Strong and weak convergence rates for slow-fast stochastic differential equations driven by $\alpha$-stable process

XIAOBIN SUN$^{1,*}$ LONGJIE XIE$^{1,**}$ and YINGCHAO XIE$^{1,†}$

$^1$School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, China, 221000.
E-mail: $^*\text{xbsun@jsnu.edu.cn}$; $^{**}\text{xlj.98@whu.edu.cn}$; $^{†}\text{yczie@jsnu.edu.cn}$

In this paper, we study the averaging principle for a class of stochastic differential equations driven by $\alpha$-stable processes with slow and fast time-scales, where $\alpha \in (1, 2)$. We prove that the strong and weak convergence order are $1 - 1/\alpha$ and 1 respectively. We show, by a simple example, that $1 - 1/\alpha$ is the optimal strong convergence rate.

Keywords: averaging principle, $\alpha$-stable process, slow-fast system, convergence rates.

1. Introduction

Multiscale models involving “slow” and “fast” components appear naturally in various fields, such as nonlinear oscillations, chemical kinetics, biology, climate dynamics, etc, see, e.g., [3,12,22,33] and the references therein. The averaging principle of multiscale models describes the asymptotic behavior of the slow components as the scale parameter $\epsilon \rightarrow 0$.

In [23], Khasminskii considered a class of multiscale stochastic differential equations (SDEs for short) driven by Wiener noise, i.e.,

$$
\begin{align*}
&\begin{cases}
  dX^\epsilon_t = A(X^\epsilon_t, Y^\epsilon_t)dt + dW_t, & X^\epsilon_0 = x \in \mathbb{R}^d, \\
  dY^\epsilon_t = \frac{1}{\epsilon}B(X^\epsilon_t, Y^\epsilon_t)dt + \frac{1}{\sqrt{\epsilon}}dW_t, & Y^\epsilon_0 = y \in \mathbb{R}^d,
\end{cases}
\end{align*}
$$

where $W_t$ is a $d$-dimensional Brownian motion, $\epsilon$ is a small and positive parameter which describes the ratio of the time scale between the slow component $X^\epsilon_t$ and fast component $Y^\epsilon_t$, the coefficients $A, B : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ are assumed to be Lipschitz continuous. It is assumed that there exists a map $\bar{A} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that

$$
\left| \frac{1}{T} \int_0^T \mathbb{E}A(x, Y^{x,y}_t)dt - \bar{A}(x) \right| \leq \alpha(T)(1 + |x|^2)
$$

for some function $\alpha(T)$ vanishing as $T$ goes to infinity, where $Y^{x,y}_t$ is the unique solution of the corresponding frozen equation

$$
dY^{x,y}_t = B(x, Y^{x,y}_t)dt + dW_t, \quad Y^{x,y}_0 = y \in \mathbb{R}^d.
$$
Then Khasminskii proved that the slow component \( X_t^\epsilon \) converges to \( \bar{X}_t \) in probability, where \( \bar{X}_t \) is the unique solution of the corresponding averaged equation,

\[
\bar{X}_t = \bar{A}(\bar{X})dt + dW_t, \quad \bar{X}_0 = x \in \mathbb{R}^d.
\]

Since this pioneering work, many people studied the averaging principle for various stochastic systems driven by Wiener noise, see, e.g., [23–28, 30, 39] for the averaging principle of SDEs and [7–9, 11, 13–19, 40] for the averaging principle of stochastic partial differential equations (SPDEs for short).

All the papers mentioned above considered stochastic systems with continuous path. However, in many applications, systems driven by discontinuous noises appear naturally. There have been many papers devoted to study the averaging principle for slow-fast stochastic systems driven by jump noises, see, e.g., [21, 29, 42–45]. But in these papers, the noises are assumed to have second order moments in order to obtain the usual energy estimates. This excludes the important \( \alpha \)-stable noise with \( \alpha \in (0, 2) \). The discontinuity and the heavy tail property make the \( \alpha \)-stable noise a useful driving process in models arising in physics, telecommunication networks, finance and other fields, see e.g., [1, 2, 35] and the references therein for more backgrounds. Slow-fast stochastic systems driven by \( \alpha \)-stable noises have attracted more attentions recently. Zhang et al. [46] studied data assimilation and parameter estimation for a multiscale stochastic system with \( \alpha \)-stable noise. Zulfiqar et al. [47] studied slow manifolds of a slow-fast stochastic evolutionary system with stable Lévy noise. Bao et al. [2] studied the strong averaging principle for two-time scale SPDEs driven by \( \alpha \)-stable noise. In [38] and [10], the first named authors and his collaborators studied the strong averaging principle for stochastic Ginzburg-Landau equation and stochastic Burgers equations driven by \( \alpha \)-stable processes, respectively.

There are also many works devoted to study the rate of convergence in the averaging principle. The main motivation comes from the well-known Heterogeneous Multi-scale Methods used to approximate the slow component, see e.g., [6, 12]. Furthermore, the rate of convergence is also known to be very important for functional limit theorems in probability theory and homogenization, see e.g., [25, 31, 32, 40]. Strong and weak convergence rates for slow-fast stochastic systems with Wiener noise have been studied extensively, e.g., see [20, 28, 29, 36, 45] for the finite dimension case and [4, 5, 11, 15] for the infinite dimension case. Usually, Khasminskii’s time discretization technique is used to study the strong convergence rate while the method of asymptotic expansion of solutions of Kolmogorov equations is used to study the weak convergence rate. We mention that in [2, 10, 38], because of the time discretization method used therein, no satisfactory convergence rates were obtained.

In this paper, we consider the following multiscale SDEs driven by \( \alpha \)-stable processes:

\[
\begin{align*}
\dot{X}_t^\epsilon &= b(X_t^\epsilon, Y_t^\epsilon)dt + dL_t^1, \quad X_0^\epsilon = x \in \mathbb{R}^{d_1}, \\
\dot{Y}_t^\epsilon &= \frac{1}{\epsilon} f(X_t^\epsilon, Y_t^\epsilon)dt + \frac{1}{\epsilon^{1/\alpha}} dL_t^2, \quad Y_0^\epsilon = y \in \mathbb{R}^{d_2},
\end{align*}
\]

where \( \{L_t^1\}_{t \geq 0} \) and \( \{L_t^2\} \) are independent \( d_1 \) and \( d_2 \) dimensional isotropic \( \alpha \)-stable processes with \( \alpha \in (1, 2) \), \( b : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_1} \) and \( f : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_2} \) are Borel functions.
whose regularity conditions will be stated below.

It is well known that for systems driven by Wiener noise, the optimal strong convergence rate is $1/2$ and the weak convergence rate is 1. A natural question is that: for systems driven by $\alpha$-stable noises, what are the optimal strong and weak convergence rates? The purpose of this paper is to establish both the strong and the weak convergence rates for the stochastic system (1.1). We will prove that the weak convergence order is still 1, but the strong convergence order is $1 - 1/\alpha$. We show, by a simple example, that the strong convergence rate $1 - 1/\alpha$ is optimal.

The main technique used in this paper is the Poisson equation, which is inspired from [5] (see also [31,32,36,37]). In contrast to the classical Khasminskii’s time discretization, it has great advantage in obtaining the rates of convergence. More precisely, we need to study the following Poisson equation in $\mathbb{R}^{d_2}$:

$$-\mathcal{L}_2(x,y)\Phi(x,y) = b(x,y) - \bar{b}(x), \quad y \in \mathbb{R}^{d_2},$$

where $x \in \mathbb{R}^{d_1}$ is a parameter and $\mathcal{L}_2(x,y)$ is the non-local operator defined by (3.3) below. The main difficulty lies in analyzing the regularity of the solution $\Phi(x,y)$ of (1.2). In the case of Wiener noise, one needs the $C^2$-regularity of $\Phi$ with respect to $x$ in order to apply Itô’s formula with respect to $X^\epsilon_t$. However, in the case of $\alpha$-stable noise, $C^{\alpha+}$-regularity of $\Phi$ with respect to $x$ is enough for the application of Itô’s formula, where $\varphi \in C^{\alpha+}(\mathbb{R}^{d_1})$ means there exists a constant $\gamma > \alpha$ such that $\varphi \in C^\gamma(\mathbb{R}^{d_1})$ (the detailed definition is given below). Thus, in this paper we will only assume that $b$ has $C^{\alpha+}$-regularity with respect to the $x$ variable.

The organization of this paper is as follows. In the next section, we introduce some notation and state our main results. Section 3 is devoted to study the regularity of the solution for the Poisson equation. The strong and weak convergence rates are proved in Subsections 4.1 and 4.2 respectively. Finally, some basic properties for the $\alpha$-stable process as well as some a priori estimates for the solutions of the system (1.1), the frozen equation (2.4) and averaged equation (2.3) are given in the Appendix. Throughout this paper, $C$ and $C_T$ stand for constants whose value may change from line to line, and $C_T$ is used to emphasize that the constant depends on $T$.

2. Notations and Main results

We first introduce some notation throughout this paper. $\mathbb{R}^d$ stands for the $d$-dimensional Euclidean space and $\mathbb{N}_+$ stands for the collection of all the positive integers. We will use $|\cdot|$ and $\langle\cdot,\cdot\rangle$ to denote the Euclidean norm and Euclidean inner product respectively. We use $\|\cdot\|$ to denote the matrix norm. For any $k \in \mathbb{N}_+$ and $\delta \in (0,1)$, we define

$$C^k(\mathbb{R}^d) := \{u : \mathbb{R}^d \to \mathbb{R} : u \text{ and all its partial derivatives up to order } k \text{ are continuous}\},$$

$$C^k_b(\mathbb{R}^d) := \{u \in C^k(\mathbb{R}^d) : \text{ for } 1 \leq i \leq k, \text{ the } i\text{-the partial derivatives of } u \text{ are bounded}\},$$

$$C^k_b^{\delta}(\mathbb{R}^d) := \{u \in C^k_b(\mathbb{R}^d) : \text{ all the } k\text{-th order partial derivatives of } u \text{ are } \delta\text{-Hölder continuous}\}.$$
For any $k \in \mathbb{N}_+$, $\delta \in (0, 1)$, the spaces $C^k_b(\mathbb{R}^d)$ and $C^{k+\delta}_b(\mathbb{R}^d)$ when equipped with the usual norm $\| \cdot \|_{C^k_b}$ and $\| \cdot \|_{C^{k+\delta}_b}$, are Banach spaces.

