Precise large deviations for dependent subexponential variables

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In this paper we study precise large deviations for the partial sums of a stationary sequence with a subexponential marginal distribution. Our main focus is on distributions which either have a regularly varying or a lognormal-type tail. We apply the results to prove limit theory for the maxima of the entries large sample covariance matrices.

Keywords: large deviation probability, subexponential distribution, maximum domain of attraction, Gumbel distribution, Fréchet distribution, regular variation, stationary sequence.

1. Introduction

We consider a (strictly) stationary real-valued sequence $(X_i)$ with generic element $X$ and distribution function $F$ with finite first moment. The corresponding centered partial sums are given by

$$S_0 = 0, \quad S_n = X_1 + \cdots + X_n - n \mathbb{E}[X], \quad n \geq 1. \quad (1.1)$$

To ease notation we will always assume that $X$ is centered. We also assume that $F$ is subexponential.

1.1. Subexponential distributions

For the moment assume $(X_i)$ are iid. Following the classical definition of Čistyakov (1964) (cf. Embrechts et al. (1997), p. 39), $F$ is subexponential if $X$ is non-negative and has the tail-equivalence property for convolutions, i.e.,

$$\mathbb{P}(S_n > x) \sim n (1 - F(x)) = n \bar{F}(x), \quad n \geq 2, \quad x \to \infty; \quad (1.2)$$

we write $F \in \mathcal{S}^+$. Here $f(x) \sim g(x)$ for positive functions $f, g$ means that $f(x)/g(x) \to 1$ as $x \to \infty$. In this paper we will consider two-sided subexponential distribution functions, i.e., $X^+ = X \vee 0$ has a subexponential distribution and a tail balance condition holds

$$\lim_{x \to \infty} \frac{\mathbb{P}(X > x)}{\mathbb{P}(|X| > x)} = p_+,$$

$$\lim_{x \to \infty} \frac{\mathbb{P}(-X > x)}{\mathbb{P}(|X| > x)} = p_-,$$

$$\lim_{x \to \infty} \frac{\mathbb{P}(|X| > x)}{\mathbb{P}(|X| > x)} = 1 \quad (1.3)$$
for some $p_+ > 0, p_- > 0$, and we write $\mathcal{S}$ for this enlarged class of distributions. The property (1.2) has the interpretation that $S_n$ and $M_n = \max(X_1, \ldots, X_n)$ are tail-equivalent for every $n$. Therefore it is considered a very natural class of heavy-tailed distributions which has multiple applications in insurance mathematics, telecommunications, queuing and branching theory. Textbook treatments can be found in Embrechts et al. (1997), Rolski et al. (1999), and Asmussensen (2003).

The class $\mathcal{S}_+$ covers a wide range of tail behaviors from power laws with certain moments infinite to semi-exponential tails such that $X$ has all moments finite but no moment generating function. We will mainly be interested in two sub-classes of distribution functions $F \in \mathcal{S}$:

- **RV($\alpha$).** We say that $X$ and its distribution $F$ are regularly varying with index $\alpha > 0$ ($F \in \text{RV}(\alpha)$) if $F(x) = 1 - F(x) = L(x)x^{-\alpha}$ for some slowly varying function $L$.
- **LN.** This class consists of subexponential distributions $F$ such that $F(x) = \exp(-S(x))$ where $S$ is a slowly varying function such that $S(x)/\log x \to \infty, \quad x \to \infty$.

Well-known representatives $F \in \text{RV}(\alpha)$ with positive tail index $\alpha$ are the Pareto, Burr, student distributions. A representative of LN is the (standard) lognormal distribution with tail

$$
P(X > x) \sim \frac{e^{-(\log x)^2/2}}{\sqrt{2\pi \log x}} = e^{-(\log x)^2/2 + \log(\sqrt{2\pi \log x})}.
$$

(1.4)

An interesting third subclass of $\mathcal{S}_+$ are the Weibull-type distributions with tail $F(x) = \exp(-x^\alpha L(x))$ for a slowly varying function $L$ and $\alpha \in (0,1)$. Unfortunately, the techniques developed in this paper fail for these distributions, see Remark 2.3 below.

**1.2. Precise large deviations of subexponential type in the iid case**

Early on, it was discovered that the defining property of a subexponential distribution (1.2) extends to situations when $n \to \infty$ and $x = x_n \to \infty$. To be more precise, a relation of the type

$$
\sup_{x > t_n} \left| \frac{P(S_n > x)}{n F(x)} - 1 \right| \to 0, \quad n \to \infty,
$$

(1.5)

holds for a suitable sequence $(t_n)$; we call it a separating sequence, and (1.5) a (precise) large deviation of subexponential type. As a matter of fact, Cline and Hsing (1998) discovered that $F \in \mathcal{S}$ is an “almost” necessary and sufficient condition for (1.5) to hold. Pioneering work on large deviations of type (1.5) is due to A.V. Nagaev (1969a,b, 1977), S.V. Nagaev (1965, 1979), Rozovskii (1994); see also Cline and Hsing (1998), Denisov et
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al. (2008). Large deviations for the sample paths of a Lévy process and random walks with regularly varying increments were considered by Hult et al. (2005), Rhee et al. (2019).

The perhaps best known result in this context is due to S.V. Nagaev (1979). For \( F \in \text{RV}(\alpha) \) and \( \alpha > 2 \), assuming \( E[X] = 0 \) and \( \text{var}(X) = 1 \), he proved that (1.5) holds for \( x > t_n = \sqrt{(\alpha - 2)n \log n} \), while for \( x < t_n \) one has

\[
\sup_{x < t_n} \left| \frac{P(S_n > x)}{\Phi(x/\sqrt{n})} - 1 \right| \to 0, \quad n \to \infty, \tag{1.6}
\]

where \( \Phi \) is the standard normal distribution function.

Results of the types of (1.5) and (1.6) are also valid for various other distributions in \( \mathcal{S} \). In particular, the lognormal distribution with tail (1.4) satisfies (1.5) for \( x \gg t_n \) and (1.6) for \( x \ll t_n \) where \( t_n = \sqrt{n \log n} \). Rozovskii (1994) found that the separating sequences \((t_n)\) in (1.5) and (1.6) have to be distinct if \( F \) is lighter than the tail of a lognormal distribution.

Extensions of large deviations of subexponential type to stationary sequences only exist in a few cases. Mikosch and Samorodnitsky (2000) proved large deviations of subexponential type for regularly varying linear processes driven by iid regularly varying noise. The main difference to the iid case is that the limit of \( \frac{P(S_n > x)}{n \Phi(x)} \) converges uniformly for \( x \gg t_n \) to a constant depending on the coefficients of the linear process and the tail index of the noise. This fact shows that extremal clustering in the \( X \)-sequence causes that exceedances of \( S_n \) above high thresholds \( x \) appear in clumps and not separated from each other, and the limiting constant is a measure of the size of these clumps. Solutions to affine stochastic recurrence equations \( X_t = A_t X_{t-1} + B_t, \ t \in \mathbb{Z} \), for an iid sequence \((A_t, B_t), \ t \in \mathbb{Z}\), may have power-law tails \( P(\pm X > x) \sim c_\pm x^{-\alpha} \) for some \( \alpha > 0 \) either due to regular variation of \( B_1 \) with index \( \alpha \) and \( E[|A_1|^{\alpha}] < 1 \) (the so-called Grincevičius-Grey case) or due to the condition \( E[|A_1|^{\alpha}] = 1 \) (the so-called Kesten-Goldie case); see Section 3.4.2 in Buraczewski et al. (2016) for an overview. Buraczewski et al. (2013) proved large deviation results of subexponential type in the Kesten-Goldie case, and Konstantinides and Mikosch (2005) in the Grincevičius-Grey case. Mikosch and Wintenberger (2013, 2016) derived large deviation results for regularly varying Markov chains and \( m \)-dependent processes and applied these results to get bounds for ruin probabilities.

1.3. Goals of this paper

In this paper we aim at proving analogs of the subexponential large deviation results for a stationary dependent sequence \((X_t)\). In most cases, we have to restrict ourselves to an \( m \)-dependent sequence, i.e., the dependence ranges only over \( m \) lags. We work under the heavy-tail assumption \( F \in \mathcal{S} \) which is a natural condition, as we explained in Section 1.2. We also have to impose an asymptotic tail independence condition on the distributions of the pairs \((X_0, X_h)\) for \( 1 \leq h \leq m \). Under the aforementioned conditions and for \( F \in \text{RV}(\alpha) \) and \( F \in \text{LN} \) we prove results of the type (1.5). The strong asymptotic tail
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independence conditions ensure that (1.5) is valid for suitable sequences \((t_n)\). Based on
the \(m\)-dependence of \((X_t)\) we make heavy use of the known large deviation results in the
iid case. This is the topic of Section 3.

In Section 4 we study subexponential large deviations for a linear process driven by
an iid noise sequence with a common subexponential distribution \(F\) in the class \(\text{LN}\). In
this case, a result of type (1.5) does in general not hold but the denominator \(n \overline{F}(x)\) has
to replaced by \(n \overline{F}(x/m_0)\) for some number \(m_0\) which depends on the coefficients of the
linear process. The proof makes heavy use of the linear structure and exploits the known
large deviation results for an iid sequence. We also mention that the distribution of \(X\) is
tail-equivalent to the subexponential noise distribution.

In Section 5 we show how large deviations of subexponential type can be applied to
determine the limits of the maxima of the diagonal or off-diagonal entries of a large
sample covariance matrix with row-wise dependent entries.

2. Preliminaries

2.1. Maximum domains of attraction

Assume that \((X_i)\) is iid with common distribution \(F\).

The condition \(F \in \text{RV}(\alpha)\) for \(\alpha > 0\) is equivalent to membership of \(F\) in the maximum
domain of attraction of the Fréchet distribution \(\Phi_\alpha\) \((F \in \text{MDA}(\Phi_\alpha))\). This means that
there exist constants \(a_n > 0\) such that
\[
\mathbb{P}(a_n^{-1} M_n \leq x) \to \Phi_\alpha(x) = e^{-x^{-\alpha}}, \quad x \geq 0, \quad n \to \infty.
\]

For \(F \in \text{LN} \cap \mathcal{S}\) we also require that it is a member of the maximum domain of attraction
of the Gumbel distribution \(\Lambda\) \((F \in \text{MDA}(\Lambda))\), i.e., there exist constants \(c_n > 0, d_n \in \mathbb{R}\)
such that
\[
\mathbb{P}(c_n^{-1}(M_n - d_n) \leq x) \to \Lambda(x) = e^{-e^{-x}}, \quad x \in \mathbb{R}, \quad n \to \infty.
\]

According to the Pickands-Balkema-de Haan Theorem \((\text{Pickands (1975); Balkema and
de Haan (1974)})\), cf. Theorem 3.4.5 in Embrechts et al. (1997)) a distribution with infinite
right endpoint \(F \in \text{MDA}(\Lambda)\) if and only if there exists a positive function \(a\) with Lebesgue
density \(a'\) such that \(a'(x) \to 0\) as \(x \to \infty\) and
\[
\frac{\overline{F}(x + ya(x))}{\overline{F}(x)} \to e^{-y}, \quad x \to \infty, \quad y \in \mathbb{R}.
\] (2.1)

The auxiliary function \(a\) can be chosen as the mean-excess function of \(F\)
\[
a(x) = \int_x^\infty \frac{\overline{F}(y)}{\overline{F}(x)} \, dy, \quad x > 0;
\]
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cf. Resnick (1987), Proposition 1.9. We have $F \in \text{MDA}(\Lambda)$ if and only if

$$F(x) = c(x) \exp \left(- \int_z^x \frac{1}{a(t)} \, dt\right), \quad x > z,$$

(2.2)

for some $z$ and $c(x) \to c > 0$ as $x \to \infty$.

