Stability for gains from large investors' strategies in $\mathrm{M}_1/\mathrm{J}_1$ topologies

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We prove continuity of a controlled SDE solution in Skorokhod's M_1 and J_1 topologies and also uniformly, in probability, as a non-linear functional of the control strategy. The functional comes from a finance problem to model price impact of a large investor in an illiquid market. We show that M_1-continuity is the key to ensure that proceeds and wealth processes from (self-financing) càdlàg trading strategies are determined as the continuous extensions for those from continuous strategies. We demonstrate by examples how continuity properties are useful to solve different stochastic control problems on optimal liquidation and to identify asymptotically realizable proceeds.

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1. Introduction

A classical theme in the theory of stochastic differential equations is how stably the solution process behaves, as a functional of its integrand and integrator processes, see e.g. [KP96] and [Pro04, Chapter V.4]. A typical question is how to extend such a functional sensibly to a larger class of input processes. Continuity is a key property to address such problems, cf. e.g. in [Mar81] for his canonical extension of Stratonovich SDEs.

In singular control problems for instance, the non-linear objective functional may initially be only defined for finite variation or even absolutely continuous control strategies.

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Existence of an optimizer might require a continuous extension of the functional to a more general class of controls, e.g. semimartingale controls for the problem of hedging. Herein the question of which topology to embrace arises, and this depends on the problem at hand, see e.g. [Kar13] for an example of utility maximization in a frictionless financial market where the Emery topology turns out to be useful for the existence of an optimal wealth process. For our application we need suitable topologies on the Skorokhod space of càdlàg functions. The two most common choices here are the uniform topology and Skorokhod J_1 topology; they share the property that a jump in a limiting process can only be approximated by jumps of comparable size at the same time or, respectively, at nearby times. But this can be overly restrictive for such applications, as we have in mind, where a jump may be approximated sensibly by many small jumps in fast succession or by continuous processes such as Wong-Zakai-type approximations. The M_1 topology by Skorokhod [Sko56] captures such approximations of *unmatched jumps*. We will take this as a starting point to identify the relevant non-linear objective functional for càdlàg controls as a continuous extension from (absolutely) continuous controls. See [Whi02] for a profound survey on the M_1 topology.

We demonstrate how the old subject of stability of SDEs with jumps, when considered with respect to the M_1 topology, has applications for recent problems in mathematical finance. Our application context is that of an illiquid financial market for trading a single risky asset. A large investor's trading causes transient price impact on some exogenously given fundamental price which would prevail in a frictionless market. Such could be seen as a non-linear (non-proportional) transaction cost with intertemporal impact also on subsequent prices. Our framework is quite general; it accommodates for instance models of additive impact that are most common in the literature on transient impact, see Example 2.1. An original aspect of our framework is that it also permits for multiplicative impact which appears to fit better to multiplicative price evolutions as e.g. in models of Black-Scholes type, cf. [BBF17a, Example 5.4]; In comparison, it moreover ensures positivity of asset prices, which is desirable from a theoretical point of view, relevant for applications whose time horizon is not short (as they can occur e.g. for large institutional trades [CL95, KMS17], or for hedging problems with longer maturities).

Having specified the evolution for an endogenous price process at which trading (of infinitesimal quantities) would occur, one still has, even for a simple block trade, to define the variations in the bank account by which the trades in risky asset are financed, i.e. the so-called self-financing condition. The large trader's feedback effect on prices causes the proceeds (negative expenses) to be a non-linear functional of her control strategy for dynamic trading in risky assets. Choosing a seemingly sensible, but ad-hoc, definition could lead to surprising and undesirable consequences, in that the large investor can evade her liquidity costs entirely by using continuous finite variation strategies to approximate her target control strategy, cf. Example 3.2. Optimal trade execution proceeds or superreplication prices may be only approximately attainable in such models. Indeed, the analysis in [BB04, ÇJP04] shows that approximations by continuous strategies of finite variation play a particular role. This is, of course, a familar theme in stochastic analysis, at least since [WZ65]. However, in the models in [BB04, ÇJP04] the aforementioned strategies have zero liquidity costs, permitting the large trader to avoid those costs entirely by simply approximating more general strategies. This appears not desirable from an application point of view, and it seems also mathematically inconvenient to distinguish between proceeds and asymptotically realizable proceeds. To settle this issue, a stability analysis for proceeds for a class of price impact models should address in particular the M_1 topology, in which continuous finite variation strategies are dense in the space of càdlàg strategies (in contrast to the uniform or J_1 topologies), see Remark 3.5.

We contribute a systematic study on stability of the proceeds functional. Starting with an unambiguous definition (2.4) for continuous finite-variation strategies, we identify the approximately realizable gains for a large set of controls. A mathematical challenge for stability of the stochastic integral functional is that both the integrand and the integrator depend on the control strategy. Our main Theorem 3.7 shows continuity of this non-linear controlled functional in the uniform, J_1 and M_1 topologies, in probability, on the space of (predictable) semimartingale or càdlàg strategies which are bounded in probability. A particular consequence is a Wong-Zakai approximation result, that could alternatively be shown by adapting results from [KPP95] on the Marcus canonical equation to our setup, cf. Section 3.3. Another direct implication of M_1 continuity is that proceeds of general (optimal) strategies can be approximated by those of simple strategies with only small jumps. Whereas the former property is typical for common stochastic integrals, it is far from obvious for our non-linear controlled SDE functional (3.9).

The topic of stability for the stochastic process of proceeds from dynamically trading risky assets in illiquid markets, where the dynamics of the wealth and of the proceeds for a large trader are non-linear in her strategies because of her market impact, is showing up at several places in the literature. But the mathematical topic appears to have been touched mostly in-passing so far. The focus of few notable investigations has been on the application context and on different topologies, see e.g. [RS13, Prop. 6.2] for uniform convergence in probability (ucp). In [LS13, Lem. 2.5] a cost functional is extended from simple strategies to semimartingales via convergence in ucp. [Roc11, Def. 2.1] and [ÇJP04, Sect. A.2] use particular choices of approximating sequences to extend their definition of self-financing trading strategies from simple processes to semimartingales by limits in ucp. Trading gains of semimartingale strategies are defined in [BLZ16, Prop. 1.1–1.2] as L^2-limits of gains from simple trading strategies via rebalancing at discrete times and large order split. In contrast, we contribute a study of M_1-, J_1- and ucp-stability for general approximations of càdlàg strategies in a class of price impact models with transient impact (2.3), driven by quasi-left continuous martingales (2.1).

As a further contribution, and also to demonstrate the relevance and scope of the theoretical results, we discuss in the case of multiplicative impact a variety of examples where continuity properties play a role. In Section 5.1 we establish existence of an optimal monotone liquidation strategy in finite time horizon using relative compactness and continuity of the proceeds functional in M_1. Section 5.2 shows how to solve the optimal liquidation problem in infinite time horizon with non-negative bounded semimartingale strategies by approximating their proceeds via bounded variation strategies, here the M_1-stability being needed. Section 5.3 solves the liquidation problem for an original extension of the model with a stochastic impact process, with time horizon being bounded in expectation. This relies on M_1 convergence to define the trading proceeds. It provides an example of a liquidation problem where the optimum of singular controls is not attained in a class of finite variation strategies, but a suitable extension to semimartingale strategies is needed. Section 5.4 incorporates partially instantaneous recovery of price impact to our model. Herein, the M_1 topology plays the key role to identify (asymptotically realizable) proceeds as a continuous functional. Last but not least, Section 4 proves absence of arbitrage for the large trader within a fairly large class of trading strategies.

The paper is organized as follows. Section 2 sets the model and defines the proceeds functional for finite variation strategies. In Section 3 we extend this definition to a more general set of strategies and prove our main Theorem 3.7. In the remaining Sections 4 and 5 we concentrate on the case of multiplicative impact. We show absence of arbitrage opportunities for the large investor in Section 4 as a basis for a sensible financial model. The examples related to optimal liquidation are investigated in Section 5.

2. A model for transient multiplicative price impact

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is assumed to satisfy the usual conditions of right-continuity and completeness, with $\mathcal{F}_0$ being the trivial σ-field. Paths of semimartingales are taken to be càdlàg. Let also $\mathcal{F}_{0-}$ denote the trivial σ-field. We consider a market with a single risky asset and a riskless asset (bank account) whose price is constant at 1. Without activity of large traders, the unaffected (discounted) price process of the risky asset would evolve according to the stochastic differential equation

$$\mathrm{d}\overline{S}_t = \overline{S}_{t-}(\xi_t \,\mathrm{d}\langle M\rangle_t + \mathrm{d}M_t)\,, \qquad \overline{S}_0 > 0, \tag{2.1}$$

where M is a locally square-integrable martingale that is quasi-left continuous (i.e. for any finite predictable stopping time τ, $\Delta M_\tau := M_\tau - M_{\tau-} = 0$ a.s.) with $\Delta M > -1$ and ξ is a predictable and bounded process. In particular, the predictable quadratic variation process $\langle M\rangle$ is continuous [JS03, Thm. I.4.2], and the unaffected (fundamental) price process $\overline{S} > 0$ can have jumps. We moreover assume that $\langle M\rangle = \int_0^\cdot \alpha_s \,\mathrm{d}s$ with a (locally) bounded density α, and that the martingale part of $\overline{S}$ is square integrable on compact time intervals. The assumptions on M are satisfied e.g. for $M = \int \sigma \,\mathrm{d}W$, where W is a Brownian motion and σ is a predictable stochastic volatility process that is bounded, or for Lévy processes M satisfying some integrability and lower bound on jumps.

To model the impact that trading strategies by a single large trader have on the risky asset price, let us denote by $(\Theta_t)_{t\geq 0}$ her risky asset holdings throughout time and Θ_{0-} be the number of shares the she holds initially. The process Θ is the control strategy of the large investor who executes $\mathrm{d}\Theta_t$ market orders at time t (buy orders if Θ is increasing, sell orders if it is decreasing). We will assume throughout that strategies Θ are predictable processes. The large trader is faced with illiquidity costs because her trading has an adverse impact on the prices at which her orders are executed as follows. A *market impact process* Y (called volume effect process in [PSS11]) captures the impact from a predictable strategy Θ with càdlàg paths on the price of the risky asset, and is defined as the càdlàg adapted solution Y to

$$dY_t = -h(Y_t)\,d\langle M\rangle_t + d\Theta_t \tag{2.2}$$

for some initial condition $Y_{0-} \in \mathbb{R}$. We assume that $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz with $h(0) = 0$ and $h(y)\,\mathrm{sgn}(y) \geq 0$ for all $y \in \mathbb{R}$. The Lipschitz assumption on h guarantees existence and uniqueness of Y in a pathwise sense, see [PTW07, proof of Thm. 4.1] and Proposition A.1 below. The sign assumption on h gives *transience* of the impact which recovers towards 0 (if $h(y) \neq 0$ for $y \neq 0$) when the large trader is inactive. The function h gives the speed of resilience at any level of Y_t and we will refer to it as *the resilience function*. For example, when $h(y) = \beta y$ for some constant $\beta > 0$, the market recovers at exponential rate (as in [OW13, AFS10, Løk14]). Note that we also allow for $h \equiv 0$ in which case the impact is permanent as in [BB04]. Clearly, the process Y depends on Θ, and sometimes we will indicate this dependence as a superscript $Y = Y^\Theta$. Some of the results in this paper could be extended with no additional work when considering additional noise in the market impact process, see the discussion in Section 5.3.

The actual price S observed in the market at which (continuous, cf. (2.4) below) trading occurs, is affected by the large trader's strategy Θ via (2.2) according to

$$S_t := g(\overline{S}_t, Y_t) \tag{2.3}$$

where the *price impact function* $(x, y) \mapsto g(x, y) \in C^{2,1}(\mathbb{R}^2)$ is such that g_{xx} is locally Lipschitz in y, meaning that on every compact interval $I \subset \mathbb{R}$ there exists $K > 0$ such that $|g_{xx}(x, y) - g_{xx}(x, z)| \leq K|y - z|$ for all $x, y, z \in I$. Moreover, we assume $g(x, y)$ to be non-decreasing in y. Thus, selling (buying) by the large trader has a decreasing (increasing) impact on the price S, which is transient due to (2.2).

Example 2.1. [BB04] consider a family of semimartingales $(S^\theta)_{\theta\in\mathbb{R}}$ being parametrized by the large trader's risky asset position θ. In our setup, this corresponds to general price impact function g and $h \equiv 0$, meaning that impact is permanent. A common example in the literature on transient market impact is additive price impact where $g(\overline{S}, Y) = \overline{S} + f(Y)$. For instance, [OW13] have $f(y) = \lambda y$, motivated from a block-shaped limit order book. For later generalizations to non-linear increasing $f : \mathbb{R} \to [0, \infty)$, see [AFS10, PSS11].

Another example, which has the conceptual advantage that it guarantees non-negative asset prices, is multiplicative price impact $g(\overline{S}, Y) = f(Y)\overline{S}$, with f being strictly positive and non-decreasing. The function f in this case can again be interpreted as resulting from the shape of a limit order book, see [BBF17a, Sect. 2.1]. This model will be our prime example in Sections 4 and 5. To fit our assumptions on g, we require $f \in C^1$.

While impact and resilience are given by general non-parametric functions, note that these are static. Considering such a model as a low (rather than high) frequency model, we do consider approximations by continuous and finite variation strategies to be relevant. To start, let Θ be a continuous process of finite variation (f.v., being adapted). Then, the cumulative proceeds (negative expenses), denoted by $L(\Theta)$, that are the variations in the bank account to finance buying and selling of the risky asset according to the strategy, can be defined (pathwise) in an unambiguous way. Indeed, proceeds over period $[0, T]$ from a strategy Θ that is continuous should be (justified also by Lemma 3.1)

$$L_T(\Theta) := -\int_0^T S_u \,\mathrm{d}\Theta_u = -\int_0^T g(\overline{S}_u, Y_u) \,\mathrm{d}\Theta_u. \tag{2.4}$$

Our main task is to extend by stability arguments the model from continuous to more general trading strategies, in particular such involving block trades and even more general ones with càdlàg paths, assuming transient price impact but no further frictions, like e.g. bid-ask spread (cf. Remark 4.4). To this end, we will adopt the following point of view: approximately similar trading behavior should yield similar proceeds. The next section will make precise what we mean by "similar" by considering different topologies on the càdlàg path space. It turns out that the natural extension of the functional L from the space of continuous f.v. paths to the space of càdlàg f.v. paths which makes the functional L continuous in all of the considered topologies is as follows: for discontinuous trading we take the proceeds from a block market buy or sell order of size $|\Delta\Theta_\tau|$, executed immediately at a predictable stopping time $\tau < \infty$, to be given by

$$-\int_0^{\Delta\Theta_\tau} g(\overline{S}_{\tau-}, Y_{\tau-} + x) \,\mathrm{d}x, \tag{2.5}$$

and so the proceeds up to T from a f.v. strategy Θ with continuous part Θ^c are

$$L_T(\Theta) := -\int_0^T g(\overline{S}_u, Y_u) \,\mathrm{d}\Theta_u - \sum_{\substack{\Delta\Theta_t \neq 0 \\ 0 \le t \le T}} \int_0^{\Delta\Theta_t} g(\overline{S}_{t-}, Y_{t-} + x) \,\mathrm{d}x. \tag{2.6}$$

Note that a block sell order means that $\Delta\Theta_t < 0$, so the average price per share for this trade satisfies $S_t \le -\frac{1}{\Delta\Theta_t} \int_0^{\Delta\Theta_t} g(\overline{S}_t, Y_{t-} + x) \,\mathrm{d}x \le S_{t-}$. Similarly, the average price per share for a block buy order, $\Delta\Theta_t > 0$, is between S_{t-} and S_t. The expression in (2.5) could be justified from a limit order book perspective for some cases of g, as noted in Example 2.1. But we will derive it in the next section using stability considerations.

Remark 2.2. The aim to define a model for trading under price impact for general strategies is justified by applications in finance, which encompass trade execution, utility optimization and hedging. While also e.g. [BB04, BR17, ÇJP04] define proceeds for semimartingale strategies, their definitions are not continuous with respect to the M_1 topology, in contrast to Theorem 3.7. Another difference to [BB04, BR17] is that our presentation is not going to rely on non-linear stochastic integration theory due to Kunita or, respectively, Carmona and Nualart.

3. Continuity of the proceeds in various topologies

In this section we will discuss questions about continuity of the proceeds process $\Theta \mapsto L_\cdot(\Theta)$ with respect to various topologies: the ucp topology and the Skorokhod J_1

and (in particular) M_1 topologies. Each one captures different stability features, the suitability of which may vary with application context.

Let us observe that for a continuous bounded variation trading strategy Θ the proceeds from trading should be given by (2.4). To this end, let us make just the assumption that

$$\begin{gathered}\text{a block order of a size } \Delta \text{ at some (predictable) time } t \text{ is executed at some}\\ \text{average price per share which is between } S_{t-} = g(\overline{S}_t, Y_{t-}) \text{ and } g(\overline{S}_t, Y_{t-} + c\Delta)\end{gathered} \tag{3.1}$$

for some constant $c \geq 0$. The assumption looks natural for $c = 1$ where $Y_t = Y_{t-} + c\Delta$, stating that a block trade is executed at an average price per share that is somewhere between the asset prices observed immediately before and after the execution. The more general case $c \geq 0$ is just technical at this stage but will be needed in Section 5.4. Assumption (3.1) means that proceeds by a simple strategy as in (3.3) are

$$L_t(\Theta^n) = -\sum_{k:\ t_k \leq t} \xi_k(\Theta_{t_k} - \Theta_{t_{k-1}}) \tag{3.2}$$

for some random variable ξ_k between $g(\overline{S}_{t_k}, Y^{\Theta^n}_{t_k-})$ and $g(\overline{S}_{t_k}, Y^{\Theta^n}_{t_k-} + c\Delta Y^{\Theta^n}_t)$. Note that at this point we have not specified the proceeds (negative expenses) from block trades, but we only assume that they satisfy some natural bounds. Yet, this is indeed already sufficient to derive the functional (2.4) for continuous strategies as a limit of simple ones.

