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Martingale decompositions and weak differential subordination in UMD Banach spaces

IVAN S. YAROSLAVTSEV

Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands.
E-mail: I.S.Yaroslavtsev@tudelft.nl

In this paper we consider Meyer-Yoeurp decompositions for UMD Banach space-valued martingales. Namely, we prove that X is a UMD Banach space if and only if for any fixed $p \in (1, \infty)$, any X-valued L^p-martingale M has a unique decomposition $M = M^d + M^c$ such that M^d is a purely discontinuous martingale, M^c is a continuous martingale, $M_0^c = 0$ and

$$\mathbb{E}\|M_\infty^d\|^p + \mathbb{E}\|M_\infty^c\|^p \leq c_{p,X}\mathbb{E}\|M_\infty\|^p.$$

An analogous assertion is shown for the Yoeurp decomposition of a purely discontinuous martingales into a sum of a quasi-left continuous martingale and a martingale with accessible jumps.

As an application we show that X is a UMD Banach space if and only if for any fixed $p \in (1, \infty)$ and for all X-valued martingales M and N such that N is weakly differentially subordinated to M, one has the estimate $\mathbb{E}\|N_\infty\|^p \leq C_{p,X}\mathbb{E}\|M_\infty\|^p$.

Keywords: differential subordination, weak differential subordination, UMD Banach spaces, Burkholder function, stochastic integration, Brownian representation, Meyer-Yoeurp decomposition, Yoeurp decomposition, purely discontinuous martingales, continuous martingales, quasi-left continuous, accessible jumps, canonical decomposition of martingales
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1. Introduction

It is well-known from the fundamental paper of Itô [20] on the real-valued case, and several works [1, 2, 34, 13, 5] on the vector-valued case, that for any Banach space X, any centered X-valued Lévy process has a unique decomposition $L = W + \widetilde{N}$, where W is an X-valued Wiener process, and $\widetilde{N}$ is an X-valued weak integral with respect to a certain compensated Poisson random measure. Moreover, W and $\widetilde{N}$ are independent, and therefore since W is symmetric, for each $1 < p < \infty$ and $t \geq 0$,

$$\mathbb{E}\|\widetilde{N}_t\|^p \leq \mathbb{E}\|L_t\|^p. \tag{1.1}$$

The natural generalization of this result to general martingales in the real-valued setting was provided by Meyer in [29] and Yoeurp in [43]. Namely, it was shown that any real-valued martingale M can be uniquely decomposed into a sum of two martingales M^d and M^c such that M^d is purely discontinuous (i.e. the quadratic variation $[M^d]$ has a pure jump version), and M^c is continuous with $M^c_0 = 0$. The reason why they needed such a decomposition is a further decomposition of a semimartingale, and finding an exponent of a semimartingale (we refer the reader to [23] and [43] for the details on this approach). In the present article we extend Meyer-Yoeurp theorem to the vector-valued setting, and provide extension of (1.1) for a general martingale (see Subsection 3.1). Namely, we prove that for any UMD Banach space X and any $1 < p < \infty$, an X-valued L^p-martingale M can be uniquely decomposed into a sum of two martingales M^d and M^c such that M^d is purely discontinuous (i.e. $\langle M^d, x^*\rangle$ is purely discontinuous for each $x^* \in X^*$), and M^c is continuous with $M^c_0 = 0$. Moreover, then for each $t \geq 0$,

$$(\mathbb{E}\|M^d_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \quad (\mathbb{E}\|M^c_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \tag{1.2}$$

where $\beta_{p,X}$ is the UMD$_p$ constant of X (see Subsection 2.1). Theorem 3.32 shows that such a decomposition together with L^p-estimates of type (1.2) is possible if and only if X has the UMD property.

The purely discontinuous part can be further decomposed: in [43] Yoeurp proved that any real-valued purely discontinuous M^d can be uniquely decomposed into a sum of a purely discontinuous quasi-left continuous martingale M^q (analogous to the "compensated Poisson part", which does not jump at predictable stopping times), and a purely discontinuous martingale with accessible jumps M^a (analogous to the "discrete part", which jumps only at certain predictable stopping times). In Subsection 3.2 we extend this result to a UMD space-valued setting with appropriate estimates. Namely, we prove that for each $1 < p < \infty$ the same type of decomposition is possible and unique for an X-valued purely discontinuous L^p-martingale M^d, and then for each $t \geq 0$,

$$(\mathbb{E}\|M^q_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M^d_t\|^p)^{\frac{1}{p}}, \quad (\mathbb{E}\|M^a_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M^d_t\|^p)^{\frac{1}{p}}. \tag{1.3}$$

Again as Theorem 3.32 shows, the (1.3)-type estimates are a possible only in UMD Banach spaces.

Even though the Meyer-Yoeurp and Yoeurp decompositions can be easily extended from the real-valued case to a Hilbert space case, the author could not find the corresponding estimates of type (1.2)-(1.3) in the literature, so we wish to present this special issue here. If H is a Hilbert space, $M : \mathbb{R}_+ \times \Omega \to H$ is a martingale, then there exists a unique decomposition of M into a continuous part M^c, a purely discontinuous quasi-left continuous part M^q, and a purely discontinuous part M^a with accessible jumps. Moreover, then for each $1 < p < \infty$, and for $i = c, q, a$,

$$(\mathbb{E}\|M^i_t\|^p)^{\frac{1}{p}} \leq (p^* - 1)(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \tag{1.4}$$

where $p^* = \max\{p, \frac{p}{p-1}\}$. Notice that though (1.4) follows from (1.2)-(1.3) since $\beta_{p,H} = p^* - 1$, it can be easily derived from the differential subordination estimates for Hilbert space-valued martingales obtained by Wang in [38].

Both the Meyer-Yoeurp and Yoeurp decompositions play a significant rôle in stochastic integration: if $M = M^c + M^q + M^a$ is a decomposition of an H-valued martingale M into continuous, purely discontinuous quasi-left continuous and purely discontinuous with accessible jumps parts, and if $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$ is elementary predictable for some UMD Banach space X, then the decomposition $\Phi \cdot M = \Phi \cdot M^c + \Phi \cdot M^q + \Phi \cdot M^a$ of a stochastic integral $\Phi \cdot M$ is a decomposition of the martingale $\Phi \cdot M$ into continuous, purely discontinuous quasi-left continuous and purely discontinuous with accessible jumps parts, and for any $1 < p < \infty$ we have that

$$\mathbb{E}\|(\Phi \cdot M)_\infty\|^p \eqsim_{p,X} \mathbb{E}\|(\Phi \cdot M^c)_\infty\|^p + \mathbb{E}\|(\Phi \cdot M^q)_\infty\|^p + \mathbb{E}\|(\Phi \cdot M^a)_\infty\|^p.$$

The corresponding Itô isomorphism for $\Phi \cdot M^c$ for a general UMD Banach space X was derived by Veraar and the author in [37], while Itô isomorphisms for $\Phi \cdot M^q$ and $\Phi \cdot M^a$ have been shown by Dirksen and the author in [14] for the case $X = L^r(S)$, $1 < r < \infty$.

The major underlying techniques involved in the proofs of (1.2) and (1.3) are rather different from the original methods of Meyer in [29] and Yoeurp in [43]. They include the results on the differentiability of the Burkholder function of any finite dimensional Banach space, which have been proven recently in [41] and which allow us to use Itô formula in order to show the desired inequalities in the same way as it was demonstrated by Wang in [38].

The main application of the Meyer-Yoeurp decomposition are L^p-estimates for weakly differentially subordinated martingales. The weak differential subordination property was introduced by the author in [41], and can be described in the following way: an X-valued martingale N is weakly differentially subordinated to an X-valued martingale M if for each $x^* \in X^*$ a.s. $|\langle N_0, x^*\rangle| \leq |\langle M_0, x^*\rangle|$ and for each $t \geq s \geq 0$

$$[\langle N, x^*\rangle]_t - [\langle N, x^*\rangle]_s \leq [\langle M, x^*\rangle]_t - [\langle M, x^*\rangle]_s.$$

If both M and N are purely discontinuous, and if X is a UMD Banach space, then by [41], for each $1 < p < \infty$ we have that $\mathbb{E}\|N_\infty\|^p \leq \beta_{p,X}^p \mathbb{E}\|M_\infty\|^p$. Section 4 is devoted to the generalization of this result to continuous and general martingales. There we show that if both M and N are continuous, then $\mathbb{E}\|N_\infty\|^p \leq c_{p,X}^p \mathbb{E}\|M_\infty\|^p$, where the least admissible $c_{p,X}$ is within the interval $[\beta_{p,X}, \beta_{p,X}^2]$. Furthermore, using the Meyer-Yoeurp decomposition and estimates (1.2) we show that for general X-valued martingales M and N such that N is weakly differentially subordinated to M the following holds

$$(\mathbb{E}\|N_\infty\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2(\beta_{p,X} + 1)(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}.$$

The weak differential subordination as a stronger version of the differential subordination is of interest in Harmonic Analysis. For instance, it was shown in [41] that sharp L^p-estimates for weakly differentially subordinated purely discontinuous martingales imply sharp estimates for the norms of a broad class of Fourier multipliers on $L^p(\mathbb{R}^d; X)$. Also there is a strong connection between the weak differential subordination of continuous martingales and the norm of the Hilbert transform on $L^p(\mathbb{R}; X)$ (see [41] and Remark 4.6).

Alternative approaches to Fourier multipliers for functions with values in UMD spaces have been constructed from the differential subordination for purely discontinuous martingales (see Bañuelos and Bogdan [4], Bañuelos, Bogdan and Bielaszewski [3], and recent work [41]), and for continuous martingales (see McConnell [26] and Geiss, Montgomery-Smith and Saksman [18]). It remains open whether one can combine these two approaches using the general weak differential subordination theory.

2. Preliminaries

In the sequel we will omit proofs of some statements marked with a star (e.g. Lemma*, Theorem*, etc.) Please find the corresponding proofs in the supplement.

We set the scalar field to be $\mathbb{R}$. We will use the *Kronecker symbol* δ_{ij}, which is defined in the following way: $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. For each $p \in (1,\infty)$ we set $p' \in (1,\infty)$ and $p^* \in [2,\infty)$ to be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $p^* = \max\{p,p'\}$. We set $\mathbb{R}_+ := [0,\infty)$.

2.1. UMD Banach spaces

A Banach space X is called a *UMD space* if for some (equivalently, for all) $p \in (1,\infty)$ there exists a constant $\beta > 0$ such that for every $n \geq 1$, every martingale difference sequence $(d_j)_{j=1}^n$ in $L^p(\Omega;X)$, and every $\{-1,1\}$-valued sequence $(\varepsilon_j)_{j=1}^n$ we have

$$\Bigl(\mathbb{E}\Bigl\|\sum_{j=1}^n \varepsilon_j d_j\Bigr\|^p\Bigr)^{\frac{1}{p}} \leq \beta\Bigl(\mathbb{E}\Bigl\|\sum_{j=1}^n d_j\Bigr\|^p\Bigr)^{\frac{1}{p}}.$$

The least admissible constant β is denoted by $\beta_{p,X}$ and is called the *UMD constant*. It is well-known (see [19, Chapter 4]) that $\beta_{p,X} \geq p^*-1$ and that $\beta_{p,H} = p^*-1$ for a Hilbert space H. We refer the reader to [10, 19, 35, 32] for details.

The following proposition is a vector-valued version of [11, Theorem 4.1].