2.2. Long-tailed distributions

A distribution function $F$ is said to be long-tailed if

$$\lim_{x \to \infty} \frac{F(x + y)}{F(x)} = 1, \quad \text{for any } y > 0.$$

For the properties of long-tailed distributions we refer to Foss et al. (2013). In particular, $F \in S$ implies long-tailedness of $F$; see Lemma 3.4 in Foss et al. (2013). Moreover, for each long-tailed distribution $F$ there exists a non-decreasing function $h$ with $h(x) \uparrow \infty$ as $x \to \infty$ such that

$$\lim_{x \to \infty} \frac{F(x + h(x))}{F(x)} = 1,$$

and $F$ is called $h$-insensitive. In particular, $F \in \text{MDA}(\Lambda)$ satisfies (2.1) for some auxiliary function $a(x) \to \infty$. Hence we can choose $h(x) = o(a(x))$, and if $F \in \text{MDA}(\Phi_\alpha)$ we can take any function $h$ with $h(x) = o(x)$ as $x \to \infty$.

2.3. Condition (C)

We consider a stationary sequence $(X_i)$ with mean zero and partial sum process $(S_n)$ given in (1.1). In this section we assume that $F \in \text{MDA}(\Lambda) \cap \text{LN}$. Hence, in particular, $F \in S$, $F$ has infinite right endpoint and $S(x) = -\log F(x)$ is slowly varying such that $S(x)/\log x \to \infty$ as $x \to \infty$. Characterizations of MDA($\Lambda$) are given in Section 2.1.

In what follows, we introduce and discuss a set of conditions which will be assumed in our main result, Theorem 3.1. A crucial object in this context is a positive function $g$ which describes the region $(t_n, \infty)$ where the large deviation results hold.

**Condition (C)**

$$g(x) \uparrow \infty \text{ as } x \to \infty \text{ and there is } C > 0 \text{ such that for large } x,$$

$$g(x) \leq C \frac{x}{S(x)}.$$

(2.3)
C2 There is a sequence $t_n \to \infty$ such that for any $\delta > 0$,

$$\sup_{x > t_n, \delta} \left| \frac{S(x)}{S(g(x))} - 1 \right| \to 0 \quad \text{and} \quad \frac{g(t_n)}{\sqrt{n}} \to \infty, \quad n \to \infty,$$

(2.4)

and for an iid sequence $(X'_i)$ with common distribution $F$ and partial sums $S'_n = X'_1 + \cdots + X'_n$ we have the large deviation result

$$\lim_{n \to \infty} \sup_{x > t_n, \delta} \left| \frac{\P(S'_n > x)}{nF(x)} - 1 \right| = 0, \quad \text{for any } \delta > 0.$$  
(2.5)

C3 $(X_i)$ is $m$-dependent for some $m \geq 1$, and for any $\varepsilon > 0$,

$$\lim_{x \to \infty} \frac{\P(|X_0| > \varepsilon g(x), |X_h| > \varepsilon x)}{F(x)} = 0, \quad h = 1, \ldots, m.$$  
(2.6)

The size of $g(x)$. It follows from the monotone density theorem (cf. Theorem 1.7. in Bingham et al. (1987)) and (2.2) that

$$\frac{a(x)S(x)}{x} \to \infty, \quad x \to \infty.$$

Therefore $g(x) = o(a(x))$ in agreement with condition (2.3) which also implies that $g(x)/x \to 0$ since $S(x) \to \infty$ for $F \in \text{MDA}(\Lambda)$ with infinite right endpoint. Moreover, we conclude from (2.1) that for any $c \in \R$,

$$\lim_{x \to \infty} \frac{\P(X > x - cg(x))}{\P(X > x)} = 1,$$

(2.7)

i.e., $F$ is $(cg)$-insensitive for any $c \in \R$. The latter condition will be frequently used in the remainder of this paper. On the other hand, the first condition in (2.4) ensures that $g(x)$ increases not too slowly.

Lemma 2.1. If (2.4) holds then for any $\varepsilon > 0$,

$$\lim_{n \to \infty} \sup_{x > t_n} \frac{n\overline{F}(\varepsilon x)\overline{F}(\varepsilon g(x))}{\overline{F}(x)} = 0.$$  
(2.8)

Proof. Since $F \in \text{LN} \cap \text{MDA}(\Lambda)$ we have $\lim_{x \to \infty} S(x)/\log x = \infty$. It follows from (2.4) uniformly for $x > t_n$,

$$S(x) \sim S(g(x)) \geq S(g(t_n)) \geq S(\sqrt{n}) \gg \log n.$$

Hence by slow variation of $S(x)$ and (2.4), uniformly for $x > t_n$,

$$\frac{n\overline{F}(\varepsilon x)\overline{F}(\varepsilon g(x))}{\overline{F}(x)} = \exp \left\{ \log n - S(x)(1 + o(1)) \right\} \to 0.$$
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Example 2.2. If we choose \( g(x) = x/S(x) \) and \( S(x) = f(\log x) \) for a differentiable regularly varying function \( f \) with index \( \alpha > 1 \) then (2.3) and (2.4) are satisfied. Indeed, if \( T(x) \) is slowly varying then, according to Bojanic and Seneta (1971), the condition

\[
\lim_{x \to \infty} \frac{xT'(x)}{T(x)} \log T(x) = 0
\]

implies that for any \( \rho \in \mathbb{R} \)

\[
\lim_{x \to \infty} \frac{T(x T^\rho(x))}{T(x)} = 1
\]

holds and \( T(x) = S(x) \) satisfies (2.9). In particular, one can choose \( S_1(x) = c(\log x)^\alpha (1 + o(1)) \), or \( S_2(x) = \exp(c(\log x)^\alpha)(1 + o(1)) \) for \( c > 0 \), and Lemma 2.1 applies.

Remark 2.3. Note that Weibull-type distributions do not satisfy conditions \( C_1 - C_2 \). Indeed, if a distribution \( F \) has a tail \( \mathcal{F}(x) = \exp(-x^\alpha L(x)) \) for a slowly varying function \( L \) and \( \alpha \in (0,1) \), then \( C_1 \) implies that \( g(x) \leq C x^{1-\alpha}/L(x) \) as \( x \to \infty \). Thus, \( S(x) / S(g(x)) \to \infty \) as \( x \to \infty \) and (2.4) is not satisfied.

2.4. Time series models satisfying (C)

In the previous section we verified conditions \( C_1 - C_2 \) on some examples. These conditions depended only on the marginal distribution \( F \) of \( (X_i) \). In this section we provide some examples of time series for which we can verify condition \( C_3 \) which depends on the pairwise dependence structure of \( (X_0, X_h) \), \( h = 1, \ldots, m \). Here and in what follows, \( c \) denotes any positive constant whose value is not of interest.

Example 2.4. Let \( Y = (Y_i) \) be a Gaussian \( m \)-dependent stationary sequence with mean \( \mu \), variance \( \sigma^2 > 0 \) and correlation function \( \rho(h) < 1 \) for \( h \neq 0 \). Consider a stationary sequence \( X = e^{b(Y)} = (e^{b(Y_i)}) \), where \( b(x) = \text{sign}(x) |x|^\alpha \), \( \alpha \in (0,2) \). We observe that for large \( x \)

\[
S(x) = \frac{1}{2\sigma^2} (\log x)^{2/\alpha}(1 + o(1)),
\]

thus \( S(x) \) satisfies (2.9) by Example 2.2 and then \( g(x) = x/S(x) \) satisfies \( C_1 \). The conditions \( g(t_n) / \sqrt{n} \to \infty \) and (2.5) hold with \( t_n \gg \sqrt{n}(\log n)^{2/\alpha} \). Indeed, according to Rozovskii (1994), the large deviation result (2.5) holds with \( t_n \gg \sqrt{n}(\log n)^{2/\alpha - 1} \) for \( \alpha \in (0,1] \) and with \( t_n \gg \sqrt{n}(\log n)^{1/\alpha} \) for \( \alpha \in (1,2) \). Note also that for \( \alpha = 1 \) the random vector \( (X_1, \ldots, X_d) \), \( d \in \mathbb{N} \), has a multivariate lognormal distribution in the sense of Asmussen and Rojas-Nandayapa (2008).
Next we verify $C_3$. We assume $\mu = 0$ and observe that $\rho(h) = 0$ for $h > m$. An adapted version of Shibuya's classical estimate, Shibuya (1960), and the tail-balance condition \((1.3)\) yield for $\varepsilon > 0$ and large $x$,

$$
P(|X_0| > \varepsilon x, |X_h| > \varepsilon g(x)) \leq cP(X_0 > \varepsilon g(x), X_h > \varepsilon g(x)) \leq cP(\min(Y_0, Y_h) > (\log(\varepsilon g(x)))^{1/\alpha}) \leq c\Phi\left(\frac{2(\log(\varepsilon g(x)))^{1/\alpha}}{\sqrt{1 + \rho(h)}}\right) = o\left(\Phi\left(\frac{(\log x)^{1/\alpha}}{\sigma}\right)\right),$$

where $\Phi$ is the standard normal distribution function. In the last step we used the facts that $\rho(h) < 1$ and

$$
\frac{2(\log(\varepsilon g(x)))^{1/\alpha}}{\sqrt{1 + \rho(h)}} = \frac{2}{\sqrt{1 + \rho(h)}}(1 + o(1)).
$$

We conclude that for $h \geq 1$,

$$
P(|X_0| > \varepsilon \sqrt{x}, |X_h| > \varepsilon g(x)) = o(F(x)), \quad x \to \infty,
$$

and thus $C_3$ is satisfied.