Lemma 3.1. *For $T > 0$, approximate a continuous f.v. process $(\Theta_t)_{t\in[0,T]}$ by a sequence $(\Theta^n_t)_{t\in[0,T]}$ of simple trading strategies given as follows: For a sequence of partitions $\{0 = t_0 < t_1 < \cdots < t_{m_n} = T\}$, $n \in \mathbb{N}$, with $\sup_{1\leq k\leq m_n}|t_k - t_{k-1}| \to 0$ for $n \to \infty$, let*

$$\Theta^n_t := \Theta_0 + \sum_{k=1}^{n}\big(\Theta_{t_k} - \Theta_{t_{k-1}}\big)\mathbb{1}_{[t_k,T]}(t)\,, \quad t \in [0, T]. \tag{3.3}$$

Assume (3.1) *holds for some $c \geq 0$. Then $\sup_{0\leq t\leq T}|L_t(\Theta^n) + \int_0^t S_u \,\mathrm{d}\Theta_u| \xrightarrow{n\to\infty} 0$ a.s.*

Proof. Note that $\sup_{u\in[0,T]}|\Theta^n_u - \Theta_u| \to 0$ as $n \to \infty$. The solution map $\Theta \mapsto Y^\Theta$ is continuous with respect to the uniform norm, see Proposition A.1. Therefore,

$$\sup_{u\in[0,T]} |Y^{\Theta^n}_u - Y^\Theta_u| \to 0 \quad \text{a.s. for } n \to \infty. \tag{3.4}$$

Note that for $\Delta\Theta_{t_k} := \Theta_{t_k} - \Theta_{t_{k-1}}$ and ξ_k between $g(\overline{S}_{t_k}, Y^{\Theta^n}_{t_k-})$ and $g(\overline{S}_{t_k}, Y^{\Theta^n}_{t_k-} + c\Delta\Theta_{t_k})$ and $Y := Y^\Theta$ we have

$$\begin{aligned}|\xi_k - g(\overline{S}_{t_k}, Y_{t_k})| &\leq L_g(\overline{S}_{t_k}, \omega)\max\big\{\big|Y_{t_k} - Y^{\Theta^n}_{t_k-} - c\Delta\Theta_{t_k}\big|, \big|Y_{t_k} - Y^{\Theta^n}_{t_k-}\big|\big\}\\ &\leq \tilde{c}L_g(\overline{S}_{t_k}, \omega)\big(\big|Y_{t_k} - Y^{\Theta^n}_{t_k}\big| + |\Delta\Theta_{t_k}|\big),\end{aligned}$$

where $\tilde{c} > 0$ is a universal constant, $L_g(x, \omega)$ denotes the Lipschitz constant of $y \mapsto g(x, y)$ on a compact set, depending on the (bounded) realizations for $\omega \in \Omega$ of Y^Θ and Y^{Θ^n}, $n \in \mathbb{N}$, on the interval $[0, T]$; such a compact set exits since Θ is continuous and

$\sup_{u\in[0,T]}\left|Y_u^{\Theta} - Y_u^{\Theta^n}\right|$ can be bounded by a factor times the uniform distance between Θ and Θ^n on $[0,T]$, cf. [PTW07, proof of Thm. 4.1]. Hence,

$$L_t(\Theta^n) = -\sum_{k:\ t_k\le t} g(\overline{S}_{t_k}, Y^{\Theta}_{t_k})\big(\Theta_{t_k} - \Theta_{t_{k-1}}\big) + \mathcal{E}^n_t\,, \tag{3.5}$$

$$\text{where } |\mathcal{E}^n_t| \le \tilde{c}\Big(\sup_{u\in[0,T]} L_g(\overline{S}_u,\omega)\Big)\sum_{k=1}^{n}\big(\big|Y_{t_k} - Y^{\Theta^n}_{t_k}\big| + |\Delta\Theta_{t_k}|\big)|\Delta\Theta_{t_k}| \tag{3.6}$$

$$\le C(\omega)\Big(\sup_{1\le k\le n}\big|Y_{t_k} - Y^{\Theta^n}_{t_k}\big|\Big)|\Theta(\omega)|_{\mathrm{TV}} + C(\omega)\sum_{k=1}^{n}|\Delta\Theta_{t_k}|^2 \tag{3.7}$$

$$\to 0 \quad \text{a.s. for } n\to\infty \text{ (uniformly in } t), \tag{3.8}$$

thanks to (3.4) and the fact that Θ has continuous paths of finite variation. The claim follows since by dominated convergence the Riemann-sum process in (3.5) converges a.s. to the Stieltjes-integral process $-\int_0^{\cdot} S_u\,\mathrm{d}\Theta_u$ uniformly on $[0,T]$. □

Example 3.2 (Continuity issues for an alternative "ad-hoc" definition of proceeds)**.** Consider the problem of optimally liquidating $\Theta_{0-}=1$ risky asset in time $[0,T]$ while maximizing expected proceeds. In view of assumption (3.1), an alternative but possibly "ad-hoc" definition for proceeds $\tilde{L}_T$ of simple strategies could be to consider just some price for each block trade, similarly to [BB04, Section 3] or [HH11, Example 2.4]. For multiplicative impact $g(\overline{S},Y) = \overline{S}f(Y)$, taking e.g. the price directly after the impact would yield for simple strategies Θ^n that trade at times $\{0 = t^n_0 < t^n_1 < \cdots < t^n_n = T\}$ the proceeds $\tilde{L}_T(\Theta^n) = -\sum_{k=0}^{n}\overline{S}_{t^n_k} f(Y^{\Theta^n}_{t^n_k})\Delta\Theta^n_{t^n_k}$. The family $(\Theta^n)_n$ of strategies which liquidate an initial position of size 1 until time $1/n$ in n equidistant blocks of uniform size is given by $\Theta^n_t := \sum_{k=1}^{n}\frac{n-k+1}{n}\mathbb{1}_{[\frac{k-1}{n^2},\frac{k}{n^2})}(t)$. With unaffected price $\overline{S}_t = e^{-\delta t}\widetilde{M}_t$ for a continuous martingale $\widetilde{M}$, and permanent impact ($h\equiv 0$), i.e. $Y_t = \Theta_t - 1$, this yields $\mathbb{E}[\tilde{L}_T(\Theta^n)] \to \int_0^1 f(-y)\,\mathrm{d}y$ for $n\to\infty$. Given $\delta\ge 0$, for any non-increasing simple strategy $\Theta = \sum_{k=1}^{n}\Theta_{\tau_k}\mathbb{1}_{[\![\tau_{k-1},\tau_k[\![}$ with $\Theta_{0-}=1$ holds that $\mathbb{E}[\tilde{L}(\Theta)] \le \int_0^1 f(-y)\,\mathrm{d}y$ with strict inequality for $\delta>0$. So the control sequence (Θ^n) is only asymptotically optimal among all simple monotone liquidation strategies.

Remark 3.3. Note that Example 3.2 is a toy example, since for permanent impact the optimal strategy (considering asymptotically realizable proceeds) is trivial and in case $\delta = 0$ any strategy is optimal, cf. [GZ15, Prop. 3.5(III) and the comment preceding it]. Nevertheless, this example shows that the object of interest are *asymptotically realizable* proceeds, an insight due to [BB04]. For analysis, it thus appears convenient and sensible not to make a formal distinction of (sub-optimal) realizable and asymptotically realizable proceeds, but to consider the latter and interpret strategies accordingly. Investigating asymptotically realizable proceeds can help to answer questions on modeling issues, e.g. whether the large investor could sidestep liquidity costs entirely and in effect act as a small investor, cf. [BB04, ÇJP04]. One could impose, like [ÇST10], additional constraints on strategies to avoid such issues; But in such tweaked models one could not investigate the effects from some given illiquidity friction alone, in isolation from other constraints, because results from an analysis will be consequences of the combination of both frictions.

By using integration-by-parts, we can obtain the following alternative representation of the functional in (2.4) for continuous f.v. strategies:

$$\begin{aligned}L(\Theta) = &\int_0^{\cdot} G_x(\overline{S}_{u-}, Y^{\Theta}_{u-})\,\mathrm{d}\overline{S}_u + \int_0^{\cdot} \Big(\tfrac{1}{2} G_{xx}(\overline{S}_u, Y^{\Theta}_u)\overline{S}^2_u - g(\overline{S}_u, Y^{\Theta}_u)h(Y^{\Theta}_u)\Big)\,\mathrm{d}\langle M\rangle_u \\ &- \big(G(\overline{S}_{\cdot}, Y^{\Theta}_{\cdot}) - G(\overline{S}_0, Y^{\Theta}_{0-})\big) \\ &+ \sum_{\substack{\Delta\overline{S}_u\neq 0\\ 0\le u\le \cdot}} \big(G(\overline{S}_u, Y^{\Theta}_u) - G(\overline{S}_{u-}, Y^{\Theta}_u) - G_x(\overline{S}_{u-}, Y^{\Theta}_u)\Delta\overline{S}_u\big)\,,\end{aligned} \tag{3.9}$$

where $G(x,y) := \int_c^y g(x,z)\,\mathrm{d}z$ for constant c, and using that $\overline{S}$ and Y have no common jumps. The advantage of this representation is that the right-hand side of (3.9) makes sense for any predictable process Θ with càdlàg paths in contrast to the term in (2.4) This form of the proceeds will turn out to be helpful for the stability analysis. We will show that the right-hand side in (3.9) is continuous in the control Θ when the path-space of Θ, the càdlàg path space, is endowed with various topologies. Hence, it can be used to define the proceeds for general trading strategies by continuity. Next section is going to discuss the topologies that will be of interest.

3.1. The Skorokhod space and its M_1 and J_1 topologies

We are going to derive a continuity result (Theorem 3.7) for the functional L in different topologies on the space $D \equiv D([0,T]) := D([0,T];\mathbb{R})$ of real-valued càdlàg paths on the time interval $[0,T]$. Following the convention by [Sko56], we take each element in $D[0,T]$ to be left-continuous at time T.[1] One could also consider initial and terminal jumps by extending the paths, see Remark 3.6. At this point, let us remark that finite horizon T is not essential for the results below, whose analysis carries over to the time interval $[0,\infty)$ because the topology on $D([0,\infty))$ is induced by the topologies of $D([0,T])$ for $T \geq 0$. More precisely, for the topologies we are interested in, $x_n \to x$ as $n \to \infty$ in $D([0,\infty))$ if $x_n \to x$ in $D([0,t])$ for the restrictions of x_n, x on $[0,t]$, for any t being a continuity point of x, see [Whi02, Sect. 12.9].

Convergence in the uniform topology is rather strong, in that approximating a path with a jump is only possible if the approximating sequence has jumps of comparable size at the same time. If one is interested in stability with respect to slight shift of the execution in time, then a familiar choice that also makes D separable, the Skorokhod J_1 topology, might be appropriate; for comprehensive study, see [Bil99, Ch. 3]. However, also here an approximating sequence for a path with jumps needs jumps of comparable size, if only at nearby times. To capture the occurrence of the so-called *unmatched jumps*, i.e. jumps that appear in the limit of continuous processes, another topology on

[1] This is implicitly assumed also in [Whi02], see the compactness criterion in Thm. 12.12.2 which is borrowed from [Sko56].

D is more appropriate, the Skorokhod M_1 topology. Recall that $x_n \to x$ in (D, d_{M_1}) if $d_{M_1}(x_n, x) \to 0$ as $n \to \infty$, with

$$d_{M_1}(x_n, x) := \inf\big\{ \|u - u_n\| \vee \|r - r_n\| \;\big|\; (u, r) \in \Pi(x), (u_n, r_n) \in \Pi(x_n) \big\}, \quad (3.10)$$

where $\|\cdot\|$ denotes the uniform norm on $[0, 1]$ and $\Pi(x)$ is the set of all *parametric representations* $(u, r) : [0, 1] \to \Gamma(x)$ of the completed graph (with vertical connections at jumps) $\Gamma(x)$ of $x \in D$, see [Whi02, Sect. 3.3]. In essence, two functions $x, y \in D$ are near to each other in M_1 if one could run continuously a particle on each graph $\Gamma(x)$ and $\Gamma(y)$ from the left endpoint toward the right endpoint such that the two particles are nearby in time and space. In particular, it is easy to see that a simple jump path could be approximated in M_1 by a sequence of absolutely continuous paths, in contrast to the uniform and the J_1 topologies. More precisely, we have the following

Proposition 3.4. *Let $x \in D([0, T])$ and consider the Wong-Zakai approximation sequence $(x_n) \subset D([0, T])$ defined by $x_n(t) := n \int_{t-1/n}^{t} x(s)\,\mathrm{d}s$, $t \in [0, T]$. Then*

$$x_n \to x \quad \text{for } n \to \infty, \quad \text{in } (D([0, T]), M_1).$$

Proof. To ease notation, we embed a path x in $D([0, \infty))$ and consider the corresponding approximating sequence for the extended path on $[0, \infty)$. The claim follows by restricting to the domain $[0, T]$, as 0 and T are continuity points of x, cf. [Whi02, Sect. 12.9]. The idea is to construct explicitly parametric representations of $\Gamma(x)$ and $\Gamma(x_n)$ that are close enough. For this purpose, we need to add "fictitious" time to be able to parametrize the segments that connect jump points of x. Indeed, let (a_k) be a fixed convergent series of strictly positive numbers and let $t_1, t_2, \dots$ be the jump times of x ordered such that $|\Delta x(t_1)| \geq |\Delta x(t_2)| \geq \dots$ and $t_k < t_{k+1}$ if $|\Delta x(t_k)| = |\Delta x(t_{k+1})|$. Set $\delta(t) := \sum_k a_k \mathbb{1}_{\{t_k \leq t\}}$, the total "fictitious" time added to parametrize the jumps of x up to time t.

Consider the time-changes $\gamma_n(t) := n \int_{t-1/n}^{t} (\delta(u) + u)\,\mathrm{d}u$ and $\gamma_0(t) := \delta(t) + t$, $t \geq 0$, together with their continuous inverses $\gamma_n^{-1}(s) := \inf\{u > 0 \mid \gamma_n(u) > s\}$ for $s \geq 0$, $n \geq 0$. It is easy to check that we have

$$\gamma_n^{-1}(s) - 1/n < \gamma_0^{-1}(s) < \gamma_n^{-1}(s) < \infty \quad \text{for } s \geq 0, \quad (3.11)$$

because $\gamma_n(t) < \gamma_0(t) < \gamma_n(t + 1/n)$, cf. [KPP95, Lemma 6.1]. Consider the sequence $u_n(s) := x_n(\gamma_n^{-1}(s))$ for $s \geq 0$ and let

$$u(s) := \begin{cases} x(\gamma_0^{-1}(s)) & \text{if } \eta_1(s) = \eta_2(s), \\ x(\gamma_0^{-1}(s)) \cdot \frac{s - \eta_1(s)}{\eta_2(s) - \eta_1(s)} + x(\gamma_0^{-1}(s)-) \cdot \frac{\eta_2(s) - s}{\eta_2(s) - \eta_1(s)} & \text{if } \eta_1(s) \neq \eta_2(s), \end{cases}$$

where $[\eta_1(s), \eta_2(s)]$ is the "fictitious" time added for a jump at time $t = \gamma_0^{-1}(s)$, i.e. $\eta_1(s) := \sup\{\tilde{s} \mid \gamma_0^{-1}(\tilde{s}) < \gamma_0^{-1}(s)\}$ and $\eta_2(s) := \inf\{\tilde{s} \mid \gamma_0^{-1}(\tilde{s}) > \gamma_0^{-1}(s)\}$, as in [KPP95, p. 368]. Then [KPP95, Lemma 6.2] gives $\lim_{n\to\infty} u_n = u$, uniformly on bounded intervals; our setup corresponds to $f \equiv 1$ there, so our u_n, u correspond to $V^{1/n}, V$ there.

Now the claim follows by observing that (u_n, γ_n^{-1}) is a parametric representation of the completed graph of x_n, i.e. $(u_n, \gamma_n^{-1}) \in \Pi(x_n)$, and $(u, \gamma_0^{-1}) \in \Pi(x)$ which are arbitrarily close when n is big. $\square$

Remark 3.5. A direct corollary of Proposition 3.4 is that $D([0,T])$ is the closure of the set of absolutely continuous functions in the Skorokhod M_1 topology, in contrast to the uniform or Skorokhod J_1 topologies where a jump in the limit can only be approximated by jumps of comparable sizes.

Remark 3.6 (Extended paths)**.** To include trading strategies that could additionally have initial and terminal jumps in our analysis, one may embed the paths of such strategies in the slightly larger space $D([-\varepsilon, T+\varepsilon];\mathbb{R})$ for some $\varepsilon > 0$, e.g. $\varepsilon = 1$, by setting $x(s) = x(0-)$ for $s \in [-\varepsilon, 0)$ and $x(s) = x(T+)$ for $s \in (T, T+\varepsilon]$; we will refer to thereby embedded paths as *extended paths*. This extension is relevant when trying to approximate jumps at terminal time by absolutely continuous strategies in a non-anticipative way as e.g. in Proposition 3.4 where it is clear that a bit more time could be required after a jump occurs in order to approximate it. In particular, by considering extended paths the result of Proposition 3.4 holds if one allows for initial and terminal jumps of x, but convergence holds in the extended paths space.

3.2. Main stability results

Our main result is stability of the functional L defined by the right-hand side of (3.9) for processes Θ with càdlàg paths.