For $k_1, k_2 \in \mathbb{N}_+, 0 \leq \delta_1, \delta_2 < 1$ and a real-valued function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, the notation $u \in C^{k_1+\delta_1, k_2+\delta_2}_b$ means that (i) for all $d_1$-tuple $\beta$ and $d_2$-tuple $\gamma$ with $0 \leq |\beta| \leq k_1$, $0 \leq |\gamma| \leq k_2$ and $|\beta| + |\gamma| \geq 1$, the partial derivative $\partial^\beta x \partial^\gamma y u$ is bounded continuous; and (ii) for any $|\beta| = k_1$ and $0 \leq |\gamma| \leq 1$, $\partial^\beta x \partial^\gamma_2 u$ is $\delta_1$-Hölder continuous with respect to $x$ with index $\delta_1$ uniformly in $y$, and for any $|\gamma| = k_2$, $\partial^\beta_2 \partial^\gamma_1 u$ is $\delta_2$-Hölder continuous with respect to $y$ with index $\delta_2$ uniformly in $x$.

Our first result is as follows.

**Theorem 2.1.** (Strong convergence) Suppose that $b \in C^{1+\gamma,2+\delta}_b$ and $f \in C^{1+\gamma,2+\gamma}_b$ with some $\gamma \in (\alpha - 1, 1)$ and $\delta \in (0, 1)$, and that there exists $\beta > 0$ such that for any $x \in \mathbb{R}^{d_1}$, $y_1, y_2 \in \mathbb{R}^{d_2}$,

$$
\sup_{x \in \mathbb{R}^{d_1}} |f(x, 0)| < \infty, \quad \langle f(x, y_1) - f(x, y_2), y_1 - y_2 \rangle \leq -\beta |y_1 - y_2|^2.
$$

(2.1)

Then for any initial value $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $T > 0$ and $p \in [1, \alpha)$, we have

$$
\mathbb{E} \left( \sup_{t \in [0, T]} |X^x_t - \bar{X}_t|^p \right) \leq Ce^{p(1-1/\alpha)},
$$

(2.2)

where $C$ is a positive constant depending on $p, T, |x|, |y|$, and $\bar{X}$ is the solution of the following averaged equation:

$$
d\bar{X}_t = \bar{b}(\bar{X}_t)dt + dL^1_t, \quad \bar{X}_0 = x,
$$

(2.3)

where $\bar{b}(x) = \int_{\mathbb{R}^{d_2}} b(x, y) \mu(x)(dy)$, and $\mu(x)$ denotes the unique invariant measure for the transition semigroup of the corresponding frozen equation

$$
dY_t = f(x, Y_t)dt + dL^2_t, \quad Y_0 = y.
$$

(2.4)

**Example 2.2.** The estimate (2.2) implies that the strong convergent order is $1 - 1/\alpha$. Following [28, Section 2], we show via an simple example that $1 - 1/\alpha$ is the optimal order. Consider

$$
\begin{cases}
    dX^x_t = Y^x_t dt + dL^1_t, & X^x_0 = x \in \mathbb{R}, \\
    dY^x_t = -\frac{1}{\epsilon} Y^x_t dt + \frac{1}{\epsilon^{1/\alpha}} dL^2_t, & Y^x_0 = 0 \in \mathbb{R},
\end{cases}
$$

where $\{L^1_t\}_{t \geq 0}$ and $\{L^2_t\}_{t \geq 0}$ are independent 1-dimensional symmetric $\alpha$-stable process. The solution of the equation above is given by

$$
\begin{cases}
    X^x_t = x + \int_0^t Y^x_s ds + L^1_t, \\
    Y^x_t = \frac{1}{\epsilon^{1/\alpha}} \int_0^t e^{-(t-s)/\epsilon} dL^2_s,
\end{cases}
$$
Note that the corresponding frozen equation is
\[ dY_t = -Y_t dt + dL^2_t, \quad Y_0 = 0, \]
which has a unique solution \( Y^y_t = \int_0^t e^{-(t-s)} dL^2_s \). Then it is easy to prove \( \{Y^y_t, t \geq 0\} \) has a unique invariant measure with mean zero. Thus, the corresponding averaged equation is given by
\[ \bar{X}_t = x + L^1_t. \]
As a result, we have for \( 0 < p < \alpha \),
\[ E|X^y_t - \bar{X}_t|^p = E \left| \int_0^t Y^\epsilon_s ds \right|^p. \]
Put \( Z^\epsilon_t := \int_0^t Y^\epsilon_s ds \), then it is easy to see that
\[ Z^\epsilon_t = \frac{1}{\epsilon^{1/\alpha}} \int_0^t \int_0^s e^{-\frac{1}{\alpha}(s-r)} dL^2_r dr = \frac{1}{\epsilon^{1/\alpha}} \int_0^t \left[ \int_r^t e^{-\frac{1}{\alpha}(s-r)} ds \right] dL^2_r. \]
Refer to [35, (2.4)], for any continuous function \( f : [0,t] \rightarrow \mathbb{R} \), we have
\[ E \left[ e^{ih \int_0^t f_s dL^2_s} \right] = \exp \left\{ -\int_0^t \psi(f_s h) ds \right\}, \quad h \in \mathbb{R}, \]
where \( \psi(x) = -C_\alpha |x|^{\alpha} \). As a result, the characteristic function of \( Z^\epsilon_t \) is given by
\[ \varphi_{Z^\epsilon}(h) := E \left[ e^{ihZ^\epsilon_t} \right] = \exp \left\{ -\int_0^t \psi \left( \frac{h}{\epsilon^{1/\alpha}} \int_r^t e^{-\frac{1}{\alpha}(s-r)} ds \right) dr \right\} \]
\[ = \exp \left\{ -\int_0^t C_\alpha (1 - e^{-z})^\alpha dr \left( e^{1-1/\alpha} \right)^{\alpha} |h|^\alpha \right\}, \quad h \in \mathbb{R}. \]
Refer to [35, (3.2)], for any symmetric real \( \alpha \)-stable random variable \( X \) has the characteristic function:
\[ E \left[ e^{ihX} \right] = \exp \{ -\sigma |h|^{\alpha} \}, \]
for some \( \sigma \geq 0 \), then we have
\[ E|X|^p = C_{\alpha, p} \sigma^p, \quad 0 < p < \alpha. \]
Thus, it is easy to see that
\[ E \left| \int_0^t Y^\epsilon_s ds \right|^p = C_{\alpha, p} \left[ \int_0^t (1 - e^{-z})^\alpha dr \right]^{p/\alpha} \left( e^{1-1/\alpha} \right)^p, \]
which implies the desired result.
The following is our second main result about the weak convergence rate.

**Theorem 2.3.** (Weak convergence) Suppose that the assumptions in Theorem 2.1 holds. Assume further that \( b \) is uniformly bounded and \( b, f \in C^{2+\gamma} \) with \( \gamma \in (\alpha - 1, 1) \). Then for any \( \phi \in C^{2+\gamma} \), initial value \((x, y) \in \mathbb{R}^d \times \mathbb{R}^{d_2}\) and \( T > 0 \), we have

\[
\sup_{t \in [0,T]} |\mathbb{E}\phi(X_t^\epsilon) - \mathbb{E}\phi(\bar{X}_t)| \leq C\epsilon, \tag{2.5}
\]

where \( C \) is a positive constant depending on \( T, ||\phi||_{C^{2+\gamma}}, |x| \) and \( |y| \), and \( \bar{X} \) is the solution of the averaged equation (2.3).

### 3. Non-local Poisson equation

This section is devoted to study the Poisson equation. In Subsection 3.1, we prove the exponential ergodicity for the transition semigroup of the frozen equation (2.4). In Subsection 3.2, we prove the well-posedness as well as the regularities of the solution for the Poisson equation (1.2).

#### 3.1. The frozen equation

Let \( Y_t^{x,y} \) satisfies the frozen equation

\[
dY_t = f(x, Y_t)dt + dL_t^2, \quad Y_0 = y \in \mathbb{R}^{d_2}.
\]

Some moment estimates for \( Y_t^{x,y} \) are collected in Lemma 5.2 in Appendix. Note that for any \( \epsilon > 0 \), define \( \tilde{L}_t^\epsilon := \frac{1}{\epsilon^{1/\alpha}} L^2_{\epsilon t}, t \geq 0 \), then \( \tilde{L}_t^\epsilon \) is again an \( \alpha \)-stable process and

\[
Y_t^\epsilon = y + \frac{1}{\epsilon} \int_0^t f(X_s^\epsilon, Y_s^\epsilon) ds + \frac{1}{\epsilon^{1/\alpha}} L_t^\epsilon = y + \int_0^t f(X_s^\epsilon, Y_s^\epsilon) ds + \tilde{L}_t^\epsilon. \tag{3.1}
\]

This explains the scaling \( \epsilon^{1/\alpha} \) in the fast component of (1.1). Here, we prove the following results, which will be used below.

**Lemma 3.1.** Assume that \( f \in C^{1,1}_b \) and condition (2.1) holds. Then for any \( t \geq 0 \), \( x_i \in \mathbb{R}^{d_i} \), and \( y_i \in \mathbb{R}^{d_2}, i = 1, 2 \), we have

\[
|Y_t^{x_1,y_1} - Y_t^{x_2,y_2}| \leq e^{-\beta t} |y_1 - y_2| + C |x_1 - x_2|,
\]

where \( C \) is a constant independent of \( t \).

**Proof.** Note that

\[
d(Y_t^{x_1,y_1} - Y_t^{x_2,y_2}) = [f(x_1, Y_t^{x_1,y_1}) - f(x_2, Y_t^{x_2,y_2})] dt, \quad Y_0^{x_1,y_1} - Y_0^{x_2,y_2} = y_1 - y_2.
\]
Multiplying both sides by $2(Y_t^{x_1,y_1} - Y_t^{x_2,y_2})$, we obtain
\[
\frac{d}{dt}|Y_t^{x_1,y_1} - Y_t^{x_2,y_2}|^2 = 2\langle f(x_1,Y_t^{x_1,y_1}) - f(x_2,Y_t^{x_2,y_2}), Y_t^{x_1,y_1} - Y_t^{x_2,y_2}\rangle.
\]

Then by condition (2.1) and Young’s inequality, we get
\[
\frac{d}{dt}|Y_t^{x_1,y_1} - Y_t^{x_2,y_2}|^2 \leq 2\langle f(x_1,Y_t^{x_1,y_1}) - f(x_2,Y_t^{x_2,y_2}), Y_t^{x_1,y_1} - Y_t^{x_2,y_2}\rangle
\]
\[
+ 2\langle f(x_1,Y_t^{x_2,y_2}) - f(x_2,Y_t^{x_2,y_2}), Y_t^{x_1,y_1} - Y_t^{x_2,y_2}\rangle
\]
\[
\leq -2\beta |Y_t^{x_1,y_1} - Y_t^{x_2,y_2}|^2 + C|x_1 - x_2||Y_t^{x_1,y_1} - Y_t^{x_2,y_2}|
\]
\[
\leq -\beta |Y_t^{x_1,y_1} - Y_t^{x_2,y_2}|^2 + C|x_1 - x_2|^2.
\]

As a result of the comparison theorem, we have that for any $t \geq 0$,
\[
|Y_t^{x_1,y_1} - Y_t^{x_2,y_2}|^2 \leq e^{-\beta t}|y_1 - y_2|^2 + C|x_1 - x_2|^2.
\]

The proof is complete.