**Example 2.5.** Let $(Y_i)$ be an iid sequence with common distribution given by

$$
P(Y > x) = \exp\left(-\left(\frac{\log x}{\alpha}\right)^{\alpha}\right), \quad x > 1, \quad \text{(2.11)}
$$

for some $\alpha > 1$. The sequence

$$
X_i = \min(a_0 Y_i, a_1 Y_{i+1}, \ldots, a_m Y_{i+m})
$$

for some positive $a_0, \ldots, a_m$ is $m$-dependent, stationary and has tail

$$
P(X > x) = \exp\left(-\sum_{i=0}^{m} S(x/a_i)\right) = \exp\left(-m S(x)(1 + o(1))\right).
$$

Thus the distribution of $X$ is also subexponential. This follows by checking Pitman’s condition, Pitman (1980): integrability of the function $\exp(xF'(x)/F(x))F'(x)$ on $(0, \infty)$. We verify that \((C)\) holds with $g(x) = x/(\log x)^{\alpha}$. $C_1$ is immediate. $C_2$ follows by virtue of Example 2.2. It remains to verify $C_3$. Direct calculation yields for $\varepsilon > 0$ and $h = 1, \ldots, m$,

$$
\frac{P(X_0 > \varepsilon g(x), X_h > \varepsilon x)}{P(x)} \leq \frac{P(\min(a_0 Y_0, \ldots, a_{h-1} Y_{h-1}) > \varepsilon g(x), \min(a_0 Y_h, \ldots, a_m Y_{m+h}) > \varepsilon x)}{P(\min(a_0 Y_0, \ldots, a_m Y_m) > x)} \leq \exp\left(\sum_{i=0}^{m} S(x/a_i) - \sum_{i=0}^{h-1} S(\varepsilon g(x)/a_i) - \sum_{i=0}^{m} S(\varepsilon x/a_i)\right) = \exp\left((1 + o(1))S(x)((m + 1) - (m + 1 + h))\right) \to 0, \quad x \to \infty.
$$
Example 2.6. Consider the stochastic volatility model

$$X_i = \sigma_i Y_i,$$

where \((\sigma_i)\) is a stationary sequence with \(P(a \leq \sigma_1 \leq b) = 1, 0 < a < 1 < b,\) and \((Y_i)\) is an iid sequence with common distribution function \(F_Y(x) = 1 - e^{-S_Y(x)},\) such that \(F_Y \in \text{MDA}(A) \cap S,\) it satisfies the tail-balance condition (1.3), and (2.9) holds for \(S_Y.\) We also assume that the distribution \(F\) of \(X\) is subexponential. This is not automatic even though it is easily verified that \(S(x) = S_Y(x)(1 + o(1)),\) hence \(S(x)\) is slowly varying, but this fact does not necessarily imply subexponentiality of \(F:\) see comments on p. 52 in Embrechts et al. (1997). Subexponentiality of \(F\) can be verified in simple situations, e.g. if \(\sigma\) has a binomial distribution on \((a,b),\) by using Pitman’s aforementioned condition.

We choose as before \(g(x) = x/S(x)\) and assume that it increases. Hence \(C_1 - C_2\) are satisfied. It remains to show \(C_3.\) Applying the slow variation of \(S(x),\) the tail-balance condition (1.3) and \(C_2,\) we have for \(h = 1, \ldots, m\) and \(\varepsilon > 0,

$$\frac{P(|X_0| > \varepsilon g(x), |X_h| > \varepsilon x)}{F(x)} \leq \frac{P(|Y_0| > g(x)/b, |Y_h| > x/b)}{F_Y(x/a)} \leq \frac{e^{-S_Y(x)(1 + o(1))}}{F_Y(x/a)} \to 0, \quad x \to \infty.$$

2.5. Regularly varying stationary sequences

A random vector \(X\) with values in \(\mathbb{R}^d\) and its distribution are regularly varying with index \(\alpha > 0\) if

$$P\left(\left(\frac{X}{x}, \frac{X}{|X|}\right) \in \cdot \mid |X| > x\right) \overset{w}{\to} P((Y, \Theta) \in \cdot), \quad x \to \infty,$$

where \(Y\) is Pareto distributed, \(P(Y > x) = x^{-\alpha}, x > 1,\) independent of \(\Theta;\) see Resnick (1987, 2007) for some reading on multivariate regular variation. Davis and Hsing (1995) introduced regularly varying stationary sequences \((X_t)\) by assuming that each lagged vector \((X_0, \ldots, X_h), h \geq 0,\) is regularly varying with index \(\alpha.\) Basrak and Segers (2009) characterized such sequences by showing that regular variation of \((X_t)\) is equivalent to the existence of a spectral tail process \((\Theta_t)\) defined via the limit relations

$$P(x^{-1}(X_0, \ldots, X_h) \in \cdot \mid |X_0| > x) \overset{w}{\to} P(Y(\Theta_0, \ldots, \Theta_h) \in \cdot), \quad h \geq 0, x \to \infty.$$

where \(Y\) is Pareto distributed and independent of \((\Theta_t).\) Obviously, \(|\Theta_0| = 1.\) If \(\Theta_t = 0\) a.s. for \(t \neq 0\) then \((X_t)\) is called asymptotically independent. In Section 3.2 we will work under this assumption.

We will work under the following set of conditions.
Condition (RV)

**RV$_1$** The separating sequence $(t_n)$ satisfies

$$
\lim_{n \to \infty} \frac{t_n}{\sqrt{n \log n}} = \infty.
$$

(2.13)

**RV$_2$** $F \in \text{RV}(\alpha)$ for some $\alpha > 2$ and satisfies the tail-balance condition (1.3).

**RV$_3$** $(X_t)$ is $m$-dependent for some $m \geq 1$ and satisfies

$$
\lim_{x \to \infty} P(|X_h| > x \mid |X_0| > x) = 0, \quad h = 1, \ldots, m.
$$

(2.14)

Condition RV$_2$ implies in particular that $E[|X|^{2+\delta}] < \infty$ for $0 < \delta < \alpha - 2$. Moreover, $S(x) = -\log F(x) = \alpha \log x - \log L(x)$ for some slowly varying function $L$. We conclude that any function $g$ satisfying $g(x)/x \to 0$ as $x \to \infty$ has the property

$$
\lim_{x \to \infty} \frac{F(x + g(x))}{F(x)} = 1.
$$

Condition RV$_3$ implies the asymptotic independence of the sequence $(X_t)$, i.e., $\Theta_t = 0$ a.s., $t \neq 0$, in (2.12). In particular, $(X_t)$ is regularly varying with index $\alpha$. By regular variation we can rewrite (2.14) in the form

$$
\lim_{x \to \infty} P(|X_h| > \varepsilon x \mid |X_0| > \varepsilon x) = 0, \quad h = 1, \ldots, m, \quad \varepsilon > 0.
$$

Condition RV$_3$ is slightly stronger than the corresponding one in Mikosch and Wintenberger (2016) who proved their large deviation result under the assumption that all $\Theta_t$, $t = 1, \ldots, m$, have an atom at zero. However, the proof in this paper is direct in contrast to Mikosch and Wintenberger (2016) who use techniques from the theory of regularly varying processes. In Section 1.2 we mentioned that the best separating sequence in the iid regularly varying case is $t_n = \sqrt{(\alpha - 2)n \log n}$. Thus RV$_1$ is not too far away from the latter growth condition.

**Example 2.7.** We consider the stochastic volatility model $X_t = \sigma_t Z_t$, $t \in \mathbb{Z}$, where $(\sigma_t)$ is a positive stationary sequence independent of the iid regularly varying sequence $(Z_t)$ with index $\alpha > 0$. If $E[\sigma^{\alpha+\delta}] < \infty$ for some $\delta > 0$ then it is not difficult to see that $(X_t)$ is regularly varying with index $\alpha$. Moreover, it is asymptotically independent. Condition (2.14) can be verified as follows: for $h \geq 1$,

$$
P(|X_h| > x \mid |X_0| > x) = P(\min(\sigma_h |Z_h|, \sigma_0 |Z_0|) > x)
\leq P((\sigma_h \vee \sigma_0) (|Z_0| \wedge |Z_h|) > x) =: I(x).
$$

We observe that $|Z_0| \wedge |Z_h|$ has regularly varying tail with index $-2\alpha$. By a result of Breiman (1965) we have

$$
I(x) \sim \mathbb{E}[(\sigma_h \vee \sigma_0)^{2\alpha}] P(|Z_0| \wedge |Z_h| > x) = \mathbb{E}[(\sigma_h \vee \sigma_0)^{2\alpha}] [P(|Z| > x)]^2.
$$
provided $\mathbb{E}[\sigma^{2\alpha+\delta}] < \infty$ for some $\delta$. The latter condition is satisfied e.g. if $\sigma$ has a lognormal distribution. This is a standard assumption in financial time series analysis; see Andersen et al. (2009). Since we also have $\mathbb{P}(|X| > x) \sim \mathbb{E}[\sigma^\alpha] \mathbb{P}(|Z| > x)$ relation (2.14) is immediate.

3. Main results

3.1. $X$ has semi-exponential tails

The following result is our main precise large deviation result for a stationary sequence with semi-exponential tails. The proof is given in Section 6.

**Theorem 3.1.** Consider an $m$-dependent stationary process $(X_i)$ with marginal distribution $F \in \text{MDA}(\Lambda) \cap \text{LN}$ for some $m \geq 1$. Assume condition (C). Then we have

$$\lim_{n \to \infty} \sup_{x > t_n, \delta} \left| \frac{\mathbb{P}(S_n > x)}{n \mathcal{F}(x)} - 1 \right| = 0, \quad \text{for any } \delta > 0. \quad (3.1)$$

An inspection of the proof of Theorem 3.1 shows that it can be generalized in various directions. Instead of $F \in \text{MDA}(\Lambda) \cap \text{LN}$ we may require $\mathbb{E}[|X|^{2+\delta}] < \infty$ for some $\delta > 0$, that $S(x)$ is slowly varying, $g$ satisfies (C) and $F$ is $(\varepsilon g)$-insensitive for any $\varepsilon > 0$. Hence we may also take into consideration distributions with infinite moments, in particular the class $\text{RV}(\alpha)$ for some $\alpha > 2$. However, the method of proof does not allow one to get an “almost” optimal separating sequence $t_n \gg \sqrt{n \log n}$ under $\text{RV}(\alpha)$. For this reason, we provide Theorem 3.2 under the latter condition which proves (3.1) for a best possible separating sequence.

3.2. $X$ has regularly varying tails

The following theorem complements the large deviation result for $m$-dependent stationary regularly varying sequences by Mikosch and Wintenberger (2016). The methods of proof are distinct and do not make direct use of techniques for regularly varying sequences.

**Theorem 3.2.** Assume $(X_i)$ is an $m$-dependent stationary sequence which is regularly varying with index $\alpha > 2$ and condition (RV) is satisfied. Then the large deviation result (3.1) holds.

The proof of this result is given in Section 7.
4. Linear process with subexponential noise

Assume that $Z$ has a subexponential distribution $F_Z$ ($F_Z \in S$) in the sense that $Z_+$ has a subexponential distribution and a tail-balance condition holds:

\[ \frac{P(Z > x)}{P(|Z| > x)} \to p_+, \quad \frac{P(-Z > x)}{P(|Z| > x)} \to p_-, \quad x \to \infty, \quad (4.1) \]

for some $p_+ > 0$, $p_- \geq 0$ such that $p_+ + p_- = 1$. Throughout this section we assume $F_Z \in \text{MDA}(\Lambda) \cap S$. Consider real coefficients $(\psi_j)$ such that $\psi_j = 0$ for $j < 0$, $\max_j |\psi_j| = 1$ and

\[ \sum_{j=0}^{\infty} |\psi_j|^\delta < \infty \quad \text{for some} \ \delta \in (0,1). \quad (4.2) \]

Let $k_\pm = \# \{j : \psi_j = \pm 1\}$, $m_0 = \sum_{j=0}^{\infty} \psi_j$ and $m_1 = \sum_{j=1}^{\infty} |\psi_j|$ which are finite in view of (4.2). Then the infinite series

\[ X = \sum_{j=0}^{\infty} \psi_j Z_j \]

converges a.s. provided $(Z_t)$ is an iid sequence with generic element $Z$. Indeed, $F_Z \in \text{MDA}(\Lambda) \cap S$ implies that $Z$ has finite first moment and therefore $E[\sum_{j=0}^{\infty} |\psi_j Z_j|] = (m_1 + |\psi_0|)E[|Z|] < \infty$.