Proposition 2.1. *Let X be a Banach space, $p \in (1,\infty)$. Then X has the UMD property if and only if there exists $C > 0$ such that for each $n \geq 1$, for every martingale difference sequence $(d_j)_{j=1}^n$ in $L^p(\Omega;X)$, and every sequence $(\varepsilon_j)_{j=1}^n$ such that $\varepsilon_j \in \{0,1\}$ for each $j = 1,\dots,n$ we have*

$$\Bigl(\mathbb{E}\Bigl\|\sum_{j=1}^n \varepsilon_j d_j\Bigr\|^p\Bigr)^{\frac{1}{p}} \leq C\Bigl(\mathbb{E}\Bigl\|\sum_{j=1}^n d_j\Bigr\|^p\Bigr)^{\frac{1}{p}}.$$

If this is the case, then the least admissible C is in the interval $[\frac{\beta_{p,X}-1}{2}, \beta_{p,X}]$

2.2. Martingales and stopping times in continuous time

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ which satisfies the usual conditions. Then $\mathbb{F}$ is right-continuous, and the following proposition holds (see [41]):

Proposition 2.2. *Let X be a Banach space. Then any martingale $M : \mathbb{R}_+ \times \Omega \to X$ has a* càdlàg *version*

Let $1 \leq p \leq \infty$. A martingale $M : \mathbb{R}_+ \times \Omega \to X$ is called an *L^p-martingale* if $M_t \in L^p(\Omega; X)$ for each $t \geq 0$, there exists an a.s. limit $M_\infty := \lim_{t\to\infty} M_t$, $M_\infty \in L^p(\Omega; X)$ and $M_t \to M_\infty$ in $L^p(\Omega; X)$ as $t \to \infty$. We will denote the space of all X-valued L^p-martingales on Ω by $\mathcal{M}_X^p(\Omega)$. For brevity we will use $\mathcal{M}_X^p$ instead. Notice that $\mathcal{M}_X^p$ is a Banach space with the given norm: $\|M\|_{\mathcal{M}_X^p} := \|M_\infty\|_{L^p(\Omega;X)}$ (see [23, 21] and [19, Chapter 1]).

Proposition 2.3. *Let X be a Banach space with the Radon-Nikodým property (e.g. reflexive), $1 < p < \infty$. Then $(\mathcal{M}_X^p)^* = \mathcal{M}_{X^*}^{p'}$, and $\|M\|_{(\mathcal{M}_X^p)^*} = \|M\|_{\mathcal{M}_{X^*}^{p'}}$ for each $M \in \mathcal{M}_{X^*}^{p'}$.*

A random variable $\tau : \Omega \to \mathbb{R}_+$ is called an *optional stopping time* (or just a *stopping time*) if $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$. With an optional stopping time τ we associate a σ-field $\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t, t \in \mathbb{R}_+\}$. Note that M_τ is strongly $\mathcal{F}_\tau$-measurable for any local martingale M. We refer to [23, Chapter 7] for details.

Due to the existence of a càdlàg version of a martingale $M : \mathbb{R}_+ \times \Omega \to X$, we can define an X-valued random variables $M_{\tau-}$ and ΔM_τ for any stopping time τ in the following way: $M_{\tau-} = \lim_{\varepsilon\to 0} M_{(\tau-\varepsilon)\vee 0}$, $\Delta M_\tau = M_\tau - M_{\tau-}$.

2.3. Quadratic variation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ that satisfies the usual conditions, H be a Hilbert space. Let $M : \mathbb{R}_+ \times \Omega \to H$ be a local martingale. We define a *quadratic variation* of M in the following way:

$$[M]_t := \mathbb{P} - \lim_{\text{mesh}\to 0} \sum_{n=1}^{N} \|M(t_n) - M(t_{n-1})\|^2, \tag{2.1}$$

where the limit in probability is taken over partitions $0 = t_0 < \ldots < t_N = t$. Note that $[M]$ exists and is nondecreasing a.s. The reader can find more on quadratic variations in [27, 28] for the vector-valued setting, and in [23, 33, 28] for the real-valued setting.

For any martingales $M, N : \mathbb{R}_+ \times \Omega \to H$ we can define a *covariation* $[M,N] : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ as $[M,N] := \frac{1}{4}([M+N] - [M-N])$. Since M and N have càdlàg versions, $[M,N]$ has a càdlàg version as well (see [22, Theorem I.4.47] and [27]).

Remark 2.4 ([27]). *The process $\langle M,N\rangle - [M,N]$ is a local martingale.*

2.4. Continuous martingales

Let X be a Banach space. A martingale $M : \mathbb{R}_+ \times \Omega \to X$ is called *continuous* if M has continuous paths.

Remark 2.5 ([23, 28]). *If X is a Hilbert space, $M, N : \mathbb{R}_+ \times \Omega \to X$ are continuous martingales, then $[M,N]$ has a continuous version.*

Let $1 \leq p \leq \infty$. We will denote the linear space of all continuous X-valued L^p-martingales on Ω which start at zero by $\mathcal{M}_X^{p;c}(\Omega)$. For brevity we will write $\mathcal{M}_X^{p;c}$ instead of $\mathcal{M}_X^{p;c}(\Omega)$ since Ω is fixed. Analogously to [23, Lemma 17.4] by applying Doob's maximal inequality [19, Theorem 3.2.2] one can show the following proposition.

Proposition 2.6. *Let X be a Banach space, $p \in (1,\infty)$. Then $\mathcal{M}_X^{p;c}$ is a Banach space with the following norm: $\|M\|_{\mathcal{M}_X^{p;c}} := \|M_\infty\|_{L^p(\Omega;X)}$.*

2.5. Purely discontinuous martingales

An increasing càdlàg process $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is called *pure jump* if a.s. for each $t \geq 0$, $A_t = A_0 + \sum_{s=0}^t \Delta A_s$. A local martingale $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is called *purely discontinuous* if $[M]$ is a pure jump process. The reader can find more on purely discontinuous martingales in [22, 23]. We leave the following evident lemma without proof.

Lemma 2.7. *Let $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be an increasing adapted càdlàg process such that $A_0 = 0$. Then there exist unique up to indistinguishability increasing adapted càdlàg processes $A^c, A^d : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ such that A^c is continuous a.s., A^d is pure jump a.s., $A_0^c = A_0^d = 0$ and $A = A^c + A^d$.*

Remark 2.8. *According to the works [29] by Meyer and [43] by Yoeurp (see also [23, Theorem 26.14]), any martingale $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ can be uniquely decomposed into a sum of a purely discontinuous local martingale M^d and a continuous local martingale M^c such that $M_0^c = 0$. Moreover, $[M]^c = [M^c]$ and $[M]^d = [M^d]$, where $[M]^c$ and $[M]^d$ are defined as in Lemma 2.7.*

Corollary 2.9. *Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a martingale which is both continuous and purely discontinuous. Then $M = M_0$ a.s.*

Proposition* 2.10. *A martingale $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is purely discontinuous if and only if MN is a martingale for any continuous bounded martingale $N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ with $N_0 = 0$.*

Note that some authors take this equivalent condition as the definition of a purely discontinuous martingale, see e.g. [22, Definition I.4.11] and [21, Chapter I].

Definition 2.11. *Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a local martingale. Then M is called* purely discontinuous *if for each $x^* \in X^*$ the local martingale $\langle M, x^*\rangle$ is purely discontinuous.*

Remark 2.12. *Let X be finite dimensional. Then similarly to Remark 2.8 any martingale $M : \mathbb{R}_+ \times \Omega \to X$ can be uniquely decomposed into a sum of a purely discontinuous local martingale M^d and a continuous local martingale M^c such that $M^c_0 = 0$.*

Remark 2.13. *Analogously to Proposition 2.10, a martingale $M : \mathbb{R}_+ \times \Omega \to X$ is purely discontinuous if and only if $\langle M, x^*\rangle N$ is a martingale for any $x^* \in X^*$ and any continuous bounded martingale $N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that $N_0 = 0$.*

Let $p \in [1, \infty]$. We will denote the linear space of all purely discontinuous X-valued L^p-martingales on Ω by $\mathcal{M}_X^{p,d}(\Omega)$. Since Ω is fixed, we will use $\mathcal{M}_X^{p,d}$ instead. The scalar case of the next result have been presented in [21, Lemme I.2.12].

Proposition 2.14. *Let X be a Banach space, $p \in (1, \infty)$. Then $\mathcal{M}_X^{p,d}$ is a Banach space with a norm defined as follows: $\|M\|_{\mathcal{M}_X^{p,d}} := \|M_\infty\|_{L^p(\Omega;X)}$.*

Proof. Let $(M^n)_{n\geq 1}$ be a sequence of purely discontinuous X-valued L^p-martingales such that $(M^n_\infty)_{n\geq 1}$ is a Cauchy sequence in $L^p(\Omega; X)$. Let $\xi \in L^p(\Omega; X)$ be such that $\lim_{n\to\infty} M^n_\infty = \xi$. Define a martingale $M : \mathbb{R}_+ \times \Omega \to X$ as follows: $M = (M_s)_{s\geq 0} = (\mathbb{E}(\xi|\mathcal{F}_s))_{s\geq 0}$. Let us show that $M \in \mathcal{M}_X^{p,d}$. First notice that $\|M_\infty\|_{L^p(\Omega;X)} = \|\xi\|_{L^p(\Omega;X)} < \infty$. Further for each $x^* \in X^*$ by [21, Lemme I.2.12] we have that $\langle M, x^*\rangle$ as a limit of real-valued purely discontinuous martingales $(\langle M^n, x^*\rangle)_{n\geq 1}$ in $\mathcal{M}^p_{\mathbb{R}}$ is purely discontinuous. Therefore M is purely discontinuous by the definition. □

Lemma 2.15. *Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a martingale such that M is both continuous and purely discontinuous. Then $M = M_0$ a.s.*

Proof. Follows analogously Corollary 2.9. □

2.6. Time-change

A nondecreasing, right-continuous family of stopping times $\tau = (\tau_s)_{s\geq 0}$ is called a *random time-change*. If $\mathbb{F}$ is right-continuous, then according to [23, Lemma 7.3] the same holds true for the *induced filtration* $\mathbb{G} = (\mathcal{G}_s)_{s\geq 0} = (\mathcal{F}_{\tau_s})_{s\geq 0}$ (see more in [23, Chapter 7]). Let X be a Banach space. A martingale $M : \mathbb{R}_+ \times \Omega \to X$ is said to be *τ-continuous* if M is an a.s. constant on every interval $[\tau_{s-}, \tau_s]$, $s \geq 0$, where we let $\tau_{0-} = 0$.

Theorem* 2.16. *Let $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be a strictly increasing continuous predictable process such that $A_0 = 0$ and $A_t \to \infty$ as $t \to \infty$ a.s. Let $\tau = (\tau_s)_{s\geq 0}$ be a random time-change defined as $\tau_s := \{t : A_t = s\}$, $s \geq 0$. Then $(A \circ \tau)(t) = (\tau \circ A)(t) = t$ a.s.*

for each $t \geq 0$. Let $\mathbb{G} = (\mathcal{G}_s)_{s\geq 0} = (\mathcal{F}_{\tau_s})_{s\geq 0}$ be the induced filtration. Then $(A_t)_{t\geq 0}$ is a random time-change with respect to $\mathbb{G}$ and for any $\mathbb{F}$-martingale $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ the following holds

(i) $M \circ \tau$ is a continuous $\mathbb{G}$-martingale if and only if M is continuous, and
(ii) $M \circ \tau$ is a purely discontinuous $\mathbb{G}$-martingale if and only if M is purely discontinuous.

2.7. Stochastic integration

Let X be a Banach space, H be a Hilbert space. For each $h \in H$, $x \in X$ we denote the linear operator $g \mapsto \langle g, h\rangle x$, $g \in H$, by $h \otimes x$. The process $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$ is called *elementary progressive* with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ if it is of the form

$$\Phi(t, \omega) = \sum_{k=1}^{K}\sum_{m=1}^{M} \mathbf{1}_{(t_{k-1},t_k]\times B_{mk}}(t, \omega) \sum_{n=1}^{N} h_n \otimes x_{kmn}, \quad t \geq 0, \omega \in \Omega, \tag{2.2}$$

where $0 \leq t_0 < \ldots < t_K < \infty$, for each $k = 1, \ldots, K$ the sets $B_{1k}, \ldots, B_{Mk}$ are in $\mathcal{F}_{t_{k-1}}$ and the vectors $h_1, \ldots, h_N$ are orthogonal. Let $M : \mathbb{R}_+ \times \Omega \to H$ be a martingale. Then we define the *stochastic integral* $\Phi \cdot M : \mathbb{R}_+ \times \Omega \to X$ of Φ with respect to M as follows:

$$(\Phi \cdot M)_t = \sum_{k=1}^{K}\sum_{m=1}^{M} \mathbf{1}_{B_{mk}} \sum_{n=1}^{N} \langle (M(t_k \wedge t) - M(t_{k-1} \wedge t)), h_n\rangle x_{kmn}, \ \ t \geq 0. \tag{2.3}$$

We will need the following lemma on stochastic integration (see [41]).