Theorem 3.7. *Let a sequence of predictable processes (Θ^n) converge to the predictable process Θ in (D,ρ), in probability, where ρ denotes the uniform topology, the Skorokhod J_1 or M_1 topology, being generated by a suitable metric d. Assume that (Θ^n) is bounded in $L^0(\mathbb{P})$, i.e. there exists $K \in L^0(\mathbb{P})$ such that $\sup_{0\le t\le T}|\Theta^n_t| \le K$ for all n. Then the sequence of processes $L(\Theta^n)$ converges to $L(\Theta)$ in (D,ρ) in probability, i.e.*

$$\mathbb{P}\big[d\big(L(\Theta^n), L(\Theta)\big) \ge \varepsilon\big] \to 0 \qquad \text{for } n \to \infty \text{ and } \varepsilon > 0. \tag{3.12}$$

In particular, there is a subsequence $L(\Theta^{n_k})$ that converges a.s. to $L(\Theta)$ in (D,ρ).

Proof. By considering subsequences, one could assume that the sequence (Θ^n) converges to Θ in (D,ρ) a.s. The idea for the proof is to show that each summand in the definition of L is continuous. But as D endowed with J_1 or M_1 is not a topological vector space, since addition is not continuous in general, further arguments will be required. Addition is continuous (and hence also multiplication) if for instance the summands have no common jumps, see [JS03, Prop. VI.2.2] for J_1 and [Whi02, Cor. 12.7.1] for M_1. In our case however, there are three terms in L that can have common jumps, namely the stochastic integral process $\int_0^\cdot G_x(\overline{S}_{u-}, Y_{u-})\,\mathrm{d}\overline{S}_u$, the sum of jumps $\Sigma := \sum_{u\le\cdot}\big(G(\overline{S}_u, Y_u) - G(\overline{S}_{u-}, Y_u) - G_x(\overline{S}_{u-}, Y_u)\Delta\overline{S}_u\big)$ and the term $-G(\overline{S}, Y)$. At jump times of Θ (i.e. of Y) which are predictable stopping times, $\overline{S}$ does not jump since it is quasi-left continuous. Hence the only common jump times can be jumps times of $\overline{S}$ which are totally inaccessible. If $\Delta\overline{S}_\tau \ne 0$, then $\Delta\big(\int_0^\cdot G_x(\overline{S}_{u-}, Y_{u-})\,\mathrm{d}\overline{S}_u\big)_\tau = G_x(\overline{S}_{\tau-}, Y_\tau)\Delta\overline{S}_\tau$ and $\Delta(-G(\overline{S}, Y))_\tau = -\big(G(\overline{S}_\tau, Y_\tau) - G(\overline{S}_{\tau-}, Y_\tau)\big)$, because $\Delta Y_\tau = 0$ a.s. Since moreover $\Delta\Sigma_\tau = G(\overline{S}_\tau, Y_\tau) - G(\overline{S}_{\tau-}, Y_\tau) - G_x(\overline{S}_{\tau-}, Y_\tau)\Delta\overline{S}_\tau$, one has cancellation of jumps at jump times of $\overline{S}$. However, these are times of continuity for Y and this will be crucial below to deduce continuity of addition on the support of $\left(\int_0^\cdot G_x(\overline{S}_{u-}, Y_{u-})\,\mathrm{d}\overline{S}_u, \Sigma, -G(\overline{S}, Y)\right)$ in $(D,\rho)\times(D,\rho)\times(D,\rho)$.

First consider the case of uniformly bounded sequence (Θ^n). Then the processes

$$\mathrm{d}Y_t^n = -h(Y_t^n)\,\mathrm{d}\langle M\rangle_t + \mathrm{d}\Theta_t^n, \quad Y_{0-}^n = y\,,$$

are uniformly bounded, so we can assume w.l.o.g. that h, gh, G, G_x and G_{xx} are ω-wise Lipschitz continuous in the y coordinate and bounded (it is so on the range of all Y^n, Y, which is contained in a compact subset of $\mathbb{R}$). By Proposition A.1 we have $Y^n \to Y$ in (D,ρ), almost surely. Hence, by the Lipschitz property of G, G_{xx} and gh, we get

$$G(\overline{S}, Y^n) \to G(\overline{S}, Y) \quad \text{in } (D,\rho), \text{ a.s.},$$

and also $\frac{1}{2}G_{xx}(\overline{S}, Y^n) - g(\overline{S}, Y^n)h(Y^n) \to \frac{1}{2}G_{xx}(\overline{S}, Y) - g(\overline{S}, Y)h(Y)$ in (D,ρ), a.s., by absence of common jumps of $\overline{S}$ and Y, cf. [JS03, Prop. VI.2.2b] for J_1 and [Whi02, Thm. 12.6.1 and 12.7.1] for M_1. An application of the dominated convergence theorem (cf. [Whi02, Thm. 11.5.1] for a similar argument) gives

$$\int_0^\cdot \tfrac{1}{2}G_{xx}(\overline{S}_u, Y_u^n) - g(\overline{S}_u, Y_u^n)h(Y_u^n)\,\mathrm{d}\langle M\rangle_u \to \int_0^\cdot \tfrac{1}{2}G_{xx}(\overline{S}_u, Y_u) - g(\overline{S}_u, Y_u)h(Y_u)\langle M\rangle_u$$

uniformly on $[0,T]$, a.s., here being crucial that $\langle M\rangle$ is (a.s.) absolutely continuous w.r.t. Lebesgue measure. Hence these two summands in the definition of L, see (3.9), are continuous in Θ (pathwise).

Now we handle the stochastic integral and jump terms in (3.9). By the above arguments we can also deal with the drift in the process $\overline{S}$. Thus we may assume w.l.o.g. that $\overline{S}$ is a martingale. In particular, up to a localization argument (see below for details), we can assume that $\overline{S}$ is bounded and therefore the stochastic integral is a true martingale, since the integrand is bounded. To obtain convergence of the stochastic integrals, we would like to derive convergence of Y^n to Y a.e. in the space $(\Omega\times[0,T], \mathbb{P}\otimes\mathrm{Leb}([0,T]))$; that would be sufficient to conclude convergence of the stochastic integrals in the uniform topology, in probability. Note that jump times of Θ coincide with jump times of Y and form a random countable subset of $[0,T]$. Moreover, convergence in (D,ρ) implies local uniform convergence at continuity points of the limit; for a proof when ρ is the M_1 topology, see [Whi02, Lemma 12.5.1], for the J_1 topology see [JS03, Prop. VI.2.1]. Hence, $Y_t^n \to Y_t$ for almost all $t\in[0,T]$, $\mathbb{P}$-a.s., and using dominated convergence on $\big([0,T], \mathrm{Leb}([0,T])\big)$ we obtain

$$\int_0^T (Y_{u-}^n - Y_{u-})^2\,\mathrm{d}\langle\overline{S}\rangle_u \to 0, \quad \text{as } n\to\infty,\ \mathbb{P}\text{-a.s.}$$

Since Y^n, Y are uniformly bounded one gets, again by dominated convergence, that

$$\mathbb{E}\Big[\int_0^T (Y_{u-}^n - Y_{u-})^2\,\mathrm{d}\langle\overline{S}\rangle_u\Big] \to 0, \quad \text{as } n\to\infty,$$

i.e. $Y^n_- \to Y_-$ in $L^2(\Omega \times [0,T], \mathrm{d}\mathbb{P} \otimes \mathrm{d}\langle \overline{S} \rangle)$. Now, by Itô's isometry

$$\mathbb{E}\Big[\Big(\int_0^T G_x(\overline{S}_{u-}, Y^n_{u-}) - G_x(\overline{S}_{u-}, Y_{u-}) \,\mathrm{d}\overline{S}_u\Big)^2\Big] \to 0 \quad \text{as } n \to \infty.$$

Doob's martingale inequality finally implies

$$\mathbb{P}\Bigg[\sup_{0\le t\le T}\Big|\int_0^t G_x(\overline{S}_{u-}, Y^n_{u-}) \,\mathrm{d}\overline{S}_u - \int_0^t G_x(\overline{S}_{u-}, Y_{u-}) \,\mathrm{d}\overline{S}_u\Big| \ge \varepsilon\Bigg] \to 0 \quad \text{as } n \to \infty. \tag{3.13}$$

For the sum of jumps Σ^n (defined like Σ, but with Y^n instead of Y) we have a.s. uniform convergence $\Sigma^n \to \Sigma$ by Lemma A.4. Hence $\int_0^t G_x(\overline{S}_{u-}, Y^n_{u-}) \,\mathrm{d}\overline{S}_u + \Sigma^n$ converges in ucp. To conclude on the proceeds, note that at jump times of $\overline{S}$, when cancellation of jumps occurs, one has continuity of Y and hence local uniform convergence of the sequence Y^n. For our setup, Lemmas A.2 and A.3 show continuity of addition on the support of $\big(\int_0^\cdot G_x(\overline{S}_{u-}, Y_{u-}) \,\mathrm{d}\overline{S}_u + \Sigma, -G(\overline{S}, Y)\big)$ (along the support of $\big(\int_0^\cdot G_x(\overline{S}_{u-}, Y^n_{u-}) \,\mathrm{d}\overline{S}_u + \Sigma^n, -G(\overline{S}, Y^n)\big)$) for the J_1 and M_1 topologies, respectively. So the continuous mapping theorem [Kal02, Lem. 4.3] yields the claim for the proceeds functional L (the uniform topology being stronger than ρ).

It remains to investigate the more general case of $\overline{S}$ and (Θ^n) being only bounded in $L^0(\mathbb{P})$. Note that the continuity of all terms except the stochastic integral in the definition of L was proven ω-wise; in this case $\sup_n \sup_{0\le t\le T}|\Theta^n_t(\omega)| < \infty$ (by the a.s. convergence of Θ^n to Θ in (D, ρ)) and hence the same arguments carry over here by restricting our attention to compact sets (depending on ω). Hence refinement of the argument above is only needed for the stochastic integral term. The bound on $\overline{S}$ and (Θ^n) means that for every $\varepsilon > 0$ there exists $\Omega_\varepsilon \in \mathcal{F}$ with $\mathbb{P}(\Omega_\varepsilon) > 1-\varepsilon$ and a positive constant K_ε which is a uniform bound for the sequence (together with the limit Θ) on Ω_ε. For the stopping time $\tau := \inf \tau_n$, where $\tau_n := \inf\{t \ge 0 \mid |\Theta^n_t| \vee |\overline{S}_t| > K_\varepsilon\} \wedge T$ (τ is a stopping time because the filtration is right-continuous by our assumptions), we then have that $\tau = T$ on Ω_ε. By the arguments above we conclude that $d\big(\int_0^{\cdot\wedge\tau} G_x(\overline{S}_{u-}, Y^n_{u-}) \,\mathrm{d}\overline{S}_u, \int_0^{\cdot\wedge\tau} G_x(\overline{S}_{u-}, Y_{u-}) \,\mathrm{d}\overline{S}_u\big) \to 0$ in probability. Since $\int_0^{\cdot\wedge\tau} G_x(\overline{S}_{u-}, Y^n_{u-}) \,\mathrm{d}\overline{S}_u = \int_0^\cdot G_x(\overline{S}_{u-}, Y^n_{u-}) \,\mathrm{d}\overline{S}_u$ on Ω_ε, we conclude

$$\mathbb{P}\Big[d\Big(\int_0^\cdot G_x(\overline{S}_{u-}, Y^n_{u-}) \,\mathrm{d}\overline{S}_u, \int_0^\cdot G_x(\overline{S}_{u-}, Y_{u-}) \,\mathrm{d}\overline{S}_u\Big) \ge \varepsilon\Big] \le 2\varepsilon$$

for all n large enough, and this finishes the proof since ε was arbitrary. □

Remark 3.8. Inspection of the proof above reveals that predictability of the strategies is only needed to show why the addition map is continuous when there is cancellation of jumps in (3.9); indeed, for predictable Θ the processes Y^Θ and $\overline{S}$ will have no common jump and this was sufficient for the arguments. However, in the case when M (and thus $\overline{S}$) is continuous, only one term in (3.9) might have jumps, namely $G(\overline{S}, Y^\Theta)$. Hence, in this case the conclusion of Theorem 3.7 even holds under the relaxed assumption that the càdlàg strategies are merely adapted, instead of being predictable.

An important consequence of Theorem 3.7 is a stability property for our model. It essentially implies that we can approximate each strategy by a sequence of absolutely continuous strategies, corresponding to small intertemporal shifts of reassigned trades, whose proceeds will approximate the proceeds of the original strategy. More precisely, if we restrict our attention to the class of monotone strategies, then we can restate this stability in terms of the Prokhorov metric on the pathwise proceeds (which are monotone and hence define measures on the time axis). This result on stability of proceeds with respect to small intertemporal Wong-Zakai-type re-allocation of orders may be compared to seminal work by [HHK92] on a different but related problem, who required that for economic reason the utility should be a continuous functional of cumulative consumption with respect to the Lévy-Prokhorov metric d_{LP}, in order to satisfy the sensible property of intertemporal substitution for consumption. Recall for convenience of the reader the definition of d_{LP} in our context: for increasing càdlàg paths on $[0,\tilde{T}]$, $x,y:[0,\tilde{T}]\to\mathbb{R}$ with $x(0-)=y(0-)$ and $x(\tilde{T})=y(\tilde{T})$,

$$d_{\mathrm{LP}}(x,y):=\inf\{\varepsilon>0 \mid x(t)\le y((t+\varepsilon)\wedge\tilde{T})+\varepsilon,\ \ y(t)\le x((t+\varepsilon)\wedge\tilde{T})+\varepsilon\ \ \forall t\in[0,\tilde{T}]\}.$$

Corollary 3.9. *Let Θ be a predictable process with càdlàg paths defined on the time interval $[0,T]$ (with possible initial and terminal jumps) that is extended to the time interval $[-1,T+1]$ as in Remark 3.6. Consider the sequence of f.v. processes (Θ^n) where*

$$\Theta^n_t := n\int_{t-1/n}^{t}\Theta_s\,\mathrm{d}s,\quad t\ge 0, \tag{3.14}$$

and let $L:=L(\Theta), L^n:=L(\Theta^n)$ be the proceeds processes from the respective trading. Then $L^n_t\to L_t$ at all continuity points $t\in[0,T+1]$ of L as $n\to\infty$, in probability. In particular, for any bounded monotone strategy Θ the Borel measures $L^n(\mathrm{d}t;\omega)$ and $L(\mathrm{d}t;\omega)$ on $[0,T+1]$ are finite (a.s.) and converge in the Lévy-Prokhorov metric $d_{LP}(L^n(\omega),L(\omega))$ in probability, i.e. for any $\varepsilon>0$,

$$\mathbb{P}\big[d_{LP}(L^n(\omega),L(\omega))>\varepsilon\big]\to 0\quad \text{as } n\to\infty.$$

Proof. An application of Proposition 3.4 together with Theorem 3.7 gives

$$d_{M1}(L^n,L)\xrightarrow{\mathbb{P}}0.$$

The first part of the claim now follows from the fact that convergence in M_1 implies local uniform convergence at continuity points of the limit, see [Whi02, Lemma 12.5.1]. The same property implies the claim about the Lévy-Prokhorov metric because convergence in this metric is equivalent to weak convergence of the associated measures which on the other hand is equivalent to convergence at all continuity points of the cumulative distribution function (together with the total mass). $\square$

Note that the sequence (Θ^n) from Corollary 3.9 satisfies $\Theta^n\equiv\Theta_T$ on $[T+1/n,T+1]$ for all n, i.e. the approximating strategies arrive at the position Θ_T, however by requiring a bit more time to execute. Based on the Wong-Zakai approximation sequence from

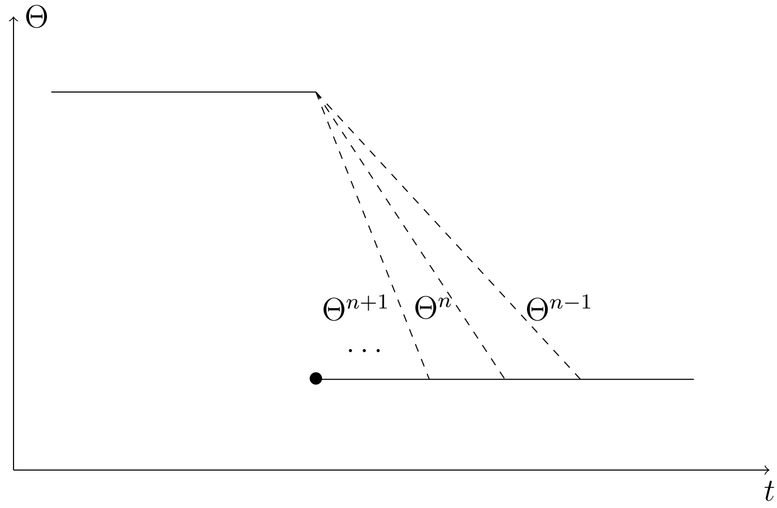

Figure 1: The Wong-Zakai approximation in (3.14) for a single jump process.

(3.14), we next show that each semimartingale strategy on the time interval $[0,T]$ can be approximated by simple adapted strategies with uniformly small jumps that, however, again need slightly more time to be executed.

Proposition 3.10. *Let $(\Theta_t)_{t\in[0,T]}$ be a predictable process with càdlàg paths extended to the time interval $[0,T+1]$ as in Remark 3.6. Then there exists a sequence $(\Theta^n_t)_{t\in[0,T+1]}$ of simple predictable càdlàg processes with jumps of size not more than $1/n$ such that $d_{M1}(L(\Theta^n),L(\Theta)) \xrightarrow{\mathbb{P}} 0$ as $n\to\infty$, where d_{M1} is the Skorokhod M_1 metric on $D([0,T+1];\mathbb{R})$. Moreover, if Θ is continuous, the same convergence holds true in the uniform metric on $[0,T]$ instead.*

Proof. Consider the Wong-Zakai approximation sequence $\widetilde{\Theta}^n$ from Corollary 3.9 for which $d_{M_1}(L(\widetilde{\Theta}^n),L(\Theta)) \xrightarrow{\mathbb{P}} 0$, where the Skorokhod M_1 topology is considered for the extended paths on time-horizon $[0,T+1]$. Now we approximate each (absolutely) continuous process $\widetilde{\Theta}^n$ by a sequence of simple processes as follows.

