PERPETUAL INTEGRALS VIA RANDOM TIME CHANGES

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ABSTRACT. Let $(X_t)_{t\geq 0}$ be a d-dimensional Feller process with symbol q, and let $f:\mathbb{R}^d \to (0,\infty)$ be a continuous function. In this paper we establish a growth condition in terms of q and f such that the perpetual integral

$$\int_0^\infty f(X_s)\,ds$$

is infinite almost surely. The result applies, in particular, if $(X_t)_{t\geq 0}$ is a Lévy process. The key idea is to approach perpetuals integrals via random time changes. As a by-product of the proof, a sufficient condition for the non-explosion of solutions to martingale problems is obtained. Moreover, we establish a condition which ensures that the random time change of a Feller process is a conservative C_b-Feller process.

1. Introduction

Let $(X_t)_{t\geq 0}$ be a d-dimensional Markov process and $f:\mathbb{R}^d \to (0,\infty)$ a Borel measurable function. We are interested in finding sufficient conditions such that the perpetual integral

$$\int_0^\infty f(X_s)\,ds \tag{1}$$

is infinite almost surely. Perpetual integrals are particular examples of additive functionals and appear naturally both in theory and applications, see e. g. [4, 16, 20] for further discussion. In this paper we will take advantage of the fact that perpetual integrals are closely linked to random time changes. As a by-product, we obtain a sufficient condition for a random time change to be conservative which is an interesting result on its own.

For diffusion processes $(X_t)_{t\geq 0}$ the study of perpetual integrals has a long history, but for jump processes there are only few results in the literature and these are concerned with the particular case that $(X_t)_{t\geq 0}$ is a Lévy process. Let us give a brief overview on the existing literature.

(i) If $X_t = B_t + \mu t$ for a one-dimensional Brownian motion $(B_t)_{t\geq 0}$ and $\mu \neq 0$, then

$$\int_0^\infty f(X_s)\,ds < \infty \iff \int_0^\infty f(x)\,dx < \infty \tag{2}$$

for any locally integrable function $f > 0$, cf. [3, 17].

(ii) If $(X_t)_{t\geq 0}$ is a one-dimensional Lévy process which is spectrally negative (that is, the support of the Lévy measure is contained in $(-\infty, 0)$) and which drifts almost surely to infinity, then the 0-1 law (2) holds for any locally integrable function $f > 0$, cf. [9] and [20, Example 3.8].

(iii) Döring & Kyprianou [2] showed that the equivalence (2) holds for any one-dimensional Lévy processes $(X_t)_{t\geq 0}$ which has local times, finite mean $\mathbb{E}(X_t) \in (0,\infty)$ and which is not a compound Poisson process.

(iv) For non-increasing functions $f:\mathbb{R} \to (0,\infty)$ and one-dimensional Lévy processes $(X_t)_{t\geq 0}$ drifting almost surely to infinity, Erickson & Maller [4] obtained a necessary and sufficient condition for the perpetual integral to be (in)finite almost surely.

Let us remark that the integral test (2) fails to hold if the Lévy process has infinite mean. cf. [4] and Example 1.4(iii) below. Moreover, we would like to point out that all the above results are restricted to the one-dimensional framework. It is far from obvious how to generalize the statements to higher dimensions since the dimension of the state space plays an important role; for instance if $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion, then $(B_t)_{t\geq 0}$ is recurrent and so

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$\int_0^\infty f(B_s)\,ds = \infty$ for any $f > 0$; in contrast, if $(B_t)_{t\geq 0}$ is a 3-dimensional Brownian motion, then $(B_t)_{t\geq 0}$ is transient and therefore we cannot expect $\int_0^\infty f(B_s)\,ds = \infty$ a. s. without further growth assumptions on f.

The idea of this paper is to approach perpetual integrals via random time changes. The key observation is that $\zeta := \int_{(0,\infty)} f(X_s)\,ds$ is the lifetime of a stochastic process $(Y_t)_{t\geq 0}$ which is obtained from $(X_t)_{t\geq 0}$ by a random time change (i. e. $Y_t = X_{\alpha_t}$ for some random mapping α_t). This means that $(Y_t)_{t\geq 0}$ does almost surely not explode in finite time if, and only if, $\int_{(0,\infty)} f(X_s)\,ds = \infty$ almost surely. If we can establish a condition which prevents the time-changed process $(Y_t)_{t\geq 0}$ from exploding in finite time, this will allow us to deduce that the perpetual integral is infinite almost surely.
It is known that the random time change $(Y_t)_{t\geq 0}$ of a strong Markov process $(X_t)_{t\geq 0}$ is again Markovian, cf. [21], but since there is no general criterion for the non-explosion of Markov processes, this is not enough to give a sufficient condition for the non-explosion of $(Y_t)_{t\geq 0}$. It is therefore necessary to study the process $(Y_t)_{t\geq 0}$ in more detail. We will first show that if $(X_t)_{t\geq 0}$ is a "nice" Feller process, then the time-changed process $(Y_t)_{t\geq 0}$ satisfies Dynkin's formula

$$\mathbb{E}^x u(Y_{t\wedge\tau_r^x}) - u(x) = \mathbb{E}^x\left(\int_{(0,t\wedge\tau_r^x)} Lu(Y_s)\,ds\right), \qquad u \in C_c^\infty(\mathbb{R}^d),$$

for a certain operator L where

$$\tau_r^x := \inf\{t > 0; |Y_t - x| > r\}$$

denotes the first exit time from the closed ball $\overline{B(x,r)}$. Using a similar reasoning as in [1] and [22], this allows us to establish estimates for the first exit times from compact sets and then to derive a sufficient condition for the non-explosion of $(Y_t)_{t\geq 0}$. As a by-product, we obtain a criterion for the non-explosion of solutions to martingale problems, cf. Corollary 3.2. Moreover, we will establish a sufficient condition which ensures that the random time change of a conservative Feller process $(X_t)_{t\geq 0}$ is a C_b-Feller process, cf. Theorem 4.3. Both results are of independent interest.

The following statement is one of our main results; the required definitions will be explained in Section 2.

