Testing for simultaneous jumps in case of asynchronous observations

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June 22, 2017

Abstract

This paper proposes a novel test for simultaneous jumps in a bivariate Itô semimartingale when observation times are asynchronous and irregular. Inference is built on a realized correlation coefficient for the squared jumps of the two processes which is estimated using bivariate power variations of Hayashi-Yoshida type without an additional synchronization step. An associated central limit theorem is shown whose asymptotic distribution is assessed using a bootstrap procedure. Simulations show that the test works remarkably well in comparison with the much simpler case of regular observations.

Keywords and Phrases: Asynchronous observations; common jumps; high-frequency statistics; Itô semimartingale; stable convergence

AMS Subject Classification: 62G10, 62M05 (primary); 60J60, 60J75 (secondary)

1 Introduction

Understanding the jump behaviour of a continuous time process is of importance in econometrics, as many decisions in finance are based on knowledge of the path properties of the underlying asset prices. For this reason, a large amount of research over the last decade was concerned with the estimation of certain jump characteristics or with the construction of tests regarding the existence and the nature of the jumps in the respective processes. Quite naturally, the focus was on the univariate setting for most cases, and we refer to the recent monographs Jacod and Protter (2012) and Aït-Sahalia and Jacod (2014) as well as to the references cited therein for an overview on statistical methods for (univariate) semimartingales observed in discrete time.

On the other hand, when it comes to portfolio management and diversification issues there is a clear need for statistical methods which help deciding whether jumps in a specific asset are of idiosyncratic nature or are accompanied by jumps in other assets as well. Starting with Barndorff-Nielsen and Shephard (2006), authors therefore have developed tests for simultaneous jumps in a multivariate framework,

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but these tests are typically based on the assumption that all components of the multivariate process can be observed synchronously and in a regular fashion. See for example Jacod and Todorov (2009), Liao and Anderson (2011) and Mancini and Gobbi (2012).

A remarkable exception is the test for co-jumps from Bibinger and Winkelmann (2015) which is designed for observations including additional noise and works in more general sampling schemes than just regular ones. The authors have shown that the effect of microstructure noise asymptotically dominates asynchronicity, whereas our focus will be on irregular observation schemes and their specific contribution to the asymptotic theory when jumps are the quantities of interest. Allowing for such models is much more realistic when it comes to practical applications, as even in the univariate setting observations do not come at equidistant times, and in the case of multivariate processes it is typically the case that not any observation of one component coincides with observations of all the others. For this reason, there has always been some interest in the generalization of methods for regular sampling schemes to more realistic frameworks. This includes in particular the (simpler) case of continuous Itô semimartingales. See for example Hayashi et al. (2011) or Mykland and Zhang (2012) for the asymptotic properties of power variations in the univariate setting, or Hayashi and Yoshida (2005) and Hayashi and Yoshida (2008) on estimation of covariation for bivariate processes.

Even more complicated is the situation when the underlying processes contain jumps. In this case, the (few) existing results involving irregular observations have mostly focused on the univariate situation. Consistency results for certain power variations can be found in Chapter 3 of Jacod and Protter (2012), but associated central limit theorems are only given in the case where jumps do not play a role asymptotically. See also Mancini and Gobbi (2012) for consistency of a truncated Hayashi-Yoshida type estimator for integrated covariation. On the other hand, Bibinger and Vetter (2015) provide a central limit theorem which involves non-trivial parts related to jumps, but only in the relatively simple case of realized volatility.

The aim of the present work therefore is twofold: First, we extend results from Jacod and Todorov (2009), providing a feasible test for simultaneous jumps of a bivariate process $X = (X^{(1)}, X^{(2)})$ over $[0, T]$, when observation times are asynchronous and irregular. As they discriminate between joint and disjoint jumps by estimating an empirical correlation coefficient for the squares of the two jump processes, namely

$$\Phi_T^{(d)} = \frac{\sum_{s \le T} \big(\Delta X_s^{(1)}\big)^2 \big(\Delta X_s^{(2)}\big)^2}{\sqrt{\sum_{s \le T} \big(\Delta X_s^{(1)}\big)^4} \sqrt{\sum_{s \le T} \big(\Delta X_s^{(2)}\big)^4}}, \tag{1.1}$$

we need an extension of the results from Bibinger and Vetter (2015) to a multidimensional framework in order to estimate $\Phi_T^{(d)}$ from irregular sampling schemes as well. Our technique here utilizes the heuristics behind the standard Hayashi-Yoshida estimator for realized covariation in order to identify joint jumps, and we believe that our results are of independent interest as quantities such as $\Phi_T^{(d)}$ also play a central role in various other situations related to inference on jump processes.

Second, under the null hypothesis of no joint jumps we provide an associated central limit theorem for our estimator of $\Phi_T^{(d)}$. As the limiting variable not only depends in a complicated way on the characteristics of X, but also on unknown variables which are due to the fine structure of the sampling scheme, we provide a bootstrap procedure in order to estimate critical values of our final test statistic. An extensive simulation study shows that our test has a similar finite sample behaviour as the standard test by Jacod and Todorov (2009) when the (random) number of observations in both components equals on average the fixed number of observations in the simple regular case. This is remarkable when it comes to practical applications, as no additional synchronization step is necessary which inevitably causes a loss of data and therefore leads to a loss in efficiency.

The remainder of the paper is organized as follows: Section 2 deals with the formal setting in this work, and we introduce our estimator for $\Phi_T^{(d)}$ as well as minor assumptions under which consistency holds. In Section 3 we need stronger conditions, as we are interested in the associated central limit theorem. The bootstrap procedure leading to the final test statistic is introduced in Section 4, while its finite sample properties are investigated in Section 5. All proofs are gathered in the Appendix, which is Section 6.

2 Setting and test statistic

Our goal in the sequel is to derive a statistical test based on high-frequency observations which allows to decide whether two processes do jump at a common time or not. We consider the following model for the process and the observation times: Let $X = (X^{(1)}, X^{(2)})^*$ be a two-dimensional Itô semimartingale on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}^2} \delta(s,z) \mathbb{1}_{\{\|\delta(s,z)\| \leq 1\}} (\mu - \nu)(ds, dz) + \int_0^t \int_{\mathbb{R}^2} \delta(s,z) \mathbb{1}_{\{\|\delta(s,z)\| > 1\}} \mu(ds, dz), \quad (2.1)$$

where $W = (W^{(1)}, W^{(2)})^*$ is a two-dimensional standard Brownian motion, μ is a Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}^2$, and its predictable compensator satisfies $\nu(ds, dz) = ds \otimes \lambda(dz)$ for some σ-finite measure λ on $\mathbb{R}^2$ endowed with the Borelian σ-algebra. b is a two-dimensional adapted process,

$$\sigma_s = \begin{pmatrix} \sigma_s^{(1)} & 0 \\ \rho_s \sigma_s^{(2)} & \sqrt{1-\rho_s^2}\sigma_s^{(2)} \end{pmatrix}$$

is a (2×2)-dimensional process and δ is a two-dimensional predictable process on $\Omega \times \mathbb{R}^+ \times \mathbb{R}^2$. $\sigma_s^{(1)}$, $\sigma_s^{(2)}$ and $\rho_s \in [-1, 1]$ are all univariate adapted. We write $\Delta X_s = X_s - X_{s-}$ with $X_{s-} = \lim_{t \nearrow s} X_t$ for a possible jump of X in s.

The observation times are given by

$$\pi_n = \big\{ \big(t_{i,n}^{(1)}\big)_{i \in \mathbb{N}_0}, \big(t_{i,n}^{(2)}\big)_{i \in \mathbb{N}_0} \big\}, \quad n \in \mathbb{N},$$

where $\big(t_{i,n}^{(l)}\big)_{i \in \mathbb{N}_0}$, $l = 1, 2$, are increasing sequences of stopping times with $t_{0,n}^{(l)} = 0$. By

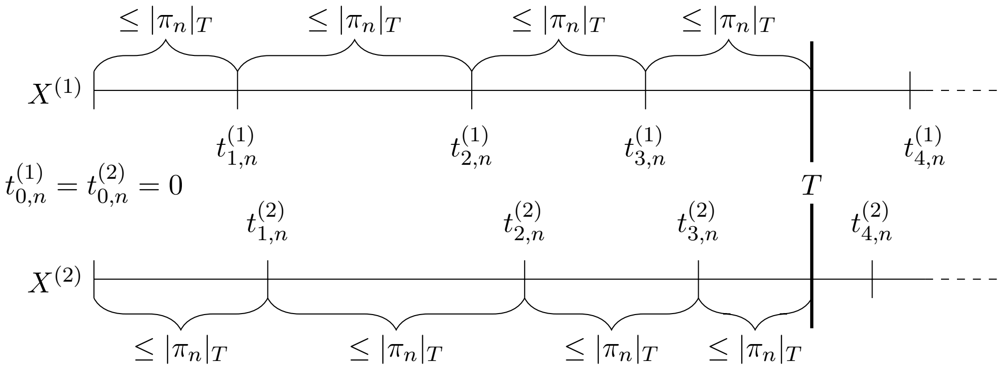

Figure 1: A realization of the observation scheme π_n restricted to $[0, T]$.

$$|\pi_n|_T = \sup\left\{t^{(l)}_{i,n} \wedge T - t^{(l)}_{i-1,n} \wedge T \middle| i \geq 1,\ l = 1, 2\right\}$$

we denote the mesh of the observation times up to T. Throughout the paper we use n as an unobservable variable governing the observations and the asymptotics which does not appear in the statistics used later on.

We introduce the following subsets of Ω to formalize the hypotheses:

$$\begin{aligned}
\Omega^{(d)}_T &= \{\omega \in \Omega : \exists s_1, s_2 \in [0, T] \text{ with } \Delta X^{(1)}_{s_1} \neq 0 \text{ and } \Delta X^{(2)}_{s_2} \neq 0, \\
&\qquad\qquad\qquad\qquad \text{but } \Delta X^{(1)}_s \Delta X^{(2)}_s = 0\ \forall s \in [0, T]\}, \\
\Omega^{(j)}_T &= \{\omega \in \Omega : \exists s \in [0, T] \text{ with } \Delta X^{(1)}_s \Delta X^{(2)}_s \neq 0\}, \\
\Omega^{(c)}_T &= \{\omega \in \Omega : \Delta X^{(1)}_s = 0\ \forall s \in [0, T] \text{ or } \Delta X^{(2)}_s = 0\ \forall s \in [0, T]\}.
\end{aligned}$$

Hence $\Omega^{(d)}_T$ is the set where $X^{(1)}$ and $X^{(2)}$ are both discontinuous on $[0, T]$ but do not jump together, $\Omega^{(j)}_T$ is the set where $X^{(1)}$ and $X^{(2)}$ have at least one common jump in $[0, T]$, and $\Omega^{(c)}_T$ is the set where at least one of the processes $X^{(1)}$ or $X^{(2)}$ is continuous on $[0, T]$. Our goal in this paper is to find a testing procedure for deciding whether an observation is from $\Omega^{(d)}_T$ or from $\Omega^{(j)}_T$. This means in particular that we focus on a specific path of X, and it might be that the underlying model allows for joint jumps but none of them occurs on the observed path up to time T. In such a case the hypothesis of joint jumps should be rejected. Also, it is reasonable to apply a test for jumps in any of the processes (like the one from Aït-Sahalia and Jacod (2009)) prior to the analysis, as one does not know a priori whether $\omega \in \Omega^{(c)}_T$ or not.