Lemma 2.17. *Let d be a natural number, H be a d-dimensional Hilbert space, $p \in (1, \infty)$, $M, N : \mathbb{R}_+ \times \Omega \to H$ be L^p-martingales, $F : H \to H$ be a measurable function such that $\|F(h)\| \leq C\|h\|^{p-1}$ for each $h \in H$ and some $C > 0$. Define $N_- : \mathbb{R}_+ \times \Omega \to H$ by $(N_-)_t = N_{t-}$, $t \geq 0$. Then $F(N_-) \cdot M$ is a martingale and for each $t \geq 0$*

$$\mathbb{E}|(F(N_-) \cdot M)_t| \lesssim_{p,d} C(\mathbb{E}\|N_t\|^p)^{\frac{p-1}{p}}(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}. \tag{2.4}$$

2.8. Multidimensional Wiener process

Let d be a natural number. $W : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ is called a *standard d-dimensional Wiener process* if $\langle W, h\rangle$ is a standard Wiener process for each $h \in \mathbb{R}^d$ such that $\|h\| = 1$. The following lemma is a multidimensional variation of [24, (3.2.19)].

Lemma 2.18. *Let $X = \mathbb{R}$, $d \geq 1$, W be a standard d-dimensional Wiener process, $\Phi, \Psi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(\mathbb{R}^d, \mathbb{R})$ be elementary progressive. Then for all $t \geq 0$ a.s.*

$$[\Phi \cdot W, \Psi \cdot W]_t = \int_0^t \langle \Phi^*(s), \Psi^*(s)\rangle \,\mathrm{d}s.$$

The reader can find more on stochastic integration with respect to a Wiener process in the Hilbert space case in [12], in the case of Banach spaces with a martingale type 2 in [7], and in the UMD case in [30]. Notice that the last mentioned work provides sharp L^p-estimates for stochastic integrals for the broadest till now known class of spaces.

2.9. Brownian representation

The following theorem can be found in [24, Theorem 3.4.2] (see also [36, 39]).

Theorem 2.19. *Let $d \geq 1$, $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ be a continuous martingale such that $[M]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. Then there exist an enlarged probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ with an enlarged filtration $\widetilde{\mathbb{F}} = (\widetilde{F}_t)_{t\geq 0}$, a d-dimensional standard Wiener process $W : \mathbb{R}_+ \times \widetilde{\Omega} \to \mathbb{R}^d$ which is defined on the filtration $\widetilde{\mathbb{F}}$, and an $\widetilde{\mathbb{F}}$-progressively measurable $\Phi : \mathbb{R}_+ \times \widetilde{\Omega} \to \mathcal{L}(\mathbb{R}^d)$ such that $M = \Phi \cdot W$.*

2.10. Lebesgue measure

Let X be a finite dimensional Banach space. Then according to Theorem 2.20 and Proposition 2.21 in [16] there exists a unique translation-invariant measure λ_X on X such that $\lambda_X(\mathbb{B}_X) = 1$ for the unit ball $\mathbb{B}_X$ of X. We will call λ_X the *Lebesgue measure*.

3. UMD Banach spaces and martingale decompositions

Let X be a Banach space, $1 < p < \infty$. In this section we will show that the Meyer-Yoeurp and Yoeurp decompositions for X-valued L^p-martingales take place if and only if X has the UMD property.

3.1. Meyer-Yoeurp decomposition in UMD case

This subsection is devoted to the generalization of Meyer-Yoeurp decomposition (see Remark 2.8) to the UMD Banach space case:

Theorem 3.1 (Meyer-Yoeurp decomposition)**.** *Let X be a UMD Banach space, $p \in (1,\infty)$, $M : \mathbb{R}_+ \times \Omega \to X$ be an L^p-martingale. Then there exist unique martingales $M^d, M^c : \mathbb{R}_+ \times \Omega \to X$ such that M^d is purely discontinuous, M^c is continuous, $M_0^c = 0$ and $M = M^d + M^c$. Moreover, then for all $t \geq 0$*

$$(\mathbb{E}\|M_t^d\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \quad (\mathbb{E}\|M_t^c\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}. \tag{3.1}$$

The proof of the theorem consists of several steps. First we introduce the main tool of our proof – the Burkholder function.

Definition 3.2. *Let E be a linear space with a scalar field $\mathbb{R}$.*

(i) *A function $f : E \times E \to \mathbb{R}$ is called* biconcave *if for each $x, y \in E$ one has that the mappings $e \mapsto f(x, e)$ and $e \mapsto f(e, y)$ are concave.*
(ii) *A function $f : E \times E \to \mathbb{R}$ is called* zigzag-concave *if for each $x, y \in E$ and $\varepsilon \in \mathbb{R}$ such that $|\varepsilon| \leq 1$, the function $z \mapsto f(x + z, y + \varepsilon z)$ is concave.*

The following theorem is a small variation of [9] and [19, Theorem 4.5.6], and has been proven in [41].

Theorem 3.3 (Burkholder)**.** *For a Banach space X the following are equivalent*

1. *X is a UMD Banach space;*
2. *for each $p \in (1, \infty)$ there exists a constant β and a zigzag-concave function $U : X \times X \to \mathbb{R}$ such that*

$$U(x, y) \geq \|y\|^p - \beta^p \|x\|^p, \quad x, y \in X. \tag{3.2}$$

The smallest admissible β for which such U exists is $\beta_{p,X}$.

Remark 3.4. *Fix a UMD space X and $p \in (1, \infty)$. A special zigzag-concave function U from Theorem 3.3 have been obtained in [19, Theorem 4.5.6]. We will call this function* the Burkholder function. *For the convenience of the reader we leave out the construction of the Burkholder function. The following properties of the Burkholder function U were demonstrated in [41, Section 3]:*

(A) *$U(\alpha x, \alpha y) = |\alpha|^p U(x, y)$ for all $x, y \in X$, $\alpha \in \mathbb{R}$.*
(B) *$U(x, \alpha x) \leq 0$ for all $x \in X$, $\alpha \in [-1, 1]$.*
(C) *U is continuous.*

Remark 3.5. *Fix a UMD space X and $p \in (1, \infty)$. Let the Burkholder function U be as in Remark 3.4. Then there exists a biconcave function $V : X \times X \to \mathbb{R}$ such that*

$$V(x, y) = U\Big(\frac{x - y}{2}, \frac{x + y}{2}\Big), \quad x, y \in X. \tag{3.3}$$

In [41, Section 3] the following properties of V have been explored:

(A) *For each $x, y \in X$ and $a, b \in \mathbb{R}$ such that $|a + b| \leq |a - b|$ one has that the function*

$$z \mapsto V(x + az, y + bz) = U\Big(\frac{x - y}{2} + \frac{(a - b)z}{2}, \frac{x + y}{2} + \frac{(a + b)z}{2}\Big)$$

is concave.
(B) *V is continuous.*

(C) *Let X be finite dimensional. Then $x \mapsto V(x,y)$ and $y \mapsto V(x,y)$ are a.s. Fréchet-differentiable with respect to the Lebesgue measure λ_X, and for a.a. $(x,y) \in X \times X$ for each $u,v \in X$ there exists the directional derivative $\frac{\partial V(x+tu,y+tv)}{\partial t}$. Moreover,*

$$\frac{\partial V(x+tu,y+tv)}{\partial t} = \langle \partial_x V(x,y), u\rangle + \langle \partial_y V(x,y), v\rangle, \tag{3.4}$$

where $\partial_x V$ and $\partial_y V$ are the corresponding Fréchet derivatives with respect to the first and the second variable.

(D) *Let X be finite dimensional. Then for a.e. $(x,y) \in X \times X$, for all $z \in X$ and real-valued a and b such that $|a+b| \leq |a-b|$*

$$\begin{aligned} V(x+az,y+bz) \leq V(x,y) + \frac{\partial V(x+atz,y+btz)}{\partial t} \\ = V(x,y) + a\langle \partial_x V(x,y), z\rangle + b\langle \partial_y V(x,y), z\rangle. \end{aligned} \tag{3.5}$$

(E) *Let X be finite dimensional. Then there exists $C>0$ which depends only on V such that for a.e. pair $x,y \in X$, $\|\partial_x V(x,y)\|, \|\partial_y V(x,y)\| \leq C(\|x\|^{p-1} + \|y\|^{p-1})$.*

Definition 3.6. *Let d be a natural number, E be a d-dimensional linear space, $(e_n)_{n=1}^d$ be a basis of E. Then $(e_n^*)_{n=1}^d \subset E^*$ is called the* corresponding dual basis *of $(e_n)_{n=1}^d$ if $\langle e_n, e_m^*\rangle = \delta_{nm}$ for each $m,n = 1,\dots,d$.*

Note that the corresponding dual basis is uniquely determined. Moreover, if $(e_n^*)_{n=1}^d$ is the corresponding dual basis of $(e_n)_{n=1}^d$, then, the other way around, $(e_n)_{n=1}^d$ is the corresponding dual basis of $(e_n^*)_{n=1}^d$ (here we identify E^{**} with E in the natural way).

The following Itô formula is a version of [23, Theorem 26.7] that does not use the Euclidean structure of a finite dimensional Banach space. The proof can be found in [41].

Theorem 3.7 (Itô formula)**.** *Let d be a natural number, X be a d-dimensional Banach space, $f \in C^2(X)$, $M : \mathbb{R}_+ \times \Omega \to X$ be a martingale. Let $(x_n)_{n=1}^d$ be a basis of X, $(x_n^*)_{n=1}^d$ be the corresponding dual basis. Then for each $t \geq 0$*

$$\begin{aligned} f(M_t) = f(M_0) &+ \int_0^t \langle \partial_x f(M_{s-}), \mathrm{d}M_s\rangle \\ &+ \frac{1}{2}\int_0^t \sum_{n,m=1}^d f_{x_n,x_m}(M_{s-})\, \mathrm{d}[\langle M, x_n^*\rangle, \langle M, x_m^*\rangle]_s^c \\ &+ \sum_{s\leq t} (\Delta f(M_s) - \langle \partial_x f(M_{s-}), \Delta M_s\rangle). \end{aligned} \tag{3.6}$$

Proposition 3.8. *Let X be a finite dimensional Banach space, $p \in (1,\infty)$. Let $Y = X \oplus \mathbb{R}$ be a Banach space such that $\|(x,r)\|_Y = (\|x\|_X^p + |r|^p)^{\frac{1}{p}}$. Then $\beta_{p,Y} = \beta_{p,X}$. Moreover, if $M : \mathbb{R}_+ \times \Omega \to X$ is a martingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, then there exists a sequence $(M^m)_{m\geq 1}$ of Y-valued martingales on an enlarged probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ with an enlarged filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\geq 0}$ such that*

1. *M_t^m has absolutely continuous distributions with respect to the Lebesgue measure on Y for each $m \geq 1$ and $t \geq 0$;*
2. *$M_t^m \to (M_t, 0)$ pointwise as $m \to \infty$ for each $t \geq 0$;*
3. *if for some $t \geq 0$ $\mathbb{E}\|M_t\|^p < \infty$, then for each $m \geq 1$ one has that $\mathbb{E}\|M_t^m\|^p < \infty$ and $\mathbb{E}\|M_t^m - (M_t, 0)\|^p \to 0$ as $m \to \infty$;*
4. *if M is continuous, then $(M^m)_{m\geq 1}$ are continuous as well,*
5. *if M is purely discontinuous, then $(M^m)_{m\geq 1}$ are purely discontinuous as well.*

Proof. The proof of (1)-(3) follows from [41], while (4) and (5) follow from the construction of M^m and N^m given in [41]. □

Remark 3.9. *Notice that the construction in [41] also allows us to sum these approximations for different martingales. Namely, if M and N are two X-valued martingales, then we can construct the corresponding Y-valued martingales $(M^m)_{m\geq 1}$ and $(N^m)_{m\geq 1}$ as in Proposition 3.8 in such a way that $M_t^m + N_t^m$ has an absolutely continuous distribution for each $t \geq 0$ and $m \geq 1$.*

Proof of Theorem 3.1. *Step 1: finite dimensional case.* Let X be finite dimensional. Then M^d and M^c exist due to Remark 2.12. Without loss of generality $\mathcal{F}_t = \mathcal{F}_\infty$, $M_t^d = M_\infty^d$ and $M_t^c = M_\infty^c$. Let d be the dimension of X.

