The class of multivariate max-id copulas with $\ell_1$-norm symmetric exponent measure

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ABSTRACT

Members of the well-known family of bivariate Galambos copulas can be expressed in a closed form in terms of the univariate Fréchet distribution. This formula extends to any dimension and can be used to define a whole new class of tractable multivariate copulas that are generated by suitable univariate distributions. This paper gives necessary and sufficient conditions on the underlying univariate distribution which ensure that the resulting copula exists. It is also shown that these new copulas are in fact dependence structures of certain max-id distributions with $\ell_1$-norm symmetric exponent measure. The basic dependence properties of this new class of multivariate exchangeable copulas is investigated, and an efficient algorithm is provided for generating observations from distributions in this class.

Keywords: Clayton copula; Completely monotone function; exponent measure; Galambos copula; Laplace transform; $\ell_1$-norm symmetric max-id distributions.

1 Introduction

Copula models are ubiquitous in multivariate data analysis and the topic has been the object of several books and innumerable articles in the past 30 years; see, e.g., Joe (2015) for a review. The popularity of copula models stems from the flexibility they provide in capturing the behavior of any random vector $(X_1, \ldots, X_d) \in \mathbb{R}^d$ by coupling parametric distributions $F_1, \ldots, F_d$ for its marginals with a copula $C$, so that, for all $x_1, \ldots, x_d \in \mathbb{R}$,

$$\Pr(X_1 \leq x_1, \ldots, X_d \leq x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\}.$$  

Common choices of parametric families for $C$ include Archimedean, Archimax, elliptical, and extreme-value copulas, as described, e.g., in Nelsen (2006), Genest and Nešlehová (2012) and Charpentier et al. (2014). Even greater flexibility can be achieved by resorting to vine, hierarchical, and factor copula constructions; see, e.g., Kurowicka and Joe (2011), Mai and Scherer (2012) and Krupskii and Joe (2013). Nevertheless, there is still a need to expand the supply of multivariate copula structures, particularly with models that are tractable, interpretable, and relatively easy to simulate.
The purpose of this paper is to propose such a class of copulas and to explore its properties. Each member of this new class is generated by a continuous univariate distribution function \( F \) with support \([0, x_F] \) for some \( 0 < x_F \leq \infty \). In the bivariate case, the copula \( C_F \) generated by \( F \) is defined, for all \( u_1, u_2 \in (0, 1) \), by

\[
C_F(u_1, u_2) = \frac{u_1 u_2}{F^{-1}(u_1) + F^{-1}(u_2)},
\]

where \( F^{-1} \) is the generalized inverse of \( F \) defined, at all \( u \in (0, 1] \), by

\[
F^{-1}(u) = \inf\{ t \in \mathbb{R} : F(t) \geq u \}, \quad \text{and at } u = 0 \text{ by } F^{-1}(0) = \sup\{ t \in \mathbb{R} : F(t) = 0 \}.
\]

We propose to call copulas in this new class reciprocal Archimedean because in dimension \( d = 2 \), their form is reminiscent of Archimedean copulas (Genest and MacKay, 1986; McNeil and Nešlehová, 2009) defined, for all \( u_1, u_2 \in (0, 1) \), by

\[
C_\psi(u_1, u_2) = \psi\{\psi^{-1}(u_1) + \psi^{-1}(u_2)\}
\]

for some suitable choice of generator \( \psi : [0, \infty) \to [0, 1] \). Bivariate reciprocal Archimedean copulas were first explored briefly by Khoudraji (1995) who observed, e.g., that the Galambos and Clayton copulas with positive dependence are included in this class.

To define a reciprocal Archimedean copula with generator \( F \) in any dimension \( d \geq 2 \), let

\[
\begin{align*}
P_{d,o} & = \{ A \subseteq \{1, \ldots, d\} : A \neq \emptyset \text{ and } |A| \text{ is odd} \}, \\
P_{d,e} & = \{ A \subseteq \{1, \ldots, d\} : A \neq \emptyset \text{ and } |A| \text{ is even} \},
\end{align*}
\]

and set, for all \( u_1, \ldots, u_d \in (0, 1) \),

\[
C_F(u_1, \ldots, u_d) = \prod_{A \in P_{d,o}} F\left(\sum_{k \in A} F^{-1}(u_k)\right) \bigg/ \prod_{A \in P_{d,e}} F\left(\sum_{k \in A} F^{-1}(u_k)\right).
\]

As the numerator and denominator of (3) have \( 2^{d-1} \) and \( 2^{d-1} - 1 \) factors respectively, the expression for \( C_F \) is more involved than that of a multivariate Archimedean copula with generator \( \psi \), viz.

\[
C_\psi(u_1, \ldots, u_d) = \psi\left\{ \sum_{k=1}^d \psi^{-1}(u_k) \right\}.
\]

Nonetheless there turn out to be surprising parallels between these two classes of copulas. First, it will be shown here that a continuous distribution function \( F \) with support \([0, x_F] \) generates a reciprocal Archimedean copula in dimension \( d \) if and only if \( \Lambda = -\ln(F) \) is \( d \)-monotone. This resembles the result that \( C_\psi \) given in (4) is a copula if and only if \( \psi \) is a \( d \)-monotone Archimedean generator (Malov, 2001; McNeil and Nešlehová, 2009).

Second, and more importantly, it will be seen that reciprocal Archimedean copulas have a stochastic representation that parallels that of Archimedean copulas. As shown by McNeil and Nešlehová (2009), the function given, for all \( x_1, \ldots, x_d \in [0, \infty) \), by

\[
C_\psi\{\psi(x_1), \ldots, \psi(x_d)\} = \psi(x_1 + \cdots + x_d)
\]
is the survival function of a $d$-variate $\ell_1$-norm symmetric distribution, whose radial variable is in one-to-one relationship with $\psi$. The surprising result established here is that the function given, for all $x_1, \ldots, x_d \in [0, \infty)$, by

$$H_F(x_1, \ldots, x_d) = C_F\{F(x_1), \ldots, F(x_d)\} = \prod_{A \in \mathcal{P}_{d,e}} F\left(\sum_{i \in A} x_i\right) / \prod_{A \in \mathcal{P}_{d,o}} F\left(\sum_{i \in A} x_i\right)$$  \hspace{1cm} (5)$$

is the cumulative distribution function of a max infinitely divisible (max-id) distribution whose exponent measure $\mu$ has an $\ell_1$-norm symmetric structure. The radial measure $\nu$ of $\mu$, which plays an analogous role as the radial variable of an $\ell_1$-norm symmetric distribution, is in one-to-one relationship with the reciprocal Archimedean generator $F$. Furthermore, similarly to the Archimedean case, $\nu$ can be calculated explicitly using the inverse Williamson $d$-transform of $\Lambda$.

Third, just as $\psi$ must be completely monotone for (4) to be an Archimedean copula in every dimension, $\Lambda = -\ln(F)$ must be completely monotone for $F$ to generate a reciprocal Archimedean copula for all $d \geq 2$. In this special case, which was briefly investigated by Lmoudden (2011), further parallels can be drawn between reciprocal Archimedean copulas and the dependence structures of multiplicative hazard models (Marshall and Olkin, 1988).

The layout of this article is as follows. Preliminary observations are given in Section 2, consolidating the initial investigations of Khoudraji (1995) and Lmoudden (2011). The key insight provided in Section 3 for the first time is that reciprocal Archimedean copulas are precisely the dependence structures of max-id distributions with $\ell_1$-norm symmetric exponent measure. As an implication, it is established therein that $C_F$ is a copula in a given dimension $d$ if and only if $\Lambda = -\ln(F)$ is $d$-monotone on $(0, \infty)$. A geometric insight into the structure of $\ell_1$-norm symmetric exponent measures is provided in Section 4, which leads, in Section 5, to a stochastic representation for reciprocal Archimedean copulas in terms of Poisson point processes, and to an effective simulation algorithm. Section 6 provides a number of new examples and illustrations. The case of completely monotone $\Lambda$ is briefly discussed in Section 7 and Section 8 provides a roadmap of the case where $F^{-1}(0) > 0$, which includes, e.g., Ali–Mikhail–Haq copulas with positive dependence. Section 9 contains brief concluding remarks. Lengthy proofs are relegated to a series of Appendices.

In what follows, $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ represent vectors in $\mathbb{R}^d$; furthermore, $\mathbf{0}$ stands for the origin in $\mathbb{R}^d$ and $\infty = (\infty, \ldots, \infty)$. Unless stated otherwise, all expressions such as $x + y$, $\max(x, y)$ or $x \leq y$ are understood as component-wise operations. The set $[x, y]$ refers to $(x_1, y_1] \times \cdots \times (x_d, y_d]$ and similarly for $[-\infty, x)$, $[0, \infty]$, etc. Furthermore, $\|x\|_1 = |x_1| + \cdots + |x_d|$ denotes the $\ell_1$-norm of $x$. Finally, $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{R}_+^d = [0, \infty)^d$.

2 Elementary properties

This section surveys some of the basic properties of the class of reciprocal Archimedean copulas. To this end, let $\mathcal{F}_d$ be the set of continuous distribution functions $F$ with support
[0, x_F] for some 0 < x_F ≤ ∞ such that C_F defined in (3) is a bona fide d-dimensional copula; the notation \( \mathcal{F}_d \) reflects the fact that a reciprocal Archimedean generator may generate a copula in certain dimensions only. Let also \( \mathcal{C}_d = \{ C_F : F \in \mathcal{F}_d \} \) be the set of all reciprocal Archimedean copulas in dimension \( d \geq 2 \).

First note that it is immediate from (3) that any reciprocal Archimedean copula \( C_F \) is exchangeable, i.e., for any permutation \( \pi \) of \( \{1, \ldots, d\} \) and \( u_1, \ldots, u_d \in [0, 1] \), \( C_F(u_\pi(1), \ldots, u_\pi(d)) = C_F(u_1, \ldots, u_d) \). Furthermore, all lower-dimensional margins of \( C_F \in \mathcal{C}_d \) are again reciprocal Archimedean with the same generator \( F \). Consequently, \( \mathcal{F}_2 \supseteq \mathcal{F}_3 \supseteq \cdots \).

Observe that the generator of a reciprocal Archimedean copula is not unique. However, as stated below and proved in Appendix A, all generators must be members of the same scale family. Reciprocal Archimedean generators could thus be made unique by an additional scaling condition, such as \( F(1) = 1/2 \); this option is not considered here, however.

**Lemma 1.** Two distributions \( F, G \in \mathcal{F}_d \) generate the same reciprocal Archimedean copula \( C_F = C_G \) if and only if there exists \( \kappa > 0 \) such that, for all \( t \in \mathbb{R} \), \( G(t) = F(\kappa t) \).

Distributions \( F \in \mathcal{F}_d \) are most easily characterized in terms of higher-order monotonicity of the map \( \Lambda : (0, \infty) \to [0, \infty) \) defined, for all \( t > 0 \), by

\[
\Lambda(t) = -\ln\{F(t)\}.
\]

Clearly, \( \Lambda \) is non-negative and non-increasing; moreover, \( \Lambda(t) \to \infty \) as \( t \to 0 \) and \( \Lambda(t) \to 0 \) as \( t \to \infty \). Khoudraji (1995) further observed that if \( F \in \mathcal{F}_2 \) has derivative \( f \) on \( (0, x_F) \), then the reverse hazard rate \( r = f/F \) must be non-increasing, which implies that \( \Lambda \) must be convex. More generally, it can be shown from first principles that in dimension \( d \geq 2 \), \( \Lambda \) must be \( d \)-monotone. This is stated formally below and proved in Appendix A, where a formal definition of \( d \)-monotonicity is also recalled.

**Proposition 1.** If \( F \in \mathcal{F}_d \), then \( \Lambda = -\ln(F) \) is \( d \)-monotone. Furthermore, \( \Lambda(t) \to \infty \) as \( t \to 0 \) and \( \Lambda(t) \to 0 \) as \( t \to \infty \).

An easy but important consequence of this result is that in the bivariate case, the distribution \( H_F \) given by (5) is \( TP_2 \), i.e., totally positive of order 2, as proved in Appendix A.

**Corollary 1.** When \( d = 2 \), \( H_F \) is \( TP_2 \), i.e., for all \( x_1, x_2, y_1, y_2 \in (0, \infty) \),

\[
x_1 < y_1 \text{ and } x_2 < y_2 \Rightarrow H_F(x_1, x_2)H_F(y_1, y_2) \geq H_F(x_1, y_2)H_F(y_1, x_2).
\]

Accordingly, all bivariate reciprocal Archimedean copulas are \( TP_2 \).

The \( TP_2 \) concept is a very strong notion of positive dependence originally introduced by Lehmann (1966). In particular, it implies that any measure of concordance in the sense of Scarsini (1984) is non-negative. For completeness, we repeat below simple formulas given by Lmoudden (2011) for three such classical dependence measures.
Lemma 2. Let $F \in \mathcal{F}_2$ with density $f$ and reverse hazard rate $r = f/F$. If $C_F$ is defined by (1), then Blomqvist’s beta and Spearman’s rho are respectively given by

$$\beta(C_F) = -1 + 1/F(2F^{-1}(1/2)),$$

$$\rho(C_F) = 3 \int_0^\infty \int_0^\infty H^2(x_1, x_2)\{r(x_1) + r(x_2)\}f(x_1 + x_2)dx_2dx_1.$$ 

Furthermore, if $f$ is differentiable, then Kendall’s tau can be expressed as

$$\tau(C_F) = -2 \int_0^\infty \int_0^\infty H^2(x_1, x_2)r'(x_1 + x_2)dx_2dx_1.$$ 

Next, the tail dependence coefficients (Joe, 1993) of bivariate reciprocal Archimedean copulas are computed here for the first time. To this end, recall that a function $g : (0, \infty) \to (0, \infty)$ is regularly varying at infinity of degree $\eta$, denoted $g \in RV_\eta$, if and only if

$$\forall x > 0 \lim_{t \to \infty} \frac{1 - g(xt)}{1 - g(t)} = x^\eta.$$ 

Recall also that $g : (0, \infty) \to (0, \infty)$ is called rapidly varying at infinity if

$$\lim_{t \to \infty} \frac{1 - g(xt)}{1 - g(t)} = \begin{cases} 0 & \text{if } x > 1, \\ \infty & \text{if } 0 < x < 1. \end{cases}$$ 

In what follows, $\bar{F} = 1 - F$ denotes the survival function corresponding to $F$, and $F^*$ refers to the survival function defined, for all $t > 0$, by $F^*(t) = F(1/t)$.

Lemma 3. Let $C_F$ be a bivariate reciprocal Archimedean copula with generator $F \in \mathcal{F}_2$.

(i) If $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$, then the upper tail dependence coefficient of $C_F$ is

$$\lambda_U(C_F) = 2 - \lim_{u \uparrow 1} \frac{1 - C_F(u, u)}{1 - u} = 2^{-\alpha}$$

while $\lambda_U(C_F) = 0$ either when $x_F < \infty$ or when $x_F = \infty$ and $\bar{F}$ is rapidly varying.

(ii) If $F^* \in RV_{-\beta}$ for some $\beta > 0$, then the lower tail dependence coefficient of $C_F$ is

$$\lambda_L(C_F) = \lim_{u \downarrow 0} \frac{C_F(u, u)}{u} = 2^{-\beta},$$

while $\lambda_L(C_F) = 0$ whenever $F^*$ is rapidly varying.

Remark 1. The results in Lemma 3 are reminiscent of the fact that the coefficients of lower and upper tail dependence of a bivariate Archimedean copula are related to the regular variation of its generator $\psi$ and of the function $\psi^*$ defined, for all $t > 0$, by $\psi^*(t) = \psi(\bar{F}(t))$. 