All our test statistics are based on the increments

$$\Delta^{(l)}_{i,n} X = X^{(l)}_{t^{(l)}_{i,n}} - X^{(l)}_{t^{(l)}_{i-1,n}}, \quad i \geq 1, \quad l = 1, 2,$$

and we denote by $\mathcal{I}^{(l)}_{i,n} = \left(t^{(l)}_{i-1,n}, t^{(l)}_{i,n}\right]$, $l = 1, 2$, the corresponding observation intervals. For a function $f : \mathbb{R}^2 \to \mathbb{R}$ we set

$$V(f, \pi_n)_T = \sum_{i,j: t^{(1)}_{i,n} \wedge t^{(2)}_{j,n} \leq T} f\left(\Delta^{(1)}_{i,n} X, \Delta^{(2)}_{j,n} X\right) \mathbb{1}_{\{\mathcal{I}^{(1)}_{i,n} \cap \mathcal{I}^{(2)}_{j,n} \neq \emptyset\}}$$

in the style of the Hayashi-Yoshida estimator for the quadratic covariation (Hayashi and Yoshida (2005)), and for a function $g : \mathbb{R} \to \mathbb{R}$ we define

$$V^{(l)}(g, \pi_n)_T = \sum_{i:t_{i,n}^{(l)} \leq T} g\big(\Delta_{i,n}^{(l)} X\big), \quad l = 1, 2.$$

In particular, as we are interested in estimating $\Phi_T^{(d)}$ from (1.1), we consider these expressions for the functions $f(x) = (x_1 x_2)^2$ and $g(x) = x^4$. Then our main statistic becomes

$$\widetilde{\Phi}_{n,T}^{(d)} = \frac{V(f, \pi_n)_T}{\sqrt{V^{(1)}(g, \pi_n)_T V^{(2)}(g, \pi_n)_T}},$$

whose asymptotics we are going to study and which will be used to construct an asymptotic test.

In order to describe the asymptotics of $\widetilde{\Phi}_{n,T}^{(d)}$ we set

$$B_T = \sum_{s \leq T} \big(\Delta X_s^{(1)}\big)^2 \big(\Delta X_s^{(2)}\big)^2, \quad B_T^{(l)} = \sum_{s \leq T} \big(\Delta X_s^{(l)}\big)^4 \text{ for } l = 1, 2,$$

so that

$$\Phi_T^{(d)} = \frac{B_T}{\sqrt{B_T^{(1)} B_T^{(2)}}}.$$

Obviously, $\Phi_T^{(d)}$ is well-defined on the complement of $\Omega_T^{(c)}$ only, and in this case it can be interpreted as the correlation between the squared jumps of $X^{(1)}$ and $X^{(2)}$: $\Phi_T^{(d)}$ is always in $[0, 1]$, and it is equal to 0 if and only if there are no common jumps and equal to 1 if and only if there exists a constant $c > 0$ with $\big(\Delta X_s^{(1)}\big)^2 = c\big(\Delta X_s^{(2)}\big)^2$ for all $s \leq T$. Note that working with the correlation of squared jumps is convenient as the corresponding power variations in the definition of $\widetilde{\Phi}_{n,T}^{(d)}$ have limits which do not include the volatility. This is in general not the case for smaller powers.

In order to derive results on the asymptotic behaviour of $\widetilde{\Phi}_{n,T}^{(d)}$, we require the following assumptions on the process X and the observation scheme π_n.

Condition 2.1. The process b_s is locally bounded and the processes $\sigma_s^{(1)}, \sigma_s^{(2)}, \rho_s$ are càdlàg. Furthermore, there exists a locally bounded process Γ_s with $\|\delta(\omega, s, z)\| \leq \Gamma_s(\omega)\gamma(z)$ for some deterministic bounded function γ which satisfies $\int (\gamma(z)^2 \wedge 1)\lambda(dz) < \infty$. The sequence of observation schemes $(\pi_n)_n$ fulfills

$$|\pi_n|_T \xrightarrow{\mathbb{P}} 0.$$

The assumption on the components of X is close to condition (H) in earlier work of Jacod (compare Assumption (H) in Jacod and Todorov (2009) or Assumption 4.4.2 in Jacod and Protter (2012)) and not very restrictive, as it covers a variety of models studied in financial mathematics. In fact, we are able to incorporate various dependencies between the price process X_t and the stochastic volatility process σ_t as well as dependence between jumps in the price and in the volatility process. Assuming additionally that $\sigma_s^{(1)}, \sigma_s^{(2)}, \rho_s$ have continuous paths as in Bibinger and Vetter (2015) would simplify the structure of the asymptotics and the proofs. However, there is empirical evidence that volatility jumps do exist and that they even occur at common times with jumps in the price (see e.g. Jacod and Todorov (2010) or Todorov and Tauchen (2011)). This is why we construct a statistical test that works also within this setting.

Regarding the observation scheme, we are able to work in the general setting of increasing stopping times with vanishing mesh in order to derive consistency of the estimator $\widetilde{\Phi}^{(d)}_{n,T}$. This is a minimal condition since we consider properties like the presence of jumps in the observed path which depend on knowledge of the entire path in continuous time. The result itself might be of its own interest, as it generalizes results from Section 3 of Jacod and Protter (2012) to the case of asynchronicity. However, for the construction of a central limit theorem in Section 3 we are not able to work within this general setting. Although in practice a theory for endogeneous observation times might be desirable, previous research shows that even in simple situations it is difficult to derive central limit theorems (see Fukasawa and Rosenbaum (2012) or Vetter and Zwingmann (2017)). For this reason we restrict ourselves in Section 3 to exogeneous observation times which still cover a lot of random and irregular sampling schemes. We will see that already in this setting the asymptotic theory becomes significantly more difficult compared to the framework of equidistant observations.

Speaking of consistency only, we are able to prove

$$V(f,\pi_n)_T \xrightarrow{\mathbb{P}} B_T, \tag{2.2}$$

$$V^{(l)}(g,\pi_n)_T \xrightarrow{\mathbb{P}} B^{(l)}_T, \quad l=1,2, \tag{2.3}$$

whenever Condition 2.1 holds. Note that (2.3) already follows from Theorem 3.3.1 in Jacod and Protter (2012) while the first statement (2.2) needs a generalization of this theorem to the setting of asynchronous observations.

Theorem 2.2. *Let X be an Itô semimartingale of the form (2.1) and $(\pi_n)_n$ be a sequence of observation schemes such that Condition 2.1 is fulfilled. Then we have*

$$\widetilde{\Phi}^{(d)}_{n,T} \xrightarrow{\mathbb{P}} \Phi^{(d)}_T$$

on the complement of $\Omega^{(c)}_T$.

Theorem 2.2 states that $\widetilde{\Phi}^{(d)}_{n,T}$ converges to 0 on the set $\Omega^{(d)}_T$ and to a strictly positive limit on $\Omega^{(j)}_T$. So a natural test for the null $\omega \in \Omega^{(d)}_T$ against $\omega \in \Omega^{(j)}_T$ makes use of a critical region of the form

$$\mathcal{C}_n = \{\widetilde{\Phi}^{(d)}_{n,T} > c_n\} \tag{2.4}$$

for a suitable, possibly random sequence $(c_n)_{n\in\mathbb{N}}$. In order to choose c_n such that the test has a certain level α we need knowledge of the asymptotic behaviour of $\widetilde{\Phi}^{(d)}_{n,T}$ on $\Omega^{(d)}_T$, which will be developed in form of a central limit theorem in the next section.

3 Central limit theorem

In order to derive a central limit theorem we first have to specify the asymptotics of the observation scheme. The methodology and hence the notation in this section are inspired by the results in Section 4 of Bibinger and Vetter (2015). We start by defining the following two functions for $n \in \mathbb{N}$

$$G_n(t) = n \sum_{i,j: t_{i,n}^{(1)} \wedge t_{j,n}^{(2)} \leq t} \big| \mathcal{I}_{i,n}^{(1)} \cap \mathcal{I}_{j,n}^{(2)} \big|^2,$$

$$H_n(t) = n \sum_{i,j: t_{i,n}^{(1)} \wedge t_{j,n}^{(2)} \leq t} |\mathcal{I}_{i,n}^{(1)}||\mathcal{I}_{j,n}^{(2)}| \mathbb{1}_{\{\mathcal{I}_{i,n}^{(1)} \cap \mathcal{I}_{j,n}^{(2)} \neq \emptyset\}}.$$

Here, $|A|$ denote the Lebesgue measure of a Borel set A.

Let $i_n^{(l)}(s)$ denote the index of the observation interval of $X^{(l)}$ containing s, i.e. $i_n^{(l)}(s)$ is defined via

$$s \in \mathcal{I}_{i_n^{(l)}(s),n}^{(l)}.$$

Set $\overline{W}_t = (W_t^{(1)}, \rho_t W_t^{(1)} + \sqrt{1-\rho_t^2} W_t^{(2)})^*$ such that $\overline{W}^{(l)}$ is the Brownian motion driving the process $X^{(l)}$ for $l = 1,2$. We denote

$$\begin{aligned} \eta_{n,-}^{(l)}(s) &= \sum_{j: \mathcal{I}_{j,n}^{(l)} \leq T} (\Delta_{j,n}^{(l)} \overline{W})^2 \mathbb{1}_{\left\{\mathcal{I}_{j,n}^{(l)} \cap \mathcal{I}_{i_n^{(3-l)}(s),n}^{(3-l)} \neq \emptyset \wedge j < i_n^{(l)}(s)\right\}}, \\ \eta_{n,+}^{(l)}(s) &= \sum_{j: \mathcal{I}_{j,n}^{(l)} \leq T} (\Delta_{j,n}^{(l)} \overline{W})^2 \mathbb{1}_{\left\{\mathcal{I}_{j,n}^{(l)} \cap \mathcal{I}_{i_n^{(3-l)}(s),n}^{(3-l)} \neq \emptyset \wedge j > i_n^{(l)}(s)\right\}}, \end{aligned} \tag{3.1}$$

for $l = 1,2$. Following Bibinger and Vetter (2015) we denote by

$$\tau_{n,-}^{(l)}(s) = \sup\{t_{i,n}^{(l)} | t_{i,n}^{(l)} < s\}, \quad \tau_{n,+}^{(l)}(s) = \inf\{t_{i,n}^{(l)} | t_{i,n}^{(l)} \geq s\}, \quad l = 1,2,$$

the observation times immediately before and after time s. Using this notation we set

$$\delta_{n,-}^{(l)}(s) = s - \tau_{n,-}^{(l)}(s), \quad \delta_{n,+}^{(l)}(s) = \tau_{n,+}^{(l)}(s) - s, \quad l = 1,2.$$

Then we have the identity

$$\begin{aligned} \sum_{j: \mathcal{I}_{j,n}^{(l)} \cap \mathcal{I}_{i_n^{(3-l)}(s),n}^{(3-l)} \neq \emptyset} (\Delta_{j,n}^{(l)} \overline{W})^2 &= \eta_{n,-}^{(l)}(s) + \big[(\delta_{n,-}^{(l)}(s))^{1/2}((\overline{W}_s^{(l)} - \overline{W}_{\tau_{n,-}^{(l)}(s)}^{(l)})/(\delta_{n,-}^{(l)}(s))^{1/2}) \\ &+ (\delta_{n,+}^{(l)}(s))^{1/2}((\overline{W}_{\tau_{n,+}^{(l)}(s)}^{(l)} - \overline{W}_s^{(l)})/(\delta_{n,+}^{(l)}(s))^{1/2})\big]^2 + \eta_{n,+}^{(l)}(s) \end{aligned} \tag{3.2}$$

for the sum over the squared increments of the Brownian motions driving the processes $X^{(l)}$, $l = 1,2$, over intervals $\mathcal{I}_{j,n}^{(l)}$ which overlap with the observation interval $\mathcal{I}_{i_n(s),n}^{(3-l)}$ containing s. See Figure 3 for an illustration. We distinguish between increments of $\overline{W}^{(l)}$ before and after s to allow for different volatilities immediately before and after s due to a volatility jump at time s, and we write

$$Z_n^{(l)}(s) = \big(n\eta_{n,-}^{(l)}(s), n\eta_{n,+}^{(l)}(s), n\delta_{n,-}^{(l)}(s), n\delta_{n,+}^{(l)}(s)\big)^*$$

to shorten notation. Even though the driving Brownian motions $\overline{W}^{(l)}$ are in general dependent, we will see that the limiting variables of $Z_n^{(1)}(s)$ and $Z_n^{(2)}(s)$ can be chosen to be independent, as under the null hypothesis both variables never occur at the same time in the limit.

