Compound Poisson approximation for regularly varying fields with application to sequence alignment

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The article determines the asymptotic shape of the extremal clusters in stationary regularly varying random fields. To deduce this result, we present a general framework for the Poisson approximation of point processes on Polish spaces which appears to be of independent interest. We further introduce a novel and convenient concept of anchoring of the extremal clusters for regularly varying sequences and fields. Together with the Poissonian approximation theory, this allows for a concise description of the limiting behavior of random fields in this setting. We apply this theory to shed entirely new light on the classical problem of evaluating local alignments of biological sequences.

Keywords: compound Poisson approximation, random fields, regular variation, tail process, point process, local sequence alignment, Gumbel distribution.

1. Introduction

Developments in the theory of stationary regularly varying sequences have broadened our understanding of several key time series models, see for instance [8, 22, 29] and references therein. This theory extends to regularly varying random fields in a relatively straightforward manner, the main technical difficulty being the absence of a natural ordering on the higher-dimensional integer lattice. In parallel to the one-dimensional case, the extreme values in such a random field typically exhibit local clustering. Characterizing the limiting behavior of those extreme clusters is one of the main goals of our study.

In order to deal with this question, we first present a new theory of Poisson approximation for point processes on general Polish spaces which seems of independent interest. Next, we introduce a novel concept of anchoring. This notion is original, and we think, illuminating and bound to be useful even in the well understood time series setting. Using it, we deduce several results concerning compound Poisson limit approximations for extremes of stationary regularly varying random fields.

Finally, these methods allow us to revisit the classical problem of local sequence alignments. In particular, we give a new geometric interpretation for the asymptotic behavior of the scores in local alignments of i.i.d. sequences. Our main result in this context is given as Theorem 1.3 below.

1.1. Regularly varying random fields

We say that a real-valued random field $Y = (Y_{i,j} : i, j \in \mathbb{Z})$ represents the tail field (or the tail process) of a (strictly) stationary real-valued random field $(X_{i,j} : i, j \in \mathbb{Z})$ if it appears as the limit

$$ (u^{-1}X_{i,j})_{i,j \in \{-m, \ldots, m\}} \mid |X_{0,0}| > u \xrightarrow{d} (Y_{i,j})_{i,j \in \{-m, \ldots, m\}}, $$

for every $m \in \mathbb{N}$ as $u \to \infty$. Note that in this introduction we consider random fields indexed over the two-dimensional integer lattice, while we actually develop the theory for integer lattices of arbitrary dimension $d \in \mathbb{N}$. 

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The notion of the tail process for stationary time series was introduced in [8]. In Section 3.1 we extend this theory to random fields. This extension is relatively straightforward but some issues arise due to the absence of a natural ordering on \( \mathbb{Z}^2 \), see Section 3.1.1. As in the one-dimensional case, the existence of the tail process is equivalent to \((X_{i,j})\) being regularly varying, that is, to having all of its finite-dimensional distributions multivariate regularly varying.

One of our main goals is to describe the limiting extremal behavior of \((X_{i,j})_{i,j\in\{1,...,n\}}\) as \( n \to \infty \) relying on the theory of point processes; cf. Section 2.1 where we recall the definition of a point process on a general state space and the related notion of vague convergence. The limiting extremal behavior can be deduced easily if \( X \) is a point process on a general state space. This space is denoted by \( \tilde{l}_0 \) and can be seen as a quotient space, see Section 3.2.1 for a precise definition where we also endow \( \tilde{l}_0 \) with the metric generated by the norm \( \|(x_{i,j})_{i,j}\| = \max_{i,j} |x_{i,j}| \).

In Theorem 3.9 we show that under some standard weak dependence conditions on the field \((X_{i,j})\) and for a sequence of positive numbers \((a_n)\) satisfying \( \lim_{n \to \infty} n^2 \mathbb{P}(|X_{0,0}| > a_n) = 1 \),

\[
\sum_{i\in I_n} \delta(i/k_n, X_{n,i}/a_n) \xrightarrow{d} \sum_{k\in \mathbb{N}} \delta((T_k, P_k(Q_k^i)), i,j\in \mathbb{Z}) \quad n \to \infty, \tag{1.3}
\]

in the space of point measures on \([0,1]^2 \times (\tilde{l}_0 \setminus \{0\})\) where \( \mathbf{0} \) is the array consisting only of 0’s, and

\[
\begin{align*}
(1) & \quad \sum_{k\in \mathbb{N}} \delta((T_k, P_k)) \text{ is a Poisson point process on } [0,1]^2 \times (0,\infty) \text{ with intensity measure } \vartheta dt \times \alpha y^{-\alpha-1} dy \text{ for some constant } \vartheta > 0; \\
(2) & \quad (Q_k^i)_{i,j\in \mathbb{Z}}, \quad k \in \mathbb{N} \text{ is a sequence of i.i.d. random fields independent of } \sum_{k\in \mathbb{N}} \delta((T_k, P_k)).
\end{align*}
\]

As usual, the vague topology used in (1.3) controls only the blocks \( X_{n,i} \) whose maximal value \( \|X_{n,i}\| \) exceeds a sufficiently high threshold, see Section 2.1 and Section 3.2 for the technical details. For a schematic representation of the limit in (1.3) on a particular class of regularly varying fields see the right side of Figure 1 and the discussion after Theorem 1.3. Note that the spatial location of the block \( X_{n,i} \) in (1.3) satisfies \( i/k_n \approx i r_n/n \) for large \( n \) with \( i r_n \) being the upper-right end index in \( J_{n,i} \) from (1.2).

In the time series setting, the limit in (1.3) appeared already in [7, Theorem 3.6]. The novelty of our paper in this context is twofold. First, the link between the tail process \( Y \) and the key ingredients of the limit in (1.3), constant \( \vartheta \) and the distribution of \( (Q_k^i)_{i,j\in \mathbb{Z}} \), is described in detail using the novel notion of anchoring, see Section 3.2.3. We think that this notion sheds new light even on known results in the time series setting. Second, we show that the convergence in (1.3) can be seen in the light of the classical Poisson convergence principle going back to Grigelionis. For that purpose, in Section 2 we present a general Poissonian approximation theorem for point processes on Polish spaces constructed from points which satisfy a suitable asymptotic (in)dependence condition. Moreover, we give sufficient conditions for this theorem to hold in the spirit of [2]. These results seem to be of independent interest and related to those obtained by Schuhmacher [35] using the Chen-Stein method. We, however, rely on the Laplace functionals of point processes.
Finally, the continuous mapping theorem and (1.3) jointly yield
\[ \sum_{i,j=1}^{n} \delta_{(i,j)/n,X_{i,j}/a_{n}} \xrightarrow{d} \sum_{k \in \mathbb{N}} \sum_{i,j \in \mathbb{Z}} \delta_{(T_{k},P_{k}Q_{i,j})}, \quad n \to \infty, \tag{1.4} \]
in the simpler (and more familiar) space of point measures on \([0,1]^2 \times (\mathbb{R}\setminus\{0\})\), see Corollary 3.10. Observe that the limit in (1.4) has a form of a Poisson cluster (or a compound Poisson) process.

### 1.2. Local sequence alignment

Because of its importance in molecular biology, the local alignment problem was studied extensively both from a probabilistic and applied perspective, see for instance [2, 15, 19] and references therein. Since it represents one of the main motivations for our study, we explain here its key ingredients and our main result in that context.

Let \((A_{i})_{i \in \mathbb{N}}\) and \((B_{i})_{i \in \mathbb{N}}\) be two independent i.i.d. sequences taking values in a finite alphabet \(E\). Also, let \(A\) and \(B\) be independent random variables distributed as \(A_{1}\) and \(B_{1}\), respectively. For a fixed score function \(s : E \times E \to \mathbb{R}\) and for all \(i,j \in \mathbb{N}\) and \(m = 0, 1, \ldots, i \wedge j\) (where \(i \wedge j := \min\{i, j\}\)), let
\[ S_{i,j}^m = \sum_{k=0}^{m-1} s(A_{i-k}, B_{j-k}) \]
be the score of aligning segments \(A_{i-m+1}, \ldots, A_{i}\) and \(B_{j-m+1}, \ldots, B_{j}\). Further, for all \(i,j \in \mathbb{N}\) define
\[ S_{i,j} = \max\{S_{i,j}^m : 0 \leq m \leq i \wedge j\}. \tag{1.5} \]

From a biological perspective it is essential to understand the extremal distributional properties of the random matrix \((S_{i,j} : 1 \leq i, j \leq n)\) as \(n \to \infty\). The following simple assumption is standard in this context, cf. Dembo et al. [15].

**Assumption 1.1.** The distribution of \(s(A, B)\) is nonlattice, i.e. \(\mathbb{P}(s(A, B) \in \delta\mathbb{Z}) < 1\) for all \(\delta > 0\), and satisfying
\[ \mathbb{E}[s(A, B)] < 0 \quad \text{and} \quad \mathbb{P}(s(A, B) > 0) > 0. \tag{1.6} \]

The lattice case is excluded for simplicity in the sequel. It is known to be conceptually similar, although technically more involved. Note further that, like [15] and [19], we consider only gapless local alignments.

Denote by \(\mu_{A}\) and \(\mu_{B}\) the distributions of \(A\) and \(B\), respectively and assume for simplicity that \(\mu_{A}(e), \mu_{B}(e) > 0\) for each letter \(e\) in the alphabet \(E\). By Assumption 1.1 there exists a unique strictly positive solution \(\alpha^{*}\) of the Lundberg equation
\[ m(\alpha^{*}) := \mathbb{E}[e^{\alpha^{*}s(A, B)}] = 1. \]

Let \(\mu^{*}\) be the (exponentially tilted) probability measure on \(E \times E\) given by
\[ \mu^{*}(a,b) = e^{\alpha^{*}s(a,b)} \mu_{A}(a) \mu_{B}(b), \quad a, b \in E. \tag{1.7} \]

For two probability measures \(\mu\) and \(\nu\) on a finite set \(F\), denote by \(H(\nu | \mu)\) the relative entropy of \(\nu\) with respect to \(\mu\), i.e.
\[ H(\nu | \mu) = \sum_{x \in F} \nu(x) \log \frac{\nu(x)}{\mu(x)}. \]

Dembo et al. [15] introduce one final condition on the tilted probability measure \(\mu^{*}\).
Assumption 1.2 (Condition (E’) in [15]). It holds that
\[ H(\mu^*|\mu_A \times \mu_B) > 2 \{ H(\mu_A^*|\mu_A) \lor H(\mu_B^*|\mu_B) \}, \tag{1.8} \]
where \( \mu_A^* \) and \( \mu_B^* \) denote the marginals of \( \mu^* \).

Note that (1.8) holds automatically if \( \mu_A = \mu_B \) and if the score function \( s \) is symmetric (i.e. \( s(a, b) = s(b, a) \)) but not of the form \( s(a, b) = s(a) + s(b) \), see [14, Section 3].

Under Assumptions 1.1 and 1.2, Dembo et al. [15] (see also Hansen [19]) showed that the distribution of the maximal local alignment score \( M_n = \max_{1 \leq i,j \leq n} S_{i,j} \), asymptotically follows a Gumbel distribution. More precisely, as \( n \to \infty \), for a certain constant \( K^* > 0 \),
\[ \Pr \left( M_n - \frac{2 \log(n)}{\alpha^*} \leq x \right) \to e^{-K^* e^{-\alpha^* x}}, \quad x \in \mathbb{R}. \tag{1.9} \]

Observe that the field \( (S_{i,j}) \) consists of dependent random variables. For instance, simple arguments can be given (cf. (1.12) below) showing that any extreme score, i.e. score exceeding a given large threshold, will be followed by a run of extreme scores along the diagonal. This phenomenon is illustrated in Figure 1. The approach of [15] is based on showing that the number of such extreme clusters, as both the sample size and the threshold tend to infinity, becomes asymptotically Poisson distributed.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{heatmap.png}
\caption{Heatmap of the local scores \( S_{i,j}, i,j = 1,\ldots,n \), exceeding a prespecified threshold for two simulated sequences of length \( n = 500 \) (on the left). Schematic representation of the limit as \( n \to \infty \) with the clusters of values above the given threshold collapsing to a single point marked with the corresponding tail field (on the right), see the discussion after Theorem 1.3.}
\end{figure}

In the sequel, we show that one can give a much more detailed information about the structure within the extreme clusters. In particular, following the method below one can deduce the asymptotic distribution of arbitrary functionals of the upper order statistics of the field \( (S_{i,j}) \).

Observe first that for each \( i, j \in \mathbb{N} \), \( S_{i,j} \) can be seen as the maximum of a truncated random walk \( (S^m_{i,j})_{m=0,\ldots,i,j} \) which by (1.6) has negative drift. It can be rigorously shown, see Remark 4.1, that in all our asymptotic considerations this truncation and the related edge effects can be ignored. Therefore we assume throughout that the sequences \( (A_i) \) and \( (B_i) \) extend over all integers \( i \in \mathbb{Z} \). This makes scores \( S^m_{i,j} \) well defined for all \( i,j \in \mathbb{Z} \) and \( m \geq 0 \), and consequently we update the original field of scores \( (S_{i,j}) \) as follows
\[ S_{i,j} = \sup \{ S^m_{i,j} : m \geq 0 \}, \quad i,j \in \mathbb{Z}. \tag{1.10} \]

By construction, the field \( (S_{i,j}) \) is stationary. Moreover, by the classical Cramér-Lundberg theory, Assumption 1.1 implies that the tail of \( S_{i,j} \) is asymptotically exponential, or more precisely
\[ \Pr(S_{i,j} > u) \sim Ce^{-\alpha^* u}, \quad \text{as } u \to \infty, \tag{1.11} \]
for some $C > 0$. Note that, in the language of extreme value theory, marginal distribution of the field $(S_{i,j})$ belongs to the maximum domain of attraction of the Gumbel distribution. In this light, the limiting result (1.9) may not be very surprising, but its proof remains quite involved due to the clustering of extremal scores of the field $(S_{i,j})$. Observe that the field $(S_{i,j})_{i,j \in \mathbb{Z}}$ satisfies the following simple (Lindley) recursion along any diagonal, namely
\[
S_{i,j} = (S_{i-1,j-1} + \varepsilon_{i,j})_+, \tag{1.12}
\]
where random variables $\varepsilon_{i,j} = s(A_i, B_j)$ have negative mean.

Our main result in this context strengthens (1.9) to a convergence in distribution of point processes based on the $S_{i,j}$'s. The key observation is that under Assumptions 1.1 and 1.2 the transformed field
\[
X_{i,j} = e^{S_{i,j}}, i, j \in \mathbb{Z}
\]
admits a tail process $(Y_{i,j} : i, j \in \mathbb{Z})$, hence it is regularly varying; see Proposition 4.1. Its tail process satisfies
\[
Y_{i,j} = 0, i \neq j.
\]
Moreover, the distribution of $Y_{m,m}$'s can be described in detail using two auxiliary independent i.i.d. sequences $(\varepsilon_i)_{i \geq 1}$ and $(\varepsilon_i^*)_{i \geq 1}$ whose distributions correspond to the distributions of $s(A, B)$ under the product measure $\mu_A \times \mu_B$ and under the tilted measure $\mu^*$ from (1.7), respectively: if $S_0 = 0$ and
\[
S_m^\varepsilon = \begin{cases} 
\sum_{i=1}^m \varepsilon_i, & m \geq 1, \\
-\sum_{i=1}^{-m} \varepsilon_i^*, & m \leq -1,
\end{cases}
\]
then
\[
Y_{m,m} = Y_{0,0} e^{S_m^\varepsilon}, m \in \mathbb{Z},
\]
where $Y_{0,0}$ is Pareto distributed with index $\alpha^*$, i.e.
\[
\mathbb{P}(Y_{0,0} > y) = y^{-\alpha^*} \quad \text{for all } y \geq 1,
\]
and independent of $(S_m^\varepsilon)_{m}$. To state our main result denote by $\Theta_{i,j} = Y_{i,j}/Y_{0,0}$, $i, j \in \mathbb{Z}$, the so-called spectral tail field of $(X_{i,j})$, so that
\[
\Theta_{m,m} = e^{S_m^\varepsilon} \quad \text{for } m \in \mathbb{Z}, \quad \text{and} \quad \Theta_{i,j} = 0 \quad \text{for } i \neq j. \tag{1.13}
\]
Take an arbitrary sequence of positive integers $(r_n)$ such that $\lim_{n \to \infty} r_n = \infty$ and $\lim_{n \to \infty} r_n/n^\varepsilon \to 0$ for all $\varepsilon > 0$ and recall the blocks $X_{n,i}$ defined in (1.1).