1.1. **Theorem** *Let $(X_t)_{t\geq 0}$ be a d-dimensional Feller process such that the smooth functions with compact support $C_c^\infty(\mathbb{R}^d)$ are contained in the domain of the infinitesimal generator of $(X_t)_{t\geq 0}$. Denote by q the symbol of the Feller process and assume that $q(\cdot,0) = 0$. If $f : \mathbb{R}^d \to (0,\infty)$ is a continuous mapping and*

$$\liminf_{R\to\infty} \sup_{|y|\leq 4R} \sup_{|\xi|\leq R^{-1}} \left(\frac{1}{f(y)} + 1\right) |q(y,\xi)| < \infty, \tag{3}$$

then

$$\int_{(0,\infty)} f(X_s)\,ds = \infty \qquad \mathbb{P}^x\text{-a.s. for all } x \in \mathbb{R}^d. \tag{4}$$

Any Lévy process is a Feller process with a rich domain, and therefore we obtain the following corollary.

1.2. **Corollary** *Let $(X_t)_{t\geq 0}$ be a d-dimensional Lévy process with characteristic exponent $\psi : \mathbb{R}^d \to \mathbb{C}$ such that $\psi(0) = 0$, and let $f : \mathbb{R}^d \to (0,\infty)$ be a continuous function. If either Spitzer's condition*

$$\int_{|\xi|<1} \mathrm{Re}\left(\frac{1}{\psi(\xi)}\right) d\xi = \infty \tag{5}$$

is satisfied or

$$\liminf_{R\to\infty}\left(\sup_{|y|\leq 4R} \frac{1}{f(y)} \sup_{|\xi|\leq R^{-1}} |\psi(\xi)|\right) < \infty, \tag{6}$$

then

$$\int_{(0,\infty)} f(x + X_s)\,ds = \infty \quad \mathbb{P}\text{-a. s. for any } x \in \mathbb{R}^d. \tag{7}$$

Since (5) is equivalent to saying that $(L_t)_{t\geq 0}$ is recurrent, cf. [18, Section 37], it is clear that (5) implies (7); the implication (6) $\implies$ (7) follows from Theorem 1.1. In contrast to the results mentioned at the very beginning of this paper, Corollary 1.2 is not restricted to dimension $d = 1$. Let us illustrate Corollary 1.2 with some examples.

1.3. **Example** (Brownian motion with drift) For a d-dimensional Brownian motion $(B_t)_{t\geq 0}$ and $\mu \in \mathbb{R}^d$ denote by $X_t := B_t + \mu t$ the Brownian motion with drift. Let $f : \mathbb{R}^d \to (0,\infty)$ be a continuous function. We consider the cases $\mu \neq 0$ and $\mu = 0$ separately.

(i) $\mu \neq 0$: Corollary 1.2 gives $\int_{(0,\infty)} f(x + X_s)\,ds = \infty$ almost surely for any f such that $f(y) \geq c/(1 + |y|)$, $y \in \mathbb{R}^d$, for some $c > 0$.
Discussion: In dimension $d = 1$ it is known (see [3, 17]) that

$$\int_{(0,\infty)} f(x + X_s)\,ds = \infty \text{ a.s.} \iff \int_{(0,\infty)} f(x)\,dx = \infty$$

which shows that our growth condition on f is not sharp but not much stronger than the optimal one.

(ii) $\mu = 0$: Corollary 1.2 shows $\int_{(0,\infty)} f(x + X_s)\,ds = \infty$ almost surely if either $d \in \{1, 2\}$ or $d \geq 3$ and $f(y) \geq c/(1 + |y|^2)$, $y \in \mathbb{R}^d$, for some constant $c > 0$.

1.4. **Example** (Lévy jump process) (i) For a Poisson process $(N_t)_{t\geq 0}$ Corollary 1.2 gives

$$\left\{\forall n \geq 1 : f(n) \geq \frac{c}{1+n}\right\} \implies \int_0^\infty f(N_s)\,ds = \infty \text{ a.s.} \tag{8}$$

On the other hand, it is not difficult to see from elementary considerations that

$$\sum_{n\geq 1} f(n) = \infty \iff \int_0^\infty f(N_s)\,ds = \infty \text{ a.s.}$$

for any function $f > 0$ which shows that (8) is close to the optimal condition.

(ii) Let $(L_t)_{t\geq 0}$ be an isotropic α-stable Lévy process, $\alpha \in (0, 2)$, and set $X_t := L_t + \mu t$ for some $\mu \in \mathbb{R}^d$. Applying Corollary 1.2 we find that $\int_{(0,\infty)} f(x + X_s)\,ds$ is almost surely infinite for any $x \in \mathbb{R}^d$ in each of the following cases:
   (a) $d = 1$, $\gamma = 0$, $\alpha \geq 1$,
   (b) $\gamma = 0$, $f(y) \geq c/(1 + |y|^\alpha)$ for some absolute constant $c > 0$,
   (c) $\gamma \neq 0$, $f(y) \geq c/(1 + |y|^{\min\{\alpha,1\}})$ for some absolute constant $c > 0$.

Discussion: Condition (a) corresponds to $(X_t)_{t\geq 0}$ being recurrent. For the particular case that $d = 1$, $\alpha > 1$ and $\gamma \neq 0$ it follows from the 0-1 law by Döring & Kyprianou [2] that

$$\int_0^\infty f(x + X_s)\,ds = \infty \text{ a.s.} \iff \int f(x)\,dx = \infty;$$

our growth condition (c) reads in this special case $f(y) \geq c/(1 + |y|)$ which is slightly stronger than $\int f(x)\,dx = \infty$.

(iii) Let $(L_t)_{t\geq 0}$ be a one-dimensional pure-jump Lévy process with Lévy measure $\nu(dy) := |y|^{-1-\alpha}\mathbb{1}_{(0,\infty)}(y)\,dy$ for some $\alpha \in (0, 1)$. By Erickson & Maller [4] the equivalence

$$\int_0^\infty x^\alpha f(x)\,\frac{dx}{x} = \infty \iff \int_0^\infty f(L_s)\,ds = \infty \text{ a.s.}$$

holds for any non-increasing function $f > 0$. If we apply Corollary 1.2, we obtain

$$f(x) \geq \frac{c}{1 + |x|^\alpha} \implies \int_0^\infty f(L_s)\,ds = \infty \text{ a.s.}$$

for any continuous function $f > 0$; this is close to the optimal condition.