Let $\{P^y_t\}_{t \geq 0}$ be the transition semigroup of $Y_t^{x,y}$, i.e., for any bounded measurable function $\varphi : \mathbb{R}^d \to \mathbb{R}$,
\[
P^y_t \varphi(y) := \mathbb{E} \varphi(Y_t^{x,y}), \quad y \in \mathbb{R}^d, \quad t \geq 0.
\]

Then condition (2.1) ensures that $P^y_t$ has a unique invariant measure $\mu^x$ (see e.g. [41, Theorem 1.1]). Moreover, in view of (5.12) in the Appendix, we have for any $p \in (0, \alpha)$,
\[
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d^2} |z|^p \mu^x(dz) < \infty.
\]

The following result will play an important role below.

**Proposition 3.2.** Assume that $f(x, \cdot) \in C^1_b$ and condition (2.1) holds. Then for any function $g \in C_b$, there exists a positive constant $C$ such that for any $t \geq 0$ and $y \in \mathbb{R}^d$,
\[
\sup_{x \in \mathbb{R}^d} |P^y_t g(y) - \mu^x(g)| \leq C\|g\|_1 e^{-\frac{\beta t}{\tau}} (1 + |y|), \quad \text{(3.2)}
\]

where $\|g\|_1 := \sup_{x \neq y \in \mathbb{R}^d} \left|\frac{g(x) - g(y)}{x - y}\right|$.

**Proof.** By the definition of invariant measure and Lemma 3.1, for any $t \geq 0$ we have
\[
|\mathbb{E} g(Y_t^{x,y}) - \mu^x(g)| = \left|\mathbb{E} g(Y_t^{x,y}) - \int_{\mathbb{R}^d^2} g(z) \mu^x(dz)\right|
\]
\[
\leq \left|\int_{\mathbb{R}^d^2} [\mathbb{E} g(Y_t^{x,y}) - \mathbb{E} g(Y_t^{z,y})] \mu^x(dz)\right|
\]
\[
\leq \|g\|_1 \int_{\mathbb{R}^d^2} \mathbb{E} |Y_t^{x,y} - Y_t^{z,y}| \mu^x(dz)
\]
\[
\leq \|g\|_1 e^{-\frac{\beta t}{\tau}} \int_{\mathbb{R}^d^2} |y - z| \mu^x(dz) \leq C\|g\|_1 e^{-\frac{\beta t}{\tau}} (1 + |y|).
\]

The proof is complete. \qed
3.2. Poisson equation

Let \( \mathcal{L}_2(x,y) \) be the generator of \( Y_t^{x,y} \), i.e.,

\[
\mathcal{L}_2(x,y)\Phi(x,y) := -(-\Delta_y)^{\alpha/2}\Phi(x,y) + \langle f(x,y), \nabla_y \Phi(x,y) \rangle.
\]  

(3.3)

The following is the main result of this subsection.

**Proposition 3.3.** Suppose that the assumptions in Theorem 2.1 hold. Define

\[
\Phi(x,y) := \int_0^\infty \left[ \mathbb{E} b(x,Y_t^{x,y}) - \bar{b}(x) \right] dt.
\]  

(3.4)

Then \( \Phi(x,y) \) is a solution of the Poisson equation (1.2). Moreover, we have \( \Phi(\cdot, y) \in C^1(\mathbb{R}^{d_1}, \mathbb{R}^{d_1}) \), \( \Phi(x, \cdot) \in C^2(\mathbb{R}^{d_2}, \mathbb{R}^{d_1}) \), and there exists \( C > 0 \) such that

\[
\sup_{x \in \mathbb{R}^{d_1}} |\Phi(x,y)| \leq C(1 + |y|), \quad \sup_{x \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2}} \|\nabla_y \Phi(x,y)\| \leq C,
\]  

(3.5)

and for any \( \theta \in (0,1] \), there exists \( C_\theta > 0 \) such that for any \( x_1, x_2 \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2} \),

\[
\sup_{x \in \mathbb{R}^{d_1}} \|\nabla_x \Phi(x,y)\| \leq C_\theta (1 + |y|^{\theta}),
\]  

(3.6)

\[
\|\nabla_x \Phi(x_1,y) - \nabla_x \Phi(x_2,y)\| \leq C\|x_1 - x_2\|^{\gamma}(1 + |x_1 - x_2|^{1-\gamma})(1 + |y|),
\]  

(3.7)

where \( \gamma \in (\alpha - 1, 1) \) is the constant in Theorem 2.1.

**Proof.** The assertion that (3.4) is a solution of the Poisson equation (1.2) follows by Itô’s formula. Moreover, by straightforward computation, we can see that \( \Phi(\cdot, y) \in C^1(\mathbb{R}^{d_1}, \mathbb{R}^{d_1}) \) and \( \Phi(x, \cdot) \in C^2(\mathbb{R}^{d_2}, \mathbb{R}^{d_1}) \). Below, we mainly focus on the regularity estimates (3.5)-(3.7). By Proposition 3.2, we have

\[
|\Phi(x,y)| \leq \int_0^\infty |\mathbb{E}[b(x,Y_t^{x,y})] - \bar{b}(x)| dt
\leq C(1 + |y|) \int_0^\infty e^{-\frac{\beta t}{\tau}} dt \leq C_\beta(1 + |y|),
\]

which implies the first estimate in (3.5). Note that

\[
\nabla_y \Phi(x,y) = \int_0^\infty \mathbb{E} [\nabla_y b(x,Y_t^{x,y}) \cdot \nabla_y Y_t^{x,y}] dt,
\]

where \( \nabla_y Y_t^{x,y} \) satisfies

\[
\left\{
\begin{array}{l}
d\nabla_y Y_t^{x,y} = \nabla_y f(x,Y_t^{x,y}) \cdot \nabla_y Y_t^{x,y} dt, \\
\nabla_y Y_0^{x,y} = I.
\end{array}
\right.
\]

(3.8)
Then by Lemma 3.1, we have
\[ \sup_{x \in \mathbb{R}^d, y \in \mathbb{R}^{d_2}} \| \nabla_y \Phi(x, y) \| \leq C \varepsilon^{-\frac{\beta}{2}}. \] (3.9)

Thus, by the boundedness of \( \| \nabla_y b(x, y) \| \), there exists \( C > 0 \) such that
\[ \sup_{x \in \mathbb{R}^d, y \in \mathbb{R}^{d_2}} \| \nabla_y \Phi(x, y) \| \leq C, \]
which implies the second estimate in (3.5).

Now, we define
\[ \hat{b}_{t_0}(x, y, t) := \hat{b}(x, y, t) - \hat{b}(x, y, t + t_0), \]
where \( \hat{b}(x, y, t) := \mathbb{E} b(x, Y_t^{x,y}) \). We claim that for any \( \theta \in (0, 1] \), there exist \( C_{\theta} > 0 \) and \( \eta > 0 \) such that for any \( t_0 > 0, t > 0, x \in \mathbb{R}^d \) and \( y \in \mathbb{R}^{d_2} \), we have
\[ \| \nabla_x \hat{b}_{t_0}(x, y, t) \| \leq C_{\theta} e^{-\eta t} (1 + |y|^{\theta}), \] (3.10)
\[ \| \nabla_x \hat{b}_{t_0}(x_1, y, t) - \nabla_x \hat{b}_{t_0}(x_2, y, t) \| \leq C e^{-\eta t} |x_1 - x_2|^\gamma (1 + |x_1 - x_2|^{1-\gamma})(1 + |y|). \] (3.11)

On the other hand, Proposition 3.2 implies that
\[ \lim_{t_0 \to +\infty} \hat{b}_{t_0}(x, y, t) = \mathbb{E}[b(x, Y_{t_0}^{x,y})] - \hat{b}(x) \]
and thus
\[ \lim_{t_0 \to +\infty} \nabla_x \hat{b}_{t_0}(x, y, t) = \nabla_x \left[ \mathbb{E}[b(x, Y_{t_0}^{x,y})] - \hat{b}(x) \right]. \]

Consequently, we have estimates (3.6) and (3.7) hold. Below, we provide the proof of estimates (3.10) and (3.11) separately. \( \square \)

**Proof of (3.10).** By the Markov property, we write
\[ \hat{b}_{t_0}(x, y, t) = \hat{b}(x, y, t) - \mathbb{E} b(x, Y_{t_0}^{x,y}) \]
\[ = \hat{b}(x, y, t) - \mathbb{E} \{ \mathbb{E}[b(x, Y_{t_0}^{x,y})] | \mathcal{F}_{t_0} \} \]
\[ = \hat{b}(x, y, t) - \mathbb{E} b(x, Y_{t_0}^{x,y}, t). \]

Thus
\[ \nabla_x \hat{b}_{t_0}(x, y, t) = \nabla_x \hat{b}(x, y, t) - \mathbb{E} \nabla_x \hat{b}(x, Y_{t_0}^{x,y}, t) - \mathbb{E} \left[ \nabla_y \hat{b}(x, Y_{t_0}^{x,y}, t) \cdot \nabla_x Y_{t_0}^{x,y} \right]. \] (3.12)

where \( \nabla_x Y_{t}^{x,y} \) satisfies
\[ d \nabla_x Y_{t}^{x,y} = \nabla_x f(x, Y_{t}^{x,y}) dt + \nabla_y f(x, Y_{t}^{x,y}) \cdot \nabla_x Y_{t}^{x,y} dt, \quad \nabla_x Y_{0}^{x,y} = 0. \]
Obviously, Lemma 3.1 implies
\[
\sup_{t \geq 0, x \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2}} \| \nabla_x Y^{x,y}_t \| \leq C. \tag{3.13}
\]

Note that \( \nabla_y \hat{b}(x, y, t) = \mathbb{E}[\nabla_y b(x, Y^{x,y}_t) \cdot \nabla_y Y^{x,y}_t] \), combining this with (3.9) and the boundedness of \( \| \nabla_y b(x, y, t) \| \), we get
\[
\sup_{x \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2}} \| \nabla_y \hat{b}(x, y, t) \| \leq Ce^{-\frac{\alpha t}{2}}, \tag{3.14}
\]
Next, we can show that for any \( \theta \in (0, 1] \), there exists \( C_\theta > 0 \) such that for any \( t \geq 0, x \in \mathbb{R}^{d_1} \) and \( y_1, y_2 \in \mathbb{R}^{d_2} \),
\[
\| \nabla_x \hat{b}(x, y_1, t) - \nabla_x \hat{b}(x, y_2, t) \| \leq C_\theta e^{-\frac{\alpha t}{2}} |y_1 - y_2|^\theta. \tag{3.15}
\]

Then, by (3.12), estimates (3.13)-(3.15) and (5.12), we have
\[
\| \nabla_x \hat{b}_0(x, y, t) \| \leq C_\theta e^{-\frac{\alpha t}{2}} \mathbb{E}[|y - Y^{x,y}_t|^\theta] + Ce^{-\frac{\beta t}{2}} \leq C_\theta e^{-\frac{\beta t}{2}} (1 + |y|^\theta),
\]
which proves (3.10).