$1 - \psi(1/t)$; see, e.g., Charpentier and Segers (2009) or Larsson and Nešlehová (2011). It is further interesting to note that if $Z$ is a random variable with distribution $F$, Lemma 3 and classical univariate extreme-value theory (Resnick, 2008) imply that $\lambda_U(C_F) = 2^{-\alpha}$ holds if $Z$ is in the maximum domain of attraction (MDA) of the Fréchet distribution $\Phi_\alpha$ with parameter $\alpha > 0$, while if $Z$ is in the Weibull or the Gumbel domain of attraction, $\lambda_U(C_F) = 0$. Similarly, $\lambda_L(C_F) = 2^{-\beta}$ if $1/Z$ is in the maximum domain of attraction of the Fréchet distribution $\Phi_\beta$. If $1/Z$ is in the Gumbel domain, $\lambda_L(C_F) = 0$. Note that the conditions on the support of $F$ imply that the upper endpoint of the distribution of $1/Z$ is infinity, and hence $1/Z$ cannot be in the Weibull domain.

Using Lemma 3, one can easily recover the well-known expressions of the tail dependence coefficients of the classical bivariate Galambos and Clayton copulas with positive dependence, both of which belong to the class $C_2$ (Khoudraji, 1995).

**Example 1.** The reciprocal Archimedean generator of the bivariate Galambos copula with parameter $\theta > 0$ is given, for all $t > 0$, by

$$F_G(t) = \Phi_{1/\theta}(t) = e^{-t^{1/\theta}},$$

i.e., the Fréchet distribution $\Phi_{1/\theta}$ with parameter $1/\theta$. It is easily seen that $\bar{F}_G \in RV_{-1/\theta}$ and that $F_G^*$ is rapidly varying, implying that $\lambda_L = 0$ and $\lambda_U = 2^{-1/\theta}$. ♦

**Example 2.** The reciprocal Archimedean generator of the bivariate Clayton copula with parameter $\theta > 0$ is given, for all $t > 0$, by

$$F_C(t) = (1 - e^{-t})^{1/\theta},$$

i.e., a Lehmann alternative to the unit exponential distribution. Using L’Hospital’s rule, one finds $F_C \in RV_{-1/\theta}$ and that $F_C^*$ is rapidly varying. Hence, $\lambda_L = 2^{-1/\theta}$ and $\lambda_U = 0$. ♦

As a final global comment, note that the independence (or product) copula defined, for all $u_1, \ldots, u_d \in [0, 1]$, by $\Pi(u_1, \ldots, u_d) = u_1 \times \cdots \times u_d$, is not a member of the class $C_d$. It may represent a limiting case, however, as when $\theta \to 0$ in the bivariate Galambos or Clayton families discussed above. The proof that $\Pi \not\in C_F$ is by contradiction. If $C_F = \Pi$ held for some $F \in \mathcal{F}_d$, it would follow from (5) that, for all $x_1, x_2 > 0$, $F(x_1)F(x_2) = F(x_1)F(x_2)/F(x_1 + x_2)$ and consequently $F(x_1 + x_2) = 1$. This is clearly impossible, given that $F$ is nondegenerate. If the endpoint $x_F$ of $F$ is finite, however, then for all $u_1, \ldots, u_d \in [0, 1]$ such that for some $j < k \in \{1, \ldots, d\}$, $F^{-1}(u_j) + F^{-1}(u_k) \geq x_F$, we have

$$C_F(u_1, \ldots, u_d) = u_j u_k C_F(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{k-1}, u_{k+1}, \ldots, u_d).$$

(7)

This means that reciprocal Archimedean copulas can coincide with the independence copula on a subset of $[0, 1]^d$. This is illustrated, e.g., in Example 9 in Section 6.
Remark 2. Note that for any \( F \in \mathcal{F}_d \), it necessarily holds that \( F^{-1}(0) = \sup \{ t \in \mathbb{R} : F(t) = 0 \} = 0 \). This constraint could be relaxed, in principle. The first observation, made by Khoudraji (1995), is that \( F^{-1}(0) < 0 \) is not possible. To see this, it suffices to consider the case \( d = 2 \). If \( F^{-1}(0) \) were negative but finite, one could find \( x_1, x_2 \in (F^{-1}(0), \infty) \) with \( x_1 + x_2 < F^{-1}(0) \), so that \( F(x_1 + x_2) = 0 \) and \( H_F(x_1, x_2) = F(x_1)/F(x_2)\). This is infinite, which is a contradiction. If \( F^{-1}(0) = -\infty \), fix \( x_1 < 0 \) and note that \( F(x_2)/F(x_1 + x_2) \geq 1 \) for all \( x_2 \in \mathbb{R} \) because \( F \) is non-decreasing. Thus \( H_F(x_1, x_2) = F(x_1)/F(x_2) \) does not tend to 0 as \( x_2 \to -\infty \), which is also a contradiction.

In contrast, \( F^{-1}(0) > 0 \) can happen. For example, setting, for an arbitrary \( \theta \in (0, 1) \), \( F(t) = 1 - e^{-t}/\theta \) whenever \( t \geq -\ln(\theta) \) and \( F(t) = 0 \) otherwise, gives \( F^{-1}(0) = -\ln(\theta) > 0 \). It can be verified that this \( F \) indeed generates a bona fide copula in \( d = 2 \) through (1); in fact, \( C_F \) coincides with the Ali–Mikhail–Haq Archimedean copula with parameter \( \theta \). However, Proposition 1 does not hold when \( F^{-1}(0) > 0 \) and a stochastic representation of the corresponding reciprocal Archimedean copula seems rather cumbersome. For this reason, only generators with support of the form \([0, x_F]\) are considered in this paper; partial results for generators with \( F^{-1}(0) > 0 \) are provided in Section 8.

3 Connection with max-id distributions

There is more to the TP\(_2\) property of bivariate reciprocal Archimedean copulas established in Corollary 1. As shown by Marshall and Olkin (1990), any bivariate distribution with this property is in fact max-id, and vice versa. Consequently, when \( d = 2 \), the distribution function \( H_F \) given by (5) is max-id. As will be shown in this section, the concept of max infinite divisibility is in fact central to reciprocal Archimedean copulas in any dimension.

When \( d \geq 2 \), a \( d \)-variate continuous distribution function \( G \) is said to be max-id if and only if \( C^p \) is a distribution function for any \( p > 0 \) (Resnick, 2008). This is equivalent to the condition that the copula of \( G \) is max-id, in the sense defined below.

**Definition 1.** A copula \( C \) is said to be max-id if and only if for all \( p > 0 \), there exists a copula \( C^p \) such that for all \( u_1, \ldots, u_d \in (0, 1) \), \( C^p(u_1^{1/p}, \ldots, u_d^{1/p}) = C_p(u_1, \ldots, u_d) \).

We will establish that a reciprocal Archimedean copula \( C_F \) with \( F \in \mathcal{F}_d \) is max-id by showing that the distribution function \( H_F \) given in (5) has this property. This requires a deeper insight into the structure of max-id distributions. The concepts needed for the study of \( H_F \) are recalled next; for a detailed treatment, see, e.g., Resnick (2008).

Let \( E_d = [0, \infty) \setminus \{0\} \subset [0, \infty]^d \) and observe that relatively compact subsets of \( E_d \) are sets that are bounded away from the origin. An exponent measure on \( E_d \) is defined as follows.

**Definition 2.** A measure \( \mu \) on \( E_d \) is said to be an exponent measure if \( \mu \) is Radon, i.e., \( \mu(B) < \infty \) for any relatively compact set \( B \subseteq E_d \), and if \( \mu \) is such that

\[
\mu \left( \bigcup_{k=1}^{d} \{ x \in E_d : x_k = \infty \} \right) = 0.
\]
As explained in the discussion surrounding Example 5.7 in Section 5.3 of Resnick (2008), an exponent measure \( \mu \) can be conveniently characterized as follows.

**Lemma 4.** A measure \( \mu \) on \( E_d \) is an exponent measure if \( \mu\left[\infty, x\right]^c < \infty \) for all \( x > 0 \) and if \( \mu\left[-\infty, x\right]^c \to 0 \) as \( x \to \infty \), where \( ^c \) denotes set complement within the space \( E_d \).

If \( \mu \) is an exponent measure on \( E_d \), it follows from Proposition 5.8 in Resnick (2008) that the function \( H_\mu \) defined, for all \( x \geq 0 \), by

\[
H_\mu(x) = e^{-\mu\left[\infty, x\right]^c}
\]

is a max-id distribution. In the present context, it suffices to focus on max-id distributions whose exponent measure has the additional property that, for all \( k \in \{1, \ldots, d\} \),

\[
\mu\{x \in E_d : x_k = 0\} = 0,
\]

i.e., \( \mu \) places no mass on \( E_d \setminus (0, \infty) = \bigcup_{k=1}^d \{x \in E_d : x_k = 0 \text{ or } x_k = \infty\} \) because

\[
\mu(E_d \setminus (0, \infty)) \leq \sum_{k=1}^d \mu\{x \in E_d : x_k = 0\} + \mu\left(\bigcup_{k=1}^d \{x \in E_d : x_k = \infty\}\right) = 0.
\]

Let \( \mathcal{M}_d \) denote the class of exponent measures on \( E_d \) with property (9). As stated in Lemma 5 below and proved in Appendix B, members of this class are conveniently characterized through the following concept, which extends the notion of a survival function.

**Definition 3.** The generalized survival function \( S_\mu : E_d \to [0, \infty) \) of an exponent measure \( \mu \in \mathcal{M}_d \) is defined, for all \( x \in E_d \), by

\[
S_\mu(x) = \mu(x, \infty).
\]

Note that \( S_\mu \) is well defined as for all \( x \in E_d \), the set \( (x, \infty] \) is relatively compact and hence \( \mu(x, \infty] < \infty \). In what follows, let also \( S_d = \{S_\mu : \mu \in \mathcal{M}_d\} \).

**Lemma 5.** If \( S_\mu : E_d \to [0, \infty) \) for some \( \mu \in \mathcal{M}_d \), then \( S_\mu \) has the following properties:

(i) \( S_\mu(x) \to 0 \) when \( x_k \to \infty \) for at least one \( k \in \{1, \ldots, d\} \).

(ii) \( S_\mu \) is right-continuous, i.e., for all \( x \in E_d \),

\[
\forall \varepsilon > 0 \exists \delta > 0 \forall y \geq x \quad \|y - x\|_1 < \delta \quad \Rightarrow \quad |S_\mu(y) - S_\mu(x)| < \varepsilon.
\]

(iii) For all \( x, y \in E_d \) such that \( x \leq y \), the \( S_\mu \)-volume of \([x, y]\) is non-negative, i.e.,

\[
\sum_{c \in S_{x,y}} \text{sign}(c)S_\mu(c) \geq 0,
\]

where \( S_{x,y} = \{c \in E_d : \forall k \in \{1, \ldots, d\} \ c_k \in \{x_k, y_k\}\} \) and

\[
\text{sign}(c) = \begin{cases} 
1 & \text{if } c_k = y_k \text{ for an even number of integers } k \in \{1, \ldots, d\}, \\
-1 & \text{if } c_k = y_k \text{ for an odd number of integers } k \in \{1, \ldots, d\}.
\end{cases}
\]
Conversely, if a function \( S : E_d \to [0, \infty) \) satisfies the above three conditions, it is the generalized survival function of a unique exponent measure \( \mu \in \mathcal{M}_d \).

Finally, we introduce the notion of \( \ell_1 \)-norm symmetric exponent measure, which is key to the study of distributions of the form (5) and which parallels the notion of an \( \ell_1 \)-norm symmetric distribution (Fang et al., 1990; McNeil and Neslehová, 2009).

**Definition 4.** An exponent measure \( \mu \in \mathcal{M}_d \) is called \( \ell_1 \)-norm symmetric if there exists a map \( \Lambda : (0, \infty) \to [0, \infty) \) such that \( \Lambda(t) \to 0 \) as \( t \to \infty \) and, for all \( \mathbf{x} \in E_d \),

\[
S_{\mu}(x_1, \ldots, x_d) = \Lambda_{\mu}(x_1 + \cdots + x_d).
\]

The map \( \Lambda_{\mu} \), which is unique, is called the generator of \( \mu \).

To generate an \( \ell_1 \)-norm symmetric exponent measure \( \mu \), the map \( \Lambda \) must be such that \( S_{\mu} \) satisfies conditions (i)–(iii) of Lemma 5. The following result, proved in Appendix B, characterizes the class \( \mathcal{L}_d \) of generators of \( \ell_1 \)-norm symmetric exponent measures \( \mu \in \mathcal{M}_d \).

**Proposition 2.** A map \( \Lambda : (0, \infty) \to [0, \infty) \) belongs to \( \mathcal{L}_d \) if and only if \( \Lambda \) is \( d \)-monotone and \( \Lambda(t) \to 0 \) as \( t \to \infty \).

Now suppose that \( \mu \) is an \( \ell_1 \)-norm symmetric exponent measure with generator \( \Lambda_{\mu} \). It then emerges from (B3) in the proof of Lemma 5 that for any \( \mathbf{x} > 0 \),

\[
\mu[\mathbf{0}, \mathbf{x}] = \sum_{k=1}^{d} (-1)^{k+1} \sum_{A \subseteq \{1, \ldots, d\}, |A| = k} \Lambda_{\mu}\left(\sum_{j \in A} x_j\right).
\]

The max-id distribution \( H_{\mu} \) with exponent measure \( \mu \) defined in (8) is thus precisely of the form (5) with \( F = F_{\mu} \), where

\[
F_{\mu}(t) = \begin{cases} 
 e^{-\Lambda_{\mu}(t)} & \text{if } t > 0, \\
 0 & \text{otherwise}.
\end{cases}
\]

If in addition \( \Lambda_{\mu}(t) \to \infty \) as \( t \to 0 \), then \( F_{\mu} \) is continuous, and so is \( H_{\mu} \). The unique underlying copula of \( H_{\mu} \) is then reciprocal Archimedean with generator \( F_{\mu} \). Furthermore, the generator of \( \mu \) satisfies \( \Lambda_{\mu} = -\ln(F_{\mu}) \). This insight allows for the complete characterization of the class \( \mathcal{F}_d \) of reciprocal Archimedean generators in dimension \( d \geq 2 \).

**Theorem 1.** A continuous distribution function \( F \) with support \([0, x_F]\), \( 0 < x_F \leq \infty \), belongs to \( \mathcal{F}_d \) if and only if the map \( \Lambda = -\ln(F) \) on \((0, \infty)\) is \( d \)-monotone and satisfies \( \Lambda(t) \to 0 \) as \( t \to \infty \), and \( \Lambda(t) \to \infty \) as \( t \to 0 \), i.e., \( \Lambda \in \mathcal{L}_d \) and \( \Lambda(t) \to \infty \) as \( t \to 0 \). The class of all such functions \( \Lambda \) is denoted \( \mathcal{L}^0_d \).

**Proof.** The necessity of the conditions on \( \Lambda \) was established in Proposition 1. Sufficiency follows from the discussion preceding Theorem 1 given that \( \Lambda \in \mathcal{L}_d \) by Proposition 2, i.e., \( \Lambda = \Lambda_{\mu} \) is the generator of an \( \ell_1 \)-norm symmetric exponent measure \( \mu \in \mathcal{M}_d \). □
Corollary 2. A distribution function $H_F$ of the form (5) for some $F \in \mathcal{F}_d$ is a continuous max-id distribution with $\ell_1$-norm symmetric exponent measure $\mu_F$ generated by $\Lambda = -\ln(F)$ with $\Lambda \in \mathcal{L}^0_d$, and vice versa.

Thus for any $F \in \mathcal{F}_d$, the reciprocal Archimedean copula $C_F$ is precisely the dependence structure of a continuous max-id distribution whose exponent measure is $\ell_1$-norm symmetric and generated by $\Lambda = -\ln(F)$. In particular, therefore, reciprocal Archimedean copulas are multivariate total positive in the sense of Alzaid and Proschan (1994). As they are also associated, they are both positive lower and upper orthant dependent; see, e.g., Theorem 8.6 of Joe (2015).