Let $|||\cdot|||$ be a Euclidean norm on X. Then $(X, |||\cdot|||)$ is a Hilbert space, and by Remark 2.5 the quadratic variation $[M^c]$ exists and has a continuous version. Let us show that without loss of generality we can suppose that $[M^c]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. Let $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be as follows: $A_t = [M^c]_t + t$. Then A is strictly increasing continuous, $A_0 = 0$ and $A_\infty = \infty$ a.s. Let the time-change $\tau = (\tau_s)_{s\geq 1}$ be defined as in Theorem 2.16. Then by Theorem 2.16, $M^c \circ \tau$ is a continuous martingale, $M^d \circ \tau$ is a purely discontinuous martingale, $(M^c \circ \tau)_0 = 0$, $(M^d \circ \tau)_0 = M_0^d$ and due to the Kazamaki theorem [23, Theorem 17.24], $[M^c \circ \tau] = [M^c] \circ \tau$. Therefore for all $t > s \geq 0$ by Theorem 2.16 and the fact that $\tau_t \geq \tau_s$ a.s.

$$
\begin{aligned}
[M^c \circ \tau]_t - [M^c \circ \tau]_s = [M^c]_{\tau_t} - [M^c]_{\tau_s} &\leq [M^c]_{\tau_t} - [M^c]_{\tau_s} + (\tau_t - \tau_s) \\
&= ([M^c]_{\tau_t} + \tau_t) - ([M^c]_{\tau_s} + \tau_s) \\
&= A_{\tau_t} - A_{\tau_s} = t - s.
\end{aligned}
$$

Hence $[M^c \circ \tau]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. Moreover, $(M^i \circ \tau)_\infty = M_\infty^i$, $i \in \{c, d\}$, so this time-change argument does not affect (3.1). Hence we can redefine $M^c := M^c \circ \tau$, $M^d := M^d \circ \tau$, $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0} := \mathbb{G} = (\mathcal{F}_{\tau_s})_{s\geq 0}$.

Since $[M^c]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$ and thanks to Theorem 2.19, we can extend Ω and find a d-dimensional Wiener process $W : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ and a stochastically integrable progressively measurable function $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(\mathbb{R}^d, X)$ such that $M^c = \Phi \cdot W$.

Let $U : X \times X \to \mathbb{R}$ be the Burkholder function that was discussed in Remark 3.4 and Remark 3.5. Let us show that $\mathbb{E}U(M_t, M_t^d) \leq 0$.

Due to Proposition 3.8 and Remark 3.9 we can assume that M_s^c, M_s^d and $M_s = M_s^d + M_s^c$ have absolutely continuous distributions with respect to the Lebesgue measure λ_X on X for each $s \geq 0$. Let $(x_n)_{n=1}^d$ be a basis of X, $(x_n^*)_{n=1}^d$ be the corresponding dual basis of X^* (see Definition 3.6). By the Itô formula (3.6),

$$\begin{aligned}\mathbb{E}U(M_t, M_t^d) = \mathbb{E}U(M_0, M_0^d) + \mathbb{E}\int_0^t \langle \partial_x U(M_{s-}, M_{s-}^d),\, \mathrm{d}M_s\rangle \\ + \mathbb{E}\int_0^t \langle \partial_y U(M_{s-}, M_{s-}^d),\, \mathrm{d}M_s^d\rangle + \mathbb{E}I_1 + \mathbb{E}I_2,\end{aligned} \tag{3.7}$$

where

$$I_1 = \sum_{0<s\leq t} [\Delta U(M_s, M_s^d) - \langle \partial_x U(M_{s-}, M_{s-}^d), \Delta M_s\rangle - \langle \partial_y U(M_{s-}, M_{s-}^d), \Delta M_s^d\rangle],$$

$$\begin{aligned}I_2 &= \frac{1}{2}\int_0^t \sum_{i,j=1}^d U_{x_i,x_j}(M_{s-}, M_{s-}^d)\, \mathrm{d}[\langle M, x_i^*\rangle, \langle M, x_j^*\rangle]_s^c \\ &= \frac{1}{2}\int_0^t \sum_{i,j=1}^d U_{x_i,x_j}(M_{s-}, M_{s-}^d) \langle \Phi^*(s)x_i^*, \Phi^*(s)x_j^*\rangle\, \mathrm{d}s.\end{aligned}$$

(Recall that by (3.3) and Remark 3.5(C), U is Fréchet-differentiable a.s. on $X\times X$, hence $\partial_x U$ and $\partial_y U$ are well-defined. Moreover, U is zigzag-concave, so U is concave in the first variable, and therefore the second-order derivatives U_{x_i,x_j} in the first variable are well-defined and exist a.s. on $X \times X$ by the Alexandrov theorem [15, Theorem 6.4.1].) The last equality holds due to Theorem 3.7 and the fact that by Lemma 2.18 for all $s \geq 0$ a.s.

$$\begin{aligned}[\langle M, x_i^*\rangle, \langle M, x_j^*\rangle]_s^c = [\langle \Phi\cdot W, x_i^*\rangle, \langle \Phi\cdot W, x_j^*\rangle]_s = [(\Phi^* x_i^*)\cdot W, (\Phi^* x_j^*)\cdot W]_s \\ = \int_0^s \langle \Phi^*(r)x_i^*, \Phi^*(r)x_j^*\rangle\, \mathrm{d}r.\end{aligned}$$

Let us first show that $I_1 \leq 0$ a.s. Indeed, since M^d is a purely discontinuous part of M, then by Definition 2.11 $\langle M^d, x^*\rangle$ is a purely discontinuous part of $\langle M, x^*\rangle$, and due to Remark 2.8 a.s. for each $t \geq 0$

$$\Delta|\langle M^d, x^*\rangle|_t^2 = \Delta[\langle M^d, x^*\rangle]_t = \Delta[\langle M, x^*\rangle]_t = \Delta|\langle M, x^*\rangle|_t^2$$

for each $x^* \in X^*$. Thus for each $s \geq 0$ by (3.4) and (3.5) $\mathbb{P}$-a.s.

$$\begin{aligned}&\Delta U(M_s, M_s^d) - \langle \partial_x U(M_{s-}, M_{s-}^d), \Delta M_s\rangle - \langle \partial_y U(M_{s-}, M_{s-}^d), \Delta M_s^d\rangle \\ &= V(M_{s-} + M_{s-}^d + 2\Delta M_s, M_{s-}^d - M_{s-}) - V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}) \\ &\quad - \langle \partial_x V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}), 2\Delta M_s\rangle \leq 0,\end{aligned}$$

so $I_1 \leq 0$ a.s., and $\mathbb{E}I_1 \leq 0$. Now we show that

$$\mathbb{E}\Big(\int_0^t \langle \partial_x U(M_{s-}, M_{s-}^d),\, \mathrm{d}M_s\rangle + \int_0^t \langle \partial_y U(M_{s-}, M_{s-}^d),\, \mathrm{d}M_s^d\rangle\Big) = 0.$$

Indeed,

$$\begin{aligned}\int_0^t \langle \partial_x U(M_{s-}, M^d_{s-}), \,\mathrm{d}M_s\rangle + \int_0^t \langle \partial_y U(M_{s-}, M^d_{s-}), \,\mathrm{d}M^d_s\rangle \\ = \int_0^t \langle \partial_x V(M_{s-} + M^d_{s-}, M^d_{s-} - M_{s-}), \,\mathrm{d}(M_s + M^d_s)\rangle \\ + \int_0^t \langle \partial_y V(M_{s-} + M^d_{s-}, M^d_{s-} - M_{s-}), \,\mathrm{d}(M^d_s - M_s)\rangle\end{aligned}$$

so by Lemma 2.17 and Remark 3.5(E) it is a martingale which starts at zero, hence its expectation is zero.

Finally let us show that $I_2 \leq 0$ a.s. Fix $s \in [0,t]$ and $\omega \in \Omega$. Then $x^* \mapsto \|\Phi^*(s,\omega)x^*\|^2$ defines a nonnegative definite quadratic form on X^*, and since any nonnegative quadratic form defines a Euclidean seminorm, there exists a basis $(\tilde{x}^*_n)_{n=1}^d$ of X^* and a $\{0,1\}$-valued sequence $(a_n)_{n=1}^d$ such that

$$\langle \Phi^*(s,\omega)\tilde{x}^*_n, \Phi^*(s,\omega)\tilde{x}^*_m\rangle = a_n\delta_{mn}, \quad m,n = 1,\dots,d.$$

Let $(\tilde{x}_n)_{n=1}^d$ be the corresponding dual basis of X as it is defined in Definition 3.6. Then due to Lemma S.1 (see Supplement section) and the linearity of Φ and directional derivatives of U (we skip s and ω for the simplicity of the expressions)

$$\begin{aligned}\sum_{i,j=1}^d U_{x_i,x_j}(M_{s-}, M^d_{s-})\langle \Phi^* x^*_i, \Phi^* x^*_j\rangle &= \sum_{i,j=1}^d U_{\tilde{x}_i,\tilde{x}_j}(M_{s-}, M^d_{s-})\langle \Phi^*\tilde{x}^*_i, \Phi^*\tilde{x}^*_j\rangle \\ &= \sum_{i=1}^d U_{\tilde{x}_i,\tilde{x}_i}(M_{s-}, M^d_{s-})\|\Phi^*\tilde{x}^*_i\|^2.\end{aligned}$$

Recall that U is zigzag-concave, so $t \mapsto U(x + t\tilde{x}_i, y)$ is concave for each $x, y \in X$, $i = 1,\dots,d$. Therefore $U_{\tilde{x}_i,\tilde{x}_i}(M_{s-}, M^d_{s-}) \leq 0$ a.s., and a.s.

$$\sum_{i=1}^d U_{\tilde{x}_i,\tilde{x}_i}(M_{s-}(\omega), M^d_{s-}(\omega))\|\Phi^*(s,\omega)\tilde{x}^*_i\|^2 \leq 0.$$

Consequently, $I_2 \leq 0$ a.s., and by (3.7), Remark 3.4(B) and the fact that $M^d_0 = M_0$

$$\mathbb{E}U(M_t, M^d_t) \leq \mathbb{E}U(M_0, M_0) \leq 0.$$

By (3.2), $\mathbb{E}\|M^d_t\|^p - \beta^p_{p,X}\mathbb{E}\|M_t\|^p \leq \mathbb{E}U(M_t, M^d_t) \leq 0$, so the first part of (3.1) holds.

The second part of (3.1) follows from the same machinery applied for V. Namely, one can analogously show that

$$\mathbb{E}\|M^c_t\|^p - \beta^p_{p,X}\mathbb{E}\|M_t\|^p \leq \mathbb{E}U(M_t, M^c_t) = \mathbb{E}V(M^d + 2M^c, -M^d) \leq 0$$

by using a V-version of (3.7), inequality (3.5), and the fact that V is concave in the first variable a.s. on $X \times X$.

Step 2: general case. Without loss of generality we set $\mathcal{F}_\infty = \mathcal{F}_t$. Let $M_t = \xi$. If ξ is a simple function, then it takes its values in a finite dimensional subspace X_0 of X, and therefore $(M_s)_{s\geq 0} = (\mathbb{E}(\xi|\mathcal{F}_s))_{s\geq 0}$ takes its values in X_0 as well, so the theorem and (3.1) follow from Step 1.

Now let ξ be general. Let $(\xi_n)_{n\geq 1}$ be a sequence of simple $\mathcal{F}_t$-measurable functions in $L^p(\Omega; X)$ such that $\xi_n \to \xi$ as $n \to \infty$ in $L^p(\Omega; X)$. For each $n \geq 1$ define $\mathcal{F}_t$-measurable ξ_n^d and ξ_n^c such that

$$
\begin{aligned}
M^{d,n} &= (M_s^{d,n})_{s\geq 0} = (\mathbb{E}(\xi_n^d|\mathcal{F}_s))_{s\geq 0},\\
M^{c,n} &= (M_s^{c,n})_{s\geq 0} = (\mathbb{E}(\xi_n^c|\mathcal{F}_s))_{s\geq 0}
\end{aligned}
\tag{3.8}
$$

are the respectively purely discontinuous and continuous parts of martingale $M^n = (\mathbb{E}(\xi_n|\mathcal{F}_s))_{s\geq 0}$ as in Remark 2.12. Then due to Step 1 and (3.1), $(\xi_n^d)_{n\geq 1}$ and $(\xi_n^c)_{n\geq 1}$ are Cauchy sequences in $L^p(\Omega; X)$. Let $\xi^c := L^p - \lim_{n\to\infty} \xi_n^c$ and $\xi^d := L^p - \lim_{n\to\infty} \xi_n^d$. Define the X-valued L^p-martingales M^d and M^c by

$$
M^d = (M_s^d)_{s\geq 0} := (\mathbb{E}(\xi^d|\mathcal{F}_s))_{s\geq 0}, \quad M^c = (M_s^c)_{s\geq 0} := (\mathbb{E}(\xi^c|\mathcal{F}_s))_{s\geq 0}.
$$

Thanks to Proposition 2.14, M^d is purely discontinuous, and due to Proposition 2.6 M^c is continuous and $M_0^c = 0$, so $M = M^d + M^c$ is the desired decomposition.