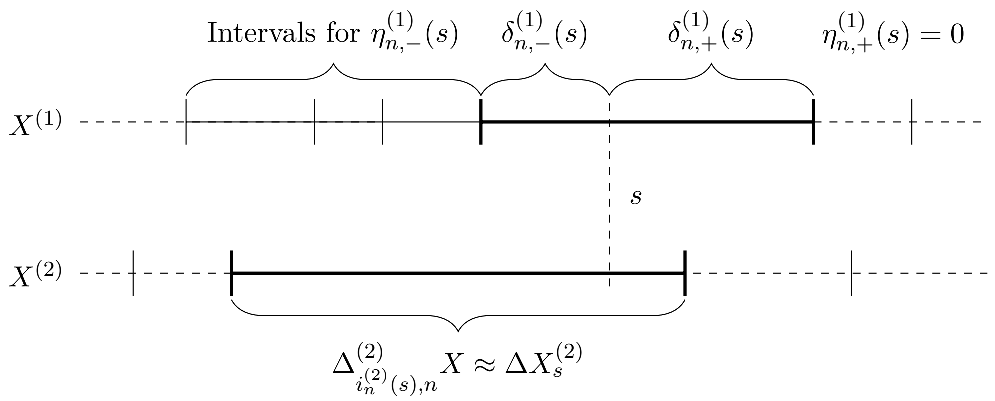

Figure 2: Illustration of $Z^{(1)}_n(s)$ at a time s where $X^{(2)}$ jumps.

The following condition comprises the assumptions on the asymptotics of the sequence of observation schemes $(\pi_n)_n$ which are needed for the derivation of a central limit theorem. While the first one is a rather mild assumption on the mesh of the sampling scheme, the other two conditions ensure a kind of local regularity which is needed to deduce convergence both of the purely continuous part and the cross part in the limit. Analogous conditions are needed in Bibinger and Vetter (2015) to derive a central limit theorem for the Hayashi-Yoshida estimator for the covariation process with jumps, and assumptions similar to Condition 3.1(i) and (ii) have also occured in Hayashi and Yoshida (2011) when deriving the asymptotics of the Hayashi-Yoshida estimator for the covariation process without jumps.

Condition 3.1. The process X and the sequence of observation schemes $(\pi_n)_n$ fulfill Condition 2.1, and the observation times are exogeneous, i.e. independent of the process X and its components.

(i) It holds

$$\mathbb{E}[(|\pi_n|_T)^2] = o(n^{-1}).$$

(ii) The functions $G_n(t)$ and $H_n(t)$ converge pointwise on $[0,T]$ in probability to strictly increasing continuous functions $G, H : [0,\infty) \to [0,\infty)$.

(iii) For all integers k_1, k_2 and all bounded continuous functions $g : \mathbb{R}^{k_1+k_2} \to \mathbb{R}$ and $h^{(l)}_p : \mathbb{R}^4 \to \mathbb{R}$, $p = 1, \ldots, k_l$, $l = 1, 2$, the integral

$$\int_{[0,T]^{k_1+k_2}} g(x_1,\ldots,x_{k_1},x'_1,\ldots,x'_{k_2})\mathbb{E}\Bigg[\prod_{p=1}^{k_1} h^{(1)}_p\big(Z^{(1)}_n(x_p)\big) \times \prod_{p=1}^{k_2} h^{(2)}_p\big(Z^{(2)}_n(x'_p)\big)\Bigg] dx_{k_1}\ldots dx_1 dx'_{k_2}\ldots dx'_1 \quad (3.3)$$

converges to

$$\int_{[0,T]^{k_1+k_2}} g(x_1,\dots,x_{k_1},x'_1,\dots,x'_{k_2}) \prod_{p=1}^{k_1} \int_{\mathbb{R}} h_p^{(1)}(y)\Gamma^{(1)}(x_p,dy)$$
$$\times \prod_{p=1}^{k_2} \int_{\mathbb{R}} h_p^{(2)}(y')\Gamma^{(2)}(x'_p,dy')dx_{k_1}\dots dx_1 dx'_{k_2}\dots dx'_1 \quad (3.4)$$

as $n \to \infty$. Here, $\Gamma^{(l)}(\cdot,dy)$, $l=1,2$, are families of probability measures on $[0,\infty)^2 \times (0,\infty)^2$ such that the first moments are uniformly bounded.

Because of the exogeneity of the observation times we may assume in the following that the probability space has the form

$$(\Omega,\mathcal{F},\mathbb{P}) = (\Omega_{\mathcal{X}} \times \Omega_{\mathcal{S}}, \mathcal{X}\otimes\mathcal{S}, \mathbb{P}_{\mathcal{X}}\otimes\mathbb{P}_{\mathcal{S}}),$$

where $\mathcal{X}$ denotes the σ-algebra generated by X and its components and $\mathcal{S}$ denotes the σ-algebra generated by the observation schemes $(\pi_n)_n$.

As usual when power variations for orders higher than two are considered, the limiting term in the central limit theorem will be comprised of a continuous term and a cross term which contains the continuous part of one process and the jumps of the other process. The term originating from the continuous part is given by

$$\widetilde{C}_T = \int_0^T 2(\rho_s\sigma_s^{(1)}\sigma_s^{(2)})^2 dG(s) + \int_0^T (\sigma_s^{(1)}\sigma_s^{(2)})^2 dH(s)$$

where the integrals are well defined as Lebesgue-Stieltjes integrals because G,H are increasing (as G_n,H_n are increasing) and continuous and hence define measures on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. Here, the differentials dG and dH are a measure for the asymptotic density of observation times in a given time interval. Two different functions are needed because the products of increments over overlapping and non-overlapping observation intervals have different variances.

The limiting quantity originating from the cross terms of the continuous part and the jumps is given by

$$\widetilde{D}_T = \sum_{p:S_p\le T} \big((\Delta X_{S_p}^{(1)})^2 R^{(2)}(S_p) + (\Delta X_{S_p}^{(2)})^2 R^{(1)}(S_p)\big),$$

where $(S_p)_{p\ge 0}$ is an enumeration of the jump times of X. Here $R^{(l)}(s)$ is given by

$$\begin{aligned} R^{(l)}(s) =& (\sigma_{s-}^{(2)})^2\eta_-^{(l)}(s) + \big(\sigma_{s-}^{(l)}(\delta_-^{(l)}(s))^{1/2}U_-^{(l)}(s) + \sigma_s^{(l)}(\delta_+^{(l)}(s))^{1/2}U_+^{(l)}(s)\big)^2 \\ &+ (\sigma_s^{(2)})^2\eta_+^{(l)}(s), \quad s\in[0,T], \quad l=1,2, \end{aligned} \quad (3.5)$$

where $Z^{(l)}(s) = (\eta_-^{(l)}(s),\eta_+^{(l)}(s),\delta_-^{(l)}(s),\delta_+^{(l)}(s))^*$ are random variables defined on an extended probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$. Their distribution is given by

$$\widetilde{\mathbb{P}}^{Z^{(l)}(x)}(dy) = \Gamma^{(l)}(x,dy),$$

where the $U_-^{(l)}(s), U_+^{(l)}(s)$ are i.i.d. $\mathcal{N}(0,1)$ random variables defined on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ as well. The $Z^{(l)}(s)$ and $U_-^{(l)}(s)$, $U_+^{(l)}(s)$ are independent of each other and independent of the process X and its components. It is worth mentioning that we do not consider common jumps, since we derive the central limit theorem under the null hypothesis of no common jumps. This leads to independent $Z^{(l)}(s)$ which simplifies the structure of the limiting variables compared to Bibinger and Vetter (2015). On the other hand, the form of the limiting object becomes more complex due to adding volatility jumps.

Using the above notation we derive the following central limit theorem on $\Omega_T^{(d)}$.

Theorem 3.2. *If Condition 3.1 is fulfilled, we have the $\mathcal{X}$-stable convergence*

$$n\widetilde{\Phi}_{n,T}^{(d)} \xrightarrow{\mathcal{L}-s} \widetilde{\Psi}_T = \frac{\widetilde{C}_T + \widetilde{D}_T}{\sqrt{B_T^1 B_T^2}} \tag{3.6}$$

on the set $\Omega_T^{(d)}$.

The central limit theorem states that $n\widetilde{\Phi}_{n,T}^{(d)}$ converges $\mathcal{X}$-stably in law on the set $\Omega_T^{(d)}$ to a random variable $\widetilde{\Psi}_T$ on an extended probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ which means that we have

$$\mathbb{E}\big[g\big(n\widetilde{\Phi}_{n,T}^{(d)}\big)Y\mathbb{1}_{\Omega_T^{(d)}}\big] \to \widetilde{\mathbb{E}}\big[g\big(\widetilde{\Psi}_T\big)Y\mathbb{1}_{\Omega_T^{(d)}}\big]$$

for all bounded and continuous functions g and all $\mathcal{X}$-measurable bounded random variables Y. For more background information on stable convergence in law we refer to Jacod and Protter (2012), Jacod and Shiryaev (2002) and Podolskij and Vetter (2010).

Example 3.3. Let us discuss the standard setting of equidistant and synchronous observations times. In this case, $t_{i,n}^{(l)} = i/n$, so we have $|\pi_n|_T = n^{-1}$. Hence Condition 2.1 and Condition 3.1(i) are trivially fulfilled. Furthermore,

$$H_n(t) = G_n(t) = n \sum_{i=1}^{\lfloor t/n \rfloor} (1/n)^2 \to t,$$

which yields Condition 3.1(ii). We also have $\eta_{n,-}^{(l)}(s) = \eta_{n,+}^{(l)}(s) = 0$ and $\delta_{n,-}^{(l)}(s) + \delta_{n,+}^{(l)}(s) = 1$ for all $s \in [0,T]$, $l = 1,2$.

Note that the outer integral in (3.3) can be interpreted as the expectation with regard to $k_1 + k_2$ independent uniformly distributed random variables S_p on $[0,T]$. Then, as the variables $(n\delta_{n,-}^{(l)}(S_p), n\delta_{n,+}^{(l)}(S_p))$ are distributed like $(\kappa, 1-\kappa)$ with $\kappa \sim \mathcal{U}[0,1]$, we obtain the limiting distribution as $Z^{(l)}(x) \sim (0, 0, \kappa^{(l)}(x), 1-\kappa^{(l)}(x))$ for independent $\kappa^{(l)}(x) \sim \mathcal{U}[0,1]$. Standard arguments further show that $Z_n^{(l)}(S_p)$ and $Z_n^{(l')}(S_{p'})$ are asymptotically independent for $p \neq p'$. Hence Condition 3.1(iii) is satisfied and we have (3.6) with

$$\begin{aligned}
\widetilde{C}_T &= \int_0^T \big(2(\rho_s\sigma_s^{(1)}\sigma_s^{(2)})^2 + (\sigma_s^{(1)}\sigma_s^{(2)})^2\big)ds, \\
\widetilde{D}_T &= \sum_{p:S_p \le T} \Big(\big(\Delta X_{S_p}^{(1)}\big)^2\big(\sigma_{S_p-}^{(2)}(\kappa_p^{(2)})^{1/2}U_{p,-}^{(2)} + \sigma_{S_p}^{(2)}(1-\kappa_p^{(2)})^{1/2}U_{p,+}^{(2)}\big)^2 \\
&\qquad + \big(\Delta X_{S_p}^{(2)}\big)^2\big(\sigma_{S_p-}^{(1)}(\kappa_p^{(1)})^{1/2}U_{p,-}^{(1)} + \sigma_{S_p}^{(1)}(1-\kappa_p^{(1)})^{1/2}U_{p,+}^{(1)}\big)^2\Big),
\end{aligned}$$

for independent standard normal distributed random variables $U^{(l)}_{p,-}, U^{(l)}_{p,+}$ and independent $\mathcal{U}[0,1]$ random variables $\kappa^{(l)}$, $l = 1, 2$. Of course, these terms are identical to the corresponding terms C_T and $\widetilde{D}_T$ in (3.12) and (3.14) of Jacod and Todorov (2009), and Theorem 3.2 becomes Theorem 4.1(a) of Jacod and Todorov (2009) in this setting.