Example 1.3 and 1.4 show that the conditions presented in Corollary 1.2 are not sharp, but close to the necessary ones. Let us close this section with an application of Theorem 1.1.

1.5. **Example** (Lévy-driven SDE) Let $(L_t)_{t\geq 0}$ be an isotropic α-stable Lévy process, $\alpha \in (0, 2)$, and let $\sigma : \mathbb{R}^k \to \mathbb{R}^{k\times d}$ be a continuous function which is at most of linear growth (i. e. there exists

$M > 0$ such that $|\sigma(x)| \leq M(1+|x|)$ for all $x \in \mathbb{R}^k$). Assume that the Lévy-driven stochastic differential equation (SDE)

$$dX_t = \sigma(X_{t-})\, dL_t, \qquad X_0 = x,$$

gives rise to a Feller process $(X_t)_{t\geq 0}$ and that the domain of the infinitesimal generator of $(X_t)_{t\geq 0}$ contains the smooth functions with compact support, see [11] for sufficient and necessary conditions. If $f : \mathbb{R}^k \to (0,\infty)$ is a continuous function such that

$$f(y) \geq c\frac{|\sigma(y)|^\alpha}{1+|y|^\alpha}, \qquad y \in \mathbb{R}^k,$$

for some constant $c > 0$, then

$$\int_0^\infty f(X_s)\, ds = \infty \quad \mathbb{P}^x\text{-a.s. for any } x \in \mathbb{R}^k.$$

This is a direct consequence of Theorem 1.5 and the fact that the symbol of $(X_t)_{t\geq 0}$ is given by $q(x,\xi) = |\sigma(x)^T\xi|^\alpha|$, $x,\xi \in \mathbb{R}^d$, cf. [11]; here $\sigma(x)^T$ denotes the transpose of the matrix $\sigma(x)$.

The remaining part of the paper is organized as follows. Basic definitions and notation are introduced in Section 2. In Section 3 we establish a sufficient condition for the conservativeness of a class of stochastic processes, including Feller processes and solutions to martingale problems. Section 4 is on random time changes of Feller processes. Using the results from Section 3 we establish a sufficient condition which ensures that the random time change of a Feller process is a conservative C_b-Feller process, cf. Theorem 4.3. At the end of Section 4 we prove Theorem 1.1 and Corollary 1.2, cf. p. 9.

2. **Preliminaries**

We consider the Euclidean space $\mathbb{R}^d$ with its canonical scalar product $x\cdot y = \sum_{j=1}^d x_j y_j$ and its Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. By $B(x,r)$ we denote the open ball of radius r centered at x and by $\overline{B(x,r)}$ its closure. We use $\mathbb{R}^d_\Delta$ to denote the one-point compactification of $\mathbb{R}^d$ and extend functions $f : \mathbb{R}^d \to \mathbb{R}$ to $\mathbb{R}^d_\Delta$ by setting $f(\Delta) := 0$. If $\tau : \Omega \to [0,\infty]$ is a stopping time with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ on a measurable space $(\Omega,\mathcal{A})$, then we denote by

$$\mathcal{F}_\tau := \{A \in \mathcal{F}_\infty; \forall t \geq 0 : A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$$

the σ-algebra associated with τ where $\mathcal{F}_\infty := \sigma(\mathcal{F}_t; t \geq 0)$ is the smallest σ-algebra containing $\mathcal{F}_t$, $t \geq 0$. A stochastic process $(X_t)_{t\geq 0}$ on a probability space $(\Omega,\mathcal{A},\mathbb{P})$ *does almost surely not explode in finite time* if the life-time $\zeta := \inf\{t > 0; X_t = \Delta\}$ is $\mathbb{P}$-almost surely infinite.
An E-valued Markov process $(\Omega,\mathcal{A},\mathbb{P}^x, x \in E, X_t, t \geq 0)$ with càdlàg (right-continuous with left-hand limits) sample paths is called a *Feller process* if the associated semigroup $(T_t)_{t\geq 0}$ defined by

$$T_t f(x) := \mathbb{E}^x f(X_t), \quad x \in E, f \in \mathcal{B}_b(E) := \{f : E \to \mathbb{R}; f \text{ bounded, Borel measurable}\}$$

has the *Feller property* and $(T_t)_{t\geq 0}$ is *strongly continuous at* $t = 0$, i. e. $T_t f \in C_\infty(E)$ for all $C_\infty(E)$ and $\|T_t f - f\|_\infty \xrightarrow{t\to 0} 0$ for any $f \in C_\infty(E)$. Here, $C_\infty(E)$ denotes the space of continuous functions vanishing at infinity. Following [19] we call a Markov process $(X_t)_{t\geq 0}$ with càdlàg sample paths a *C_b-Feller process* if $T_t(C_b(E)) \subseteq C_b(E)$ for all $t \geq 0$. We will always consider $E = \mathbb{R}^d$ or $E = \mathbb{R}^d_\Delta$. An $\mathbb{R}^d_\Delta$-valued Markov process with semigroup $(T_t)_{t\geq 0}$ is *conservative* if $T_t \mathbb{1}_{\mathbb{R}^d} = \mathbb{1}_{\mathbb{R}^d}$ for all $t \geq 0$. If the smooth functions with compact support $C_c^\infty(\mathbb{R}^d)$ are contained in the domain of the generator $(L,\mathcal{D}(L))$ of the C_∞-semigroup of a Feller process $(X_t)_{t\geq 0}$, then we speak of a *rich* Feller process. A result due to von Waldenfels and Courrège, cf. [1, Theorem 2.21], states that the generator L of an $\mathbb{R}^d$-valued rich Feller process is, when restricted to $C_c^\infty(\mathbb{R}^d)$, a pseudo-differential operator with negative definite symbol:

$$Lf(x) = -\int_{\mathbb{R}^d} e^{i\,x\cdot\xi} q(x,\xi)\hat{f}(\xi)\, d\xi, \qquad f \in C_c^\infty(\mathbb{R}^d),\ x \in \mathbb{R}^d$$

where $\hat{f}(\xi) := \mathcal{F}f(\xi) := (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-ix\xi} f(x)\, dx$ denotes the Fourier transform of f and

$$q(x,\xi) = q(x,0) - ib(x)\cdot\xi + \frac{1}{2}\xi\cdot Q(x)\xi + \int_{\mathbb{R}^d\setminus\{0\}} (1 - e^{iy\cdot\xi} + iy\cdot\xi\mathbb{1}_{(0,1)}(|y|))\,\nu(x,dy). \tag{9}$$