The uniqueness of the decomposition follows from Lemma 2.15. For estimates (3.1) we note that by Step 1, (3.1) applied for Step 1, and [19, Proposition 4.2.17] for each $n \geq 1$

$$
(\mathbb{E}\|\xi_n^d\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|\xi_n\|^p)^{\frac{1}{p}}, \quad (\mathbb{E}\|\xi_n^c\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|\xi_n\|^p)^{\frac{1}{p}},
$$

and it remains to let $n \to \infty$. □

Remark 3.10. *Let X be a UMD Banach space, $1 < p < \infty$, $M : \mathbb{R}_+ \times \Omega \to X$ be continuous (resp. purely discontinuous) L^p-martingale. Then there exists a sequence $(M^n)_{n\geq 1}$ of continuous (resp. purely discontinuous) X-valued L^p-martingales such that M^n takes its values is a finite dimensional subspace of X for each $n \geq 1$ and $M_\infty^n \to M_\infty$ in $L^p(\Omega; X)$ as $n \to \infty$. Such a sequence can be provided e.g. by (3.8).*

We have proven the Meyer-Yoeurp decomposition in the UMD setting. Next we prove a converse result which shows the necessity of the UMD property.

Theorem 3.11. *Let X be a finite dimensional Banach space, $p \in (1, \infty)$, $\delta \in (0, (\beta_{p,X} - 1) \wedge 1)$. Then there exist a purely discontinuous martingale $M^d : \mathbb{R}_+ \times \Omega \to X$, a continuous martingale $M^c : \mathbb{R}_+ \times \Omega \to X$ such that $\mathbb{E}\|M_\infty^d\|^p, \mathbb{E}\|M_\infty^c\|^p < \infty$, $M_0^d = M_0^c = 0$, and for $M = M^d + M^c$ and $i \in \{c, d\}$ the following hold*

$$
(\mathbb{E}\|M_\infty^i\|^p)^{\frac{1}{p}} \geq \Big(\frac{\beta_{p,X} - 1}{2} - \delta\Big)(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}.
\tag{3.9}
$$

Recall that by [19, Proposition 4.2.17] $\beta_{p,X} \geq \beta_{p,\mathbb{R}} = p^* - 1 \geq 1$ for any UMD Banach space X and $1 < p < \infty$.

Definition 3.12. *A random variable $r : \Omega \to \{-1, 1\}$ is called a* Rademacher variable *if $\mathbb{P}(r = 1) = \mathbb{P}(r = -1) = \frac{1}{2}$.*

Lemma* 3.13. *Let $\varepsilon > 0$, $p \in (1,\infty)$. Then there exists a continuous martingale $M : [0,1] \times \Omega \to [-1,1]$ with a symmetric distribution such that* sign M_1 *is a Rademacher random variable and*

$$\|M_1 - \operatorname{sign} M_1\|_{L^p(\Omega)} < \varepsilon. \tag{3.10}$$

We will need a definition of a Paley-Walsh martingale.

Definition 3.14 (Paley-Walsh martingales)**.** *Let X be a Banach space. A discrete X-valued martingale $(f_n)_{n\geq 0}$ is called a* Paley-Walsh martingale *if there exist a sequence of independent Rademacher variables $(r_n)_{n\geq 1}$, a function $\phi_n : \{-1,1\}^{n-1} \to X$ for each $n \geq 2$ and $\phi_1 \in X$ such that $df_n = r_n \phi_n(r_1, \dots, r_{n-1})$ for each $n \geq 2$ and $df_1 = r_1\phi_1$.*

Remark 3.15. *Let X be a UMD space, $1 < p < \infty$, $\delta > 0$. Then using Proposition 2.1 one can construct a martingale difference sequence $(d_j)_{j=1}^n \in L^p(\Omega; X)$ and a $\{-1,1\}$-valued sequence $(\varepsilon_j)_{j=1}^n$ such that*

$$\Bigl(\mathbb{E}\Bigl\|\sum_{j=1}^n \frac{\varepsilon_j \pm 1}{2} d_j\Bigr\|^p\Bigr)^{\frac{1}{p}} \geq \frac{\beta_{p,X} - \delta - 1}{2}\Bigl(\mathbb{E}\Bigl\|\sum_{j=1}^n d_j\Bigr\|^p\Bigr)^{\frac{1}{p}}.$$

Proof of Theorem 3.11. Denote $\frac{\beta_{p,X}-\delta-1}{2}$ by $\gamma^{\delta}_{p,X}$. By Proposition 2.1 there exists a natural number $N \geq 1$, a discrete X-valued martingale $(f_n)_{n=0}^N$ such that $f_0 = 0$, and a sequence of scalars $(\varepsilon_n)_{n=1}^N$ such that $\varepsilon_n \in \{0,1\}$ for each $n = 1, \dots, N$, such that

$$\Bigl(\mathbb{E}\Bigl\|\sum_{n=1}^N \varepsilon_n df_n\Bigr\|^p\Bigr)^{\frac{1}{p}} \geq \gamma^{\delta}_{p,X}(\mathbb{E}\|f_N\|^p)^{\frac{1}{p}}. \tag{3.11}$$

According to [19, Theorem 3.6.1] we can assume that $(f_n)_{n=0}^N$ is a Paley-Walsh martingale. Let $(r_n)_{n=1}^N$ be a sequence of Rademacher variables and $(\phi_n)_{n=1}^N$ be a sequence of functions as in Definition 3.14, i.e. be such that $f_n = \sum_{k=2}^n r_k\phi_k(r_1, \dots, r_{k-1}) + r_1\phi_1$ for each $n = 1, \dots, N$. Without loss of generality we assume that

$$(\mathbb{E}\|f_N\|^p)^{\frac{1}{p}} \geq 2. \tag{3.12}$$

For each $n = 1, \dots, N$ define a continuous martingale $M^n : [0,1] \times \Omega \to [-1,1]$ as in Lemma 3.13, i.e. a martingale M^n with a symmetric distribution such that sign M^n_1 is a Rademacher variable and

$$\|M^n_1 - \operatorname{sign} M^n_1\|_{L^p(\Omega)} < \frac{\delta}{KL}, \tag{3.13}$$

where $K = \beta_{p,X} N \max\{\|\phi_1\|, \|\phi_2\|_\infty, \dots, \|\phi_N\|_\infty\}$, and $L = 2\beta_{p,X}$. Without loss of generality suppose that $(M^n)_{n=1}^N$ are independent. For each $n = 1, \dots, N$ set $\sigma_n = \operatorname{sign} M_1^n$. Define a martingale $M : [0, N+1] \times \Omega \to X$ in the following way:

$$M_t = \begin{cases} 0, & \text{if } \ 0 \le t < 1; \\ M_{n-} + M_{t-n}^n \phi_n(\sigma_1, \dots, \sigma_{n-1}), & \text{if } \ t \in [n, n+1) \ \text{ and } \ \varepsilon_n = 0; \\ M_{n-} + \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}), & \text{if } \ t \in [n, n+1) \ \text{ and } \ \varepsilon_n = 1. \end{cases}$$

Let $M = M^d + M^c$ be the decomposition of Theorem 3.1. Then

$$\begin{aligned} M_{N+1}^c &= \sum_{n=1}^N M_1^n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \mathbf{1}_{\varepsilon_n = 0}, \\ M_{N+1}^d &= \sum_{n=1}^N \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \mathbf{1}_{\varepsilon_n = 1} = \sum_{n=1}^N \varepsilon_n \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}). \end{aligned}$$

Notice that $(\sigma_n)_{n=1}^N$ is a sequence of independent Rademacher variables, so by (3.11) and the discussion thereafter

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n \, \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1})\Big\|^p\Big)^{\frac{1}{p}} \ge \gamma_{p,X}^{\delta} \Big(\mathbb{E}\Big\|\sum_{n=1}^N \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1})\Big\|^p\Big)^{\frac{1}{p}}. \tag{3.14}$$

Let us first show (3.9) with $i = d$. Note that by the triangle inequality, (3.12) and (3.13)

$$\begin{aligned} (\mathbb{E}\|M_{N+1}\|^p)^{\frac{1}{p}} &\ge (\mathbb{E}\|f_N\|^p)^{\frac{1}{p}} - \sum_{n=1}^N \Big(\mathbb{E}\Big\|(M_1^n - \sigma_n)\phi_n(\sigma_1, \dots, \sigma_{n-1})\Big\|^p\Big)^{\frac{1}{p}} \\ &\ge 2 - \frac{\delta}{KL} \cdot N \cdot \max\{\|\phi_1\|, \|\phi_2\|_\infty, \dots, \|\phi_N\|_\infty\} > 1. \end{aligned} \tag{3.15}$$

Therefore,

$$\begin{aligned} (\mathbb{E}\|M_{N+1}^d\|^p)^{\frac{1}{p}} &= \Big(\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n \, \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1})\Big\|^p\Big)^{\frac{1}{p}} \overset{(i)}{\ge} \gamma_{p,X}^{\delta} \Big(\mathbb{E}\Big\|\sum_{n=1}^N \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1})\Big\|^p\Big)^{\frac{1}{p}} \\ &\overset{(ii)}{\ge} \gamma_{p,X}^{\delta} \Big(\mathbb{E}\Big\|\sum_{n=1}^N \mathbf{1}_{\varepsilon_n = 1} \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) + \sum_{n=1}^N \mathbf{1}_{\varepsilon_n = 0} M_1^n \phi_n(\sigma_1, \dots, \sigma_{n-1})\Big\|^p\Big)^{\frac{1}{p}} \\ &\quad - \gamma_{p,X}^{\delta} \sum_{n=1}^N \Big(\mathbb{E}\Big\|(M_1^n - \sigma_n)\phi_n(\sigma_1, \dots, \sigma_{n-1})\Big\|^p\Big)^{\frac{1}{p}} \\ &\overset{(iii)}{\ge} \gamma_{p,X}^{\delta} (\mathbb{E}\|M_{N+1}\|^p)^{\frac{1}{p}} - \frac{\delta}{L} \overset{(iv)}{\ge} \Big(\frac{\beta_{p,X} - 1}{2} - \delta\Big) (\mathbb{E}\|M_{N+1}\|^p)^{\frac{1}{p}}, \end{aligned}$$

where (i) follows from (3.14), (ii) holds by the triangle inequality, (iii) holds by (3.13), and (iv) follows from (3.15). By the same reason and Remark 3.15, (3.9) holds for $i = c$. □

Let $p \in (1,\infty)$. Recall that $\mathcal{M}_X^p$ is a space of all X-valued L^p-martingales, $\mathcal{M}_X^{p,d}, \mathcal{M}_X^{p,c} \subset \mathcal{M}_X^p$ are its subspaces of purely discontinuous martingales and continuous martingales that start at zero respectively (see Subsection 2.2, 2.4, and 2.5).

Theorem* 3.16. *Let X be a Banach space. Then X is UMD if and only if for some (or, equivalently, for all) $p \in (1,\infty)$, for any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with any filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ that satisfies the usual conditions, $\mathcal{M}_X^p = \mathcal{M}_X^{p,d} \oplus \mathcal{M}_X^{p,c}$, and there exist projections $A^d, A^c \in \mathcal{L}(\mathcal{M}_X^p)$ such that ran $A^d = \mathcal{M}_X^{p,d}$, ran $A^c = \mathcal{M}_X^{p,c}$, and for any $M \in \mathcal{M}_X^p$ the decomposition $M = A^dM + A^cM$ is the Meyer-Yoeurp decomposition from Theorem 3.1. If this is the case, then*

$$\|A^d\| \leq \beta_{p,X} \quad \text{and} \quad \|A^c\| \leq \beta_{p,X}. \tag{3.16}$$

Moreover, there exist $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ such that

$$\|A^d\|, \|A^c\| \geq \frac{\beta_{p,X} - 1}{2} \vee 1. \tag{3.17}$$

Corollary 3.17. *Let X be a UMD Banach space, $p \in (1,\infty)$. Let $i \in \{c, d\}$. Then $(\mathcal{M}_X^{p,i})^* \simeq \mathcal{M}_{X^*}^{p',i}$, and for each $M \in \mathcal{M}_{X^*}^{p',i}$ and $N \in \mathcal{M}_X^{p,i}$*

$$\langle M, N\rangle := \mathbb{E}\langle M_\infty, N_\infty\rangle, \quad \|M\|_{(\mathcal{M}_X^{p,i})^*} \eqsim_{p,X} \|M\|_{\mathcal{M}_{X^*}^{p',i}}.$$

To prove the corollary above we will need the following lemma.