Now, we proceed to prove (3.15). Indeed,
\[
\| \nabla_x \hat{b}(x, y_1, t) - \nabla_x \hat{b}(x, y_2, t) \| = \| \nabla_x \mathbb{E}[b(x, Y^{x,y}_t) - \nabla_x \mathbb{E}[b(x, Y^{x,y}_t)]
\]
\[
\leq \mathbb{E} \| \nabla_x b(x, Y^{x,y}_t) - \nabla_x b(x, Y^{x,y}_t) \| + \mathbb{E} \| \nabla_x b(x, Y^{x,y}_t) \cdot \nabla_x Y^{x,y}_t - \nabla_y b(x, Y^{x,y}_t) \cdot \nabla_x Y^{x,y}_t \|
\]
\[
\leq \mathbb{E} \| \nabla_x b(x, Y^{x,y}_t) - \nabla_x b(x, Y^{x,y}_t) \| + \mathbb{E} \| \nabla_y b(x, Y^{x,y}_t) \cdot \nabla_x Y^{x,y}_t - \nabla_y b(x, Y^{x,y}_t) \cdot \nabla_x Y^{x,y}_t \|
\]
\[
\leq C_\theta e^{-\frac{\alpha t}{2}} |y_1 - y_2|^\theta. \tag{3.16}
\]

By the boundedness of \( \| \nabla_x b(x, y) \|, \| \nabla_y \nabla_x b(x, y) \| \) and \( \| \partial_y^2 b(x, y) \| \), we have for any \( \theta \in (0, 1] \),
\[
S_1 \leq C_\theta E[Y^{x,y}_t - Y^{x,y}_t]^\theta \leq C_\theta e^{-\frac{\alpha t}{2}} |y_1 - y_2|^\theta, \tag{3.17}
\]
and by the boundedness of \( \| \nabla_y^2 b(x, y) \| \) and Lemma 3.1,
\[
S_2 \leq C_\theta E[Y^{x,y}_t - Y^{x,y}_t]^\theta \leq C_\theta e^{-\frac{\alpha t}{2}} |y_1 - y_2|^\theta. \tag{3.18}
\]

For the term \( S_3 \), by the assumption \( f \in C^{1+\gamma,2}_b \), condition (2.1) and a straightforward computation, we have that for any \( x_1, x_2 \in \mathbb{R}^{d_1}, y_1, y_2 \in \mathbb{R}^{d_2},\)
\[
\| \nabla_x Y^{x_1,y_1}_t - \nabla_x Y^{x_2,y_2}_t \|^2 \leq C(\|x_1 - x_2\|^\gamma + \|x_1 - x_2\|^2) + e^{-\frac{\alpha t}{2}} |y_1 - y_2|^2. \tag{3.19}
\]

Then by (3.13) and (3.18), it is easy to see
\[
S_3 \leq C_\theta E \| \nabla_x Y^{x,y}_t - \nabla_x Y^{x,y}_t \|^\theta \leq C_\theta e^{-\frac{\alpha t}{2}} |y_1 - y_2|^\theta. \tag{3.19}
\]
Now (3.15) follows directly from (3.16)-(3.19). \( \square \)
Proof of (3.11). Recall that
\[ \nabla_x \tilde{b}_0(x, y, t) = \nabla_x \hat{b}(x, y, t) - \mathbb{E} \nabla_x \hat{b}(x, Y_{t_0}^{x,y}, t) - \mathbb{E} \left[ \nabla_y \hat{b}(x, Y_{t_0}^{x,y}, t) \cdot \nabla_x Y_{t_0}^{x,y} \right]. \]

Then we get for any \( x_1, x_2 \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2} \) and \( t, t_0 > 0 \),
\[
\begin{align*}
& \| \nabla_x \tilde{b}_0(x_1, y, t) - \nabla_x \tilde{b}_0(x_2, y, t) \| \\
& = \| \nabla_x \hat{b}(x_1, y, t) - \mathbb{E} \nabla_x \hat{b}(x, Y_{t_0}^{x,y}, t) - \left[ \nabla_x \hat{b}(x_2, y, t) - \mathbb{E} \nabla_x \hat{b}(x_2, Y_{t_0}^{x,y}, t) \right] \\
& \quad - \mathbb{E} \left[ \nabla_y \hat{b}(x_1, Y_{t_0}^{x,y}, t) \cdot \nabla_x Y_{t_0}^{x,y} \right] + \mathbb{E} \left[ \nabla_y \hat{b}(x_2, Y_{t_0}^{x,y}, t) \cdot \nabla_x Y_{t_0}^{x,y} \right] \| \\
& \leq \| \nabla_x \hat{b}(x_1, y, t) - \mathbb{E} \nabla_x \hat{b}(x, Y_{t_0}^{x,y}, t) - \left[ \nabla_x \hat{b}(x_2, y, t) - \mathbb{E} \nabla_x \hat{b}(x_2, Y_{t_0}^{x,y}, t) \right] \| \\
& \quad + \| \mathbb{E} \nabla_x \hat{b}(x_2, Y_{t_0}^{x,y}, t) - \mathbb{E} \nabla_x \hat{b}(x_2, Y_{t_0}^{x_1,y}, t) \| \\
& \quad + \| \mathbb{E} \left[ \nabla_y \hat{b}(x_1, Y_{t_0}^{x,y}, t) \cdot \nabla_x Y_{t_0}^{x,y} \right] - \mathbb{E} \left[ \nabla_y \hat{b}(x_2, Y_{t_0}^{x_2,y}, t) \cdot \nabla_x Y_{t_0}^{x_2,y} \right] \| = \sum_{i=1}^{3} Q_i.
\end{align*}
\]

(i) For the term \( Q_1 \), recall that
\[ \nabla_x \hat{b}(x, y, t) = \mathbb{E} \left[ \nabla_x b(x, Y_t^{x,y}) \right] + \mathbb{E} \left[ \nabla_y b(x, Y_t^{x,y}) \cdot \nabla_x Y_t^{x,y} \right], \]
which implies
\[
\begin{align*}
& \| \nabla_x \hat{b}(x_1, y_1, t) - \nabla_x \hat{b}(x_2, y_2, t) - \left[ \nabla_x \hat{b}(x_1, y_1, t) - \nabla_x \hat{b}(x_2, y_2, t) \right] \| \\
& = \| \mathbb{E} \nabla_x b(x_1, Y_t^{x_1,y_1}) + \mathbb{E} \nabla_y b(x_1, Y_t^{x_1,y_1}) \cdot \nabla_x Y_t^{x_1,y_1} \| \\
& \quad - \mathbb{E} \nabla_y b(x_1, Y_t^{x_1,y_1}) - \mathbb{E} \nabla_x b(x_1, Y_t^{x_1,y_1}) \cdot \nabla_x Y_t^{x_1,y_1} \\
& \quad - \mathbb{E} \nabla_y b(x_2, Y_t^{x_2,y_2}) - \mathbb{E} \nabla_x b(x_2, Y_t^{x_2,y_2}) \cdot \nabla_x Y_t^{x_2,y_2} \\
& \quad + \mathbb{E} \left[ \nabla_y b(x_2, Y_t^{x_2,y_2}) \cdot \nabla_x Y_t^{x_2,y_2} \right] \| \leq \sum_{i=1}^{3} Q_{1i},
\end{align*}
\]

where
\[
Q_{11} := \| \mathbb{E} \nabla_x b(x_1, Y_t^{x_1,y_1}) - \nabla_x b(x_1, Y_t^{x_1,y_1}) - \left( \nabla_x b(x_2, Y_t^{x_1,y_1}) - \nabla_x b(x_2, Y_t^{x_1,y_2}) \right) \|,
\]
\[
Q_{12} := \| \mathbb{E} \nabla_x b(x_2, Y_t^{x_1,y_1}) - \nabla_x b(x_2, Y_t^{x_2,y_2}) - \left( \nabla_x b(x_2, Y_t^{x_1,y_2}) - \nabla_x b(x_2, Y_t^{x_2,y_2}) \right) \|,
\]
\[
Q_{13} := \| \mathbb{E} \nabla_y b(x_1, Y_t^{x_1,y_1}) \cdot \nabla_x Y_t^{x_1,y_1} - \nabla_y b(x_2, Y_t^{x_2,y_1}) \cdot \nabla_x Y_t^{x_2,y_1} \\
& \quad - \mathbb{E} \nabla_y b(x_1, Y_t^{x_1,y_1}) \cdot \nabla_x Y_t^{x_2,y_2} - \nabla_y b(x_2, Y_t^{x_2,y_2}) \cdot \nabla_x Y_t^{x_2,y_2} \|.
\]

By the assumption that \( \nabla_x \nabla_y b(x, y) \) is Hölder continuous with respect to \( x \) with index \( \gamma \) and Lemma 3.1, we get that
\[
Q_{11} \leq \mathbb{E} \left\| \int_0^1 \left[ \nabla_x \nabla_y b(x_1, \xi Y_t^{x_1,y_1} + (1 - \xi) Y_t^{x_1,y_2}) \right] d\xi \cdot (Y_t^{x_1,y_1} - Y_t^{x_1,y_2}) \right\|.
\]
\[-\int_0^1 \left[ \nabla_x \nabla_y b(x, \xi Y_t^{x_1, y_1} + (1 - \xi) Y_t^{x_2, y_2}) \right] d\xi \cdot (Y_t^{x_1, y_1} - Y_t^{x_2, y_2}) \leq C|x_1 - x_2|^7 |y_1 - y_2| e^{-\frac{\beta t}{2}}. \] (3.20)

By the boundedness of \(|\nabla_x \partial^2_y b(x, y)|\) and \(|\nabla_x \nabla_y b(x, y)|\), we obtain

\[
Q_{12} \leq \mathbb{E} \left[ \int_0^1 \left[ \nabla_x \nabla_y b(x, \xi Y_t^{x_1, y_1} + (1 - \xi) Y_t^{x_2, y_2}) \right] d\xi \cdot \nabla_x \nabla_y b(x, \xi Y_t^{x_1, y_1} + (1 - \xi) Y_t^{x_2, y_2}) \right] \leq C|x_1 - x_2|^7 |y_1 - y_2| e^{-\frac{\beta t}{2}}, \] (3.21)

where the last inequality is due to Lemma 3.1 and estimate (3.18). Note that

\[
Q_{13} \leq \mathbb{E} \left[ \left| \nabla_y b(x_1, Y_t^{x_1, y_1}) - \nabla_y b(x_2, Y_t^{x_1, y_1}) \right| \cdot \nabla_x Y_t^{x_1, y_1} \right] \leq C|x_1 - x_2| |y_1 - y_2| e^{-\frac{\beta t}{2}}. \]