Theorem 1.3. Under Assumptions 1.1 and 1.2,
\[
\sum_{i \in T_n} \delta_{(i/k_n, X_{n,i}/n^\varepsilon^*)} \overset{d}{\to} \sum_{k \in \mathbb{N}} \delta_{(T_k, P_k(Q_{i,k})_{i,j \in \mathbb{Z}})} \tag{1.14}
\]
in the space of point measures on $[0, 1]^2 \times \left(\{0\} \setminus \{0\}\right)$ where
(i) $\sum_{k \in \mathbb{N}} \delta_{(T_k, P_k)}$ is a Poisson point process on $[0, 1]^2 \times (0, \infty)$ with intensity measure $\vartheta Cdt \times \alpha^* y^{-\alpha^*-1}dy$ where $C$ is the constant from (1.11) and
\[
\vartheta = \mathbb{P}(\sup_{m \geq 1} S_m^\varepsilon + \Gamma \leq 0),
\]
for an exponential random variable $\Gamma$ with parameter $\alpha^*$ independent of $(S_m^\varepsilon)$;
(ii) $(Q_{i,k})_{i,j \in \mathbb{Z}}$, $k \in \mathbb{N}$ are i.i.d. random fields independent of $\sum_{k \in \mathbb{N}} \delta_{(T_k, P_k)}$ and with common distribution equal to the distribution of $(\Theta_{i,j})_{i,j \in \mathbb{Z}}$ in (1.13), but conditionally on the underlying random walk $(S_m^\varepsilon)_{m}$ being negative for $m < 0$ and nonpositive for $m > 0$. 
An interpretation of the theorem can be given through Figure 1. On the left, we plot the scores exceeding a prespecified threshold for two simulated independent sequences of length $n = 500$ from the uniform distribution on a four letter alphabet. The grey dots correspond to the scores exceeding 50% of the maximal score $M_n$, while the other dots represent points over 75% $M_n$ (they are colored from red to black, with the darker color indicating a higher score). In this simulation, for illustration purposes, we score a match by $\sqrt{3}$ and a mismatch by $-1$. The picture on the right schematically illustrates the limit of the leading clusters of (exponentially transformed) high scores grouped into blocks which, after a rescaling, collapse to a single point (at position $T_k$ say) which is then marked by its maximum and the shape of the cluster (denoted by $P_k$ and $(Q^k_{i,j})_{i,j \in \mathbb{Z}}$ say), see also the discussion after Remark 3.10. In this case, the random fields $(Q^k_{i,j})$ are concentrated on the diagonal because of (1.13).

Taking logarithms, from (1.14) one can deduce the convergence

$$
\sum_{i,j=1}^{n} \delta^{\left(\frac{(i,j)}{n}, S_{i,j} - \frac{2 \log(n)}{\alpha}\right)} \overset{d}\rightarrow \sum_{k \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \delta^{(T_k, \log(P_k)+\log(Q^k_{m,m}))}
$$

in the space of point measures on $[0,1]^2 \times \mathbb{R}$ with a suitable vague topology, see Corollary 4.8 for details. In particular, this yields (1.9) at once with the following new expression for the key constant therein

$$K^* = \vartheta C.$$

Note that $\vartheta$ is the so-called extremal index of the field $(S_{i,j})$, cf. Remark 3.11. The same expression for $\vartheta$ appears in a different context in de Haan et al. [13, Section 3] together with a suggested algorithm for its numerical computation. Moreover, the constant $C$ arising from (1.11) is frequently encountered in the literature; for various expressions of $C$ we refer to [3, Part C, XIII.5]. Thus, in principle, for i.i.d. sequences (as in Altschul et al. [1] for instance) the constants $K^*$ and $\alpha^*$ in (1.9) do not have to be estimated since they can be directly determined from the marginal distribution of the letters and the scoring function $s$. Note also that the distribution of random walks conditioned to stay negative (or positive) is discussed in detail by Tanaka [41] and Biggins [9].

Finally, Theorem 1.3 has some specific implications for the interpretation of real biological sequence alignments. First of all, observe that the number of $\log(P_k)$’s above a given threshold $x$ in (1.15) is Poisson distributed, while the overshoots of $\log(P_k) - x$ are i.i.d. and have an exponential distribution. This fact gives a theoretical underpinning to the use of the peaks-over-a-threshold approach to the modeling of local alignments in which the number of clusters (islands) of scores above a high threshold is modeled by a Poisson random variable and where the local extremes of these clusters exceed a given threshold by a random amounts which are independent and exponentially distributed. For an application of this idea in two different contexts see Altschul et al. [1] and Hansen [20]. Moreover, if one connects the $k$ leading nonoverlapping clusters of high scores in the direction of the alignment, one can incorporate gaps into the alignment and approximate $p$-values of such extended and possibly penalized local alignments (this would go into the direction of Siegmund and Yakir [38, 39], cf. also Metzler et al. [28] where our deduced limit is simply assumed). Finally, zooming in into individual clusters, the theorem allows one to study the structure of subsequences $(A_{i-k}, \ldots, A_i)$ and $(B_{j-k}, \ldots, B_j)$ in a cluster of high scores, to see if it agrees with the predicted theoretical distribution of such a cluster given a very close alignment. Each of these issues arguably deserves a detailed study and a real–life data illustration, but that would exceed the scope of our paper.

1.3. Organization of the paper

The rest of the article is organized as follows — in Section 2, we present a general type of a Poissonian approximation theorem which allows one to study point processes constructed from general random fields with values in a Polish space under an appropriate dependence assumption. We also find sufficient conditions for such a dependence assumption to hold. Section 3 presents
the point process convergence theory for stationary regularly varying random fields indexed over \( \mathbb{Z}^d \) with \( d \in \mathbb{N} \), complementing and extending the theory from the case \( d = 1 \). In particular, we introduce the notion of the tail field/process and point out at the subtleties of this extension arising from the fact that there is no unique natural ordering of the points in the \( d \)-dimensional lattice, for \( d \geq 2 \). Moreover, a special attention is dedicated to the notion of anchoring which clarifies the link between the tail process and the components \( \vartheta \) and \( (Q_{i,j}^b) \) of the limiting point process from (1.3). Section 4 is entirely dedicated to the alignment problem and the proof of Theorem 1.3. Finally, in Section 5 we give the proofs of Theorem 3.1 from Section 3 and several auxiliary results used in Section 4. Some proofs and arguments which are straightforward generalizations of the existing results can be found in [31].

2. On (compound) Poisson approximation in general Polish spaces

For the general theory of point processes on Polish spaces and the so-called vague convergence see e.g. Kallenberg [23] or Resnick [33]. Note that even though the latter reference considers only point processes on a locally compact state space, most of the results transfer directly to the general Polish case. However, as proposed in [6], we use a slight modification of the definition of vague convergence.

2.1. Basic setup and the notion of vague convergence

Let \( X \) be a Polish space. Denote by \( B(X) \) the Borel \( \sigma \)-field on \( X \) and choose a subfamily \( B_b(X) \subseteq B(X) \) of sets, called bounded (Borel) sets of \( X \). When there is no fear of confusion, we will simply write \( B \) and \( B_b \). We say that a Borel measure \( \mu \) on \( X \) is locally (or boundedly) finite if \( \mu(B) < \infty \) for all \( B \in B_b \). The space of all such measures is denoted by \( \mathcal{M}(X) = \mathcal{M}(X,B_b) \).

For measures \( \mu, \mu_1, \mu_2, \ldots \in \mathcal{M}(X) \), we say that \( \mu_n \) converge vaguely to \( \mu \) and denote this by \( \mu_n \rightharpoonup \mu \), if as \( n \to \infty \),

\[
\mu_n(f) = \int f d\mu_n \to \int f d\mu = \mu(f),
\]

for all bounded and continuous real-valued functions \( f \) on \( X \) with support being a bounded set. Denote by \( CB_b(X) \) the family of all such functions and by \( CB_b^+(X) \) the subset of all nonnegative functions in \( CB_b(X) \).

In the sequel we assume that the family of bounded sets \( B_b \) satisfies the following properties and in that case say that \( B_b \) properly localizes \( X \).

(i) \( A \subseteq B \in B_b \) for a Borel set \( A \subseteq X \) implies \( A \in B_b \), and \( A, B \in B_b \) implies \( A \cup B \in B_b \).

(ii) For each \( B \in B_b \) there exists an open set \( U \in B_b \) such that \( \overline{B} \subseteq U \), where \( \overline{B} \) denotes the closure of \( B \) in \( X \).

(iii) There exists a sequence \( (K_m)_{m \in \mathbb{N}} \) of bounded Borel sets which cover \( X \) and such that every \( B \in B_b \) is contained in \( K_m \) for some \( m \in \mathbb{N} \).

Moreover, the sequence \( (K_m)_{m \in \mathbb{N}} \) can always be chosen to consist of open sets satisfying

\[
X_m \subseteq K_{m+1}, \text{ for all } m \in \mathbb{N}.
\]

Any such sequence \( (K_m) \) is called a proper localizing sequence.

By the theory of Hu [21, Section V.5], properties (i)-(iii) are equivalent to the existence of a metric on \( X \) which generates the topology of \( X \) and such that the corresponding family of metrically bounded Borel subsets of \( X \) is precisely \( B_b \). Since this is exactly the framework of [23, Chapter 4], the theory developed therein directly applies. In particular, by [23, Theorem 4.2], the topology on \( \mathcal{M}(X) \) inducing the notion of vague convergence, called the vague topology, is again Polish, see also [6, Section 3].
Note that by choosing a different family of bounded sets one changes the space of locally finite measures and the related notion of vague convergence.

**Example 2.1.** Let \((X', d')\) be a complete and separable metric space and \(C \subseteq X'\) a closed set. Assume that \(X\) is of the form \(X = X' \setminus C\) equipped with the subspace topology and set \(B\) to be the class of all Borel sets \(B \subseteq X\) such that for some \(\epsilon > 0\), \(d'(x, C) > \epsilon\) for all \(x \in B\), where \(d'(x, C) = \inf\{d'(x, z) : z \in C\}\). In words, \(B\) is bounded if it is bounded away from \(C\). Such \(B\) properly localizes \(X\) and one can take \(K_m = \{x \in X : d'(x, C) > 1/m\}\), \(m \in \mathbb{N}\), as a proper localizing sequence. The corresponding notion of convergence coincides with the so-called \(M_0\)-convergence from Lindskog et al. [26] and is frequently used in extreme value theory.

Denote by \(\delta_x\) the Dirac measure concentrated at \(x \in X\). A (locally finite) point measure on \(X\) is a locally finite measure \(\mu \in \mathcal{M}(X)\) which is of the form \(\mu = \sum_{i=1}^{K} \delta_{x_i}\) for some \(K \in \{0, 1, \ldots\} \cup \{\infty\}\) and (not necessarily distinct) points \(x_1, x_2, \ldots, x_K\) in \(X\). Denote by \(\mathcal{M}_p(X)\) the space of all point measures on \(X\) and endow it with the vague topology. Vague convergence of point measures is equivalent to the convergence of points in (almost) all bounded Borel sets of \(X\), see [6, Proposition 2.8] for details.

A point process on \(X\) is a random element of the space \(\mathcal{M}_p(X)\) with respect to the Borel \(\sigma\)-algebra. We denote convergence in distribution by \(\xrightarrow{d}\). Recall, for point processes \(N, N_1, N_2, \ldots\), convergence of Laplace functionals \(\mathbb{E}[e^{-N(f)}] \to \mathbb{E}[e^{-N(f)}]\) for all \(f \in CB_0^+(X)\) is equivalent to \(N_n \xrightarrow{d} N\) in \(\mathcal{M}_p(X)\), see [23, Theorem 4.11].

**Definition 2.1.** We say that a family \(\mathcal{F} \subseteq CB_0^+(X)\) is (point process) convergence determining if, for any point processes \(N, N_1, N_2, \ldots\), convergence \(\mathbb{E}[e^{-N_n(f)}] \to \mathbb{E}[e^{-N(f)}]\) for all \(f \in \mathcal{F}\) implies that \(N_n \xrightarrow{d} N\) in \(\mathcal{M}_p(X)\).

For example, one can take the subfamily \(\mathcal{F} \subseteq CB_0^+(X)\) of functions which are Lipschitz continuous with respect to a suitable metric, see [6, Proposition 4.1].

### 2.2. General Poisson approximation

Let \((I_n)_{n \in \mathbb{N}}\) be a sequence of finite index sets but such that \(\lim_{n \to \infty} |I_n| = \infty\), where \(|I_n|\) denotes the number of elements in \(I_n\). For each \(n \in \mathbb{N}\), let \((X_{n,i} : i \in I_n)\) be a family of random elements in a topological space \(X'\). Assume that there exists a Polish subset \(X\) of \(X'\) (e.g. as in Example 2.1) with a family of bounded Borel sets \(B = B_{B(X)}\) such that, as \(n \to \infty\),

\[
\sup_{i \in I_n} \mathbb{P}(X_{n,i} \in B) \to 0, \ B \in B_{B(X)}.
\]  

(2.1)

The central theme of this section is convergence in distribution in \(\mathcal{M}_p(X)\) of the point processes

\[
N_n = \sum_{i \in I_n} \delta_{X_{n,i}}, \ n \in \mathbb{N},
\]

restricted to the space \(X\). For a locally finite measure \(\lambda\) on \(X\) denote by \(\text{PPP}(\lambda)\) the distribution of a Poisson point process on \(X\) with intensity measure \(\lambda\).

Observe that if for each \(n \in \mathbb{N}\), \((X_{n,i} : i \in I_n)\) were independent, (2.1) would imply that measures \(\delta_{X_{n,i}}\) on \(X, n \in \mathbb{N}, i \in I_n\) form a null-array (see [23, p. 129]) and by the so-called Grigelionis theorem (see [23, Corollary 4.25]), for \(\lambda \in \mathcal{M}(X)\), convergence \(N_n \xrightarrow{d} N \sim \text{PPP}(\lambda)\) holds in \(\mathcal{M}_p(X)\) if and only if

\[
\mathbb{E}[N_n(\cdot)] = \sum_{i \in I_n} \mathbb{P}(X_{n,i} \in \cdot) \xrightarrow{d} \lambda
\]

in \(\mathcal{M}(X)\).
In general, one can still obtain the same Poisson limit if the asymptotic distributional behavior of $N_n$’s is indistinguishable from its independent version.

More precisely, let for each $n \in \mathbb{N}$, $(X_{n,i}^* : i \in I_n)$ be independent random elements such that for all $i \in I_n$, $X_{n,i}^*$ is distributed as $X_{n,i}$, and denote by $N_{n}^* = \sum_{i \in I_n} \delta X_{n,i}$ the corresponding point processes on $X$. Further, let $\mathcal{F}$ be a class of measurable and nonnegative functions on $X$ with bounded support. We say that the family $(X_{n,i} : n \in \mathbb{N}, i \in I_n)$ is asymptotically $\mathcal{F}$-independent ($AI(\mathcal{F})$) if

\[
\| \mathbb{E} \left[ e^{-N_n(f)} \right] - \mathbb{E} \left[ e^{-N_n^*(f)} \right] \| = \| \mathbb{E} \left[ e^{-\sum_{i \in I_n} f(X_{n,i})} \right] - \prod_{i \in I_n} \mathbb{E} \left[ e^{-f(X_{n,i})} \right] \| \to 0, \text{ as } n \to \infty,
\]

for all $f \in \mathcal{F}$, where we set $f(x) = 0$ for all $x \in X \setminus X'$. To obtain meaningful results we will require that the functions in $\mathcal{F}$ determine convergence in distribution in $\mathcal{M}_p(X)$ in the sense of Definition 2.1. Since $N_{n}^* \overset{d}{\to} N \sim \text{PPP}(\lambda)$ implies convergence $\mathbb{E}[e^{-N_n^*(f)}] \to \mathbb{E}[e^{-N(f)}]$ for all $f \in CB_0^+(X)$, the following result is now immediate.

**Theorem 2.2.** Assume that (2.1) holds and that there exists a measure $\lambda \in \mathcal{M}(X)$ such that, as $n \to \infty$,

\[
\sum_{i \in I_n} \mathbb{P}(X_{n,i} \in \cdot) \overset{d}{\to} \lambda. \tag{2.2}
\]

Then for any convergence determining family $\mathcal{F} \subseteq CB_0^+(X)$, $N_n \overset{d}{\to} N \sim \text{PPP}(\lambda)$ in $\mathcal{M}_p(X)$ if and only if $(X_{n,i} : n \in \mathbb{N}, i \in I_n)$ is $AI(\mathcal{F})$.

**Remark 2.1.** Observe that we have assumed that $\mathcal{F}$ consists only of continuous functions. However, $N_{n}^* \overset{d}{\to} N$ actually implies $\mathbb{E}[e^{-N_n^*(f)}] \to \mathbb{E}[e^{-N(f)}]$ for all nonnegative and bounded functions $f$ with bounded support for which $N(\text{disc}(f)) = 0$ almost surely, where $\text{disc}(f)$ denotes the set of all discontinuity points of $f$ (see [23, Lemma 4.12]). Consequently, if (2.2) holds and $N \sim \text{PPP}(\lambda)$, in the necessary and sufficient condition for $N_n \overset{d}{\to} N$, one can allow $\mathcal{F}$ to be a sufficiently rich class of functions $f$ which are not necessarily continuous, e.g. $\mathcal{F}$ could consist of nonnegative simple functions with bounded support, see [23, Theorem 4.11] for details.

**Remark 2.2.** Assume that (2.1) holds and that $X_{n,i}$’s are $AI(\mathcal{F})$ for some convergence determining family $\mathcal{F}$. In this case, if $N_{n}$ converge in distribution to some limit, $N$ say, then $N$ is necessarily a Poisson process. Indeed, since also $N_{n}^* \overset{d}{\to} N$, by [23, Theorem 4.22] $N$ is infinitely divisible and moreover, by the construction of $N_{n}^*$, its so-called Lévy measure (see [23, p. 89]) is concentrated on the set $\{ \delta_x : x \in X \}$ which implies that $N$ is Poisson.

Observe that the assumption $AI(\mathcal{F})$ implies that $X_{n,i}$, $i \in I_n$ asymptotically behave as if they were independent, but only on the bounded sets of the space $X$. The key fact here is that all functions in $\mathcal{F}$ have bounded support, so for every fixed $f \in \mathcal{F}$, $N_n(f)$ is unaffected by the behavior of $X_{n,i}$’s outside of a fixed bounded set. Sufficient condition for $AI(\mathcal{F})$ to hold is given in Proposition 2.4 below.