Lemma 3.18. *Let X be a UMD Banach space, $p \in (1,\infty)$, $M \in \mathcal{M}_X^{p,d}$, $N \in \mathcal{M}_{X^*}^{p',c}$. Then $\mathbb{E}\langle M_\infty, N_\infty\rangle = 0$.*

Proof. First suppose that N_∞ takes it values in a finite dimensional subspace Y of X^*. Let $d \geq 1$ be the dimension of Y, $(y_k)_{k=1}^d$ be the basis of Y. Then there exist $N^1, \ldots, N^d \in \mathcal{M}_{\mathbb{R}}^{p',c}$ such that $N = \sum_{k=1}^d N^k y_k$. Hence

$$E\langle M_\infty, N_\infty\rangle = E\Big\langle M_\infty, \sum_{k=1}^d N_\infty^k y_k\Big\rangle = \sum_{k=1}^d \mathbb{E}\langle M_\infty, y_k\rangle N_\infty^k \overset{(*)}{=} 0, \tag{3.18}$$

where $(*)$ holds due to Proposition 2.10.

Now turn to the general case. By Remark 3.10 for each $N \in \mathcal{M}_{X^*}^{p',c}$ there exists a sequence $(N^n)_{n\geq 1}$ of continuous martingales such that each of N^n is in $\mathcal{M}_{X^*}^{p',c}$ and takes its valued in a finite dimensional subspace of X^*, and $N_\infty^n \to N_\infty$ in $L^{p'}(\Omega; X^*)$ as $n \to \infty$. Then due to (3.18), $E\langle M_\infty, N_\infty\rangle = \lim_{n\to\infty} E\langle M_\infty, N_\infty^n\rangle = 0$, so the lemma holds. □

Proof of Corollary 3.17. We will show only the case $i = d$, the case $i = c$ can be shown analogously.

$\mathcal{M}^{p',d}_{X^*} \subset (\mathcal{M}^{p,d}_X)^*$ and $\|M\|_{(\mathcal{M}^{p,d}_X)^*} \leq \|M\|_{\mathcal{M}^{p',d}_{X^*}}$ for each $M \in \mathcal{M}^{p',d}_{X^*}$ thanks to the Hölder inequality. Now let us show the inverse. Let $f \in (\mathcal{M}^{p,d}_X)^*$. Since due to Proposition 2.14 $\mathcal{M}^{p,d}_X$ is a closed subspace of $\mathcal{M}^p_X$, by the Hahn-Banach theorem and Proposition 2.3 there exists $L \in \mathcal{M}^{p'}_{X^*}$ such that $\mathbb{E}\langle L_\infty, N_\infty\rangle = f(N)$ for any $N \in \mathcal{M}^{p,d}_X$, and $\|L\|_{\mathcal{M}^{p'}_{X^*}} = \|f\|_{(\mathcal{M}^{p,d}_X)^*}$. Let $L = L^d + L^c$ be the Meyer-Yoeurp decomposition of L as in Theorem 3.1. Then by (3.1)

$$\|L^d\|_{\mathcal{M}^{p',d}_{X^*}} \lesssim_{p,X} \|L\|_{\mathcal{M}^{p'}_{X^*}} = \|f\|_{(\mathcal{M}^{p,d}_X)^*}$$

and $\mathbb{E}\langle L^d_\infty, N_\infty\rangle = \mathbb{E}\langle L_\infty, N_\infty\rangle$, so the theorem holds. □

3.2. Yoeurp decomposition of purely discontinuous martingales

As Yoeurp shown in [43], one can provide further decomposition of a purely discontinuous martingale into two parts: a martingale with accessible jumps and a quasi-left continuous martingale. This subsection is devoted to the generalization of this result to a UMD case.

Definition 3.19. *Let τ be a stopping time. Then τ is called a* predictable stopping time *if there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\tau_n < \tau$ a.s. on $\{\tau > 0\}$ for each $n \geq 1$ and $\tau_n \nearrow \tau$ a.s.*

Definition 3.20. *Let τ be a stopping time. Then τ is called a* totally inaccessible stopping time *if $\mathbb{P}\{\tau = \sigma < \infty\} = 0$ for each predictable stopping time σ.*

Definition 3.21. *Let $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be an adapted càdlàg process. A has* accessible jumps *if $\Delta A_\tau = 0$ a.s. for any totally inaccessible stopping time τ. A is called* quasi-left continuous *if $\Delta A_\tau = 0$ a.s. for any predictable stopping time τ.*

For the further information on the definitions given we refer the reader to [23].

Remark 3.22. *According to [23, Proposition 25.17] one can show that for any pure jump increasing adapted càdlàg process $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ there exist unique increasing adapted càdlàg processes $A^a, A^q : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that A^a has accessible jumps, A^q is quasi-left continuous, $A^q_0 = 0$ and $A = A^a + A^q$.*

The following decomposition theorem was shown by Yoeurp in [43] (see also [23, Corollary 26.16]):

Theorem 3.23. *Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a purely discontinuous martingale. Then there exist unique purely discontinuous martingales $M^a, M^q : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that M^a is has accessible jumps, M^q is quasi-left continuous, $M^q_0 = 0$ and $M = M^a + M^q$. Moreover, then $[M^a] = [M]^a$ and $[M^q] = [M]^q$.*

Corollary 3.24. *Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a purely discontinuous martingale which is both with accessible jumps and quasi-left continuous. Then $M = M_0$ a.s.*

Proposition* 3.25. *Let $1 < p < \infty$, $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a purely discontinuous L^p-martingale. Let $(M^n)_{n\geq 1}$ be a sequence of purely discontinuous martingales such that $M^n_\infty \to M_\infty$ in $L^p(\Omega)$. Then the following assertions hold*

- *(a) if $(M^n)_{n\geq 1}$ have accessible jumps, then M has accessible jumps as well;*
- *(b) if $(M^n)_{n\geq 1}$ are quasi-left continuous martingales, then M is quasi-left continuous as well.*

Definition 3.26. *Let X be a Banach space. A martingale $M : \mathbb{R}_+ \times \Omega \to X$ has* accessible jumps *if $\Delta M_\tau = 0$ a.s. for any totally inaccessible stopping time τ. A martingale $M : \mathbb{R}_+ \times \Omega \to X$ is called* quasi-left continuous *if $\Delta M_\tau = 0$ a.s. for any predictable stopping time τ.*

Lemma* 3.27. *Let X be a reflexive Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a purely discontinuous martingale.*

- *(i) M has accessible jumps if and only if for each $x^* \in X^*$ the martingale $\langle M, x^*\rangle$ has accessible jumps;*
- *(ii) M is quasi-left continuous if and only if for each $x^* \in X^*$ the martingale $\langle M, x^*\rangle$ is quasi-left continuous.*

Definition 3.28. *Let X be a Banach space, $p \in (1,\infty)$. Then we define $\mathcal{M}^{p,q}_X \subset \mathcal{M}^{p,d}_X$ as a linear space of all X-valued purely discontinuous quasi-left continuous L^p-martingales which start at 0. We define $\mathcal{M}^{p,a}_X \subset \mathcal{M}^{p,d}_X$ as a linear space of all X-valued purely discontinuous L^p-martingales with accessible jumps.*

Proposition* 3.29. *Let X be a Banach space, $1 < p < \infty$. Then $\mathcal{M}^{p,q}_X$ and $\mathcal{M}^{p,a}_X$ are closed subspaces of $\mathcal{M}^{p,d}_X$.*

The following lemma follows from Corollary 3.24.

Lemma 3.30. *Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a purely discontinuous martingale. Let M be both with accessible jumps and quasi-left continuous. Then $M = M_0$ a.s. In other words, $\mathcal{M}^{p,q}_X \cap \mathcal{M}^{p,a}_X = 0$.*

The main theorem of this subsection is the following UMD variant of Theorem 3.23.

Theorem 3.31. *Let X be a UMD Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a purely discontinuous L^p-martingale. Then there exist unique purely discontinuous martingales $M^a, M^q : \mathbb{R}_+ \times \Omega \to X$ such that M^a has accessible jumps, M^q is quasi-left continuous, $M^q_0 = 0$ and $M = M^a + M^q$. Moreover, if this is the case, then for $i \in \{a, q\}$*

$$(\mathbb{E}\|M^i_\infty\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \tag{3.19}$$

Proof. *Step 1: finite dimensional case.* First assume that X is finite dimensional. Then M^a and M^q exist and unique due to coordinate-wise applying of Theorem 3.23. Let $M = M^a + M^q$, $N = M^a$. Then for any $x^* \in X^*$, $t \geq 0$ by Theorem 3.23 and Lemma 3.27 a.s.

$$[\langle M, x^*\rangle]_t = [\langle M, x^*\rangle]_t^a + [\langle M, x^*\rangle]_t^q = [\langle M^a, x^*\rangle]_t + [\langle M^q, x^*\rangle]_t,$$

and

$$[\langle N, x^*\rangle]_t = [\langle N, x^*\rangle]_t^a + [\langle N, x^*\rangle]_t^q = [\langle M^a, x^*\rangle]_t.$$

Therefore a.s.

$$[\langle N, x^*\rangle]_t - [\langle N, x^*\rangle]_s \leq [\langle M, x^*\rangle]_t - [\langle M, x^*\rangle]_s, \quad 0 \leq s < t.$$

Moreover $M_0 = N_0$. Hence N is weakly differentially subordinated to M (see Section 4), and (3.19) for $i = a$ follows from [41]. By the same reason and since $M_0^q = 0$, (3.19) holds true for $i = q$.

Step 2: general case. Now let X be general. Let $\xi = M_\infty$. Without loss of generality we set $\mathcal{F}_\infty = \mathcal{F}_t$. Let $(\xi_n)_{n\geq 1}$ be a sequence of simple $\mathcal{F}_t$-measurable functions in $L^p(\Omega; X)$ such that $\xi_n \to \xi$ as $n \to \infty$ in $L^p(\Omega; X)$. For each $n \geq 1$ define $\mathcal{F}_t$-measurable ξ_n^d and ξ_n^c such that $M^{d,n} = (\mathbb{E}(\xi_n^d|\mathcal{F}_s))_{s\geq 0}$ and $M^{c,n} = (\mathbb{E}(\xi_n^c|\mathcal{F}_s))_{s\geq 0}$ are respectively purely discontinuous and continuous parts of a martingale $(\mathbb{E}(\xi_n|\mathcal{F}_s))_{s\geq 0}$ as in Remark 2.12. Then thanks to Theorem 3.1, $\xi_n^d \to \xi$ and $\xi_n^c \to 0$ in $L^p(\Omega; X)$ as $n \to \infty$ since M is purely discontinuous.

Since for each $n \geq 1$ the random variable ξ_n^d takes its values in a finite dimensional space, by Theorem 3.23 there exist $\mathcal{F}_t$-measurable $\xi^a, \xi^q \in L^p(\Omega; X)$ such that purely discontinuous martingales $M^{a,n} = (\mathbb{E}(\xi_n^a|\mathcal{F}_s))_{s\geq 0}$ and $M^{q,n} = (\mathbb{E}(\xi_n^q|\mathcal{F}_s))_{s\geq 0}$ are respectively with accessible jumps and quasi-left continuous, $\mathbb{E}(\xi_n^q|\mathcal{F}_0) = 0$, and the decomposition $M^{d,n} = M^{a,n} + M^{q,n}$ is as in Theorem 3.23. Since $(\xi_n^d)_{n\geq 1}$ is a Cauchy sequence in $L^p(\Omega; X)$, by Step 1 both $(\xi_n^a)_{n\geq 1}$ and $(\xi_n^q)_{n\geq 1}$ are Cauchy in $L^p(\Omega; X)$ as well. Let ξ^a and ξ^q be their limits. Define martingales $M^a, M^q : \mathbb{R}_+ \times \Omega \to X$ in the following way:

$$M_s^a := \mathbb{E}(\xi^a|\mathcal{F}_s), \quad M_s^q := \mathbb{E}(\xi^q|\mathcal{F}_s), \quad s \geq 0.$$

By Proposition 3.29 M^a is a martingale with accessible jumps, M^q is quasi-left continuous, $M_0^q = 0$ a.s., and therefore $M = M^a + M^q$ is the desired decomposition. Moreover, by Step 1 for each $n \geq 1$ and $i \in \{a, q\}$, $(\mathbb{E}\|\xi_n^i\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|\xi_n^d\|^p)^{\frac{1}{p}}$, and hence the estimate (3.19) follows by letting n to infinity.