Since \(|\nabla_x \nabla_y b(x, y)|\), \(|\nabla_y b(x, y)|\), \(|\nabla_x \partial^2_y b(x, y)|\) are bounded and \(\nabla_y b(x, y)\) is Hölder continuous with respect to \(y\) with index \(\delta\), we have

\[
Q_{131} \leq \mathbb{E} \left[ \left| \nabla_x \nabla_y b(x_1 + (1 - \xi) x_2, Y_t^{x_1, y_1}) \right| \cdot (x_1 - x_2, \nabla_x Y_t^{x_1, y_1}) \right] \leq C|x_1 - x_2| |y_1 - y_2| e^{-\frac{\beta t}{2}}. \] (3.22)

and

\[
Q_{132} \leq \mathbb{E} \left[ \int_0^1 \nabla_y^2 b(x_2, \xi Y_t^{x_1, y_1} + (1 - \xi) Y_t^{x_2, y_2}) d\xi \cdot (Y_t^{x_1, y_1} - Y_t^{x_2, y_2}, \nabla_x Y_t^{x_1, y_1}) \right] \leq C|x_1 - x_2| |y_1 - y_2| e^{-\frac{\beta t}{2}}. \]
Combining (3.22), (3.23) and (3.25), we get
\[ \gamma x_{\text{spect}} \]

The assumption \( f \in C_b^{1+\gamma,2+\gamma} \) implies that \( \nabla_x \nabla_y f(x, y) \) is Hölder continuous with respect to \( x \) with index \( \gamma \) and \( \nabla_y^2 f(x, y) \) is Hölder continuous with respect to \( y \) with index \( \gamma \), and \( \| \nabla_x \nabla_y^2 f(x, y) \| \) is uniformly bounded. By a straightforward computation, we get

\[ \sup_{y \in \mathbb{R}^{d_2}} \mathbb{E} \| \nabla_y \nabla_x Y^{x,y}_t - \nabla_y \nabla_x Y^{x,y}_0 \| \leq C e^{-\frac{\beta}{2} \gamma (1 + |x_1 - x_2|^{1-\gamma})}, \]  

(3.24)

and \( \nabla_y \nabla_x Y^{x,y}_t \) satisfies

\[ d \nabla_y \nabla_x Y^{x,y}_t = \nabla_y \nabla_x f(x, Y^{x,y}_t) \cdot \nabla_y Y^{x,y}_t dt + \partial_y^2 f(x, Y^{x,y}_t) \cdot (\nabla_y Y^{x,y}_t, \nabla_x Y^{x,y}_t) dt \]

\[ + \nabla_y f(x, Y^{x,y}_t) \cdot \nabla_y \nabla_x Y^{x,y}_t dt, \quad \nabla_y \nabla_x Y^{x,y}_0 = 0. \]

Using (3.18) and (3.24), we get

\[ Q_{13} \leq \mathbb{E} \| \nabla_y b(x_2, Y^{x,y}_t) - \nabla_y b(x_2, Y^{x,y}_0) \| + (\nabla_x Y^{x,y}_t - \nabla_x Y^{x,y}_0) \| \]

\[ + \mathbb{E} \| \nabla_y b(x_2, Y^{x,y}_0) \cdot (\nabla_x Y^{x,y}_t - \nabla_x Y^{x,y}_0) \| \]

\[ \leq C e^{-\frac{\beta}{2} \gamma (1 + |x_1 - x_2|^{1-\gamma}) |y_1 - y_2|}. \]  

(3.25)

Combining (3.22), (3.23) and (3.25), we get

\[ Q_{13} \leq C e^{-\frac{(\beta+\delta+\gamma)}{4} |x_1 - x_2|^{1-\gamma} (1 + |y_1 - y_2|^{1-\gamma})}. \]  

(3.26)

Finally, (3.20), (3.21) and (3.26) together imply

\[ Q_1 \leq C e^{-\frac{(\beta+\delta+\gamma)}{4} |x_1 - x_2|^{1-\gamma} (1 + |y_1 - y_2|^{1-\gamma})} E(|y - Y^{x,y}_{t_0}| + |y - Y^{x,y}_{t_0}|^\delta) \]

\[ \leq C e^{-\frac{(\beta+\delta+\gamma)}{4} |x_1 - x_2|^{1-\gamma} (1 + |y_1 - y_2|^{1-\gamma})} (1 + |y|). \]  

(3.27)

(ii) For the term \( Q_2 \), note that

\[ \nabla_y \nabla_x b(x, y, t) = \nabla_y \mathbb{E} [\nabla_x b(x, Y^{x,y}_t) \cdot \nabla_y Y^{x,y}_t] = \mathbb{E} [\nabla_x \nabla_y b(x, Y^{x,y}_t) \cdot \nabla_y Y^{x,y}_t] + \mathbb{E} [\nabla_y b(x, Y^{x,y}_t) \cdot (\nabla_x Y^{x,y}_t, \nabla_y Y^{x,y}_t)] + \mathbb{E} [\nabla_x \nabla_y b(x, Y^{x,y}_t) \cdot \nabla_x \nabla_y Y^{x,y}_t]. \]

(3.18) implies

\[ \sup_{x \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2}} \| \nabla_x \nabla_y Y^{x,y}_t \| \leq C e^{-\frac{\beta}{4} \gamma}. \]
Combining this with (3.13) and (3.9), we get
\[ \sup_{x \in \mathbb{R}^d_1, y \in \mathbb{R}^d_2} \| \nabla_x \nabla_y b(x, y, t) \| \leq Ce^{-\frac{\beta t}{4}}. \] (3.28)

Thus
\[ Q_2 \leq Ce^{-\frac{\beta t}{4}} E[Y_{t_0}^{x_2, y} - Y_{t_0}^{x_1, y}] \leq Ce^{-\frac{\beta t}{4}} |x_1 - x_2|. \] (3.29)

(iii) For the term \( Q_3 \), by a similar argument as in the proof of (3.28), we have
\[ \sup_{x \in \mathbb{R}^d_1, y \in \mathbb{R}^d_2} \| \nabla^2_y \hat{b}(x, y, t) \| \leq Ce^{-\frac{\beta t}{4}}, \]
which, together with (3.28), implies
\[ Q_3 \leq Ce^{-\frac{\beta t}{4}} |x_1 - x_2|. \] (3.30)

Combining (3.27), (3.29) and (3.30), we get (3.11). The proof is complete.

4. Proof of main results

In this section, we give the proofs of Theorem 2.1 and Theorem 2.3. Our arguments are based on the Poisson equation and are inspired by [5] (see also [31–33, 36, 37]).

4.1. Proof of Theorem 2.1

Note that
\[ X^*_t - \bar{X}_t = \int_0^t \left[ b(X^*_s, Y^*_s) - \bar{b}(\bar{X}_s) \right] ds \]
\[ = \int_0^t \left[ b(X^*_s, Y^*_s) - \bar{b}(X^*_s) \right] ds + \int_0^t \left[ \bar{b}(X^*_s) - \bar{b}(\bar{X}_s) \right] ds. \]

Using the Lipschitz continuity of \( \bar{b} \), one can easily show that for any \( p \in [1, \alpha) \),
\[ \mathbb{E} \left( \sup_{t \in [0,T]} |X^*_t - \bar{X}_t|^p \right) \leq C_p \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t b(X^*_s, Y^*_s) - \bar{b}(X^*_s) ds \right|^p \right] + C_{p,T} \mathbb{E} \int_0^T |X^*_s - \bar{X}_s|^p dt. \]

Grownall’s inequality implies
\[ \mathbb{E} \left( \sup_{t \in [0,T]} |X^*_t - \bar{X}_t|^p \right) \leq C_{p,T} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t b(X^*_s, Y^*_s) - \bar{b}(X^*_s) ds \right|^p \right]. \] (4.1)
By Proposition 3.3, there exists a function $\Phi(x, y)$ such that $\Phi(\cdot, y) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\Phi(x, \cdot) \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ and

$$-\mathcal{L}_2(x, y)\Phi(x, y) = b(x, y) - \bar{b}(x).$$

(4.2)

Moreover, the estimates (3.5) and (3.6) hold. By Itô’s formula (see [1, Theorem 4.4.7]), we have

$$\Phi(X^\epsilon_t, Y^\epsilon_t) = \Phi(x, y) + \int_0^t \mathcal{L}_1(Y^\epsilon_r)\Phi(X^\epsilon_r, Y^\epsilon_r)dr + \frac{1}{\epsilon} \int_0^t \mathcal{L}_2(X^\epsilon_r, Y^\epsilon_r)\Phi(x, y)dr + M^{\epsilon,1}_t + M^{\epsilon,2}_t,$$

where $\mathcal{L}_2(x, y)\Phi(x, y)$ is defined by (3.3) and

$$\mathcal{L}_1(y)\Phi(x, y) := -(-\Delta_x)^{\alpha/2} \Phi(x, y) + \langle b(x, y), \nabla_x \Phi(x, y) \rangle,$$

and $M^{\epsilon,1}_t, M^{\epsilon,2}_t$ are two $\mathcal{F}_t$-martingales defined by

$$M^{\epsilon,1}_t := \int_0^t \int_{\mathbb{R}^d_1} \Phi(X^\epsilon_{r-} + x, Y^\epsilon_{r-}) - \Phi(X^\epsilon_{r-}, Y^\epsilon_{r-})\hat{N}^1(dr, dx),$$

$$M^{\epsilon,2}_t := \int_0^t \int_{\mathbb{R}^d_2} \Phi(X^\epsilon_{r-}, Y^\epsilon_{r-} + \epsilon^{-1/\alpha} y) - \Phi(X^\epsilon_{r-}, Y^\epsilon_{r-})\hat{N}^2(dr, dy),$$

and $\hat{N}^i (i = 1, 2)$ are defined by (5.1). Consequently, we have

$$\int_0^t -\mathcal{L}_2(X^\epsilon_r, Y^\epsilon_r)\Phi(X^\epsilon_r, Y^\epsilon_r)dr = -\epsilon \Phi(x, y) - \Phi(X^\epsilon_t, Y^\epsilon_t)$$

$$+ \int_0^t \mathcal{L}_1(Y^\epsilon_r)\Phi(X^\epsilon_r, Y^\epsilon_r)dr + M^{\epsilon,1}_t + M^{\epsilon,2}_t.$$ 