For $\varepsilon>0$, consider the sequence of stopping times with $\sigma_0^{\varepsilon,n}:=0$ and

$$\sigma_{k+1}^{\varepsilon,n} := \inf\big\{t \;\big|\; t>\sigma_k^{\varepsilon,n} \text{ and } |\widetilde{\Theta}^n_t-\widetilde{\Theta}^n_{\sigma_k^{\varepsilon,n}}|\geq\varepsilon\big\}\wedge(\sigma_k^{\varepsilon,n}+1/n) \quad \text{for } k\geq 0.$$

Note that $\sigma_k^{\varepsilon,n}$ are predictable as hitting times of continuous processes and $\sigma_k^{\varepsilon,n}\nearrow\infty$ as $k\to\infty$ because the process $\widetilde{\Theta}^n$ is continuous. When $\varepsilon\to 0$, we have $\Theta^{\varepsilon,n}\xrightarrow{ucp}\widetilde{\Theta}^n$ for

$$\Theta^{\varepsilon,n} := \widetilde{\Theta}^n_0+\sum_{k=1}^{\infty}\big(\widetilde{\Theta}^n_{\sigma_k^{\varepsilon,n}}-\widetilde{\Theta}^n_{\sigma_{k-1}^{\varepsilon,n}}\big)\mathbb{1}_{[\![\sigma_k^{\varepsilon,n},\infty[\![}.$$

Moreover, if for each integer $m\geq 1$ we define the (predictable) process $\Theta^{\varepsilon,n,m}$ by

$$\Theta^{\varepsilon,n,m} := \widetilde{\Theta}^n_0+\sum_{k=1}^{m}\big(\widetilde{\Theta}^n_{\sigma_k^{\varepsilon,n}}-\widetilde{\Theta}^n_{\sigma_{k-1}^{\varepsilon,n}}\big)\mathbb{1}_{[\![\sigma_k^{\varepsilon,n},\infty[\![},$$

then for each fixed ε and n we have $\Theta^{\varepsilon,n,m} \xrightarrow{ucp} \Theta^{\varepsilon,n}$ when $m \to \infty$. Hence, we can choose $\varepsilon = \varepsilon(n)$ small enough and $m = m(n)$ big enough such that

$$d(\widetilde{\Theta}^n, \Theta^{\varepsilon(n),n,m(n)}) < 2^{-n},$$

with $d(\cdot,\cdot)$ denoting a metric that metrizes ucp convergence (cf. e.g. [Pro04, p. 57]). Thus, $\Theta^n := \Theta^{\varepsilon(n),n,m(n)}$ will be close to Θ in the Skorokhod M_1 topology, in probability, because the uniform topology is stronger than the M_1 topology.

Note that if Θ is already continuous, no intermediate Wong-Zakai approximation would be needed, and so we obtain uniform convergence in probability in that case. □

The previous theorem provided a general result on convergence in probability which relies solely on topological closeness of strategies. Differently in spirit, an approximation idea due to [BB04] shows that one can actually approximate the proceeds of any strategy almost surely by some cleverly constructed continuous f.v. strategies which can be implemented within the same time interval, if the base price $\overline{S}$ is continuous.

Proposition 3.11 (Almost sure uniform approximation *à la* Bank-Baum by continuous f.v. strategies)**.** *Suppose that $\overline{S}$ is continuous and $g(x,\cdot)$ and h are continuously differentiable with locally Hölder-continuous derivatives for some index $\delta > 0$. For any predictable càdlàg process Θ on $[0,T]$ and any $\varepsilon > 0$, there exists a continuous process Θ^ε with f.v. paths such $Y_T^\Theta = Y_T^{\Theta^\varepsilon}$, $\Theta_0^\varepsilon = \Theta_{0-}$ and $|L_T(\Theta) - L_T(\Theta^\varepsilon)| \vee |\Theta_T - \Theta_T^\varepsilon| \le \varepsilon$, $\mathbb{P}$-a.s.*

Proof. Note that $K(y,t) := G(\overline{S}, y) - h(y)\int_0^t g(\overline{S}_u, y)\,\mathrm{d}\langle M\rangle_u$ and $\widetilde{K}(y,t) := h(y)\langle M\rangle_t$ define smooth families of semimartingales in the sense of [BB04, Def. 2.2] and

$$L_T(\Theta) = \int_0^T K(Y_{s-}, \mathrm{d}s) - \big(G(\overline{S}_T, Y_T) - G(\overline{S}_{0-}, Y_{0-})\big). \tag{3.15}$$

Predictability of Θ implies predictability of Y and hence Y_T is $\mathcal{F}_{T-}$ measurable. By the multidimensional version of [BB04, Thm. 4.4] for the non-linear integrator $(K, \widetilde{K})$ (extending the proof to this multidimensional setup is straightforward), for every $\varepsilon > 0$ there exists a predictable process Y^ε with continuous paths of finite variation, such that $Y_0^\varepsilon = Y_{0-}$, $Y_T^\varepsilon = Y_T$ and $\mathbb{P}$-a.s.

$$\sup_{0\le t\le T}\Big\{\Big|\int_0^t K(Y_{s-}, \mathrm{d}s) - \int_0^t K(Y^\varepsilon_{s-}, \mathrm{d}s)\Big| \vee \Big|\int_0^t h(Y_s)\,\mathrm{d}\langle M\rangle_s - \int_0^t h(Y_s^\varepsilon)\,\mathrm{d}\langle M\rangle_s\Big|\Big\} \le \varepsilon.$$

The process Y^ε corresponds to a predictable process Θ^ε with continuous f.v. paths, namely $\Theta^\varepsilon = Y^\varepsilon - Y_{0-} + \Theta_{0-} + \int_0^\cdot h(Y^\varepsilon)\,\mathrm{d}\langle M\rangle_u$, that satisfies $|\Theta_T - \Theta_T^\varepsilon| \le \varepsilon$, and with reference to (3.15), also satisfies $|L_T(\Theta) - L_T(\Theta^\varepsilon)| \le \varepsilon$. □

3.3. Connection to the Marcus canonical equation

Here we explain briefly, how our proceeds functional connects with an interesting SDE which is known as the Marcus canonical equation [Mar81]. Stability in the sense of Wong-Zakai approximations for this kind of equations has been studied in [KPP95]. Their techniques offer an alternative way to derive the approximation result of Corollary 3.9.

Definition (Marcus canonical equation)**.** Let $\Phi : \mathbb{R}^d \to \mathbb{R}^{d\times k}$ be continuously differentiable and Z be a k-dimensional semimartingale. Then the notation

$$X_t = X_{0-} + \int_0^t \Phi(X_s) \circ \mathrm{d}Z_s \tag{3.16}$$

means that X satisfies the stochastic integral equation

$$\begin{aligned} X_t =& X_{0-} + \int_0^t \Phi(X_{s-})\,\mathrm{d}Z_s + \frac{1}{2}\sum_{j,m=1}^{k}\sum_{\ell=1}^{d}\int_0^t \frac{\partial \Phi_{\cdot,j}}{\partial x_\ell}(X_{s-})\Phi_{\ell,m}(X_{s-})\,\mathrm{d}[Z^j, Z^m]_s^c \\ &+ \sum_{0\le s\le t\,,\,\Delta Z_s \neq 0} \big(\varphi(\Phi(\cdot)\Delta Z_s, X_{s-}) - X_{s-} - \Phi(X_{s-})\Delta Z_s\big), \end{aligned} \tag{3.17}$$

where $\Phi_{\cdot,j}$ is the j^{th} column of Φ, Z^j is the j^{th} entry of Z and $\varphi(\xi, x)$ denotes the value $y(1)$ of the solution to

$$y'(u) = \xi(y(u)) \quad \text{with} \quad y(0) = x. \tag{3.18}$$

The quadratic (co-)variation process is denoted by $[\cdot] = [\cdot]^c + [\cdot]^d$, it decomposes into a continuous part (appearing in (3.17)) and a discontinuous part. The next lemma gives a representation of the impact and proceeds processes of our model in terms of a Marcus canonical equation for the case $h \in C^1$. To this end, let the function $\Phi : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ for $X = (X^1, X^2, X^3)^{tr} \in \mathbb{R}^3$ be given by

$$\Phi(X) := \begin{pmatrix} -g(X^3, X^2) & 0 & 0 \\ 1 & 0 & -h(X^2) \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.19}$$

Lemma 3.12. *Let Θ be a càdlàg process with paths of finite total variation, and L be defined by* (2.6) *be the process describing the evolution of proceeds generated by Θ. Set $X_t := \big(L_t, Y_t, \overline{S}_t\big)^{tr}$, so $X_{0-} = \big(0, Y_{0-}, \overline{S}_{0-}\big)^{tr}$, and $Z_t := \big(\Theta_t, \overline{S}_t, \langle M\rangle_t\big)^{tr}$. Then the process X is the solution to the Marcus canonical equation*

$$X_t = X_{0-} + \int_0^t \Phi(X_s) \circ \mathrm{d}Z_s\,.$$

For the proof see Appendix A. Following [KPP95, Sect. 6], we now derive a Wong-Zakai-type approximation result in our setup. For a bounded semimartingale process Θ and $\varepsilon > 0$ consider the approximating absolutely continuous processes defined by

$$\Theta_t^\varepsilon := \frac{1}{\varepsilon}\int_{t-\varepsilon}^t \Theta_s\,\mathrm{d}s, \quad t \ge 0, \tag{3.20}$$

with the convention that $\Theta_t = \Theta_{0-}$ for $t < 0$. See Figure 1, where $\varepsilon = 1/n$.

Let $Z_t^\varepsilon := (\Theta_t^\varepsilon, \overline{S}_t, \langle M\rangle_t)^{tr}$ and X^ε be a solution to the following SDE in the Itô sense

$$\mathrm{d}X_t^\varepsilon = \Phi(X_t^\varepsilon)\,\mathrm{d}Z_t^\varepsilon, \qquad X_0^\varepsilon = X_{0-}. \tag{3.21}$$

The next result on Wong-Zakai convergence is based on the theory from [KPP95, Sect. 5]. See [BBF15, Thm. 6.2] for a proof in the case of $\overline{S}$ being geometric Brownian motion and $g(x,y) = xf(y)$, which however generalizes easily to continuous $\overline{S}$ and general impact function g.

Theorem 3.13. *Suppose that $\overline{S}$ is continuous and let $(\Theta_t)_{t\geq 0}$ be a bounded semimartingale. For $\varepsilon > 0$, let Θ^ε be the Wong-Zakai-type approximations from (3.20). Let X^ε be defined by (3.21) for $Z^\varepsilon_t := (\Theta^\varepsilon_t, \overline{S}_t, \langle M\rangle_t)^{tr}$ and Φ as in (3.19). For time-changes $\gamma_\varepsilon(t) := \frac{1}{\varepsilon}\int_{t-\varepsilon}^t ([\Theta]^d_s + s)\,\mathrm{d}s$, consider the processes $(\mathcal{X}^\varepsilon_t)_{t\geq 0}$ defined by $\mathcal{X}^\varepsilon_t := X^\varepsilon_{\gamma_\varepsilon^{-1}(t)}$. For $\varepsilon \to 0$ the processes $\mathcal{X}^\varepsilon$ then converge in probability in the compact uniform topology to a process $(\mathcal{X}^0_t)_{t\geq 0}$, such that $X_t = (X^1_t, X^2_t, X^3_t)^{tr} := \mathcal{X}^0_{\gamma_0(t)}$ is a solution of*

$$X_t = X_{0-} + \int_0^t \Phi(X_s) \circ \mathrm{d}Z_s - \Big(\frac{1}{2}\int_0^t g_x(\overline{S}_s, X^2_{s-})\,\mathrm{d}[\overline{S},\Theta]_s,\ 0,\ 0\Big)^{tr}, \tag{3.22}$$

where $X_{0-} = (0, Y_{0-}, \overline{S}_0)^{tr}$ and $\gamma_0(t) := [\Theta]^d_t + t$.

Theorem 3.13 directly gives, noting $X^1 = L$, that for a bounded semimartingale strategy Θ, the proceeds $L = L(\Theta)$ of this strategy up to $T < \infty$ take the form

$$\begin{aligned} L_T = &-\int_0^T g(\overline{S}_{t-}, Y^\Theta_{t-})\,\mathrm{d}\Theta_t - \frac{1}{2}\int_0^T g_y(\overline{S}_{t-}, Y^\Theta_{t-})\,\mathrm{d}[\Theta]^c_t - \int_0^T g_x(\overline{S}_t, Y^\Theta_{t-})\,\mathrm{d}[\overline{S},\Theta]_t \\ &- \sum_{\substack{\Delta\Theta_t\neq 0\\ t\leq T}} \Big(\int_0^{\Delta\Theta_t} g(\overline{S}_t, Y^\Theta_{t-} + x)\,\mathrm{d}x - g(\overline{S}_t, Y^\Theta_{t-})\Delta\Theta_t\Big), \end{aligned} \tag{3.23}$$

where the stochastic integral is understood in Itô's sense and Y^Θ is given as in (2.2). It is straightforward to see that (3.23) coincides with (3.9).

Remark 3.14. a) Note that boundedness of Θ implies that X^2 is bounded. Localizing along $\overline{S}$ (the variable X^3), we can assume that g is globally Lipschitz continuous. This implies absolute convergence of the infinite sum in (3.17), see [KPP95, p. 356]. In particular, (3.23) is well-defined.

b) The additional covariation term in the limiting equation (3.22) arises since only the strategies Θ are approximated in a Wong-Zakai sense, but not also unaffected price $\overline{S}$ and clock $\langle M\rangle$. For strategies Θ being of finite variation (as it would be natural under proportional transaction costs), this additional covariation term clearly vanishes.

c) Note that Theorem 3.13 implies the results in Corollary 3.9 for bounded semimartingale processes Θ. Indeed, Theorem 3.13 gives for the first components $L^\varepsilon = X^{\varepsilon,1}$, $L = X^{0,1}$ that for any $\eta > 0$ and any horizon $T \in [0,\infty)$ we have $\mathbb{P}\big[\sup_{t\leq T}\big|L^\varepsilon_{\gamma_\varepsilon^{-1}(\gamma_0(t))} - L_t\big| \leq \eta\big] \to 1$ for $\varepsilon \to 0$. Since $\gamma_\varepsilon^{-1}(\gamma_0(t)) \to t$ at continuity points of γ_0 (which are the continuity points of Θ and thus of L) it follows that $\mathbb{P}[\Omega^\eta_\varepsilon] \to 1$ as $\varepsilon \to 0$ with

$$\Omega^\eta_\varepsilon := \{\omega \mid \forall t \text{ with } \Delta L_t(\omega) = 0 : |L^\varepsilon_t(\omega) - L_t(\omega)| \leq \eta\}.$$

d) The proof of Theorem 3.13 could be adapted to the case when M is quasi-left continuous if the bounded semimartingale Θ is assumed to be predictable.

4. Absence of arbitrage for the large trader

On the one hand the large trader is faced with adverse price reaction to her trades. On the other hand, her market influence might give her opportunities to manipulate price dynamics in her favor. It is therefore relevant to show that the model does not permit arbitrage opportunities for the large trader in a (fairly large) set of trading strategies. For this section we consider a multiplicative price impact model where $g(\overline{S}, Y) = f(Y)\overline{S}$ with a non-negative, increasing and continuously differentiable function f, cf. Example 2.1.[2] Consider a portfolio (β_t, Θ_t) of the large investor, where β_t represents holdings in the bank account (riskless numéraire with discounted value 1) and Θ_t denotes holdings in the risky asset S at time t. We will consider bounded càdlàg strategies Θ on the full time horizon $[0, \infty)$ although our results below will deal with a finite but arbitrary horizon. For the strategy (β, Θ) to be self-financing, the bank account evolves according to

$$\beta_t = \beta_{0-} + L_t(\Theta)\,, \quad t \geq 0, \tag{4.1}$$

with $L(\Theta)$ as in (3.9). In order to define the wealth dynamics induced by the large trader's strategy, we have to specify the dynamics of the value of the risky asset position in the portfolio. If the large trader were to unwind her risky asset position at time t immediately by selling Θ_t shares (meaning to buy shares in case of a short position $\Theta_t < 0$), the resulting change in the bank account would be given by a term of the form (2.5). In this sense, let the *instantaneous liquidation value* process of her position be

$$V_t^\Theta = \beta_t + \overline{S}_t \int_0^{\Theta_t} f(Y_t^\Theta - x)\,\mathrm{d}x\,, \quad t \geq 0. \tag{4.2}$$

This corresponds to the asymptotically realizable real wealth process in [BB04]. Its dynamics (4.3) are mathematically tractable and relevant, e.g. to study no-arbitrage. For $F(x) := \int_0^x f(y)\,\mathrm{d}y$ we have $\overline{S}_t \int_0^{\Theta_t} f(Y_t^\Theta - x)\,\mathrm{d}x = \overline{S}_t\big(F(Y_t^\Theta) - F(Y_t^\Theta - \Theta_t)\big)$. By (3.9) and (4.1), noting that $Y^\Theta - \Theta$ and $\langle M\rangle$ are absolutely continuous processes, we have

$$\begin{aligned}
\mathrm{d}V_t^\Theta &= F(Y_{t-}^\Theta)\,\mathrm{d}\overline{S}_t - \overline{S}_t(fh)(Y_{t-}^\Theta)\,\mathrm{d}\langle M\rangle_t - \mathrm{d}\big(\overline{S}_\cdot F(Y_\cdot^\Theta - \Theta_\cdot)\big)_t \\
&= \big(F(Y_{t-}^\Theta) - F(Y_{t-}^\Theta - \Theta_{t-})\big)\,\mathrm{d}\overline{S}_t - \overline{S}_t\big(F'(Y_{t-}^\Theta) - F'(Y_{t-}^\Theta - \Theta_{t-})\big)h(Y_{t-}^\Theta)\,\mathrm{d}\langle M\rangle_t \\
&= \big(F(Y_{t-}^\Theta) - F(Y_{t-}^\Theta - \Theta_{t-})\big)\overline{S}_{t-}(\mu_t\,\mathrm{d}\langle M\rangle_t + \mathrm{d}M_t),
\end{aligned} \tag{4.3}$$

with $\mu_t := \xi_t - h(Y_{t-}^\Theta)\cdot\dfrac{F'(Y_{t-}^\Theta) - F'(Y_{t-}^\Theta - \Theta_{t-})}{F(Y_{t-}^\Theta) - F(Y_{t-}^\Theta - \Theta_{t-})}\mathbb{1}_{\{\Theta_{t-}\neq 0\}}$ and $V_0^\Theta = \beta_0 + \displaystyle\int_0^{\Theta_0} f(Y_0 + x)\,\mathrm{d}x$.