4.1. Tail behavior of $X$

The following result was proved by Davis and Resnick (1988).

**Lemma 4.1.** If (4.2) and $F \in \text{MDA}(\Lambda) \cap S$ hold then

\[ P(X > x) \sim k_+ P(Z > x) + k_- P(Z < -x) \sim (k_+ p_+ + k_- p_-) P(|Z| > x). \quad (4.3) \]

We may conclude that the distribution of $X$ is tail-equivalent to $F_Z$. Hence it inherits subexponentiality.

4.2. Large deviations of linear processes

We consider the causal linear process

\[ X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad t \in \mathbb{Z}, \]
Proposition 4.2. Consider a causal linear process \( g \) with distribution \( F \) sequence \( (Z_t) \) with generic element \( X \) in the sense of (4.3). Assume that for a separating sequence \( (t_n) \) and if

\[
|\psi_j| < \infty
\]

for any small \( \delta > 0 \), we have

\[
\limsup_{n \to \infty} \sup_{x \in \Lambda_n} \frac{\mathbb{P}(m_1 |Z| > g(x))}{n \mathbb{P}(|m_0| |Z| > x)} = 0.
\]

If \( m_0 > 0 \) we assume that for any small \( \delta > 0 \),

\[
\sup_{x > m_0(1+\delta) t_n} \left| \frac{\mathbb{P}(S_n, Z > x)}{\mathbb{P}(Z > x)} - 1 \right| \to 0,
\]

and if \( m_0 < 0 \) and \( 0 < p_+ < 1 \),

\[
\sup_{x > |m_0| t_n(1+\delta)} \left| \frac{\mathbb{P}(-S_n, Z > x)}{\mathbb{P}(Z \leq -x)} - 1 \right| \to 0.
\]

Then

\[
\limsup_{n \to \infty} \sup_{x \in \Lambda_n} \left| \frac{\mathbb{P}(S_n > x)}{n \mathbb{P}(|Z| > x/|m_0|)} - p_+ \mathbb{I}_{(0,\infty)}(|m_0|) \right| = 0.
\]

2. Assume \( \psi_j = 0 \), \( j > m \), for some \( m \geq 1 \), and (4.6) or (4.7) hold according as \( m_0 > 0 \) or \( m_0 < 0 \). Moreover, assume that there is a set \( \Lambda_n \) such that \( \Lambda_n \subset (|m_0| t_n(1+\delta), \infty) \) for any \( \delta > 0 \) and, for \( m_0' = \sum_{i=0}^m |\psi_i| \),

\[
\lim_{n \to \infty} \sup_{x \in \Lambda_n} \left[ \frac{\mathbb{P}(m_0' |Z| > x)}{n \mathbb{P}(|m_0| |Z| > x)} + \frac{\mathbb{P}(m_0' |Z| > g(x))^2}{\mathbb{P}(|m_0| |Z| > x)} \right] = 0.
\]

Then (4.8) holds.
Remark 4.3. We observe that $m'_0 = |m_0|$ if either $\psi_j \geq 0$ for all $j$ or $\psi_j \leq 0$ for all $j$. In both situations, the first ratio in (4.9) vanishes for $\Lambda_n = (|m_0|t_n(1 + \delta), \infty)$. The second ratio vanishes if $-2S(g(x)/m_0) + S(x/m_0) \to -\infty$ as $x \to \infty$. This condition holds if we can ensure that $\sup_{x \geq t_n} |S(g(x))/S(x) - 1| \to 0$. The latter condition is satisfied for lognormal $Z$ if we choose $g(x) = x/S(x)$ and $t_n \gg \sqrt{n \log n}$.

Example 4.4. Condition (4.5) is quite restrictive. We illustrate this for an iid sequence $(Z_i)$ with distribution given by

$$F_Z(x) = \mathbb{P}(Z > x) = \exp(-\log x)^{\alpha}, \quad x > 1,$$

for $\alpha > 1$ and $g(x) = \varepsilon a(x)$, where $$\varepsilon = \varepsilon(x) \to 0 \text{ and } \varepsilon(x) \log x \to \infty, \quad x \to \infty.$$ Calculation yields $a(x) \sim cx/\log x^{\alpha - 1}$ for some $c > 0$. For convenience, we assume $m_0 > 0$. We have

$$\frac{\mathbb{P}(m_1 Z > g(x))}{n \mathbb{P}(m_0 | Z > x)} = \exp \left( - (\log(\varepsilon a(x)/m_1))^{\alpha} + (\log(x/m_0))^{\alpha} - \log n \right),$$

$$= \exp \left( (\alpha - 1)(\log x)^{\alpha - 1} \log x (1 + o(1)) - \log n \right).$$

For $\alpha \geq 2$ one can choose $t_n \gg \sqrt{n(\log n)^{1/2}}$ in (4.7); see the discussion in Example 2.4. In this case $\Lambda_n$ is empty. For $\alpha \in (1, 2)$ we can choose

$$\Lambda_n = (c_n, b_n), \quad c_n \gg \sqrt{n(\log n)\alpha}, \quad b_n = \exp \left( \left( \frac{(1 - \delta) \log n}{(\alpha - 1) \log \log n} \right)^{1/(\alpha - 1)} \right)$$

for arbitrarily small $\delta > 0$. In particular, $c n \in \Lambda_n$ for any $c > 0$.

Example 4.5. We assume $m'_0 > m_0 > 0$ in (4.9). In this case (4.9) is as restrictive as (4.5). To illustrate this, choose $F_Z$ as in (4.10). As mentioned in Remark 4.3, the second summand in (4.9) vanishes for $x > t_n$ if $\sup_{x > t_n} |S(g(x))/S(x) - 1| \to 0$. We investigate the first summand. We have

$$\frac{\mathbb{P}(m'_0 Z > x)}{n \mathbb{P}(m_0 | Z > x)} = \exp \left( - (\log x - \log m'_0)^{\alpha} + (\log x - \log m_0)^{\alpha} - \log n \right),$$

$$= \exp \left( \alpha \log(m'_0/m_0)(\log x)^{\alpha - 1}(1 + o(1)) - \log n \right).$$

Thus we get similar restrictions as in Example 4.4. The set $\Lambda_n$ is empty for $\alpha > 2$. For $1 < \alpha < 2$ we can choose

$$\Lambda_n = (c_n, b_n), \quad c_n \gg \sqrt{n(\log n)\alpha}, \quad b_n = \exp \left( \left( \frac{(1 - \delta) \log n}{\alpha \log(m'_0/m_0)} \right)^{1/(\alpha - 1)} \right)$$

for arbitrarily small $\delta > 0$, and we observe that $c n \in \Lambda_n$ for any $c > 0$. If $\alpha = 2$, $\Lambda_n$ is not empty if $m'_0/m_0 < c$ and contains the sequence $c n$ for any $c > 0$ if $m'_0/m_0 < \sqrt{\alpha}$. 
By independence of $S_n$, applying Lemma 4.1, we obtain $S_n = \sum_{j=-\infty}^{0} Z_j \beta_{n,j} + \sum_{j=1}^{\infty} Z_j \beta_{n,j} =: S_{n,1} + S_{n,2}$, where $\beta_{n,j} = \sum_{i=1-j}^{\infty} \psi_i$.

We have
\[
\mathbb{P}(S_{n,1} > x) \leq \mathbb{P}\left(\sum_{j=0}^{\infty} |Z_j| \sum_{i=1+j}^{n+j} |\psi_i| > x\right)
\leq \mathbb{P}\left(\sum_{j=0}^{\infty} |Z_j| \sum_{i=1+j}^{\infty} |\psi_i| > x\right) \leq c \mathbb{P}(|Z| m_1 > x),
\]
where we used Lemma 4.1 in the last step. Indeed, the conditions of this lemma are satisfied by virtue of (4.4). We have
\[
S_{n,2} = \sum_{j=1}^{n} Z_j \sum_{i=0}^{n-j} \psi_i = \sum_{j=1}^{n} Z_j \sum_{i=0}^{j-1} \psi_j = m_0 S_{n,z} - \sum_{j=1}^{n} Z_j \sum_{i=0}^{\infty} \psi_j = m_0 S_{n,z} - S_{n,21}.
\]
Applying Lemma 4.1, we obtain
\[
\mathbb{P}(|S_{n,21}| > x) \leq \mathbb{P}\left(\sum_{j=1}^{\infty} |Z_j| \sum_{i=0}^{\infty} |\psi_i| > x\right) \leq c \mathbb{P}(|Z| m_1 > x).
\]

By independence of $S_{n,1}$ and $S_{n,2}$ we observe that
\[
\mathbb{P}(S_n > x) \leq \mathbb{P}(S_{n,1} > x - g(x)) + \mathbb{P}(S_{n,2} > x - g(x)) + \mathbb{P}(S_{n,1} > g(x)) \mathbb{P}(S_{n,2} > g(x)).
\]

Hence for $m_0 > 0$, $x > t_n m_0 (1 + \delta)$ and sufficiently large $n$,
\[
\mathbb{P}(S_n > x) \leq c \mathbb{P}(m_1 |Z| > x - g(x)) + \mathbb{P}(S_{n,2} > x - g(x)) + c \mathbb{P}(S_{n,2} > g(x)) \mathbb{P}(m_1 |Z| > g(x))
\leq c \mathbb{P}(m_1 |Z| > x - g(x)) + \mathbb{P}(m_0 S_{n,z} > x - 2g(x)) + \mathbb{P}(-S_{n,21} > g(x)) + c \mathbb{P}(m_0 S_{n,z} > g(x)/2) + \mathbb{P}(-S_{n,21} > g(x)/2) \mathbb{P}(m_1 |Z| > g(x)).
\]

We conclude that
\[
\limsup_{n \to \infty} \sup_{x \in \Lambda_n} \frac{\mathbb{P}(S_n > x)}{n \mathbb{P}(m_0 |Z| > x)} \leq \limsup_{n \to \infty} \sup_{x \in \Lambda_n} \left[ \frac{\mathbb{P}(m_0 Z > x)}{\mathbb{P}(m_0 |Z| > x)} + c \frac{\mathbb{P}(m_1 |Z| > g(x))}{n \mathbb{P}(m_0 |Z| > x)} \right]
= p_+ + c \limsup_{n \to \infty} \sup_{x \in \Lambda_n} \frac{\mathbb{P}(m_1 |Z| > g(x))}{n \mathbb{P}(m_0 |Z| > x)} = p_+.
\]
Hence, under (4.9), the large deviation result (4.6) and the tail balance condition (4.1).