(4.3)

Combining (4.1), (4.2) and (4.3), we get

$$\mathbb{E} \left( \sup_{t \in [0,T]} |X^\epsilon_t - \bar{X}^\epsilon_t|^p \right) \leq C_{p,T} \mathbb{E} \left[ \int_0^t -\mathcal{L}_2(X^\epsilon_r, Y^\epsilon_r)\Phi(X^\epsilon_r, Y^\epsilon_r)dr \right]^p \leq C_{p,T} \mathbb{E} \left[ \sup_{t \in [0,T]} |\Phi(x, y) - \Phi(X^\epsilon_t, Y^\epsilon_t)|^p \right] + \mathbb{E} \int_0^T |\mathcal{L}_1(Y^\epsilon_r)\Phi(X^\epsilon_r, Y^\epsilon_r)|^p dr$$

$$+ \mathbb{E} \left( \sup_{t \in [0,T]} |M^{\epsilon,1}_t|^p \right) + \mathbb{E} \left( \sup_{t \in [0,T]} |M^{\epsilon,2}_t|^p \right).$$

(4.4)

By (3.5) and (5.20), we have

$$\mathbb{E} \left( \sup_{t \in [0,T]} |\Phi(x, y) - \Phi(X^\epsilon_t, Y^\epsilon_t)|^p \right) \leq C(1 + |y|^p) + \mathbb{E} \left( \sup_{t \in [0,T]} |Y^\epsilon_t|^p \right)$$

(4.5)
It follows from (3.6) and (3.7) that

$$\mathbb{E} \int_0^T |\mathcal{L}_1(Y_s^\varepsilon)\Phi(X_s^\varepsilon, Y_s^\varepsilon)|^p \, dr \leq C_{p,T} (1 + |y|^p) e^{-p/\alpha}. \quad (4.5)$$

where $p < p' < \alpha$ and $\theta$ is small enough such that $\frac{\theta p'}{p' - p} \vee (p + \theta) < \alpha$.

Using Burkholder-Davis-Gundy's inequality (see e.g. [34, Lemma 8.22]) and (3.6), we get for any $\theta \in (0, 1/2)$,

$$\mathbb{E} \left( \sup_{t \in [0,T]} |M_t^{\varepsilon,1}|^p \right) \leq C_p \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \int_{|z| \leq 1} \Phi(X_s^\varepsilon + x, Y_s^\varepsilon) - \Phi(X_s^\varepsilon, Y_s^\varepsilon) |\mathcal{N}_1(ds, dx) \right|^p \right]$$

$$+ C_p \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \int_{|z| > 1} \Phi(X_s^\varepsilon + x, Y_s^\varepsilon) - \Phi(X_s^\varepsilon, Y_s^\varepsilon) |\mathcal{N}_1(ds, dx) \right|^p \right]$$

$$\leq C_p \mathbb{E} \left| \int_0^T \int_{|z| \leq 1} |\Phi(X_s^\varepsilon + x, Y_s^\varepsilon) - \Phi(X_s^\varepsilon, Y_s^\varepsilon)|^2 \mathcal{N}_1(ds, dx) \right|^p$$

$$+ C_p \mathbb{E} \left| \int_0^T \int_{|z| > 1} |\Phi(X_s^\varepsilon + x, Y_s^\varepsilon) - \Phi(X_s^\varepsilon, Y_s^\varepsilon)|^p \nu_1(dx) ds \right|$$

$$\leq C_p \left[ \mathbb{E} \int_0^T \int_{|z| \leq 1} |z|^2 \mathcal{N}_1(dx)(1 + |Y_s^\varepsilon |^2 \theta) ds \right]^{p/2}$$
and by (3.5), we have
\[ + C_p \mathbb{E} \int_0^T \int_{|x|>1} |x|^p \nu_1(dx)(1 + |Y_s^\epsilon|^p\theta)ds \leq C_{p,T}(1 + |y|^p), \tag{4.7} \]

and by (3.5), we have
\[
\begin{align*}
\mathbb{E} \left( \sup_{t \in [0,T]} |M_t|^{\alpha} \right)^{\frac{p}{\alpha}} &\leq C_p \mathbb{E} \left[ \sup_{t \in [0,T]} \left( \int_0^t \int_{|y| \leq 1} \Phi(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-1/\alpha} y) - \Phi(X_s^\epsilon, Y_s^\epsilon) \mathbb{N}^2(ds, dy) \right)^p \right] \\
&+ C_p \mathbb{E} \left[ \sup_{t \in [0,T]} \left( \int_0^t \int_{|y| > 1} \Phi(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-1/\alpha} y) - \Phi(X_s^\epsilon, Y_s^\epsilon) \mathbb{N}^2(ds, dy) \right)^p \right] \\
&\leq C_p \mathbb{E} \left[ \int_0^T \int_{|y| \leq 1} \left| \Phi(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-1/\alpha} y) - \Phi(X_s^\epsilon, Y_s^\epsilon) \right|^2 \mathbb{N}^2(ds, dy) \right]^{p/2} \\
&+ C_p \mathbb{E} \left[ \int_0^T \int_{|y| > 1} \left| \Phi(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-1/\alpha} y) - \Phi(X_s^\epsilon, Y_s^\epsilon) \right|^p \nu_2(dy)ds \right] \\
&\leq C_p \epsilon^{-p/\alpha} \left\{ \left[ \int_0^T \int_{|y| \leq 1} |y|^p \nu_2(dy)ds \right]^{p/2} + \int_0^T \int_{|y| > 1} |y|^p \nu_2(dy)ds \right\} \leq C_{p,T} \epsilon^{-p/\alpha}, \tag{4.8} \end{align*}
\]

where we have also used the fact that \( \Phi(x, \cdot) \in C^1_b(\mathbb{R}^d) \). Hence, (4.4)-(4.8) imply that
\[
\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\epsilon - \bar{X}_t|^{p} \right) \leq C_{p,T}(1 + |x|^p + |y|^p)e^{p(1-1/\alpha)}.
\]

The proof is complete.

4.2. The Proof of Theorem 2.3

We consider the following Kolmogorov equation:
\[
\begin{cases}
\nabla_t u(t, x) = \tilde{\mathcal{L}}_1 u(t, x), & t \in [0, T], \\
u(0, x) = \phi(x),
\end{cases}
\tag{4.9}
\]

where \( \phi \in C^{2+\gamma}_b(\mathbb{R}^d) \) and \( \tilde{\mathcal{L}}_1 \) is the infinitesimal generator of the transition semigroup of the averaged equation \( (2.3) \), which is given by
\[
\tilde{\mathcal{L}}_1 \phi(x) := -(\Delta_x)^{\gamma/2} \phi(x) + \langle b(x), \nabla_x \phi(x) \rangle.
\]

Since \( b, f \in C^{2+\gamma, 2+\gamma}_b \), one can check by straightforward computation that \( \bar{b} \in C^{2+\gamma}_b(\mathbb{R}^d) \). Thus, equation \( (4.9) \) has a unique solution \( u \) which is given by
\[
u(t, x) = \mathbb{E}\phi(\bar{X}_t(x)), \quad t \in [0, T].
Furthermore, \( u(t, \cdot) \in C_b^{2+\gamma}(\mathbb{R}^d_1), \nabla_x u(\cdot, x) \in C^1([0, T]) \) and there exists \( C_T > 0 \) such that

\[
\sup_{t \in [0, T]} \| u(t, \cdot) \|_{C_b^{2+\gamma}} \leq C_T, \quad \sup_{t \in [0, T], x \in \mathbb{R}^d_1} \| \nabla_x (\nabla_x u(t, x)) \| \leq C_T. \tag{4.10}
\]

Now we are in a position to give:

**Proof of Theorem 2.3.** For fixed \( t > 0 \), let \( \tilde{u}^t(s, x) := u(t - s, x), s \in [0, t] \). By Itô’s formula, we have

\[
\tilde{u}^t(t, X_t^s) = \tilde{u}^t(0, x) + \int_0^t \nabla_s \tilde{u}^t(s, X_s^x) ds + \int_0^t \mathcal{L}_1(Y_s^x) \tilde{u}^t(s, X_s^x) ds + \tilde{M}_t,
\]

where \( \tilde{M}_t \) is a \( \mathcal{F}_t \)-martingale defined by,

\[
\tilde{M}_t := \int_0^t \int_{\mathbb{R}^d_1} \tilde{u}^t(s, X_s^x + x) - \tilde{u}^t(s, X_s^x) \tilde{N}(dx, ds).
\]

Note that \( \tilde{u}^t(t, X_t^s) = \phi(X_t^s), \tilde{u}^t(0, x) = \mathbb{E}\phi(\bar{X}_t(x)) \) and \( \nabla_s \tilde{u}^t(s, X_s^x) = -\mathcal{L}_1 \tilde{u}^t(s, X_s^x) \), we have

\[
\left| \mathbb{E}\phi(X_t^s) - \mathbb{E}\phi(\bar{X}_t) \right| = \left| \mathbb{E} \int_0^t -\mathcal{L}_1 \tilde{u}^t(s, X_s^x) ds + \mathbb{E} \int_0^t \mathcal{L}_1(Y_s^x) \tilde{u}^t(s, X_s^x) ds \right|
\]

\[
= \left| \mathbb{E} \int_0^t \langle b(X_s^x, Y_s^x) - \bar{b}(X_s^x), \nabla_x \tilde{u}^t(s, X_s^x) \rangle ds \right|. \tag{4.11}
\]

For any \( s \in [0, t], x \in \mathbb{R}^d_1, y \in \mathbb{R}^d_2 \), define

\[
F^t(s, x, y) := (b(x, y), \nabla_x \tilde{u}^t(s, x))
\]

and \( \tilde{F}^t(s, x) := \int_{\mathbb{R}^d_2} F^t(s, x, y) \mu^x(dy) = \langle b(x), \nabla_x \tilde{u}^t(s, x) \rangle \). Since \( b \) is bounded, \( b \in C_b^{2+\gamma, 2+\gamma} \) and \( \tilde{u}^t(s, \cdot) \in C_b^{2+\gamma}(\mathbb{R}^d_1) \), we have

\[
F^t(s, \cdot, \cdot) \in C_b^{1+\gamma, 2+\gamma}, \quad \nabla_s F^t(s, x, \cdot) \in C_b^1(\mathbb{R}^d_2).
\]

Using (4.10) and an argument similar to that used in the proof of Proposition 3.3, we can get that

\[
\tilde{\Phi}^t(s, x, y) := \int_0^\infty \mathbb{E}F^t(s, x, Y_r^x, y) - \tilde{F}^t(s, x) dr
\]

is a solution of the following Poisson equation:

\[-\mathcal{L}_2(x, y) \tilde{\Phi}^t(s, x, y) = F^t(s, x, y) - \tilde{F}^t(s, x), \quad s \in [0, t]. \tag{4.12}\]