In order to illustrate the theory laid out above we also want to discuss a truly irregular and random setting. Specifically, we consider observation times which are given by the jump times of Poisson processes, but our conditions cover various other sampling schemes as well. Note that Poisson sampling has been discussed frequently in the literature; see e.g. Bibinger and Vetter (2015) and Hayashi and Yoshida (2008).

Example 3.4. Let the observation times of $X^{(1)}$ and $X^{(2)}$ be given by the jump times of independent Poisson processes with intensities $n\lambda_1$ and $n\lambda_2$. Lemma 8 from Hayashi and Yoshida (2008) states

$$\mathbb{E}\big[(|\pi_n|_T)^q\big] = o(n^{-\alpha}) \tag{3.7}$$

for any $0 \leq \alpha < q$, so both Condition 2.1 and Condition 3.1(i) are satisfied. In addition, Proposition 1 in Hayashi and Yoshida (2008) gives Condition 3.1(ii) via

$$G_n(t) \xrightarrow{\mathbb{P}} \frac{2}{\lambda_1 + \lambda_2} t,$$
$$H_n(t) \xrightarrow{\mathbb{P}} (\frac{2}{\lambda_1} + \frac{2}{\lambda_2}) t.$$

Finally, we show that Condition 3.1(iii) is satisfied. Note first that the distributions of the sampling scheme π_1 and the rescaled $n\pi_n$ are identical. Therefore, the distributions of $Z^{(l)}_n(s)$ and $Z^{(l)}_1(ns)$ are identical, and the distribution of the latter only depends on s through the fact that the backward waiting times for the previous observations are bounded by ns. This effect becomes asymptotically irrevelant as n grows, thus $Z^{(l)}_n(s)$ converges. Note also that the $Z^{(l)}_n(s)$ are asymptotically independent because the Wiener process W and the Poisson processes have independent increments and the $Z^{(l)}_n(S_p)$ overlap asymptotically with diminishing probability. Therefore the factorization of the expectations in (3.4) holds.

By symmetry we focus on $Z^{(1)}(s)$ only which can be constructed from elementary distributions. Let $E^{(1)}_{k,-}, E^{(1)}_{k,+} \sim Exp(\lambda_1)$, $k \in \mathbb{N}$, and $E^{(2)}_{-}, E^{(2)}_{+} \sim Exp(\lambda_2)$ be independent exponentially distributed random variables. Then, after rescaling, the lengths of the intervals around s are by the memorylessness of the exponential distribution asymptotically distributed like $E^{(1)}_{1,-} + E^{(1)}_{1,+}$ and $E^{(2)}_{-} + E^{(2)}_{+}$ while the other intervals in $\eta^{(1)}_{-}(s)$ and $\eta^{(1)}_{+}(s)$ are $Exp(\lambda_1)$-distributed. Then, if $(U_{k,-})_{k\in\mathbb{N}}$, $(U_{k,+})_{k\in\mathbb{N}}$ are i.i.d. $\mathcal{N}(0,1)$ random variables, it is easy to deduce that

$$Z^{(1)}(s) \overset{\mathcal{L}}{=} (\eta^{(1)}_{-}(s), \eta^{(1)}_{+}(s), E^{(1)}_{1,-}, E^{(1)}_{1,+})^* \tag{3.8}$$

holds, where we set

$$\eta^{(1)}_{-}(s) = \sum_{k=2}^{\infty} E^{(1)}_{k,-}(U_{k,-})^2 \mathbb{1}_{\{\sum_{j=1}^{k-1} E^{(1)}_{k,-} < E^2_{-}\}},$$
$$\eta^{(1)}_{+}(s) = \sum_{k=2}^{\infty} E^{(1)}_{k,+}(U_{k,+})^2 \mathbb{1}_{\{\sum_{j=1}^{k-1} E^{(1)}_{k,+} < E^2_{+}\}}.$$

4 Testing for disjoint jumps

We will introduce a test which makes use of a critical region of the form (2.4). In Section 3 we have derived a central limit theorem for $\widetilde{\Phi}^{(d)}_{n,T}$. However, this result can not directly be applied for determining c_n, since the law of the limiting variable in Theorem 3.2 is itself random and not known to the statistician. Hence, in order to develop a statistical test we need to estimate the law of the limiting variable $\widetilde{\Psi}_T$.

Estimating the continuous term $\widetilde{C}_T$ in $\widetilde{\Psi}_T$ boils down to estimating the continuous part of X. This can be done using truncated increments as e.g. in (4.5) of Jacod and Todorov (2009). With $\beta > 0$ and $\varpi \in (0, 1/2)$ we set

$$A_{n,T} = n \sum_{i,j: t^{(1)}_{i,n} \wedge t^{(2)}_{j,n} \leq T} (\Delta^{(1)}_{i,n} X^{(1)})^2 (\Delta^{(2)}_{j,n} X^{(2)})^2 \\ \times \mathbb{1}_{\{|\Delta^{(1)}_{i,n} X^{(1)}| \leq \beta |\mathcal{I}^{(1)}_{i,n}|^{\varpi} \wedge |\Delta^{(2)}_{j,n} X^{(2)}| \leq \beta |\mathcal{I}^{(2)}_{j,n}|^{\varpi}\}} \mathbb{1}_{\{\mathcal{I}^{(1)}_{i,n} \cap \mathcal{I}^{(2)}_{j,n} \neq \emptyset\}}.$$

In order to estimate the law of $\widetilde{D}_T$ in $\widetilde{\Psi}_T$ we need to estimate the law of the $Z^{(l)}(s)$ first, which is not known in practice unless one imposes knowledge on the nature of the sampling scheme. In principle, we would like to introduce a Monte Carlo approach and simulate the quantiles of $\widetilde{D}_T$, and a first approach obviously is to replace the increments of the Brownian motion in (3.1) and (3.2) by appropriately scaled realizations of standard normal random variables. However, we have to scale by the lengths of the observation intervals which follow an unknown distribution. Also the distribution of $n\delta^{(l)}_{n,-}(s), n\delta^{(l)}_{n,+}(s)$ (and therefore of $\delta^{(l)}_{-}(s), \delta^{(l)}_{+}(s)$) is unknown. To circumvent this issue we use a bootstrap method and estimate the distribution of the observation intervals as well. The idea here is to estimate their distribution around time s by using $Z^{(l)}_n(u)$ for u close to s. In order for this procedure to work we will introduce a local homogeneity condition later, in the sense that $Z^{(l)}_n(s)$ and $Z^{(l)}_n(s')$ have similar distributions for small values of $|s - s'|$ but remain asymptotically independent.

To formalize, let $(K_n)_n$ and $(M_n)_n$ denote deterministic sequences of integers which tend to infinity. Here, $2K_n + 1$ equals the number of intervals $\mathcal{I}^{(3-l)}_{i^{(3-l)}_n(s)+k,n}$, $|k| \leq K_n$, from which we pick u to bootstrap M_n realizations of $Z^{(l)}_n(s)$. For any $s \in (0, T)$ we define the random variables

$$\hat{\eta}^{(l)}_{n,m,-}(s) = n \sum_{i: \mathcal{I}^{(l)}_{i,n} \leq T} |\mathcal{I}^{(l)}_{i,n}| (U^{(l)}_{n,i,m})^2 \mathbb{1}_{\{\mathcal{I}^{(l)}_{i,n} \cap \mathcal{I}^{(3-l)}_{i^{(3-l)}_n(s)+V^{(l,3-l)}_{n,m}(s),n} \neq \emptyset \wedge i < i^{(l)}_n(s) + V^{(l,l)}_{n,m}(s)\}},$$

$$\hat{\eta}^{(l)}_{n,m,+}(s) = n \sum_{i: \mathcal{I}^{(l)}_{i,n} \leq T} |\mathcal{I}^{(l)}_{i,n}| (U^{(l)}_{n,i,m})^2 \mathbb{1}_{\{\mathcal{I}^{(l)}_{i,n} \cap \mathcal{I}^{(3-l)}_{i^{(3-l)}_n(s)+V^{(l,3-l)}_{n,m}(s),n} \neq \emptyset \wedge i > i^{(l)}_n(s) + V^{(l,l)}_{n,m}(s)\}},$$

$$\hat{\delta}^{(l)}_{n,m,-}(s) = n\Big(\kappa^{(l)}_{n,m}(s) |\mathcal{I}^{(l)}_{i^{(l)}_n(s)+V^{(l,l)}_{n,m}(s),n} \cap \mathcal{I}^{(3-l)}_{i^{(3-l)}_n(s)+V^{(l,3-l)}_{n,m}(s),n}| \\ + \big(t^{(3-l)}_{i^{(3-l)}_n(s)+V^{(l,3-l)}_{n,m}(s)-1,n} - t^{(l)}_{i^{(l)}_n(s)+V^{(l,l)}_{n,m}(s)-1,n}\big)^+\Big),$$

$$\hat{\delta}^{(l)}_{n,m,+}(s) = n|\mathcal{I}^{(l)}_{V^{(l,l)}_{n,m}(s),n}| - \hat{\delta}^{(l)}_{n,m,-}(s),$$

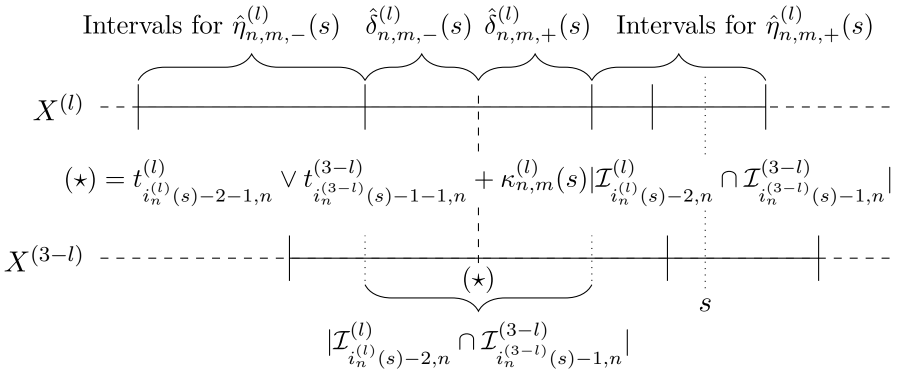

Figure 3: Realization of $\widehat{Z}^{(l)}_{n,m}(s)$ for $V^{(l,l)}_{n,m} = -2$, $V^{(l,3-l)}_{n,m}(s) = -1$.