We call q the *symbol* of the rich Feller process $(X_t)_{t\geq 0}$ and $-q$ the symbol of the pseudo-differential operator; (b,Q,ν) are the *characteristics* of the symbol q. For each fixed $x\in\mathbb{R}^d$, $(b(x),Q(x),\nu(x,dy))$ is a Lévy triplet, i.e. $b(x)\in\mathbb{R}^d$, $Q(x)\in\mathbb{R}^{d\times d}$ is a symmetric positive semidefinite matrix and $\nu(x,dy)$ a σ-finite measure on $(\mathbb{R}^d\backslash\{0\},\mathcal{B}(\mathbb{R}^d\backslash\{0\}))$ satisfying $\int_{y\neq 0}\min\{|y|^2,1\}\,\nu(x,dy)<\infty$. We say that a rich Feller process with symbol q has *bounded coefficients* if

$$\sup_{x\in\mathbb{R}^d}\left(|q(x,0)|+|b(x)|+|Q(x)|+\int_{\mathbb{R}^d\backslash\{0\}}|y|^2\wedge 1\,\nu(x,dy)\right)<\infty.$$

Let us remark that Feller processes are sometimes also called *Lévy-type processes.*
A *Lévy process* $(L_t)_{t\geq 0}$ is a rich Feller process whose symbol q does not depend on x. This is equivalent to saying that $(L_t)_{t\geq 0}$ has stationary and independent increments and càdlàg sample paths. The symbol $q=q(\xi)$ (also called *characteristic exponent*) and the Lévy process $(L_t)_{t\geq 0}$ are related through the Lévy–Khintchine formula:

$$\mathbb{E}^x e^{i\xi\cdot(L_t-x)}=e^{-tq(\xi)}\qquad\text{for all}\quad t\geq 0,\,x,\xi\in\mathbb{R}^d.$$

Following [18] we call a Lévy process $(L_t)_{t\geq 0}$ *recurrent* if $\liminf_{t\to\infty}|L_t|=0$ almost surely and *transient* if $\lim_{t\to\infty}|L_t|=\infty$ almost surely. It is known that any Lévy process is either recurrent or transient, cf. [18, Theorem 35.4]. A result by Spitzer shows that a Lévy process with characteristic exponent q is transient if, and only if,

$$\int_{B(0,1)}\mathrm{Re}\left(\frac{1}{q(\xi)}\right)d\xi<\infty,$$

cf. [18, Section 37]. Our standard reference for Lévy processes is the monograph [18] by Sato.
Let $(A,\mathcal{D})$ be a linear operator with domain $\mathcal{D}\subseteq\mathcal{B}_b(\mathbb{R}^d)$ and μ a probability measure on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$. An $\mathbb{R}^d_\Delta$-valued stochastic process $(X_t)_{t\geq 0}$ with càdlàg sample paths is a *solution to the* $(A,\mathcal{D})$*-martingale problem with initial distribution* μ if $X_0\sim\mu$ and

$$M_t^f:=f(X_t)-f(X_0)-\int_0^t Af(X_s)\,ds,\qquad t\geq 0,$$

is a martingale with respect to the canonical filtration of $(X_t)_{t\geq 0}$ for any $f\in\mathcal{D}$. (Here we use again the convention that $g(\Delta):=0$ for any mapping $g:\mathbb{R}^d\to\mathbb{R}$.) The $(A,\mathcal{D})$-martingale problem is *well-posed* if for any initial distribution μ there exists a unique (in law) solution to the $(A,\mathcal{D})$-martingale problem with initial distribution μ. For a comprehensive study of martingale problems see [5, Chapter 4].

3. **Non-Explosion of Feller processes and solutions to martingale problems**

In this section we establish a sufficient condition for the non-explosion of a class of stochastic processes, including Feller processes and solutions of martingale problems. It will be used in the next section to prove that the random time change of a conservative Feller process does not explode in finite time, see Theorem 4.2 for the precise statement. We start with the following auxiliary result.

3.1. **Lemma** *Let $(X_t)_{t\geq 0}$ be an $\mathbb{R}^d_\Delta$-valued stochastic process with càdlàg sample paths and $x\in\mathbb{R}^d$ such that*

$$\mathbb{E}^x u(X_{t\wedge\tau_r^x})-u(x)=\mathbb{E}^x\left(\int_{(0,t\wedge\tau_r^x)}Au(X_s)\,ds\right),\qquad t\geq 0,r>0 \tag{10}$$

for all $u\in C_c^\infty(\mathbb{R}^d)$ where $\tau_r^x:=\inf\{t\geq 0;|X_t-x|>r\}$ denotes the first exit time from the closed ball $\overline{B(x,r)}$ and

$$Au(z):=-\int_{\mathbb{R}^d}p(z,\xi)e^{iz\cdot\xi}\hat{u}(\xi)\,d\xi,\qquad z\in\mathbb{R}^d$$

for a family of continuous negative definite functions $(p(z,\cdot))_{z\in\mathbb{R}^d}$. Suppose that $p(\cdot,0)=0$ and that for any compact set $K\subseteq\mathbb{R}^d$ there exists a constant $c>0$ such that $|p(z,\xi)|\leq c(1+|\xi|^2)$ for all $z\in K$, $\xi\in\mathbb{R}^d$. If

$$\liminf_{r\to\infty}\sup_{|z-x|\leq 2r}\sup_{|\xi|\leq r^{-1}}|p(z,\xi)|<\infty,$$

then

$$\lim_{r\to\infty} \mathbb{P}^x\left(\sup_{s\le t}|X_s|\ge r\right) = 0 \qquad \textit{for all} \quad t\ge 0.$$