The uniqueness of the decomposition follows from Lemma 3.30. □

The following theorem, as Theorem 3.11, illustrates that the decomposition in Theorem 3.31 takes place only in the UMD space case.

Theorem 3.32. *Let X be a finite dimensional Banach space, $p \in (1, \infty)$, $\delta \in \big(0, \frac{\beta_{p,X}-1}{2}\big)$. Then there exist purely discontinuous martingales $M^a, M^q : \mathbb{R}_+ \times \Omega \to X$ such that M^a has accessible jumps, M^q is quasi-left continuous, $\mathbb{E}\|M_\infty^a\|^p, \mathbb{E}\|M_\infty^q\|^p < \infty$, $M_0^a = M_0^q = 0$, and for $M = M^a + M^q$ and $i \in \{a, q\}$ the following holds*

$$(\mathbb{E}\|M_\infty^i\|^p)^{\frac{1}{p}} \geq \Big(\frac{\beta_{p,X}-1}{2} - \delta\Big)(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \tag{3.20}$$

For the proof we will need the following lemma.

Lemma 3.33. *Let $\varepsilon \in \big(0, \frac{1}{2}\big)$, $p \in (1,\infty)$. Then there exist martingales $M, M^a, M^q : [0,1]\times\Omega \to [-1-\varepsilon, 1+\varepsilon]$ with symmetric distributions such that M^a is a martingale with accessible jumps, $\|M_1^a\|_{L^p(\Omega)} < \varepsilon$, M^q is a quasi-left continuous martingale, $M_0^q = 0$ a.s., $M = M^a + M^q$, $\operatorname{sign} M_1$ is a Rademacher random variable and*

$$\|M_1 - \operatorname{sign}\, M_1\|_{L^p(\Omega)} < \varepsilon. \tag{3.21}$$

Proof. Let $N^+, N^- : [0,1]\times\Omega \to \mathbb{R}$ be independent Poisson processes with the same intensity λ_ε such that $\mathbb{P}(N_1^+ = 0) = \mathbb{P}(N_1^- = 0) < \frac{\varepsilon^p}{2^p}$ (such λ_ε exists since N_1^+ and N_1^- have Poisson distributions, see [25]). Define a stopping time τ in the following way:

$$\tau = \inf\{t : N_t^+ \geq 1\} \wedge \inf\{t : N_t^- \geq 1\} \wedge 1.$$

Let $M_t^q := N_{t\wedge\tau}^+ - N_{t\wedge\tau}^-$, $t \in [0,1]$. Then M^q is quasi-left continuous with a symmetric distribution. Let r be an independent Rademacher variable, $M_t^a = \frac{\varepsilon}{2} r$ for each $t \in [0,1]$. Then M^a is a martingale with accessible jumps and symmetric distribution, and $\|M_1^a\|_{L^p(\Omega)} = \frac{\varepsilon}{2} < \varepsilon$. Let $M = M^a + M^q$. Then a.s.

$$M_1 \in \Big\{-1-\frac{\varepsilon}{2}, -1+\frac{\varepsilon}{2}, -\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2}\Big\}, \tag{3.22}$$

so $\mathbb{P}(M_1 = 0) = 0$, and therefore $\operatorname{sign} M_1$ is a Rademacher random variable. Let us prove (3.21). Notice that due to (3.22) if $|M_1^q| = 1$, then $|M_1 - \operatorname{sign}\, M_1| < \frac{\varepsilon}{2}$, and if $|M_1^q| = 0$, then $|M_1 - \operatorname{sign}\, M_1| < 1$. Therefore

$$\begin{aligned}\mathbb{E}|M_1 - \operatorname{sign}\, M_1|^p &= \mathbb{E}|M_1 - \operatorname{sign}\, M_1|^p \mathbf{1}_{|M_1^q|=1} + \mathbb{E}|M_1 - \operatorname{sign}\, M_1|^p \mathbf{1}_{|M_1^q|=0}\\ &< \frac{\varepsilon^p}{2^p} + \frac{\varepsilon^p}{2^p} < \varepsilon^p,\end{aligned}$$

so (3.21) holds. □

Proof of Theorem 3.32. The proof is analogous to the proof of Theorem 3.11, while one has to use Lemma 3.33 instead of Lemma 3.13. □

Theorem 3.32 yields the following characterization of the UMD property.

Theorem 3.34. *Let X be a Banach space. Then X is a UMD Banach space if and only if for some (equivalently, for all) $p \in (1,\infty)$ there exists $c_{p,X} > 0$ such that for any L^p-martingale $M := \mathbb{R}_+ \times \Omega \to X$ there exist unique martingales $M^c, M^q, M^a : \mathbb{R}_+ \times \Omega \to X$ such that $M_0^c = M_0^q = 0$, M^c is continuous, M^q is purely discontinuous quasi-left continuous, M^a is purely discontinuous with accessible jumps, $M = M^c + M^q + M^a$, and*

$$(\mathbb{E}\|M_\infty^c\|^p)^{\frac{1}{p}} + (\mathbb{E}\|M_\infty^q\|^p)^{\frac{1}{p}} + (\mathbb{E}\|M_\infty^a\|^p)^{\frac{1}{p}} \leq c_{p,X} (\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \tag{3.23}$$

If this is the case, then the least admissible $c_{p,X}$ is in the interval $\big[\frac{3\beta_{p,X}-3}{2} \vee 1, 3\beta_{p,X}\big]$.

The decomposition $M = M^c + M^q + M^a$ is called the *canonical decomposition* of the martingale M (see [23, 43, 14]).

Proof. The "if and only if" part follows from Theorem 3.16, Theorem 3.31 and Theorem 3.32. The estimate $c_{p,X} \leq 3\beta_{p,X}$ follows from (3.1) and (3.19). The estimate $c_{p,X} \geq \frac{3\beta_{p,X}-3}{2} \vee 1$ follows from (3.9) and (3.20). □

Corollary 3.35. *Let X be a Banach space. Then X is a UMD Banach space if and only if $\mathcal{M}_X^{p,d} = \mathcal{M}_X^{p,a} \oplus \mathcal{M}_X^{p,q}$ and $\mathcal{M}_X^{p} = \mathcal{M}_X^{p,c} \oplus \mathcal{M}_X^{p,q} \oplus \mathcal{M}_X^{p,a}$ for any filtration that satisfies the usual conditions.*

Proof. The corollary follows from Theorem 3.31, Theorem 3.32 and Theorem 3.34. □

3.3. Stochastic integration

The current subsection is devoted to application of Theorem 3.34 to stochastic integration with respect to a general martingale.

Proposition* 3.36. *Let H be a Hilbert space, X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to H$ be a martingale, $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$ be elementary progressive. Then*

(i) if M is continuous, then $\Phi \cdot M$ is continuous;
(ii) if M is purely discontinuous, then $\Phi \cdot M$ is purely discontinuous;
(iii) if M has accessible jumps, then $\Phi \cdot M$ has accessible jumps;
(iv) if M is quasi-left continuous, then $\Phi \cdot M$ is quasi-left continuous.

Proposition 3.37. *Let H be a Hilbert space, $M : \mathbb{R}_+ \times \Omega \to H$ be a local martingale. Then there exist unique martingales $M^c, M^q, M^a : \mathbb{R}_+ \times \Omega \to H$ such that M^c is continuous, M^q and M^a are purely discontinuous, M^q is quasi-left continuous, M^a has accessible jumps, $M_0^c = M_0^q = 0$ a.s., and $M = M^c + M^q + M^a$.*

Proof. Analogously to Theorem 26.14 and Corollary 26.16 in [23]. □

Theorem 3.38. *Let H be a Hilbert space, X be a UMD Banach space, $p \in (1, \infty)$, $M : \mathbb{R}_+ \times \Omega \to H$ be a local martingale, $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$ be elementary progressive. Let $M = M^c + M^q + M^a$ be the canonical decomposition from Proposition 3.37. Then*

$$\mathbb{E}\|(\Phi \cdot M)_\infty\|^p \eqsim_{p,X} \mathbb{E}\|(\Phi \cdot M^c)_\infty\|^p + \mathbb{E}\|(\Phi \cdot M^q)_\infty\|^p + \mathbb{E}\|(\Phi \cdot M^a)_\infty\|^p. \tag{3.24}$$

and if $(\Phi \cdot M)_\infty \in L^p(\Omega; X)$, then $\Phi \cdot M = \Phi \cdot M^c + \Phi \cdot M^q + \Phi \cdot M^a$ is the canonical decomposition from Theorem 3.34.

Proof. The statement that $\Phi \cdot M = \Phi \cdot M^c + \Phi \cdot M^q + \Phi \cdot M^a$ is the canonical decomposition follows from Proposition 3.36, Theorem 3.34 and the fact that a.s. $(\Phi \cdot M)_0 = (\Phi \cdot M^c)_0 = (\Phi \cdot M^q)_0 = 0$. (3.24) follows then from (3.23) and the triangle inequality. □

Remark 3.39. *Notice that the Itô isomorphism for the term $\Phi \cdot M^c$ from (3.24) was explored in [37]. It remains open what to do with the other two terms, but positive results in this direction were obtained in the case of $X = L^q(S)$ in [14].*

4. Weak differential subordination and general martingales

This subsection is devoted to the generalization of the main theorem in work [41]. Namely, here we show the L^p-estimates for general X-valued weakly differentially subordinated martingales.

Definition 4.1. *Let X be a Banach space, $M, N : \mathbb{R}_+ \times \Omega \to X$ be local martingales. Then N is* weakly differentially subordinated *to M if $[\langle M, x^*\rangle] - [\langle N, x^*\rangle]$ is an increasing process a.s. for each $x^* \in X^*$.*

The following theorem have been proven in [41].

Theorem 4.2. *Let X be a Banach space. Then X has the UMD property if and only if for some (equivalently, for all) $p \in (1, \infty)$ there exists $\beta > 0$ such that for each pair of purely discontinuous martingales $M, N : \mathbb{R}_+ \times \Omega \to X$ such that N is weakly differentially subordinated to M one has that*

$$(\mathbb{E}\|N_\infty\|^p)^{\frac{1}{p}} \leq \beta(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}.$$

If this is the case, then the least admissible β is the UMD constant $\beta_{p,X}$.

The main goal of the current section is to prove the following generalization of Theorem 4.2 to the case of arbitrary martingales.

Theorem 4.3. *Let X be a UMD Banach space, $M, N : \mathbb{R}_+ \times \Omega \to X$ be two martingales such that N is weakly differentially subordinated to M. Then for each $p \in (1, \infty)$, $t \geq 0$,*

$$(\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2(\beta_{p,X} + 1)(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}. \tag{4.1}$$

The proof will be done in several steps. First we show an analogue of Theorem 4.2 for continuous martingales.

Theorem* 4.4. *Let X be a Banach space. Then X is a UMD Banach space if and only if for some (equivalently, for all) $p \in (1, \infty)$ there exists $c > 0$ such that for any continuous martingales $M, N : \mathbb{R}_+ \times \Omega \to X$ such that N is weakly differentially subordinated to M, $M_0 = N_0 = 0$, one has that*

$$(\mathbb{E}\|N_\infty\|^p)^{\frac{1}{p}} \leq c_{p,X}(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \tag{4.2}$$

If this is the case, then the least admissible $c_{p,X}$ is in the segment $[\beta_{p,X}, \beta_{p,X}^2]$.

For the proof we will need the following proposition, which demonstrates that one needs a slightly weaker assumption rather then in Theorem 4.4 so that the estimate (4.2) holds in a UMD Banach space.

Proposition 4.5. *Let X be a UMD Banach space, $1 < p < \infty$, $M, N : \mathbb{R}_+ \times \Omega \to X$ be continuous L^p-martingales s.t. $M_0 = N_0 = 0$ and for each $x^* \in X^*$ a.s. for each $t \geq 0$*

$$[\langle N, x^*\rangle]_t \leq [\langle M, x^*\rangle]_t. \tag{4.3}$$

Then for each $t \geq 0$

$$(\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}. \tag{4.4}$$

Proof. Without loss of generality by a stopping time argument we assume that M and N are bounded and that $M_\infty = M_t$ and $N_\infty = N_t$.