We also have
\[
\begin{align*}
P(S_n > x) & \geq P(S_n > x, S_{n,1} \leq g(x)) \geq P(S_{n,2} > x + g(x), S_{n,1} \leq g(x)) \\
& = P(S_{n,2} > x + g(x)) (1 - P(S_{n,1} > g(x))) = P(S_{n,2} > x + g(x))(1 + o(1)).
\end{align*}
\]

Thus it suffices to find a lower bound for
\[
P(S_{n,2} > x + g(x)) \geq P(m_0 S_{n,Z} - S_{n,21} > x + g(x), |S_{n,21}| \leq g(x)) \geq P(m_0 S_{n,Z} > x, |S_{n,21}| \leq g(x)) \geq P(m_0 S_{n,Z} > x) - P(|S_{n,21}| > g(x)) \geq n P(m_0 Z > x)(1 + o(1)) - c P(m_1 |Z| > g(x)).
\]

Therefore
\[
\lim_{n \to \infty} \sup_{x \in \Lambda_n} \frac{P(S_n > x)}{n P(m_0 |Z| > x)} \geq \lim_{n \to \infty} \sup_{x \in \Lambda_n} \frac{P(m_0 Z > x)}{n P(m_0 |Z| > x)} - c \lim_{n \to \infty} \sup_{x \in \Lambda_n} \frac{P(m_1 |Z| > g(x))}{n P(m_0 |Z| > x)} = p_+.
\]

2. We again assume \(m_0 > 0\). In this case we have
\[
S_n = \sum_{j=-m+1}^{0} Z_j \sum_{i=1-j}^{m} \psi_i + \sum_{j=n-m}^{n} Z_j \sum_{i=0}^{n-j} \psi_i + \sum_{j=1}^{n-m-1} Z_j m_0 =: T_{n,1} + T_{n,2} + T_{n,3}.
\]

Hence,
\[
P(S_n > x) \leq P(T_{n,1} + T_{n,2} > x - g(x)) + P(T_{n,3} > x - g(x)) + P(T_{n,1} + T_{n,2} > g(x)) P(T_{n,3} > g(x)).
\]

We have by Lemma 4.1 for sufficiently large \(x\),
\[
P(T_{n,1} + T_{n,2} > x) \leq \mathbb{P}\left(\sum_{j=-m+1}^{0} |Z_j| \sum_{i=1-j}^{m} |\psi_i| + \sum_{j=1}^{n-m} |Z_j| \sum_{i=0}^{j-1} |\psi_i| > x\right) \leq c \mathbb{P}(m_0 |Z| > x).
\]

Under (4.9),
\[
\limsup_{n \to \infty} \sup_{x \in \Lambda_n} \frac{P(S_n > x)}{n P(m_0 |Z| > x)} \leq \limsup_{n \to \infty} \sup_{x \in \Lambda_n} \left[ \frac{P(T_{n,1} + T_{n,2} > x - g(x))}{n P(m_0 |Z| > x)} + \frac{P(T_{n,3} > x - g(x))}{n P(m_0 |Z| > x)} \right]
\]
\[
+ \frac{P(T_{n,1} + T_{n,2} > g(x)) P(T_{n,3} > g(x))}{n P(m_0 |Z| > x)} \leq \limsup_{n \to \infty} \sup_{x \in \Lambda_n} \left[ \frac{P(m_0 |Z| > x)}{n P(m_0 |Z| > x)} + \frac{P(m_0 Z > x)}{P(m_0 |Z| > x)} + \frac{P(m_0 |Z| > g(x))^2}{P(m_0 |Z| > x)} \right] = p_+.
\]
As regards the lower bound, we have uniformly for \( x > m_0 t_n (1 + \delta) \),

\[
P(S_n > x) \geq P(T_{n,3} > x + g(x)) P(T_{n,1} + T_{n,2} > -g(x))
= P(T_{n,3} > x + g(x)) (1 - o(1))
\sim n P(m_0 Z > x + g(x)) \sim n P(m_0 |Z| > x).
\]

\[\square\]

5. Application to a large sample covariance matrix

Consider a real-valued field \((X_{it})\). We assume that the rows \((X_{it})\), \(i = 1, 2, \ldots\), constitute iid stationary \(m\)-dependent sequences. We observe the matrix \(X = (X_{it})_{i=1,\ldots,p; t=1,\ldots,n}\).

The corresponding sample covariance matrix is given by

\[
XX^\top = \left( \sum_{t=1}^{n} X_{it} X_{jt} \right)_{i,j=1,\ldots,p} =: (S^{(n)}_{ij})_{i,j=1,\ldots,p}.
\]

We assume that \(p = p_n \to \infty\). In what follows, \(X\) stands for a generic element of the field with distribution \(F\), and we also write \((X_i)\) for an iid sequence with common distribution \(F\).

5.1. The case \(F \in \text{RV}(\alpha)\)

**Lemma 5.1.** Assume the following conditions:

- \(X \in \text{RV}(\alpha)\) for some \(\alpha > 4\), in particular there is \((c_n)\) such that \(n P(X^2 > c_n) \to 1\) and \(c_n^{-1} \max_{i=1,\ldots,n} X_i^2 \xrightarrow{d} Y \sim \Phi_{\alpha/2}\).
- The asymptotic tail relations are valid:

\[
P(S^{(n)}_{11} - n E[X^2] > c_{np} x) \sim n P(X^2 > c_{np} x), \quad x > 0,
\]

\[
P(|S^{(n)}_{12} - n (E[X])^2| > c_{np} x) \leq c_n n P(|X_1 X_2| > c_{np} x) = o(p^{-2}), \quad x > 0.
\]

Then the following limit relations hold:

\[
c_{np}^{-1} \max_{1 \leq i < j \leq p} |S^{(n)}_{ij} - n (E[X])^2| \xrightarrow{p} 0,
\]

\[
c_{np}^{-1} \max_{i=1,\ldots,p} (S^{(n)}_{ii} - n E[X^2]) \xrightarrow{d} Y.
\]

**Proof of Lemma 5.1.** By assumption (5.1) we have for any \(x > 0\),

\[
p P(S^{(n)}_{11} - n E[X^2] > c_{np} x) \sim (np) P(X^2 > c_{np} x) \to x^{-\alpha/2}.
\]
The random variables \( (S_{ii}^{(n)}) \) are iid and therefore (5.4) holds if and only if the latter relation does.

Next we show that (5.3) holds. We have for any positive \( x \),
\[
\mathbb{P}
\left(
\sum_{1 \leq i < j \leq p}
\left|S_{ij}^{(n)} - n \mathbb{E}[X]^2\right|
> x
\right)
\leq \mathbb{P}
\left(
\max_{1 \leq i < j \leq p}
\left|S_{ij}^{(n)} - n \mathbb{E}[X]^2\right|
> c_{np}x
\right)
+ \mathbb{P}
\left(
\max_{1 \leq i < j \leq p}
\left(-S_{ij}^{(n)} + n \mathbb{E}[X]^2\right)
> c_{np}x
\right) =: I_1 + I_2.
\]

We restrict ourselves to prove \( I_1 \to 0 \). We have by assumption
\[
I_1 \leq n^2 \mathbb{P}(S_{12}^{(n)} - n \mathbb{E}[X]^2 > c_{np}x) \leq c p^2 \mathbb{P}(|X_1X_2| > c_{np}x) \to 0.
\]

\[\square\]

**Example 5.2.** Assume that \( X \in \text{RV}(\alpha) \) for some \( \alpha > 4 \) and the conditions of Theorem 3.2 are satisfied for the sequences \((X_{ij}^{(n)})\) and \((X_{12})\) with the same separating sequence \( (t_n) \) satisfying \( t_n \gg \sqrt{n \log n} \). Thus, \( X^2 \) is regularly varying with index \( \alpha/2 > 2 \) and \( \mathbb{P}(X_1X_2 > x) \sim x^{-\alpha/2} \) for some slowly varying \( \ell \); see Embrechts et al. (1980). We choose \( (c_n) \) such that \( n \mathbb{P}(X^2 > c_n) \to 1 \), i.e., \( c_n = n^{2/\alpha} \ell(n) \) for some slowly varying \( \ell \). We take \( (p_n) \) such that \( p = n^\beta \) with \( \beta > \alpha/4 - 1 \), then we have \( c_{np} \gg t_n \). An application of Theorem 3.2 yields for \( x > 0 \),
\[
\mathbb{P}(S_{11}^{(n)} - n \mathbb{E}[X]^2 > c_{np}x) \sim n \mathbb{P}(X^2 > c_{np}x).
\]
This is the desired relation (5.1). Next we consider
\[
q_n = \mathbb{P}(\left|S_{12}^{(n)} - n \mathbb{E}[X]^2\right| > c_{np}x).
\]

By Theorem 3.2 we have for some slowly varying \( \tilde{\ell} \),
\[
q_n \sim c n \mathbb{P}(|X_1X_2| > c_{np}x) = c n (np)^{-\tilde{\ell}(np)} = n^{-1} (np)^{-\tilde{\ell}(np)} = o(p^{-2})
\]
provided \( \ell(np)/n \to 0 \). This condition is satisfied since we chose \( p = n^\beta \). Thus we have the desired relation (5.2). We conclude that the limit relations (5.3) and (5.4) for the maxima of the diagonal and off-diagonal terms \( S_{ii}^{(n)} \) and \( S_{ij}^{(n)}, i \neq j \), hold.

### 5.2. The case \( F \in \text{MDA}(\Lambda) \cap \mathcal{S} \)

**Lemma 5.3.** Assume the following conditions:
- The distribution of \( X^2 \) is in \( \text{MDA}(\Lambda) \), i.e., there exist constants \( c_n > 0 \) and \( d_n \in \mathbb{R} \) such that \( c_n^{-1}(\max_{i=1,...,n} X_i^2 - d_n) \xrightarrow{d} Y \) with standard Gumbel limit.
The asymptotic tail relations are valid:

\[ \mathbb{P}(S_{11}^{(n)} - n \mathbb{E}[X^2] > c_{np} x + d_{np}) \sim n \mathbb{P}(X^2 > c_{np} x + d_{np}), \quad x \in \mathbb{R}, \]  
\[ \mathbb{P}(S_{12}^{(n)} - n (\mathbb{E}[X])^2 > c_{np} x + d_{np}) \leq c n \mathbb{P}(X_1 X_2 > c_{np} x + d_{np}) = o(p^{-2}), \quad x \in \mathbb{R}. \]  

Then the following limit relations hold:

\[ c_{np}^{-1} \max_{1 \leq i < j \leq p} ((S_{ij}^{(n)} - n (\mathbb{E}[X])^2) - d_{np}) \overset{\mathbb{P}}{\to} -\infty, \quad i \neq j, \]  
\[ c_{np}^{-1} \max_{i=1, \ldots, p} ((S_{ii}^{(n)} - n \mathbb{E}[X^2]) - d_{np}) \overset{d}{\to} Y. \]  

**Proof of Lemma 5.3.** By assumption (5.5) we have for any \( x \),

\[ p \mathbb{P}(S_{11}^{(n)} - n \mathbb{E}[X^2] > c_{np} x + d_{np}) \sim (np) \mathbb{P}(X^2 > c_{np} x + d_{np}) \to e^{-x}. \]

The random variables \( S_{ii}^{(n)} \) are iid and therefore (5.8) holds if and only if the latter relation does.