Remark 3. Theorem 1 and its corollary resemble the characterization of $d$-variate Archimedean copulas. As shown by Malov (2001) and McNeil and Neslehová (2009), a function $\psi: [0, \infty) \to [0, \infty)$ generates an Archimedean copula in dimension $d$ if and only if $\psi$ is $d$-monotone and such that $\psi(0) = 1$ and $\psi(t) \to 0$ as $t \to \infty$. Furthermore, McNeil and Neslehová (2009) established that Archimedean copulas are precisely the dependence structures of continuous $\ell_1$-norm symmetric distributions, i.e., distributions whose survival function is of the form $S(x) = \psi(x_1 + \cdots + x_d)$ for all $x \in \mathbb{R}^d$.

For later use, illustrations of the above notions are presented below. They pertain to the bivariate Galambos and Clayton copulas with positive dependence.

Example 3. For the generator $F_G$ with parameter $\theta > 0$ given in Example 1, one has $\Lambda_G(t) = -\ln\{F_G(t)\} = t^{-1/\theta}$ for all $t > 0$. Clearly, $\Lambda_G$ satisfies the limit conditions at 0 and $\infty$. For any $k \in \mathbb{N}$, the $k$th derivative of $\Lambda_G$ is given, for all $t > 0$, by

$$
\Lambda_G^{(k)}(t) = (-1)^k t^{-k-1/\theta} \prod_{\ell=0}^{k-1} (\ell + 1/\theta).
$$

In particular, therefore, $\Lambda_G$ is $d$-monotone for any $d \geq 2$ and $F_G$ can generate a reciprocal Archimedean copula in any dimension. This $d$-variate copula is the so-called multivariate Galambos extreme-value copula (Joe, 1990).

Example 4. For the generator $F_C$ with parameter $\theta > 0$ given in Example 2, one has $\Lambda_C(t) = -\ln\{F_C(t)\} = -\ln(1 - e^{-t})/\theta$ for all $t > 0$. Obviously, $\Lambda_C$ satisfies the limit conditions at 0 and $\infty$. For all $t > 0$, one has $\Lambda_C^{(1)}(t) = -e^{-t}/\{\theta(1 - e^{-t})\}$ and can further show by induction that, for any integer $k \geq 2$,

$$
\Lambda_C^{(k)}(t) = (-1)^k \frac{e^{-t}}{\theta(1 - e^{-t})^k} \sum_{\ell=0}^{k-2} A_{k-1,\ell} e^{-\ell t},
$$

where, for each $k \in \mathbb{N}$ and $\ell \in \{0, \ldots, k-1\}$, $A_{k,\ell}$ is the Eulerian number, i.e., the number of permutations of $\{1, \ldots, k\}$ having $\ell$ permutation ascents (Graham et al., 1994, Section 6.2). These coefficients are such that $A_{k,0} = A_{k,k-1} = 1$ and for all $\ell \in \{1, \ldots, k-2\}$,

$$
A_{k,\ell} = (\ell + 1)A_{k-1,\ell} + (k - \ell)A_{k-1,\ell-1}.
$$
Writing \( A_k(t) = A_{k,0} + A_{k,1} t + \cdots + A_{k,k-1} t^{k-1} \) for the \( k \)th Eulerian polynomial with the convention that \( A_0(t) = 1 \), one also has, for all \( k \in \mathbb{N} \),

\[
A^{(k)}_C(t) = (-1)^k \frac{e^{-t}}{\theta(1-e^{-t})^k} A_{k-1}(e^{-t}).
\]

Because the Eulerian numbers are all strictly positive, \( \Lambda_C \) is \( d \)-monotone for any \( d \geq 2 \) and hence \( F_C \) can generate a reciprocal Archimedean copula in any dimension. As mentioned in Example 1 and first observed by Khoudraji (1995), \( F_C \) is a Clayton copula in dimension \( d = 2 \). Interestingly, however, \( C_{F_C} \), referred here as the reciprocal Clayton, is not Clayton when \( d \geq 3 \). A numerical verification of this fact is given by Lmoudden (2011).

4 Geometric structure of \( \ell_1 \)-norm exponent measures

We next show that \( \ell_1 \)-norm symmetric exponent measures have a geometric structure that parallels the stochastic representation of \( \ell_1 \)-norm symmetric distributions. From the work of McNeil and Neslehová (2009), the latter are known to be the distributions of random vectors \( R \times S \), where \( R \) is a strictly positive random variable, independent of the \( d \)-variate random vector \( S \) uniformly distributed on the simplex

\[
S_d = \{ s \in [0,1]^d : \|s\|_1 = 1 \}.
\]

The following result provides an analogous representation of \( \ell_1 \)-norm symmetric exponent measures. Its proof is in Appendix C, along with the characterization of \( d \)-monotone functions due to Williamson (1956) on which it relies.

**Theorem 2.** A measure \( \mu \) on \( E_d \) is an \( \ell_1 \)-norm symmetric exponent measure if and only if the image measure of \( \mu \) by the polar coordinate transformation

\[
T : E_d \rightarrow (0, \infty] \times S_d : (x_1, \ldots, x_d) \mapsto \left( \| x \|_1, \frac{x_1}{\| x \|_1}, \ldots, \frac{x_d}{\| x \|_1} \right)
\]

is of the form \( \nu_{\mu} \times \sigma_d \), where \( \sigma_d \) is the uniform probability distribution on \( S_d \) and \( \nu_{\mu} \) is a Radon measure on \( (0, \infty] \) such that \( \nu_{\mu}\{\infty\} = 0 \), termed the radial measure. Furthermore, the generator \( \Lambda_{\mu} \) of \( \mu \) is the Williamson \( d \)-transform of \( \nu_{\mu} \), i.e., for any \( t > 0 \),

\[
\Lambda_{\mu}(t) = \mathfrak{M}_d(\nu_{\mu})(t) \equiv \int_0^\infty (1 - t/r)^{d-1} d\nu_{\mu}(r),
\]

and conversely \( \nu_{\mu} \) is determined through the inverse Williamson \( d \)-transform of \( \Lambda_{\mu} \), i.e., for any \( r > 0 \), the generalized survival function \( S_{\nu_{\mu}}(r) = \nu_{\mu}(r, \infty) \) of \( \nu_{\mu} \) satisfies

\[
S_{\nu_{\mu}}(r) = \mathfrak{M}_d^{-1}(\Lambda_{\mu})(r) \equiv \sum_{k=0}^{d-2} \frac{(-1)^k \Lambda^{(k)}_{\mu}(r)}{k!} r^k + \frac{(-1)^{d-1} \Lambda_{\mu_+}^{(d-1)}(r)}{(d-1)!} r^{d-1},
\]

where \( \Lambda_{\mu_+}^{(d-1)} \) denotes the right-hand derivative of \( \Lambda_{\mu}^{(d-2)} \).
The above result is an analogue of Theorem 3.1 of McNeil and Nešlehová (2009) and justifies the term “$\ell_1$-norm symmetric exponent measure.” The radial measure $\nu_\mu$ parallels the notion of the radial distribution of an $\ell_1$-norm symmetric distribution. Before proceeding, we compute the latter for the Galambos and the reciprocal Clayton copulas.

**Example 5.** Consider the generator $F_G$ of the Galambos copula given in Example 1 and let $\mu_G$ denote the $\ell_1$-norm symmetric exponent measure generated by $\Lambda_G = -\ln(F_G)$. The generalized survival function $S_{\nu_G}$ of the radial measure $\nu_G$ of $\mu_G$ is easily computed using the derivatives of $\Lambda_G$ given in Example 3. For all $r > 0$,

$$S_{\nu_G}(r) = r^{-1/\theta} \sum_{k=0}^{d-1} \frac{1}{k!} \prod_{\ell=0}^{k-1} (\ell + 1/\theta) = \frac{\theta}{B(d, 1/\theta)} r^{-1/\theta},$$

where $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$ denotes Euler’s Beta function at any $\alpha, \beta > 0$.

**Example 6.** Consider the generator $F_C$ of the reciprocal Clayton copula given in Example 2 and let $\mu_C$ denote the $\ell_1$-norm symmetric exponent measure generated by $\Lambda_C = -\ln(F_C)$. Calculations in Example 4 imply that the generalized survival function $S_{\nu_C}$ of the radial measure $\nu_C$ of $\mu_C$ satisfies, for all $r > 0$,

$$S_{\nu_C}(r) = \frac{1}{\theta} \{ -\ln(1 - e^{-r}) + \sum_{k=1}^{d-1} \frac{e^{-r}r^k}{k!(1 - e^{-r})^k} A_{k-1}(e^{-r}) \},$$

where $A_k$ is the Eulerian polynomial defined in Example 4.

The geometric structure of $\ell_1$-norm symmetric exponent measures resembles that of exponent measures corresponding to multivariate extreme-value distributions with unit Fréchet margins. Indeed, as detailed, e.g., in Section 5.4.1 of Resnick (2008), a distribution function $G$ is multivariate extreme-value with $\Phi_1$ margins if and only if it is a max-id distribution of the form (8), whose exponent measure $\mu \equiv \mu_G$ is such that for any Borel set $B \subset E_d$ and $t > 0$, $\mu_G(B) = t\mu_G(tB)$. Thus the image measure of $\mu_G$ by $T$ is $\nu_G \times \varsigma_d$, where for any $r > 0$, $\nu_G(r, \infty] = d/r$ and $\varsigma_d$ is a probability distribution on $S_d$ with mean $1/d$.

While for max-id distributions with $\ell_1$-norm symmetric exponent measure the measure on $S_d$ is fixed and the radial measure can vary, for multivariate extreme-value distributions with $\Phi_1$ margins we have the exact opposite: The radial measure is fixed while the measure on $S_d$ can vary. Because the mean of the uniform distribution $\sigma_d$ is $1/d$, the only distribution which is in both classes is the max-id distribution $H_F$ with $F = \Phi_1$. Indeed, as can be deduced from Example 5, $\nu_F(r, \infty] = d/r$.

This observation is not in contradiction with the fact that all Galambos copulas are both reciprocal Archimedean and extreme-value as seen in Example 3. This is because the above discussion was limited to extreme-value distributions with unit Fréchet margins. However, it begs the question of whether any other extreme-value copula can be expressed in the form (3) for an appropriate choice of $F \in \mathcal{F}_d$. The answer is negative, as stated below and proved in Appendix C.
Proposition 3. A copula $C_F \in \mathcal{C}_d$ for some $F \in \mathcal{F}_d$ is extreme-value if and only if $F$ is a Fréchet distribution, i.e., there exists $\theta > 0$ such that, for all $t > 0$, $F(t) = \Phi_{1/\theta}(t) = e^{-t^{-1/\theta}}$. The Galambos copulas are thus the only extreme-value copulas in the class $\mathcal{C}_d$.

The result in Proposition 3 is thus parallel to the characterization of Gumbel copulas as the only Archimedean copulas that are extreme-value (Genest and Rivest, 1989).

5 Stochastic representation and simulation

It will now be shown that any max-id distribution $H_F$ with $F \in \mathcal{F}_d$ has a convenient stochastic representation that leads to an algorithm for generating observations from a reciprocal Archimedean copula. Recall that $H_F$ has an $\ell_1$-norm symmetric exponent measure $\mu_F$ generated by $\Lambda = -\ln(F)$, as established in Corollary 2. The stochastic representation of $H_F$ can then be obtained by combining Theorem 2 with results from Balkema and Resnick (1977) and Gerritse (1986) described, e.g., in Example 5.7 in Resnick (2008). In what follows, $\varepsilon$ denotes the Dirac measure and $a \lor b = \max(a,b)$.

Proposition 4. Given any $F \in \mathcal{F}_d$, let $H_F$ be of the form (5) and let $\nu_F$ be the radial measure corresponding to $\Lambda = -\ln(F)$ through (11). Further, let $\zeta = \sum_k \varepsilon_{(r_k, s_k)}$ be a Poisson random measure (PRM) on $(0, \infty] \times \mathbb{S}_d$ with mean measure $\nu_F \times \sigma_d$. Then $H_F$ is the distribution function of the vector

$$Y = (Y_1, \ldots, Y_d) = \max_k (r_k \times s_k) \lor 0,$$

where the maximum is understood as a component-wise operation.

Proof. By Theorem 2, the image measure of $\nu_F \times \sigma_d$ by $T^{-1}$ is the $\ell_1$-norm symmetric exponent measure $\mu_F$ generated by $\Lambda$. Thus, for any $y \in E_d$,

$$\Pr(Y \leq y) = \Pr[\zeta(T([-\infty, y]^c])] = 0 = \exp\{-\mu_F[-\infty, y]^c\} = H_F(y),$$

while $\Pr(Y \leq 0) = \Pr[\zeta(T(E_d))] = 0 = 0$ because $\mu_F(E_d) = \infty$. \hfill \Box

Now suppose that $C_F$ is a $d$-variate reciprocal Archimedean copula generated by $F \in \mathcal{F}_d$. From Sklar’s representation and Corollary 2, we know that $C_F$ is the copula of the max-id distribution $H_F$ whose exponent measure $\mu_F$ is $\ell_1$-norm symmetric with radial measure $\nu_F$. This also means that $C_F$ is the copula of the random vector $Y$ in (12). In principle, a random sample from $C_F$ can thus be obtained by generating observations from $Y$.

The difficulty with this approach is that $\nu_F \times \sigma_d\{[0, \infty] \times \mathbb{S}_d\} = \mu_F(E_d) = \infty$, which means that realizations of the PRM $\zeta$ cannot be generated explicitly. To circumvent this problem, consider a decreasing sequence of constants $\infty = \Delta_0 > \Delta_1 > \Delta_2 > \cdots > 0$ such that $\Delta_k \to 0$ as $k \to \infty$ and for every integer $k \in \mathbb{N}$, set $D_k = (\Delta_k, \Delta_{k-1}] \times \mathbb{S}_d$ so that
\[ \zeta = \sum_{k=1}^{\infty} \zeta_k, \text{ where } \zeta_k \text{ is a PRM on } D_k \text{ with mean measure } \nu_F \times \sigma_d \text{ restricted to } D_k. \]

Thus we propose to approximate \( \omega \) This implies that for every \( Y = \max(S) \) where \( S \) is a decreasing sequence such that \( 0 \leq Y_{k_j}(\omega) \leq \Delta_{k-1} \) for all \( j \in \{1, \ldots, d\} \) and \( \ell \in \mathbb{N} \). Thus for any \( k \in \mathbb{N} \), the realizations \( Y_j(\omega) \) and \( W_{k_j}(\omega) \) are equal as soon as \( W_{k_j}(\omega) > \Delta_k \). If \( W_{k_j}(\omega) \leq \Delta_k \), then \( Y_j(\omega) \leq \Delta_k \), so that \( |Y_j(\omega) - W_{k_j}(\omega)| \leq \Delta_k \).

This implies that for every \( \omega \in \Omega \), as \( k \to \infty \), \( W_k(\omega) \uparrow Y(\omega) \).

Thus we propose to approximate \( Y \) by \( W_K \) for some suitable \( K \in \mathbb{N} \). This leads to the following algorithm for generating samples from the reciprocal Archimedean copula \( C_F \).

**Algorithm 1.** Let \( C_F \) be a \( d \)-variate reciprocal Archimedean copula with generator \( F \in \mathcal{F}_d \). Let \( \Lambda = -\ln(F) \) and \( \nu_F \) be the radial measure of the \( \ell_1 \)-norm symmetric exponent measure \( \mu_F \) of \( H_F \), as given in (11). Let also \( \infty = \Delta_0 > \Delta_1 > \cdots \) be a decreasing sequence such that \( \Delta_k \to 0 \) as \( k \to \infty \) and let \( K \in \mathbb{N} \) be a fixed integer. To generate an observation \( U \) from \( C_F \), proceed as follows.

\begin{enumerate}
  \item **Step 1.** (Initialization) Set \( k = 0 \) and \( W_0 = 0 \).
  \item **Step 2.** (Iteration)
    \begin{enumerate}
      \item Set \( k = k + 1 \) and compute \( \lambda_k = \nu_F(\Delta_k, \Delta_{k-1}) = S_{\nu_F}(\Delta_k) - S_{\nu_F}(\Delta_{k-1}). \)
      \item Generate \( N_k \) from the Poisson distribution with parameter \( \lambda_k \).
    \end{enumerate}
\end{enumerate}
2c. Draw $N_k$ independent observations $R_{k1}, \ldots, R_{kN_k}$ from the distribution with survival function (13).

