also possible, and all of them have the same distance from \( y \) as \( x_0 \)). It should be clear by now that the worst situation is when all such a neighbors are leaves (if not we have probability at least \( 1/2 \) to increase further the distance from \( y \), similarly to the case \( \deg_T(x_0) \geq 2 \)) and therefore we are going to consider only this case. With probability \( 1/2(s-r)/2 \), we have that \( \text{dist}_{T(l+1)}(X_{(l+1)s}, y) = \text{dist}_T(x_0, y) \). Thus, with probability at least \( \kappa \) the walker attaches a leaf on the vertex it resides on at time \( (l+1)s \) and with probability \( 1/2 \) it jumps to the new leaf. This proves that

\[
P_{T,x_0,s,\xi}(\text{dist}_{T(l+1)s+1}(X_{(l+1)s+1}, y) = \text{dist}_T(x_0, y) + 1 \mid X_{ls+r} = x_0, \deg_T(x_0) = 1) 
\geq \frac{\kappa}{2^{(s-r)/2+1}}.
\]

- **\( s - r \) is odd:** this case is similar to the previous with the only difference that at time \( (l+1)s \) the walker cannot resides on vertices with the same distance than \( y \) as \( x_0 \), and it is necessary to take some extra steps. Note that with probability at least \( 1/2(s-r-1)/2 \) we have that \( X_{(l+1)s} = x_1 \). Then, taking other \( s \) steps, regardless the value of \( \xi_{(l+1)s} \), we have probability at least \( 1/2(s+1)/2 \) of landing on \( x_0 \) or on one of the other leaves attached to \( x_1 \). This proves that

\[
P_{T,x_0,s,\xi}(X_{(l+2)s} \text{ is a leaf of } x_1 \mid X_{ls+r} = x_0, \deg_T(x_0) = 1) 
\geq \frac{1}{2^s}.
\]

Finally, with probability at least \( \kappa \) we add leaves to \( X_{(l+2)s} \), and with probability \( 1/2 \) we jump to it. Then,

\[
P_{T,x_0,s,\xi}(\text{dist}_{T(l+2)s+1}(X_{(l+2)s+1}, y) = \text{dist}_T(x_0, y) + 1 \mid X_{ls+r} = x_0, \deg_T(x_0) = 1) 
\geq \frac{\kappa}{2^{s+1}}.
\]
Now we are able to prove Lemma 4.5.

**Proof of Lemma 4.5.** Denote by \( y_* \) the vertex on the unique path from \( x_0 \) to \( y \) with \( \text{dist}_T(x_0, y_*) = \lceil \log^M n \rceil - 1 \). By condition (R)_M, there exists \( n_0 \) depending only on \( s, M \) and \( \xi \) such that:

\[
\forall n \geq n_0 : \mathbb{P}_{T,x_0,s,\xi}(\exists t \leq n : \text{dist}_{T_t}(X_t, y_*) \geq \log^M n) \geq 1 - e^{-n^{1/4}}.
\]

Let \( F \) denote the event that the walker has failed to increase its distance from \( y_\) i.e, the event that \( \text{dist}_{T_t}(X_t, y) \leq \text{dist}_T(x_0, y) \) for all \( t \leq n \). Let \( \tau_{y_*} \) be the hitting time of \( y_* \)

\[
\tau_{y_*} := \inf \{ t \in \mathbb{N} : X_t = y_* \} \in \mathbb{N} \cup \{ +\infty \}.
\]

We also define inductively \( \tau_{x_0}^{+k} \) as the \( k \)-th return time to \( x_0 \). Setting \( \tau_{x_0}^{+0} := 0 \), \( \tau_{x_0}^{+k} \) is defined as

\[
\tau_{x_0}^{+k} := \begin{cases} 
+\infty, & \text{if } \tau_{x_0}^{+(k-1)} = +\infty, \\
\inf \{ t > \tau_{x_0}^{+(k-1)} : X_t = x_0 \} \in \mathbb{N} \cup \{ +\infty \}, & \text{otherwise}.
\end{cases}
\]

Observe that, for each \( k \), we bound the probability of \( F \) as follows

\[
\mathbb{P}_{T,x_0,s,\xi}(F) \leq \mathbb{P}_{T,x_0,s,\xi}(F \cap \{ \tau_{y_*} > n \}) + \mathbb{P}_{T,x_0,s,\xi}(F \cap \{ \tau_{x_0}^{+k} < \tau_{y_*} \leq n \}) + \mathbb{P}_{T,x_0,s,\xi}(F \cap \{ \tau_{x_0}^{+k} \geq \tau_{y_*} \}) \tag{14}
\]