First we state a stationary version of the previous result, cf. [33, Proposition 3.21]. For $d \in \mathbb{N}$ consider the space $[0,1]^d \times X$ with respect to the product topology and with $B' \subseteq B([0,1]^d \times X)$ being bounded if the set $\{ x \in X : (t,x) \in B' \text{ for some } t \in [0,1] \}$ is bounded in $X$.

**Corollary 2.3.** Assume that $I_n = \{1,2,\ldots,k_n \} \subseteq \mathbb{Z}^d$ for some $d \in \mathbb{N}$ with $k_n \to \infty$ and that $(X_{n,i} : i \in I_n)$ are identically distributed for every $n \in \mathbb{N}$. If there exists a measure $\nu \in \mathcal{M}(X)$ such that, as $n \to \infty$,

\[
k_n \mathbb{P}(X_{n,1} \in \cdot) \overset{d}{\to} \nu, \tag{2.3}
\]

then for any convergence determining family $\mathcal{F}'$ on $[0,1]^d \times X$,

\[
N_n' = \sum_{i \in I_n} \delta(i/k_n,X_{n,i}) \overset{d}{\to} N' \sim \text{PPP}(\text{Leb} \times \nu)
\]
in $\mathcal{M}_p([0,1]^d \times \mathbb{X})$ if and only if $(i/k_n, X_{n,i}) : n \in \mathbb{N}, i \in I_n$ is $AI(F')$, where Leb denotes the Lebesgue measure on $[0,1]^d$.

**Proof.** We simply apply Theorem 2.2 to random elements $X'_{n,i} := (i/k_n, X_{n,i}), n \in \mathbb{N}, i \in I_n$. Take an arbitrary $B' \in \mathcal{B}_0([0,1]^d \times \mathbb{X})$ and define $B = \{x \in \mathbb{X} : (t,x) \in B'$ for some $t \in [0,1]\}$. Since $B \in \mathcal{B}_0(\mathbb{X})$, (2.3) and [23, Lemma 4.1(iv)] imply that

$$\limsup_{n \to \infty} \sum_{i \in I_n} \Pr(X'_{n,i} \in B') = \limsup_{n \to \infty} k_n^d \Pr(X_{n,1} \in B) \leq \nu(B) < +\infty.$$

Hence, (2.1) holds since $k_n \to \infty$.

Further, note that for arbitrary $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ in $[0,1]^d$ such that $a_j \leq b_j$ for all $j = 1, \ldots, d$ and a set $B \in \mathcal{B}_b$ such that $\nu(\partial B) = 0$, (2.3) implies that as $n \to \infty$,

$$\sum_{i \in I_n} \Pr(X'_{n,i} \in (a, b) \times B) = \frac{1}{k_n^d} \prod_{j=1}^d \left| k_n(b_j - a_j) \right| \cdot k_n^d \Pr(X_{n,1} \in B) \to \prod_{j=1}^d (b_j - a_j) \cdot \nu(B).$$

By [23, Lemma 4.1], this implies that $\sum_{i \in I_n} \Pr(X'_{n,i} \in \cdot) \xrightarrow{d} \text{Leb} \times \nu$ in $\mathcal{M}_p([0,1]^d \times \mathbb{X})$, i.e. (2.2) holds with $\lambda = \text{Leb} \times \nu$. \hfill $\Box$

### 2.3. Sufficient conditions for asymptotic $\mathcal{F}$-independence

For each $i \in I_n$, choose a subset of the index set $B_n(i) \subseteq I_n$ containing $i$, and call it the neighborhood of dependence of $i$. Intuitively, it will be beneficial to choose $B_n(i)$ as small as possible, but such that $X_{n,i}$ is (nearly) independent of all $X_{n,j}$ for $j \notin B_n(i)$.

Select an arbitrary ordering of the elements in $I_n$. Without loss of generality, we will assume that $I_n = \{1, 2, \ldots, m_n\}$ where $m_n \to \infty$ as $n \to \infty$. For all $i \in I_n$, partition $\{i+1, \ldots, m_n\}$ into $\hat{B}_n(i) := \{j \in B_n(i) : j > i\}$ and $\check{B}_n(i) := \{j \notin B_n(i) : j > i\}$. Further, fix an arbitrary sequence $(K_m)_{m \in \mathbb{N}} \subseteq \mathcal{B}_0$ of $\mathcal{B}_b$ such that for every $B \in \mathcal{B}_b$, $B \subseteq K_m$ for some $m \in \mathbb{N}$.

For a given neighborhood structure $(B_n(i) : n \in \mathbb{N}, i \in I_n)$ and for all $m, n \in \mathbb{N}$ define

$$b_{n,1}^m = \sum_{i \in I_n} \sum_{j \in \check{B}_n(i)} \Pr(X_{n,i} \in K_m) \cdot \Pr(X_{n,j} \in K_m),$$

$$b_{n,2}^m = \sum_{i \in I_n} \sum_{j \in \hat{B}_n(i)} \Pr(X_{n,i} \in K_m, X_{n,j} \in K_m).$$

Furthermore, for all $n \in \mathbb{N}$ and an arbitrary nonnegative measurable function $f$ on $\mathbb{X}$ define

$$b_{n,3}(f) = \sum_{i \in I_n} \frac{\mathbb{E}[e^{-f(X_{n,i})} \prod_{j \in \check{B}_n(i)} e^{-f(X_{n,j})} - \mathbb{E}[e^{-f(X_{n,i})}] \cdot \mathbb{E}[\prod_{j \in \check{B}_n(i)} e^{-f(X_{n,j})}]]}{\mathbb{E}[e^{-\sum_{i \in I_n} f(X_{n,i})} \prod_{j \in \check{B}_n(i)} e^{-f(X_{n,j})}]}.$$

**Proposition 2.4.** Let $f$ be a nonnegative measurable function on $\mathbb{X}$ with bounded support. If $m \in \mathbb{N}$ is such that the support of $f$ is contained in $K_m$, then for all $n \in \mathbb{N},$

$$|\mathbb{E}\left[e^{-\sum_{i \in I_n} f(X_{n,i})} \prod_{i \in I_n} e^{-f(X_{n,i})}\right] - \prod_{i \in I_n} \mathbb{E}[e^{-f(X_{n,i})}]| \leq b_{n,1}^m + b_{n,2}^m + b_{n,3}(f)$$

for all $n \in \mathbb{N}$. In particular, if there exists a neighborhood structure $(B_n(i) : n \in \mathbb{N}, i \in I_n)$ such that for all $m \in \mathbb{N}$ and every $f \in \mathcal{F}$,

$$\lim_{n \to \infty} b_{n,1}^m = \lim_{n \to \infty} b_{n,2}^m = \lim_{n \to \infty} b_{n,3}(f) = 0,$$

then the family $(X_{n,i} : n \in \mathbb{N}, i \in I_n)$ is $AI(\mathcal{F})$. \hfill $\Box$
Proof. The proof is an adaptation of argument in Nakhapetyan [30, Lemma 3], though the main idea goes back to [4, Theorem 4]. Since $e^{-f}$ is positive and bounded by 1 it follows that

$$|E\left[e^{-\sum_{i\in I_n} f(X_{n,i})} \right] - \prod_{i\in I_n} E\left[e^{-f(X_{n,i})}\right]| \leq \sum_{i=1}^{m_n-1} \left|E\left[e^{-f(X_{n,i})} \prod_{j=i+1}^{m_n} e^{-f(X_{n,j})}\right] - E\left[e^{-f(X_{n,i})}\right] \cdot E\left[\prod_{j=i+1}^{m_n} e^{-f(X_{n,j})}\right]\right| := \sum_{i=1}^{m_n-1} \varepsilon_i.$$

Fix now an arbitrary $i \in \{1, \ldots, m_n - 1\}$. After writing

$$\prod_{j=1}^{m_n} e^{-f(X_{n,j})} = \prod_{j \in B_n(i)} e^{-f(X_{n,j})} \prod_{j \in B_n^c(i)} e^{-f(X_{n,j})},$$

one can easily check that

$$\varepsilon_i \leq \left|E\left[e^{-f(X_{n,i})} \cdot \left(\prod_{j \in B_n(i)} e^{-f(X_{n,j})} - 1\right) \prod_{j \in B_n^c(i)} e^{-f(X_{n,j})}\right] - E\left[e^{-f(X_{n,i})}\right] \cdot E\left[\left(\prod_{j \in B_n(i)} e^{-f(X_{n,j})} - 1\right) \prod_{j \in B_n^c(i)} e^{-f(X_{n,j})}\right]\right|$$

$$+ \left|E\left[e^{-f(X_{n,i})} \prod_{j \in B_n^c(i)} e^{-f(X_{n,j})}\right] - E\left[e^{-f(X_{n,i})}\right] \cdot E\left[\prod_{j \in B_n^c(i)} e^{-f(X_{n,j})}\right]\right|.$$ 

Note that the first summand on the right hand side of the previous inequality equals

$$\left|E\left[(e^{-f(X_{n,i})} - 1) \cdot \left(\prod_{j \in B_n(i)} e^{-f(X_{n,j})} - 1\right) \prod_{j \in B_n^c(i)} e^{-f(X_{n,j})}\right] - E\left[(e^{-f(X_{n,i})} - 1) \cdot \left(\prod_{j \in B_n(i)} e^{-f(X_{n,j})} - 1\right) \prod_{j \in B_n^c(i)} e^{-f(X_{n,j})}\right]\right|,$$

and since $e^{-\sum f(x_k)} - 1 \neq 0$ implies that $f(x_k) > 0$, and hence $x_k \in K_m$, for at least one $k$, we obtain that

$$\varepsilon_i \leq P\left(X_{n,i} \in K_m, \bigcup_{j \in B_n(i)} \{X_{n,j} \in K_m\}\right) + P\left(X_{n,i} \in K_m\right) \cdot P\left(\bigcup_{j \in B_n(i)} \{X_{n,j} \in K_m\}\right) + \left|E\left[e^{-f(X_{n,i})} \prod_{j \in B_n^c(i)} e^{-f(X_{n,j})}\right] - E\left[e^{-f(X_{n,i})}\right] \cdot E\left[\prod_{j \in B_n^c(i)} e^{-f(X_{n,j})}\right]\right|.$$ 

Hence,

$$\left|E\left[e^{-\sum_{i\in I_n} f(X_{n,i})} \right] - \prod_{i\in I_n} E\left[e^{-f(X_{n,i})}\right]\right| \leq \sum_{i=1}^{m_n-1} \varepsilon_i \leq b_{n,1} + b_{n,2} + b_{n,3}(f).$$

□

Remark 2.3. Recall, $(X^{*}_{n,i} : i \in I_n)$ are independent random elements such that for all $i \in I_n$, $X^{*}_{n,i}$ is distributed as $X_{n,i}$. Further, let $(X^{*}_{n,i} : i \in I_n)$ and $(X_{n,i} : i \in I_n)$ be defined on the same probability space and independent. We can then bound $b_{n,3}(f)$ by

$$b_{n,3}(f) \leq \sum_{i \in I_n} E|E\left[e^{-f(X_{n,i})} - e^{-f(X^{*}_{n,i})} \mid \sigma(X_{n,j} : j \in B_n^c(i))\right]|$$

$$= \sum_{i \in I_n} E|E\left[e^{-f(X_{n,i})} \mid \sigma(X_{n,j} : j \in B_n^c(i))\right] - E\left[e^{-f(X_{n,i})}\right]|.$$
Since for any $f \in CB^+_b(\mathcal{X})$ the function $1 - e^{-f}$ is also an element $CB^+_b(\mathcal{X})$ and further bounded by 1, it follows that
\[
\sum_{i \in I_n} \mathbb{E}[\mathbb{E}[f(X_{n,i}) | \sigma(X_{n,j} : j \in \tilde{B}_{r_n}(i))]] - \mathbb{E}[f(X_{n,i})] \to 0
\]
for all $f \in CB^+_b(\mathcal{X})$ which are bounded by 1 implies that $b_{n,3}(f) \to 0$ for all $f \in CB^+_b(\mathcal{X})$.

**Remark 2.4.** The concept of neighborhoods implicitly appears already in Banys [4, Theorem 4]. There, essentially the same sufficient conditions for convergence of $N_n$ to a Poisson point process are given but with, in our notation, neighborhoods of the form $\tilde{B}_n(i) = \{i + 1, \ldots, i + r_n\}$ and $\tilde{B}_{r_n}(i) = \{i + r_n + 1, \ldots, m_n\}$ for all $i \in I_n$ where $(r_n)_{n \in \mathbb{N}}$ is a sequence of nonnegative integers. The proof is similar to ours and even though it is stated only for the case when $\mathcal{X}$ is locally compact, it transfers directly to the case of a general Polish space.

**Remark 2.5.** Similar results were also obtained by Schuhmacher [35, Theorem 2.1], but with a completely different approach, using the Chen-Stein method. As a consequence, Schuhmacher even provides bounds on the convergence in the so-called Barbour-Brown distance $d_2$. However, this result does not directly imply our results, see [35, Remark 2.4(b)] for the comparison to the result of Banys [4] which is also relevant to our case.

**Example 2.2.** For Bernoulli random variables $X_{n,i}$ such that $\lim_{n \to \infty} \sup_{i \in I_n} \mathbb{P}(X_{n,i} = 1) = 0$ and $\lim_{n \to \infty} \sum_{i \in I_n} \mathbb{P}(X_{n,i} = 1) = \lambda \in (0, \infty)$, one can set $\mathcal{X}' = \{0, 1\}$ and $\mathcal{X} = K_{\lambda} = \{1\}$ for all $m \in \mathbb{N}$. Using Theorem 2.2 together with Proposition 2.4 and Remark 2.3, we recover the result of Arratia et al. [2, Theorem 1] on convergence in distribution of $\sum_{i \in I_n} X_{n,i}$ to a Poisson random variable with intensity $\lambda$, but without the bound on the distance in total variation.

### 3. Regularly varying fields

#### 3.1. The tail field

Consider a (strictly) stationary $\mathbb{R}$-valued random field $\mathbf{X} = (X_i : i \in \mathbb{Z}^d)$ with $d \in \mathbb{N}$. For every finite and nonempty subset of indices $I \subseteq \mathbb{Z}^d$, denote by $\mathbf{X}_I$ the $\mathbb{R}^{|I|}$-valued random vector $(X_i : i \in I)$, i.e. $\mathbf{X}_I$’s represent finite-dimensional distributions of $\mathbf{X}$.

We say that a random field $\mathbf{Y} = (Y_i : i \in \mathbb{Z}^d)$ is the tail field (or tail process) of $\mathbf{X}$, if for all finite and nonempty $I \subseteq \mathbb{Z}^d$,
\[
\lim_{u \to \infty} \mathbb{P}(|X_I| > u) = \lim_{u \to \infty} \mathbb{P}(|Y_I| > u) = 0,
\]
where $0 = (0, \ldots, 0) \in \mathbb{Z}^d$. Here and in the rest of the paper, $A(u) \mid B(u) \overset{d}{\to} C$ as $u \to \infty$ for a family of random elements $A(u), C$ and events $B(u), u > 0$, means that the law of $A(u)$ conditionally on $B(u)$ converges weakly as $u \to \infty$ to the law of $C$.

Note that in (3.1) we implicitly assume that $\mathbb{P}(|X_0| > u) > 0$ for all $u > 0$. Observe, taking $I = \{0\}$ in (3.1) yields that $\lim_{u \to \infty} \mathbb{P}(|X_0| > u) = 0$, and so for all except at most countably many $y \in [1, \infty)$. By standard arguments (see [1, Theorem 1.4.1] and the discussion before it), this implies that $u \mapsto \mathbb{P}(|X_0| > u)$ is a regularly varying function with index $-\alpha$ for some $\alpha > 0$, i.e.
\[
\lim_{u \to \infty} \frac{\mathbb{P}(|X_0| > uy)}{\mathbb{P}(|X_0| > u)} = y^{-\alpha}, y > 0.
\]

In particular, $\mathbb{P}(|Y_0| > y) = y^{-\alpha}$ for all $y \geq 1$, i.e. $|Y_0|$ is Pareto distributed with index $\alpha$. 
Remark 3.1. For notational convenience, in this paper we only consider \( \mathbb{R} \)-valued random fields. All the results in this section extend easily to the case of \( \mathbb{R}^n \)-valued random fields with \( n \in \mathbb{N} \) by simply replacing the absolute value \( | \cdot | \) with an arbitrary norm \( \| \cdot \| \) on \( \mathbb{R}^n \).

3.1. Existence of the tail field

A family of indices \( I \subseteq \mathbb{Z}^d \) is said to be encompassing if for every finite and nonempty \( I \subseteq \mathbb{Z}^d \) there exists at least one \( i^* \in I \) such that \( I - i^* \subseteq I \). Note that necessarily \( \emptyset \in I \).

If \( d = 1 \), the set of nonnegative (or nonpositive) integers is an example of such family. More generally, assume that \( \leq \) is an arbitrary total order on \( \mathbb{Z}^d \) which is translation-invariant in the sense that for all \( i, j \) and \( k \) in \( \mathbb{Z}^d \), \( i \preceq j \) implies \( i + k \preceq j + k \). Then the set \( \mathbb{Z}^d_{\geq} = \{i \in \mathbb{Z}^d : i \succeq 0\} \) is clearly encompassing. Indeed, simply set \( i^* \in I \) to be the (unique) minimal element of the finite set \( I \) with respect to \( \leq \). We refer to such orders as group orders on \( \mathbb{Z}^d \).

In particular, the lexicographic order on \( \mathbb{Z}^d \), denoted by \( \preceq \), is a group order. Recall, for indices \( i = (i_1, \ldots, i_d), j = (j_1, \ldots, j_d) \in \mathbb{Z}^d \), \( i \prec j \) if \( i_k < j_k \) for the first \( k \) where \( i_k \) and \( j_k \) differ, and \( i \preceq j \) if \( i \prec j \) or \( i = j \).