We will prove a no-arbitrage theorem for the large trader essentially for models that do not permit arbitrage opportunities for small investors in the absence of trading by the large trader. More precisely, for this section we assume for the driving noise M the

Assumption 4.1. For every predictable and bounded process μ and every $T \geq 0$, there exists a probability measure $\mathbb{P}^\mu \approx \mathbb{P}$ on $\mathcal{F}_T$ such that the process $M + \int_0^\cdot \mu_s\,\mathrm{d}\langle M\rangle_s$ is a $\mathbb{P}^\mu$-local martingale on $[0, T]$.

[2] For additive dynamics of $\overline{S}$ instead of (2.1), one could carry out the analysis in this section also in the case of additive impact $g(\overline{S}, Y) = \overline{S} + f(Y)$

Example 4.2 (Models satisfying assumption Assumption 4.1). a) If M is continuous, then under our model assumptions from Section 2, for every predictable and bounded process μ the probability measure $\mathrm{d}\mathbb{P}^\mu = \mathcal{E}(-\int_0^\cdot \mu_s \,\mathrm{d}M_s)\,\mathrm{d}\mathbb{P}$ is well-defined (thanks to Novikov's condition) and satisfies Assumption 4.1.

b) Let M be a Lévy process that is a martingale with $\Delta M > -1$ and $\mathbb{E}[M_1^2] < \infty$. In this case, it is a special semimartingale with characteristic triplet $(0, \sigma, K)$ (w.r.t. the identity truncation function), and we have the decomposition $M = \sqrt{\sigma} W + x * (\mu^M - \nu^{\mathbb{P}})$, where W is a $\mathbb{P}$-Brownian motion (or null if $\sigma = 0$), μ^M is the jump measure of M and $\nu^{\mathbb{P}}(\mathrm{d}x, \mathrm{d}t) = K(\mathrm{d}x)\,\mathrm{d}t$ is the $\mathbb{P}$-predictable compensator of μ^M. We have $\langle M \rangle_t = \lambda t$, $t \geq 0$, for some $\lambda \geq 0$. In the case $\sigma > 0$, Assumption 4.1 is clearly satisfied. Indeed, an equivalent change of measure by the standard Girsanov's theorem with respect to the non-vanishing (scaled) Brownian motion M^c can be done such that $M^c + \int \mu \,\mathrm{d}\langle M \rangle$ becomes a martingale, without changing the Lévy measure.

Otherwise, in case of $\sigma = 0$, M is a pure jump Lévy process. For this case, let us restrict our consideration to the situation of two-sided jumps, since pure-jump Lévy processes of such type appear more relevant to the modeling of financial returns than those ones with one-sided jumps only; examples are the exponential transform of the variance-gamma process or the so-called CGMY-process (suitably compensated to give a martingale exponential transform), cf. [KS02, CGMY02] for the relevant notions and models respectively. Here, it turns out that $K((-\infty, 0)) > 0$ and $K((0, +\infty)) > 0$ is already a sufficient condition for Assumption 4.1 to hold, i.e. possibility for jumps occurring in both directions. Indeed, a suitable change of measure can then be constructed as follows. Let $n > 0$ be such that $K([1/n, n]) > 0$ and $K([-n, -1/n]) > 0$. Denote $C^+ := \int_{[1/n,n]} x^2 K(\mathrm{d}x) > 0$ and $C^- := \int_{[-n,-1/n]} x^2 K(\mathrm{d}x) > 0$. Define functions $Y^\pm : \mathbb{R} \to \mathbb{R}$ by $Y^+ := 1$ on $[1/n, n]^c$, $Y^+(x) - 1 := x/C^+$ on $[1/n, n]$, and by $Y^- := 1$ on $[-n, -1/n]^c$, $Y^-(x) - 1 := -x/C^-$ on $[-n, -1/n]$, respectively. Thus $\int_{\mathbb{R}} x(Y^\pm(x) - 1) K(\mathrm{d}x) = \pm 1$ and hence , with $\eta := \lambda\mu$, the bounded previsible process

$$Y(\omega, t, x) := \eta_t^-(\omega)(Y^+(x) - 1) + \eta_t^+(\omega)(Y^-(x) - 1) + 1$$

satisfies $\int_{\mathbb{R}} x(Y(x) - 1) K(\mathrm{d}x) = -\eta$. The stochastic exponential $Z := \mathcal{E}((Y-1)*(\mu^L - \nu^{\mathbb{P}}))$ is a strictly positive $\mathbb{P}$-martingale, cf. [ES05, Prop. 5]. So for $T \geq 0$ there is a measure $d\mathbb{P}^\mu = Z_T d\mathbb{P}$ with density process $(Z_t)_{t \leq T}$. By Girsanov's theorem [JS03, Thm. III.3.11], $M - 1/Z_- \cdot \langle M, Z \rangle = M + \int_0^\cdot \mu_u \,\mathrm{d}\langle M \rangle_u$ is a $\mathbb{P}^\mu$-local martingale on $[0, T]$.

The set of *admissible trading strategies* that we consider is

$$\begin{aligned} \mathcal{A} := \big\{ (\Theta_t)_{t \geq 0} \mid & \text{ bounded, predictable, càdlàg, with } V^\Theta \text{ bounded from below,} \\ & \Theta_{0-} = 0, \text{ and such that } \Theta_t = 0 \text{ for } t \in [T, \infty) \text{ for some } T < \infty \big\}. \end{aligned}$$

Note that for such a strategy Θ it clearly holds $V^\Theta = \beta$ on $[T, \infty)$, i.e. beyond some bounded horizon $T < \infty$ the liquidation value coincides with the cash holdings β_T. Boundedness from below for V^Θ has a clear economical meaning, while the boundedness of Θ may be viewed as a more technical requirement. It ensures under Assumption 4.1 the existence of a strategy-dependent measure $\mathbb{Q}^\Theta \approx \mathbb{P}$ (on $\mathcal{F}_T$) so that V^Θ is a $\mathbb{Q}^\Theta$-local martingale on $[0, T]$. This relies on (4.3) and is at the key idea for the proof for

Theorem 4.3. *Under Assumption 4.1, the model is free of arbitrage up to any finite time horizon $T \in [0, \infty)$, in the sense that there exists no $\Theta \in \mathcal{A}$ with $\Theta_t = 0$ on $t \in [T, \infty)$ such that for the corresponding self-financing strategy (β, Θ) with $\beta_{0-} = 0$ we have*

$$\mathbb{P}[V_T^\Theta \geq 0] = 1 \text{ and } \mathbb{P}[V_T^\Theta > 0] > 0\,. \tag{4.4}$$

Proof. Recall the SDE (4.3) which describes the liquidation value process V, and note that $V_0 = 0$. For each $\Theta \in \mathcal{A}$ we have that (Θ, Y^Θ) is bounded. Thus, the drift μ is bounded as well because, in the case of $\Theta_{t-} \neq 0$, by the mean value theorem we have

$$\frac{F'(Y^\Theta_{t-}) - F'(Y^\Theta_{t-} - \Theta_{t-})}{F(Y^\Theta_{t-}) - F(Y^\Theta_{t-} - \Theta_{t-})} = \frac{f'(z_1)}{f(z_2)} \quad \text{for some } z_{1,2} \text{ between } Y^\Theta_{t-} \text{ and } Y^\Theta_{t-} - \Theta_{t-},$$

and this is bounded from above because f, f' are continuous and $f > 0$ (so it is bounded away from zero on any compact set). Hence, Assumption 4.1 guarantees the existence of $\mathbb{P}^\mu \approx \mathbb{P}$ on $\mathcal{F}_T$ such that V^Θ is a $\mathbb{P}^\mu$-local martingale on $[0, T]$, and since it is also bounded from below, it is a $\mathbb{P}^\mu$-supermartingale, so $E^\mu[V_T^\Theta] \leq V_0^\Theta = 0$. This rules out arbitrage opportunities, as described in (4.4), under any probability $\mathbb{P}$ equivalent to $\mathbb{P}^\mu$ on $\mathcal{F}_T$, for any $T \in [0, \infty)$. □

Remark 4.4 (Extension to bid-ask spread)**.** Absence of arbitrage in the model with zero bid-ask spread naturally implies no arbitrage for model extensions with spread, at least when the admissible trading strategies have paths of finite variation. To make this precise, let us model different impact processes Y^{Θ^-} and Y^{Θ^+} from selling and buying, respectively, according to (2.2), and best bid and ask price processes $(S^b, S^a) := \big(f(Y^{\Theta^-})\overline{S}^b, f(Y^{\Theta^+})\overline{S}^a\big)$ with $S^b \leq S^a$ for non-increasing Θ^- and non-decreasing Θ^+. Then, the proceeds from implementing (Θ^-, Θ^+) on $[0, T]$ would be

$$-\int_0^T S_t^b \,\mathrm{d}\Theta_t^{-,c} - \int_0^T S_t^a \,\mathrm{d}\Theta_t^{+,c} - \sum_{\substack{0 \leq t \leq T \\ \Delta\Theta_t^- > 0}} \overline{S}_t^b \int_0^{\Delta\Theta_t^-} f(Y_{t-}^{\Theta^-} + x)\,\mathrm{d}x - \sum_{\substack{0 \leq t \leq T \\ \Delta\Theta_t^+ > 0}} \overline{S}_t^a \int_0^{\Delta\Theta_t^+} f(Y_{t-}^{\Theta^+} + x)\,\mathrm{d}x.$$

Now for $\Theta := \Theta^- + \Theta^+$, the initial relation $Y_{0-}^{\Theta^-} \leq Y_{0-}^{\Theta} \leq Y_{0-}^{\Theta^+}$ implies $Y^{\Theta^-} \leq Y^\Theta \leq Y^{\Theta^+}$. Hence $S^b \leq S \leq S^a$ and the proceeds above for the model with non-vanishing spread would be dominated (a.s.) by those that we get in (2.6), i.e. in the model without bid-ask spread. In an alternative but different variant, one could extend the zero bid-ask spread model to a one-tick-spread model, motivated by insights in [CDL13], by letting $(S^b, S^a) := (S, S + \delta)$ for some $\delta > 0$. Again, proceeds in this model would be dominated by those in the zero-spread model. In either variant, absence of arbitrage opportunities in the zero bid-ask spread model implies the same for an extended model with spread.

Remark 4.5 (Extension to càglàd strategies)**.** For any càglàd (left continuous with right limits) $(\Theta_t)_{t \geq 0}$ (with $\Theta_{0-} = \Theta_0$) the unique càglàd solution Y^Θ to the integral equation $Y_t - Y_s = \int_s^t h(Y_u)\alpha_u \,du + \Theta_t - \Theta_s$ ($0 \leq s < t$, with $Y_0 = Y_{0-}$), corresponding to (2.2), can be defined pathwise (cf. proof of [PTW07, Thm. 4.1]); statements on càdlàg paths ($\bar{\Theta}$,$Y^{\bar{\Theta}}$) translate to càglàd paths (Θ,Y^{Θ}) by relations $\bar{\Theta}_{t-} = \Theta_t$ and $Y^{\bar{\Theta}}_{t-} = Y^{\Theta}_t$, $t \geq 0$. Using this, we can define the dynamics of the liquidation wealth process V for any strategy Θ which is adapted with càglàd paths or predictable with càdlàg paths, and hence locally bounded, by the the unique (strong) solution to the SDE (4.3) for given initial condition $V_0 \in \mathbb{R}$. Thereby, the result on absence of arbitrage can be extended to a larger set of strategies, which contains the set $\mathcal{A}$ and in addition all bounded adapted and càglàd (left-continuous with right limits) processes $(\Theta_t)_{t\geq 0}$ with $\Theta_{0-} = \Theta_0 = 0$ for which there exists some $T < \infty$ such that $\Theta_t = 0$ for $t \in [T, \infty)$ holds. Indeed, the same lines of proof show that such Θ cannot give an arbitrage opportunity in the sense of Theorem 4.3.

5. Application examples

In this section, we present four examples in the framework of multiplicative impact $g(\overline{S}, Y) = f(Y)\overline{S}$, cf. Example 2.1, that highlight different questions in which our stability results are helpful. Section 5.1 shows, by compactness argument, the existence of an optimal control by an application of our continuity result in Theorem 3.7. For this, it is rather easy to check that the set of controls is compact for the M_1 topology. In Section 5.2 we identify the solution of an optimal liquidation problem with the already known optimizer in a smaller class of admissible controls, by approximating semimartingale strategies with strategies of bounded variation, where stability of the proceeds functional plays a crucial role.

Sections 5.3 and 5.4 illustrate modifications of the price impact model by changing the impact process to allow stochastic, respectively partially instantaneous, impact, to which the analysis in Section 3 carries over. Herein, the M_1 topology is again key for identifying the (asymptotically realizable) proceeds and thus extending the models to a larger class of trading strategies. This is particularly crucial in Section 5.3, where the optimal liquidation problem with stochastic liquidity can be solved explicitly by a convexity argument if the price process is a martingale. In this case, any finite-variation strategy turns out to be suboptimal. We construct an optimal singular control of infinite variation.

5.1. Optimal liquidation problem on finite time horizon

In this example, using continuity of the proceeds in the M_1 topology we will show that the optimal liquidation problem over monotone strategies on a finite time horizon admits an optimal strategy. For $\theta \geq 0$ shares to be liquidated, the problem is to

$$\text{maximize} \quad \mathbb{E}[L_T(\Theta)] \quad \text{over } \Theta \in \mathcal{A}_{\text{mon}}(\theta), \tag{5.1}$$

over the set of all decreasing adapted càdlàg Θ with $\Theta_{0-} = \theta$ and $\Theta \mathbb{1}_{[T,\infty)} = 0$. We consider the situation when the unaffected price process has constant drift, i.e. $\overline{S}_t = e^{\mu t} M_t$ for $t \geq 0$, where $\mu \in \mathbb{R}$ and M is a non-negative continuous martingale that is locally square integrable. Existence and (explicit) structural description of the optimal strategy is already known in the following two cases: a) $\mu = 0$ and any time horizon $T \geq 0$, cf. [PSS11, Løk12]; or: b) $\mu < 0$ and sufficiently big time horizon $T \geq T(\theta, \mu)$ under additional assumptions on f and h, cf. [BBF17a]. There M can be taken even quasi-left continuous in which case the set of admissible strategies should be restricted to predictable processes.

In the general case, the following compactness argument proves existence of an optimizer - without providing any structural description for it, of course. First, it suffices to optimize over deterministic strategies and thus to take $M \equiv 1$ by a change of measure argument, see [BBF17a, Remark 3.9]. Now, for some fixed $\varepsilon > 0$ consider the optimization problem over the set of strategies

$$\widetilde{\mathcal{A}}_{\mathrm{mon}}(\theta) = \{\widetilde{\Theta} \in D[-\varepsilon, T+\varepsilon] \mid \widetilde{\Theta} \text{ is the } \textit{extended path} \text{ of some determ. } \Theta \in \mathcal{A}_{\mathrm{mon}}(\theta)\}.$$

Endowing $\widetilde{\mathcal{A}}_{\mathrm{mon}}(\theta)$ with the Skorokhod M_1 topology makes it relatively compact, which is straightforward to check using [Whi02, Thm. 12.12.2]; the compactness criterion in [Whi02, Thm. 12.12.2] is trivial for such monotone strategies because the M_1 oscillation function is zero and all the paths are constant in neighborhoods of the end points. Thus, if $(\widetilde{\Theta^n}) \subset \widetilde{\mathcal{A}}_{\mathrm{mon}}(\theta)$ is a maximizing sequence (of extended paths) for the problem (5.1), then it (or some subsequence) converges to $\widetilde{\Theta^*} \in D[-\varepsilon, T+\varepsilon]$. By continuity of the proceeds functional L in the M_1 topology (Theorem 3.7) we obtain

$$\sup_{\Theta \in \mathcal{A}_{\mathrm{mon}}(\theta)} L_T(\Theta) = \lim_{n\to\infty} L_{T+\varepsilon}(\widetilde{\Theta^n}) = L_{T+\varepsilon}(\widetilde{\Theta^*}). \tag{5.2}$$

Since on $[-\varepsilon, 0)$ (resp. $(T, \varepsilon]$) each $\widetilde{\Theta^n}$ is constant θ (resp. 0) and convergence in M_1 implies local uniform convergence at continuity points of the limit, cf. [Whi02, Lemma 12.5.1], there exists $\Theta^* \in \mathcal{A}_{\mathrm{mon}}(\theta)$ such that $\widetilde{\Theta^*}$ is its extended path in $D[-\varepsilon, T+\varepsilon]$. Thus $L_{T+\varepsilon}(\widetilde{\Theta^*}) = L_T(\Theta^*)$ and Θ^* is an optimal liquidation strategy by (5.2).