2d. Draw $N_k$ independent $d$-variate random vectors $E_{k1}, \ldots, E_{kN_k}$ whose components are independent unit exponential variables and for each $\ell \in \{1, \ldots, N_k\}$, set $S_{k\ell} = E_{k\ell}/(E_{k\ell,1} + \cdots + E_{k\ell,d})$.

2e. For each $\ell \in \{1, \ldots, N_k\}$, set $S_{k\ell} = E_{k\ell}/(E_{k\ell,1} + \cdots + E_{k\ell,d})$.

2f. Set $X_{k\ell} = R_{k\ell} \times S_{k\ell}$.

2g. Set $W_k = \max(Y_k, W_{k-1})$.

Step 3. (Stopping criterion). If $k = K$ or $W_{kj} > \Delta_k$ for each $j \in \{1, \ldots, d\}$, then set $Y = W_k$ and go to Step 4. Otherwise, go to Step 2.

Step 4. (Output) Set $U = (F(Y_1), \ldots, F(Y_d))$.

The performance of Algorithm 1 clearly depends on the choice of the sequence $(\Delta_k)$ and the integer $K$ that bounds the number of iterations. From our experiments and the simulations used in this paper, it suffices to set $K = 2$ provided that $\Delta_1$ and $\Delta_2$ are well chosen. When selecting $\Delta_1$, a trade-off between speed and accuracy takes place. When $\Delta_1$ is small, chances are high that all components of $W_1$ will be above $\Delta_1$, in which case $W_1$ is a realization of the vector $Y$ and there is no need for further iterations. However, a small value of $\Delta_1$ also means that the Poisson parameter $\lambda_1$ is large. High values of $N_1$ are then likely, which in turn may result in unnecessarily large samples in Steps 2c–d. This is wasteful and can slow down the algorithm considerably.

The choice of $\Delta_1$ is further complicated by the fact that the meaning of “small” depends on the generator $F$ and the dimension $d$. To see how to proceed, first note that the probability that Algorithm 1 with $K > 1$ requires only one iteration each is at least $p$.

**Lemma 6.** For arbitrary $F \in \mathcal{F}_d$ and $\Delta > 0$, let $\zeta^* = \sum_k e(r_k, s_k)$ be a PRM on $(\Delta, \infty) \times S_d$ with mean measure $\nu_F \times \sigma_d$. Let also $Y^* = \max_k(r_k \times s_k) \lor 0$. Then

$$\Pr(Y^*_1 > \Delta, \ldots, Y^*_d > \Delta) = \tilde{H}_F(\Delta, \ldots, \Delta) \geq (1 - F(\Delta))^d,$$

where $\tilde{H}_F$ denotes the survival function of $H_F$.

This result suggests the following procedure for selecting $\Delta_1$ in order to generate a random sample of size $n$ from a reciprocal Archimedean copula $C_F$.

**Rule of Thumb 1.** Choose $p \in (0, 1)$ and let $\Delta_1$ be the largest $\Delta$ for which the probability of $n$ consecutive runs of Algorithm 1 with $K > 1$ requiring one iteration each is at least $p$.

Because $\tilde{H}_F(\Delta, \ldots, \Delta)$ as a function of $\Delta$ may be cumbersome to invert, especially when $d$ is high, we suggest to use the inequality in Lemma 6, viz.

$$p = \{\Pr(Y_{11} > \Delta, \ldots, Y_{1d} > \Delta)\}^n \geq (1 - F(\Delta))^n d \quad (14)$$
so that \( \Delta_1 \) is given by

\[
\Delta_1 = F^{-1}\{1 - p^{1/(nd)}\}.
\]  

(15)

In what follows, Eq. (15) was used with \( p = 0.99 \). To catch the cases where Algorithm 1 requires more than one iteration, taking \( \Delta_2 = \Delta_1 \times 10^{-4} \) seemed sufficient.

As an illustration of Algorithm 1 and the implementation of Rule of Thumb 1, its performance is explored below in the case of the Galambos and the reciprocal Clayton family.

**Example 7.** To generate a random sample of size \( n \) from the \( d \)-variate Galambos copula with parameter \( \theta > 0 \), Algorithm 1 must be implemented with measure \( \nu_G \) defined in Example 5. In the first iteration, one must draw \( N_1 \sim \mathcal{P}(\lambda_1) \) with \( \lambda_1 = \theta \Delta_1^{-1/\theta} / B(d, 1/\theta) \) and \( R_1 \) from a Pareto distribution with shape parameter \( 1/\theta \) and scale parameter \( \Delta_1 \), i.e., for all \( r > \Delta_1 \), \( \Pr(R_1 > r) = (\Delta_1/r)^{1/\theta} \). Based on Eq. (15), \( \Delta_1 = [-\ln\{1 - p^{1/(nd)}\}]^{-\theta} \).

The left panel of Figure 1 shows how \( \Delta_1 \) varies as a function of Kendall’s \( \tau \) (which is the same for all pairs of components) when \( p = 0.99 \), \( n = 1000 \) and \( d \in \{2, 5, 10\} \). The top panel of Figure 2 displays samples of this size resulting from Algorithm 1 when \( \tau = 0.5 \).

**Example 8.** To generate a random sample from the \( d \)-variate reciprocal Clayton copula with parameter \( \theta > 0 \), Algorithm 1 must be implemented with measure \( \nu_C \) defined in Example 6. This requires numerical inversion for the generation of \( R_1 \). Based on Eq. (15), \( \Delta_1 = -\ln[1 - \{1 - p^{1/(nd)}\}^\theta] \). The right panel of Figure 1 shows how \( \Delta_1 \) varies as a function of Kendall’s \( \tau \) (which is the same for all pairs of components) when \( p = 0.99 \), \( n = 1000 \) and \( d \in \{2, 5, 10\} \). The bottom panel of Figure 2 displays samples resulting from Algorithm 1 when \( \tau = 0.5 \) and \( d \in \{2, 3\} \), and compares the trivariate sample to a sample of the same size from the trivariate Clayton Archimedean copula with \( \tau = 0.5 \).

Additional examples of reciprocal Archimedean copulas are discussed in Section 5. In our experience, the values of \( \Delta_1 \) given by Rule of Thumb 1 tend to be much too conservative.
Figure 2: Upper panel: Random samples of size 1000 from the bivariate (upper left) and trivariate (upper right) Galambos copula with dependence $\tau = 0.5$ between all pairs. Lower panel: Random samples of size 1000 from the bivariate Clayton (lower left) and trivariate (upper right) Clayton copula (red) and reciprocal Clayton (blue) copula with dependence $\tau = 0.5$ between all pairs.

As evidenced by Figure 1, the value of $\Delta_1$ decreases as the dimension of the vector and the dependence between its components increase. This suggests that to improve the performance of Algorithm 1, the lower bound given in (14) would need to be refined by taking into account this dependence. We leave this issue aside for the time being.

6 Illustrations

This section explores mechanisms by which new classes of multivariate reciprocal Archimedean copulas can be constructed. Two approaches are considered, depending on whether it is preferred to specify a parametric class of reciprocal Archimedean generators directly or through the corresponding radial measure. Before proceeding, note that:

(i) By construction, all reciprocal Archimedean copulas are exchangeable and max-id.
Therefore, these copulas are multivariate total positive and associated (Alzaid and Proschan, 1994; Joe, 2015) and capable of modeling positive association only.

(ii) As already mentioned at the end of Section 2, the independence copula \( \Pi \) does not belong to the class \( \mathcal{C}_d \) in any dimension \( d \geq 2 \). This is also easily seen from Condition (9), which prevents the \( \ell_1 \)-norm symmetric exponent measure \( \mu \) of a reciprocal Archimedean copula from placing any mass on the axes going through \( 0 \). In contrast, the exponent measure of the max-id distribution with independent, identical margins \( F \) places all its mass on the axes going through the origin.

6.1 Specifying the radial measure

The easiest way to construct the reciprocal Archimedean generator \( F \in \mathcal{F}_d \) is by constructing the radial measure \( \nu \). To accomplish this, one must first choose \( S \in \mathcal{S}_1 \), i.e., a non-increasing, right-continuous function \( S : (0, \infty) \to [0, \infty) \) such that \( S(t) \to 0 \) as \( t \to \infty \) and \( S(t) \to \infty \) as \( t \to 0 \). Then one sets \( \nu(t, \infty] = S(t) \) for all \( t > 0 \). By Theorem 2, the generator of the corresponding \( \ell_1 \)-norm symmetric exponent measure \( \mu \in \mathcal{M}_d \) is given by the Williamson \( d \)-transform of \( \nu \), i.e., for any \( t > 0 \),

\[
\Lambda(t) = \int_t^\infty (1 - t/r)^{d-1} \, d\nu(r).
\]

By construction, we then have that \( \Lambda \in \mathcal{L}_d \), as well as \( \Lambda(t) \to 0 \) as \( t \to \infty \) and \( \Lambda(t) \to \infty \) as \( t \to 0 \), so that, by Theorem 1, the distribution function given by

\[
F(t) = \begin{cases} 
  e^{-\Lambda(t)} & \text{if } t > 0, \\
  0 & \text{otherwise}
\end{cases}
\]

is a valid reciprocal Archimedean generator in dimension \( d \), i.e., \( F \in \mathcal{F}_d \).

This approach is not only advantageous because the conditions on \( S \) are easily met but also because \( S \) is explicit, which comes in handy in Step 2c of Algorithm 1. However, this often comes at the cost of an intractable formula for \( \Lambda \) and \( F \), because the Williamson \( d \)-transform of \( \nu \) may not be explicit. Nonetheless, the integral it involves is one-dimensional, so Step 4 of Algorithm 1 can in principle be carried out using numerical integration.

Perhaps the most natural (though already known) example of such a construction is the Galambos copula, for which \( S(t) \propto t^{-1/\theta} \) for all \( t > 0 \), as computed in Example 5. Below, we expand on it and explore what happens when \( S \) vanishes beyond some point \( \kappa > 0 \).

**Example 9.** Consider the “truncated” version of the generalized survival function corresponding to the \( d \)-variate Galambos copula, given, for \( \theta > 0 \) and \( \kappa > 0 \), by

\[
S_{\theta, \kappa}(r) = \begin{cases} 
  \frac{\theta}{B(d, 1/\theta)} r^{-1/\theta} & \text{if } r \in (0, \kappa], \\
  0 & \text{otherwise}.
\end{cases}
\]
Figure 3: Random samples of size 1000 from the bivariate truncated Galambos copula from Example 9 with $\theta = 1.28$ and $\kappa = 1$ (left), $\kappa = 2$ (middle) and $\kappa = 10$ (right). The curve $F^{-1}(u_1) + F^{-1}(u_2) = \kappa$ is indicated in red.

Note that $S_{\theta,\kappa}$ is discontinuous at $t = \kappa$. For any $t \in (0, \kappa)$, we have

$$\Lambda_{\theta,\kappa}(t) = \frac{1}{B(d, 1/\theta)} \left\{ \theta(1 - t/\kappa)^{d-1} \kappa^{-1/\theta} + \int_t^\kappa (1 - t/r)^{d-1} r^{-1/\theta - 1} dr \right\}$$

while $\Lambda_{\theta,\kappa}(t) \equiv 0$ for all $t \geq \kappa$. Elementary calculations lead to the alternative expression

$$\Lambda_{\theta,\kappa}(t) = t^{-1/\theta} + \frac{\kappa^{-1/\theta}}{B(d, 1/\theta)} \sum_{\ell=1}^{d-1} \frac{\theta}{\ell + 1/\theta} B(\ell, d - \ell) \left( -\frac{t}{\kappa} \right)^\ell,$$

valid for all $t < \kappa$. An explicit expression for the reciprocal Archimedean generator $F_{\theta,\kappa}$ then follows from the fact that, for all $t > 0$, $F_{\theta,\kappa}(t) = e^{-\Lambda_{\theta,\kappa}(t)}$.

When $d = 2$, one has, for all $t \in (0, \kappa)$, $\Lambda_{\theta,\kappa}(t) = t^{-1/\theta} - t\kappa^{-1/\theta - 1}$ and hence

$$F_{\theta,\kappa}(t) = \begin{cases} \exp(-t^{-1/\theta} + t\kappa^{-1/\theta - 1}) & \text{if } t \in (0, \kappa], \\ 1 & \text{otherwise}. \end{cases}$$

The fact that $F_{\theta,\kappa}$ has a finite upper endpoint $x_F = \kappa$ implies that when $x_1 + x_2 \geq \kappa$, the bivariate distribution $H_F$ reduces to independence, viz. $H_F(x_1, x_2) = F(x_1)F(x_2)$. Consequently, at any $u_1, u_2 \in [0, 1]$ such that $F^{-1}(u_1) + F^{-1}(u_2) \geq \kappa$, the reciprocal Archimedean copula $C_F$ generated by $F$ reduces to the independence copula, i.e., $C_F(u_1, u_2) = u_1 u_2$. This is illustrated in Figure 3, which shows samples of size $n = 1000$ for $\theta = 1$ and various values of $\kappa$. The plots also suggest that there is a singularity along the curve $F^{-1}(u_1) + F^{-1}(u_2) = \kappa$, indicated in red. This is indeed the case and due to the fact that $S_{\theta,\kappa}$ is discontinuous at $\kappa$, as explained in Remark 4 below. Finally, note that $C_F$ converges to the independence copula when $\kappa \to 0$ while $C_F$ converges to the Galambos copula with parameter $\theta$ when $\kappa \to \infty$.  

\[\Box\]
Remark 4. Given $F \in \mathcal{F}_d$, let $H_F$ be the radial measure corresponding to $\Lambda = -\ln(F)$ through (11). Let also $U = (U_1, \ldots, U_d)$ be distributed as the reciprocal Archimedean copula $C_F$. If $\nu_F$ has an atom at $\kappa > 0$, then
\[ \Pr\{F^{-1}(U_1) + \cdots + F^{-1}(U_d) = \kappa\} > 0, \]
implying that $C_F$ has a singular component. To see why this holds, let $Y$ be the random vector distributed as $H_F$ and $\zeta$ be the PRM with mean measure $\nu_F \times \sigma_d$, as in Proposition 4. Now fix some arbitrary $0 < \varepsilon < 1/\{d(d-1)\}$ and set
\[ \mathcal{B}_\varepsilon = \{ s \in \mathbb{S}_d : \forall i \in \{1, \ldots, d-1\} s_i \in [1/d - \varepsilon, 1/d + \varepsilon] \} \]
Clearly, if $s \in \mathcal{B}_\varepsilon$, then $s_d \in [1/d - (d-1)\varepsilon, 1/d + (d-1)\varepsilon] \subset [0, 1]$ by the choice of $\varepsilon$. As a consequence, $\sigma_d(\mathcal{B}_\varepsilon) > 0$. Now set $\delta = \kappa\{1/d - (d-1)\varepsilon\}$, so that $0 < \delta < \kappa$, and introduce the following partition of $(0, \infty) \times \mathbb{S}_d$:
\[ A_1 = (\kappa, \infty] \times \mathbb{S}_d, \quad A_2 = \{ \kappa \} \times \mathcal{B}_\varepsilon, \quad A_3 = \{ \kappa \} \times \mathbb{B}_\varepsilon^c, \quad A_4 = (\delta, \kappa] \times \mathbb{S}_d, \quad A_5 = (0, \delta] \times \mathbb{S}_d. \]
The event $\{Y_1 + \cdots + Y_d = \kappa\}$ occurs in particular if $\zeta(A_j) = 0$ for $j \in \{1, 3, 4\}$ and $\zeta(A_2) = 1$. Given that $\nu_F \times \sigma_d(A_j) \in (0, \infty)$ for $j \in \{1, 2, 3, 4\}$, one has
\[ \Pr(Y_1 + \cdots + Y_d = \kappa) \geq \nu_F(\{ \kappa \}) \times \sigma_d(\mathcal{B}_\varepsilon) \times \prod_{j=1}^4 e^{-\nu_F \times \sigma_d(A_j)} > 0, \]
as claimed. The exact computation of $\Pr(Y_1 + \cdots + Y_d = \kappa)$ seems cumbersome, however.