The following result extends [8, Theorem 2.1] which treats the case \( d = 1 \); the proof is postponed to Section 5.1.

Theorem 3.1. For a stationary random field \( X = (X_i : i \in \mathbb{Z}^d) \) and \( \alpha > 0 \), the following three statements are equivalent:

(i) All finite-dimensional distributions of \( X \) are multivariate regularly varying with index \( \alpha \);
(ii) The field \( X \) has a tail field \( Y = (Y_i : i \in \mathbb{Z}^d) \) with \( \mathbb{P}(|Y_0| \geq y) = y^{-\alpha} \) for \( y \geq 1 \).
(iii) There exists an encompassing \( I \subseteq \mathbb{Z}^d \) and a family of random variables \( (Y_i : i \in I) \) with \( \mathbb{P}(|Y_0| \geq y) = y^{-\alpha} \) for \( y \geq 1 \), such that for all finite and nonempty \( I \subseteq \mathbb{Z}^d \),

\[
u^{-1}X_I \mid |X_0| > u \overset{d}{\rightarrow} (Y_i)_{i \in I} , \text{ as } u \to \infty .\tag{3.2}
\]

Recall that for finite \( I \subseteq \mathbb{Z}^d \), \( X_I \) is multivariate regularly varying with index \( \alpha > 0 \) if for some norm \( \| \cdot \| \) on \( \mathbb{R}^{|I|} \) there exists a random vector on \( \mathbb{R}^{|I|} \), say \( \Theta^{(I)} \), such that \( \|\Theta^{(I)}\| = 1 \) and

\[
\langle u^{-1}\|X_I\|, \|X_I^{-1} X_I\rangle \|X_I\| > u \overset{d}{\rightarrow} (Y, \Theta^{(I)}) , \text{ as } u \to \infty ,
\]

where \( Y \) is independent of \( \Theta^{(I)} \) and satisfies \( \mathbb{P}(Y > y) = y^{-\alpha} \) for \( y \geq 1 \).

The equivalence between (i) and (ii) explains why fields admitting a tail process will simply be called regularly varying. We refer to the corresponding \( \alpha \) as the (tail) index of the field.

Remark 3.2. While writing the paper, we learned of a parallel study by Wu and Samorodnitsky [42] who also consider regularly varying fields but with the emphasis on the various notions of the "extremal indices" in this context and the application of the theory to the Brown-Resnick random fields. They show by an example that for \( d \geq 2 \) existence of the limit of \( u^{-1}X_I \mid |X_0| > u \) for all finite \( I \subseteq \mathbb{Z}^d \) when \( \mathbb{I} \) is an orthant in \( \mathbb{Z}^d \), is not sufficient for regular variation of \( X \) and hence existence of the tail field. This made us reconsider an earlier (incorrect) version of Theorem 3.1 and eventually led to a proper extension of [8, Theorem 2.1(ii)].

3.1.2. The spectral tail field

Consider now the space \( \mathbb{R}^{2d} \) equipped with the product topology and the corresponding Borel \( \sigma \)-algebra. One can then rephrase (3.1) simply as

\[
u^{-1}X \mid |X_0| > u \overset{d}{\rightarrow} Y \text{ in } \mathbb{R}^{2d} ,
\]

see e.g. [10, p. 19]. The spectral tail field \( \Theta = (\Theta_i : i \in \mathbb{Z}^d) \) of \( X \) is defined by \( \Theta_i = Y_i/|Y_0|, i \in \mathbb{Z}^d \). Note that \( |\Theta_0| = 1 \). Moreover, the spectral field \( \Theta \) is independent of \( |Y_0| \) and satisfies

\[
|X_0|^{-1}X \mid |X_0| > u \overset{d}{\rightarrow} \Theta \text{ in } \mathbb{R}^{2d} ,
\]
see [31, Proposition 2.2.3].

Even though the tail field is typically not stationary, regular variation and stationarity of the underlying random field $X$ yield specific distributional properties of $\Theta$ (and hence of $Y$) summarized by the so-called time-change formula: for every integrable (in the sense that one of the expectations below exists) or nonnegative measurable function $h: \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ and all $j \in \mathbb{Z}^d$,

$$
\mathbb{E} \left[ h \left( (\Theta_{i-j})_{i \in \mathbb{Z}^d} \right) 1\{\Theta_{-j} \neq 0\} \right] = \mathbb{E} \left[ h \left( (\Theta_{i}/1_{\mathbb{Z}^d})_{i \in \mathbb{Z}^d} \right) 1\{\Theta_{j} \neq 0\} \right].
$$

(3.3)

In the case of time series, (3.3) appears in [8] and the proof is easily extended to the case of random fields, see [42, Theorem 3.2]. Alternatively, one can arrive at (3.3) following the approach of [32] who use the so-called tail measure of $X$ introduced in [34], see also [17].

**Remark 3.3.** Let $X$ be a stationary random field and $\alpha > 0$. If $\lim_{n \to \infty} \mathbb{P}(|X_0| > uy)/\mathbb{P}(|X_0| > u) \to y^{-\alpha}$ for all $y > 0$ and for some encompassing $I \subseteq \mathbb{Z}^d$ there exist random variables $(\Theta_i : i \in I)$ such that for all finite and nonempty $I \subseteq \mathbb{Z}^d$, $|X_0^{-1}X_I| > u \lim_{n \to \infty} (\Theta_i)_{i \in I}$, then $X$ is regularly varying with index $\alpha$; combine the proof of [8, Corollary 3.2] and Theorem 3.1. If $I \neq \mathbb{Z}^d$, the distribution of the whole spectral process $\Theta$ is then determined by (3.3) and the tail field of $X$ is given by $Y = Y \Theta$ where $Y$ is independent of $\Theta$ and satisfies $\mathbb{P}(Y \geq y) = y^{-\alpha}$ for $y \geq 1$.

### 3.2. Convergence to a compound Poisson process

Denote by $\leq$ the componentwise order on $\mathbb{Z}^d$, thus for $i = (i_1, \ldots, i_d), j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$, $i \leq j$ if $i_k \leq j_k$ for all $k = 1, \ldots, d$. Take a sequence of positive integers $(r_n)$ such that $\lim_{n \to \infty} r_n = \infty$ and let $k_n = \lfloor n/r_n \rfloor$. For each $n \in \mathbb{N}$, decompose $\{1, \ldots, n\}^d$ into blocks $J_{n,i}$, $i \in I_n := \{1, \ldots, k_n\}^d$, of size $r_n^d$ by

$$
J_{n,i} = \{j \in \mathbb{Z}^d : (i - 1) \cdot r_n + 1 \leq j \leq i \cdot r_n\}.
$$

(3.4)

In this section we apply the Poisson approximation theory from Section 2 to the point processes based on the (increasing) blocks $X_{n,i} := X_{J_{n,i}}, \ i \in I_n$.

Following [7], we first introduce a suitable space for the $X_{n,i}$’s; for details see [31, Subsection 2.3.1].

#### 3.2.1. A space for blocks - $l_0$

Let $l_0$ be the space of all $\mathbb{R}$-valued arrays on $\mathbb{Z}^d$ converging to zero in all directions, i.e. $l_0 = \{(x_i)_{i \in \mathbb{Z}^d} : \lim_{|i| \to \infty} |x_i| = 0\}$, where $|i| = \max_{k=1,\ldots,d} |i_k|$ for $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$. On $l_0$ consider the uniform norm

$$
\|x\| = \max_{i \in \mathbb{Z}^d} |x_i|, \ x = (x_i)_{i \in \mathbb{Z}^d},
$$

which makes $l_0$ into a separable Banach space. Also, denote by $0 \in l_0$ the array consisting only of 0’s.

Introduce an equivalence relation $\sim$ on $l_0$ by letting $x \sim y$ for $x, y \in l_0$ if for some $j \in \mathbb{Z}^d$, $y_i = x_{i+j}$ for all $i \in \mathbb{Z}^d$. In the sequel, we consider the quotient space $\tilde{l}_0 = l_0/\sim$ of shift-equivalent arrays. Observe, for $\tilde{x} \in \tilde{l}_0$ and an arbitrary $x = (x_i)_i \in \tilde{x}, \tilde{x} = \{(x_{i+j})_i : j \in \mathbb{Z}^d\}$. Further, metric $d : \tilde{l}_0 \times \tilde{l}_0 \to [0, \infty)$ defined by

$$
d(\tilde{x}, \tilde{y}) = \inf \{\|x - y\| : x \in \tilde{x}, y \in \tilde{y}\}, \tilde{x}, \tilde{y} \in \tilde{l}_0,
$$

(3.5)

makes $\tilde{l}_0$ a separable and complete metric space. Note that for $\tilde{x}, \tilde{x}_1, \tilde{x}_2, \ldots \in \tilde{l}_0$, $d(\tilde{x}_n, \tilde{x}) \to 0$ as $n \to \infty$ if and only if for some, and then for every, $x \in \tilde{x}$ there exists $x_n \in \tilde{x}_n, n \in \mathbb{N}$, such that $\|x_n - x\| \to 0$. 
In what follows, on $l_0$ and $\tilde{l}_0$ consider their respective Borel $\sigma$-algebras $\mathcal{B}(l_0)$ and $\mathcal{B}(\tilde{l}_0)$. Call a function $h$ on $l_0$ shift-invariant if $h((x_{i+j})_i) = h((x_i)_i)$ for all $(x_i)_i \in l_0, j \in \mathbb{Z}^d$. Note that $\mathcal{B}(l_0)$ coincides with the trace $\sigma$-algebra of $l_0$ in $\mathbb{R}^{d_0}$ considered with respect to its cylindrical $\sigma$-algebra, and a function $\hat{h}$ on $l_0$ is measurable if and only if the function $x \mapsto \hat{h}(x)$ is a (shift-invariant) measurable function on $l_0$.

### 3.2.2. The point process of blocks

Consider now the space $\tilde{l}_{0,0} := \tilde{l}_0 \setminus \{0\}$ with a Borel subset $B \subseteq \tilde{l}_{0,0}$ being bounded if for some $\epsilon > 0$, $\|x\| > \epsilon$ for all $x \in B$. In other words, bounded sets are those which are bounded away from 0 w.r.t. the metric $\tilde{d}$ defined in (3.5).

We will consider the finite block $X_{n,i}$ as an element of $\tilde{l}_0$ by simply adding infinitely many zeros around $X_{n,i}$ and then mapping the resulting element of $l_0$ into its equivalence class in $\tilde{l}_0$.

**Remark 3.4.** One can regard blocks as elements of the simpler space $l_0$ but since we are interested in clusters of high-threshold exceedances in these blocks one would also need to specify a reference exceedance (an anchor, see Section 3.2.3 below) around which the block is centered, that is, which exceedance is put at position 0. This introduces additional technical difficulties. For example, one natural choice for the anchor is the first maximum of the block (e.g. w.r.t. the lexicographic order on $\mathbb{Z}^d$). In this case one encounters continuity issues since it is possible that the limiting cluster can with positive probability have two exceedances of the exactly same magnitude (e.g. take a moving average process from Example 3.1 below which has at least two identical non-zero coefficients). Consequently, to deduce the limiting behavior of the extremal clusters with this choice of an anchor one would need to exclude such cases by e.g. imposing a suitable condition on the tail process.

Another choice for the anchor could be the first exceedance over a (high) threshold but this (i) is dependent on the choice of the threshold, and (ii) in this case one does not have the nice polar decomposition of the limiting cluster, see Lemma 3.7 and Remark 3.7 below.

On the other hand, the use of $\tilde{l}_0$ is immune to these issues and allows one to develop a general point process convergence theory while keeping all the relevant information about the structure within the extremal clusters, see [6] for an application of the theory to the study of sums and records times of regularly varying time series.

Define the point process of blocks

$$N'_n = \sum_{x \in I_n} \delta_{(x/k_n, X_{n,i}/a_n)}, n \in \mathbb{N},$$

in $\mathcal{M}([0,1]^d \times \tilde{l}_{0,0})$, where the sequence $(a_n)$ is chosen such that

$$\lim_{n \to \infty} n^d \mathbb{P}(|X_0| > a_n) = 1.$$

To obtain the convergence of $N'_n$ we will apply Corollary 2.3.

For each $n \in \mathbb{N}$, denote $J_{r_n} := \{1, \ldots, r_n\}^d = J_{n,1}$ and let $X_{r_n} := X_{J_{r_n}}$ represent the common distribution of the blocks $X_{n,i}, i \in I_n$. Under this notation, the condition (2.3) reduces to the existence of a measure $\nu$ in $\mathcal{M}(l_{0,0})$ satisfying

$$k_n^d \mathbb{P}(a_n^{-1} X_{r_n} \in \cdot) \Rightarrow \nu, \text{ as } n \to \infty. \quad (3.6)$$

Property (3.7) below provides one sufficient condition for this convergence to hold. It appears in the time series literature under the name finite mean cluster size condition or the anticlustering condition.

**Assumption 3.2.** There exists a sequence of positive integers $(r_n)_n$ satisfying $r_n \to \infty$, $r_n/n \to 0$, and for every $u > 0$,

$$\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(\max_{m < |i| \leq r_n} |X_i| > a_n u, |X_0| > a_n u\right) = 0. \quad (3.7)$$
As shown in Proposition 3.8 below, for a sequence \((r_n)\) satisfying (3.7), the convergence in (3.6) holds with the limiting measure \(\nu\) of the form

\[
\nu(\cdot) = \vartheta \int_0^\infty P(y\tilde{Q} \in \cdot)\alpha y^{-\alpha-1}dy,
\]

for some \(\vartheta \in (0,1]\) and \(\tilde{Q}\) being a random element in \(\tilde{I}_0\) satisfying \(\|\tilde{Q}\| = 1\) almost surely. In the following we first describe \(\vartheta\) and \(\tilde{Q}\) in terms of the tail field of \(X\) using the concept of anchoring.

### 3.2.3. Anchoring the tail process

From now on we will restrict our attention to tail fields \(Y = (Y_i)_{i \in \mathbb{Z}^d}\) which satisfy

\[
P(\lim_{|i| \to \infty} |Y_i| = 0) = P(Y \in I_0) = 1.
\]

For example, this is true whenever the underlying random field \(X = (X_i)_{i \in \mathbb{Z}^d}\) satisfies Assumption 3.2 (cf. [8, Proposition 4.2]).

We say that a measurable function \(A : \{x \in I_0 : \|x\| > 1\} \to \mathbb{Z}^d\) is an anchoring function if

1. \(A((x_i)_{i \in \mathbb{Z}^d}) = j\) for some \(j \in \mathbb{Z}^d\) implies that \(|x_j| > 1\);
2. For each \(j \in \mathbb{Z}^d\), \(A((x_{i+j})_i) = A((x_i)_i) - j\).

In words, \(A\) picks one of the finitely many \(x_j\)'s which are larger than one in absolute value in a way which is translation covariant. Observe, for an arbitrary group order on \(\mathbb{Z}^d\), the following are examples of an anchoring function.

- first exceedance: \(A^{fe}((x_i)_i) = \min\{j \in \mathbb{Z}^d : |x_j| > 1\}\),
- last exceedance: \(A^{le}((x_i)_i) = \max\{j \in \mathbb{Z}^d : |x_j| > 1\}\),
- first maximum: \(A^{fm}((x_i)_i) = \min\{j \in \mathbb{Z}^d : |x_j| = \|X_i\|\}\).

We will exploit the following property of the tail field which is implied solely by the stationarity of \(X\), and can be seen as a special case of the time-change formula.

**Lemma 3.3.** For every bounded measurable function \(h : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}\) and all \(j \in \mathbb{Z}^d\),

\[
E[h((Y_i)_{i \in \mathbb{Z}^d}) \1 \{|Y_j| > 1\}] = E[h((Y_{i-j})_{i \in \mathbb{Z}^d}) \1 \{|Y_{-j}| > 1\}]. \tag{3.8}
\]

**Proof.** Assume in addition that \(h\) is continuous with respect to the product topology on \(\mathbb{R}^{\mathbb{Z}^d}\). Then, since \(P(|Y_j| = 1) = P(|Y_0| = |\Theta_j| = 1) = 0\) for all \(j \in \mathbb{Z}^d\), the definition of the tail process and stationarity of \((X_i)\) imply

\[
E[h((Y_i)_i) \1 \{|Y_j| > 1\}] = \lim_{u \to \infty} E[h((u^{-1}X_i)_i) \1 \{|X_j| > u\} \mid |X_0| > u] = \lim_{u \to \infty} \frac{E[h((u^{-1}X_i)_i) \1 \{|X_j| > u, |X_0| > u\}]}{P(|X_0| > u)} = \lim_{u \to \infty} \frac{E[h((u^{-1}X_{i-j})_i) \1 \{|X_0| > u, |X_{-j}| > u\}]}{P(|X_0| > u)} = E[h((Y_{i-j})_i) \1 \{|Y_{-j}| > 1\}] .
\]

Since finite Borel measures on a metric space are determined by integrals of continuous and bounded functions, this yields (3.8). \(\square\)

**Remark 3.5.** Using the already mentioned tail measure of \(X\), one can give a one-line proof of the previous result, see [32, Lemma 2.2].