Proof. The first part of the proof is similar to the proof of the maximal inequality for Feller processes, cf. [1, Corollary 5.2] or [13, 14, Lemma 1.29]. Pick $\chi\in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}\chi\subseteq B(0,1)$ and $0\le\chi\le 1=\chi(1)$. Set $\chi_r^x(z) := \chi((z-x)/r)$ for fixed $x\in\mathbb{R}^d$, $r>0$. Since $\chi_r^x\in C_c^\infty(\mathbb{R}^d)$, we have

$$1-\mathbb{E}^x(\chi_r^x(X_{t\wedge\tau_r^x})) = \mathbb{E}^x\left(\int_{(0,\tau_r^x)} A\chi_r^x(X_s)\,ds\right).$$

Using that

$$\mathbb{P}^x\left(\sup_{s\le t}|X_s-x|>r\right)\le \mathbb{P}^x(\tau_r^x\le t)\le \mathbb{E}^x(1-\chi_r^x(X_{t\wedge\tau_r^x}))$$

and

$$\begin{aligned}A\chi_r^x(z) &= -\int_{\mathbb{R}^d} p(z,\xi)\hat\chi_r^x(\xi)e^{iz\cdot\xi}\,d\xi = -\int_{\mathbb{R}^d} p(z,\xi)r^d\hat\chi(r\xi)e^{i(z-x)\cdot\xi}\,d\xi\\ &= -\int_{\mathbb{R}^d} p(z,\xi/r)\hat\chi(\xi)e^{i(z-x)\cdot\xi/r}\,d\xi\end{aligned}$$

we find

$$\begin{aligned}\mathbb{P}^x\left(\sup_{s\le t}|X_s-x|>r\right) &\le \left|\mathbb{E}^x\left(\int_{(0,\tau_r^x)}\left[\mathbb{1}_{|z-x|\le r}\int_{\mathbb{R}^d}p(z,\xi/r)\hat\chi(\xi)e^{i(z-x)\cdot\xi/r}\,d\xi\right]\Bigg|_{z=X_s}ds\right)\right|\\ &\le \mathbb{E}^x\left(\int_0^t\left[\mathbb{1}_{|z-x|\le r}\int_{\mathbb{R}^d}|p(z,\xi/r)|\cdot|\hat\chi(\xi)|\,d\xi\right]\Bigg|_{z=X_s}ds\right). \qquad (11)\end{aligned}$$

Pick a cut-off function $\kappa\in C_c^\infty(\mathbb{R}^d)$ such that $\mathbb{1}_{B(0,1)}\le\kappa\le\mathbb{1}_{B(0,2)}$. If we set

$$g_r(z) := \kappa((z-x)/r)\int_{\mathbb{R}^d}|p(z,\xi/r)|\cdot|\hat\chi(\xi)|\,d\xi$$

then the above estimate shows

$$\mathbb{P}^x\left(\sup_{s\le t}|X_s-x|>r\right)\le\int_0^t\mathbb{E}^x g_r(X_s)\,ds.$$

As $\sup_{|z|\le 2r}|p(z,\xi)|\le c(1+|\xi|^2)$ an application of the dominated convergence theorem gives $g_r(z)\xrightarrow{r\to\infty}0$ for all $z\in\mathbb{R}^d$. Since

$$g_r(z)\le c'\sup_{|y-x|\le 2r}\sup_{|\eta|\le r^{-1}}|p(y,\eta)|\int_{\mathbb{R}^d}(1+|\xi|^2)|\hat\chi(\xi)|\,d\xi,$$

cf. [1, Proposition 2.17d)], there exists, by assumption, a sequence $(r_k)_{k\in\mathbb{N}}\subseteq(0,\infty)$ such that $r_k\to\infty$ and $\sup_k\sup_z g_{r_k}(z)<\infty$. Applying the dominated convergence theorem yields

$$\lim_{k\to\infty}\mathbb{P}^x\left(\sup_{s\le t}|X_s-x|>r_k\right)=0. \qquad \square$$

Lemma 3.1 applies, in particular, to solutions of martingale problems.

3.2. **Corollary** *Let A be a pseudo-differential operator with continuous negative definite symbol p, i. e.*

$$Af(x) = -\int_{\mathbb{R}^d}e^{ix\cdot\xi}p(x,\xi)\hat f(\xi)\,d\xi,\qquad f\in C_c^\infty(\mathbb{R}^d), x\in\mathbb{R}^d.$$

Suppose that $p(\cdot,0)=0$ and that for any compact set $K\subseteq\mathbb{R}^d$ there exists a constant $c>0$ such that $|p(z,\xi)|\le c(1+|\xi|^2)$ for all $z\in K$, $\xi\in\mathbb{R}^d$. Let $(X_t)_{t\ge0}$ be an $\mathbb{R}^d_\Delta$-valued solution to the $(A,C_c^\infty(\mathbb{R}^d))$-martingale problem with initial distribution $\mu=\delta_x$. If

$$\liminf_{r\to\infty}\sup_{|z-x|\le 2r}\sup_{|\xi|\le r^{-1}}|p(z,\xi)|<\infty,$$

then $(X_t)_{t\ge0}$ does almost surely not explode in finite time.