5.2. Optimal liquidation problem with general strategies

Consider the problem from [BBF17a, Sect. 5] to liquidate a risky asset optimally, posed over the set of bounded variation strategies $\mathcal{A}_{\mathrm{bv}}(\theta)$ with no shortselling, for some initial position $\theta \geq 0$, i.e. $\max_{\Theta \in \mathcal{A}_{\mathrm{bv}}(\theta)} \mathbb{E}[L_\infty(\Theta)]$; Recall that in the setup there the fundamental price process is $\overline{S}_t = e^{-\delta t} M_t$ for some $\delta > 0$ and a non-negative locally square integrable quasi-left continuous martingale M, and $\mathrm{d}\langle M\rangle_t$ in the dynamics of Y in (2.2) is replaced by $\mathrm{d}t$. By [BBF17a, Thm. 5.1], the optimal bounded variation strategy Θ^* is deterministic and liquidates in some finite time $T-1$ (which depends on the model parameters).

Now consider the optimal liquidation problem over the larger set of admissible strategies

$$\mathcal{A}_{\mathrm{semi}}(\theta) := \{\Theta \mid \text{bounded predictable semimartingale}, \Theta \geq 0, \Theta_{0-} = \theta, \Theta_t = \Theta_{t\wedge(T-1)}\}.$$

Note that for any admissible strategy $\Theta \in \mathcal{A}_{\mathrm{semi}}(\theta)$, the (martingale part of the) stochastic integral in equation (3.9) is a true martingale and will vanish in expectation, yielding

$$\mathbb{E}[L_T(\Theta)] = \mathbb{E}\Big[-\int_0^T e^{-\delta t} M_t((fh)(Y_t^\Theta) + \delta F(Y_t^\Theta))\,\mathrm{d}t - (e^{-\delta T} M_T F(Y_T^\Theta) - M_{0-} F(Y_{0-}^\Theta))\Big],$$

where $F(x) = \int_0^x f(y)\,\mathrm{d}y$. A change of measure argument as in [BBF17a, Rem. 3.9] shows that we can take w.l.o.g. $M \equiv 1$ and thus it suffices to optimize the proceeds over the set $\mathcal{A}_{\text{càdlàg}}(\theta)$ of all deterministic non-negative càdlàg paths having square-summable jumps, starting at time $0-$ at θ and being zero after time $T-1$. For each such $\Theta \in \mathcal{A}_{\text{càdlàg}}(\theta)$ and every $\varepsilon > 0$, we can find a deterministic bounded variation strategy $\Theta^\varepsilon \in \mathcal{A}_{\text{bv}}(\theta)$ that executes until time T and gives proceeds that are at most ε-away from the proceeds of Θ. Indeed, this follows from Corollary 3.9 where the approximating sequence is indeed of bounded variation continuous processes (since Θ is bounded), and noting that the probabilistic nature of the stability results in Section 3.2 is due to the presence of the (intrinsically probabilistic) stochastic integral in (3.9), cf. the proof of Theorem 3.7, which would be immaterial here in the case of constant M. In particular,

$$\sup_{\Theta \in \mathcal{A}_{\text{semi}}(\theta)} \mathbb{E}[L_T(\Theta)] \le \sup_{\mathcal{A}_{\text{càdlàg}}(\theta)} \mathbb{E}[L_T(\Theta)] = \sup_{\Theta \in \mathcal{A}_{\text{bv}}(\theta)} \mathbb{E}[L_T(\Theta)] = \mathbb{E}[L_T(\Theta^*)],$$

meaning that Θ^* is optimal also within in the (larger) set $\mathcal{A}_{\text{semi}}(\theta)$.

5.3. Stochastic liquidity and constrained liquidation horizon

Let us investigate an optimal liquidation problem for a variant of the price impact model which features *stochastic liquidity.* The singular control problem exhibits two interesting properties: it still permits an explicit description for the optimal strategy under a new constraint on the *expected* time to (complete) liquidation, but the optimal control is not of finite variation. So the set of admissible strategies needs to accommodate for infinite variation controls. As it is clear how to define the proceeds functional for (continuous) strategies of finite variation (cf. (2.4)), and we want (and need) to admit for jumps in the (optimal) control, the M_1 topology is a natural choice to extend the domain continuously. We consider no discounting or drift in the unaffected price process, letting $\overline{S}_t = \overline{S}_0 \mathcal{E}(\sigma W)_t$ with constant $\sigma > 0$. This martingale case will permit to apply convexity arguments in spirit of [PSS11] to construct an optimal control, see Theorem 5.2 below. In (2.2), the dynamics of market impact Y (called volume effect in [PSS11]) was deterministic in the large trader's strategy Θ. To model liquidity which is stochastic (e.g. by transient imbalances from other large 'noise' traders), now let the impact process Y^Θ solve

$$\mathrm{d}Y_t^\Theta = -\beta Y_t^\Theta\,\mathrm{d}t + \hat{\sigma}\,\mathrm{d}B_t + \mathrm{d}\Theta_t\,, \quad \text{with} \quad Y_{0-}^\Theta = Y_{0-} \in \mathbb{R} \text{ given}, \tag{5.3}$$

for constants $\beta, \hat{\sigma} > 0$ and a Brownian motion B that is independent of W. For the impact function $f \in C^3(\mathbb{R})$, giving the observed price by $S_t = f(Y_t)\overline{S}_t$, we require $f, f' > 0$ with $f(0) = 1$ and that $\lambda(y) := f'(y)/f(y)$ is bounded away from 0 and ∞, i.e. for constants $0 < \lambda_{\min} \le \lambda_{\max}$ we have $\lambda_{\min} \le \lambda(y) \le \lambda_{\max}$ for all $y \in \mathbb{R}$, with bounded derivative λ'. Moreover, we assume that $k(y) := \frac{\hat{\sigma}^2}{2}\frac{f''(y)}{f(y)} - \beta - \beta y \frac{f'(y)}{f(y)}$ is strictly decreasing. An example satisfying these conditions is $f(y) = e^{\lambda y}$ with constant $\lambda > 0$. Let $F(x) := \int_{-\infty}^x f(y)\,\mathrm{d}y$, which is positive and of exponential growth due to the bounds on λ: $0 < F(x) \le \left(e^{\lambda_{\min}} + e^{\lambda_{\max}}\right)/\lambda_{\min}$. The liquidation problem on infinite horizon *with* discounting and *without* intermediate buying in this model has been solved in [BBF17b].

For our problem here, proceeds of general semimartingale strategies Θ should be

$$L_T(\Theta) = \int_0^T \overline{S}_t \psi(Y_{t-}^{\Theta})\,\mathrm{d}t + \overline{S}_0 F(Y_{0-}) - \overline{S}_T F(Y_T^{\Theta}) + \int_0^T F(Y_{t-}^{\Theta})\,\mathrm{d}\overline{S}_t + \hat{\sigma}\int_0^T \overline{S}_t f(Y_{t-}^{\Theta})\,\mathrm{d}B_t\,, \tag{5.4}$$

with $\psi(y) := -\beta y f(y) + \frac{\hat{\sigma}^2}{2} f'(y)$, because (5.4) is the continuous extension (in M_1 in probability, as in Theorem 3.7) of the functional $L(\Theta^c) = -\int_0^T S_u\,\mathrm{d}\Theta_u^c$ from continuous f.v. Θ^c to semimartingales Θ that are *bounded in probability* on $[0,\infty)$: The proof of Theorem 3.7 carries over as for such Θ, impact Y and thus $\psi(Y)$ and $F(Y)$ are then also bounded in probability and the stochastic $\mathrm{d}B$-integral in (5.4) converges by a similar argument as in (3.13) for the $\mathrm{d}\overline{S}$ integral, using $\langle \overline{S}\rangle_t = \sigma^2\int_0^t \overline{S}_u^2\,\mathrm{d}u = \sigma^2\langle\int_0^\cdot \overline{S}_u\,\mathrm{d}B_u\rangle_t$.

Our goal is to maximize expected proceeds $\mathbb{E}[L_\infty(\Theta)]$ over some suitable set of admissible strategies that we specify now. From an application point of view, it makes sense to impose some bound on the time horizon within which liquidation is to be completed. Indeed, since our control objective here involves no discounting, one needs to restrict the horizon to get a non-trivial solution. Let some $\eta_{\max} \geq 0$ be given. A semimartingale Θ that is bounded in probability on $[0,\infty)$ will be called an *admissible strategy*, if

> there exists a stopping time τ with $\mathbb{E}[\tau] \leq \eta_{\max}$ such that $\Theta_t = \Theta_t \mathbb{1}_{t\leq\tau}$, with $\mathbb{E}[\tau\overline{S}_\tau] < \infty$, $\big(L_\tau(\Theta)\big)^- \in L^1(\mathbb{P})$ and such that the processes $\overline{S}_{\cdot\wedge\tau}$, $(\overline{S}B)_{\cdot\wedge\tau}$, $\displaystyle\int_0^{\cdot\wedge\tau} \overline{S}_t F(Y_{t-}^{\Theta})\,\mathrm{d}W_t$ and $\displaystyle\int_0^{\cdot\wedge\tau} \overline{S}_t f(Y_{t-}^{\Theta})\,\mathrm{d}B_t$ are uniformly integrable (UI).

The integrability conditions ensure $L_\tau(\Theta) \in L^1(\mathbb{P})$. Indeed, for admissible Θ it suffices to check $\big(\int_0^\tau \overline{S}_t\psi(Y_{t-}^{\Theta})\,\mathrm{d}t\big)^+ \in L^1(\mathbb{P})$. We will show in the proof of Theorem 5.2 that ψ attains a maximum $\psi(y^*)$. Thus we can bound $\int_0^\tau \overline{S}_t\psi(Y_{t-}^{\Theta})\,\mathrm{d}t$ from above by $\psi(y^*)\int_0^\tau \overline{S}_t\,\mathrm{d}t$, which is integrable by optional projection [DM82, Thm. VI.57] since $\mathbb{E}[\tau\overline{S}_\tau] < \infty$.

Let $\mathcal{A}_{\eta_{\max}}$ be the set of all admissible strategies with given fixed initial value Θ_{0-}, where $|\Theta_{0-}|$ is the number of shares to be liquidated (sold) if $\Theta_{0-} > 0$, resp. acquired (bought) if $\Theta_{0-} < 0$. The definition of $\mathcal{A}_{\eta_{\max}}$ involves several technical conditions. But the set $\mathcal{A}_{\eta_{\max}}$ is not small, for instance it clearly contains all strategies of finite variation which liquidate until some bounded stopping times τ with $\mathbb{E}[\tau] \leq \eta_{\max}$, and also strategies of infinite variation (see below). Note that intermediate short selling is permitted, and that $\mathcal{A}_0$ contains only the trivial strategy to sell (resp. buy) everything immediately.

We will show that optimal strategies are *impact fixing*. For $\tilde{\Upsilon}, \Upsilon \in \mathbb{R}$ an *impact fixing strategy* $\Theta = \Theta^{\tilde{\Upsilon},\Upsilon}$ is a strategy with liquidation time τ (i.e. $\Theta_t = 0$ for $t \geq \tau$), such that $Y = Y^{\Theta^{\tilde{\Upsilon},\Upsilon}}$ satisfies $Y_t = \tilde{\Upsilon}$ on $[\![0,\tau[\![$ and $Y_\tau = \Upsilon$. More precisely, $\Theta_0 = \Theta_{0-} + \tilde{\Upsilon} - Y_{0-}$, $\mathrm{d}\Theta_t = \beta\tilde{\Upsilon}\,\mathrm{d}t - \hat{\sigma}\,\mathrm{d}B_t$ on $]\!]0,\tau[\![$ until $\tau = \tau^{\tilde{\Upsilon},\Upsilon} := \inf\{t > 0 \mid \Theta_{t-} = \tilde{\Upsilon} - \Upsilon\}$, with final block trade of size $\Delta\Theta_\tau = -\Theta_{\tau-} = \Upsilon - \tilde{\Upsilon}$ and $\Theta = 0$ on $[\![\tau,\infty[\![$. We have the following properties of impact fixing strategies (for proof, see Appendix A).

Lemma 5.1 (Admissibility of impact fixing strategies)**.** *The liquidation time $\tau = \tau^{\tilde{\Upsilon},\Upsilon}$ of an impact fixing strategy $\Theta^{\tilde{\Upsilon},\Upsilon}$ has expectation $\mathbb{E}[\tau] = (Y_{0-} - \Theta_{0-} - \Upsilon)/(\beta\tilde{\Upsilon})$ if $(Y_{0-} - \Theta_{0-} - \Upsilon)\tilde{\Upsilon} > 0$, and $\mathbb{E}[\tau] = 0$ if $\Upsilon = Y_{0-} - \Theta_{0-}$, otherwise $\mathbb{E}[\tau] = \infty$. Moreover, if $\mathbb{E}[\tau^{\tilde{\Upsilon},\Upsilon}] \le \eta_{\max}$ then $\Theta^{\tilde{\Upsilon},\Upsilon} \in \mathcal{A}_{\eta_{\max}}$.*

Using convexity arguments we construct the solution for the optimization problem in

Theorem 5.2. *For every $\eta_{\max} \in [0, \infty)$ there exist $\hat{\eta} \in [0, \eta_{\max}]$ and $\tilde{\Upsilon}$, $\Upsilon \in \mathbb{R}$ such that the associated impact fixing strategy $\hat{\Theta} := \Theta^{\tilde{\Upsilon},\Upsilon}$ generates maximal expected proceeds in expected time $\mathbb{E}[\tau^{\tilde{\Upsilon},\Upsilon}] = \hat{\eta}$ among all admissible strategies, i.e.*

$$\mathbb{E}[L_\infty(\hat{\Theta})] = \max\big\{\mathbb{E}[L_\infty(\Theta)] \bigm| \Theta \in \mathcal{A}_{\eta_{\max}}\big\}\,.$$

Moreover, if $f(y) = e^{\lambda y}$ with $\lambda \in (0, \infty)$, then we have $\hat{\eta} = \eta_{\max}$ and the optimal strategy is unique.

The proof will also show that optimal strategies have to be *impact fixing*. In particular, any non-trivial admissible strategy of finite variation is suboptimal.

Proof. Since f'/f and $(f'/f)'$ are bounded, then f''/f is also bounded and hence there is a unique $y^* \in \mathbb{R}$ with $k(y^*) = 0$. So ψ is strictly increasing on $(-\infty, y^*)$ and decreasing on (y^*, ∞), since $\psi'(y) = f(y)k(y)$. Note that ψ is strictly concave on $[y^*, \infty)$ and $\psi(y) > 0$ for $y < 0$. Hence, the concave hull of ψ is

$$\hat{\psi}(y) := \inf\{\ell(y) \mid \ell \text{ is an affine function with } \ell(x) \ge \psi(x)\ \forall x\} = \psi(y \vee y^*)\,.$$

Let $\Theta \in \mathcal{A}_{\eta_{\max}}$ with liquidation time τ. Denote by $\mathbb{Q}$ the measure with $\mathrm{d}\mathbb{Q} = \big(\overline{S}_\tau/\overline{S}_0\big)\,\mathrm{d}\mathbb{P}$. Then by optional projection, as in [DM82, Thm. VI.57], we obtain (taking w.l.o.g. $\overline{S}_0 = 1$):

$$\begin{aligned}
\mathbb{E}[L_\infty] = \mathbb{E}[L_\tau] &= \mathbb{E}\Big[\int_0^\tau \overline{S}_t \psi(Y_t)\,\mathrm{d}t\Big] + F(Y_{0-}) - \mathbb{E}\big[\overline{S}_\tau F(Y_\tau)\big] \\
&= F(Y_{0-}) + \mathbb{E}_{\mathbb{Q}}\Big[\int_0^\tau \psi(Y_t)\,\mathrm{d}t\Big] - \mathbb{E}_{\mathbb{Q}}\big[F(Y_\tau)\big] \\
&= F(Y_{0-}) + \int_{\Omega\times[0,\infty)} \psi(Y_t(\omega))\mu(\mathrm{d}\omega, \mathrm{d}t) - \mathbb{E}_{\mathbb{Q}}\big[F(Y_\tau)\big]\,, \qquad (5.5)
\end{aligned}$$

for the finite measure μ given by $\mu(A \times B) := \int_A \int_0^{\tau(\omega)} \mathbb{1}_B(t)\,\mathrm{d}t\mathbb{Q}[\mathrm{d}\omega]$ with total mass $\mu(\Omega, [0, \infty)) = \mathbb{E}_{\mathbb{Q}}[\tau] = \mathbb{E}[\tau\overline{S}_\tau] < \infty$. For $\tau \ne 0$, Jensen's inequality for $\hat{\psi}$ and F gives

$$\begin{aligned}
\mathbb{E}[L_\infty] &\le F(Y_{0-}) + \int_{\Omega\times[0,\infty)} \hat{\psi}(Y_t(\omega))\mu(\mathrm{d}\omega, \mathrm{d}t) - \mathbb{E}_{\mathbb{Q}}\big[F(Y_\tau)\big] & (5.6) \\
&\le F(Y_{0-}) + \mathbb{E}_{\mathbb{Q}}[\tau]\hat{\psi}\bigg(\frac{1}{\mathbb{E}_{\mathbb{Q}}[\tau]}\int_{\Omega\times[0,\infty)} Y_t(\omega)\mu(\mathrm{d}\omega, \mathrm{d}t)\bigg) - \mathbb{E}_{\mathbb{Q}}\big[F(Y_\tau)\big] & (5.7) \\
&= F(Y_{0-}) + \mathbb{E}_{\mathbb{Q}}[\tau]\hat{\psi}\bigg(\frac{1}{\beta\mathbb{E}_{\mathbb{Q}}[\tau]}\mathbb{E}_{\mathbb{Q}}\Big[\int_0^\tau \beta Y_t\,\mathrm{d}t\Big]\bigg) - \mathbb{E}_{\mathbb{Q}}\big[F(Y_\tau)\big] & (5.8)
\end{aligned}$$