Moreover, \( \tilde{\Phi}^t(\cdot, x, y) \in C^1([0, t]), \tilde{\Phi}^t(s, \cdot, y) \in C^1(\mathbb{R}^d_1), \tilde{\Phi}^t(s, x, \cdot) \in C^2(\mathbb{R}^d_2) \) and for any \( T > 0, t \in [0, T], \theta \in (0, 1) \), there exist \( C_T, C_T, T > 0 \) such that the following estimates hold:

\[
\sup_{s \in [0, t], x \in \mathbb{R}^d_1} \left[ |\tilde{\Phi}^t(s, x, y)| + |\nabla_s \tilde{\Phi}^t(s, x, y)| \right] \leq C_T(1 + |y|). \tag{4.13}
\]
\[
\sup_{s \in [0,t], x \in \mathbb{R}^d} |\nabla_x \tilde{\Phi}^t(s, x, y)| \leq C_T, (1 + |y|^\theta), \quad (4.14)
\]

Using Itô’s formula and taking expectation on both sides, we get
\[
\mathbb{E} \tilde{\Phi}^t(t, X_t, Y_t) = \tilde{\Phi}^t(0, x, y) + \mathbb{E} \int_0^t \nabla_s \tilde{\Phi}^t(s, X_s^t, Y_s^t) ds + \mathbb{E} \int_0^t L_1(Y_s^t) \tilde{\Phi}^t(s, X_s^t, Y_s^t) ds
\]

which implies
\[
- \mathbb{E} \int_0^t L_2(X_s^t, Y_s^t) \tilde{\Phi}^t(s, X_s^t, Y_s^t) ds = \epsilon [\tilde{\Phi}^t(0, x, y) - \mathbb{E} \tilde{\Phi}^t(t, X_t^t, Y_t^t)]
\]

Combining (4.11), (4.12) and (4.16), we get
\[
\sup_{t \in [0,T]} |\mathbb{E} \phi(X_t^t) - \mathbb{E} \phi(\tilde{X}_t)| = \sup_{t \in [0,T]} \left| \mathbb{E} \int_0^t L_2(X_s^t, Y_s^t) \tilde{\Phi}^t(s, X_s^t, Y_s^t) ds \right|
\]

Finally, using (4.13), (4.14), (4.15) and an argument similar to that used in the proof of (4.6), we easily get
\[
\sup_{t \in [0,T]} |\mathbb{E} \phi(X_t^t) - \mathbb{E} \phi(\tilde{X}_t)| \leq C \epsilon,
\]

where \(C\) is a constant depending on \(T, x\) and \(y\). The proof is complete. \(\square\)

5. Appendix

Let us first recall some facts about the isotropic \(\alpha\)-stable processes. For \(k = 1, 2\), let \(L_k^t\) be two isotropic \(\alpha\)-stable processes in \(\mathbb{R}^{d_k}\). The associated Poisson random measure are
defined by (see e.g. [1])

\[ N^k(t, \Gamma) = \sum_{s \leq t} 1_{\Gamma}(L^k_s - L^k_{s-}), \quad \forall \Gamma \in \mathcal{B}(\mathbb{R}^{d_k}), \]

where \( \mathcal{B}(\mathbb{R}^{d_k}) \) are the Borel \( \sigma \)-algebra of \( \mathbb{R}^{d_k} \). The corresponding compensated Poisson measure are given by

\[ \widetilde{N}^k(t, \Gamma) = N^k(t, \Gamma) - t\nu_k(\Gamma), \quad (5.1) \]

where \( \nu_k(dy) := \frac{c_{\alpha,d_k}}{|y|^{d_k+\alpha}}dy \) are the Lévy measures, and \( c_{\alpha,d_k} > 0 \) are constants. By Lévy-Itô’s decomposition, one has

\[ L^k_t = \int_{|x| \leq \varepsilon} x\widetilde{N}^k(t, dx) + \int_{|x| > \varepsilon} xN^k(t, dx). \quad (5.2) \]

Concerning the multiscale system (1.1), we have the following result.

**Lemma 5.1.** Suppose the assumptions in Theorem 2.1 hold. Then for any \( \varepsilon > 0 \), initial value \( x \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2} \), there exists a unique strong solution \( \{ (X^\varepsilon_t, Y^\varepsilon_t), t \geq 0 \} \) to system (1.1). Moreover, for any \( p \in [1, \alpha) \) and \( T > 0 \), there exist constants \( C_p \) and \( C_{p,T} > 0 \) such that

\[ \sup_{\varepsilon \in (0,1)} \mathbb{E} \left( \sup_{t \in [0,T]} |X^\varepsilon_t|^p \right) \leq C_{p,T}(1 + |x|^p + |y|^p) \quad (5.3) \]

and

\[ \sup_{\varepsilon \in (0,1)} \sup_{t \geq 0} \mathbb{E}|Y^\varepsilon_t|^p \leq C_p(1 + |y|^p). \quad (5.4) \]

**Proof.** Since \( b, f \) are globally Lipschitz continuous with respect to \( (x, y) \), by [1, Theorems 6.2.3 and Theorem 6.2.11], there exists a unique solution \( \{(X^\varepsilon_t, Y^\varepsilon_t), t \geq 0\} \) to the system (1.1). By Burkholder-Davis-Gundy’s inequality (see e.g. [34, Lemma 8.22]), we have for any \( 1 \leq p < \alpha \),

\[ \mathbb{E} \left( \sup_{t \in [0,T]} |L^1_t|^p \right) \leq C_p \mathbb{E} \left[ \int_{|x| \leq 1} |x|^2 N^1(T, dx) \right]^{p/2} \]
Then for any $y$

Thus, Grownall’s inequality yields

It is easy to see

Next we estimate $Y_t^\epsilon$. Define

Then for any $y \in \mathbb{R}^d$, it is easy to check that

and

Note that by (5.2), $Y_t^\epsilon$ can be rewritten as

Using Itô formula and taking expectation on both sides, we get

$$
\mathbb{E}U(Y_t^\epsilon) = U(y) + \frac{1}{\epsilon} \mathbb{E} \int_0^t \langle f(X_s^\epsilon, Y_s^\epsilon), DU(Y_s^\epsilon) \rangle ds
$$

$$
+ \mathbb{E} \int_0^t \int_{|z| < \epsilon^{1/\alpha}} \left[ U(Y_s^\epsilon + \epsilon^{-1/\alpha} z) - U(Y_s^\epsilon) - \langle DU(Y_s^\epsilon), \epsilon^{-1/\alpha} z \rangle \right] \nu_2(dz) ds
$$
\[ + \mathbb{E} \int_0^t \int_{|z| > \epsilon^{1/\alpha}} [U(Y_s^\epsilon + \epsilon^{-1/\alpha} z) - U(Y_s^\epsilon)] \nu_2(dz) ds, \]

which implies
\[
\frac{d\mathbb{E}U(Y_t^\epsilon)}{dt} = \frac{1}{\epsilon} \mathbb{E}\langle f(X_t^\epsilon, Y_t^\epsilon), DU(Y_t^\epsilon) \rangle \\
+ \mathbb{E} \int_{|z| \leq \epsilon^{1/\alpha}} [U(Y_t^\epsilon + \epsilon^{-1/\alpha} z) - U(Y_t^\epsilon) - \langle DU(Y_t^\epsilon), \epsilon^{-1/\alpha} z \rangle] \nu_2(dz) \\
+ \mathbb{E} \int_{|z| > \epsilon^{1/\alpha}} [U(Y_t^\epsilon + \epsilon^{-1/\alpha} z) - U(Y_t^\epsilon)] \nu_2(dz) := \sum_{i=1}^3 J_i(t). \]

For the term \( J_1(t) \), by condition (2.1), there exists \( \eta > 0 \) such that
\[
\langle f(X_t^\epsilon, Y_t^\epsilon), DU(Y_t^\epsilon) \rangle = \frac{\langle f(X_t^\epsilon, Y_t^\epsilon) - f(X_t^\epsilon, 0), pY_t^\epsilon \rangle + \langle f(X_t^\epsilon, 0), pY_t^\epsilon \rangle}{(|Y_t^\epsilon|^2 + 1)^{p/2}} \\
\leq -\frac{p \beta |Y_t^\epsilon|^2 + C_p |Y_t^\epsilon|}{(|Y_t^\epsilon|^2 + 1)^{1-p/2}} \\
\leq -\eta (|Y_t^\epsilon|^2 + 1)^{p/2} + C_p.
\]

As a consequence,
\[
J_1(t) \leq \frac{-\eta \mathbb{E}U(Y_t^\epsilon)}{\epsilon} + \frac{C_p}{\epsilon}. \tag{5.8}
\]

For the term \( J_2(t) \), by changing variable \( y = \epsilon^{-1/\alpha} z \) and (5.7), we obtain
\[
J_2(t) \leq \frac{1}{\epsilon} \mathbb{E} \int_{|y| \leq 1} [U(Y_t^\epsilon + y) - U(Y_t^\epsilon) - \langle DU(Y_t^\epsilon), y \rangle] \nu_2(dy) \\
\leq \frac{C_p}{\epsilon} \mathbb{E} \int_{|y| \leq 1} |y|^p \nu_2(dy) \leq \frac{C_p}{\epsilon}, \tag{5.9}
\]

and by (5.6), we have
\[
J_3(t) \leq \frac{1}{\epsilon} \mathbb{E} \int_{|y| > 1} [U(Y_t^\epsilon + y) - U(Y_t^\epsilon)] \nu_2(dy) \\
\leq \frac{C_p}{\epsilon} \mathbb{E} \int_{|y| > 1} (|Y_t^\epsilon|^p + |y|^{p-1}) \nu_2(dy) \leq \frac{\eta \mathbb{E}U(Y_t^\epsilon)}{2\epsilon} + \frac{C_p}{\epsilon}. \tag{5.10}
\]

Combining (5.8)-(5.10), we get
\[
\frac{d\mathbb{E}U(Y_t^\epsilon)}{dt} \leq \frac{-\eta \mathbb{E}U(Y_t^\epsilon)}{2\epsilon} + \frac{C_p}{\epsilon}.
\]
Now the comparison theorem implies that for any $t \geq 0$,

$$
\mathbb{E} (Y_t^x) \leq e^{-\frac{\mu t}{2}} (|y|^2 + 1)^{p/2} + \frac{C_p}{\epsilon} \int_0^t e^{-(t-s)\eta} ds \leq C_p (1 + |y|^p),
$$

(5.11)

which implies (5.4) holds. Estimate (5.3) follows immediately from (5.5) and (5.11).

Concerning the frozen equation (2.4), we have the following result.