Next, note the following fact, whose proof is straightforward and therefore omitted.

**Lemma 7.** Let $S \in \mathcal{S}_1$ be a generalized survival function of a radial measure $\nu$ with Williamson $d$-transform $\Lambda$. Then for any $\theta > 0$, $\theta S \in \mathcal{S}_1$ is the generalized survival function of $\nu_{\theta} = \theta \nu$. Furthermore, the Williamson $d$-transform of $\nu_{\theta}$ is $\theta \Lambda$.

In view of Lemma 7, parametric families of reciprocal Archimedean copulas can be created by first choosing a baseline reciprocal Archimedean generator $F$ through a generalized survival function $S$ as outlined above, and then considering the class of reciprocal Archimedean copulas generated by the Lehmann alternatives of $F$, viz. $\{F^\theta : \theta > 0\}$.

**Example 10.** For any $t > 0$ and $\theta > 0$, set $S_\theta(t) = \theta(t^{1/t} - 1)$. Straightforward calculations lead to the following expression for the generator of the corresponding $\ell_1$-norm symmetric exponent measure, valid for all $t > 0$:
\[ A_\theta(t) = \theta \int_0^\infty (1 - t/r)^{d-1} e^{1/r}(1/r)^2 dr = \theta e^{1/t}r^d \gamma(d, 1/t), \]
where for any $x > 0$, $\gamma(d, x) = \int_0^x z^{d-1} e^{-z} dz$ denotes the lower incomplete Gamma function. Consequently, the reciprocal Archimedean generator of this “reciprocal Gamma” family is given, for all $t > 0$, by
\[ F_\theta(t) = \exp\{-\theta e^{1/t}r^d \gamma(d, 1/t)\}. \]
Figure 4 shows samples of size $n = 1000$ from this copula for various values of $\theta$ when $d = 2$. As the plots suggest, this family can only capture low degrees of dependence. ◦

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Figure 4: Random samples of size 1000 from the bivariate reciprocal Gamma family from Example 10 with \( \theta = 0.1 \) (left), \( \theta = 1 \) (middle) and \( \theta = 100 \) (right).

### 6.2 Specifying the exponent measure generator \( \Lambda \)

An alternative strategy for the construction of reciprocal Archimedean copulas consists of specifying the generator directly through a choice of \( \Lambda \in \mathcal{L}_0^d \), i.e., a \( d \)-monotone function on \((0, \infty)\) such that \( \Lambda(t) \to 0 \) as \( t \to \infty \) and \( \Lambda(t) \to \infty \) as \( t \to 0 \). By Theorem 2, the generalized survival function \( S_\nu \) of the corresponding radial measure \( \nu \) is then given, for all \( t > 0 \), by the inverse Williamson \( d \)-transform of \( \Lambda \), viz.

\[
S_\nu(t) = \sum_{k=0}^{d-2} \frac{(-1)^k \Lambda^{(k)}(t)}{k!} t^k + \frac{(-1)^{d-1} \Lambda^{(d-1)}(t)}{(d-1)!} t^{d-1}.
\]

The main advantage of this strategy is the explicit form of the reciprocal Archimedean generator \( F = e^{-\Lambda} \) and consequently of the reciprocal Archimedean copula itself. This makes Step 4 of Algorithm 1 easy to carry out. However, there are two major disadvantages. First, choosing a valid \( \Lambda \) is not straightforward, especially in higher dimensions, because the \( d \)-monotonicity condition is not easily met or verified. More importantly, the formula for \( S_\nu \) is likely to be cumbersome, because it involves a number of higher order derivatives of \( \Lambda \) that may be tedious to compute explicitly. This in turn complicates matters in Step 2c of Algorithm 1, as the random generation therein requires the inverse of \( S_\nu \).

A prototype of this construction strategy is the reciprocal Clayton copula, already detailed in Examples 2, 4 and 6. The following example shows how it could be generalized.

**Example 11.** Let \( \varphi : [0, 1] \to [0, \infty) \) be a completely monotone function such that \( \varphi(1) = 0 \) and \( \varphi(t) \to \infty \) as \( t \to 0 \). Consider the function defined, for all \( t \in \mathbb{R}_+ \), by

\[
\Lambda(t) = \varphi(1 - e^{-t}).
\]

Then \( \Lambda \) is clearly decreasing and such that \( \Lambda(t) \to 0 \) as \( t \to \infty \) while \( \Lambda(t) \to \infty \) as \( t \to 0 \). Through an application of Faà di Bruno’s formula, one can also see that the \( k \)th derivative
of \( \Lambda \) is given, for all \( t > 0 \), by

\[
\Lambda^{(k)}(t) = \sum^{*} \frac{k!}{m_1! \cdots m_k!} \varphi^{(m)}(1 - e^{-t}) \prod_{j=1}^{k} \left\{ \frac{(-1)^{j+1}}{j!} e^{-t} \right\}^{m_j},
\]

where \( m = m_1 + \cdots + m_k \) and \( \Sigma^* \) denotes a sum over all \( k \)-tuples of non-negative integers \( (m_1, \ldots, m_k) \) such that \( m_1 + 2m_2 + \cdots + km_k = k \). In this expression, \( (-1)^{j+1} e^{-t} \) is the \( j \)th derivative of \( 1 - e^{-t} \). Elementary manipulations lead to the simplified form

\[
\Lambda^{(k)}(t) = (-1)^k \sum^{*} \frac{k!}{\prod_{j=1}^{k} m_j!(j!)^{m_j}} (-1)^m \varphi^{(m)}(1 - e^{-t}) e^{-mt},
\]

which shows that \( (-1)^k \Lambda^{(k)} \) is non-negative, owing to the fact that \( \varphi \) is completely monotone and hence \( (-1)^m \varphi^{(m)} \geq 0 \) for any integer \( m \geq 0 \). Consequently, \( \Lambda \in \mathcal{L}_d \) for any \( d \geq 2 \) and \( F = e^{-\Lambda} \) can generate a reciprocal Archimedean copula in any dimension.

A trivial example of completely monotone function \( \varphi \) is given, for fixed \( \theta > 0 \) and all \( t \in \mathbb{R} \), by \( \varphi(t) = -\theta \ln(t) \). The corresponding generator \( \Lambda \) is that of the reciprocal Clayton copula already considered in Examples 2, 4 and 6. Another example is provided by defining, for fixed \( \theta > 0 \) and all \( t > 0 \), \( \varphi(t) = t^{-\theta} - 1 \). One then has, for all \( t > 0 \), \( \Lambda(t) = (1 - e^{-t})^{-\theta} - 1 \) and \( F(t) = \exp\{1 - (1 - e^{-t})^{-\theta}\} \).

Next, observe that the class \( \mathcal{L}_d \) of valid generators \( \Lambda \) is closed under sums and products.

**Lemma 8.** Let \( \Lambda_1, \Lambda_2 \) be arbitrary elements in \( \mathcal{L}_d \). Then \( \Lambda_1 \Lambda_2 \in \mathcal{L}_d \) and \( \Lambda_1 + \Lambda_2 \in \mathcal{L}_d \).

**Proof.** To verify that \( \Lambda_1 \Lambda_2 \) and \( \Lambda_1 + \Lambda_2 \) satisfy the limit conditions at 0 and \( \infty \) and that \( \Lambda_1 + \Lambda_2 \) is \( d \)-monotone is straightforward. The fact that \( \Lambda_1 \Lambda_2 \) is \( d \)-monotone follows directly from Theorem 5 of Williamson (1956). \( \Box \)

The final example of this section shows how Lemma 8 can be used to create new families of reciprocal Archimedean copulas.

**Example 12.** For \( t > 0 \) and \( \theta_1, \theta_2 > 0 \), let \( \Lambda_{\theta_1}(t) = t^{-1/\theta_1} \) and \( \Lambda_{\theta_2}(t) = -\ln(1 - e^{-t})/\theta_2 \) be the exponent measure generators corresponding to the Galambos and reciprocal Clayton copulas, respectively (see Examples 1–2). Consider the exponent measure generator \( \Lambda_{\theta_1+\theta_2} = \Lambda_{\theta_1} + \Lambda_{\theta_2} \) whose corresponding reciprocal Archimedean generator is \( F_{\theta_1,\theta_2} = F_{\theta_1} F_{\theta_2} \), where \( F_{\theta_j} = e^{-\Lambda_{\theta_j}} \) for \( j = 1, 2 \).

To draw a random sample from the family generated by \( F_{\theta_1,\theta_2} \) one can use the fact that \( H_{F_{\theta_1,\theta_2}} = H_{F_{\theta_1}} H_{F_{\theta_2}} \) is the distribution of \( Y_{\theta_1} \lor Y_{\theta_2} \), where for \( j = 1, 2 \), \( Y_{\theta_j} \) is a random vector distributed as \( H_{F_{\theta_j}} \). Alternatively, Algorithm 1 can be used directly. To this end, it is particularly convenient that by Theorem 2, the generalized survival function of the radial measure corresponding to \( \Lambda_{\theta_1,\theta_2} \) is of the form \( S_{\theta_1,\theta_2} = S_{\theta_1} + S_{\theta_2} \), where for \( j = 1, 2 \) \( S_{\theta_j} \) is the generalized survival function of the radial measure corresponding to \( \Lambda_{\theta_j} \). Eq. (13)
Figure 5: Random samples of size 1000 from the bivariate reciprocal mixture copula from Example 12 with \((\theta_1, \theta_2) = (1.28, 0.1)\) (left), \((5, 1)\) (middle) and \((8, 2)\) (right).

then implies that the survival function of the radial variable \(R_{k,i}\) that needs to be sampled from in Step 2c is a mixture, i.e., for all \(r > 0\),

\[
\Pr(R_{k,i} > r) = \alpha_k \Pr(R^1_{k,i} > r) + (1 - \alpha_k) \Pr(R^2_{k,i} > r),
\]

where the mixing probability is given by

\[
\alpha_k = \frac{S_{\theta_1}(\Delta_k) - S_{\theta_1}(\Delta_{k-1})}{S_{\theta_1}(\Delta_k) + S_{\theta_2}(\Delta_k) - S_{\theta_1}(\Delta_{k-1}) - S_{\theta_2}(\Delta_{k-1})}
\]

and for \(j = 1, 2\), \(\Pr(R^j_{k,i} > r)\) is the survival function of the radial variable corresponding to \(S_{\theta_j}\), i.e., as in (13) with \(S_{\nu_F}\) replaced by \(S_{\theta_j}\).

Figure 5 shows samples of size 1000 for various values of \(\theta_1\) and \(\theta_2\) when \(d = 2\). Although the right-hand picture suggests that there is both upper- and lower-tail dependence, it can be checked using Lemma 3 that \(\lambda_U > 0\) but \(\lambda_L = 0\).

\[\Diamond\]

7 The completely monotone case

As mentioned, e.g., by Mai and Scherer (2012) in the case of finance, extendible dependence structures are often useful in high-dimensional problems. Reciprocal Archimedean copulas can be used in this context provided that the corresponding generator \(F\) belongs to \(\mathcal{F}_d\) for all \(d \geq 2\). The Galambos and reciprocal Clayton copulas satisfy this requirement, as do the families considered in Examples 11 and 12.

Theorem 1 implies that if \(F \in \mathcal{F}_d\) for every \(d \geq 2\), \(F\) must be logarithmically completely monotone on \((0, \infty)\), i.e., \(\Lambda = -\ln(F)\) must be completely monotone on \((0, \infty)\). By the well-known Hausdorff–Bernstein–Widder Theorem (Widder, 1941, Theorem 12b, p. 161), \(\Lambda\) is then the Laplace transform of a non-negative measure \(\omega_F\) on \([0, \infty)\), i.e., for all \(t > 0\),

\[
\Lambda(t) = \int_0^\infty e^{-st}d\omega_F(s). \tag{16}
\]

Because \(\Lambda(t) \to 0\) when \(t \to \infty\) and \(\Lambda(t) \to \infty\) when \(t \to 0\), the measure \(\omega_F\) satisfies \(\omega_F\{0\} = 0\) and \(\omega_F[0, \infty) = \infty\).
Example 13. For the Galambos copula with parameter $\theta > 0$, $\Lambda_G(t) = t^{-1/\theta}$ for all $t > 0$; see Example 1. Setting, for all $s \geq 0$, $\omega_{FG}[0,s] = s^{1/\theta}/\Gamma(1+1/\theta)$, one finds
\[
\int_0^\infty e^{-st}\omega_{FG}(s) = \frac{1/\theta}{\Gamma(1+1/\theta)} \int_0^\infty e^{-st}s^{1/\theta-1}ds = t^{-1/\theta} = \Lambda_G(t).
\]

Example 14. For the reciprocal Clayton copula with parameter $\theta > 0$, $F_C(t) = (1-e^{-t})^{1/\theta}$ for all $t > 0$; see Example 2. Expanding $\Lambda_C = -\ln(F_C)$ in Taylor series, one can see that the non-negative measure $\omega_{FC}$ whose Laplace transform is $\Lambda_{FC}$ satisfies, for all $t > 0$,
\[
\Lambda_C(t) = -\frac{1}{\theta} \ln(1-e^{-t}) = \frac{1}{\theta} \sum_{k=1}^\infty \frac{e^{-kt}}{k} = \int_0^\infty e^{-ts}d\omega_{FC}(s)
\]
Therefore, $\omega_{FC}$ is a discrete measure assigning mass $1/(\theta k)$ to every $k \in \mathbb{N}$.

Because of Eq. (16), reciprocal Archimedean copulas having a logarithmically completely monotone generator have an alternative form, as stated next and proved in Appendix E.

Proposition 5. Suppose that $F \in \mathcal{F}_d$ is such that $\Lambda = -\ln(F)$ is completely monotone on $(0,\infty)$. Then, for all $u_1, \ldots, u_d \in (0,1)$,
\[
C_F(u_1, \ldots, u_d) = \exp\left[\int_0^\infty \prod_{k=1}^d \{1 - e^{-s_F^{-1}(u_k)}\} - 1\right]d\omega_F(s)
\]
The following result shows how the measure $\omega_F$ is related to the radial measure $\nu_F$. It also gives an alternative stochastic representation of reciprocal Archimedean copulas with logarithmically completely monotone generators.

Proposition 6. Suppose that $F \in \mathcal{F}_d$ is such that $\Lambda = -\ln(F)$ is completely monotone on $(0,\infty)$. Let $\nu_F$ be the radial measure corresponding to $\Lambda$ through Eq. (11) and $\omega_F$ be such that Eq. (16) holds for all $t > 0$. Then the following statements hold.

(i) Let $g : (0,\infty) \times [0,\infty) \to (0,\infty)$ be the mapping defined, for all $x \in (0,\infty)$ and $w \geq 0$, by $g(x,w) = x/w$. Let also $\epsilon_d$ denote the Erlang distribution with parameter $d \in \mathbb{N}$. Then the image measure of $\epsilon_d \times \omega_F$ by $g$ is $\nu_F$.

(ii) Let $h : (0,\infty)^d \times [0,\infty) \to E_d$ be the mapping defined, for all $s \in (0,\infty)^d$ and $t \geq 0$, by $h(x,t) = x/t$. Let also $\epsilon_1$ denote the unit exponential distribution and $\xi = \sum_k \epsilon(x_k,w_k)$ be a PRM on $(0,\infty)^d \times [0,\infty)$ with mean measure $\epsilon_1 \times \cdots \times \epsilon_1 \times \omega_F$. Then $H_F$ is the distribution function of the random vector
\[
Z = \max_k (x_k/w_k) \lor 0.
\]
Again, there is a clear parallel between this representation and that of Archimedean copulas with completely monotone generators in terms of multiplicative hazard models. In the present context, the measure \( \omega_F \) plays the same role as the frailty distribution.