$l = 1, 2$. The $U^{(l)}_{n,i,m}$ are i.i.d. $\mathcal{N}(0,1)$, the $\kappa^{(l)}_{n,m}(s)$ are i.i.d. $\mathcal{U}[0,1]$ random variables and the $V^{(l,l)}_{n,m}(s)$, $V^{(l,3-l)}_{n,m}(s)$ are distributed according to

$$\widetilde{\mathbb{P}}\big((V^{(l,l)}_{n,m}(s), V^{(l,3-l)}_{n,m}(s)) = (k_1,k_2)\big|\mathcal{S}\big) = |\mathcal{I}^{(l)}_{i_n^{(l)}(s)+k_1,n} \cap \mathcal{I}^{(3-l)}_{i_n^{(3-l)}(s)+k_2,n}|$$
$$\times \Big(\sum_{j_1\in\mathbb{Z}, |j_2|\le K_n} |\mathcal{I}^{(l)}_{i_n^{(l)}(s)+j_1,n} \cap \mathcal{I}^{(3-l)}_{i_n^{(3-l)}(s)+j_2,n}|\Big)^{-1}, \quad (k_1,k_2) \in \mathbb{Z} \times \{-K_n, \ldots, K_n\},$$

where the $(V^{(l,l)}_{n,m}(s), V^{(l,3-l)}_{n,m}(s))$ are $\mathcal{S}$-conditionally independent as $m = 1, \ldots, M_n$ varies. All newly introduced random variables are defined on $(\widetilde\Omega, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ as well. By construction,

$$\widehat{Z}^{(l)}_{n,m}(s) = (\hat\eta^{(l)}_{n,m,-}(s), \hat\eta^{(l)}_{n,m,+}(s), \hat\delta^{(l)}_{n,m,-}(s), \hat\delta^{(l)}_{n,m,+}(s))^*$$

then equals a mixture of the $Z^{(l)}_n(u)$ for $u \in [t^{(3-l)}_{i_n^{(3-l)}(s)-K_n-1,n}, t^{(3-l)}_{i_n^{(3-l)}(s)+K_n,n}]$ where in $\eta^{(l)}_{n,-}(u), \eta^{(l)}_{n,+}(u)$ the rescaled increments of the Brownian motion are replaced by independent normally distributed random variables. By the choice of the distribution for $(V^{(l,l)}_{n,m}(s), V^{(l,3-l)}_{n,m}(s))$ and $\kappa^{(l)}_{n,m}(s)$ we obtain further that u in the mixture is chosen uniformly from the interval $[t^{(3-l)}_{i_n^{(3-l)}(s)-K_n-1,n}, t^{(3-l)}_{i_n^{(3-l)}(s)+K_n,n}]$. This makes sense intuitively, as the jump times of an Itô semimartingale (2.1) are also uniformly distributed in time.

Consistent estimators for the jumps $\Delta X^{(l)}_s$, the volatility $(\sigma^{(l)}_s)^2$ and for $(\sigma^{(l)}_{s-})^2$, $l = 1, 2$, are given by

$$\begin{aligned} \widehat\Delta_n X^{(l)}(s) &= (\Delta^{(l)}_{i_n^{(l)}(s),n} X)\mathbb{1}_{\{|\Delta^{(l)}_{i_n^{(l)}(s),n}X| > \beta|\mathcal{I}^{(l)}_{i_n^{(l)}(s),n}|^{\varpi}\}}, \\ \big(\hat\sigma^{(l)}_n(s,+)\big)^2 &= \frac{1}{b_n}\sum_{i:t^{(l)}_{i-1,n}\in[s,s+b_n]} \big(\Delta^{(l)}_{i,n}X\big)^2, \\ \big(\hat\sigma^{(l)}_n(s,-)\big)^2 &= \frac{1}{b_n}\sum_{i:t^{(l)}_{i,n}\in[s-b_n,s)} \big(\Delta^{(l)}_{i,n}X\big)^2, \end{aligned} \tag{4.1}$$

where $\beta > 0$ and $\varpi \in (0, 1/2)$, and b_n is a sequence with $b_n \to 0$ and $|\pi_n|_T/b_n \xrightarrow{\mathbb{P}} 0$.

Using these estimators we define

$$\widehat{D}_{T,n,m} = \sum_{l=1,2} \sum_{i:t_{i,n}^{(l)} \leq T} \big(\Delta_{i,n}^{(l)} X\big)^2 \mathbb{1}_{\{|\Delta_{i,n}^{(l)} X| > \beta |\mathcal{I}_{i,n}^{(l)}|^{\varpi}\}} \widehat{R}_{n,m}^{(3-l)}(t_{i,n}^{(l)})$$

with

$$\widehat{R}_{n,m}^{(l)}(s) = \big(\hat{\sigma}_n^{(l)}(s,-)\big)^2 \hat{\eta}_{n,m,-}^{(l)}(s) + \big(\hat{\sigma}_n^{(l)}(s,-)(\hat{\delta}_{n,m,-}^{(l)}(s))^{1/2} U_{n,m,-}^{(l)}(s)$$
$$+ \hat{\sigma}_n^{(l)}(s,+)(\hat{\delta}_{n,m,+}^{(l)}(s))^{1/2} U_{n,m,+}^{(l)}(s)\big)^2 + \big(\hat{\sigma}_n^{(l)}(s,+)\big)^2 \hat{\eta}_{n,m,+}^{(l)}, \quad s \in [0,T], \quad l = 1,2,$$

and for $\alpha \in [0,1]$ we set

$$\widehat{Q}_{n,T}(\alpha) = \widehat{Q}_{\alpha}\big(\{\widehat{D}_{T,n,m} | m = 1, \ldots, M_n\}\big)$$

where $\widehat{Q}_{\alpha}(B)$ denotes the $\lfloor \alpha N \rfloor$-th largest element of a set B with $N \in \mathbb{N}$ elements. We will see that these expressions consistently estimate the $\mathcal{X}$-conditional α quantile of $\widetilde{D}_T$ on $\Omega_T^{(d)}$ which is defined as the $\mathcal{X}$-measurable random variable $Q(\alpha) \in [0,\infty]$ fulfilling

$$\mathbb{P}(\widetilde{D}_T \leq Q(\alpha) | \mathcal{X})(\omega) = \alpha \quad \text{for almost all } \omega \in \Omega_T^{(d)} \tag{4.2}$$

and we set $(Q(\alpha))(\omega) = 0$ for $\omega \in (\Omega_T^{(d)})^C$. Note that $Q(\alpha)$ is well defined for $\alpha \in [0,1]$ as the $\mathcal{X}$-conditional distribution of $\widetilde{D}_T$ restricted to $\Omega_T^{(d)}$ is continuous and has strictly positive density on $[0,\infty)$ by Condition 3.1(iii) and Condition 4.1.

The following condition summarizes all additional assumptions we need in order to obtain an asymptotic test. It ensures in particular that the volatility does not vanish, which yields $\widetilde{D}_T > 0$ almost surely, and by (4.3) that the empirical distribution of the $\widehat{Z}_{n,m}^{(l)}(s) = (\hat{\eta}_{n,m,-}^{(l)}(s), \eta_{n,m,+}^{(l)}(s), \delta_{n,m,-}^{(l)}(s), \delta_{n,m,+}^{(l)}(s))^*$ for $m = 1, \ldots, M_n$ converges to the non-degenerate distribution of $Z^{(l)}(s)$ which is essential for the bootstrap method to work.

Condition 4.1. The process X and the sequence of observation schemes $(\pi_n)_n$ satisfy Condition 3.1, and $\{s \in [0,T] : \sigma_s^{(1)} \sigma_s^{(2)} = 0\}$ is a Lebesgue null set. $(b_n)_n$ fulfills $|\pi_n|_T / b_n \xrightarrow{\mathbb{P}} 0$, $(K_n)_n$ and $(M_n)_n$ are sequences of integers converging to infinity, and $|\pi_n|_T K_n \xrightarrow{\mathbb{P}} 0$. Additionally,

$$\widetilde{\mathbb{P}}\big(\big|\widetilde{\mathbb{P}}\big(\widehat{Z}_{n,1}^{(l_j)}(s_j) \leq x_j,\ j = 1, \ldots, J \big| \mathcal{S}\big) - \widetilde{\mathbb{P}}\big(Z^{(l_j)}(s_j) \leq x_j,\ j = 1, \ldots, J\big)\big| > \varepsilon\big) \to 0 \tag{4.3}$$

as $n \to \infty$, for all $\varepsilon > 0$, $J \in \mathbb{N}$, $x = (x_1, \ldots, x_J) \in \mathbb{R}^{J \times 4}$, $l_j \in \{1,2\}$ and $s_j \in (0,T)$, $j = 1, \ldots, J$, with $s_i \neq s_j$ for $i \neq j$.

Theorem 4.2. *If Condition 4.1 is satisfied, the test defined in (2.4) with*

$$c_n = \frac{A_{n,T} + \widehat{Q}_{n,T}(1-\alpha)}{n\sqrt{V^{(1)}(g,\pi_n)_T V^{(2)}(g,\pi_n)_T}}, \quad \alpha \in [0,1],$$

has asymptotic level α in the sense that we have

$$\widetilde{\mathbb{P}}\big(\widetilde{\Phi}_{n,T}^{(d)} > c_n \big| F^{(d)}\big) \to \alpha \tag{4.4}$$

for all $F^{(d)} \subset \Omega_T^{(d)}$ *with* $\mathbb{P}(F^{(d)}) > 0$*. Because of*

$$\widetilde{\mathbb{P}}\big(\widetilde{\Phi}_{n,T}^{(d)} > c_n \big| F^{(j)}\big) \to 1 \tag{4.5}$$

for all $F^{(j)} \subset \Omega_T^{(j)}$ *with* $\mathbb{P}(F^{(j)}) > 0$ *it is consistent as well.*

Although n appears in the definition of c_n it is not used for the computation of c_n as it also occurs linearly in $A_{n,T}$ and $\widehat{Q}_{n,T}(1-\alpha)$. It enters indirectly through the choice of b_n, K_n, M_n, however, for which usually just a rough idea of the magnitude of n is needed.

Example 4.3. If the sampling scheme is deterministic, then (4.3) holds in all situations where a minimal local regularity is assumed. This is in particular the case for the setting of synchronous equidistant observation times as in Example 3.3 where our estimator $\widehat{Q}_{n,T}(1-\alpha)$ equals the estimator $Z_n^{(d)}(\alpha)$ defined in (5.10) of Jacod and Todorov (2009) for $N_n = M_n$ and any choice of K_n (not necessarily converging to infinity).

Example 4.4. Regarding the Poisson setting from Example 3.4, $|\pi_n|_T / b_n \xrightarrow{\mathbb{P}} 0$ follows from (3.7) for every $b_n = O(n^{-\alpha})$ with $\alpha \in (0,1)$. Showing that (4.3) holds, however, is rather tedious and postponed to Section 6.

5 Simulation results

We conduct a simulation study to verify the finite sample properties of the introduced methods. Our benchmark model is the one from Section 6 of Jacod and Todorov (2009), as we use the same configuration as in their paper to compare our approach to the case of equidistant and synchronous observations. The model for X is given by

$$dX_t^{(1)} = X_t^{(1)}\sigma_1 dW_t^{(1)} + \alpha_1 \int_{\mathbb{R}} X_{t-}^{(1)} x_1 \mu_1(dt, dx_1) + \alpha_3 \int_{\mathbb{R}} X_{t-}^{(1)} x_3 \mu_3(dt, dx_3),$$
$$dX_t^{(2)} = X_t^{(2)}\sigma_2 dW_t^{(2)} + \alpha_2 \int_{\mathbb{R}} X_{t-}^{(2)} x_2 \mu_2(dt, dx_2) + \alpha_3 \int_{\mathbb{R}} X_{t-}^{(2)} x_3 \mu_3(dt, dx_3),$$

where $[W^{(1)}, W^{(2)}]_t = \rho t$ and the Poisson measures μ_i are independent of each other and have predictable compensators ν_i of the form

$$\nu_i(dt, dx_i) = \kappa_i \frac{\mathbb{1}_{[-h_i,-l_i]\cup[l_i,h_i]}(x_i)}{2(h_i - l_i)} dt dx_i$$

where $0 < l_i < h_i$ for $i = 1, 2, 3$, and the initial values are $X_0 = (1,1)^T$. We consider the same twelve parameter settings which were discussed in Jacod and Todorov (2009) of which six allow for common jumps and six do not. In the case where common jumps are possible, we only use the simulated paths which contain common jumps. For the parameters we set $\sigma_1^2 = \sigma_2^2 = 8 \times 10^{-5}$ in all scenarios and choose the parameters for the Poisson measures such that the contribution of the jumps to the total variation remains approximately constant and matches estimations from real financial data (see Huang and Tauchen (2006)). The parameter settings are summarized in Table 1 (compare Table 1 in Jacod and Todorov (2009)).