**Lemma 3.4.** Assume that \(P(Y \in I_0) = 1\). Then for every anchoring function \(A\)

\[
P(A(Y) = 0) > 0 .
\]
Proof. Assume that $\mathbb{P}(A(Y) = 0) = 0$. Applying (3.8) yields
\[
1 = \sum_{j \in \mathbb{Z}^d} \mathbb{P}(A(Y) = j) = \sum_{j \in \mathbb{Z}^d} \mathbb{P}(A(Y) = j, |Y_j| > 1)
\]
\[
= \sum_{j \in \mathbb{Z}^d} \mathbb{P}(A((Y_i - j)_i) = j, |Y_j| > 1) = \sum_{j \in \mathbb{Z}^d} \mathbb{P}(A(Y) = 0, |Y_j| > 1) = 0.
\]
Hence, $\mathbb{P}(A(Y) = 0) > 0$.

If $\mathbb{P}(Y \in l_0) = 1$, for any anchoring function $A$ we define the anchored tail process of $Y$ (with respect to $A$) as any random element of $l_0$, denoted by $Z^A = (Z^A_i : i \in \mathbb{Z}^d)$, which satisfies
\[
Z^A \overset{d}{=} Y \mid A(Y) = 0.
\]
Also, define $Q^A = (Q^A_i : i \in \mathbb{Z}^d)$ by $Q^A = Z^A/\|Z^A\|$ and call it the anchored spectral tail process (with respect to $A$).

Lemma 3.5. Assume that $\mathbb{P}(Y \in l_0) = 1$ and let $A, A'$ be two anchoring functions. Then
\[
\mathbb{P}(A(Y) = 0) = \mathbb{P}(A'(Y) = 0)
\]
and
\[
Z^A \overset{d}{=} Z^{A'} \text{ in } l_0.
\]
Proof. Let $h : l_0 \to [0, \infty)$ be an arbitrary measurable and shift-invariant function. Using (3.8) and shift-invariance of $h$ we obtain
\[
\mathbb{E}[h(Y)1\{A(Y) = 0\}] = \sum_{j \in \mathbb{Z}^d} \mathbb{E}[h(Y)1\{A(Y) = 0, A'(Y) = j, |Y_j| > 1\}]
\]
\[
= \sum_{j \in \mathbb{Z}^d} \mathbb{E}[h(Y)1\{A(Y) = -j, A'(Y) = 0\}]
\]
\[
= \mathbb{E}[h(Y)1\{A'(Y) = 0\}].
\]
Taking $h \equiv 1$ yields the first statement, and then the second one follows immediately by the construction of the space $l_0$.

If $\mathbb{P}(Y \in l_0) = 1$, denote by $\vartheta$ the common value of $\mathbb{P}(A(Y) = 0)$, i.e. for an arbitrary anchoring function $A$ set
\[
\vartheta = \mathbb{P}(A(Y) = 0).
\]
In particular, for any group order $\preceq$ on $\mathbb{Z}^d$, using the first/last exceedance as an anchor yields,
\[
\vartheta = \mathbb{P}(\sup_{j \preceq 0} |Y_j| \leq 1) = \mathbb{P}(\sup_{j \preceq 0} |Y_j| \leq 1)
\]
since $\mathbb{P}(|Y_0| > 1) = 1$. Also,
\[
\vartheta = \mathbb{P}(A^{fm}(Y) = 0) = \mathbb{P}(A^{fm}(\Theta) = 0).
\]
Observe here that the function $A^{fm}$ remains well defined on the whole set $l_0$ without $0$. Under suitable dependence conditions, $\vartheta$ turns out to be the extremal index of the field $(|X_j|)_j$, see Remark 3.11 below (cf. also Remark 3.9).
Furthermore, second part of the previous result shows that the distribution of the anchored tail process, when viewed as an element in $\tilde{l}_0$, does not depend on the anchoring function. Hence, there exists a random element in $\tilde{l}_0$, denoted by $\tilde{Z}$, which satisfies

$$\tilde{Z} \overset{d}{=} Z^A$$

for all anchoring functions $A$; simply take your favorite anchoring function $A$ and let $\tilde{Z}$ be the equivalence class of $Z^A$ in $\tilde{l}_0$. Moreover, let $\tilde{Q} = \tilde{Z}/\|Z\|$, so in particular $\tilde{Q} \overset{d}{=} Q^A$ in $\tilde{l}_0$ for any anchor $A$. We will also refer to $\tilde{Z}$ and $\tilde{Q}$ as the anchored tail process and the anchored spectral tail process, respectively.

Under an appropriate assumption, the distribution of the anchored tail process $\tilde{Z}$ represents the distribution of the asymptotic cluster of exceedances of the underlying field $X$ and one can think of it as a "typical" cluster of exceedances; see Remark 3.9 below. On the other hand, due to the conditioning, the distribution of the tail process $Y$ exhibits bias towards clusters with more exceedances. This Palm-like relationship between the typical cluster and the tail process is made formal in the following result and has links with the recent work of Sigman and Whitt [40] who studied Palm distributions of marked point processes on $Z$.

A random element $R = (R_i)_{i \in \mathbb{Z}^d}$ in $l_0$ is called a representative of a random element $\tilde{R}$ in $\tilde{l}_0$ if $R \overset{d}{=} \tilde{R}$ in $l_0$. In particular, for any anchoring function $A$, $Z^A$ and $Q^A$ become representatives of $\tilde{Z}$ and $\tilde{Q}$, respectively.

**Proposition 3.6.** Assume that $\mathbb{P}(Y \in l_0) = 1$ and let $Z = (Z_i)_{i \in \mathbb{Z}^d}$ be any representative of $\tilde{Z}$. Then for every measurable and shift-invariant function $h : l_0 \to [0, \infty)$,

$$\mathbb{E}[h(Y)] = \vartheta \mathbb{E} \left[ h(Z) \cdot \sum_{k \in \mathbb{Z}^d} \mathbb{1}\{|Z_k| > 1\} \right]. \quad (3.11)$$

**Remark 3.6.** Taking $h \equiv 1$ in (3.11) yields

$$\vartheta = \frac{1}{\mathbb{E} \left[ \sum_{k \in \mathbb{Z}^d} \mathbb{1}\{|Z_k| > 1\} \right]},$$

and since $\vartheta > 0$ this implies $\mathbb{E} \left[ \sum_{k \in \mathbb{Z}^d} \mathbb{1}\{|Z_k| > 1\} \right] < \infty$.

**Proof of Proposition 3.6.** Fix an arbitrary anchoring function $A$. By the definition of $Z^A$, (3.8) and shift-invariance of $h$ we get

$$\vartheta \mathbb{E} \left[ h(Z^A) \cdot \sum_{k \in \mathbb{Z}^d} \mathbb{1}\{|Z^A_k| > 1\} \right] = \sum_{k \in \mathbb{Z}^d} \mathbb{E}[h(Y)\mathbb{1}\{|Y_k| > 1, A(Y) = 0\}]$$

$$= \sum_{k \in \mathbb{Z}^d} \mathbb{E}[h(Y)\mathbb{1}\{|Y_{-k}| > 1, A(Y) = -k\}]$$

$$= \sum_{k \in \mathbb{Z}^d} \mathbb{E}[h(Y)] = \mathbb{E}[h(Y)].$$

The claim for an arbitrary representative $(Z_i)_i$ of $\tilde{Z}$ now follows since the function $\vartheta \mapsto h((x_i)_i) \cdot \sum_{k \in \mathbb{Z}^d} \mathbb{1}\{|x_k| > 1\}$ on $l_0$ is shift-invariant. \qed

The next result shows that the polar decomposition of the tail process carries over to the anchored tail process, and gives a representative of the anchored spectral tail process $Q$ only in terms of the original spectral tail process $\Theta$. 
Lemma 3.7. Assume that \( \mathbb{P}(Y \in l_0) = 1 \). Then \( \mathbb{P}(\|Z\| \geq y) = y^{-\alpha} \) for all \( y \geq 1 \), and \( \|\tilde{Z}\| \) and \( \tilde{Q} \) are independent. Moreover,

\[
\tilde{Q} \overset{d}{=} \Theta \big| A^{f_m}(\Theta) = 0 \big. \tag{3.12}
\]

**Proof.** Using \( A^{f_m} \) as anchor implies that in \( \tilde{l}_0 \),

\[
(\|\tilde{Z}\|, \tilde{Q}) \overset{d}{=} (\|Y\|, Y/\|Y\|) \big| A^{f_m}(Y) = 0 = (|Y_0|, \Theta) \big| A^{f_m}(\Theta) = 0.
\]

The result now follows by the properties of the tail process.

**Remark 3.7.** Let \( Z = (Z_i)_{i \in \mathbb{Z}^d} \) be any representative of \( \tilde{Z} \) and set \( Q = (Q_i)_i = (Z_i/\|Z\|)_i \). Since the function \( x \mapsto \|x\| \) on \( l_0 \) is shift-invariant, the previous result implies that \( \mathbb{P}(\|Z\| \geq y) = y^{-\alpha}, \ y \geq 1 \). On the other hand, \( \|Z\| \) and \( Q \) (as an element in \( l_0 \)) are in general not independent. Still, if \( h : l_0 \to [0, \infty) \) is measurable and shift-invariant then

\[
\mathbb{E}[h(Z)] = \mathbb{E} \left[ \int_1^\infty h(yQ)y^{-\alpha} \, dy \right].
\]

**Example 3.1.** Let \( (\xi_i : i \in \mathbb{Z}^d) \) be i.i.d. random variables with regularly varying distribution with index \( \alpha > 0 \), i.e.

\[
\lim_{u \to \infty} \frac{\mathbb{P}(|\xi_0| > uy)}{\mathbb{P}(|\xi_0| > u)} = y^{-\alpha}, \ y > 0,
\]

and for some \( p \in [0, 1] \),

\[
\lim_{u \to \infty} \mathbb{P}(\xi_0 > 0 | \xi_0 > u) = p, \ \lim_{u \to \infty} \mathbb{P}(\xi_0 < 0 | \xi_0 > u) = 1 - p.
\]

Consider the infinite order moving average process \( X = (X_i : i \in \mathbb{Z}^d) \) defined by

\[
X_i = \sum_{j \in \mathbb{Z}^d} c_j \xi_{i-j},
\]

where \( (c_j : j \in \mathbb{Z}^d) \) is a field of real numbers satisfying

\[
0 < \sum_{j \in \mathbb{Z}^d} |c_j|^\delta < \infty,
\]

for some \( \delta > 0 \) such that \( \delta < \alpha \) and \( \delta \leq 1 \). It is easily shown (see e.g. [33, Section 4.5]) that this condition ensures that the series above is absolutely convergent. Note also that \( \sum_{j \in \mathbb{Z}^d} |c_j|^\alpha < \infty \). Furthermore, it can be proved as in [33, Lemma 4.24] that

\[
\lim_{u \to \infty} \frac{\mathbb{P}(|X_0| > u)}{\mathbb{P}(|\xi_0| > u)} = \sum_{j \in \mathbb{Z}^d} |c_j|^\alpha.
\]

Moreover, extending the arguments of Meinguet and Segers [27, Example 9.2], one can show that the stationary field \( X \) is jointly regularly varying with index \( \alpha \) and the spectral tail field given by

\[
(\Theta_i)_{i \in \mathbb{Z}^d} \overset{d}{=} (Kc_{i+j}/|c_j|)_{i \in \mathbb{Z}^d}
\]

where \( K \) is a \((-1, 1)\)-valued random variable with \( \mathbb{P}(K = 1) = p \), and \( J \) an \( \mathbb{Z}^d \)-valued random variable, independent of \( K \), such that \( \mathbb{P}(J = j) = |c_j|^\alpha / \sum_{j \in \mathbb{Z}^d} |c_j|^\alpha \) for all \( j \in \mathbb{Z}^d \).

In particular, \( \mathbb{P}(\Theta \in l_0) = \mathbb{P}(Y \in l_0) = 1 \). Choosing \( A^{f_m} \) as the anchoring function (see (3.10) and (3.12)) yields that

\[
\vartheta = \max_{j \in \mathbb{Z}^d} \frac{|c_j|^\alpha}{\sum_{j \in \mathbb{Z}^d} |c_j|^\alpha}, \quad \tilde{Q} \overset{d}{=} \left( \frac{Kc_j}{\max_{i \in \mathbb{Z}^d} |c_i|} \right)_{j \in \mathbb{Z}^d} \text{ in } \tilde{l}_0.
\]
3.2.4. Intensity convergence

The following result is an extension of the case \( d = 1 \) shown in [7, Lemma 3.3]. The proof is based on [8, Theorem 4.3] and can be found in [31, Section 2.3.5] (note that \( \tilde{Q} \) is there denoted by \( Q \)).

**Proposition 3.8.** If \( (r_n)_{n \in \mathbb{N}} \) is a sequence of positive integers satisfying \( r_n \to \infty, r_n/n \to 0 \), and such that (3.7) holds, then

\[
k_n \mathbb{P}(a_n^{-1} X_{r_n} \in \cdot) \xrightarrow{\text{d}} \nu(\cdot) = \vartheta \int_0^{\infty} \mathbb{P}(y \tilde{Q} \in \cdot) \alpha y^{-\alpha-1} dy,
\]

as \( n \to \infty \) in \( \mathcal{M}(\tilde{l}_{0,0}) \), where \( \vartheta \in (0,1] \) and the anchored spectral tail process \( \tilde{Q} \) are defined in Section 3.2.3.

**Remark 3.8.** Note that \( \nu \) is a proper element of \( \mathcal{M}(\tilde{l}_{0,0}) \). Indeed, since \( \|Q\| = 1 \), \( \nu(\{x \in \tilde{l}_{0,0} : \|x\| > \varepsilon\}) = \vartheta \varepsilon^{-\alpha} < \infty \) for all \( \varepsilon > 0 \).

**Remark 3.9.** Observe, since \( \nu(\{x : \|x\| = u\}) = 0 \) for all \( u > 0 \), (3.13) implies that

\[
k_n \mathbb{P}(M_{r_n} > a_n u) \to \vartheta u^{-\alpha}, \ u > 0,
\]

as \( n \to \infty \), where \( M_{r_n} = \|X_{r_n}\| \) is the maximum of the block \( X_{r_n} \). Moreover, for every \( u > 0 \),

\[
\mathbb{P}( (a_n u)^{-1} X_{r_n} \in \cdot \mid M_{r_n} > a_n u) = \frac{k_n \mathbb{P}( (a_n u)^{-1} X_{r_n} \in \cdot, M_{r_n} > a_n u)}{k_n \mathbb{P}(M_{r_n} > a_n u)} \xrightarrow{\text{w}} \frac{u^\alpha}{\vartheta} \int_0^{\infty} \mathbb{P}(u^{-1} y \tilde{Q} \in \cdot) \alpha y^{-\alpha-1} dy
\]

\[
= \int_1^{\infty} \mathbb{P}(y \tilde{Q} \in \cdot) \alpha y^{-\alpha-1} dy = \mathbb{P}(\tilde{Z} \in \cdot),
\]

where \( \xrightarrow{\text{w}} \) denotes weak convergence of finite measures and the last line follows from Lemma 3.7. Hence, for all \( u > 0 \),

\[
(a_n u)^{-1} X_{r_n} \mid M_{r_n} > a_n u \xrightarrow{\text{d}} \tilde{Z} \text{ in } \tilde{l}_{0,0}.
\]

Thus, the distribution of the anchored tail process \( \tilde{Z} \) is the asymptotic distribution of a cluster of extremes of \( X \), i.e. block of size \( r_n^d \) with at least one exceedance over the level \( a_n u \). Also, we identify the anchored spectral process \( \tilde{Q} \) by

\[
M_{r_n}^{-1} X_{r_n} \mid M_{r_n} > a_n u \xrightarrow{\text{d}} \tilde{Q} \text{ in } \tilde{l}_{0,0}.
\]

In fact, convergences (3.14) and (3.15) imply (3.13), this is actually the approach in [7, Lemma 3.3].

3.2.5. Point process convergence

Following [7] we give a convenient convergence determining family for point processes on \([0,1]^d \times \tilde{l}_{0,0}\) (see Definition 2.1). For an element \( x \in \tilde{l}_{0,0} \) and any \( \delta > 0 \) denote by \( x^\delta \in \tilde{l}_{0,0} \) the equivalence class of the sequence \( (x_i \{ |x_i| > \delta \})_i \), where \( (x_i)_i \in \tilde{l}_{0,0} \) is an arbitrary representative of \( x \). Let \( F^0_{\tilde{l}_{0,0}} \) be the family of all functions \( f \in CB^+_{\tilde{l}_{0,0}}(\tilde{l}_{0,0}) \) such that for some \( \delta > 0 \), \( f(x) = f(x^\delta) \) for all \( x \in \tilde{l}_{0,0} \), where we set \( f(0) = 0 \), i.e. \( f \) depends only on coordinates greater than \( \delta \) in absolute value. As shown in [31, Lemma 2.5.2 and Remark 2.5.4], \( F^0_{\tilde{l}_{0,0}} \) is convergence determining in the sense of Definition 2.1.

In view of Proposition 3.8, our main result now follows by an application of Corollary 2.3.
Theorem 3.9. Let $X$ be a stationary regularly varying random field with tail index $\alpha > 0$. Assume that $(r_n)_{n \in \mathbb{N}}$ is a sequence of positive integers satisfying $r_n \to \infty$, $r_n/n \to 0$, such that (3.7) holds and the family $((i/k_n, X_{n,i}/a_n) : n \in \mathbb{N}, i \in I_n)$ is $AI(F'_0)$.

Then
\[
N'_n = \sum_{i \in I_n} \delta(i/k_n, X_{n,i}/a_n) \xrightarrow{d} N' = \sum_{i \in \mathbb{N}} \delta(T_i, P_iQ_i)
\]

in $\mathcal{M}_p([0,1]^d \times \hat{I}_0,0)$, where $N' \sim$ PPP(Leb $\times \nu$) and

(i) $\sum_{i \in \mathbb{N}} \delta(T_i, P_i)$ is a Poisson point process on $[0,1]^d \times (0,\infty)$ with intensity measure $\theta \text{Leb} \times \alpha y^{-\alpha-1} \, dy$;

(ii) $(Q_i)_{i \in \mathbb{N}}$ is a sequence of i.i.d. elements in $\hat{I}_0$, independent of $\sum_{i \in \mathbb{N}} \delta(T_i, P_i)$ and with common distribution equal to the distribution of the anchored spectral process $\hat{Q}$.