The bounds for the first and third terms in the RHS are the easiest ones. For the first one, observe that in that event the walker has not achieved distance at least \( \log^M n \) from \( y_* \) in \( n \) steps. Then, by condition (R)_M, this happens with probability at most \( \exp(-n^{1/4}) \). Whereas, for the third one, note that in this event, before the \( k \)-th visit of \( x_0 \) the walk \( X \) has reached \( y_* \). By a simple comparison with a simple random walk on the line connecting \( x_0 \) to \( y_* \) we bound the third term by \( k/(\lceil \log^M n \rceil - 1) \).

We finally consider the second term in the RHS of (14). In order for \( F \cap \{ \tau_{x_0}^{+k} < \tau_{y_*} \} \) to take place, it must be that \( X_t \) returns at least \( k \) times to \( x_0 \) before visiting \( y_* \) but never gets to jump to a neighbor of \( x_0 \) which does not belong to the unique path connecting \( x_0 \) to \( y \). Note that if \( s = 1 \) and (UE) holds, then at each visit to \( x_0 \) we have a bounded away from zero probability of jumping down. Thus, in this particular case, the second term of (14) decays exponentially fast in \( k \). To extend this idea to for general \( s \) odd we apply Lemma 4.6 in the following way: Let \( A_t \) denote the following event

\[
A_t := \{ \exists m \leq 2s, \text{dist}_{T_t+m}(X_{t+m}, y) = \text{dist}_T(x_0, y) + 1 \}.
\]

By Lemma 4.6, we have that, for all \( t \in \mathbb{N} \)

\[
\inf_{T,x_0} \mathbb{P}_{T,x_0,s,\xi}(A_t | X_t = x_0) \geq \frac{K}{2s+1}. \tag{15}
\]
Also notice that the following inclusion of events holds
\[ F \cap \{ \tau_{x_0}^+ < \tau_{y_*} \leq n \} \subset \bigcap_{j=0}^{\lfloor k/2s \rfloor} \left( A_{\tau_{x_0}^+}^c \cap \{ \tau_{x_0}^+ < n \} \right) . \]

Combining the Strong Markov Property with (15) leads to
\[ P_{T,x_0,s,\xi} \left( A_{\tau_{x_0}^+}^c \bigg| \tau_{x_0}^+ < n, \bigcap_{j=0}^{\lfloor k/2s \rfloor} \left( A_{\tau_{x_0}^+}^c \cap \{ \tau_{x_0}^+ < n \} \right) \right) \leq 1 - \frac{\kappa}{2s+1} . \]

The above bound implies that
\[ P_{T,x_0,s,\xi} \left( F \cap \{ \tau_{x_0}^+ < \tau_{y_*} \leq n \} \right) \leq \left( 1 - \frac{\kappa}{2s+1} \right)^{\lfloor k/2s \rfloor} \leq e^{\log(1-\kappa/2s+1) \lfloor k/2s \rfloor} . \]

Overall, we obtain
\[ P_{T,x_0,s,\xi}(F) \leq e^{-n^{1/4}} + \frac{k}{\log^M n - 1} + e^{\log(1-\kappa/2s+1) \lfloor k/2s \rfloor} . \]

Setting \( k = \log \log^2 n \), proves the lemma. \( \square \)

4.2.4. Iterating the argument.

In Lemma 4.3, we have proved that under \( \text{(UE)} \), and \( s \) odd, \( \text{(R)}_{1/2} \) holds. Now, in order to prove that TBRW satisfies condition \( \text{(R)}_M \), for some \( M > 1 \) (and thus condition \( \text{(R)} \)) we prove the following lemma.

**Lemma 4.7.** Consider a TBRW with \( s \) odd and independent environment process satisfying condition \( \text{(UE)} \). Then, if \( \text{(R)}_M \) holds, so does \( \text{(R)}_{M+1/2} \).

Once we have Lemma 4.5, the proof of Lemma 4.7 is similar to the proof for BGRW in [12]. A sketch of this proof is provided in the supplemental article [15], and we refer the reader to Proposition 3.4 in [12] for further details.