For Feller processes a slightly stronger statement holds true:

3.3. **Lemma** *Let $(X_t)_{t\geq 0}$ be a rich Feller process with generator $(A, \mathcal{D}(A))$ and symbol p such that $p(\cdot,0)=0$. Suppose that there exists a set $U \subseteq \mathbb{R}^d$ such that*

$$\liminf_{r\to\infty} \sup_{|z-x|\leq 2r} \sup_{|\xi|\leq r^{-1}} |p(z,\xi)| < \infty \qquad \textit{for all} \quad x \in U. \tag{12}$$

If $(x_n)_{n\in\mathbb{N}} \subseteq U$ is a sequence such that $x_n \to x \in U$, then $(X_t)_{t\geq 0}$ satisfies the compact containment condition

$$\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \mathbb{P}^{x_n}\left(\sup_{s\leq t}|X_s| > r\right) = 0 \qquad \textit{for all} \quad t\geq 0.$$

In particular, if $U=\mathbb{R}^d$, then $(X_t)_{t\geq 0}$ is conservative.

For the particular case that $x_n := x$ we recover a result by Wang [22, Theorem 2.1] which states that a rich Feller process with symbol p is conservative if

$$\liminf_{r\to\infty} \sup_{|z-x|\leq r} \sup_{|\xi|\leq r^{-1}} |p(z,\xi)| < \infty \qquad \text{for all} \quad x \in \mathbb{R}^d.$$

Let us remark that the proof of Lemma 3.3 becomes much easier if we replace (12) by the stronger assumption

$$\liminf_{r\to\infty} \sup_{|z-x|\leq 2r} \sup_{|\xi|\leq r^{-1}} |p(z,\xi)| = 0 \qquad \text{for all} \quad x \in \mathbb{R}^d; \tag{13}$$

in this case Lemma 3.3 is a direct consequence of the maximal inequality which states that

$$\mathbb{P}^x\left(\sup_{s\leq t}|X_s - x| > r\right) \leq ct \sup_{|z-x|\leq r} \sup_{|\xi|\leq r^{-1}} |p(z,\xi)|, \qquad x\in\mathbb{R}^d,$$

for an absolute constant $c>0$, cf. [1, Corollary 5.2] or [13, 14, Lemma 1.29]. If one considers, for instance, solutions of SDEs driven by a one-dimensional isotropic α-stable Lévy process $(L_t)_{t\geq 0}$

$$dX_t = \sigma(X_{t-})\,dL_t, \qquad X_0 = x,$$

then the symbol of $(X_t)_{t\geq 0}$ equals $p(x,\xi) = |\sigma(x)|^\alpha |\xi|^\alpha$, and therefore (12) allows us to consider coefficients σ of linear growth whereas (13) would restrict us to functions σ of *sub*linear growth.

Proof of Lemma 3.3. Let $(X_t)_{t\geq 0}$ be a rich Feller process with symbol p. Then the Dynkin formula (10) holds, and it follows from [1, Theorem 2.31] that the other assumption of Lemma 3.1 are satisfied. Let $(y_n)_{n\in\mathbb{N}} \subseteq U$ be a sequence such that $y_n \to y \in U$. Then $\overline{B(y_n,r)} \subseteq B(y,3r/2)$ for sufficiently large $r>0$. Pick a cut-off function $\kappa \in C_c^\infty(\mathbb{R}^d)$ such that $\mathbb{1}_{B(0,3/2)} \leq \kappa \leq \mathbb{1}_{B(0,2)}$. If we set

$$g_r(z) := \kappa((z-y)/r) \int_{\mathbb{R}^d} |p(z,\xi/r)| \cdot |\hat{\chi}(\xi)|\,d\xi$$

then (11) shows

$$\mathbb{P}^{y_n}\left(\sup_{s\leq t}|X_s - y_n| > r\right) \leq \int_0^t \mathbb{E}^{y_n} g_r(X_s)\,ds \qquad \text{for all} \quad n\in\mathbb{N}.$$

As $p(\cdot,0)=0$, we obtain from [1, Theorem 2.31] that $p(\cdot,\xi)$ is continuous for all $\xi\in\mathbb{R}^d$. Using that $\sup_{z\in K}|p(z,\xi)| \leq c(1+|\xi|^2)$ for any compact set $K\subseteq\mathbb{R}^d$, it follows from the dominated convergence theorem that $g_r \in C_b(\mathbb{R}^d)$. Since $(X_t)_{t\geq 0}$ is a conservative Feller process, $\mathbb{P}^{y_n}_{X_t} := \mathbb{P}^{y_n}(X_t \in \cdot)$ converges weakly to $\mathbb{P}^y_{X_t} := \mathbb{P}^y(X_t\in\cdot)$. Combining this with the dominated convergence theorem we obtain

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}^{y_n}\left(\sup_{s\leq t}|X_s - y_n| > r\right) \leq \limsup_{n\to\infty} \int_0^t \mathbb{E}^{y_n} g_r(X_s)\,ds &= \int_0^t \lim_{n\to\infty} \mathbb{E}^{y_n} g_r(X_s)\,ds \\ &= \int_0^t \mathbb{E}^y g_r(X_s)\,ds.\end{aligned}$$

The proof of Lemma 3.1 shows that there exists a sequence $(r_k)_{k\in\mathbb{N}} \subseteq (0,\infty)$ such that $r_k\to\infty$ and

$$\limsup_{k\to\infty} \lim_{n\to\infty} \mathbb{P}^{y_n}\left(\sup_{s\leq t}|X_s - y_n| > r_k\right) \leq \lim_{k\to\infty} \int_0^t \mathbb{E}^y g_{r_k}(X_s)\,ds = 0.$$

Using the boundedness of the sequence $(y_n)_{n\in\mathbb{N}}$, the assertion follows. $\square$