Proof. The only thing left to verify is that $N' \sim$ PPP(Leb $\times \nu$) can be represented as in (3.16) but this follows easily using standard arguments; for details see [31, Theorem 2.3.4].

Remark 3.10. If $(Q_{j}^i)_{j \in \mathbb{Z}^d}$, $i \in \mathbb{N}$ is a sequence of independent elements of $\mathbb{I}_0$ which are representatives of $\hat{Q}$ and independent of $\sum_{i \in \mathbb{N}} \delta(T_i, P_i)$, one can construct the limiting process $N'$ simply by considering $\sum_{i \in \mathbb{N}} \delta(T_i, P_i(Q_j^i)_{j \in \mathbb{Z}^d})$ as a point process on $[0,1]^d \times \hat{I}_0,0$.

To illustrate the meaning the result in Theorem 3.9 set $P_i = (\Gamma_i/\theta)^{-1/\alpha}$ where $\Gamma_i = E_1 + \cdots + E_i$, $i \in \mathbb{N}$, with $(E_i)_{i \in \mathbb{N}}$ being i.i.d. standard exponential random variables, and let $(T_i)_{i \in \mathbb{N}}$ be i.i.d. uniform random vectors in $[0,1]^d$ independent of the sequence $(P_i)_{i \in \mathbb{N}}$. Then $\sum_{i \in \mathbb{N}} \delta(T_i, P_i)$ is a PPP($\theta \text{Leb} \times \alpha y^{-\alpha-1} \, dy$) which in addition satisfies $P_1 > P_2 > \ldots$ almost surely. Consequently, if $X_{n,i}$ and $T_{n,i}$, $i = 1, 2, \ldots, k_n$, denote the original blocks $X_{n,i}$ and their positions $i/k_n$, $i \in I_n$, but relabeled so that
\[
\|X_{n,(1)}\| \geq \|X_{n,(2)}\| \geq \cdots \geq \|X_{n,(k)}\|,
\]

the continuous mapping theorem applied to (3.16) for every $k \in \mathbb{N}$ yields the convergence
\[
(T_{n,(i)}, X_{n,(i)}/a_n)_{i=1,2,\ldots,k} \xrightarrow{d} (T_i, P_i(Q_j^i)_{j \in \mathbb{Z}^d})_{i=1,2,\ldots,k}
\]
in the space $([0,1]^d \times \mathbb{R} \setminus \{0\})^k$ (to show that the corresponding mapping is a.s. continuous w.r.t. the limit in (3.16) use [6, Proposition 2.8]).

Furthermore, by applying the continuous mapping theorem to (3.16) and using similar arguments as in [24, Proposition 1.34], one obtains the following convergence of point processes on a simpler state space; the details can be found in [31, Corollary 2.3.15].

Corollary 3.10. In the notation of Remark 3.10, if there exists a sequence $r_n \to \infty$, $r_n/n \to 0$ for which (3.16) holds, then, with $J_n = \{1, \ldots, n\}^d$,
\[
\sum_{j \in J_n} \delta(j/n, X_j/a_n) \xrightarrow{d} \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{Z}^d} \delta(T_i, P_iQ_j^i)
\]
in $\mathcal{M}_p([0,1]^d \times (\mathbb{R} \setminus \{0\}))$ with bounded sets being those which are bounded away from $[0,1]^d \times \{0\}$.

Observe that in this convergence one loses the information about the structure of the cluster in the limit, see [7] for a detailed discussion.

Remark 3.11. As noted by [8, Remark 4.7], when convergence in (3.17) holds, the quantity $\hat{\theta}$ is the extremal index of the field $(|X_j|)_{j \in \mathbb{Z}^d}$ since $n^d \mathbb{P}(|X_0| > a_n u) \to u^{-\alpha}$ and
\[
\mathbb{P}(\max_{j \in J_n} |X_j| \leq a_n u) \to \mathbb{P}
\[
\left(\sum_{i \in \mathbb{N}} \mathbb{I}_{\{P_i > u\}} = 0\right) = e^{-\hat{\theta}u^{-\alpha}},
\]
as $n \to \infty$, for all $u > 0$. 

The assumptions of Theorem 3.9 are straightforward to check in the case of $m$-dependent stationary fields. In general, however, checking these assumptions is not trivial. Still, one can extend the convergence in (3.16) to fields which can be approximated by $m$-dependent fields, such as spatial infinite order moving average processes from Example 3.1 as explained in the following remark.

**Remark 3.12.** Assume that $X = (X_i : i \in \mathbb{Z}^d)$ is a stationary random field such that there exists a sequence of stationary regularly varying $m$-dependent fields $X^{(m)} = (X_i^{(m)} : i \in \mathbb{Z}^d)$, $m \in \mathbb{N}$, and two sequences of strictly positive real numbers $(b_n)$ and $(d^{(m)})$ such that for all $m \in \mathbb{N}$, $n^d \mathbb{P}(|X_0^{(m)}| > b_n) \to d^{(m)} > 0$, while also for any $u > 0$

$$\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}(\max_{1 \leq i \leq 1 \cdot n} |X_i^{(m)} - X_i| > b_n u) = 0.$$ 

Provided that the tail processes of the approximating random fields $X^{(m)}$ behave reasonably as $m \to \infty$, the process $X$ satisfies the Poissonian limiting relation in (3.16), see [31, Section 2.4.1] for details, cf. also Kulik and Soulier [25] who study the problem in the time series setting.

In Section 4 below we show that Theorem 3.9 can be applied to the random field of (exponentially transformed) scores from the sequence alignment problem. In particular, this is an example of a field with a nontrivial dependence structure, but for which the asymptotic $F'_0$-independence property can be shown to hold. For this purpose we apply Proposition 2.4 and for convenience, we rephrase it in this setting and in the form suitable for our needs.

**Corollary 3.11.** Let for each $n \in \mathbb{N}$, $(\tilde{X}_{n,i} : i \in I_n)$ be identically distributed random elements in $\tilde{l}_0$ and such that for all $\epsilon > 0$,

$$\lim_{n \to \infty} k_n^d \mathbb{P}(\|\tilde{X}_{n,1}\| > a_n \epsilon) < \infty. \quad (3.18)$$

If there exists a neighborhood structure $(B_n(i) : n \in \mathbb{N}, i \in I_n)$ such that, denoting $\|B_n\| = \max_{i \in I_n} |B_n(i)|$,

(i) As $n \to \infty$, $\|B_n\|/k_n^d \to 0$ and for all $\epsilon > 0$,

$$k_n^d \|B_n\| \max_{i \in I_n} \mathbb{P}(\|\tilde{X}_{n,i}\| > a_n \epsilon, \|\tilde{X}_{n,j}\| > a_n \epsilon) \to 0; \quad (3.19)$$

(ii) For $n$ big enough, $\tilde{X}_{n,i}$ is independent of $\sigma(\tilde{X}_{n,j} : j \notin B_n(i))$ for each $i \in I_n$. Then the family $(i/k_n, \tilde{X}_{n,i}/a_n) : n \in \mathbb{N}, i \in I_n)$ is $AI(F'_0)$.

**Proof.** First, observe that for any sequence $\epsilon_m \to 0$ sets $K'_m = [0,1]^d \times \{x \in \tilde{l}_0 : \|x\| > \epsilon_m\}$, $m \in \mathbb{N}$, form a base for the family of bounded sets of $[0,1]^d \times \tilde{l}_0$. Next, regardless of ordering of $I_n = \{1, \ldots, k_n\}$, $|\hat{B}_n(i)| \leq |B_n(i)|$ for all $i \in I_n$. Since $\tilde{X}_{n,i}$’s are identically distributed,

$$b_{m,1}^n = \sum_{i \in I_n} \sum_{j \in B_n(i)} \mathbb{P}(i/k_n, \tilde{X}_{n,i}/a_n) \in K'_m \cdot \mathbb{P}(j/k_n, \tilde{X}_{n,j}/a_n) \in K'_m) \leq k_n^d \|B_n\| \mathbb{P}(\|\tilde{X}_{n,1}\| > a_n \epsilon_m)^2.$$

In view of (3.18), $\limsup_{n \to \infty} b_{m,1}^n \leq (const.) \limsup_{n \to \infty} \|B_n\|/k_n^d = 0$ for all $m \in \mathbb{N}$. Similarly, (3.19) implies that $\lim_{n \to \infty} b_{n,2}^m = 0$ for all $m \in \mathbb{N}$, and by (ii), $b_{n,3}(f) = 0$ for every measurable function $f \geq 0$ on $[0,1]^d \times \tilde{l}_0$ and $n$ big enough. Applying Proposition 2.4 finishes the proof. \qed
4. Sequence alignment problem

This section is devoted to the proof of Theorem 1.3. We will use the notation introduced in Section 1.2 and assume throughout that Assumptions 1.1 and 1.2 hold. In particular, \((A_i)_{i \in \mathbb{Z}}\) and \((B_i)_{i \in \mathbb{Z}}\) are independent i.i.d. sequences, \(S_m^i = \sum_{k=0}^{m-1} s(A_{i-k}, B_{j-k})\) for \(i, j \in \mathbb{Z}\) and \(m \geq 0\), and \(S_{i,j} = \sup\{S_m^i : m \geq 0\}\) for \(i, j \in \mathbb{Z}\). For some of the key technical results in our analysis we are indebted to Hansen [19] who even allows sequences \((A_i)\) and \((B_i)\) to be Markov chains. In the i.i.d. setting the corresponding proofs, which rely on change of measure arguments, are much less involved. For an alternative approach based on combinatorial arguments see Dembo et. al. [15].

4.1. The tail field

Consider the positive stationary field \(X = (X_{i,j} : i, j \in \mathbb{Z})\) defined by
\[ X_{i,j} = e^{S_{i,j}}, \quad i, j \in \mathbb{Z}. \]
Observe that by (1.11), for \(\alpha^* > 0\) satisfying \(E[e^{\alpha^* s(A,B)}] = 1\),
\[ P(X_{i,j} > u) \sim Cu^{-\alpha^*}, \quad \text{as } u \to \infty, \]
i.e. the marginal distribution of \(X\) is regularly varying. Moreover, the transformed field \(X\) has a tail field and therefore fits into the framework of Section 3.

**Proposition 4.1.** The field \(X\) is regularly varying with tail index \(\alpha^*\) and with the spectral tail field \(\Theta = (\Theta_{i,j} : i, j \in \mathbb{Z})\) satisfying

(i) \(\Theta_{i,j} = 0\) for \(i, j \in \mathbb{Z}, \, i \neq j\).

(ii) \(\Theta_{m,m} = e^{S_m^*}\) for \(m \in \mathbb{Z}\), where \(S_0^* = 0\) and
\[ S_m^* = \sum_{i=1}^{m} \varepsilon_i, \quad \text{for } m \geq 1 \quad \text{and} \quad S_m^* = -\sum_{i=1}^{m} \varepsilon_i^*, \quad \text{for } m \leq -1, \]
for independent i.i.d. sequences \((\varepsilon_i)_{i \geq 1}\) and \((\varepsilon_i^*)_{i \geq 1}\) whose distributions correspond to the distributions of \(s(A,B)\) under the product measure \(\mu_A \times \mu_B\) and under the tilted measure \(\mu^*\) from (1.7), respectively.

Before proving Proposition 4.1 we give one expression for the constant \(\vartheta\) and one representative of the anchored spectral tail process \(\tilde{Q}\), both defined in Section 3.2.3.

**Corollary 4.2.** The tail field \(Y\) of \(X\) satisfies \(P(Y \in l_0) = 1\) with
\[ \vartheta = P(\Gamma + \max_{m \geq 1} S_m^* \leq 0) > 0, \]
where \(\Gamma\) is independent of \((S_m^*)_{m \geq 1}\) and satisfies \(P(\Gamma \geq x) = e^{-\alpha^* x}, \, x \geq 0\). A representative \(Q = (Q_{i,j})_{i,j \in \mathbb{Z}}\) of the anchored spectral tail process \(\tilde{Q}\) is given by
\[ Q_{i,j} = 0 \quad \text{for } i \neq j, \quad (Q_{m,m})_{m \in \mathbb{Z}} \overset{d}{=} \left( (S_m^*, m \in \mathbb{Z}) \mid \sup_{m \leq -1} S_m^* < 0, \sup_{m \geq 1} S_m^* \leq 0 \right). \]

**Proof.** The tail field \(Y = (Y_{i,j})_{i,j \in \mathbb{Z}}\) of \(X\) is given by \(Y_{i,j} = Y \cdot \Theta_{i,j}\) where \(Y\) satisfies \(P(Y \geq y) = y^{-\alpha^*}\) for \(y \geq 1\) and is independent from \(\Theta\). Observe, \(E[\varepsilon_i] = E[s(A,B)] < 0\) and since the moment generating function \(m(\alpha) = E[e^{\alpha s(A,B)}]\) is strictly convex and \(m(0) = m(\alpha^*) = 1\),
\[ E[\varepsilon_i^*] = E[s(A,B)e^{\alpha^* s(A,B)}] = \frac{dm}{d\alpha}(\alpha^*) > 0. \]
This implies that $\mathbb{P}(\lim_{|m| \to \infty} S_m^c = -\infty) = 1$ so $\Theta$ and $Y$ are elements of $l_0$ almost surely. In particular, by (3.9),

$$0 < \theta = \mathbb{P}( \sup_{(i,j) \neq (0,0)} Y_{i,j} < 1 ) = \mathbb{P}(Y \max_{m \geq 1} \Theta_{m,m} \leq 1) = \mathbb{P}(\log Y + \max_{m \geq 1} S_m^c \leq 0),$$

where $\log Y$ is a standard exponential random variable with index $\alpha^*$. This yields (4.2) and (4.3) follows directly from (3.12).

To prove Proposition 4.1 we need two auxiliary lemmas. The first one is a rough estimate using Markov inequality, see Section 5.2 for the proof.

**Lemma 4.3.** There exist a constant $c_0 > 0$ such that

$$\lim_{u \to \infty} e^{2\alpha^* u} \mathbb{P}\left( \max_{m > c_0 u} S_{m,0,0}^c \geq 0 \right) = 0.$$ 

Before we state the second lemma, observe first that, using $\mathbb{E}[e^{s(A,B)}] = 1$, for all $u \geq 0$ and any integer $m \geq 0$,

$$\mathbb{P}(S_{m,0,0}^c \geq u) = \mathbb{E}[e^{-\alpha^* s(A,B)}] e^{\alpha^* S_{m,0,0}^c} \mathbb{I}\{S_{m,0,0}^c \geq u\} \leq e^{-\alpha^* u} \mathbb{P}_* (S_{0,0}^c \geq u) \leq e^{-\alpha^* u},$$

where the tilted measure $\mathbb{P}_*$ makes pairs $(A_k, B_k)$ for $k = -m + 1, \ldots, 0$, independent and distributed according to the measure $\mu_*$. The following result is proved in [19, Lemma 5.11] using change of measure arguments and the Azuma-Hoeffding inequality for martingales. The key fact is that, whenever $\mu_* \neq \mu_0 \times \mu_B^*$ (which holds under (1.8)),

$$\mathbb{E}_{\nu_A \times \nu_B}[s(A, B)] < \mathbb{E}_\mu[s(A, B)]$$

for all $\nu_A \in \{\mu_0, \mu_A^*\}$ and $\nu_B \in \{\mu_B, \mu_B^*\}$, where $\mathbb{E}_\mu$ denotes the expectation assuming $(A, B)$ is distributed according to $\mu$, see [14, beginning of Section 3]. The proof of [19, Lemma 5.11] is much simpler in the i.i.d. setting and can be found in [31, Lemma 4.2.3].

**Lemma 4.4 ([19, Lemma 5.11]).** There exists an $0 < \epsilon_0 < 1$ such that for all $u > 0$,

$$\sup_{i,j \in \mathbb{Z}, i \neq j, m \geq 0} \mathbb{P}(S_{m,0,0}^c > u, S_{i,j}^c > u) \leq 2e^{-(1+\epsilon_0)\alpha^* u}.$$

**Proof of Proposition 4.1.** Let $\Theta$ be from the statement of the proposition. We first show that, as $u \to \infty$,

$$X_{0,0}^{-1} J I \mid X_{0,0} > u \xrightarrow{d} \Theta_I,$$

for all $I \subseteq \mathbb{Z}^2 \setminus \{(m,m) : m \leq -1\}$. Since $X_{0,0}$ is regularly varying with index $\alpha^*$, this will prove the regular variation property of $X$ and show that the spectral tail field $\Theta' = (\Theta'_{i,j})_{i,j \in \mathbb{Z}}$ of $X$ satisfies

$$(\Theta'_{i,j} : (i,j) \in \mathbb{Z}^2 \setminus \{(m,m) : m \leq -1\}) \xrightarrow{d} (\Theta_{i,j} : (i,j) \in \mathbb{Z}^2 \setminus \{(m,m) : m \leq -1\}),$$

see Remark 3.3.