$$
\begin{aligned}
&= F(Y_{0-}) + \mathbb{E}_{\mathbb{Q}}[\tau]\hat{\psi}\bigg(\frac{Y_{0-}-\Theta_{0-}-\mathbb{E}_{\mathbb{Q}}[Y_\tau]}{\beta\mathbb{E}_{\mathbb{Q}}[\tau]}\bigg) - \mathbb{E}_{\mathbb{Q}}\big[F(Y_\tau)\big] && (5.9)\\
&\le F(Y_{0-}) + \mathbb{E}_{\mathbb{Q}}[\tau]\hat{\psi}\bigg(\frac{Y_{0-}-\Theta_{0-}-\mathbb{E}_{\mathbb{Q}}[Y_\tau]}{\beta\mathbb{E}_{\mathbb{Q}}[\tau]}\bigg) - F\big(\mathbb{E}_{\mathbb{Q}}[Y_\tau]\big) && (5.10)\\
&= F(Y_{0-}) + \hat{\Psi}\big(\mathbb{E}_{\mathbb{Q}}[\tau], \mathbb{E}_{\mathbb{Q}}[Y_\tau]\big)\,, && (5.11)
\end{aligned}
$$

for $\hat{\Psi}(\eta,\Upsilon) := \eta\hat{\psi}\big(\frac{Y_{0-}-\Theta_{0-}-\Upsilon}{\beta\eta}\big) - F(\Upsilon)$ when $\eta > 0$, while for $\tau = 0$ we get that $\mathbb{E}[L_\infty]$ is given by (5.11) with $\hat{\Psi}(0,\Upsilon) := -F(\Upsilon)$. The step from (5.8) to (5.9) uses that $\mathbb{E}[\overline{S}_\tau B_\tau] = 0$, due to $(\overline{S}B)_{\cdot\wedge\tau}$ being UI, and $\int_0^t \beta Y_s\,\mathrm{d}s = \hat{\sigma}B_t + \Theta_t - \Theta_{0-} - Y_t + Y_{0-}$. Since F is strictly convex, we obtain equality in (5.10) if and only if Y_τ is concentrated at a point $\Upsilon \in \mathbb{R}$ $\mathbb{P}$-a.s. At (5.7) we obtain equality if and only if either $Y_t \in (-\infty, y^*]$ μ-a.e. (where $\hat{\psi}$ is affine) or Y_t is concentrated at a point $\tilde{\Upsilon} \in \mathbb{R}$ μ-a.e. Equality at (5.6) can only happen if $Y \ge y^*$ μ-a.e. Hence, we only get equality

$$
\mathbb{E}[L_\infty] = F(Y_{0-}) + \hat{\Psi}\big(\mathbb{E}_{\mathbb{Q}}[\tau], \mathbb{E}_{\mathbb{Q}}[Y_\tau]\big)
$$

for *impact fixing strategies* $\Theta = \Theta^{\tilde{\Upsilon},\Upsilon}$ with $\tilde{\Upsilon} \ge y^*$, where $\mathbb{E}[L_\tau] = F(Y_{0-}) + \hat{\Psi}(\mathbb{E}[\tau], \Upsilon)$. Since y^* is the largest maximizer of $\hat{\psi}$, $\lim_{y\to\infty}\psi'(y) = -\infty$ and F is strictly increasing, $\hat{\Psi}(\eta,\cdot)$ has a unique maximizer $\hat{e}(\eta) \in (-\infty, e^*)$ where $e^* = e^*(\eta) = Y_{0-} - \Theta_{0-} - \beta\eta y^*$ for $\eta > 0$ and $\hat{e}(0) = e^*(0) = Y_{0-} - \Theta_{0-}$. Because $\hat{y}(\eta) := \big(Y_{0-} - \Theta_{0-} - \hat{e}(\eta)\big)/(\beta\eta) > y^*$, the impact fixing strategy $\Theta^{\hat{y}(\eta),\hat{e}(\eta)}$ has expected time to liquidation η (cf. Lemma 5.1) and generates $F(Y_{0-}) + \hat{\Psi}\big(\eta, \hat{e}(\eta)\big)$ expected proceeds that are optimal among all impact fixing strategies with expected time to liquidation η.

Note that $\hat{e}(\eta)$ is continuous in $\eta \in (0,+\infty)$ by the implicit function theorem; recall that $\hat{e}(\eta)$ solves $0 = \hat{\Psi}_\Upsilon(\eta,\hat{e}(\eta)) = -\hat{\psi}'(\hat{y}(\eta))/\beta - f(\hat{e}(\eta))$, and $\hat{\Psi}_{\Upsilon\Upsilon}(\eta,\Upsilon) < 0$ for $\Upsilon < e^*(\eta)$. Moreover, $\hat{e}(\eta) \to \hat{e}(0)$ when $\eta \to 0$, otherwise $\hat{y}(\eta) \to +\infty$ for a subsequence giving $-\hat{\psi}'(\hat{y}(\eta))/\beta = -(fk)(\hat{y}(\eta))/\beta \to +\infty$ and therefore also $f(\hat{e}(\eta)) \to +\infty$, which would contradict $\limsup_{\eta\to 0}\hat{e}(\eta) \le \lim_{\eta\to 0} e^*(\eta) = Y_{0-} - \Theta_{0-}$.

In particular, the contradiction argument above shows that $\hat{y}(\eta)$ is contained in a compact set for small η. As a consequence, $\hat{G}(\eta,\hat{e}(\eta)) = \eta\hat{\psi}(\hat{y}(\eta)) - F(\hat{e}(\eta)) \to \hat{G}(0,\hat{e}(0))$ as $\eta \to 0$, i.e. the map $\eta \mapsto \hat{\Psi}(\eta,\hat{e}(\eta))$ is continuous on $[0,+\infty)$. Hence, it attains a maximizer $\hat{\eta} \in [0,\eta_{\max}]$ whose associated impact fixing strategy $\hat{\Theta} = \Theta^{\hat{y}(\hat{\eta}),\hat{e}(\hat{\eta})}$ generates maximal expected proceeds in expected time $\mathbb{E}[\tau^{\hat{y}(\hat{\eta}),\hat{e}(\hat{\eta})}] = \hat{\eta}$ among all admissible strategies $\mathcal{A}_{\eta_{\max}}$.

If $f(y) = e^{\lambda y}$ with $\lambda \in (0,\infty)$, one can check by direct calculations that $\hat{G}_\eta(\eta,\Upsilon) > 0$ for $\eta > 0$, $\Upsilon \in \mathbb{R}$, and thus using $\frac{\mathrm{d}}{\mathrm{d}\eta}\hat{G}(\eta,\hat{e}(\eta)) = \hat{G}_\eta(\eta,\hat{e}(\eta)) + \hat{G}_\Upsilon(\eta,\hat{e}(\eta))\hat{e}'(\eta) = \hat{G}_\eta(\eta,\hat{e}(\eta))$, the map $\eta \mapsto \hat{\Psi}(\eta,\hat{e}(\eta))$ is strictly increasing, so $\hat{\eta} = \eta_{\max}$ is its unique maximizer in $[0,\eta_{\max}]$ and hence the optimal strategy is unique. □

5.4. Price impact with partially instantaneous recovery

This example is inspired by work of [Roc11] on a different (additive impact, block-shaped LOB) price impact model; adapting his interesting idea to our setup leads to an extension of our transient impact model, where a further parameter $\eta \in (0,1]$ permits for partially instantaneous recovery of price impact. Further, this example illustrates how proceeds from trading could, at first, be given for simple strategies only, and continuity arguments are key for an extension to a larger space of strategies and its analysis.

Motivated by observations that other traders respond quickly to market orders by adding limit orders in opposite direction, [Roc11] proposed a model where impact from a block trade is partially instantaneous and partially transient. A market sell (resp. buy) order eats into the bid (resp. ask) side of a LOB and is filled at respective prices, price impact being a function of the shape of the LOB. A certain fraction $1-\eta$ $(0<\eta\le 1)$ of that impact is instantaneously recovered directly after the trade, while only the remaining η-fraction constitutes a transient impact that decays gradually over time (cf. (5.12)). As stated in [Roc11], this means that "we think of $1-\eta$ as the fraction of the order book which is renewed after a market order so that in practice the actual impact on prices is η times the full impact". In our previous model for a two-sided LOB (non-monotone strategies), with the idealizing assumption of zero bid-ask spread, the model with full impact $(\eta = 1)$ implicitly postulates that the gap between bid and ask prices after a block buy (resp. sell) order is filled up instantaneously with ask (resp. bid) orders. For one-directional trading such hypothesis is conservative, but for trading in alternating directions it may be overly optimistic. So, it appears at first sight to be an interesting generalization to postulate that the gap is closed from both sides in a certain fraction.

To incorporate this into our setup, let $\eta \in [0,1]$ and suppose that the impact directly after completion of a block trade of size $\Delta\Theta_t$ at time $t \in [0,\infty)$ is actually $Y_{t-} + \eta\Delta\Theta_t$, where Y_{t-} is the market impact immediately before the trade. Thus, the market impact process $Y^{\eta,\Theta}$ evolves according to

$$\mathrm{d}Y_t^{\eta,\Theta} = -h(Y_t^{\eta,\Theta})\,\mathrm{d}\langle M\rangle_t + \eta\,\mathrm{d}\Theta_t, \quad t \ge 0. \tag{5.12}$$

Indeed, (5.12) holds for simple strategies Θ and hence for all càdlàg trading programs Θ by continuity of $\Theta \mapsto Y^{\eta,\Theta}$ in the uniform and Skorokhod J_1 and M_1 topologies.

The case $\eta = 0$ corresponds to no (non-instantaneous) impact while $\eta = 1$ gives our previous setup with full impact. The situation where $\eta \in (0,1)$ is more delicate, in that executing a block order at once would always be suboptimal, whereas subdividing a block trade into smaller ones and executing them one after the other would lead to smaller expenses, i.e. larger proceeds, due to the instantaneous partial recovery of price impact. Thus, there would be a difference between *asymptotically realizable* proceeds from a block trade (in the terminology of [BB04]) and its direct proceeds from a LOB interpretation.

Motivated by optimization questions like the optimal trade execution problem where a trader tries to evade illiquidity costs from large (block) orders, if possible, our aim is to specify a model that is stable with respect to small intertemporal changes, in particular approximating block trades by subdividing the trade into small packages and executing them in short time intervals. Thus, the proceeds that we will derive here will be asymptotically realizable. First, let us only assume that at every time $t \ge 0$, the average price per share for a block trade of size Δ is some value between $f(Y_{t-})\overline{S}_t$ and $f(Y_{t-}+\Delta)\overline{S}_t$, where Y_{t-} is the state of the impact process right before the block trade.

Hence, the arguments in the proof of Lemma 3.1 carry over (with $c = 1/\eta$, $Y = Y^\eta/\eta$ and suitably re-scaled functions f,h) and yield that the proceeds from implementing a continuous finite variation strategy Θ should be given by $\tilde{L}_T(\Theta) = -\int_0^T \overline{S}_t f(Y_t^{\eta,\Theta})\,\mathrm{d}\Theta_t$, $T \geq 0$, irrespective of a particular initial specification for proceeds from block trades. As such was the starting point for Section 3, the analysis there for the case $\eta = 1$ carries over to the model extension for $\eta \in (0,1]$: For any continuous f.v. process Θ we obtain

$$\tilde{L}_T(\Theta) = \frac{1}{\eta}\left(\int_0^T F(Y_{u-}^{\eta,\Theta})\,\mathrm{d}\overline{S}_u - \int_0^T \overline{S}_u (fh)(Y_u^{\eta,\Theta})\,\mathrm{d}\langle M\rangle_u - \big(\overline{S}_T F(Y_T^{\eta,\Theta}) - \overline{S}_0 F(Y_{0-}^{\eta,\Theta})\big)\right). \quad (5.13)$$

By Theorem 3.7 the right-hand side of (5.13) is continuous in the predictable strategy Θ taking values in $D([0,T];\mathbb{R})$ when endowed with any of the uniform, Skorokhod J_1 and M_1 topologies. So, asymptotically realizable proceeds are given by (5.13). In particular, asymptotically realizable proceeds from a block sale of size $\Delta \neq 0$ at time t are

$$-\frac{1}{\eta}\overline{S}_t\big(F(y_{t-} + \eta\Delta) - F(y_{t-})\big) = -\frac{1}{\eta}\overline{S}_t \int_0^{\eta\Delta} f(y_{t-} + x)\,\mathrm{d}x\,,$$

where y_{t-} denotes the state of the market impact process before the trade. Note that these proceeds strictly dominate the proceeds $-\overline{S}_t \int_0^\Delta f(y_{t-} + x)\,\mathrm{d}x$ that would arise from a executing the block sale in the LOB corresponding to the price impact function f. Also this model variant is free of arbitrage in the sense of Theorem 4.3, whose proof carries over. In mathematical terms one may observe, maybe surprisingly, that the model structure (see (5.12) and (5.13)) for the extension $\eta \in (0,1]$ is like the one for the previous model (with $\eta = 1$), and is hence amenable to a likewise analysis. In finance terms however, to model partially instantaneous recovery in such a way does not lead to new qualitative features for the model, since the large investor could side-step the disadvantageous effect from block trades by trading continuously, at least in absence of further frictions (like e.g. bid-ask spread).

A. Appendix

The next proposition collects known continuity properties of the solution map $\Theta \mapsto Y^\Theta$ on $D([0,T];\mathbb{R})$ from (2.2), with the presentation being adapted to our setup.

Proposition A.1. *Assume that h is Lipschitz continuous and $\langle M\rangle = \int_0^\cdot \alpha_s\,\mathrm{d}s$ with bounded density α. Then the solution map $D([0,T];\mathbb{R}) \mapsto D([0,T];\mathbb{R})$, with $\Theta \mapsto Y^\Theta$ from* (2.2)*, is defined pathwise. The map is continuous when the space $D([0,T];\mathbb{R})$ is endowed with either the uniform topology or the Skorokhod J_1 or M_1 topology. Moreover, if Θ is an adapted càdlàg process, then the process Y^Θ is also adapted.*

Proof. The proof in the case of the uniform topology and the Skorokhod J_1 topology is given in [PTW07, proof of Thm. 4.1]; the proof there is in the case $\alpha \equiv 1$ but it clearly extends to our setup as long as α is uniformly bounded. For the M_1 topology, cf. [PW10, Thm. 1.1], where again the main argument ([PW10, proof of Thm. 1.1]) extends to our setup of more general α. That Y^Θ is adapted follows from the (pathwise) construction of Y^Θ as the (a.s.) limit (in the uniform topology) of adapted processes, the solution processes for a sequence of piecewise-constant controls Θ^n approximating uniformly Θ, cf. [PTW07, proof of Thm. 4.1]. $\square$

In general, we may have $\alpha_n \to \alpha$ and $\beta_n \to \beta$ in $D([0,T])$ endowed with J_1 (or M_1), and yet $\alpha_n + \beta_n \not\to \alpha + \beta$ when α and β have a common jump time. However, in special cases like in what follows, this does not happen.

Lemma A.2 (Allowed cancellation of jumps for J_1)**.** *Let $\alpha_n \to \alpha_0$ and $\beta_n \to \beta_0$ in $(D([0,T]), J_1)$ with the following property: for every $n \geq 0$ and every $t \in (0,T)$*

$$\Delta\alpha_n(t) \neq 0 \quad \textit{implies} \quad \Delta\beta_n(t) = -\Delta\alpha_n(t).$$

Then $\alpha_n + \beta_n \to \alpha_0 + \beta_0$ in $(D([0,T]), J_1)$.

Proof. By [JS03, Prop. VI.2.2, a] it suffices to check that for every $t \in (0,T)$ there exists a sequence $t_n \to t$ such that $\Delta\alpha_n(t_n) \to \Delta\alpha_0(t)$ and $\Delta\beta_n(t_n) \to \Delta\beta_0(t)$.

Let $t \in (0,T)$ be arbitrary and first suppose that $\Delta\alpha_0(t) \neq 0$. Then [JS03, Prop. VI.2.1, a] implies the existence of a sequence $t_n \to t$ such that $\Delta\alpha_n(t_n) \to \Delta\alpha_0(t)$. Thus, our assumption on the sequence (β_n) gives $\Delta\beta_n(t_n) \to \Delta\beta_0(t)$. For the case $\Delta\alpha_0(t) = 0$, let $t_n \to t$ be such that $\Delta\beta_n(t_n) \to \Delta\beta_0(t)$. By [JS03, Prop. VI.2.1, b.5] we conclude that $\Delta\alpha_n(t_n) \to \Delta\alpha_0(t)$ as well, finishing the proof. $\square$

Let us note that the conclusion of Lemma A.2 does not hold for the M_1 topology. Consider for example $\alpha_0 = \mathbb{1}_{[1,\infty)}$ with approximating sequence $\alpha_n(t) := n\int_t^{t+1/n} \alpha_0(s)\,\mathrm{d}s$ and $\beta_0 = 1 - \alpha_0$ with approximating sequence $\beta_n(t) := n\int_{t-1/n}^{t} \beta_0(s)\,\mathrm{d}s$. Thus we need the following refined statement.

Lemma A.3 (Allowed cancellation of jumps for M_1)**.** *Let $\alpha_n \to \alpha_0$ in $(D([0,T]), \|\cdot\|_\infty)$ and $\beta_n \to \beta_0$ in $(D([0,T]), M_1)$ with the following property: $t \in Disc(\alpha_0)$ implies $\beta_n \to \beta_0$ locally uniformly in a neighborhood of t. Then $\alpha_n + \beta_n \to \alpha_0 + \beta_0$ in $(D([0,T]), M_1)$.*

Proof. We prove the following claim that suffices to deduce M_1-convergence of $\alpha_n + \beta_n$: For any $t \in [0,T]$ and $\varepsilon > 0$ there are $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$w_s(\alpha_n + \beta_n, t, \delta) \leq w_s(\alpha_n, t, \delta) + w_s(\beta_n, t, \delta) + \varepsilon \quad \text{for all } n \geq n_0. \tag{A.1}$$

Indeed, if (A.1) holds, then the second condition in [Whi02, Thm. 12.5.1(v)] would hold, while the first condition there holds because of local uniform convergence at points of continuity of $\alpha_0 + \beta_0$: Either there is cancellation of jumps and thus local uniform convergence by our assumption, or both paths do not jump which still gives local uniform convergence because M_1-convergence implies such at continuity points of the limit.