Finally, it is worth pointing out the relation between the Laplace transform (16) and the Williamson \( d \)-transform appearing in Theorem 2. To this end, suppose that \( \nu \) is an arbitrary measure on \((0, \infty]\) such that \( \nu(0, \infty] = \infty, \nu(r, \infty] < \infty \) for any \( r > 0 \), and \( \nu\{\infty\} = 0 \). Furthermore, for any \( d \geq 2 \), let \( \nu_d \) denote the image measure of \( \nu \) by the mapping \( x \mapsto dx \).

Then the following holds, for any \( t > 0 \), by Lebesgue’s Dominated Convergence Theorem:

\[
\lim_{d \to \infty} \mathcal{W}_d(\nu_d)(t) = \lim_{d \to \infty} \int_0^\infty \left(1 - \frac{t}{r}\right)^{d-1} \nu_d(r) = \lim_{d \to \infty} \int_0^t \left(1 - \frac{t}{dr}\right)^{d-1} \nu(r) = \int_0^\infty e^{-t/r} \nu(r).
\]

In other words, the Williamson \( d \)-transform of \( \nu_d \), as \( d \to \infty \), becomes the Laplace transform of the image measure of \( \nu \) by the mapping \( x \mapsto 1/x \).

### 8 The case \( F^{-1}(0) > 0 \)

In this section, we briefly comment on the condition \( F^{-1}(0) = \sup\{x \in \mathbb{R} : F(x) = 0\} = 0 \) that was imposed throughout this article. First recall from Remark 2 that while \( F^{-1}(0) < 0 \) cannot occur, \( F^{-1}(0) > 0 \) is indeed possible. For arbitrary \( a > 0 \), let \( \mathcal{F}_a^d \) denote the set of all continuous distribution functions \( F \) such that \( F^{-1}(0) = a \) and such that \( C_F \) given by (3) is a bona fide \( d \)-dimensional copula. Let also \( C_a^d = \{C_F : F \in \mathcal{F}_a^d\} \).

As a first observation, note that for any \( d \geq 2 \) and \( a > 0 \), the independence copula \( \Pi \) is included in \( C_a^d \). Indeed, let \( F \) be an arbitrary continuous distribution function with support \([a, b]\), where \( b \leq 2a \). Then for arbitrary \( u \in [0, 1]^d \) and \( j \in \{1, \ldots, d\} \), \( F^{-1}(u_j) \geq a \).

Consequently, for any set \( A \subseteq \{1, \ldots, d\} \) with \( |A| \geq 2 \), \( F(\sum_{j \in A} F^{-1}(u_j)) = 1 \) and hence \( C_F \) given by (3) reduces to the independence copula.

Because \( C_F = \Pi \) for any continuous \( F \) whose support is \([a, b]\) with \( a > 0 \) and \( b \leq 2a \), Lemma 1 concerning the uniqueness of a reciprocal Archimedean generator no longer holds. It is also clear from this example that \( \Lambda = -\ln(F) \) does not need to be \( d \)-monotone when \( F \in \mathcal{F}_a^d \), and hence Proposition 1 does not extend to this case. Nevertheless, as stated below, \( d \)-monotonicity of \( \Lambda \) is sufficient and implies the max-id property. From the proof given in Appendix F, however, the exponent measure of \( H_F \) is no longer \( \ell_1 \)-norm symmetric.

**Proposition 7.** Let \( \Lambda \) be a \( d \)-monotone map on \((a, \infty] \) for some \( a > 0 \) such that \( \Lambda(t) \to 0 \) as \( t \to \infty \) and \( \Lambda(t) \to \infty \) as \( t \to a \). Then \( F = e^{-\Lambda} \in \mathcal{F}_a^d \) and \( C_F \) given by (3) is max-id.

For any \( F \in \mathcal{F}_d \) and any \( a > 0 \), Proposition 7 implies that \( F_a \in \mathcal{F}_d^a \), where \( F_a \) is given, for all \( t \in \mathbb{R} \), by \( F_a(t) = F(t - a) \). When \( d = 2 \), \( C_{F_a} \) is given, for all \( u_1, \ldots, u_d \in [0, 1] \), by

\[
C_{F_a}(u_1, \ldots, u_d) = \prod_{A \in \mathcal{P}_d,a} F\left\{ \sum_{k \in A} F^{-1}(u_k) + a(|A| - 1) \right\} / \prod_{A \in \mathcal{P}_d,a} F\left\{ \sum_{k \in A} F^{-1}(u_k) + a(|A| - 1) \right\}.
\]
This shows that reciprocal Archimedean copulas in \( C_d \) can be extended using a “shift” parameter \( a \). This is of interest in particular because if \( x_F < \infty \) and \( a \geq x_F \), then \( C_{F,a} = \Pi \).

9 Discussion

The main purpose of this article was to introduce a new class of multivariate copulas that are tractable, interpretable, and relatively easy to simulate. In dimension \( d \geq 2 \), these copulas are of the form (3) for some distribution function \( F \) having support \([0, x_F]\) with \( 0 < x_F \leq \infty \) whose logarithm \( \Lambda = -\ln(F) \) is \( d \)-monotone. We proposed to call them reciprocal Archimedean copulas because of their affinity with classical Archimedean copulas in the bivariate case; cf. (1) and (2). As was seen, however, the parallel between these two classes of copulas extends much further.

Given the popularity of the Archimedean class, one can easily imagine that reciprocal Archimedean copulas will gradually find applications in situations where it makes sense to use a positive, symmetric (exchangeable) dependence structure. Algorithm 1 ensures that random samples from members of this broad class of copulas can be generated with relative ease. In subsequent work, we hope to explore inferential issues associated with the use of reciprocal Archimedean copulas in multivariate data analysis. Reciprocal Archimedean copulas whose generators are such that \( F^{-1}(0) > 0 \) may also merit further research. However, given the unidentifiability issue mentioned in Section 8, inference within that class will require particular care.

Appendix A: Proofs of the results in Section 2

Proof of Lemma 1. Sufficiency is obvious. To show necessity, suppose that \( F, G \in \mathcal{F}_d \) generate the same copula \( C = C_F = C_G \in C_d \). As any bivariate margin of \( C \) is of the form (1), one then has, for all \( u_1, u_2 \in (0, 1) \),
\[
F\{F^{-1}(u_1) + F^{-1}(u_2)\} = G\{G^{-1}(u_1) + G^{-1}(u_2)\}.
\]
Equivalently, if \( s, t \) are arbitrary elements in the support of \( F \), then \( G^{-1} \circ F(s) + G^{-1} \circ F(t) = G^{-1} \circ F(s + t) \), which shows that \( G^{-1} \circ F \) satisfies Cauchy’s functional equation on the support of \( F \). Given that \( G^{-1} \circ F \) is non-decreasing, the only possible solution is of the form \( G^{-1} \circ F(t) = \kappa t \) for some \( \kappa > 0 \). This concludes the argument. \( \square \)

The proof of Proposition 1 uses several basic properties of \( d \)-monotone functions, which are summarized below for convenience, along with a definition of the concept. For additional details, see, e.g., Widder (1941), Williamson (1956) or McNeil and Nešlehová (2009).

Definition. Let \( a, b \in \mathbb{R} \) and \( f = f^{(0)} \) be a real-valued function defined on \((a, b)\). For each \( k \in \mathbb{N} \), let \( f^{(k)} \) denote the \( k \)th derivative of \( f \) when it exists.
(i) $f$ is called 1-monotone if it is non-negative and non-increasing on $(a,b)$.

(ii) $f$ is called $d$-monotone if it is differentiable up to the order $d-2$ on $(a,b)$ and if, for all $k \in \{0, \ldots, d-2\}$ and $x \in (a,b)$, one has $(-1)^k f^{(k)}(x) \geq 0$, and further if $(-1)^{d-2} f^{(d-2)}$ is non-increasing and convex in $(a,b)$.

(iii) If $f$ has derivatives of all orders in $(a,b)$ and if $(-1)^k f^{(k)}(x) \geq 0$ for all $k \in \mathbb{N}$ and $x$ in $(a,b)$, then $f$ is said to be completely monotone.

Lemma A1. Let $a, b \in \mathbb{R}$ and $f$ be a real-valued function defined on $(a,b)$. Let $\tilde{f}$ be a function defined at each $x \in (-b,-a)$ by $\tilde{f}(x) = f(-x)$. Further let $d \geq 1$ be an integer. Then the following statements are equivalent:

(i) $f$ is $d$-monotone on $(a,b)$;

(ii) $f$ is non-negative and for all $k \in \{1, \ldots, d\}$, $x \in (-b,-a)$ and $h_1, \ldots, h_k > 0$ such that $-b < x + h_1 + \cdots + h_k < -a$, one has

$$\Delta_{h_k} \cdots \Delta_{h_1} \tilde{f}(x) = \sum_{B \subseteq \{1, \ldots, k\}} (-1)^{|B|} \tilde{f} \left( x + \sum_{i \in B} h_i \right) \geq 0,$$

where $\Delta_{h_k} \cdots \Delta_{h_1}$ denotes a sequential application of the first-order difference operator $\Delta_h$ given by $\Delta_h \tilde{f}(x) \equiv \tilde{f}(x+h) - \tilde{f}(x)$ whenever $x$ and $x+h$ are in $(-b,-a)$.

Proof. This result is excerpted from Proposition A.1 in McNeil and Nešlehová (2009). What remains to be shown is that for any $k \in \{1, \ldots, d\}$, any $x \in (-b,-a)$ and any $h_1 > 0, \ldots, h_k > 0$ such that $-b < x + h_1 + \cdots + h_k < -a$, one has

$$\Delta_{h_k} \cdots \Delta_{h_1} \tilde{f}(x) = \sum_{B \subseteq \{1, \ldots, k\}} (-1)^{|B|} \tilde{f} \left( x + \sum_{i \in B} h_i \right).$$

This follows easily by induction on $k$. Indeed, for $k = 1$, the right-hand side reduces to $\tilde{f}(x+h_1) - \tilde{f}(x)$, which is precisely $\Delta_{h_1} \tilde{f}(x)$. Now suppose that the statement is true for all integers up to $k \in \{1, \ldots, d-1\}$. Then for arbitrary $x \in (-b,-a)$ and $h_1 > 0, \ldots, h_{k+1} > 0$ such that $-b < x + h_1 + \cdots + h_{k+1} < -a$, one has

$$\Delta_{h_{k+1}} \cdots \Delta_{h_1} \tilde{f}(x) = \Delta_{h_{k+1}} \sum_{B \subseteq \{1, \ldots, k\}} (-1)^{|B|} \tilde{f} \left( x + \sum_{i \in B} h_i \right)$$

$$= \sum_{B \subseteq \{1, \ldots, k\}} (-1)^{|B|} \tilde{f} \left( x + \sum_{i \in B} h_i + h_{k+1} \right) - \sum_{B \subseteq \{1, \ldots, k\}} (-1)^{|B|} \tilde{f} \left( x + \sum_{i \in B} h_i \right)$$

$$= \sum_{B \subseteq \{1, \ldots, k+1\}} (-1)^{|B|+1} \tilde{f} \left( x + \sum_{i \in B} h_i \right),$$

as claimed. $\square$
Proof of Proposition 1. First note that the limits of $\Lambda$ at 0 and infinity follow directly from the fact that $\Lambda = -\ln(F)$ and that $F(0) = 0$. Furthermore, if $C_F$ given by (3) is a copula, Sklar’s representation theorem guarantees that $H_F$ given by (5) is a $d$-variate cumulative distribution function. Consequently, all lower-dimensional margins of $H_F$ are distribution functions as well. Note that for every $k \in \{1, \ldots, d\}$, the $k$-dimensional margin of $H_F$ is of the same form as $H_F$ itself, i.e., for all $x_1, \ldots, x_k \in [0, \infty)$, one has

$$H_{F_k}(x_1, \ldots, x_k) = \lim_{\min(y_{k+1},\ldots,y_d) \to \infty} H_F(x_1, \ldots, x_k, y_{k+1}, \ldots, y_d),$$

Next, fix $k \in \{1, \ldots, d\}$, $h_1, \ldots, h_k > 0$ and $t > 0$ such that $t - (h_1 + \cdots + h_k) > 0$. Because $H_{F_k}$ is non-decreasing in its last argument, one has

$$H_F(h_1, \ldots, h_{k-1}, t - \sum_{i=1}^{k} h_i) \leq H_{F_k}(h_1, \ldots, h_{k-1}, t - \sum_{i=1}^{k-1} h_i).$$

(A1)

From the form of $H_{F_k}$, it follows directly that

$$H_{F_k}(h_1, \ldots, h_{k-1}, t - \sum_{i=1}^{k} h_i) = \frac{\prod_{A \in \mathcal{P}_{k,o}} F\left(\sum_{i \in A} h_i\right) \prod_{k \notin A} F\left(t - \sum_{i \notin A} h_i - h_k\right)}{\prod_{A \in \mathcal{P}_{k,e}} F\left(\sum_{i \in A} h_i\right) \prod_{k \notin A} F\left(t - \sum_{i \notin A} h_i - h_k\right)},$$

and similarly that

$$H_{F_k}(h_1, \ldots, h_{k-1}, t - \sum_{i=1}^{k} h_i) = \frac{\prod_{A \in \mathcal{P}_{k,e}} F\left(\sum_{i \in A} h_i\right) \prod_{k \notin A} F\left(t - \sum_{i \notin A} h_i\right)}{\prod_{A \in \mathcal{P}_{k,o}} F\left(\sum_{i \notin A} h_i\right) \prod_{k \notin A} F\left(t - \sum_{i \notin A} h_i - h_k\right)}.$$

Upon substitution into (A1) and cancellation of common terms, the inequality reduces to

$$\prod_{k \in A} F\left(t - \sum_{i \notin A} h_i\right) \prod_{k \notin A} F\left(t - \sum_{i \notin A} h_i - h_k\right) \geq 1.$$

Now observe that $\{B \subseteq \{1, \ldots, k\} : k - |B| \text{ is odd}\}$ is the same as

$$\{B \subseteq \{1, \ldots, k\} : k \notin B \text{ and } k - |B| \text{ is odd}\} \cup \{B \cup \{k\}, B \subseteq \{1, \ldots, k\} : k \notin B \text{ and } k - |B| \text{ is even}\},$$

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and that a similar identity holds for \( \{B \subseteq \{1, \ldots, k\} : k-|B| \text{ is even}\} \). Then (A1) becomes

\[
\prod_{B \subseteq \{1, \ldots, k\}, k-|B| \text{ is odd}} F\left(t - \sum_{i \in B} h_i\right) / \prod_{B \subseteq \{1, \ldots, k\}, k-|B| \text{ is even}} F\left(t - \sum_{i \in B} h_i\right) \geq 1,
\]

which is equivalent to

\[
\sum_{B \subseteq \{1, \ldots, k\}} (-1)^{k-|B|}\Lambda\left(t - \sum_{i \in B} h_i\right) = \sum_{B \subseteq \{1, \ldots, k\}} (-1)^{k-|B|}\tilde{\Lambda}\left(-t + \sum_{i \in B} h_i\right) \geq 0,
\]

where the map \( \tilde{\Lambda} : (-\infty, 0) \to [0, \infty) \) is defined, for all \( x < 0 \), by \( \tilde{\Lambda}(x) = \Lambda(-x) \). Given that this inequality holds for all \( k \in \{1, \ldots, d\} \) and arbitrary \( t > 0 \) and \( h_1, \ldots, h_k > 0 \) with \( t - (h_1 + \cdots + h_k) > 0 \), the \( d \)-monotonicity of \( \Lambda \) follows directly from Lemma A1.

\[ \square \]