Observe, by (1.12), for each $m \geq 1$,

$$X_{m,m} = \max\{X_{m-1,m-1} e^{s(A_m,B_m)}, 1\}.$$

Now since $X_{0,0}$ is regularly varying and independent of the i.i.d. sequence $(e^{s(A_k,B_k)})_{k \geq 1}$, [36, Theorem 2.3] implies that for all $m \geq 0$, as $u \to \infty$,

$$X_{0,0}^{-1} (X_{0,0}, X_{1,1}, \ldots, X_{m,m}) \mid X_{0,0} > u \xrightarrow{d} (1, e^{s(A_k,B_k)}, \ldots, \prod_{k=1}^{m} e^{s(A_k,B_k)})$$

$$\xrightarrow{d} (\Theta_{0,0}, \Theta_{1,1}, \ldots, \Theta_{m,m}).$$
Since $\Theta_{i,j} = 0$ for all $i, j \in \mathbb{Z}$, $i \neq j$, (4.4) will follow if we show that for all such $i, j$,
\[
P(X_{i,j} > X_{0,0} \eta \mid X_{0,0} > u) \leq P(X_{i,j} > \eta \mid X_{0,0} > u)
\]
\[
= P(S_{i,j} > \log u + \log \eta \mid S_{0,0} > \log u) \to 0, \quad \text{as } u \to \infty,
\]
for all $\eta \in (0, 1)$.
Fix now $i, j \in \mathbb{Z}$ such that $i \neq j$. Using (1.11) and Lemmas 4.3 and 4.4, for every $M \geq 0$,
\[
\limsup_{u \to \infty} P(S_{i,j} > u - M \mid S_{0,0} > u) = \limsup_{u \to \infty} C^{-1} e^{\alpha^* u} P(S_{0,0} > u, S_{i,j} > u - M)
\]
\[
\leq \limsup_{u \to \infty} C^{-1} e^{\alpha^* u} P\left(\max_{1 \leq m \leq c_0 u} S_{i,j}^m > u - M, \max_{1 \leq l \leq c_0 u} S_{0,0}^l > u - M\right)
\]
\[
\leq \limsup_{u \to \infty} 2C^{-1} e^{c_0 u M} (c_0 u)^2 e^{-c_0 \alpha^* u} = 0,
\]
hence (4.6) holds.

Finally, we extend (4.5) to equality in distribution on whole $\mathbb{R}^{22}$. First, fix $m \geq 1$ and note that by (3.3) and $E[e^{\alpha^* s(A,B)}] = 1$,
\[
P(\Theta_{-m,-m}^* > 0) = E[\Theta_{-m,-m}^*] = 1.
\]
Further, for arbitrary bounded measurable function $h : \mathbb{R}^{2m+1} \to \mathbb{R}$, using (3.3) and (4.5),
\[
E[h(\Theta_{-m,-m}^*, \ldots, \Theta_{m,m}^*)] = E\left[h(\Theta_{m,m}^{-1}(\Theta_{0,0}^*, \ldots, \Theta_{2m,2m}^*)))\Theta_{m,m}^*\right]
\]
\[
= E\left[h(e^{-\sum_{k=1}^{m} e_{\varepsilon_k}} e^{-\sum_{k=2}^{m} e_{\varepsilon_k}}, \ldots, e^{e_{m}}, 1, e^{e_{m+1}}, \ldots, e^{e_{2m+1}}, e_{k}) \prod_{k=1}^{m} e^{\alpha^* \varepsilon_k}\right].
\]
By definition of $(\Theta_{k,k})_{k \in \mathbb{Z}}$, this implies that
\[
E[h(\Theta_{-m,-m}^*, \ldots, \Theta_{m,m}^*)] = E[h(\Theta_{-m,-m}^*, \ldots, \Theta_{m,m}^*)].
\]
\[
\Box
\]

### 4.2. Checking the assumptions of Theorem 3.9

In view of (4.1), define the sequence $(a_n)$ by
\[
a_n = (Cn^2)^{1/\alpha^*}, \quad n \in \mathbb{N},
\]
so that $\lim_{n \to \infty} n^2 P(X_{0,0} > a_n) = 1$. The proof of the following result is postponed to Section 5.2.2.

**Proposition 4.5.** The random field $X$ satisfies Assumption 3.2 for every sequence of positive integers $(r_n)$ such that $\lim_{n \to \infty} r_n = \infty$ and $\lim_{n \to \infty} r_n/n^\epsilon = 0$ for all $\epsilon > 0$.

Take now two sequences of positive integers $(l_n)$ and $(r_n)$ such that
\[
\lim_{n \to \infty} \log n/l_n = \lim_{n \to \infty} l_n/r_n = \lim_{n \to \infty} r_n/n^\epsilon = 0
\]
for all $\epsilon > 0$ and set $k_n = \lfloor n/r_n \rfloor$. Recall the blocks of indices $J_{n,i} \subseteq \{1, \ldots, k_n r_n\}^2$ of size $r_n^2$ from (3.4) and the blocks $X_{n,i} := X_{l_n}^i$ for $i \in I_n := \{1, \ldots, k_n\}^2$. To show that the $X_{n,i}$’s satisfy the asymptotic independence condition from Theorem 3.9, we will apply Corollary 3.11. However, to use it we first need to alter the original blocks.
First, cut off the edges of the $J_{n,i}$'s by $l_n$, more precisely, define
\[ \tilde{J}_{n,i} := \{(i,j) : (i-1) \cdot r_n + 1 \leq (i,j) \leq i \cdot r_n - l_n \cdot 1, \ i \in I_n \}. \]

Further, for all $i,j \in \mathbb{Z}$ and $m \in \mathbb{N}$ let $\varepsilon_{i,j}^m$ be the empirical measure on $E^2$ of the sequence $(A_{i-k}, B_{j-k})$, $k = 0, \ldots, m-1$, i.e.
\[ \varepsilon_{i,j}^m = \frac{1}{m} \sum_{k=0}^{m-1} \delta_{(A_{i-k}, B_{j-k})}. \]

For every $\eta > 0$ denote by $B_\eta$ the set of all probability measures $\nu$ on $E^2$ satisfying $\|\nu - \mu^*\| := \sum_{(a,b) \in E} |\nu(a,b) - \mu^*(a,b)| < \eta$.

Set $b_n = \log a_n$ for all $n \in \mathbb{N}$ and for all $\eta > 0$, $i,j \in \mathbb{Z}$ define the random variable $\tilde{S}_{i,j} = \tilde{S}_{i,j}(n,\eta)$ by
\[ \tilde{S}_{i,j} = \max\{S_{i,j}^m : 1 \leq m \leq c_0 b_n, \varepsilon_{i,j}^m \in B_\eta\} \]
with $c_0 > 0$ from Lemma 4.3 and $\max 0 := 0$. Further, define the modified blocks $\tilde{X}_{n,i} = \tilde{X}_{n,i}(\eta)$ in $l_0$ by
\[ \tilde{X}_{n,i} = (e^{\tilde{S}_{i,j}} : (i,j) \in \tilde{J}_{n,i}). \]

It turns out that by restricting to the $\tilde{X}_{n,i}$'s one does not lose any relevant information. To understand the role of the $\tilde{X}_{n,i}$'s, observe that for any nonnegative and measurable function $f$ on $[0,1]^2 \times l_0, 0$
\[ |E\left[e^{-\sum_{i \in I_n} f(i/k_n, X_{n,i}/a_n)} \right] - \prod_{i \in I_n} E\left[e^{-f(i/k_n, X_{n,i}/a_n)} \right] | \leq |E\left[e^{-\sum_{i \in I_n} f(i/k_n, X_{n,i}/a_n)} \right] - E\left[e^{-\sum_{i \in I_n} f(i/k_n, X_{n,i}/a_n)} \right] | + \limsup_{n \to \infty} \|\tilde{X}_{n,i}/k_n\| = 0. \]

Recall now the convergence determining family $\mathcal{F}'_n$ from Section 3.2.5. The proof of the following result is in Section 5.2.3.

**Lemma 4.6.** For every $\eta > 0$ and every $f \in \mathcal{F}'_0$, $I_1 + I_2 \to 0$ as $n \to \infty$.

**Remark 4.1.** In particular, since $I_1 \to 0$ for all $f \in \mathcal{F}'_0$, point processes $\sum_{i \in I_n} \delta_{(i/k_n, X_{n,i})}$, which are based on the $S_{i,j}$'s, converge in distribution if and only if point processes $\sum_{i \in I_n} \delta_{(i/k_n, X_{n,i})}$, which are based on the $S_{i,j}$'s from (1.10), do, and in that case their limits coincide. Similarly, one can show that the former (and therefore the latter) convergence is equivalent to convergence of point processes of blocks based on nonstationary scores from (1.5). In particular, the point process convergence results given below hold even with the $S_{i,j}$'s from (1.10) replaced with the ones from (1.5).

By (4.8) and Lemma 4.6, to show that the $(i/k_n, X_{n,i}/a_n)$'s are A1($\mathcal{F}'_0$), it is sufficient to find at least one $\eta > 0$ such that $I_3 \to 0$ for all $f \in \mathcal{F}'_0$, i.e. that the $(i/k_n, X_{n,i}/a_n)$'s are A1($\mathcal{F}'_0$). For that purpose, we apply Corollary 3.11.

For every $i = (i_1, i_2) \in I_n$ define its neighborhood $B_n(i)$ by
\[ B_n(i) = \{j = (j_1, j_2) \in I_n : i_1 = j_1 \text{ or } i_2 = j_2\}. \]

Observe, $|B_n(i)| = 2k_n - 1$ for all $i \in I_n$ and hence $\lim_{n \to \infty} \|B_n\|/k_n^2 = 0$. Further, by (4.1) for all $\epsilon > 0$,
\[ \limsup_{n \to \infty} k_n^2 P(\|\tilde{X}_{n,1}\| > a_n \epsilon) \leq \limsup_{n \to \infty} k_n^2 \epsilon^2 P(\tilde{S}_{0,0} > a_n \epsilon) \leq \limsup_{n \to \infty} k_n^2 \epsilon^2 P(X_{0,0} > a_n \epsilon) = \epsilon^{-\alpha} < \infty. \]
Next, recall that $S_{i,j}^m = \sum_{k=0}^{m-1} s(A_{i-k}, B_{j-k})$ so by (4.7), for every $n \in \mathbb{N}$,

$$\tilde{S}_{i,j} \in \sigma(A_{i-[c_0b_n]+1}, \ldots, A_i, B_{j-[c_0b_n]+1}, \ldots, B_j).$$

By the construction of the $J_{n,i}$’s and the choice of $(l_n)$ such that, in particular, $\lim_{n \to \infty} c_0b_n/l_n = \lim_{n \to \infty} l_n/r_n = 0$, this implies that, for $n$ large enough, $X_{n,i}$ and the blocks $(X_{n,j} : j \notin B_n(i))$ are constructed from completely different sets of the $A_k$’s and the $B_k$’s, and therefore independent.

Further, when $j \in B_n(i)$, $j \neq i$, arbitrary scoring $S_{i,j}^m$ and $S_{i,j}^l$, which build blocks $X_{n,i}$ and $X_{n,j}$, respectively (i.e. $(i, j) \in J_{n,i}, (i', j') \in J_{n,j}$ and $1 \leq m, l \leq c_0b_n$), for $n$ large enough, depend on completely different sets of variables from at least one of the sequences $(A_k)$ or $(B_k)$. Thus, the following result, which is [19, Corollary 5.4], applies.

**Lemma 4.7** ([19, Corollary 5.4]). *There exist constants $c_2, \eta > 0$ such that for all $u > 0$

$$\mathbb{P}(S_{i,j}^m > u, S_{i,j}^l > u, \varepsilon^{m,0}_{i,j}, \varepsilon^{l,0}_{i,j} \in B_\eta) \leq e^{-((3/2+\epsilon_2)\alpha^* u)}$$

uniformly over all $i, j \in \mathbb{Z}$ and $m, l \in \mathbb{N}$ such that $\min\{i, j\} < -m + 1$ or $\max\{i - l, j - l\} > 0$.

**Remark 4.2.** [19, Corollary 5.4] follows from [19, Lemma 5.3] under condition (12) in [19], which, when $(A_i)$ and $(B_i)$ are i.i.d. sequences, is equivalent to Assumption 1.2, see [19, Remark 3.8]; the proof can be found in [31, Lemma 4.3.5]. For a different and, in this i.i.d. setting, probably better argument, see [15, pp. 2032–2033]. Note that the fact that $E$ is finite is here exploited.

Take now the constant $\eta > 0$ from the previous result and recall the corresponding $X_{n,i}$’s. For $n$ big enough and every $\epsilon > 0$ we get that

$$k_n^2 \|B_n\| \max_{i \notin B_n(i) \atop i = l_n} \mathbb{P}(\|X_{n,i}\| > a_n \epsilon, \|X_{n,j}\| > a_n \epsilon) \leq k_n^2 2r_0^2 (c_0b_n)^2 e^{-(3/2+\epsilon_2)\alpha^*(b_n + \log \epsilon)} \sim (\text{const.}) n^3 r_n^2 b_n \epsilon \to 0,$$

as $n \to \infty$, by the choice of $(r_n)$ and since $b_n \sim 2 \log n/\alpha^*$.

Hence by Corollary 3.11, for this $\eta$, the blocks $X_{n,i}$, and therefore the original blocks $X_{n,i}$, satisfy the asymptotic independence condition. We can now apply Theorem 3.9: the convergence

$$\sum_{i = l_n} n \delta_{(i/k_n, X_{n,i}/(Cn^{2/\alpha^*})} - \frac{d_{\vartheta}}{k_n} \sum_{k \in \mathbb{N}} \delta_{(T_k, P_k(Q_{\vartheta,i})), i,j \in \mathbb{Z}}$$

(4.9)

holds in $\mathcal{M}_p([0, 1]^2 \times [0, 0])$ where the limit is described in Theorem 3.9 and Remark 3.10, with $\vartheta$ given by (4.2) and $(Q_{\vartheta,i,j})_{i,j \in \mathbb{Z}}, k \in \mathbb{N}$ with the distribution given in (4.3).

Theorem 1.3 stated in the introduction now follows from (4.9) by an application of the continuous mapping theorem since the mapping

$$\sum_{k \in \mathbb{N}} \delta_{(t_k, x_k)} \mapsto \sum_{k \in \mathbb{N}} \delta_{(t_k, x_k, C^{1/\alpha^*})},$$

is continuous (see [6, Proposition 2.8]), and then applying standard Poisson process transformation arguments (see e.g. [33, Proposition 3.7]).

Consider now the space $\mathcal{M}_p([0, 1]^2 \times \mathbb{R})$ with a set $B \subseteq [0, 1]^2 \times \mathbb{R}$ being bounded if $B \subseteq [0, 1]^2 \times (x, \infty)$ for some $x \in \mathbb{R}$.

**Corollary 4.8.** *Under Assumptions 1.1 and 1.2,

$$\sum_{i,j=1}^{n} \delta_{(i,j, S_{i,j} - \frac{2\log(n)}{\alpha^*})} - \frac{d_{\vartheta}}{k_n} \sum_{k \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \delta_{(T_k, P_k + Q_m)},$$

in $\mathcal{M}_p([0, 1]^2 \times \mathbb{R})$ where*
(i) $\sum_{k \in \mathbb{N}} \delta_{(T_k, \bar{P}_k)}$ is a Poisson point process on $[0, 1]^2 \times \mathbb{R}$ with intensity measure $\vartheta C \text{Leb} \times \alpha^* e^{-\alpha^* u} du$.

(ii) $(Q^k_m)_{m \in \mathbb{Z}}, \ k \in \mathbb{N}$ are i.i.d. two-sided $\mathbb{R}$-valued sequences, independent of $\sum_{k \in \mathbb{N}} \delta_{(T_k, \bar{P}_k)}$ and with common distribution equal to the distribution of the random walk $(S^c_m)_m$ conditioned on staying negative for $m < 0$ and nonpositive for $m > 0$.

**Proof.** An application of Corollary 3.10 to the convergence in (4.9) yields that

$$\sum_{i,j=1}^n \delta_{((i,j)/n, X_{i,j}/(Cn^2)^{1/\alpha^*})} \xrightarrow{d} \sum_{k \in \mathbb{N}} \sum_{i,j \in \mathbb{Z}} \delta_{(T_k, P_k Q^k_{i,j})} = \sum_{k \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \delta_{(T_k, P_k Q^k_{m,m})}$$

in $\mathcal{M}_p([0, 1]^2 \times (0, \infty))$, where the last equality follows since $Q^k_{i,j} = 0$ for $i \neq j$. It is easy to see that

$$\sum_{k \in \mathbb{N}} \delta_{(t_k, x_k)} \mapsto \sum_{k \in \mathbb{N}} \delta_{(t_k, \log(x_k C^{1/\alpha^*})_k)}$$

is a well defined mapping from $\mathcal{M}_p([0, 1]^2 \times (0, \infty))$ to $\mathcal{M}_p([0, 1]^2 \times \mathbb{R})$ which is also continuous w.r.t. the vague topologies on these spaces. The result now follows easily from (4.10) via the continuous mapping theorem and using standard Poisson process transformation arguments (again, see e.g. [33, Proposition 3.7]).

\[ \Box \]

### 5. Postponed proofs

#### 5.1. Proof of Theorem 3.1

We only prove (iii)$\Rightarrow$(i) since (i)$\Rightarrow$(ii) follows as in [8, Theorem 2.1] and (ii)$\Rightarrow$(iii) is obvious. Also, since we essentially adapt the arguments of [8, Theorem 2.1], some details are omitted.

Observe first that (3.2) with $I = \{0\}$ implies that for all $\epsilon > 0$,

$$\lim_{u \to \infty} \frac{\mathbb{P}(|X_0| > u \epsilon)}{\mathbb{P}(|X_0| > u)} = e^{-\alpha},$$

and moreover that $X_0$ is a regularly varying random variable with index $\alpha$, see [8, Theorem 2.1].