To check (A.1), we have $\lim_{\delta\downarrow 0}\limsup_{n\to\infty} v(\alpha_n, \alpha_0, t, \delta) = 0$ at points $t \in [0,T]$ with $\Delta\alpha_0(t) = 0$, where for $x_1, x_2 \in D([0,T])$

$$v(x_1, x_2, t, \delta) := \sup_{0\vee(t-\delta)\leq t_1, t_2 \leq (t+\delta)\wedge T} |x_1(t_1) - x_2(t_2)|,$$

see [Whi02, Thm. 12.4.1], which implies (A.1) for small δ and large n. Now if $t \in \mathrm{Disc}(\alpha_0)$, $\alpha_n \to \alpha_0$ and $\beta_n \to \beta_0$ locally uniformly in a neighborhood of t which implies that for small δ and large n

$$w_s(\alpha_n + \beta_n, t, \delta) \leq w_s(\alpha_0 + \beta_0, t, \delta) + \varepsilon/2.$$

By possibly decreasing δ we can make $w_s(\alpha_0 + \beta_0, t, \delta)$ smaller than $\varepsilon/2$ because $\alpha_0 + \beta_0 \in D([0,T])$, which finishes the proof. $\square$

Lemma A.4 (Uniform convergence of jump term)**.** *Let $\alpha, \beta_n, \beta \in D([0,T])$ be such that $[\alpha]^d_T := \sum_{t \leq T : \Delta\alpha(t) \neq 0} |\Delta\alpha(t)|^2 < \infty$, that β_n are uniformly bounded and at every jump time $t \in [0,T]$ of α, $\Delta\alpha(t) \neq 0$, we have pointwise convergence $\beta_n(t) \to \beta(t)$. Let $G \in C^2$ such that $y \mapsto G_{xx}(x,y)$ is Lipschitz continuous on compacts. Then the sum*

$$J(\alpha, \beta_n)_t := \sum_{\substack{u \leq t \\ \Delta\alpha(t) \neq 0}} G\big(\alpha(t), \beta_n(t)\big) - G\big(\alpha(t-), \beta_n(t)\big) - G_x\big(\alpha(t-), \beta_n(t)\big)\Delta\alpha(t)$$

converges uniformly for $t \in [0,T]$ to $J(\alpha,\beta)_t$, as $n \to \infty$.

Proof. Since α, $[\alpha]^d$, β_n and β are bounded on $[0,T]$ by a constant $C \in \mathbb{R}$, we can assume w.l.o.g. that G_{xx} is globally Lipschitz in y with Lipschitz constant L. Hence $J(\alpha,\beta_n)_t < \infty$ by Taylor's theorem. Let $H(x, \Delta x, y) := G(x + \Delta x, y) - G(x,y) - G_x(x,y)\Delta x$ and denote by $\tilde{J}^{n,\pm}$ the increasing and decreasing components of $J(\alpha,\beta_n) - J(\alpha,\beta)$, respectively, i.e.

$$\tilde{J}^{n,+}_t := \sum_{\substack{u \leq t \\ \tilde{H}(\dots) > 0}} \tilde{H}\big(\alpha(u-), \Delta\alpha(u), \beta_n(u), \beta(u)\big), \quad \tilde{J}^{n,-}_t := \sum_{\substack{u \leq t \\ \tilde{H}(\dots) < 0}} \tilde{H}\big(\alpha(u-), \Delta\alpha(u), \beta_n(u), \beta(u)\big),$$

for $\tilde{H}(x, \Delta x, y, z) := H(x, \Delta x, y) - H(x, \Delta x, z)$. Moreover, take any enumeration $\{t_k \mid k \in \mathbb{N}\} = \{t \mid \Delta\alpha(t) \neq 0\}$ of the jump times of α and arbitrary $\varepsilon > 0$. Since $[\alpha]^d < \infty$, there exists $K \in \mathbb{N}$ such that $\sum_{k>K} |\Delta\alpha(t_k)|^2 < \varepsilon/(2CL)$. Moreover, we have $|\tilde{H}(x, \Delta x, y, z)| \leq \frac{1}{2}|\Delta x|^2 L|y - z|$ and thus

$$|\tilde{J}^{n,\pm}_T| \leq \frac{L}{2}\sum_{k=1}^{\infty} |\Delta\alpha(t_k)|^2 |\beta_n(t_k) - \beta(t_k)| < \frac{\varepsilon}{2} + \frac{L}{2}\Big(\max_{1 \leq k \leq K} |\beta_n(t_k) - \beta(t_k)|\Big) \sum_{k=1}^{K} |\Delta\alpha(t_k)|^2 .$$

By pointwise convergence $\beta_n(t_k) \to \beta(t_k)$ at all t_k, there exists $N \in \mathbb{N}$ such that for all $k = 1, \dots, K$ and $n \geq N$ we have $|\beta_n(t_k) - \beta(t_k)| < \varepsilon/(L[\alpha]^d_T)$ and therefore $|\tilde{J}^{n,\pm}_T| < \varepsilon$ for $n \geq N$. Hence $J^{n,\pm}_T \to 0$ as $n \to \infty$.

Since $J^{n,\pm}$ are monotone and do not cross zero, we have $\sup_{0 \leq t \leq T} |\tilde{J}^{n,\pm}_t| = |\tilde{J}^{n,\pm}_T|$ and therefore uniform convergence $\tilde{J}^{n,\pm} \to 0$ on $[0,T]$. So in particular $J(\alpha,\beta_n)$ converges to $J(\alpha,\beta)$, uniformly on $[0,T]$. $\square$

Proof of Lemma 3.12. Since Θ is of finite variation, we have $\mathrm{d}[Z^j, Z^m]_t^c = \mathrm{d}[\overline{S}]_t^c$ for $j = m = 2$, and 0 otherwise. So the $\partial\Phi_{\cdot,j}/\partial x_\ell$ terms in equation (3.17) simplify to

$$\frac{1}{2}\sum_{j,m=1}^{3}\sum_{\ell=1}^{3}\int_0^t \frac{\partial\Phi_{\cdot,j}}{\partial x_\ell}(X_{s-})\Phi_{\ell,m}(X_{s-})\,\mathrm{d}[Z^j, Z^m]_s^c = (0,\ 0,\ 0)^{tr}. \tag{A.2}$$

Jumps of Z are of the form $\Delta Z_s = \big(\Delta\Theta_s,\ \Delta\overline{S}_s,\ 0\big)^{tr}$, so for $\xi(X) := \Phi(X)\Delta Z_s$ we obtain $\xi(X) = \big(-g(X^3, X^2)\Delta\Theta_s,\ \Delta\Theta_s,\ \Delta\overline{S}_s\big)^{tr}$, which yields the solutionto (3.18) as $y(u) = V_u = (V_u^1, V_u^2, V_u^3)^{tr} \in \mathbb{R}^3$ with $V_0 = X_{s-}$,

$$\begin{aligned}
V_u^2 &= Y_{s-} + \int_0^u \Delta\Theta_s\,\mathrm{d}x = Y_{s-} + u\Delta\Theta_s\,,\\
V_u^3 &= \overline{S}_{s-} + \int_0^u \Delta\overline{S}_s\,\mathrm{d}x = \overline{S}_{s-} + u\Delta\overline{S}_s\,,\\
V_u^1 &= L_{s-} - \int_0^u g(\overline{S}_{s-} + x\Delta\overline{S}_s, Y_{s-} + x\Delta\Theta_s)\Delta\Theta_s\,\mathrm{d}x\\
&= L_{s-} - \int_0^{u\Delta\Theta_s} g(\overline{S}_{s-}, Y_{s-} + x)\,\mathrm{d}x\,,
\end{aligned}$$

since quasi-left continuity of $\overline{S}$ gives that a.s. $\Delta\overline{S}_s = 0$ whenever $\Delta\Theta_s \neq 0$ (jumps of Θ occur at predictable times). Thus the jump terms in (3.17) become

$$\begin{aligned}
&\varphi(\Phi(\cdot)\Delta Z_s, X_{s-}) - X_{s-} - \Phi(X_{s-})\Delta Z_s\\
&\quad = \Big(-\int_0^{\Delta\Theta_s} g(\overline{S}_{s-}, Y_{s-} + x)\,\mathrm{d}x + g(\overline{S}_{s-}, Y_{s-})\Delta\Theta_s,\ 0,\ 0\Big)^{tr}.
\end{aligned} \tag{A.3}$$

Furthermore, the Itô integral in (3.17) reads

$$\int_0^t \Phi(X_{s-})\,\mathrm{d}Z_s = \begin{pmatrix} -\int_0^t g(\overline{S}_{s-}, Y_{s-})\,\mathrm{d}\Theta_s \\ -\int_0^t h(Y_s)\,\mathrm{d}\langle M\rangle_s + \Theta_t - \Theta_{0-} \\ \overline{S}_t - \overline{S}_{0-} \end{pmatrix}. \tag{A.4}$$

Summing up X_{0-} and equations (A.2) to (A.4) yields the second and third components $Y_{0-} - \int_0^t h(Y_s)\,\mathrm{d}s + \Theta_t - \Theta_{0-} = Y_t$ and $\overline{S}_{0-} + \overline{S}_t - \overline{S}_{0-} = \overline{S}_t$, respectively. To complete the proof, we note that for the first component we get

$$L_{0-} - \int_0^t g(\overline{S}_{s-}, Y_{s-})\,\mathrm{d}\Theta_s + \sum_{\substack{0\le s\le t\\ \Delta\Theta_s\neq 0}}\Big(g(\overline{S}_{s-}, Y_{s-})\Delta\Theta_s - \int_0^{\Delta\Theta_s} g(\overline{S}_{s-}, Y_{s-}+x)\,\mathrm{d}x\Big) = L_t\,. \ \square$$

The following proves the technical Lemma 5.1 about admissibility of impact fixing strategies in Section 5.3.

Proof of Lemma 5.1. By [BS02, Ch. 2, Sect. 2, eq. (2.0.2) on p. 295], the law of the hitting time H_z of level z by a Brownian motion with drift μ starting in x is for

$t \in (0,\infty)$ given by $\mathbb{P}_x[H_z \in \mathrm{d}t] = h^\mu(t, z-x)\,\mathrm{d}t$ with $h^\mu(t,x) := \frac{|x|}{\sqrt{2\pi}t^{3/2}}\exp\bigl(-\frac{(x-\mu t)^2}{2t}\bigr)$ and $\mathbb{P}_x[H_z = \infty] = 1-\exp\bigl(\mu(z-x)-|\mu|\cdot|z-x|\bigr)$. With $\mu = \beta\tilde{\Upsilon}/\rho$, $x = (\Theta_{0-}+\tilde{\Upsilon}-Y_{0-})/\rho$ and $z = (\tilde{\Upsilon}-\Upsilon)/\rho$ we obtain the stated terms for $\mathbb{E}[\tau] = \mathbb{E}_x[H_z]$.

Now, let $\tilde{\Upsilon}$, Υ be such that $\mathbb{E}[\tau] \le \eta_{\max}$. Independence of τ and $\overline{S}$ gives $\mathbb{E}[\overline{S}_\tau] = \overline{S}_0$ and $\mathbb{E}[\tau\overline{S}_\tau] < \infty$. We have $\int_0^\tau \overline{S}_t f(Y_t)\,\mathrm{d}B_t = f(\tilde{\Upsilon})M_\tau$ for $M_T := \int_0^{T\wedge\tau}\overline{S}_t\,\mathrm{d}B_t$ and $\int_0^\tau \overline{S}_t F(Y_t)\,\mathrm{d}W_t = F(\tilde{\Upsilon})\sigma^{-2}\overline{S}_\tau$. Note that $[M]_\tau = \sigma^{-2}[\overline{S}]_\tau$. We will show that M, $\overline{S}_{\cdot\wedge\tau}$ and $(\overline{S}B)_{\cdot\wedge\tau}$ are in $\mathcal{H}^1$ and hence UI. By the Burkholder-Davis-Gundy inequality [Pro04, Thm. IV.4.48], there exists $C>0$ such that $\mathbb{E}\bigl[[\overline{S}]_\tau^{1/2}\bigr] \le C\mathbb{E}[\sup_{u\le\tau}|\overline{S}_u|] = C\mathbb{E}[\exp(\sigma X_\tau)]$ with $X_t := \sup_{u\le t}(W_u - \frac{\sigma}{2}u)$. Using $\{X_t > z\} = \{H_z < t\}$ for $z, t \ge 0$ with starting point $X_0 = 0$ and drift $\mu = -\sigma/2$ we first obtain

$$\begin{aligned}\mathbb{E}[\exp(\sigma X_t)] &= \int_{[0,\infty]} e^{\sigma x}\mathbb{P}[X_t \in \mathrm{d}x] = \int_{[0,\infty]} e^{\sigma x}\,\mathrm{d}\bigl(1-\mathbb{P}[X_t > x]\bigr)_x \\ &= -\int_{[0,\infty]} e^{\sigma x}\,\mathrm{d}\bigl(\mathbb{P}[X_t > x]\bigr)_x = -\int_{[0,\infty]} e^{\sigma x}\,\mathrm{d}\bigl(\mathbb{P}[H_x < t]\bigr)_x\,.\end{aligned}$$

Since $\mathbb{P}[H_\infty < t] = 0$ we can approximate the Riemann-Stieltjes integral and apply integration by parts twice to get

$$\mathbb{E}[\exp(\sigma X_t)] = -\lim_{\varepsilon\searrow 0}\int_0^t\int_\varepsilon^{1/\varepsilon} e^{\sigma x}h_x^{-\sigma/2}(u,x)\,\mathrm{d}x\,\mathrm{d}u = -\int_0^t\int_0^\infty e^{\sigma x}h_x^{-\sigma/2}(u,x)\,\mathrm{d}x\,\mathrm{d}u$$

$$\text{with}\quad h_x^{-\sigma/2}(t,x) = \frac{\mathrm{d}}{\mathrm{d}x}h^{-\sigma/2}(t,x) = -\frac{x^2-t+\frac{\sigma}{2}xt}{\sqrt{2\pi}t^{5/2}}\exp\Bigl(-\frac{(x+\frac{\sigma}{2}t)^2}{2t}\Bigr)\,.$$

So we have $e^{\sigma x}h_x^{-\sigma/2}(t,x) = h_x^{\sigma/2}(t,x) - \sigma h^{\sigma/2}(t,x)$. The contribution from the first summand of the integrand $h_x^{\sigma/2}(t,x) - \sigma h^{\sigma/2}(t,x)$ is zero, since $h^{\sigma/2}(t,x) \to 0$ for $x\to\infty$ and for $x \to 0$. Hence, $\mathbb{E}[\exp(\sigma X_t)]$ equals

$$\begin{aligned}&\sigma\int_0^t\int_0^\infty h^{\sigma/2}(u,x)\,\mathrm{d}x\,\mathrm{d}u = \sigma\int_0^t\Bigl(\frac{\exp\bigl(-\frac{\sigma^2}{8}u\bigr)}{\sqrt{2\pi u}} - \frac{\sigma}{2} + \frac{\sigma}{2}\varphi\bigl(\tfrac{\sigma}{2}\sqrt{u}\bigr)\Bigr)\mathrm{d}u \\ &= 2\varphi\bigl(\tfrac{\sigma}{2}\sqrt{t}\bigr) - 1 + \tfrac{\sigma^2}{2}t\varphi\bigl(\tfrac{\sigma}{2}\sqrt{t}\bigr) - \tfrac{\sigma^2}{2}t + \frac{\sigma\sqrt{t}}{\sqrt{2\pi}}\exp\bigl(-\tfrac{\sigma^2}{8}t\bigr) \le 1 + \frac{\sigma\sqrt{t}}{\sqrt{2\pi}}\,,\end{aligned}$$

where $\varphi(x) = \int_{-\infty}^x e^{-z^2/2}\,\mathrm{d}z/\sqrt{2\pi}$. So by independence of X and τ

$$\mathbb{E}[\exp(\sigma X_\tau)] = \mathbb{E}[(t\mapsto\mathbb{E}[e^{\sigma X_t}])(\tau)] \le \mathbb{E}\Bigl[1+\frac{\sigma}{\sqrt{2\pi}}\sqrt{\tau}\Bigr] \le 1 + \frac{\sigma}{\sqrt{2\pi}}(1+\mathbb{E}[\tau]) < \infty\,.$$

Moreover, $[\overline{S}B]_\tau = \tau[\overline{S}]_\tau$ by independence of $\overline{S}$ and B, so we can bound $\mathbb{E}\bigl[[\overline{S}B]_\tau^{1/2}\bigr]$ by $\mathbb{E}\bigl[\sqrt{\tau}[\overline{S}]_\tau^{1/2}\bigr] = \mathbb{E}\Bigl[\sqrt{t}\mathbb{E}\bigl[[\overline{S}]_t^{1/2}\bigr]\Big|_{t=\tau}\Bigr] \le C\mathbb{E}\Bigl[\sqrt{t}\mathbb{E}[\exp(\sigma X_t)]\Big|_{t=\tau}\Bigr] \le C\mathbb{E}[\sqrt{\tau} + \frac{\sigma}{\sqrt{2\pi}}\tau] < \infty$. Thus, $(\overline{S}B)_{\cdot\wedge\tau}$ is in $\mathcal{H}^1$ and hence UI.

Finally, $\bigl(L_\tau(\Theta)\bigr)^- \in L^1(\mathbb{P})$ follows from $\int_0^\tau \overline{S}_t g(Y_{t-}^\Theta)\,\mathrm{d}t = g(\tilde{\Upsilon})\int_0^\tau \overline{S}_t\,\mathrm{d}t$, which is integrable by optional projection [DM82, Thm. VI.57] since $\mathbb{E}[\tau\overline{S}_\tau] < \infty$, and integrability of $\overline{S}_\tau F(Y_\tau^\Theta) = \overline{S}_\tau F(\Upsilon)$. $\square$

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