**Proof of Corollary 1.** Given arbitrary \( x_1 < y_1 \) and \( x_2 < y_2 \), Lemma A1 implies that

\[ \Lambda(x_1 + y_2) + \Lambda(x_2 + y_1) \leq \Lambda(x_1 + x_2) + \Lambda(y_1 + y_2). \]

This is equivalent to \( F(x_1 + y_2)F(x_2 + y_1) \geq F(x_1 + x_2)F(y_1 + y_2) \) and to

\[
\frac{F(x_1)F(x_2)F(y_1)F(y_2)}{F(x_1 + x_2)F(y_1 + y_2)} \geq \frac{F(x_1)F(x_2)F(y_1)F(y_2)}{F(x_1 + y_2)F(y_1 + x_2)}.
\]

In view of the definition of \( H_F \), this is equivalent to (6).

\[ \square \]

**Proof of Lemma 2.** The formula for Blomqvist’s beta is immediate from the definition, i.e., \( \beta(C_F) = -1 + 4C_F(1/2, 1/2) \). To derive the expression for Spearman’s rho, first use integration by parts to check that

\[
\int_0^\infty \frac{F(x_2)}{F(x_1 + x_2)}dF(x_2) = \frac{1}{2} + \frac{1}{2} \int_0^\infty \frac{F^2(x_2)}{F^2(x_1 + x_2)}f(x_1 + x_2)dx_2.
\]

Upon substitution into the definition of \( \rho(C_F) \), one then gets

\[
\rho(C_F) = -3 + 12 \int_0^\infty \int_0^\infty H(x_1, x_2)dF(x_2)dF(x_1)
\]

\[
= -3 + 6 \int_0^\infty F(x_1)dF(x_1) + 6 \int_0^\infty \int_0^\infty F(x_1) \frac{F^2(x_2)}{F^2(x_1 + x_2)}f(x_1)f(x_1 + x_2)dx_2dx_1
\]

\[
= 6 \int_0^\infty H^2(x_1, x_2)r(x_1)f(x_1 + x_2)dx_2dx_1.
\]

Similarly,

\[
\rho(C_F) = 6 \int_0^\infty H^2(x_1, x_2)r(x_2)f(x_1 + x_2)dx_2dx_1.
\]

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The formula reported in the statement of the lemma is simply the average of these two expressions. Finally, to obtain the formula for Kendall’s tau, first write
\[
\tau(C_F) = -1 + 4 \int_0^\infty \int_0^\infty H(x_1, x_2) dH(x_1, x_2) = -1 + 4 \int_0^\infty \int_0^\infty H(x_1, x_2) h(x_1, x_2) dx_2 dx_1,
\]
where, for all \(x_1, x_2 \in (0, \infty),\)
\[
h(x_1, x_2) = H(x_1, x_2) r'(x_1 + x_2) + H(x_1, x_2) \{r(x_1 + x_2) - r(x_1)\}\{r(x_1 + x_2) - r(x_2)\}.
\]
Upon substitution into the definition of \(\tau(C_F),\) one can check that the result holds if
\[
2H^2(x_1, x_2)r'(x_1 + x_2) + 4H^2(x_1, x_2)\{r(x_1 + x_2) - r(x_1)\}\{r(x_1 + x_2) - r(x_2)\}
\]
integrates to 1 over \((0, \infty)^2,\) which is obvious as it is the density of \(H^2.\) \(\square\)

**Proof of Lemma 3.** (i) If \(\bar{F} \in RV_{-\alpha},\) then \(x_F = \infty.\) Consequently,
\[
\lambda_U(C_F) = 2 - \lim_{t \to \infty} \frac{1 - F^2(t)/F(2t)}{1 - F(t)} = 2 - \lim_{t \to \infty} \frac{F(2t) - F^2(t)}{F(2t)\{1 - F(t)\}}.
\]
Writing the numerator of the latter fraction as \(F(2t) - 1 + \{1 - F(t)\}\{1 + F(t)\}\) and noting that \(\{1 + F(t)\}/F(2t) \to 2\) as \(t \to \infty,\) one finds
\[
\lambda_U(C_F) = \lim_{t \to \infty} \frac{1 - F(2t)}{F(2t)\{1 - F(t)\}} = 2^{-\alpha}.
\]
By the same argument, it also follows that \(\lambda_U(C_F) = 0\) if \(x_F = \infty\) and \(\bar{F}\) is rapidly varying. Finally, if \(x_F < \infty,\) one has
\[
\lambda_U(C_F) = 2 - \lim_{t \uparrow x_F} \frac{1 + F(t)}{F(2t)} + \lim_{t \uparrow x_F} \frac{1 - F(2t)}{F(2t)\{1 - F(t)\}} = 0.
\]
(ii) First note that \(F^*\) is a survival function whose upper endpoint is \(\infty,\) from the assumption that the support of \(F\) is of the form \([0, x_F].\) By definition of \(\lambda_L(C_F),\) one has
\[
\lambda_L(C_F) = \lim_{t \downarrow 0} \frac{F(t)}{F(2t)} = \lim_{t \to \infty} \frac{F^*(t)}{F^*(t/2)}.
\]
This limit is either \(2^{-\beta}\) if \(F^* \in RV_{-\beta},\) or 0 if \(F^*\) is rapidly varying, as claimed. \(\square\)

**Appendix B: Proofs of the results in Section 3**

**Proof of Lemma 5.** Let \(S_\mu\) be the generalized survival function of an exponent measure \(\mu \in \mathcal{M}_d.\) The proof that \(S_\mu\) satisfies (i)–(iii) is analogous to establishing the well-known
properties of a survival function of a multivariate probability distribution. To show (i) it suffices, without loss of generality, to consider the case \( x_1 \to \infty \). Then for all \( x_2, \ldots, x_d \geq 0 \),

\[
\lim_{x_1 \to \infty} S_\mu(x_1, \ldots, x_d) = \mu(\{\infty\} \times (x_2, \infty] \times \cdots \times (x_d, \infty]) = 0
\]

from the fact that \( \mu \) is an exponent measure; see Definition 2. Further, the property (ii) follows directly from the fact that \( \sigma \)-finite measures are continuous from above. Finally, to see that (iii) holds, note that for any \( x \leq y \in E_d \),

\[
\sum_{c \in S_{x,y}} \text{sign}(c) S_\mu(c_1, \ldots, c_d) = \mu((x_1, y_1] \times \cdots \times (x_d, y_d]) \geq 0.
\]

Conversely, suppose that \( S : E_d \to [0, \infty) \) satisfies (i)–(iii). Let \( E \) be the collection of all finite unions of sets of the type \((x, y]\) or \( \{z \in E_d : z_k = 0, \forall i \neq k \ x_i < z_i \leq y_i\} \) for all \( x \leq y \in E_d \) and \( k \in \{1, \ldots, d\} \). Observe that \( E \) is a ring that generates the Borel \( \sigma \)-field on \( E_d \) and that Eqs. (17) and (9) define \( \mu \) on \( E \). Given that \( \mu \) is a pre-measure on \( E \) by property (ii), it can be uniquely extended by the Carathéodory process to a measure on the entire \( \sigma \)-field of \( E_d \). To show that \( \mu \) is an exponent measure, take any \( x > 0 \) and call on Poincaré’s identity to write

\[
\mu[\infty, x]^C = \mu\left( \bigcup_{k=1}^d \{z \in E_d : z_k > x_k\} \right) = \sum_{k=1}^d (-1)^{k+1} \sum_{A \subseteq \{1, \ldots, d\}, |A|=k} \mu\{z \in E_d : \forall j \in A \ z_j > x_j\}.
\]

In view of Condition (9), this may be rewritten as

\[
\mu[\infty, x]^C = \sum_{k=1}^d (-1)^{k+1} \sum_{A \subseteq \{1, \ldots, d\}, |A|=k} \mu\{z \in E_d : \forall j \in A \ z_j > x_j, \forall j \not\in A \ z_j > 0\}
\]

\[
= \sum_{k=1}^d (-1)^{k+1} \sum_{A \subseteq \{1, \ldots, d\}, |A|=k} S(x_A) < \infty,
\]

where \( x_A \in \mathbb{R}^d_+ \) is a point whose \( j \)th coordinate equals \( x_j \) if \( j \in A \) and zero otherwise. As \( S \) satisfies (i), the right-hand side tends to 0 as \( x \to \infty \). Thus \( \mu \in \mathcal{M}_d \) by Lemma 4. \( \square \)

**Proof of Proposition 2.** First suppose that \( \Lambda \) is \( d \)-monotone and that \( \Lambda(t) \to 0 \) as \( t \to \infty \). Set \( \Lambda(\infty) = 0 \) and define \( S(\mathbf{x}) = \Lambda(x_1 + \cdots + x_d) \) for all \( \mathbf{x} \in E_d \). To show that \( S \) is the generalized survival function of a measure \( \mu \in \mathcal{M}_d \), first note that \( S \) satisfies properties (i)–(ii) because \( \Lambda(t) \to 0 \) as \( t \to \infty \) and because \( \Lambda \) is continuous on \((0, \infty)\); see Lemma 2.
in McNeil and Nešlehová (2009). To establish property (iii), note that for any \( x > 0 \) and \( h_1, \ldots, h_d \in (0, \infty) \) such that \( x_k - h_k > 0 \) for all \( k \in \{1, \ldots, d\} \), Eq. (17) implies that

\[
\mu((x_1 - h_1, x_1] \times \cdots \times (x_d - h_d, x_d]) = \sum_{B \subseteq \{1, \ldots, d\}} (-1)^{d-|B|} \Lambda \left( \sum_{k=1}^{d} x_k - \sum_{k \in B} h_k \right).
\]

Next, introduce the map \( \tilde{\Lambda} : (-\infty, 0) \rightarrow [0, \infty) \) defined, for all \( t < 0 \), by \( \tilde{\Lambda}(t) = \Lambda(-t) \). In view of Lemma A1, one may conclude from the above calculation that

\[
\mu((x_1 - h_1, x_1] \times \cdots \times (x_d - h_d, x_d]) = \Delta_{h_1} \cdots \Delta_{h_d} \tilde{\Lambda} \left( -\sum_{k=1}^{d} x_k \right) \geq 0.
\]

Thus \( S \) defines a unique exponent measure \( \mu \in \mathcal{M}_d \) by Lemma 5.

Conversely, suppose that \( \mu \in \mathcal{M}_d \) is \( \ell_1 \)-norm symmetric with generator \( \Lambda_\mu \). Then

\[
\lim_{t \to \infty} \Lambda_\mu(t) = \lim_{t \to \infty} S_\mu(t/d, \ldots, t/d) = \lim_{t \to \infty} \mu((t/d, \infty] \times \cdots \times (t/d, \infty]) = 0
\]

because \( \mu \) is an exponent measure. Next, define the map \( \tilde{\Lambda}_\mu : (-\infty, 0) \rightarrow [0, \infty) \) at every \( t < 0 \) by \( \tilde{\Lambda}_\mu(t) = \Lambda_\mu(-t) \). For any \( k \in \{1, \ldots, d\} \), \( x < 0 \) and \( h_1, \ldots, h_k > 0 \) such that \( x + h_1 + \cdots + h_k < 0 \), one has

\[
\Delta_{h_1} \cdots \Delta_{h_k} \tilde{\Lambda}_\mu(x) = \sum_{B \subseteq \{1, \ldots, k\}} (-1)^{k-|B|} \Lambda_\mu \left( x + \sum_{i \in B} h_i \right)
\]

\[
= \sum_{j=0}^{k} (-1)^{k-j} \sum_{B \subseteq \{1, \ldots, k\}, |B|=j} \Lambda_\mu \left( -(k-j)x/k - \sum_{i \in B} (x/k + h_i) \right).
\]

Observing that the right-hand side is equal to

\[
\sum_{j=0}^{k} (-1)^{k-j} \sum_{B \subseteq \{1, \ldots, k\}, |B|=j} \mu \left\{ x \in E_d : \forall i \in B x_i > -x/k \text{ and } \forall i \in B x_i > -x/k - h_i \right\},
\]

one may then conclude that

\[
\Delta_{h_1} \cdots \Delta_{h_k} \tilde{\Lambda}_\mu(x) = \mu \left\{ x \in E_d : \forall i \in \{1, \ldots, k\} x_i \in (-h_i - x/k, -x/k] \right\}.
\]

As the last expression is clearly non-negative, \( \Lambda_\mu \) is \( d \)-monotone by Lemma A1. \( \Box \)

**Appendix C: Proofs of the results in Section 4**

The proof of Theorem 2 relies on Lemma 9 below, which characterizes \( d \)-monotone functions using the results of Williamson (1956). Its proof is given for completeness; it proceeds along the same lines as the proof of Proposition 3.1 in McNeil and Nešlehová (2009).
Lemma 9. A function \( f \) is \( d \)-monotone on \((0, \infty)\) if and only if there exists a unique Radon measure \( \nu \) on \((0, \infty] \) such that, for all \( t > 0 \),

\[
    f(t) = \int_t^\infty (1 - t/r)^{d-1} \, d\nu(r).
\]

Letting \( f^{(d-1)}_+ \) denote the right-hand derivative of \( f^{(d-2)} \), one also has, for any \( s > 0 \),

\[
    \nu(r, \infty] = \sum_{k=0}^{d-2} \frac{(-1)^k f^{(k)}(r)}{k!} r^k + \frac{(-1)^{d-1} f^{(d-1)}_+(r)}{(d-1)!} r^{d-1},
\]

with \( \nu(0, \infty] = \lim_{t \to 0} f(t) \) and \( \nu\{\infty\} = \lim_{t \to \infty} f(t) \).

Proof of Lemma 9. Theorems 1 and 2 of Williamson (1956) yield that \( f \) is \( d \)-monotone on \((0, \infty)\) if and only if there exists a unique measure \( \nu^* \) on \([0, \infty)\) such that

\[
    f(t) = \int_0^{1/t} (1 - rt)^{d-1} \, d\nu^*(r).
\]

Its generalized distribution function \( \gamma \) is defined, for all \( r \in \mathbb{R}_+ \), by \( \gamma(r) = \nu^*[0, r] \). Theorem 3 of Williamson (1956) implies \( \gamma(0-) = 0 \), \( \gamma(0) = f(\infty) \) and for all \( r > 0 \),

\[
    \gamma(r) = \sum_{k=0}^{d-2} \frac{(-1)^k f^{(k)}(1/r)}{k!} r^{-k} + \frac{(-1)^{d-1} f^{(d-1)}_+(1/r)}{(d-1)!} r^{-(d-1)},
\]

where \( f^{(d-1)}_+ \) denotes the left-hand derivative of \( f^{(d-2)} \). The term generalized distribution function is used because although \( \gamma \) is non-decreasing and right-continuous, it may not be that \( \gamma(r) \to 1 \) as \( r \to \infty \); in fact, it may be that \( \gamma \) is not bounded from above.

The statement of Lemma 9 then follows upon letting \( \nu \) be the image measure of \( \nu^* \) by the map \( \phi : [0, \infty) \to (0, \infty] \) defined, for all \( r \geq 0 \), by \( \phi(r) = 1/r \). Observe that \( \nu \) is a Radon measure on \((0, \infty] \) if and only if for all \( r > 0 \), \( \nu(r, \infty] < \infty \). In particular, the function \( f \) given in (10) is well defined and, for all \( r > 0 \), \( \nu(r, \infty] = \nu^*[0, 1/r] = \gamma(1/r-) \).

Finally, the fact that \( f(t) \to \nu(0, \infty] \) as \( t \to 0 \) follows from the integral representation of \( f \) by Levi’s Monotone Convergence Theorem and \( \lim_{t \to \infty} f(t) = \nu\{\infty\} \) follows from the proof of Theorem 1 in Williamson (1956).