By assumption (5.6) we have for any \( x \in \mathbb{R} \),

\[ \mathbb{P}
\left( \max_{1 \leq i < j \leq p} (S_{ij}^{(n)} - n (\mathbb{E}[X])^2) > c_{np} x + d_{np} \right) \leq p^2 \mathbb{P}(S_{12}^{(n)} - n (\mathbb{E}[X])^2 > c_{np} x + d_{np}) \leq c p^2 n \mathbb{P}(X_1 X_2 > c_{np} x + d_{np}) \to 0. \]

Relation (5.7) follows.

**Example 5.4.** Assume that \( (X_t) \) is \( m \)-dependent stationary with a lognormal generic element \( X \) and the conditions of Example 2.4 are met for \( \alpha = 1 \). We standardize the marginal distribution such that \( X \overset{d}{=} e^{N} \) for a standard normal random variable \( N \). Then, \( X^2 \overset{d}{=} e^{2N} \) and \( X_1 X_2 \overset{d}{=} e^{\sqrt{2}N} \) for independent copies \( (X_t) \) of \( X \). According to Example 2.4 we can apply Theorem 3.1 to both \( (X_t^2) \) and \( (X_t X_{jt}) \) for \( t \neq j \) and in both cases we can choose any separating sequence \( t_n \gg \sqrt{n} (\log n)^2 \).

We set

\[ c_n = 2 (2 \log n)^{-1/2} d_n, \quad d_n = \exp \left( 2 \left( \sqrt{2 \log n} - (\log (4\pi) + \log \log n)/(2 \sqrt{2 \log n}) \right) \right). \]

It is well known that \( n \mathbb{P}(X^2 > c_n x + d_n) \to e^{-x} \) for any \( x \in \mathbb{R} \); see Embrechts et al. (1997), Example 3.3.31. We take \( (p_n) \) such that \( p \gg n^{-1} \exp(C (\log n)^2) \) for some \( C > 1/16 \). Since \( c_n = o(d_n) \) we have

\[ c_{np} x + d_{np} \gg t_n \quad \text{for any negative } x. \]

Therefore, (5.5) follows from Theorem 3.1.
Next we verify (5.6). To get the first bound in this relation we apply Theorem 3.1. Again observing that \( c_n = o(d_n) \), we have for \( x \in \mathbb{R} \),

\[
\mathbb{P}(S_{12}^{(n)} - n(\mathbb{E}[X])^2 > c_{np}x + d_{np}) \sim n \mathbb{P}(X_1X_2 > c_{np}x + d_{np})
= p^2 n \mathbb{P}(N > \log(c_{np}x + d_{np})/\sqrt{2})
= p^2 n \mathbb{P}(N > 2 \sqrt{\log(np)(1 + o(1))})
\sim p^2 n e^{-2 \log(np)(1 + o(1))} \to 0.
\]

Therefore (5.6) holds. We conclude that the statements of Lemma 5.3 are valid.

6. Proof of Theorem 3.1

For later use we recall two classical inequalities. Consider a sequence \((X_i)\) of independent mean-zero random variables, set \( S_n = \sum_{i=1}^{n} X_i \) and \( \sigma_n^2 = \text{var}(S_n) \).

- **Prokhorov’s inequality** (Prokhorov (1959); see Petrov (1995), p. 77) If \( |X_i| \leq c \) a.s. for \( i = 1, \ldots, n \) and some constant \( c \) then

\[
\mathbb{P}(S_n > x) \leq \exp\left(-\frac{x}{2c} \text{arsinh}\left(\frac{ce}{2\sigma_n}\right)\right), \quad x > 0,
\]

where \( \text{arsinh}(y) = \log(y + \sqrt{y^2 + 1}) \).

- **Fuk-Nagaev’s inequality** (Fuk and S.V. Nagaev (1971, 1976); see Petrov (1995), p. 78) If \( \mathbb{E}[|X_i|^p] < \infty \) for some \( p \geq 2 \), \( i = 1, \ldots, n \), \( m_{p,n} = \sum_{i=1}^{n} \mathbb{E}[|X_i|^p] \), then for constants \( c_p, d_p > 0 \) only depending on \( p \),

\[
\mathbb{P}(S_n > x) \leq c_p m_{p,n} x^{-p} + e^{-d_p(x/\sigma_n)^2}, \quad x > 0.
\]

The lower bound

We have

\[
\{S_n > x\} \supset \bigcup_{i=1}^{n} \{ |S_n - X_i| \leq g(x), X_i > x + g(x), |X_j| \leq g(x), 1 \leq j \neq i \leq n \}.
\]
The events on the right-hand side are disjoint. Therefore
\[
\mathbb{P}(S_n > x) \geq \sum_{i=1}^{n} \mathbb{P}\left( |S_n - X_i| \leq g(x), X_i > x + g(x), \max_{j \neq i} |X_j| \leq g(x) \right)
\]
\[
= \sum_{i=1}^{n} \mathbb{P}\left( X_i > x + g(x), \max_{j \neq i} |X_j| \leq g(x) \right)
\]
\[
- \sum_{i=1}^{n} \mathbb{P}\left( |S_n - X_i| > g(x), X_i > x + g(x), \max_{j \neq i} |X_j| \leq g(x) \right)
\]
\[
= \sum_{i=1}^{n} \mathbb{P}(X_i > x + g(x)) - \sum_{i=1}^{n} \mathbb{P}\left( X_i > x + g(x), \max_{j \neq i} |X_j| > g(x) \right) (6.1)
\]
\[
- \sum_{i=1}^{n} \mathbb{P}\left( |S_n - X_i| > g(x), X_i > x + g(x), \max_{j \neq i} |X_j| \leq g(x) \right)
\]
\[
= J_1(x) - J_2(x) - J_3(x). \quad (6.2)
\]
We have \( \sup_{x > t_n} |J_1(x)/(n \sqrt{F(x)}) - 1| \to 0 \) as \( n \to \infty \) and
\[
\sup_{x > t_n} \frac{J_2(x)}{n \sqrt{F(x)}} \leq \sup_{x > t_n} \sum_{i=1}^{n} \frac{\mathbb{P}(X_i > x + g(x), \max_{j \neq i, |j-i| \leq m} |X_j| > g(x))}{n \sqrt{F(x)}}
\]
\[
+ \sup_{x > t_n} \sum_{i=1}^{n} \frac{\mathbb{P}(X > x + g(x)) \mathbb{P}\left( \max_{j \neq i, |j-i| > m} |X_j| > g(x) \right)}{n \sqrt{F(x)}}
\]
\[
\leq 2 \sup_{x > t_n} \sum_{h=1}^{m} \frac{\mathbb{P}(X_0 > x + g(x), |X_h| > g(x))}{\sqrt{F(x)}}
\]
\[
+ \sup_{x > t_n} \frac{\mathbb{P}(X > x + g(x)) \mathbb{P}\left( \max_{i=1, \ldots, n} |X_i| > g(x) \right)}{\sqrt{F(x)}}
\]
\[
\leq o(1) + \sup_{x > t_n} n \mathbb{P}(|X| > g(x)) = o(1),
\]
where the latter relation follows by \( C_3 \) and since \( g(t_n)/\sqrt{n} \to \infty \) holds. Thus, it is enough to show that \( J_3(x) \to 0 \) as \( n \to \infty \) to derive the required lower bound for \( \mathbb{P}(S_n > x) \).

In the sequel, we will use the notation,
\[
\hat{X}_j = X_j \mathbb{1}(|X_j| \leq g(x)), \quad \hat{S}_n^{(i)} = \sum_{j=1, j \neq i}^{n} \hat{X}_j. \quad (6.3)
\]
Hence,

\[ J_3(x) = \sum_{i=1}^{n} \mathbb{P}( |\tilde{\mathcal{S}}_{n}^{(i)}| > g(x), X_i > x + g(x) ) \]

\[ \leq \sum_{i=1}^{n} \mathbb{P} \left( \left| \sum_{1 \leq t \neq i \leq n, |t-i| \leq m} \tilde{X}_t \right| > g(x)/2, X_i > x + g(x) \right) \]

\[ + \sum_{i=1}^{n} \mathbb{P} \left( \left| \sum_{1 \leq t \leq n, |t-i| > m} \tilde{X}_t \right| > g(x)/2 \right) \mathbb{P}(X > x + g(x)) = J_{31}(x) + J_{32}(x). \]

We have by \( C_3 \),

\[ \frac{J_{31}(x)}{n \mathcal{F}(x)} \leq 2 \sum_{h=1}^{m} \frac{\mathbb{P}( |X_h| > g(x)/(4m), X_0 > x + g(x) )}{\mathcal{F}(x)} = o(1). \]

Finally, we deal with \( J_{32}(x) \). Since \( X \) has mean zero, we derive for arbitrary \( \delta > 0 \)

\[ n |\mathbb{E}[\tilde{X}]| = n |\mathbb{E}[\mathbb{1}(|X| < g(x))]| = n | - \mathbb{E}[\mathbb{1}(|X| > g(x))]| \leq \frac{n \mathbb{E}[|X|^{2+\delta}]}{g^{1+\delta}(x)}. \]

We deduce from (2.4) and the fact that \( \mathbb{E}[|X|^{2+\delta}] < \infty \),

\[ n |\mathbb{E}[\tilde{X}]| \leq \frac{n \mathbb{E}[|X|^{2+\delta}]}{g^{1+\delta}(t_n)} = o(g^{-1-\delta}(t_n)), \quad n \to \infty. \quad (6.4) \]

Write \( \tilde{g}_r(x) = g(x)/(2m) - \# N_r \mathbb{E}[\tilde{X}] \) where

\[ N_r = \{ 1 \leq t \leq n : t \equiv r(\text{mod } m), |t-i| > m \}, \]

and observe that \( n |\mathbb{E}[\tilde{X}]| = o(g^{-1-\delta}(x)) \) and \( |\tilde{X} - \mathbb{E}[\tilde{X}]| \leq 2g(x) \). Using the \( m \)-dependence and Prokhorov’s inequality, we have for iid copies \( \{X_i^\prime\} \) of \( X \) and large \( n \),

\[ \frac{J_{32}(x)}{n \mathcal{F}(x)} \leq \sum_{i=1}^{n} \sum_{r=1}^{m} \mathbb{P} \left( \left| \sum_{t \in N_r} \tilde{X}_t^\prime \right| > g(x)/(2m) \right) \frac{\mathbb{P}(X > x + g(x))}{n \mathcal{F}(x)} \]

\[ \leq \sum_{i=1}^{n} \sum_{r=1}^{m} \mathbb{P} \left( \left| \sum_{t \in N_r} (\tilde{X}_t^\prime - \mathbb{E}[\tilde{X}]) \right| > \tilde{g}_r(x) \right) \frac{\mathbb{P}(X > x + g(x))}{n \mathcal{F}(x)} \]

\[ \leq c \sum_{i=1}^{n} \sum_{r=1}^{m} \exp \left( -\frac{\tilde{g}_r(x)}{4g(x)} \arcsinh \left( \frac{2g(x)\tilde{g}_r(x)}{2\# N_r \text{var}(X)} \right) \right) \frac{\mathbb{P}(X > x + g(x))}{n \mathcal{F}(x)} \]

\[ \leq c m \exp \left( -\left( \frac{1}{8m} + o(1) \right) \log \left( 1 + o(1) \frac{m(g(x))^2}{2n \text{var}(X)} \right) \right) \to 0. \]