**Lemma 5.2.** Assume that $f(x, \cdot) \in C^1_b$ and condition (2.1) holds. Then we have for any $1 \leq p < \alpha$, $T \geq 1$,

$$
\sup_{t \geq 0} \mathbb{E} |Y_t^{x,y}|^p \leq C_p (1 + |y|^p),
$$

(5.12)

$$
\mathbb{E} \left( \sup_{t \in [0,T]} |Y_t^{x,y}|^p \right) \leq C_p T^{p/\alpha} + |y|^p.
$$

(5.13)

**Proof.** The estimate (5.12) can be proved easily using the same argument in the proof of (5.4), we omit the details. Now we prove (5.13). For any fixed $T \geq 1$, define

$$
U_T(y) := (|y|^2 + T^{2/\alpha})^{p/2},
$$

which satisfies

$$
|DU_T(y)| = \left| \frac{py}{(|y|^2 + T^{2/\alpha})^{1-p/2}} \right| \leq C_p |y|^{p-1},
$$

(5.14)

$$
\|D^2U_T(y)\| = \left\| \frac{pI_{d_2}}{(|y|^2 + T^{2/\alpha})^{1-p/2}} - \frac{p(p-2)y \otimes y}{(|y|^2 + T^{2/\alpha})^{2-p/2}} \right\| \leq C_p T^{\frac{p}{\alpha} (\frac{1}{p} - 1)},
$$

(5.15)

where $I_{d_2}$ is the $d_2 \times d_2$ identity matrix. By Itô’s formula and (5.2), we get

$$
U_T(Y_t^{x,y}) = U_T(y) + \int_0^t \left\{ f(x, Y_s^{x,y}) \cdot pY_s^{x,y} \right\} \frac{1}{(|Y_s^{x,y}|^2 + T^{2/\alpha})^{1-p/2}} ds
$$

+ $\int_0^t \int_{|z| \leq T^{1/\alpha}} [U_T(Y_{s-}^{x,y} + z) - U_T(Y_{s-}^{x,y})] \tilde{N}_2(ds, dz)$

+ $\int_0^t \int_{|z| \leq T^{1/\alpha}} \left[ U_T(Y_{s-}^{x,y} + z) - U_T(Y_{s-}^{x,y}) - \frac{\langle pY_s^{x,y}, z \rangle}{(|Y_s^{x,y}|^2 + T^{2/\alpha})^{1-p/2}} \right] \nu_2(dz)ds$

+ $\int_0^t \int_{|z| > T^{1/\alpha}} [U_T(Y_{s-}^{x,y} + z) - U_T(Y_{s-}^{x,y})] N_2(ds, dz) := U_T(y) + \sum_{i=1}^4 I_i(t).
$$

(5.16)
For the term $I_1(t)$, the condition (2.1) implies
\[ \langle f(x,y), y \rangle \leq \langle f(x,y) - f(x,0), y \rangle + \langle f(x,0), y \rangle \leq -\frac{\beta}{2} |y|^2 + C. \]

It is easy to see
\[
E \left[ \sup_{0 \leq t \leq T} |I_1(t)| \right] \leq \int_0^T \frac{C_p}{(|Y_{s,T}^{x,y}|^2 + T^{2/\alpha})^{1-p/2}} ds \leq C_p T^{p/\alpha + 1 - 2/\alpha}. \tag{5.17}
\]

For the term $I_2(t)$, by Burkholder-Davis-Gundy's inequality and (5.14), we have
\[
E \left[ \sup_{t \in [0,T]} |I_2(t)| \right] \leq C_E \left[ \int_0^T \int_{|z| \leq T^{1/\alpha}} \left( |Y_{s,T}^{x,y}|^2 + |z|^{2p} \right) N^2(ds,dz) \right]^{1/2}
\leq C_E \left\{ \left( \sup_{s \in [0,T]} |Y_{s,T}^{x,y}|^{p-1} \right) \left[ \int_0^T \int_{|z| \leq T^{1/\alpha}} |z|^2 N^2(ds,dz) \right]^{1/2} \right\}
+ C_E \left[ \int_0^T \int_{|z| \leq T^{1/\alpha}} |z|^{2p} N^2(ds,dz) \right]^{1/2}
\leq \frac{1}{4} E \left( \sup_{t \in [0,T]} |Y_{t}^{x,y}|^p \right) + C \left[ E \int_0^T \int_{|z| \leq T^{1/\alpha}} |z|^{2p} \nu_2(dz) ds \right]^{p/2}
+ C \left[ E \int_0^T \int_{|z| \leq T^{1/\alpha}} |z|^{2p} \nu_2(dz) ds \right]^{1/2} \leq \frac{1}{4} E \left( \sup_{t \in [0,T]} |Y_{t}^{x,y}|^p \right) + C_p T^{p/\alpha}. \tag{5.18}
\]

For the term $I_3(t)$, by Taylor’s expansion and (5.15), we have
\[
E \left[ \sup_{t \in [0,T]} |I_3(t)| \right] \leq C_p T^{\frac{\alpha}{2}} \left( \frac{\xi}{2} \right) \int_0^T \int_{|z| \leq T^{1/\alpha}} |z|^2 \nu_2(dz) ds \leq C_p T^{p/\alpha}. \tag{5.19}
\]

For the term $I_4(t)$, by (5.14) again, we obtain
\[
E \left[ \sup_{t \in [0,T]} |I_4(t)| \right] \leq C_E \int_0^T \int_{|z| > T^{1/\alpha}} |U_T(Y_{s,T}^{x,y} + z) - U_T(Y_{s,T}^{x,y})| \nu_2(dz) ds
\leq C_E \int_0^T \int_{|z| > T^{1/\alpha}} \left( |Y_{s,T}^{x,y}|^{p-1} |z| + |z|^p \right) \nu_2(dz) ds
\leq C_E \left\{ \left( \sup_{s \in [0,T]} |Y_{s,T}^{x,y}|^{p-1} \right) \left[ \int_0^T \int_{|z| > T^{1/\alpha}} |z|^{p+1} \nu_2(dz) ds \right]^{1/2} \right\}
\leq C_E \left\{ \left( \sup_{s \in [0,T]} |Y_{s,T}^{x,y}|^{p-1} \right) \left[ \int_0^T \int_{|z| > T^{1/\alpha}} |z|^p \nu_2(dz) ds \right]^{1/2} \right\}
\leq C_E \left\{ \left( \sup_{s \in [0,T]} |Y_{s,T}^{x,y}|^{p-1} \right) \left[ \int_0^T \int_{|z| > T^{1/\alpha}} |z|^{2p} \nu_2(dz) ds \right]^{1/2} \right\}
\leq C_E \left\{ \left( \sup_{s \in [0,T]} |Y_{s,T}^{x,y}|^{p-1} \right) \left[ \int_0^T \int_{|z| > T^{1/\alpha}} |z|^{2p} \nu_2(dz) ds \right]^{1/2} \right\}
\leq C_E \left\{ \left( \sup_{s \in [0,T]} |Y_{s,T}^{x,y}|^{p-1} \right) \left[ \int_0^T \int_{|z| > T^{1/\alpha}} |z|^{2p} \nu_2(dz) ds \right]^{1/2} \right\}.
\]
Combining (5.16)-(5.19), we obtain

\[ E \left( \sup_{t \in [0,T]} |Y_{t}^{x,y}|^p \right) \leq C_p T^{p/\alpha} + |y|^p. \]  

The proof is complete. \( \square \)

**Remark 5.3.** Let \( \{\tilde{Y}^{\varepsilon}_t, t \geq 0\} \) solves

\[ \tilde{Y}^{\varepsilon}_t = y + \int_0^t f(X^{\varepsilon}_{s\varepsilon}, \tilde{Y}^{\varepsilon}_s) ds + \tilde{L}^2_t, \]

where \( \tilde{L}^2_t \) is given in (3.1). Using the condition \( \sup_{x \in \mathbb{R}^d} |f(x,0)| < \infty \) and by the same argument as in the proof of (5.13), we can obtain that for any \( T \geq 1 \),

\[ E \left( \sup_{t \in [0,T]} |\tilde{Y}^{\varepsilon}_t|^p \right) \leq C_p T^{p/\alpha} + |y|^p. \]

On the other hand, note that \( Y^{\varepsilon}_t \) in (3.1) and \( \tilde{Y}^{\varepsilon}_t \) have the same law. As a consequence, we have

\[ E \left( \sup_{t \in [0,T]} |Y^{\varepsilon}_t|^p \right) = E \left( \sup_{t \in [0,T/\varepsilon]} |\tilde{Y}^{\varepsilon}_t|^p \right) \leq C_p T^{p/\alpha} - p/\alpha + |y|^p, \quad \forall \varepsilon \in (0, T]. \]  

(5.20)

Concerning the averaged equation, we have the following result.

**Lemma 5.4.** Suppose that the assumptions in Proposition 3.2 hold and that \( b \in C^{1,1}_b \). Then for any \( x \in \mathbb{R}^d \), Eq. (2.3) has a unique solution \( \bar{X}_t \). Moreover, for any \( T > 0 \), there exists a constant \( C_T > 0 \) such that for any \( 1 \leq p < \alpha \),

\[ E \left( \sup_{t \in [0,T]} |\bar{X}_t|^p \right) \leq C_T (1 + |x|^p). \]  

(5.21)
**Proof.** By Proposition 3.2 and Lemma 3.1, for any $t > 0$, $x_1, x_2 \in \mathbb{R}^d$, we have
\[
|\bar{b}(x_1) - \bar{b}(x_2)| \leq \left| \bar{b}(x_1) - \mathbb{E}b(x_1, Y_t^{x_1,0}) \right| + \left| \mathbb{E}b(x_2, Y_t^{x_2,0}) - \bar{b}(x_2) \right| \\
+ \mathbb{E} \left| b(x_1, Y_t^{x_1,0}) - b(x_2, Y_t^{x_2,0}) \right| \\
\leq Ce^{-\frac{\beta t}{2}} + C \left( |x_1 - x_2| + \mathbb{E}|Y_t^{x_1,0} - Y_t^{x_2,0}| \right) \leq Ce^{-\frac{\beta t}{2}} + C|x_1 - x_2|.
\]

Letting $t \to \infty$, we arrive at the Lipschitz continuous property of $\bar{b}$. Hence by [1, Theorems 6.2.3 and Theorem 6.2.11], there exists a unique solution $\{\bar{X}_t, t \geq 0\}$ to Eq. (2.3). Moreover, estimate (5.21) can be easily obtained by following a similar argument as in the proof of (3.3). The proof is complete.

---

**Acknowledgements**

We would like to thank Professor Renming Song for useful discussion, and the referees for carefully reading the manuscript and providing many suggestions and comments. This work is supported by the NNSF of China (11601196, 12071186, 11771187, 11931004), NSF of Jiangsu (BK20170226) and the Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

**References**