Proof of Theorem 2. First suppose that \( \mu \) is a measure on \( E_d \) whose image measure by \( \mathcal{T} \) is \( \nu_\mu \times \sigma_d \), where \( \nu_\mu \) is a Radon measure on \((0, \infty] \) such that \( \nu_\mu\{\infty\} = 0 \). Because \( \mathcal{T} \) is one-to-one, \( \mu \) is the image measure of \( \nu_\mu \times \sigma_d \) by \( \mathcal{T}^{-1} \). To see that \( \mu \) satisfies Condition (9), note that for any \( k \in \{1, \ldots, d\} \),

\[
    \mu\{x \in E_d : x_k = 0\} = \mu\left( \bigcup_{j=1}^{\infty} \{x \in E_d : \forall_{j \neq k} x_j > z_{jn} \text{ and } x_k = 0\} \right) = \lim_{n \to \infty} \mu\{x \in E_d : \forall_{j \neq k} x_j > z_{jn} \text{ and } x_k = 0\},
\]

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where \((z_n)\) is an arbitrary sequence such that \(z_n > 0\) and \(z_n \to 0\) as \(n \to \infty\). The fact that \(\mu\{x \in E_d : x_k = 0\} = 0\) now follows because

\[
\mu\{x \in E_d : \forall j \neq k \ x_j > z_{jn}, \ x_k = 0\} = \nu_\mu\left(\sum_{j \neq k} z_{jn}, \infty\right) \times \sigma_d\{s \in S_d : s_k = 0\} = 0.
\]

Furthermore, for any \(x \in E_d\),

\[
\mu(x, \infty) = \nu_\mu \times \sigma_d\left\{(r, s) : r > \sum_{k=1}^{d} x_k, \ s > x/r\right\}
\]

\[
\int_{x_1 + \cdots + x_d}^{\infty} \left(1 - \sum_{k=1}^{d} x_k/r\right)^{d-1} \, \nu_\mu(r) \equiv \Lambda_\mu(x_1 + \cdots + x_d).
\]

(C1)

Given that \(\nu_\mu(\infty) = 0\), it follows from Lemma 9 that \(\Lambda_\mu(t) \to 0\) as \(t \to \infty\). Put together, \(\mu\) is an \(\ell_1\)-norm symmetric exponent measure with generator \(\Lambda_\mu = \mathcal{W}_d(\nu_\mu)\).

Conversely, suppose that \(\mu\) is an \(\ell_1\)-norm symmetric exponent measure with generator \(\Lambda_\mu\). Proposition 2 implies that \(\Lambda_\mu\) is \(d\)-monotone on \((0, \infty)\) and such that \(\Lambda_\mu(t) \to 0\) as \(t \to \infty\).

By Lemma 9, \(\Lambda_\mu\) is the Williamson \(d\)-transform of a unique Radon measure \(\nu_\mu\) on \((0, \infty)\) such that \(\nu_\mu(\infty) = 0\) and, for all \(r > 0\), \(\nu_\mu(r, \infty) = \mathcal{W}_d^{-1}(\Lambda_\mu)(r)\). From (C1) it follows that the generalized survival function of \(\mu\) is the same as the generalized survival function of the image measure of \(\nu_F \times \sigma_d\) by \(T^{-1}\). Lemma 5 and the fact that \(T\) is one-to-one thus imply that the image measure of \(\mu\) by \(T\) is \(\nu_\mu \times \sigma_d\), as claimed.

\(\square\)

**Proof of Proposition 3.** The result follows from the fact that if \(C \in \mathcal{C}_d\) is extreme-value, \(F\) must be a univariate extreme-value distribution. Because all bivariate margins of \(C\) are identical and of the form (1) one can assume, without loss of generality, that \(d = 2\). For arbitrary \(t > 0\) and \(u, v \in (0, 1)\), one then has \(C^t(u^{1/t}, v^{1/t}) = C(u, v)\). Thus, for arbitrary \(x, y > 0\), \(H_F\) given by (5) satisfies

\[
H_F^t(x, y) = \left[C\{F(x), F(y)\}\right]^t = C^t\{\{F^t(x)\}^{1/t}, \{F^t(y)\}^{1/t}\} = C\{F^t(x), F^t(y)\}.
\]

Given the specific form of \(C\), however, this means that, for arbitrary \(t, x, y > 0\), one has

\[
\left(\frac{F^t(x)F^t(y)}{F^t(x+y)}\right) = \left(\frac{F^t(x)F^t(y)}{F^{-1} \circ F^t(x) + F^{-1} \circ F^t(y)}\right)
\]

or equivalently, \(F^{-1} \circ F^t(x + y) = F^{-1} \circ F^t(x) + F^{-1} \circ F^t(y)\). Accordingly, the map \(G_t = F^{-1} \circ F^t\) satisfies Cauchy’s functional equation on \((0, \infty)\), and hence it is linear.

Given that \(G_t\) is non-decreasing and that \(G_t(0) = 0\), one can conclude that there exists a constant \(\alpha_t > 0\) such that, for all \(x > 0\), \(G_t(x) = \alpha_t x\) or, equivalently, \(F^t(x) = F(\alpha_t x)\). As this relationship holds for all \(t > 0\), it follows that \(F\) is max-stable (Resnick, 2008), and hence an extreme-value distribution whose support is \((0, \infty)\). One can further conclude from the Fisher–Tippett Theorem that \(F\) is a Fréchet distribution \(\Phi_\theta\) for some \(\theta > 0\), i.e., \(F(t) = \Phi_\theta(t) = \exp(-t^{-1/\theta})\) for all \(t > 0\). From Example 1, the copula generated by \(F\) is the Galambos copula with parameter \(\theta\). \(\square\)
Appendix D: Proof of Lemma 6

Denote by $\nu^*_F$ the restriction of the radial measure $\nu_F$ to $(\Delta, \infty]$ and let $\mu^*_F$ be the image measure of $\nu^*_F \times \sigma_\mathbb{R}$ through the transformation $T^{-1}$. If then follows from Theorem 2 that $\mu^*_F$ is an $\ell_1$-norm symmetric exponent measure with generator given, for all $t \in (0, \infty)$, by $\Lambda^*(t) = \Lambda(t \vee \Delta)$. Proceeding as in the proof of Proposition 4, one can see that $Y^*$ has a max-id distribution with joint distribution function given, for all $x > 0$, by $H^*(x) = e^{-\mu^*_F[-\infty, x]}$. From the discussion just preceding Theorem 1, one can conclude that $H^*$ is of the form (5) with $F$ replaced by $F^* = e^{-\Lambda^*}$. As a result, one has, for all $x \geq (\Delta, \ldots, \Delta)$,

$$H^*(x) = H_F(x).$$

Finally, the inequality follows from the fact that $H_F$ is positively upper orthant dependent (PUOD); see, e.g., Theorem 8.6 of Joe (2015).

Appendix E: Proofs of the results in Section 7

Proof of Proposition 5. From the Hausdorff–Bernstein–Widder Theorem, one can write $\Lambda = -\ln(F)$ in the form (16) for some non-negative measure $\omega_F$ on $[0, \infty)$. Expression (3) then simplifies, for all $u_1, \ldots, u_d \in (0, 1)$, to

$$C_F(u_1, \ldots, u_d) = \exp \left\{ \sum_{A \in \mathcal{P}_{d,o} \cup \mathcal{P}_{d,e}} (-1)^{|A|} \int_0^\infty e^{-s \sum_{k \in A} F^{-1}(u_k)} d\omega_F(s) \right\},$$

which yields the stated result because $\mathcal{P}_{d,o} \cup \mathcal{P}_{d,e} = \{ A \subseteq \{1, \ldots, d\} : A \neq \emptyset \}$ and

$$\prod_{k=1}^d \{1 - e^{-s F^{-1}(u_k)}\} = \sum_{A \subseteq \{1, \ldots, d\}} \prod_{k \in A} \{e^{-s F^{-1}(u_k)}\}$$

by the multinomial formula.

Proof of Proposition 6. We prove part (i) by showing that the Williamson $d$-transform of the image measure of $\epsilon_d \times \omega_F$ by $g, \nu^*$, say, is $\Lambda$. Indeed, for any $t > 0$,

$$\int_t^\infty (1 - \frac{t}{r})^{d-1} d\nu^*(r) = \int_0^\infty \int_0^\infty (1 - \frac{tw}{x})^{d-1} e^{-x} \frac{dx}{\Gamma(d)} dx d\omega_F(w) = \int_0^\infty \frac{e^{-tw}}{\Gamma(d)} \int_t^\infty (x - tw)^{d-1} e^{-x+tw} dx d\omega_F(w) = \int_0^\infty e^{-tw} d\omega_F(w) = \Lambda(t).$$

The conclusion then follows from the fact, established in Lemma 9, that a Radon measure on $(0, \infty]$ is uniquely specified by its Williamson $d$-transform.

Part (ii) follows from the fact that by Proposition 3.7 in Resnick (2008), $\tilde{\xi} = \sum_k \xi_{x_k}/w_k$ is a PRM on $E_d$ with mean measure $\mu^*$, which is the image measure of $\epsilon_1 \times \cdots \times \epsilon_1 \times \omega_F$.
by \( h \). Now \( \mu^* \) clearly satisfies Eq. (9). Moreover, the generalized survival function of \( \mu^* \) is given, for any \( y \in E_d \), by
\[
S_{\mu^*}(y) = (\epsilon_1 \times \cdots \times \epsilon_1 \times \omega_F)\{x \in (0, \infty)^d, w \geq 0 : \forall_{j \in \{1, \ldots, d\}} x_j \geq wy_j\}
= \int_0^\infty \prod_{j=1}^d e^{-wy_j} d\omega_F(w) = \Lambda(y_1 + \cdots + y_d).
\]
In particular, therefore, \( \mu^* \) is identical with the \( \ell_1 \)-norm symmetric exponent measure \( \mu_F \) generated by \( \Lambda \). The rest of the argument is analogous to the proof of Proposition 4. \( \square \)

**Appendix F: Proof of Proposition 7**

Fix \( a > 0 \), let \( a = (a, \ldots, a) \in \mathbb{R}^d \) and define \( E_d^a = [a, \infty] \setminus \{a\} \). For any \( I \subset \{1, \ldots, d\} \), define the function \( S_I \) at any \( x \in (a, \infty) \) by
\[
S_I(x) = \sum_{B \subseteq I} (-1)^{|I|-|B|} \Lambda\left\{ \sum_{j \in B} x_j + a(|I| - |B|) \right\}.
\]
Now let \( \tilde{\Lambda} : (-\infty, -a) \to [0, \infty) \) be given, for all \( t < -a \), by \( \tilde{\Lambda}(t) = \Lambda(-t) \). It then follows from Lemma A1 that
\[
S_I(x) = \Delta_a \cdots \Delta_a \tilde{\Lambda}\left( -\sum_{j \notin I} x_j - a|I| \right) \geq 0.
\]
In order to construct a measure \( \mu \) on \( E_d^a \), consider the ring \( \mathcal{E}^a \) of all finite unions of sets of the type \( \{z \in E_d^a : \forall_{j \in I} z_j = a, \forall_{j \notin I} \ x_j < z_j \leq y_j\} \) for all \( I \subset \{1, \ldots, d\} \) and all \( x, y \in (a, \infty] \) with \( x \leq y \). Then \( \mathcal{E}^a \) generates the Borel \( \sigma \)-field on \( E_d^a \). For any \( I \subset \{1, \ldots, d\} \) and any \( x, y \in (a, \infty] \) with \( x \leq y \), set
\[
\mu\{z \in E_d^a : \forall_{j \in I} z_j = a, \forall_{j \notin I} x_j < z_j \leq y_j\} = \sum_{c \in S_{x,y}^I} \text{sign}(c) S_I(c),
\]
where \( S_{x,y}^I = \{z \in E_d^a : \forall_{j \in I} z_j = a, \forall_{j \notin I} \ x_j \in \{x_j, y_j\}\} \) and \( \text{sign}(c) \) is as in Lemma 5. The right-hand side can be easily rewritten as
\[
\sum_{c \in S_{x,y}^I} \text{sign}(c) \sum_{B \subseteq I} (-1)^{|I|-|B|} \Lambda\left\{ \sum_{j \in B} x_j + a(|I| - |B|) \right\}
= \sum_{A \subseteq \{1, \ldots, d\}} (-1)^{|A|-|I|} \Lambda\left\{ \sum_{j \in A \cap I^c} x_j + \sum_{j \in A \cap I^c} y_j + a(|I| - |A \cap I|) \right\}. \tag{F1}
\]
Thus if one sets \( t = \sum_{j \in I} y_j + a|I|, h_j = a \text{ if } j \in I, \text{ and } h_j = y_j - x_j \text{ if } j \in I^c \), then
\[
\sum_{j \in A \cap I^c} x_j + \sum_{j \in A \cap I^c} y_j + a(|I| - |A \cap I|) = t - \sum_{j \in A} h_j > a.
\]
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By Lemma A1 (ii) and the \(d\)-monotonicity of \(\Lambda\), the right-hand side of (F1) equals
\[
\sum_{A \subseteq \{1, \ldots, d\}} (-1)^{d-|A|} \Lambda\left(t - \sum_{j \in A} h_j\right) = \Delta h_1 \cdots \Delta h_d \tilde{\Lambda}(-t) \geq 0.
\]
Therefore, \(\mu\) is a pre-measure on \(E^a\) which can be extended to a measure on the entire \(\sigma\)-field of \(E^a\) by the Carathéodory process. Furthermore, for any \(x > a\) and \(A \subseteq \{1, \ldots, d\}\) with \(|A| = k \geq 1\), one has
\[
\mu\{z \in E^a_d : \forall j \in A \ z_j > x_j\} = \sum_{I \subseteq A^c} \mu\{z \in E^a_d : \forall j \in A \ z_j > x_j, \forall j \in I \ z_j = a\} = \sum_{I \subseteq A^c} \sum_{B \subseteq I} (-1)^{|I|-|B|} \Lambda\left\{\sum_{j \in A} x_j + a(|A^c| - |B|)\right\}.
\]
Because the summand only depends on the cardinality of the sets \(I\) and \(B\), the last expression simplifies to
\[
\sum_{\ell=0}^{d-k} \sum_{m=0}^{\ell} \binom{d-k}{\ell} \binom{\ell}{m} (-1)^{\ell-m} \Lambda\left\{\sum_{j \in A} x_j + a(d-k-m)\right\}.
\]
By interchanging the order of summation and setting \(\ell^* = \ell - m\), one gets
\[
\sum_{m=0}^{d-k} \binom{d-k}{m} \Lambda\left\{\sum_{j \in A} x_j + a(d-k-m)\right\} \sum_{\ell^*=0}^{d-k-m} \binom{d-k-m}{\ell^*} (-1)^{\ell^*}.
\]
Clearly, the second summand vanishes unless \(d-k-m = 0\). Hence,
\[
\mu\{z \in E^a_d : \forall j \in A \ z_j > x_j\} = \Lambda\left(\sum_{j \in A} x_j\right).
\]
From (B2) it then follows that for any \(x > a\),
\[
\mu([-\infty, x]^c] = \sum_{k=1}^d (-1)^{k+1} \sum_{A \subseteq \{1, \ldots, d\}, |A| = k} \Lambda\left(\sum_{j \in A} x_j\right), \quad (F2)
\]
where \(^{c}\) refers to the complement within \(E^a_d\). Because \(\Lambda(t) \to 0\) as \(t \to \infty\), \(\mu\) is an exponent measure by a straightforward extension of Lemma 4 to \(E^a_d\) stated in Section 5.3 of Resnick (2008). Note in passing that in contrast to the exponent measure constructed in the proof of Proposition 2 for a similar purpose,
\[
\mu\{E^a_d \setminus (a, \infty]\} = \infty, \quad \mu(a, \infty] = \Lambda(da) < \infty.
\]
As \(\mu\) is an exponent measure on \(E^a_d\), the distribution function defined, for all \(x \geq a\), by
\[
H_\mu(x) = e^{-\mu([-\infty, x]^c]}
\]
is max-id. Because \(\Lambda(t) \to \infty\) as \(t \to a\), the margins of \(H_\mu\) are continuous. From (F2), \(H_\mu\) is of the form (5) and its copula is reciprocal Archimedean with generator \(F\), which concludes the argument.
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