Take now an arbitrary finite $I \subseteq \mathbb{Z}^d$ such that $|I| \geq 2$ and consider the space $\mathbb{R}^{|I|} \setminus \{0\}$ with bounded sets being those which are contained in sets $B_\epsilon := \{(x_i)_{i \in I} \in \mathbb{R}^{|I|} : \sup_{i \in I} |x_i| > \epsilon\}$, $\epsilon > 0$. In view of (5.1), multivariate regular variation (with index $\alpha$) of $X_I$ is equivalent to the existence of a nonzero measure $\mu_I \in \mathcal{M}(\mathbb{R}^{|I|} \setminus \{0\})$ such that

$$\mu_I^u(\cdot) := \frac{\mathbb{P}(u^{-1} X_I \in \cdot)}{\mathbb{P}(|X_0| > u)} \xrightarrow{\nu} \mu_I, \text{ as } u \to \infty,$$

see [37, Definition 3.1, Proposition 3.1] (cf. [8, Equation (1.3)]).

Arguing exactly as in [8, Theorem 2.1] it follows that the vague limit of $\mu_I^u$, if it exists, is necessarily nonzero, and furthermore, that $\limsup_{u \to \infty} \mu_I^u(B_\epsilon) \leq |I| \epsilon^{-\alpha} < \infty$ for every $\epsilon > 0$. Since sets $\{(x_i)_{i \in I} \in \mathbb{R}^{|I|} : \sup_{i \in I} |x_i| \in [\epsilon, M]\}$ are compact for every $\epsilon, M > 0$, by [23, Theorem 4.2] it follows that the set $\{\mu_I^u : u > 0\}$ is relatively compact in the vague topology of $\mathcal{M}(\mathbb{R}^{|I|} \setminus \{0\})$.

Since $I$ is encompassing, we can take $i^* \in I$ such that $I^* := I - i^* \subseteq I$. By [8, Lemma 2.2], to show that measures $\mu_I^u$ vaguely converge as $u \to \infty$, it suffices to prove that $\lim_{u \to \infty} \mu_I^u(f)$ exists for all $f \in \mathcal{F}$ where $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \subseteq C B_\alpha^u(\mathbb{R}^{|I|} \setminus \{0\})$ with

$$\mathcal{F}_1 = \{f : \text{for some } \epsilon > 0, f((x_i)_{i \in I}) = 0 \text{ if } |x_i| \leq \epsilon\},$$

$$\mathcal{F}_2 = \{f : f((x_i)_{i \in I}) \text{ does not depend on } x_{i^*}\}.$$
Note that families \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) depend on \( I \) but we omit this in the notation.

Since \( I' \subseteq I \), stationarity, (3.2) and (5.1) imply that for every \( f \in \mathcal{F}_1 \) and \( \epsilon > 0 \) as in the definition of \( \mathcal{F}_1 \),
\[
\mu^I_u(f) = \frac{\mathbb{P}(|X_0| > u \epsilon)}{\mathbb{P}(|X_0| > u)} \cdot \mathbb{E}[f(u^{-1} X_{I'}) \mid |X_0| > u \epsilon] \to e^{-\alpha} \mathbb{E}[f(\epsilon(Y_{i \in I'}))] , \quad \text{as } u \to \infty .
\]

Further, every \( f \in \mathcal{F}_2 \) naturally induces a function \( \tilde{f} \) in \( CB^+_{0}(\mathbb{R}^{I'_{(i^*)}}) \) and by stationarity
\[
\mu^I_u(f) = \frac{\mathbb{P}(\tilde{f}(u^{-1} X_{I'\setminus\{i^*\}}))}{\mathbb{P}(|X_0| > u)} = \mu^I_u(\tilde{f}) .
\]

Hence, \( \lim_{u \to \infty} \mu^I_u(f) \) exists for all \( f \in \mathcal{F}_2 \) if \( X_{I'\setminus\{i^*\}} \) is multivariate regularly varying.

Observe, we have shown that for an arbitrary finite \( I \subseteq \mathbb{Z}^d \) such that \( |I| \geq 2 \), \( X_I \) is multivariate regularly varying if \( X_{I'\setminus\{i^*\}} \) is, where \( i^* \in I \) is such that \( I - i^* \subseteq I \). Therefore, (i) now follows by regular variation of \( X_0 \) and since \( I \) is encompassing.

### 5.2. Local sequence alignments

#### 5.2.1. Proof of Lemma 4.3

By Markov inequality, for any \( \lambda \geq 0 \) and all \( u > 0 \)
\[
\mathbb{P} \left( \max_{m > c_0u} S^m_{0,0} \geq 0 \right) \leq \sum_{l=0}^{\infty} \mathbb{P} \left( S_{0,0}^{[c_0u]+l} \geq 0 \right) \leq \sum_{l=0}^{\infty} \mathbb{E} \left[ e^{\lambda S_{0,0}^{[c_0u]+l}} \right] = \sum_{l=0}^{\infty} m(\lambda)^{[c_0u]+l} ,
\]
where \( m(\lambda) = \mathbb{E}[e^{\lambda s(A,B)}] \) is the moment generating function of \( s(A,B) \). Fix any \( 0 < \lambda_0 < \alpha^* \). By strict convexity of \( m \) and \( m(\alpha^*) = 1 \), \( 0 < m(\lambda_0) < 1 \) in particular
\[
\mathbb{P} \left( \max_{m > c_0u} S^m_{0,0} \geq 0 \right) \leq e^{c_0u \log m(\lambda_0)} \sum_{l=0}^{\infty} m(\lambda_0)^l .
\]
Since the series above is summable, taking \( c_0 \) strictly larger than \( -2\alpha^*/\log m(\lambda_0) \) finishes the proof.

#### 5.2.2. Proof of Proposition 4.5

Let \( (r_n) \) be an arbitrary sequence of positive integers satisfying \( r_n \to \infty \) and \( r_n/n^* \to 0 \) for all \( \epsilon > 0 \). We have to show that for an arbitrary \( u > 0 \)
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \max_{m < |i,j| \leq r_n} X_{i,j} > a_n u \mid X_{0,0} > a_n u \right) = 0 .
\]

We deal with the diagonal elements using arguments from [5, Lemma 4.1.4]. First, notice that by (1.12), for each \( k \geq 1 \) we can decompose
\[
X_{k,k} = \max \left\{ e^{\max_{0 \leq l \leq k} S_{k,k}^l} s_{k,k}^e, X_{0,0} e^{S_{k,k}^e} \right\} ,
\]
with \( (S_{k,k}^l)_{0 \leq l \leq k} \) being independent of \( X_{0,0} \). Hence, using stationarity,
\[
\mathbb{P} \left( \max_{m < |i| \leq r_n} X_{k,k} > a_n u \mid X_{0,0} > a_n u \right) \leq 2 \sum_{k=m+1}^{r_n} \mathbb{P}(X_{k,k} > a_n u \mid X_{0,0} > a_n u) + 2 \sum_{k=m+1}^{r_n} \mathbb{P}(X_{0,0} e^{S_{k,k}^e} > a_n u \mid X_{0,0} > a_n u) .
\]
Since $r_n/n^2 \to 0$, the choice of $(a_n)$ and (4.1) imply that
\[
2r_n \mathbb{P}(e^{\max_{0 \leq l \leq r_n} s_{l,n}^0} > a_n u) \leq 2r_n \mathbb{P}(X_{0,0} > a_n u) \to 0, \quad \text{as } n \to \infty. 
\]
For the second term, take an arbitrary $0 < \lambda_0 < \alpha^*$ so in particular $0 < m(\lambda_0) = \mathbb{E}[e^{\lambda_0 (A,B)}] < 1$ by strict convexity of $m$. Apply Markov’s inequality and use independence between $X_{0,0}$ and $S_{k,k}$ to obtain
\[
\sum_{k=m+1}^{r_n} \mathbb{P}(X_{0,0} e^{S_{k,k}^0} > a_n u \mid X_{0,0} > a_n u) \leq \frac{\mathbb{E}[X_{0,0} e^{S_{0,0}^0} 1\{X_{0,0} > a_n u\}]}{(a_n u)^{\lambda_0} \mathbb{P}(X_{0,0} > a_n u)} \sum_{k=m+1}^{r_n} m(\lambda_0)^k.
\]
A variant of Karamata’s theorem (see [12, Appendix B.4], also [11, pp. 26–28]) now implies that
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}\left( \max_{0 \leq k \leq r_n} X_{k,k} > a_n u \mid X_{0,0} > a_n u \right) \leq \frac{\alpha^*}{\alpha^* - \lambda_0} \lim_{m \to \infty} \sum_{k=m+1}^{r_n} m(\lambda_0)^k = 0.
\]
It remains to deal with the non diagonal terms. More precisely, in order to obtain (5.2), we will show that, denoting $b_n = \log a_n$ and $M = \log u$,
\[
\limsup_{n \to \infty} \mathbb{P}\left( \max_{0 \leq k \leq c_0 b_n} S_{i,j} > b_n + M \mid S_{0,0} > b_n + M \right) = C^{-1} e^{\alpha^* M} \limsup_{n \to \infty} e^{\alpha^* b_n} \mathbb{P}\left( \max_{0 \leq k \leq c_0 b_n} S_{k,k} > b_n + M, S_{0,0} > b_n + M \right) = 0.
\]
Notice that $e^{\alpha^* b_n} = C n^2$. First, since $r_n/n \to 0$, stationarity and Lemma 4.3 give
\[
\limsup_{n \to \infty} e^{\alpha^* b_n} \mathbb{P}\left( \max_{0 \leq k \leq c_0 b_n} S_{k,k} > 0 \right) \leq \limsup_{n \to \infty} e^{\alpha^* b_n} (2r_n + 1)^2 \mathbb{P}\left( \max_{0 \leq k \leq c_0 b_n} S_{0,0} > 0 \right)
\leq \limsup_{n \to \infty} \frac{(2r_n + 1)^2}{C n^2} = 0.
\]
Now by Lemma 4.4 there exist an $\epsilon_0 > 0$ such that
\[
\limsup_{n \to \infty} e^{\alpha^* b_n} \mathbb{P}\left( \max_{0 \leq k \leq c_0 b_n} S_{k,k} > b_n + M, S_{0,0} > b_n + M \right)
= \limsup_{n \to \infty} e^{\alpha^* b_n} \mathbb{P}\left( \max_{0 \leq k \leq c_0 b_n} S_{i,j} > b_n + M, \max_{0 \leq k \leq c_0 b_n} S_{k,k} > b_n + M \right)
\leq \limsup_{n \to \infty} e^{\alpha^* b_n} (2r_n + 1)^2 (c_0 b_n)^2 e^{-(1+\epsilon') \alpha^* b_n}
\leq 2c_0^2 C^{-(1+\epsilon')} \limsup_{n \to \infty} \left( \frac{2r_n + 1}{n^{\epsilon_0/2}} \right)^2 \left( \frac{b_n}{n^{\epsilon_0/2}} \right)^2 = 0,
\]
where the last equality follows by the choice of $(r_n)$ and since $b_n \sim \frac{2}{\alpha} \log n$.

5.2.3. Proof of Lemma 4.6
First, we need the following simple result proved by a change of measure argument and a large deviation bound for empirical measures, cf. the proof of [19, Lemma 5.14, Equation (54)].

**Lemma 5.1.** For all $\eta > 0$ there exists an $\epsilon_1 > 0$ such that
\[
\lim_{u \to \infty} e^{(1+\epsilon_1) C_{\eta}^2} \sup_{m \geq 1} \mathbb{P}(S_{0,0}^m > u, \varepsilon_{0,0}^m \notin B_\eta) = 0.
\]
Proof. Fix \( \eta > 0 \) and denote \( A_m(u) = \{ S_0^m > u, \varepsilon_0^m \notin B_\eta \} \) for \( m \geq 1 \) and \( u > 0 \). Note that, since \( S_0^m = \sum_{k=0}^{m-1} s(A_{-k}, B_{-k}) \), \( \mathbb{P}(A_m(u)) = 0 \) whenever \( m \leq u/\|s\| \), so for fixed \( u > 0 \) we only need to deal with \( \mathbb{P}(A_m(u)) \) for \( m > u/\|s\| \).

First, a change of measure yields
\[
\mathbb{P}(A_m(u)) = \mathbb{E} \left[ \frac{\exp(\alpha^* S_0^m)}{\exp(\alpha^* S_0^m)} \mathbb{P}_m \left( \varepsilon_0^m \notin B_\eta \right) \right] \leq e^{-\alpha^* a \mathbb{P}^*(\varepsilon_0^m \notin B_\eta)},
\]
where \( \mathbb{P}^* \) makes \( (A_{-k}, B_{-k}), k = 0, \ldots, m - 1 \), i.i.d. elements of \( E^2 \) with common distribution \( \mu^* \).

By Sanov’s theorem (see [16, Theorem 2.1.10])
\[
\limsup_{m \to \infty} \frac{1}{m} \log \mathbb{P}^*(\varepsilon_0^m \notin B_\eta) \leq - \inf_{\pi \notin B_\eta} H(\pi \mid \mu^*).
\]

Since, for a sequence of probability measures \( (\pi_n) \) on \( E^2 \), \( H(\pi_n \mid \mu^*) \to 0 \) implies that \( \|\pi_n - \mu^*\| \to 0 \), we can find a constant \( c = c(\eta) > 0 \) such that \( \inf_{\pi \notin B_\eta} H(\pi \mid \mu^*) > c \). Hence, for all \( m > u/\|s\| \) with \( u \) large enough
\[
\mathbb{P}^*(\varepsilon_0^m \notin B_\eta) \leq e^{-mc} \leq e^{-uc/\|s\|}.
\]

To finish the proof, it suffices to take \( \epsilon_1 := \frac{c}{\|s\|} > 0 \). \( \square \)

Proof of Lemma 4.6. Take an arbitrary \( f \in \mathcal{F}_0^* \subseteq CB^+(\{0, 1\}^2 \times \tilde{L}_0, 0) \) and let \( \epsilon > 0 \) be such that \( f(t, (x_{i,j}), i, j) = f(t, (x_{i,j}, k(x_{i,j}))_{i, j}) \) for all \( t \in \{0, 1\}^2 \) and \( (x_{i,j}, i, j) \in \tilde{L}_0, 0 \) with \( f(t, 0) = 0 \).

By the elementary inequality \( |\sum_{k=1}^N a_k - \sum_{k=1}^N b_k| \leq \sum_{k=1}^N |a_k - b_k| \), valid for all \( k \geq 1 \) and \( a_k, b_k \in [0, 1] \) (see e.g. [18, Lemma 3.4.3]),
\[
\left| \mathbb{E} \left[ e^{-\sum_{i \in L_n} f(i/k_n, \hat{X}_{n,i}/a_n)} \right] - \mathbb{E} \left[ e^{-\sum_{i \in L_n} f(i/k_n, \hat{X}_{n,i}/a_n)} \right] \right| \leq 2 \sum_{i \in L_n} \mathbb{E} \left[ e^{-f(i/k_n, \hat{X}_{n,i}/a_n)} - e^{-f(i/k_n, \hat{X}_{n,i}/a_n)} \right].
\]

Further, denote by \( J_{n_0} := \{1, \ldots, r_n\}^2 = J_{n, 1} \) and \( J_{n, n_0} := \{1, \ldots, r_n - l_n\}^2 = J_{n, 1} \). Using stationarity we get that
\[
\sum_{i \in L_n} \mathbb{E} \left| e^{-f(i/k_n, \hat{X}_{n,i}/a_n)} - e^{-f(i/k_n, \hat{X}_{n,i}/a_n)} \right| \leq k_n^2 (A_1 + A_2 + A_3),
\]
where
\[
A_1 = \mathbb{P}(X_{i,j} > a_n \epsilon \text{ for some } (i, j) \in J_{n_0} \setminus J_{n, 1}),
\]
\[
A_2 = \mathbb{P}(\max_{m > c_0 b_n} S_m^m > a_n \epsilon \text{ for some } (i, j) \in J_{n, 1}),
\]
\[
A_3 = \mathbb{P}(\varepsilon_i^m > a_n \epsilon \text{ and } \varepsilon_{i,j}^m \notin B_\eta \text{ for some } (i, j) \in J_{n_n, 1}, 1 \leq m \leq c_0 b_n).
\]

Observe, \( |J_{n_0} \setminus J_{n, 1}| \leq 2r_n l_n \) and \( |J_{n, n_0}| \leq r_n^2 \), and recall that \( k_n r_n \sim n \) as \( n \to \infty \), so using stationarity and then (4.1), Lemma 4.3 and Lemma 5.1, respectively,
\[
\limsup_{n \to \infty} k_n^2 A_1 \leq \limsup_{n \to \infty} 2k_n^2 r_n l_n \mathbb{P}(X_{0,0} > a_n \epsilon) = (\text{const.}) \limsup_{n \to \infty} l_n/r_n = 0,
\]
\[
\limsup_{n \to \infty} k_n^2 A_2 \leq \limsup_{n \to \infty} k_n^2 r_n \mathbb{P}(\max_{m > c_0 b_n} S_m^m > 0) \leq \limsup_{n \to \infty} n^{-2} = 0,
\]
\[
\limsup_{n \to \infty} k_n^2 A_3 \leq \limsup_{n \to \infty} k_n^2 r_n c_0 b_n \mathbb{P}(S_{0,0}^m > b_n + \log \epsilon, \varepsilon_{i,j}^m \notin B_\eta) \leq (\text{const.}) \limsup_{n \to \infty} b_n/n^{2\epsilon_1} = 0.
\]

Therefore, the right hand side, and then also the left hand side, of (5.4) tends to 0 as \( n \to \infty \), and by (5.3) this proves the lemma. \( \square \)
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