4. **Time changes of Feller processes**

4.1. **Definition** Let $(X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued stochastic process. For a measurable mapping $\varphi : \mathbb{R}^d \to (0,\infty)$ we set

$$r_n(\omega) := \int_{(0,n)} \frac{1}{\varphi(X_s(\omega))}\,ds, \qquad n \in \mathbb{N}\cup\{\infty\}$$

and denote for $t < r_\infty(\omega)$ by $\alpha_t(\omega)$ the unique number such that

$$t = \int_0^{\alpha_t(\omega)} \frac{1}{\varphi(X_s(\omega))}\,ds.$$

The process $(Y_t)_{t\geq 0}$ defined by

$$Y_t(\omega) := \begin{cases} X_{\alpha_t(\omega)}(\omega), & t < r_\infty(\omega),\\ \Delta, & t \geq r_\infty(\omega)\end{cases} \tag{14}$$

is called the *time-changed process.*

By the very definition,

$$r_\infty = \int_0^\infty \frac{1}{\varphi(X_s)}\,ds$$

is the life-time of $(Y_t)_{t\geq 0}$. This means that the perpetual integral $\int_{(0,\infty)} 1/\varphi(X_s)\,ds$ is infinite almost surely if, and only if, $(Y_t)_{t\geq 0}$ has infinite life-time with probability 1. If φ is bounded, then

$$\int_0^u \frac{1}{\varphi(X_s(\omega))}\,ds \geq \frac{u}{\|\varphi\|_\infty} \qquad \text{for all } \; u \geq 0$$

implies $r_\infty = \infty$, and so $(Y_t)_{t\geq 0}$ has infinite lifetime. This corresponds to the trivial statement that

$$\int_0^\infty f(X_s)\,ds = \infty$$

for any function $f > 0$ which is strictly bounded away from 0. We are therefore interested in finding conditions for the non-explosion of $(Y_t)_{t\geq 0}$ for *un*bounded mappings φ. Lemma 3.1 allows us to prove the following result on random time-changes of Feller processes:

4.2. **Theorem** *Let $(X_t)_{t\geq 0}$ be a rich Feller process with symbol q and generator $(A,\mathcal{D}(A))$ such that $q(x,0)=0$ for all $x \in \mathbb{R}^d$. Let $\varphi : \mathbb{R}^d \to (0,\infty)$ be a continuous mapping such that*

$$\liminf_{R\to\infty} \sup_{|y|\leq 4R} \sup_{|\xi|\leq R^{-1}} (\varphi(y)+1)|q(y,\xi)| < \infty. \tag{15}$$

(Note that (15) *implies, by Lemma 3.3, that $(X_t)_{t\geq 0}$ is conservative.) Then the time-changed process $(Y_t)_{t\geq 0}$, defined in* (14)*, does $\mathbb{P}^x$-almost surely not explode in finite time for any $x \in \mathbb{R}^d$.*

Proof. By (15) and Lemma 3.1 it suffices to show that

$$\mathbb{E}^x u(Y_{t\wedge\tau_r^x}) - u(x) = \mathbb{E}^x\left(\int_{(0,t\wedge\tau_r^x)} \varphi(Y_s)Au(Y_s)\,ds\right), \qquad x \in \mathbb{R}^d, t \geq 0,$$

for all $u \in C_c^\infty(\mathbb{R}^d)$; as usual $\tau_r^x := \inf\{t \geq 0; |Y_t - x| > r\}$ denotes the first exit time from $\overline{B(x,r)}$. Fix $u \in C_c^\infty(\mathbb{R}^d)$, and let $(\mathcal{F}_t)_{t\geq 0}$ be an admissible right-continuous filtration for $(X_t)_{t\geq 0}$, see [1, Theorem 1.20] for one possible choice. Since $(X_t)_{t\geq 0}$ is a rich Feller process, there exists a martingale $(M_t)_{t\geq 0}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$ such that

$$u(X_t) - u(x) - M_t = \int_0^t Au(X_s)\,ds;$$

By the very definition of the time change, this implies

$$u(X_{\alpha(t)\wedge n}) - u(x) - M_{\alpha(t)\wedge n} = \int_{(0,t\wedge r_n)} \varphi(Y_s)Au(Y_s)\,ds;$$

see [1, proof of Corollary 4.2] for details (recall that $r_n := \int_0^n 1/\varphi(X_s)\,ds$). For $n \in \mathbb{N}\cup\{\infty\}$ define

$$\sigma^{(n)} := \inf\left\{t \geq 0; \sup_{s\leq \alpha_t\wedge n} |X_s - x| > r\right\};$$

note that the continuity of $t \mapsto \alpha_t$ implies $\sigma^{(\infty)} = \tau_r^x$. By the optional stopping theorem, $(M_{\alpha(t)\wedge n}, \mathcal{F}_{\alpha(t)\wedge n})_{t\geq 0}$ is a martingale. Since $\sigma^{(n)}$ is an $\mathcal{F}_{\alpha(t)\wedge n}$-stopping time, another application of the optional stopping theorem yields

$$\mathbb{E}^x u(X_{\alpha(\sigma^{(n)}\wedge t)\wedge n}) - u(x) = \mathbb{E}^x \left(\int_{(0,\sigma^{(n)}\wedge t\wedge r(n))} \varphi(Y_s) Au(Y_s)\, ds \right).$$

It is not difficult to see that $\sigma^{(n)} \downarrow \sigma^{(\infty)} = \tau_r^x$ as $n \to \infty$. Hence, by the dominated convergence theorem,

$$\mathbb{E}^x u(Y_{t\wedge\tau_r^x}) - u(x) = \mathbb{E}^x \left(\int_{(0,t\wedge\tau_r^x)} \varphi(Y_s) Au(Y_s)\, ds \right)$$

where we use the convention that $f(\Delta) := 0$ for $f : \mathbb{R}^d \to \mathbb{R}$. This shows that (10) holds with $p(x,\xi) := \varphi(x) q(x,\xi)$. Applying Lemma 3.1 finishes the proof. □

We are now ready to prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. If we set $\varphi(y) := 1/f(y)$, then the assumptions of Theorem 4.2 are satisfied, and therefore the time-changed process $(Y_t)_{t\geq 0}$ has infinite life-time $\mathbb{P}^x$-almost surely. This means that

$$\infty = r_\infty = \int_0^\infty \frac{1}{\varphi(X_s)}\, ds = \int_0^\infty f(X_s)\, ds$$

$\mathbb{P}^x$-almost surely for any $x \in \mathbb{R}^d$. □

Proof of Corollary 1.2. If (5) holds, then $(L_t)_{t\geq 0}$ is recurrent, cf. [18, Section 37], and therefore (7) is trivially satisfied. On the other hand, if (6) holds, then (7) follows from Theorem 1.1 and the fact that any Lévy process is a rich Feller process. □

We close this section with a statement which follows from the results presented in Section 3 and which is of independent interest.
It is a classical result that the time-changed process $(Y_t)_{t\geq 0}$ from Theorem 4.2 is Markovian, cf. [21]. It is natural to ask whether the semigroup associated with $(Y_t)_{t\geq 0}$ inherits properties from the Feller semigroup associated with $(X_t)_{t\geq 0}$. There are several results in the literature which give sufficient conditions which ensure that the random time change of a C_b-Feller process $(X_t)_{t\geq 0}$ is a C_b-Feller process; typically, they assume that $(X_t)_{t\geq 0}$ is uniformly stochastically continuous, i. e.