To model the observation times we use the Poisson setting discussed in Example 3.4 and 4.4 for $\lambda_1 = \lambda_2 = 1$, and set $T = 1$ which amounts on average to n observations of each $X^{(1)}$ and $X^{(2)}$. We choose $n = 100$, $n = 400$ and $n = 1600$ for the simulation. In a trading day of 6.5 hours this corresponds to observing $X^{(1)}$ and $X^{(2)}$ on average every 4 minutes, every 1 minute and every 15 seconds. We set $\beta = 0.03$ and $\varpi = 0.49$ for all occuring truncations. We use $b_n = 1/\sqrt{n}$ for the local interval in the estimation of $\sigma_s^{(l)}$ and $K_n = \lfloor \ln(n) \rfloor$, $M_n = n$ in the simulation of the $\widehat{Z}_{n,m}^{(l)}(s)$. As discussed in Remark 5.5 of Jacod and Todorov (2009) the choice of the parameters β, ϖ specifying the truncation level is critical because it determines which increments are considered to be mostly driven by jumps and which are not. We choose here the same values as in Jacod and Todorov (2009) and thereby follow their recommendation to pick ϖ close to 1/2 and β to be about 3 to 4 times the magnitude of σ (which in general is unknown but can be easily estimated from the data). This choice for β is reasonable as increments $\Delta_{i,n}^{(l)} X \approx \Delta_{i,n}^{(l)} C$ where the jump part is negligible are roughly normally distributed with variance $\sigma^2 |\mathcal{I}_{i,n}^{(l)}|$. Thereby these increments are filtered out with high probability as a normal distributed random variable rarely exceeds 3 standard deviations. $b_n = 1/\sqrt{n}$ is chosen in the center of the allowed range between a constant b_n and $b_n = O(\log(n)/n)$ which balances the benefits from choosing b_n small (smaller bias in the estimation of σ if σ is not flat) and b_n large (less variance in the estimation of σ). This choice is also close to the optimal one in the sense of Theorem 13.3.3 in Jacod and Protter (2012) on the estimation of spot volatility. K_n is chosen rather small to keep the computation time low, M_n is chosen to be large enough to justify a reasonable approximation to the theoretical quantiles. In the simulation study the results were very robust to the choice of b_n, K_n, M_n.

In Figure 4 we display the results from the simulation. The plots are constructed as follows: First for different values of α the critical values are simulated according to Theorem 4.2. Then we plot the observed rejection frequencies against α.

The six plots on the left show the results for the cases where the alternative of common jumps is true. In the cases I-j, II-j and III-j there exist only joint jumps and the Brownian motions $W^{(1)}$ and $W^{(2)}$ are uncorrelated. In the cases I-m, II-m and III-m we have a mixed model which allows for disjoint and joint jumps and also the Brownian motions are positively correlated. The prefixes I, II and

	Parameters												
Case	ρ	α_1	κ_1	l_1	h_1	α_2	κ_2	l_1	h_1	α_3	κ_3	l_3	h_3
I-j	0.0	0.00				0.00				0.01	1	0.05	0.7484
II-j	0.0	0.00				0.00				0.01	5	0.05	0.3187
III-j	0.0	0.00				0.00				0.01	25	0.05	0.1238
I-m	0.5	0.01	1	0.05	0.7484	0.01	1	0.05	0.7484	0.01	1	0.05	0.7484
II-m	0.5	0.01	5	0.05	0.3187	0.01	5	0.05	0.3187	0.01	5	0.05	0.3187
III-m	0.5	0.01	25	0.05	0.1238	0.01	25	0.05	0.1238	0.01	25	0.05	0.1238
I-d0	0.0	0.01	1	0.05	0.7484	0.01	1	0.05	0.7484				
II-d0	0.0	0.01	5	0.05	0.3187	0.01	5	0.05	0.3187				
III-d0	0.0	0.01	25	0.05	0.1238	0.01	25	0.05	0.1238				
I-d1	1.0	0.01	1	0.05	0.7484	0.01	1	0.05	0.7484				
II-d1	1.0	0.01	5	0.05	0.3187	0.01	5	0.05	0.3187				
III-d1	1.0	0.01	25	0.05	0.1238	0.01	25	0.05	0.1238				

Table 1: Parameter settings for the simulation.

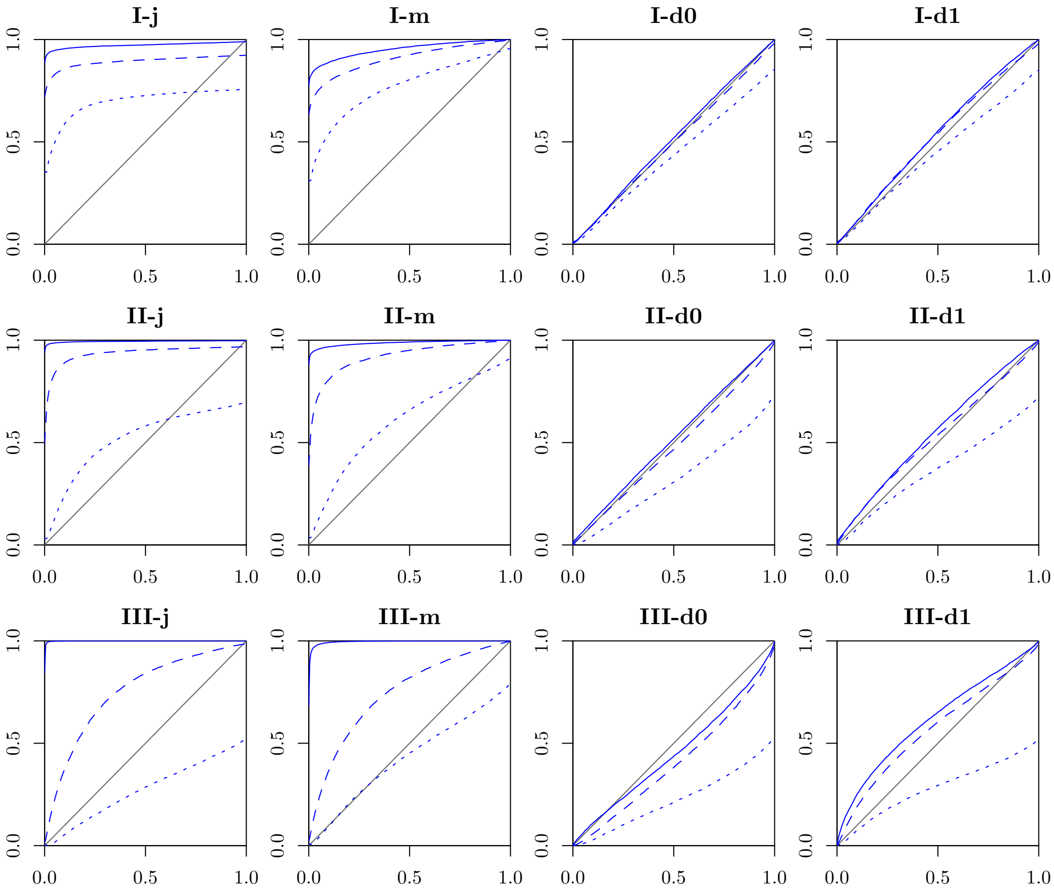

Figure 4: Empirical rejection curves from the Monte Carlo simulation for the test derived from Theorem 4.2. The dotted line represents the results for $n = 100$, the dashed line for $n = 400$ and the solid line for $n = 1600$. In each case $N = 10.000$ paths were simulated.

III indicate an increasing number of jumps present in the observed paths. Since our choice of parameters is such that the overall contribution of the jumps to the quadratic variation is roughly the same in all parameter settings, this corresponds to a decreasing size of the jumps. Hence in the cases I-* we have few big jumps while in the cases III-* we have many small jumps.

We see that the test has very good power against the alternative of common jumps. The power is greater for small n if there are less and bigger jumps as can be seen from the dotted lines for the cases I-j and I-m, because the bigger jumps are detected more easily. On the other hand the power is greater for large n if there are more and smaller jumps which can be seen from the solid lines for III-j and III-m, because then it is more probable that at least one of the common jumps is detected and one small detected common jump is sufficient for rejecting the null.

The six plots on the right in Figure 4 show the results for the cases where the null hypothesis is true. While in the cases *-d0 the Brownian motions $W^{(1)}$ and $W^{(2)}$ are uncorrelated, the Brownian motions are perfectly correlated in the cases *-d1. The prefixes I, II and III stand for an increasing number and a decreasing size of the jumps as in the first six cases.

Under the null of disjoint jumps we see that the observed rejection frequencies match the predicted asymptotic rejection probabilities from Theorem 4.2 very well in all six cases. There are slight deviations for a higher number of jumps. This is due to the fact that disjoint jumps whenever they lie close together, sometimes cannot be distinguished based on the observations which leads to over-rejection under the null hypothesis. In the cases *-d1 where the Brownian motions are perfectly correlated the rejection frequencies are systematically too high for large n. The results are worse than in the cases *-d0.

In general, the results from the Monte Carlo are very similar to the results displayed in Figure 5 (note that the values for n there are 100, 1600 and 25600) from Jacod and Todorov (2009). On a closer look we observe that the power of our test in the asynchronous setting is slightly worse than the power of the test in the equidistant and synchronous setting while under the null hypothesis the rejection levels match the asymptotic levels more closely than in Jacod and Todorov (2009). The loss in power is most pronounced for the smallest observation frequency $n = 100$ and in the range of at most a few percentage points for the more relevant frequency $n = 1600$. Our results in the cases *-d1 are better than in Jacod and Todorov (2009) because the effect of a high correlation in the Brownian motions has less influence on the test statistic due to the asynchronicity.

All in all we conclude that there is no significant backdraw of working with asynchronous observations instead of synchronous observations when testing for disjoint jumps in a bivariate process. This is of great importance, as these results demonstrate that it is possible to construct a test for disjoint jumps which works efficiently in the case of asynchronous and random observations without having to synchronize data first. Such procedures are well-known in the literature, but lead inevitably to a loss of data and, thus, power. Also, our methods are applicable in a quite universal setting without additional knowledge on the underlying observation scheme.

6 Proofs

6.1 Preliminaries

Throughout the proofs we will assume that the processes $b_s, \sigma_s^{(1)}, \sigma_s^{(2)}, \rho_s$ and Γ_s are bounded on $[0, T]$. They are all locally bounded by Condition 2.1. A localization procedure then shows that the results for bounded processes can be carried over to the case of locally bounded processes (see e.g. Section 4.4.1 in Jacod and Protter (2012)).