In the last step we used that \((g(x))^2/n \geq (g(t_n))^2/n \to \infty\); see (2.4).
The upper bound

Consider the following disjoint partition of $\Omega$:

\begin{align*}
B_1 &= \bigcup_{1 \leq i < j \leq n} \{|X_i| > g(x), |X_j| > g(x)\}, \\
B_2 &= \bigcup_{i=1}^{n} \{|X_i| > g(x), \max_{j=1,\ldots,n, i \neq j} |X_j| \leq g(x)\}, \\
B_3 &= \{\max_{j=1,\ldots,n} |X_j| \leq g(x)\}.
\end{align*}

The bound on $B_1$

We observe that for any $\xi \in (0, 1)$,

\[
P(\{S_n > x\} \cap B_1) \leq \sum_{1 \leq i < j \leq n} P(S_n > x, |X_i| > g(x), |X_j| > g(x))
\]

\[
\leq \sum_{1 \leq i < j \leq n} P(S_{ij}^{(1)} > \xi x, |X_i| > g(x), |X_j| > g(x))
\]

\[
+ \sum_{1 \leq i < j \leq n} P(S_{ij}^{(2)} > (1 - \xi) x, |X_i| > g(x), |X_j| > g(x))
\]

\[
=: R_1(x) + R_2(x),
\]

where

\[
S_{ij}^{(1)} = \sum_{h \leq n : |i - h| \land |j - h| > m} X_h, \quad S_{ij}^{(2)} = \sum_{h \leq n : |i - h| \land |j - h| \leq m} X_h,
\]

\[
(S_{ij}^{(r)})' = \sum_{h \in Q_{ij}^{(r)}} X_h, \quad Q_{ij}^{(r)} = \{h \leq n : |i - h| \land |j - h| > m, h \equiv r (\mod m)\}.
\]

For a given $r$, the summands in $(S_{ij}^{(r)})'$ are independent due to $m$-dependence and also independent of $X_i, X_j$. We have $\#Q_{ij}^{(r)} \leq n/m$ while the number of summands in $S_{ij}^{(2)}$ does not exceed $4m + 2$. Thus by the large deviation result (2.5), $m$-dependence and
stationarity,

\[
\frac{R_1(x)}{nF(x)} \leq \sum_{1 \leq i < j \leq n} \sum_{r=1}^{m} \frac{\mathbb{P}((S_{ij}^{(r)})' > \xi x/m) \mathbb{P}(|X_i| > g(x), |X_j| > g(x))}{nF(x)}
\]

\[
\approx \frac{F(x/m)}{F(x)} \sum_{1 \leq i < j \leq n} \mathbb{P}(|X_i| > g(x), |X_j| > g(x))
\]

\[
= \frac{F(x/m)}{F(x)} \sum_{h=1}^{n-1} (n-h) \mathbb{P}(|X_0| > g(x), |X_h| > g(x))
\]

\[
\leq n \frac{F(x/m)}{F(x)} \sum_{i=1}^{m} \mathbb{P}(|X_0| > g(x), |X_h| > g(x)) + \frac{F(x/m)}{F(x)} [n \mathbb{P}(|X| > g(x))]^2
\]

\[=: R_{11}(x) + R_{12}(x).\]

Applying the tail balance condition, \(C_3\) and (2.8), we have

\[
\sup_{x > t_n} R_{11}(x) \leq c \sup_{x > t_n} \frac{nF(x/m)F(g(x))}{F(x)} \rightarrow 0.
\]

Since \(g(t_n)/\sqrt{n} \rightarrow \infty\) and \(\mathbb{E}[X^2] < \infty\) we also have

\[
\sup_{x > t_n} n \mathbb{P}(|X| > g(x)) \leq \sup_{x > t_n} n \mathbb{P}(|X| > \sqrt{n}) \rightarrow 0. \tag{6.5}
\]

Hence, the tail balance condition, Lemma 2.1 and (6.5) immediately imply that \(R_{12} \rightarrow 0\).

We have

\[
R_2(x) \leq \sum_{i=1}^{n-1} \left( \sum_{j=i+1}^{(i+2m) \wedge n} + \sum_{j=1}^{(i+2m) \wedge n} \right) \mathbb{P}\left(S_{ij}^{(2)} > (1 - \xi)x, |X_i| > g(x), |X_j| > g(x)\right)
\]

\[= R_{21}(x) + R_{22}(x).\]

We restrict ourselves to the study of \(R_{21}(x); R_{22}(x)\) can be treated by similar methods.

We note that \(S_{ij}^{(2)}\) has representation

\[
S_{ij}^{(2)} = \sum_{h=(i-1)m \vee 1}^{(j+m) \wedge n} X_h.
\]

Observe that the number of summands in \(S_{ij}^{(2)}\) does not exceed \(4m + 2\). Therefore and by stationarity, taking care of the cases \(h = i\) and \(h = j\),

\[
R_{21}(x) \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{(i+2m) \wedge n} \sum_{h=(i-1)m \vee 1}^{(j+m) \wedge n} \mathbb{P}\left(|X_h| > \frac{(1 - \xi)x}{4m + 2}, |X_i| > g(x), |X_j| > g(x)\right) \tag{6.6}
\]

\[
\leq c n \sum_{h=1}^{m} \mathbb{P}\left(|X_0| > \frac{(1 - \xi)x}{4m + 2}, |X_h| > g(x)\right) + c n \frac{F\left(\frac{(1 - \xi)x}{4m + 2}\right)F(g(x))}{F(x)} \tag{6.7}
\]
By $C_3$ and (2.8) we conclude that
\[ \lim_{n \to \infty} \sup_{x > t_n} \frac{R_{21}(x)}{nF(x)} = 0. \]
Combining the previous bounds, we conclude that
\[ \lim_{n \to \infty} \sup_{x > t_n} \frac{\mathbb{P}(\{S_n > x\} \cap B_1)}{nF(x)} = 0. \]

**The bound on $B_2$**

Next we bound $\mathbb{P}(\{S_n > x\} \cap B_2)$. Recall the notation $\hat{X}_j$ and $\hat{S}_n^{(i)}$ from (6.3). Fix $b \in (0,1)$. Since $g(x)/x \to 0$ as $x \to \infty$ we have
\[
\mathbb{P}(\{S_n > x\} \cap B_2) \leq \sum_{i=1}^{n} \mathbb{P}(X_i + \hat{S}_n^{(i)} > x, |X_i| > g(x)) \\
= \sum_{i=1}^{n} \mathbb{P}(X_i + \hat{S}_n^{(i)} > x, |X_i| \in (g(x), x-bx]) \\
+ \sum_{i=1}^{n} \mathbb{P}(X_i + \hat{S}_n^{(i)} > x, |X_i| \in (x-bx, x-g(x)]) \\
+ \sum_{i=1}^{n} \mathbb{P}(X_i + \hat{S}_n^{(i)} > x, |X_i| > x-g(x)) \\
=: I_1(x) + I_2(x) + I_3(x).
\]

**Bounding $I_1(x)$**

We show that $I_1(x) = o(nF(x))$. Similarly to the bound for $R_1(x)$ estimation, using the $m$-dependence, we split $\hat{S}_n^{(i)}$ into $m$ sums of iid summands:

\[ \hat{S}_n^{(i)} = \sum_{r=1}^{m} \hat{S}_{i,r}, \quad \hat{S}_{i,r} = \sum_{h \in Q_{i,r}^*} \hat{X}_h, \quad \text{where} \quad Q_{i,r}^* = \{h \leq n : h \equiv r \mod m, h \neq i\}. \]

In view of (6.4) we have $n|\mathbb{E}[\hat{X}]| \leq g(x)$ for large $x$. Moreover, $|\hat{X} - \mathbb{E}[\hat{X}]| \leq 2g(x)$ and
\#Q_{i,r} \leq n/m. An application of Prokhorov’s inequality for large \( n \) yields

\[
I_1(x) \leq \sum_{i=1}^{n} \mathbb{P}(\hat{S}_n^{(i)} > bx) \leq \sum_{i=1}^{n} \sum_{r=1}^{m} \mathbb{P}(\hat{S}_{i,r} > bx/m) \\
\leq \sum_{i=1}^{n} \sum_{r=1}^{m} \mathbb{P}(\hat{S}_{i,r} - \#Q_{i,r} \mathbb{E}[\hat{X}] > bx/m - g(x)) \\
\leq \sum_{i=1}^{n} \sum_{r=1}^{m} \exp \left( - \frac{bx/m - g(x)}{4g(x)} \arcsinh \left( \frac{2g(x)(bx/m - g(x))}{2\#Q_{i,r} \text{var}(\hat{X})} \right) \right) \\
\leq n \sum_{i=1}^{m} \sum_{r=1}^{m} \mathbb{P}(\hat{S}_{i,r} - \mathbb{E}[\hat{X}] > \frac{bx}{m} - g(x)) \\
\leq n \sum_{i=1}^{m} \sum_{r=1}^{m} \exp \left( - \frac{x}{g(x)} \log \left( \frac{g(x)}{n} \right) \right) = o(nF(x))
\]

uniformly for \( x > t_n \). In the last step we used the bounds on \( g(x) \) in \( C_1 \) and \( C_2 \).

**Bounding \( I_2(x) \)**

Write

\[
\hat{S}_n^{(i)} = \sum_{tg \in [i-m,i+m]} \hat{X}_t.
\]

We observe that \( |\hat{X}| \leq g(x) \) and conclude by independence between \( X_i \) and \( \hat{S}_n^{(i)} \), the tail-balance condition and integration by parts that

\[
I_2(x) \leq \sum_{i=1}^{n} \mathbb{P}(X_i + \hat{S}_n^{(i)} > x - 2m g(x), |X_i| \in (x - bx, x - g(x)]) \\
\leq c \sum_{i=1}^{n} \int_{x-bx}^{x-g(x)} \mathbb{P}(\hat{S}_n^{(i)} > x - 2m g(x) - y) dF(y) \\
\leq c F(x - bx) \sum_{i=1}^{n} \mathbb{P}(\hat{S}_n^{(i)} > bx - 2mg(x)) + c \int_{g(x)}^{bx} F(x - y) \mathbb{P}(\hat{S}_n^{(i)} \in dy) \\
=: c(I_{21}(x) + I_{22}(x)).
\]

In view of (2.7), for every \( \delta > 0 \) there is \( u_3 \) such that

\[
\sup_{y \geq u_3/(1-b)} \frac{F(y)}{F(y + g(y))} \leq e^\delta.
\]

By a telescoping argument for sufficiently large \( x \),

\[
\frac{F(x - bx)}{F(x)} \leq \prod_{h=1}^{\lfloor bx/g(x) \rfloor} \frac{F(x - hg(x))}{F(x - (h-1)g(x))} \overset{\delta((bx/g(x)) + 1)}{\leq e^\delta}\frac{F(x - bx)}{F(x - g(x)(bx/g(x)))}.
\]
Precise large deviations for dependent subexponential variables

Now the same argument as for $I_1(x)$ combined with $C_1$, (2.4) and Prokhorov’s inequality yields uniformly for $x > t_n$,

$$\frac{I_{21}(x)}{n F(x)} \leq e^{e^{((bx/g(x))+1) n} \exp \left( - \frac{bx/m - 3g(x)}{4g(x)} \log \left( \frac{mg(x)(bx/m - 3g(x))}{n \text{ var}(X)} \right) \right)}$$

$$\leq \exp \left( -c \frac{x}{g(x)} \log \left( \frac{xg(x)}{n} \right) \right) \to 0, \quad n \to \infty.$$