One can also restrict to a finite dimensional case. Indeed, since X is a separable reflexive space, X^* is separable as well. Let $(Y_m)_{m\geq 1}$ be an increasing sequence of finite-dimensional subspaces of X^* such that $\overline{\bigcup_m Y_m} = X^*$ and $\|\cdot\|_{Y_m} = \|\cdot\|_{X^*|_{Y_m}}$ for each $m \geq 1$. Then for each fixed $m \geq 1$ there exists a linear operator $P_m : X \to Y_m^*$ of norm 1 defined as follows: $\langle P_m x, y\rangle = \langle x, y\rangle$ for each $x \in X, y \in Y_m$. Therefore $P_m M$ and $P_m N$ are Y_m^*-valued martingales. Moreover, (4.3) holds for $P_m M$ and $P_m N$ since there exists $P_m^* : Y_m \to X^*$, and for each $y \in Y_m$ we have that $\langle P_m M, y\rangle = \langle M, P_m y\rangle$ and $\langle P_m N, y\rangle = \langle N, P_m y\rangle$. Since Y_m is a closed subspace of X^*, [19, Proposition 4.2.17] yields $\beta_{p',Y_m} \leq \beta_{p',X^*}$, consequently again by [19, Proposition 4.2.17] $\beta_{p,Y_m^*} \leq \beta_{p,X^{**}} = \beta_{p,X}$. So if we prove the finite dimensional version, then

$$(\mathbb{E}\|P_m N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,Y_m^*}^2 (\mathbb{E}\|P_m M_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\mathbb{E}\|P_m M_t\|^p)^{\frac{1}{p}},$$

and (4.4) with $c_{p,X} = \beta_{p,X}^2$ will follow by letting $m \to \infty$.

Let d be the dimension of X, $|||\cdot|||$ be a Euclidean norm on $X \times X$. Let $L = (M, N) : \mathbb{R}_+ \times \Omega \to X \times X$ be a continuous martingale. Since $(X \times X, |||\cdot|||)$ is a Hilbert space, L has a continuous quadratic variation $[L] : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ (see Remark 2.5). Let $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be such that $A_s = [L]_s + s$ for each $s \geq 0$. Then A is continuous strictly increasing predictable. Define a random time-change $(\tau_s)_{s\geq 0}$ as in Theorem 2.16. Let $\mathbb{G} = (\mathcal{G}_s)_{s\geq 0} = (\mathcal{F}_{\tau_s})_{s\geq 0}$ be the induced filtration. Then thanks to the Kazamaki theorem [23, Theorem 17.24] $\widetilde{L} = L \circ \tau$ is a G-martingale, and $[\widetilde{L}] = [L] \circ \tau$. Notice that $\widetilde{L} = (\widetilde{M}, \widetilde{N})$ with $\widetilde{M} = M \circ \tau$, $\widetilde{N} = N \circ \tau$, and since by Kazamaki theorem [23, Theorem 17.24] $[M \circ \tau] = [M] \circ \tau$, $[N \circ \tau] = [N] \circ \tau$, and $(M \circ \tau)_0 = (N \circ \tau)_0 = 0$, we have that by (4.3) for each $x^* \in X^*$ a.s. for each $s \geq 0$

$$[\langle \widetilde{N}, x^*\rangle]_s = [\langle N, x^*\rangle]_{\tau_s} \leq [\langle M, x^*\rangle]_{\tau_s} = [\langle \widetilde{M}, x^*\rangle]_s \tag{4.5}$$

Moreover, for all $0 \leq u < s$ we have that a.s.

$$[\widetilde{L}]_s - [\widetilde{L}]_u = ([L] \circ \tau)_s - ([L] \circ \tau)_u \leq ([L] \circ \tau)_s + \tau_s - ([L] \circ \tau)_u - \tau_u$$

$$= ([L]_{\tau_s} + \tau_s) - ([L]_{\tau_u} + \tau_u) = s - u.$$

Therefore $[\widetilde{L}]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. Consequently, due to Theorem 2.19, there exists an enlarged probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ with an enlarged filtration $\widetilde{\mathbb{G}} = (\widetilde{\mathcal{G}}_s)_{s\geq 0}$, a $2d$-dimensional standard Wiener process W, which is defined on $\widetilde{\mathbb{G}}$, and a stochastically integrable progressively measurable function $f : \mathbb{R}_+ \times \widetilde{\Omega} \to \mathcal{L}(\mathbb{R}^{2d}, X \times X)$ such that $\widetilde{L} = f \cdot W$. Let $f^M, f^N : \mathbb{R}_+ \times \Omega \to \mathcal{L}(\mathbb{R}^{2d}, X)$ be such that $f = (f^M, f^N)$. Then $\widetilde{M} = f^M \cdot W$ and $\widetilde{N} = f^N \cdot W$. Let $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ be an independent probability space with a filtration $\overline{\mathbb{G}}$ and a $2d$-dimensional Wiener process $\overline{W}$ on it. Denote by $\overline{\mathbb{E}}$ the expectation on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$. Then because of the decoupling theorem [19, Theorem 4.4.1], for each $s \geq 0$

$$\begin{aligned} (\mathbb{E}\|\widetilde{N}_s\|^p)^{\frac{1}{p}} &= (\mathbb{E}\|(f^N \cdot W)_s\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\,\overline{\mathbb{E}}\|(f^N \cdot \overline{W})_s\|^p)^{\frac{1}{p}}, \\ \frac{1}{\beta_{p,X}}(\mathbb{E}\,\overline{\mathbb{E}}\|(f^M \cdot \overline{W})_s\|^p)^{\frac{1}{p}} &\leq (\mathbb{E}\|(f^M \cdot W)_s\|^p)^{\frac{1}{p}} = (\mathbb{E}\|\widetilde{M}_s\|^p)^{\frac{1}{p}}. \end{aligned} \tag{4.6}$$

Due to the multidimensional version of [23, Theorem 17.11] and (4.5) for each $x^* \in X^*$ we have that

$$s \mapsto [\langle \widetilde{M}, x^*\rangle]_s - [\langle \widetilde{N}, x^*\rangle]_s = \int_0^s (|\langle x^*, f^M(r)\rangle|^2 - |\langle x^*, f^N(r)\rangle|^2)\,\mathrm{d}r \tag{4.7}$$

is nonnegative and absolutely continuous a.s. Since X is separable, we can fix a set $\widetilde{\Omega}_0 \subset \widetilde{\Omega}$ of full measure on which the function (4.7) is nonnegative for each $s \geq 0$.

Now fix $\omega \in \widetilde{\Omega}_0$ and $s \geq 0$. Let us prove that

$$\overline{\mathbb{E}}\|(f^N(\omega) \cdot \overline{W})_s\|^p \leq \overline{\mathbb{E}}\|(f^M(\omega) \cdot \overline{W})_s\|^p.$$

Since $f^M(\omega)$ and $f^N(\omega)$ are deterministic on $\overline{\Omega}$, and since due to (4.7) for each $x^* \in X^*$

$$\begin{aligned} \overline{\mathbb{E}}|\langle (f^N(\omega) \cdot \overline{W})_s, x^*\rangle|^2 &= \int_0^s |\langle x^*, f^N(r,\omega)\rangle|^2\,\mathrm{d}r \\ &\leq \int_0^s |\langle x^*, f^M(r,\omega)\rangle|^2\,\mathrm{d}r = \overline{\mathbb{E}}|\langle (f^M(\omega) \cdot \overline{W})_s, x^*\rangle|^2, \end{aligned}$$

by [31, Corollary 4.4] we have that $\overline{\mathbb{E}}\|(f^N(\omega)\cdot\overline{W})_s\|^p \leq \overline{\mathbb{E}}\|(f^M(\omega)\cdot\overline{W})_s\|^p$. Consequently, due to (4.6) and the fact that $\widetilde{\mathbb{P}}(\Omega_0) = 1$

$$(\mathbb{E}\|\widetilde{N}_s\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\,\overline{\mathbb{E}}\|(f^N \cdot \overline{W})_s\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\,\overline{\mathbb{E}}\|(f^M \cdot \overline{W})_s\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2(\mathbb{E}\|\widetilde{M}_s\|^p)^{\frac{1}{p}}.$$

Recall that $\widetilde{M}$ and $\widetilde{N}$ are bounded, so thanks to the dominated convergence theorem one gets (4.4) with $c_{p,X} = \beta_{p,X}^2$ by letting s to infinity. □

Proof of Theorem 4.4. *The "only if" part & the upper bound of $c_{p,X}$:* The "only if" part and the estimate $c_{p,X} \leq \beta_{p,X}^2$ follows from Proposition 4.5 since (4.3) holds for M and N because N is weakly differentially subordinated to M.

The "if" part & the lower bound of $c_{p,X}$: See the supplement.

□

Remark 4.6. *Let X be a Banach space. Then according to [6, 8, 17] the Hilbert transform $\mathcal{H}_X$ can be extended to $L^p(\mathbb{R};X)$ for each $1<p<\infty$ if and only if X is a UMD Banach space. Moreover, if this is the case, then*

$$\sqrt{\beta_{p,X}} \leq \|\mathcal{H}_X\|_{\mathcal{L}(L^p(\mathbb{R};X))} \leq \beta_{p,X}^2.$$

As it was shown in [41], the upper bound $\beta_{p,X}^2$ can be also directly derived from the upper bound for $c_{p,X}$ in Theorem 4.4. The sharp upper bound for $\|\mathcal{H}_X\|_{\mathcal{L}(L^p(\mathbb{R};X))}$ remains an open question (see [19, pp. 496-497]), so the sharp upper bound for $c_{p,X}$ is of interest.

Lemma* 4.7. *Let X be a Banach space, $M^c, N^c : \mathbb{R}_+ \times \Omega \to X$ be continuous martingales, $M^d, N^d : \mathbb{R}_+ \times \Omega \to X$ be purely discontinuous martingales, $M_0^c = N_0^c = 0$. Let $M := M^c + M^d$, $N := N^c + N^d$. Suppose that N is weakly differentially subordinated to M. Then N^c is weakly differentially subordinated to M^c, and N^d is weakly differentially subordinated to M^d.*

Proof of Theorem 4.3. By Theorem 3.1 there exist martingales M^d, M^c, N^d, N^c : $\mathbb{R}_+ \times \Omega \to X$ such that M^d and N^d are purely discontinuous, M^c and N^c are continuous, $M_0^c = N_0^c = 0$, and $M = M^d + M^c$ and $N = N^d + N^c$. By Lemma 4.7, N^d is weakly differentially subordinated to M^d and N^c is weakly differentially subordinated to M^c. Therefore for each $t \geq 0$

$$\begin{aligned}(\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \overset{(i)}{\leq} (\mathbb{E}\|N_t^d\|^p)^{\frac{1}{p}} + (\mathbb{E}\|N_t^c\|^p)^{\frac{1}{p}} &\overset{(ii)}{\leq} \beta_{p,X}^2 (\mathbb{E}\|M_t^d\|^p)^{\frac{1}{p}} + \beta_{p,X} (\mathbb{E}\|M_t^c\|^p)^{\frac{1}{p}}\\ &\overset{(iii)}{\leq} \beta_{p,X}^2(\beta_{p,X}+1)(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}},\end{aligned}$$

where (i) holds thanks to the triangle inequality, (ii) follows from Theorem 4.2 and Theorem 4.4, and (iii) follows from (3.1). □

Remark 4.8. *It is worth noticing that in a view of recent results the sharp constant in* (3.1) *and* (3.19) *can be derived and equals the* $\mathrm{UMD}_p^{\{0,1\}}$ constant $\beta_{p,X}^{\{0,1\}}$*. In order to show that this is the right upper bound one needs to use a* $\{0,1\}$-Burkholder function *instead of the Burkholder function, while the sharpness follows analogously Theorem 3.11 and 3.32. See [40] for details.*

Remark 4.9. *In the recent paper [42] the existence of the canonical decomposition of a general local martingale together with the corresponding weak L^1-estimates were shown. Again existence of the canonical decomposition of any X-valued martingale is equivalent to X having the UMD property.*

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Supplementary Material

INSERT A TITLE FOR THE SUPPLEMENT
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