5. Structural knowledge: the environment growth

In this section we analyze the growth of the sequence of rooted random trees \( \{ T_n \}_{n \in \mathbb{N}} \) generated by a TBRW process. We denote by \( h \) the height functional defined for each tree \( T \) as \( h(T) = \max_x \text{dist}_T(x, \text{root}) \). From Theorem 1.3 if \( s \) is odd and the environment process satisfies condition \( \text{(UE)} \) then we have that
\[ \liminf_{n \to \infty} \frac{h(T_n)}{n} > 0 \Rightarrow P_{T_0,x_0,s,\xi} \rightarrow \text{a.s.} . \]
This means that the height of the generated trees grows linearly in time. On the other
hand, if \( s \) is even and the environment satisfies condition (I), by Theorem 1.1 item (ii),
the tree height stops growing almost surely. What does happen to the sequence of random
variables \( \{h(T_n)\}_{n \in \mathbb{N}} \) under condition (S)? To begin answering the latter question, we
recall the example from Section 1.3: the independent environment \( \xi \) with \( \xi_j \sim \text{Ber}(j^{-2}) \)
satisfies condition (S), however the sequence \( \{h(T_n)\}_{n \in \mathbb{N}} \) is almost surely finite for any
value of \( s \), since the process, eventually, stops adding new leaves. To avoid the above
situation, we also impose condition (UE) and prove the following proposition.

**Proposition 5.1.** Consider a \((2k, \xi)\)-TBRW process whose environment process \( \xi \) satisfies conditions (S) and (UE). Then, for every initial state \((T_0, x_0)\)
\[
h(T_n) \nearrow +\infty, \quad \mathbb{P}_{T_0, x_0, s, \xi} \text{almost surely.}
\]

**Proof.** Observe that a vertex which maximizes the height of the tree is necessarily a
leaf, whose parent has only leaves as children. Call this parent \( z \). Then \( z \) will be visited
infinitely many times, but we need to guarantee that it will be visited infinitely many
times with opposite parity as that of the walker. Indeed, when \( z \) is visited with opposite
parity, then with probability greater than 1/2 one of its leaves will be visited on the next
transition when the walker will have the same parity as this visited leaf. Therefore, at
any of these times when the walker visits \( z \) with opposite parity, we have a probability of
at least \( 2^{-s/2} \) to jump to one of its leaves at times multiple of \( s \), and then a probability
of at least \( \kappa \) (by (UE)) to increase by one the height of the tree. If this last event occurs
with probability one, then with probability one the height will increase indefinitely.

Under condition (S), consider a realization of the TBRW from time 0 to time \( k \), for
some fixed \( k \geq 0 \). Let \( z \in T_k \) such as before. We have to show that \( z \) is visited infinitely
many times with opposite parity as that of the walker. From now on we call this parity
the “right parity”. Put \( y \) as the neighbor of the root such that \( z \) belongs to the branch
of \( T_k \) starting at \( y \). Following our proof of recurrence, we only have to show that \( y \) is visited infinitely
many times when the walker has the right parity. By Lemma 3.1 and
Corollary 3.3 both the edges \( e_1 = (\text{root}, \text{root}) \) and \( e_2 = (\text{root}, y) \) are traversed infinite
many times. Define the sequence of Bernoulli random variables \( \{a_j\}_{j \geq 1} \) as \( a_j = 1 \) if on
the \( j \)-th crossing, after time \( k \), of either \( e_1 \) or \( e_2 \), the edge crossed is \( e_1 \), otherwise set
\( a_j = 0 \). The sequence \( \{a_j\}_{j \geq 1} \) is i.i.d. From its mixing property, \( y \) is visited on both even
and odd times infinitely often, which implies that \( y \) is visited infinitely many times when
the walker has the right parity. \( \square \)

We finish this section proving Proposition 1.2 which states that, for \( s \) even, condition (S) combined with (UE) implies null recurrence.

**Proof of Proposition 1.2.** If the environment satisfies condition (UE) we can use
Lemma 2.4, assuring that the expected hitting time of vertices sufficiently far away
from the initial state are infinite. By Proposition 5.1, under conditions (S) and (UE) the
height of the tree goes to infinity. Therefore, eventually the walker will be sufficiently far
away from any vertex. Using the strong Markov property, the proof is complete. \( \square \)
6. Final comments

We end this paper making some comments which could lead to interesting questions.