A similar argument yields

$$\frac{I_{22}(x)}{n F(x)} \leq \frac{1}{n F(x)} \sum_{k=1}^{[bx/g(x)]} \int_{g(x)k}^{g(x)(k+1)} n F(x - y) \mathbb{P}(\tilde{S}_n^{(i)} \in dy)$$

$$\leq \sum_{k=1}^{[bx/g(x)]} \frac{F(x - (k + 1)g(x))}{n F(x)} \sum_{i=1}^{n} \mathbb{P}(\tilde{S}_n^{(i)} \in g(x) (k, k + 1])$$

$$\leq \sum_{k=1}^{\infty} \frac{e^{(k+1)\delta}}{n} \sum_{i=1}^{n} \mathbb{P}(\tilde{S}_n^{(i)} > kg(x))$$

$$\leq \sum_{k=1}^{\infty} \exp \left( -c k \log \left( \frac{k g^2(x)}{n} \right) + (k + 1) \delta \right) \to 0, \quad n \to \infty.$$

Bounding $I_3(x)$

We have

$$\limsup_{x \to t_n} \frac{I_3(x)}{n F(x)} \leq \limsup_{n \to \infty} \sup_{x > t_n} \frac{1}{n F(x)} \left( \sum_{i=1}^{n} \mathbb{P}(X_i + \tilde{S}_n^{(i)} > x, X_i > x - g(x)) \right)$$

$$+ \sum_{i=1}^{n} \mathbb{P}(X_i + S_n^{(i)} > x, X_i < -x + g(x))$$

$$\leq \limsup_{n \to \infty} \sup_{x > t_n} \frac{F(x - g(x))}{F(x)} + \limsup_{n \to \infty} \sup_{x > t_n} \frac{1}{n F(x)} \sum_{i=1}^{n} \mathbb{P}(\tilde{S}_n^{(i)} > 2x - g(x))$$

$$\leq 1 + \limsup_{n \to \infty} \sup_{x > t_n} \exp \left( -c \frac{x}{g(x)} \log \left( \frac{xg(x)}{n} \right) \right) = 1.$$

The second term is bounded in the same way as $I_1$, by exploiting the $m$-dependence, $C_1$, (2.4) and Prokhorov’s inequality.

Collecting the bounds for all $I_i(x)$, we obtain the desired relation

$$\limsup_{n \to \infty} \frac{\mathbb{P} \left( \{S_n > x \} \cap B_2 \right)}{n F(x)} \leq 1.$$
The bound on $B_3$

It remains to show that $\mathbb{P}(\{S_n > x\} \cap B_3) = o(nF(x))$. We observe that $\{S_n > x\} \cap B_3 = \{\tilde{S}_n > x\}$ where $\tilde{S}_n = \sum_{i=1}^n \tilde{X}_i$ and $|\tilde{X}_i| \leq g(x)$. Now the same techniques as for bounding $I_1(x)$ apply. We omit further details. This finishes the proof of the upper bound.

7. Proof of Theorem 3.2

The proof is similar to the one of Theorem 3.1. We follow the lines of this proof and also use the same notation. We set $g(x) = g_\varepsilon(x) = \varepsilon x$ for any $\varepsilon > 0$.

The lower bound

We start with the bound (6.2): $\mathbb{P}(S_n > x) \geq J_1(x) - J_2(x) - J_3(x)$. For the first term we have

$$\lim_{n \to \infty} \sup_{x > t_n} \frac{J_1(x)}{nF(x)} = \lim_{n \to \infty} \sup_{x > t_n} \frac{nF(x + \varepsilon x)}{nF(x)} = (1 + \varepsilon)^{-\alpha},$$

and the right-hand side converges to 1 as $\varepsilon \downarrow 0$. By regular variation the bound on $J_2(x)$ turns into

$$\sup_{x > t_n} \frac{J_2(x)}{nF(x)} \leq 2 \sup_{x > t_n} \sum_{h=1}^n \frac{\mathbb{P}(X_0 > (1 + \varepsilon)x, |X_h| > \varepsilon x) F(x)}{F(x)}$$

$$+ \sup_{x > t_n} \frac{\mathbb{P}(X > (1 + \varepsilon)x) \mathbb{P}(\max_{i=1, \ldots, n} |X_i| > \varepsilon x)}{F(x)}$$

$$\leq c \sup_{x > t_n} \sum_{h=1}^n \mathbb{P}(|X_h| > \varepsilon x \mid |X_0| > \varepsilon x) + c n \mathbb{P}(|X| > t_n) \to 0.$$

Here we used condition (2.14) for the first term and the facts that $t_n \gg \sqrt{n}$ and $\text{var}(X) < \infty$ for the second term. Next we consider the bound $J_3(x) \leq J_{31}(x) + J_{32}(x)$. The relation $\lim_{n \to \infty} \sup_{x > t_n} J_{31}(x)/(nF(x)) = 0$ follows in the same way as for the first term in the last display. The negligibility of $J_{32}(x)/(nF(x))$ is again proved by Prokhorov’s inequality for iid regularly varying random variables, also observing that uniformly for $x > t_n$ and fixed $\varepsilon > 0$, by Karamata’s theorem and the choice of $t_n \gg \sqrt{n} \log n$,

$$n |\hat{E}[\hat{X}]| \leq n E(|X| \mathbb{1}(|X| > \varepsilon x)) \sim c n (\varepsilon x) \mathbb{P}(|X| > \varepsilon x) = o(x).$$
The upper bound

We start with the bound \( \mathbb{P} ( \{ S_n > x \} \cap B_1 ) \leq R_1 ( x ) + R_2 ( x ) \). Since \( x > t_n \gg \sqrt{n \log n} \) the classical Nagaev large deviation result, S.V. Nagaev (1979), applies to each of the \( \mathbb{P} ( S_n^{(t)} > \xi x / m ) \) uniformly for \( x \). Hence, uniformly for \( x > t_n \),

\[
\frac{R_2 ( x )}{n F ( x )} \leq c \sum_{h=1}^{m} \mathbb{P} ( |X_h| > \varepsilon x \mid |X_0| > \varepsilon x ) + c n \mathbb{P} ( |X| > \varepsilon x ) = o ( 1 ) .
\]

Next, having \( R_2 ( x ) = R_{21} ( x ) + R_{22} ( x ) \), we restrict ourselves to the investigation of \( R_{21} ( x ) \) as in the proof of Theorem 3.1. The relation (6.6) remains true, thus we have by \( \text{RV}_3 \)

\[
\frac{R_{21} ( x )}{n F ( x )} \leq c \sum_{i=1}^{m} \mathbb{P} \left( |X_i| > \varepsilon x \mid |X_0| > ( 1 - \xi ) x / 4m + 2 \right) + c n \mathbb{P} ( |X| > \varepsilon x ) = o ( 1 )
\]

uniformly for \( x > t_n \).

Next we comment on \( \mathbb{P} ( \{ S_n > x \} \cap B_2 ) \). For any small \( \delta > 0 \) write \( A_\delta = \cup_{i=1}^{n} \{ X_i > (1 - \delta) x \} \). Thus, we derive

\[
\mathbb{P} ( \{ S_n > x \} \cap B_2 ) = \mathbb{P} ( \{ S_n > x \} \cap B_2 \cap A_\delta ) + \mathbb{P} ( \{ S_n > x \} \cap B_2 \cap A_\delta^c )
\]

\[
\leq n F ( (1 - \delta) x ) + \sum_{i=1}^{n} \mathbb{P} ( X_i + \widehat{S}_n^{(i)} > x, |X_i| > \varepsilon x, X_i \leq (1 - \delta) x )
\]

\[
\leq n F ( (1 - \delta) x ) + \sum_{i=1}^{n} \mathbb{P} ( \widehat{S}_n^{(i)} > \delta x, |X_i| > \varepsilon x ) = C_1 ( x ) + C_2 ( x ) .
\]

Therefore,

\[
\lim_{n \to \infty} \sup_{x > t_n} \frac{C_1 ( x )}{n F ( x )} \leq (1 - \delta)^{-\alpha} ,
\]

and the right-hand side converges to 1 as \( \delta \downarrow 0 \). We have by \( m \)-dependence

\[
C_2 ( x ) \leq \sum_{i=1}^{n} \mathbb{P} \left( \sum_{j \in [m, i + m], j \neq i} \widehat{X}_j > \delta x / 2 \right)
\]

\[
+ \sum_{i=1}^{n} \mathbb{P} \left( \sum_{j \leq n, j \notin [i - m, i + m]} \widehat{X}_j > \delta x / 2 \right) \mathbb{P} ( |X| > \varepsilon x ) .
\]

The sums in the first right-hand probabilities can be bounded by \( 2m \varepsilon < \delta / 2 \) the first term vanishes. Writing \( P_{n,i}(x) \) for the summands in the second term, we have

\[
\frac{\sum_{i=1}^{n} P_{n,i}(x)}{n F(x)} \leq c \varepsilon^{-\alpha} \frac{1}{n} \sum_{i=1}^{n} \mathbb{P} \left( \sum_{j \leq n, j \notin [i - m, i + m]} \widehat{X}_j > \delta x / 2 \right).
\]
The probabilities in the sum can be bounded uniformly for \( x \) and \( i \) by splitting the sum into \( m \) sums of iid summands and then applying Prokhorov’s inequality. Moreover, this bound converges to zero since we can choose \( \varepsilon > 0 \) arbitrarily small. Thus \( \sup_{x>t_n} C_2(x)/(nF(x)) \) is negligible as \( n \to \infty \).

Finally, we bound \( \mathbb{P}(\{S_n > x\} \cap B_3) = \mathbb{P}(\hat{S}_n > x) \). We split \( \hat{S}_n \) into \( m \) independent sums and apply the Fuk-Nagaev’s inequality for \( p > \alpha \). Thus, we obtain for universal constants \( c,d > 0 \) only depending on \( p \),

\[
\mathbb{P}(\hat{S}_n > x) \leq c n E[|\hat{X}|^p] x^{-p} + \exp(-d x^2/n).
\]

We have by Karamata’s theorem uniformly for \( x > t_n \),

\[
\frac{n E[|\hat{X}/x|^p]}{n F(x)} = \frac{E[|\hat{X}|^p]}{(\varepsilon x)^p} \mathbb{P}(|X| > \varepsilon x) \frac{\varepsilon^p \mathbb{P}(|X| > \varepsilon x)}{F(x)} \to c \varepsilon^p \alpha, \quad n \to \infty,
\]

and the right-hand side vanishes as \( \varepsilon \downarrow 0 \). Moreover, for large \( x \)

\[
\frac{\exp(-d x^2/n)}{n F(x)} = \exp(-d x^2/n - \log n + S(x)) \leq \exp(-0.5 d x^2/n - \log n + \alpha \log x)
\]

and the right-hand side converges to zero as \( n \to \infty \) uniformly for \( x > t_n \gg \sqrt{n \log n} \). \( \square \)

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