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \mathbb{P}^x \left(\sup_{s\leq t} |X_s - x| > \delta \right) = 0 \qquad \text{for all } \ \delta > 0, \tag{16}$$

see e. g. Lamperti [15] or Helland [6]. This condition fails, in general, to hold for Feller processes with unbounded coefficients, and therefore it is too restrictive for our purpose. Lemma 3.3 allows us to prove the following result.

4.3. **Theorem** *Under the assumptions of Theorem 4.2 the time-changed process $(Y_t)_{t\geq 0}$ is a conservative C_b-Feller process.*

4.4. **Remark** If we assume additionally that $C_c^\infty(\mathbb{R}^d)$ is a core for the infinitesimal generator A of $(X_t)_{t\geq 0}$ (i. e. $(A, \mathcal{D}(A))$ is the closure of $(A, C_c^\infty(\mathbb{R}^d))$ with respect to the uniform norm), then it can be shown that $(Y_t)_{t\geq 0}$ is a Feller process, cf. [12]. In this case, the symbol of $(Y_t)_{t\geq 0}$ equals $p(x,\xi) := \varphi(x) q(x,\xi)$.

Proof of Theorem 4.3. We already know from Theorem 4.2 that $(Y_t)_{t\geq 0}$ has infinite life-time. Since $(Y_t)_{t\geq 0}$ is a strong Markov process, see e. g. [21], with càdlàg sample paths, it therefore suffices to prove the weak continuity:

$$\mathbb{P}_{Y_t}^{x_n} \xrightarrow[n\to\infty]{\text{weakly}} \mathbb{P}_{Y_t}^x \qquad \text{for all } \ \text{sequences } x_n \to x,\ t \geq 0. \tag{17}$$

(Here $\mathbb{P}_{Y_t}^x := \mathbb{P}^x(Y_t \in \cdot)$ denotes the distribution of Y_t under $\mathbb{P}^x$.) In the remaining part of the proof we use the canonical model, i. e. we consider $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ as mappings $X : D([0,\infty), \mathbb{R}^d) \to$

$\mathbb{R}^d$ and $Y : D([0,\infty),\mathbb{R}^d_\Delta) \to \mathbb{R}^d_\Delta$ where $D([0,\infty),E)$ denotes the space of càdlàg functions $\omega : [0,\infty) \to E$. If we define

$$f : D([0,\infty),\mathbb{R}^d) \to D([0,\infty),\mathbb{R}^d_\Delta), \quad \omega \mapsto f(\omega)(t) := \begin{cases} \omega(\alpha_t(\omega)), & t < r_\infty(\omega), \\ \Delta, & t \geq r_\infty(\omega), \end{cases}$$

then $Y_t = f(X)(t)$. In order to prove (17), we fix a sequence $x_n \to x$ and denote by $X^{(n)}$ the process started at x_n and by $X^{(0)}$ the Feller process started at x. For each $n \in \mathbb{N}_0$ the process $X^{(n)}$ induces a probability measure $\mathbb{P}^{(n)}$ on $D([0,\infty),\mathbb{R}^d)$. Clearly, (17) is equivalent to

$$f(X^{(n)})(t) \xrightarrow[n\to\infty]{d} f(X^{(0)})(t). \tag{18}$$

Since $(X_t)_{t\geq 0}$ is a Feller process, we have $X^{(n)}(t) \xrightarrow{d} X^{(0)}(t)$ for all $t \geq 0$, and by the Markov property this implies $X^{(n)} \to X^{(0)}$ in finite-dimensional distribution. On the other hand, Lemma 3.3 shows

$$\sup_{n\in\mathbb{N}_0} \mathbb{P}^{(n)}\left(\sup_{s\leq t} |X_s^{(n)}| > R\right) \xrightarrow[R\to\infty]{} 0.$$

It follows from [10, Theorem 4.9.2] that $(X^{(n)})_{n\in\mathbb{N}_0}$ is tight, and this, in turn, implies by Prohorov's theorem, cf. [5, Theorem 2.2, p. 104], relative compactness in $D([0,\infty),\mathbb{R}^d)$. Applying [5, Theorem 7.8, p. 131] we get $X^{(n)} \to X^{(0)}$ in $D([0,\infty),\mathbb{R}^d)$. Since f is $\mathbb{P}^{(0)}$-a.s. continuous, cf. [6, Theorem 2.7], the continuous mapping theorem yields

$$f(X^{(n)}) \xrightarrow{d} f(X^{(0)}).$$

As X is quasi-leftcontinuous, see [7, p. 127], and α_t is a predictable stopping time, we have

$$\mathbb{P}^{(0)}\big(\{f(X^{(0)})(t) = f(X^{(0)})(t-), t < r_\infty(X^{(0)})\}\big) = 1$$

for fixed $t > 0$. Since we already know that $(Y_t)_{t\geq 0}$ is conservative, i. e. $\mathbb{P}^{(0)}(r_\infty = \infty) = 1$, we find that the mapping $s \mapsto f(X^{(0)})(s)$ is $\mathbb{P}^{(0)}$-a.s. continuous at $s = t$. This means that the projection $y \mapsto y(t)$ is $\mathbb{P}^{(0)}$-a.s. continuous at $y = f(X^{(0)})$. Applying the continuous mapping theorem another time, we conclude

$$f(X^{(n)})(t) \xrightarrow{d} f(X^{(0)})(t) \qquad \text{for all} \quad t \geq 0. \qquad \square$$

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