6.1. Finiteness of $T_0$

Recall that in the results for $s$ even from Section 3, we required the initial tree $T_0$ to be finite. However the definition in Section 1.1 consider any initial locally finite tree. More generally, the initial state of the TBRW may be sampled according to some distribution $\nu$ over the space of pairs $(T, x)$, where $T$ is a locally finite tree, possibly infinite. Allowing infinite trees may lead to different questions from those we have addressed in this paper. For instance, in the infinite case one may not observe the trapped regime we proved for some environment conditions when $s$ is even, as is illustrated in the example below.

Example 6.1 (A heavy infinity tree). For each natural $n \geq 1$, let $d_n$ be $n^4$. Now, consider the infinity tree $T_\infty$ defined recursively in such way that all vertices at level $n$ have degree $d_n$. Also consider the deterministic environment $\xi$ defined by $\xi_{\xi j} = j^2$. Thus, $S_n \approx n^3$ and consequently the $(\xi, s)$–TBRW model satisfies condition (1) for $f(n) = n^3$. However, regardless the parity of $s$, we have the following

$$P_{T_\infty, \text{root}, s, \xi} \left( \text{dist}_{T_n}(X_n, \text{root}) = n \right) \geq \prod_{j=1}^{n} \left( 1 - \frac{1 + j^2}{j^4} \right) > 0.$$ 

Thus, the walker has positive probability of ignoring all the leaves it adds to $T_\infty$ and simply goes “down the tree”.

In the above example the probability of always going down can be made arbitrarily close to one by considering even heavier trees. Thus, for a TBRW whose initial state is sampled from a distribution $\nu$ supported on heavy infinity trees, Theorem 1.1 does not hold. A natural question is: What are the conditions on $\nu$ in order that Theorem 1.1 still holds? Our results cover the case in which $\nu$ is supported on the subset of finite trees.

6.2. Random tree process

Except from Section 5, all the results in this paper regard the walker $X$, and we have approached the TBRW as a random walk on random environment. However, one may also study the TBRW from the perspective of the random trees $\{T_n\}_{n \in \mathbb{N}}$, and see TBRW as a random graph model: starting from $T_0$, at every time $n$ multiple of $s$, a random number $\xi_n$ of new leaves are added to the current tree and attached to the vertex where the random walk $X$ resides at that time (see, Section 1.1). From the perspective of the random trees $\{T_n\}_{n \in \mathbb{N}}$ questions concerning the structure and degree distribution of the random sequence of trees $\{T_n\}_{n \geq 0}$ stand out. For instance, it would be interesting to study if TBRW is capable of generating trees whose degree distribution obeys a power
law. That is, under which assumption on \( \{\xi_n\}_{n \geq 0} \), there exist a positive constant \( C \) and positive exponent \( \gamma \) (possibly depending on \( \{\xi_n\}_{n \geq 0} \)) such that for every fixed \( d \geq 1 \)

\[
\lim_{n \to \infty} \frac{1}{|V(T_n)|} \sum_{x \in V(T_n)} 1\{\deg_{T_n}(x) = d\} = C d^{-\gamma}, \quad P_{T_0,x_0,s,\xi^-} \text{ a.s.},
\]

where, \( V(T_n) \) denotes the vertex set of \( T_n \) and \( \deg_{T_n}(x) \) the degree of \( x \) in \( T_n \).

### 6.3. Ellipticity

In the context of RWRE, efforts have been made towards dropping the uniformly elliptic condition. For instance, in [3, 4, 13] authors have obtained ballisticity criteria under elliptic condition. On the other hand, in the context of TBRW, an ellipticity condition means that the probability of adding at least one leaf is positive for each time multiple of \( s \) but it vanishes in the long run. More formally, we consider \( \lim_{n \to \infty} P(\xi_n \geq 1) = 0 \).

It is clear that if \( P(\xi_n \geq 1) \) goes fast enough to zero, Borel-Cantelli lemma implies that the process is positive recurrent since the walker stops to add new leaves to the graph eventually. The interesting question here would be to find other regimes for the decreasing rate of \( P(\xi_n \geq 1) \) for which zero speed and ballisticity are also observed.

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### Supplementary Material

Supplementary material for “Tree Builder Random Walk: recurrence, transience and ballisticity”


### References

Tree Builder Random Walk: recurrence, transience and ballisticity