We introduce the decomposition $X_t = X_0 + B(q)_t + C_t + M(q)_t + N(q)_t$ of the Itô semimartingale (2.1) with

$$\begin{aligned}
B(q)_t &= \int_0^t \big(b_s - \int (\delta(s,z)\mathbb{1}_{\{\|\delta(s,z)\|\leq 1\}} - \delta(s,z)\mathbb{1}_{\{\gamma(z)\leq 1/q\}})\lambda(dz)\big)ds,\\
C_t &= \int_0^t \sigma_s dW_s,\\
M(q)_t &= \int_0^t \int \delta(s,z)\mathbb{1}_{\{\gamma(z)\leq 1/q\}}(\mu-\nu)(ds,dz),\\
N(q)_t &= \int_0^t \int \delta(s,z)\mathbb{1}_{\{\gamma(z)>1/q\}}\mu(ds,dz).
\end{aligned}$$

Here q is a parameter which controls whether jumps are classified as small jumps or big jumps. We will make repeatedly use of the following estimates (compare Section 2.1.5 in Jacod and Protter (2012)). Throughout the proofs K and K_q will denote generic constants, the latter dependent on q, to simplify notation.

Lemma 6.1. *There exist constants $K, K_p, K_q, e_q \geq 0$ such that*

$$\|B(q)_{s+t} - B(q)_s\|^2 \leq K_q t^2, \tag{6.1}$$
$$\mathbb{E}\big[\|C_{s+t} - C_s\|^p | \mathcal{F}_s\big] \leq K_p t^{p/2}, \tag{6.2}$$
$$\mathbb{E}\big[\|M(q)_{s+t} - M(q)_s\|^2 | \mathcal{F}_s\big] \leq K t e_q, \tag{6.3}$$
$$\mathbb{E}\big[\|N(q)_{s+t} - N(q)_s\|^2 | \mathcal{F}_s\big] \leq K_q t, \tag{6.4}$$

for all $s, t \geq 0$, $q > 0$, $p \geq 1$. Here, e_q can be chosen such that $e_q \to 0$ for $q \to \infty$.

6.2 Proof of the consistency result

Proof of (2.2). We will show

$$\lim_{q\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\Big| \sum_{i,j:t^{(1)}_{i,n} \wedge t^{(2)}_{j,n} \leq T} \big(\Delta^{(1)}_{i,n} N(q) \Delta^{(2)}_{j,n} N(q)\big)^2 \mathbb{1}_{\{\mathcal{I}^{(1)}_{i,n} \cap \mathcal{I}^{(2)}_{j,n} \neq \emptyset\}} - B_T\Big| > \delta\Big) \to 0 \tag{6.5}$$

and

$$\lim_{q\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\Big|V(f,\pi_n)_T - \sum_{i,j:t^{(1)}_{i,n} \wedge t^{(2)}_{j,n} \leq T} \big(\Delta^{(1)}_{i,n} N(q) \Delta^{(2)}_{j,n} N(q)\big)^2 \mathbb{1}_{\{\mathcal{I}^{(1)}_{i,n} \cap \mathcal{I}^{(2)}_{j,n} \neq \emptyset\}}\Big| > \delta\Big) \to 0 \tag{6.6}$$

for all $\delta > 0$ from which (2.2) follows. This strategy, namely to artificially introduce auxiliary parameters such as q which eventually converge as well, will be typical for the entire section on proofs. In particular, we might even add further additional parameters.

For proving (6.5) we denote by $\Omega(n, q)$ the set on which two different jumps of $N(q)$ are further apart than $2|\pi_n|_T$. On $\Omega(n, q)$ we have

$$\sum_{i,j:t^{(1)}_{i,n} \wedge t^{(2)}_{j,n} \leq T} \big(\Delta^{(1)}_{i,n} N(q) \Delta^{(2)}_{j,n} N(q)\big)^2 \mathbb{1}_{\{\mathcal{I}^{(1)}_{i,n} \cap \mathcal{I}^{(2)}_{j,n} \neq \emptyset\}} = \sum_{s\leq T} \big(\Delta N^{(1)}(q)_s\big)^2 \big(\Delta N^{(2)}(q)_s\big)^2. \tag{6.7}$$

Note that the right hand side of (6.7) converges to B_T almost surely as $q \to \infty$. Thus, (6.5) follows since $\mathbb{P}(\Omega(n, q)) \to 1$ for $n \to \infty$.

For proving (6.6) we introduce the elementary inequality

$$\begin{aligned} &\big|(a_1 + b_1 + c_1 + d_1)^2 (a_2 + b_2 + c_2 + d_2)^2 - {d_1}^2 {d_2}^2\big| \\ &\qquad \leq c_\rho \sum_{l=1,2} \big({a_{3-l}}^2 + {b_{3-l}}^2 + {c_{3-l}}^2\big)\big({a_l}^2 + {b_l}^2 + {c_l}^2 + {d_l}^2\big) + 3\rho {d_1}^2 {d_2}^2 \end{aligned} \tag{6.8}$$

which can be proven using Cauchy-Schwarz inequality after introducing appropriate weights and which holds for real numbers $a_l, b_l, c_l, d_l \in \mathbb{R}$, $l = 1, 2$, and $\rho \in (0, 1)$ by setting $c_\rho = 9\big(1+\rho\big)^2/\rho^2$. As we are interested in the sum of the product of the squared increments of $X^{(1)}$ and $X^{(2)}$, we can simplify each summand by applying (6.8), i.e. we set $a_l = \Delta_{i,n}^{(l)} B(q), b_l = \Delta_{i,n}^{(l)} C, c_l = \Delta_{i,n}^{(l)} M(q), d_l = \Delta_{i,n}^{(l)} N(q)$.

Note that

$$3\rho \sum_{i,j:t_{i,n}^{(1)}\wedge t_{j,n}^{(2)}\le T} \big(\Delta_{i,n}^{(1)} N(q)\Delta_{j,n}^{(2)} N(q)\big)^2 \mathbb{1}_{\{\mathcal{I}_{i,n}^{(1)}\cap \mathcal{I}_{j,n}^{(2)}\neq\emptyset\}} \to 3\rho\big[N^{(1)}(q),N^{(2)}(q)\big]_T$$

where the right hand side is bounded in q by $3\rho[X,X]_t$ which tends to zero for $\rho\to 0$. Furthermore, for any $l=1,2$,

$$\begin{aligned}
&c_\rho \sum_{i,j:t_{i,n}^{(3-l)}\wedge t_{j,n}^{(l)}\le T} \big((\Delta_{i,n}^{(3-l)}B(q))^2 + (\Delta_{i,n}^{(3-l)}M(q))^2\big)\times \mathbb{1}_{\{\mathcal{I}_{i,n}^{(3-l)}\cap \mathcal{I}_{j,n}^{(l)}\neq\emptyset\}}\\
&\quad\times\big((\Delta_{j,n}^{(l)}B(q))^2 + (\Delta_{j,n}^{(l)}C)^2 + (\Delta_{j,n}^{(l)}M(q))^2 + (\Delta_{j,n}^{(l)}N(q))^2\big)\\
&\quad\le c_\rho\Big(\sum_{i:t_{i,n}^{(3-l)}\le T} \big((\Delta_{i,n}^{(3-l)}B(q))^2 + (\Delta_{i,n}^{(3-l)}M(q))^2\big)\Big)\\
&\quad\times\Big(\sum_{j:t_{i,n}^{(l)}\le T}\big((\Delta_{j,n}^{(l)}B(q))^2 + (\Delta_{j,n}^{(l)}C)^2 + (\Delta_{j,n}^{(l)}M(q))^2 + (\Delta_{j,n}^{(l)}N(q))^2\big)\\
&\quad\xrightarrow{\mathbb{P}} c_\rho\big([B^{(3-l)}(q),B^{(3-l)}(q)]_T + [M^{(3-l)}(q),M^{(3-l)}(q)]_T\big)\big[X^{(l)},X^{(l)}\big]_T
\end{aligned}$$

which tends to zero for $q\to\infty$ and any fixed ρ. For the remaining terms we set

$$K_{n,T}(l,\varepsilon) = \sup_{m,m'\in\mathbb{N},m\le m',|t_{m',n}^{(l)}-t_{m-1,n}^{(l)}|\le\varepsilon,t_{m',n}^{(l)}\le T} \sum_{k=m}^{m'} \big(C_{t_{k,n}^{(l)}}^{(l)} - C_{t_{k-1,n}^{(l)}}^{(l)}\big)^2, \quad l=1,2.$$

We have

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\mathbb{P}(K_{n,T}(l,\varepsilon)>\delta)=0 \tag{6.9}$$

for any $\delta>0$ due to the ucp convergence of realized volatility to the quadratic variation (see Theorem II.23 in Protter (2004)). In fact, on the set $\{K_{n,T}(l,\varepsilon)>\delta\}$ we have

$$\sup_{0\le s\le T}\Big|\sum_{i=1}^{\infty}\big(C_{t_{i,n}^{(l)}\wedge s}^{(l)} - C_{t_{i-1,n}^{(l)}\wedge s}^{(l)}\big)^2 - \int_0^s (\sigma_u^{(l)})^2 du\Big|\mathbb{1}_{\{K_{n,T}(l,\varepsilon)>\delta\}} \ge \frac{\delta - K^2\varepsilon}{2}$$

with $\|\sigma_s\|\le K$.

Using the fact that the total length of the observation intervals of one process which overlap with a specific observation interval of the other process is at most $3|\pi_n|_T$, we get on the set $\{3|\pi_n|_T\le\varepsilon\}$

$$\begin{aligned}
&c_\rho \sum_{i,j:t_{i,n}^{(3-l)}\wedge t_{j,n}^{(l)}\le T} (\Delta_{i,n}^{(3-l)}C^{(3-l)})^2\big((\Delta_{j,n}^{(l)}C^{(l)})^2 + (\Delta_{j,n}^{(l)}N^{(l)}(q))^2\big)\mathbb{1}_{\{\mathcal{I}_{i,n}^{(3-l)}\cap \mathcal{I}_{j,n}^{(l)}\neq\emptyset\}}\mathbb{1}_{\{|\pi_n|_T\le\varepsilon/3\}}\\
&\le c_\rho K_{n,T}(3-l,\varepsilon)\sum_{j:t_{j,n}^{(l)}\le T}\big((\Delta_{j,n}^{(l)}C^{(l)})^2 + (\Delta_{j,n}^{(l)}N^{(l)}(q))^2\big)\mathbb{1}_{\{|\pi_n|_T\le\varepsilon/3\}}.
\end{aligned}$$

As the latter sum converges to the quadratic variation of $C^{(l)}+N^{(l)}(q)$ as $n\to\infty$, which remains bounded in q, we obtain that these terms vanish by (6.9) and because of $\mathbb{P}(|\pi_n|_T\le\varepsilon)\to 1$ as $n\to\infty$ for any fixed $\varepsilon>0$. □

Proof of Theorem 2.2. This is a direct consequence of (2.2), (2.3) and the continuous mapping theorem for convergence in probability. □

6.3 Proof of the central limit theorem

We will prove the central limit theorem in three parts: We will begin with the convergence of the mixed Brownian increments to the continuous term in the limit (Proposition 6.2), followed by the convergence of the mixed term of large jumps and Brownian increments to the mixed term in the limit (Proposition 6.3), and we end with the convergence of the remaining terms to zero (Proposition 6.5).

Proposition 6.2. *If Condition 2.1 and Condition 3.1(i)-(ii) are fulfilled, we have*

$$n \sum_{i,j:t_{i,n}^{(1)}\wedge t_{j,n}^{(2)}\leq T} \big((\Delta_{i,n}^{(1)}C)^2(\Delta_{j,n}^{(2)}C)^2\big)\mathbb{1}_{\{\mathcal{I}_{i,n}^{(1)}\cap\mathcal{I}_{j,n}^{(2)}\neq\emptyset\}} \xrightarrow{\mathbb{P}} \int_0^T 2(\rho_s\sigma_s^{(1)}\sigma_s^{(2)})^2 dG(s) + \int_0^T (\sigma_s^{(1)}\sigma_s^{(2)})^2 dH(s).$$

Proof. We use a discretization of σ given via $\sigma(r)_s = \sigma_{(k-1)T/2^r}$ for $s \in [(k-1)T/2^r, kT/2^r)$, and we denote the integral of $\sigma(r)$ with respect to the Brownian motion W from (2.1) by $C(r)$. Setting

$$\begin{aligned}
Y_n &= n \sum_{i,j:t_{i,n}^{(1)}\wedge t_{j,n}^{(2)}\leq T} \big((\Delta_{i,n}^{(1)}C)^2(\Delta_{j,n}^{(2)}C)^2\big)\mathbb{1}_{\{\mathcal{I}_{i,n}^{(1)}\cap\mathcal{I}_{i,n}^{(2)}\neq\emptyset\}},\\
Y &= \int_0^T 2(\rho_s\sigma_s^{(1)}\sigma_s^{(2)})^2 dG(s) + \int_0^T (\sigma_s^{(1)}\sigma_s^{(2)})^2 dH(s),\\
Y_n(r) &= n \sum_{i,j:t_{i,n}^{(1)}\wedge t_{j,n}^{(2)}\leq T} \big((\Delta_{i,n}^{(1)}C(r))^2(\Delta_{j,n}^{(2)}C(r))^2\big)\mathbb{1}_{\{\mathcal{I}_{i,n}^{(1)}\cap\mathcal{I}_{i,n}^{(2)}\neq\emptyset\}},\\
Y(r) &= \int_0^T 2(\rho(r)_s\sigma^{(1)}(r)_s\sigma^{(2)}(r)_s)^2 dG(s) + \int_0^T (\sigma^{(1)}(r)_s\sigma^{(2)}(r)_s)^2 dH(s),
\end{aligned}$$

we will prove

$$\lim_{r\to\infty}\limsup_{n\to\infty}\mathbb{P}\big(\big|Y-Y(r)\big|+\big|Y(r)-Y_n(r)\big|+\big|Y_n(r)-Y_n\big|>\varepsilon\big)=0 \quad \forall\varepsilon>0.$$

It holds $\sigma(r)_s \to \sigma_{s-}$ as $r\to\infty$ for all $s\in[0,T]$, where σ_{s-} is well defined as σ is càdlàg. Dominated convergence then yields

$$Y(r) \to \int_0^T 2(\rho_{s-}\sigma_{s-}^{(1)}\sigma_{s-}^{(2)})^2 dG(s) + \int_0^T (\sigma_{s-}^{(1)}\sigma_{s-}^{(2)})^2 dH(s)$$

where the right hand side equals Y as G,H are continuous by Condition 3.1(ii).

In order to prove $|Y(r)-Y_n(r)| \xrightarrow{\mathbb{P}} 0$ as $n\to\infty$ we apply Lemma 2.2.12 from Jacod and Protter (2012) with

$$\xi_k^n = n \sum_{(i,j)\in L(n,k,T)} \big((\Delta_{i,n}^{(1)}C(r))^2(\Delta_{j,n}^{(2)}C(r))^2\big)\mathbb{1}_{\{\mathcal{I}_{i,n}^{(1)}\cap\mathcal{I}_{j,n}^{(2)}\neq\emptyset\}},$$

$L(n,k,T) = \{(i,j) : t^{(1)}_{i-1,n} \vee t^{(2)}_{j-1,n} \in [(k-1)T/2^{r_n}, kT/2^{r_n})\}$, $k = 1,2,\ldots,2^{r_n}$, and $\mathcal{G}^n_k = \sigma\big(\mathcal{F}_{(k-1)T/2^{r_n}} \cup \mathcal{S}\big)$. Here, r_n is a sequence of real numbers with $r_n \geq r$, $r_n \to \infty$ and

$$\begin{aligned} &2^{r_n} \sup_{s\in[0,T]} |G(s) - G_n(s)| = o_{\mathbb{P}}(1), \\ &2^{r_n} \sup_{s\in[0,T]} |H(s) - H_n(s)| = o_{\mathbb{P}}(1), \\ &2^{r_n} n(|\pi_n|_T)^2 = o_{\mathbb{P}}(1). \end{aligned} \tag{6.10}$$

Such a sequence exists, because G_n, H_n and hence G, H are nondecreasing functions and G, H are continuous such that the pointwise convergence from Condition 3.1(ii) implies uniform convergence on $[0,T]$ and because of $n(|\pi_n|_T)^2 = o_{\mathbb{P}}(1)$ by Condition 3.1(i). Elementary computations then reveal

$$\begin{aligned} \mathbb{E}\big[\xi^n_k \big| \mathcal{G}^n_{k-1}\big] = 2(\rho(r)_{(k-1)T/2^{r_n}} &\sigma^{(1)}(r)_{(k-1)T/2^{r_n}} \sigma^{(2)}(r)_{(k-1)T/2^{r_n}})^2 \\ &\times \big(G_n(kT/2^{r_n}) - G_n((k-1)T/2^{r_n})\big) \\ + (\sigma^{(1)}(r)_{(k-1)T/2^{r_n}} &\sigma^{(2)}(r)_{(k-1)T/2^{r_n}})^2 \big(H_n(kT/2^{r_n}) - H_n((k-1)T/2^{r_n})\big) \\ + O_{\mathbb{P}}(n(|\pi_n|_T)^2)&. \end{aligned}$$

In combination with the boundedness of σ the previous display implies

$$\begin{aligned} &\Big|Y(r) - \sum_{k=1}^{2^{r_n}} \mathbb{E}\big[\xi^n_k \big| \mathcal{G}^n_{k-1}\big]\Big| \\ &\quad \leq K 2^{r_n} \Big(\sup_{s\in[0,T]} |G(s) - G_n(s)| + \sup_{s\in[0,T]} |H(s) - H_n(s)|\Big) + O_{\mathbb{P}}\big(2^{r_n} n(|\pi_n|_T)^2\big) \end{aligned}$$

where the right hand side is $o_{\mathbb{P}}(1)$ by (6.10). Hence the sum over the $\mathbb{E}\big[\xi^n_k \big| \mathcal{G}^n_{k-1}\big]$ converges to $Y(r)$.

Using the Cauchy-Schwarz inequality, the definition of H_n and telescoping sums we also get

$$\begin{aligned} \sum_{k=1}^{2^{r_n}} \mathbb{E}\big[|\xi^n_k|^2 \big| \mathcal{G}^n_{k-1}\big] &\leq K \sum_{k=1}^{2^{r_n}} \Big(n \sum_{(i,j)\in L(n,k,T)} |\mathcal{I}^{(1)}_{i,n}| |\mathcal{I}^{(2)}_{j,n}| \mathbb{1}_{\{\mathcal{I}^{(1)}_{i,n} \cap \mathcal{I}^{(2)}_{i,n} \neq \emptyset\}} \Big)^2 \\ &\leq K H_n(T) \sup_{u,s\in[0,T], |u-s| \leq T2^{-r_n} + |\pi_n|_T} \big|H_n(u) - H_n(s)\big| \end{aligned}$$

where the right hand side converges to zero in probability, since H_n converges uniformly to H which is uniformly continuous on $[0,T]$. Together with

$$\sum_{k=1}^{2^{r_n}} \mathbb{E}\big[\xi^n_k \big| \mathcal{G}^n_{k-1}\big] \xrightarrow{\mathbb{P}} Y(r)$$

we obtain

$$Y_n(r) = \sum_{k=1}^{2^{r_n}} \xi^n_k \xrightarrow{\mathbb{P}} Y(r)$$

by Lemma 2.2.12 from Jacod and Protter (2012).

Finally, we prove

$$\lim_{r\to\infty}\limsup_{n\to\infty}\mathbb{P}(|Y_n(r)-Y_n|>\varepsilon)\to 0. \tag{6.11}$$

First consider the estimate

$$\begin{aligned}\big|Y_n(r)-Y_n\big| \le n \sum_{i,j:t^{(1)}_{i,n}\wedge t^{(2)}_{j,n}\le T} \Big(& \big|\Delta^{(1)}_{i,n}(C-C(r))\big|\big|\Delta^{(1)}_{i,n}(C+C(r))\big|\big|\Delta^{(2)}_{j,n}C\big|^2 \\ & + \big|\Delta^{(2)}_{j,n}(C-C(r))\big|\big|\Delta^{(2)}_{j,n}(C+C(r))\big|\big|\Delta^{(1)}_{i,n}C(r)\big|^2\Big)\mathbb{1}_{\{\mathcal{I}^{(1)}_{i,n}\cap\mathcal{I}^{(2)}_{i,n}\neq\emptyset\}}.\end{aligned}$$

Once we take conditional expectations with respect to $\mathcal{S}$ and apply Cauchy-Schwarz inequality, (6.2) and (2.1.34) from Jacod and Protter (2012), we obtain

$$\begin{aligned}&\mathbb{E}\big[|Y_n(r)-Y_n|\big|\mathcal{S}\big] \\ &\le Kn\sum_{i,j:t^{(1)}_{i,n}\wedge t^{(2)}_{j,n}\le T}\Bigg(\Big(\mathbb{E}\Big[\int_{t^{(1)}_{i-1,n}}^{t^{(1)}_{i,n}}\|\sigma_s-\sigma(r)_s\|^2ds\big|\mathcal{S}\Big]\Big)^{1/2}\big|\mathcal{I}^{(1)}_{i,n}\big|^{1/2}\big|\mathcal{I}^{(2)}_{j,n}\big| \\ &\qquad + \Big(\mathbb{E}\Big[\int_{t^{(2)}_{j-1,n}}^{t^{(2)}_{j,n}}\|\sigma_s-\sigma(r)_s\|^2ds\big|\mathcal{S}\Big]\Big)^{1/2}\big|\mathcal{I}^{(2)}_{j,n}\big|^{1/2}\big|\mathcal{I}^{(1)}_{i,n}\big|\Bigg)\mathbb{1}_{\{\mathcal{I}^{(1)}_{i,n}\cap\mathcal{I}^{(2)}_{i,n}\neq\emptyset\}} \\ &\le K\varepsilon H_n(T) \\ &\quad+\frac{K}{\varepsilon}n\sum_{l=1,2}\mathbb{E}\Big[\sum_{i,j:t^{(l)}_{i,n}\wedge t^{(3-l)}_{j,n}\le T}\int_{t^{(l)}_{i-1,n}}^{t^{(l)}_{i,n}}\|\sigma_s-\sigma(r)_s\|^2ds\big|\mathcal{I}^{(3-l)}_{j,n}\big|\mathbb{1}_{\{\mathcal{I}^{(l)}_{i,n}\cap\mathcal{I}^{(3-l)}_{i,n}\neq\emptyset\}}\Big|\mathcal{S}\Big]\end{aligned} \tag{6.12}$$

for any $\varepsilon>0$ where we used $\sqrt{ab}\le a\varepsilon+b/\varepsilon$ for $a,b,\varepsilon>0$ for the second inequality. As we have $H_n(T)\to H(T)$ for $n\to\infty$ and as ε can be chosen to be arbitrarily small it suffices for proving (6.11) that the last sum in (6.12) vanishes as $r,n\to\infty$. Consider the set $\Omega(\delta,N,K',r)$ on which there are at most N jump times $S_1,\ldots,S_N\in[0,T]$ with $\|\Delta\sigma_{S_i}\|>\delta$, $\sup_{s\in[0,T]}\|\Delta\sigma_s\|\le K'$ and it holds $\|\sigma_s-\sigma(r)_s\|\le 2\delta$ for all $s\in[0,T]\setminus\bigcup_{j=1}^N(\lfloor S_jT/2^r\rfloor 2^r/T,S_j]$ and $\|\sigma_s-\sigma(r)_s\|\le K'+\delta$ for all $s\in\bigcup_{j=1}^N(\lfloor S_jT/2^r\rfloor 2^r/T,S_j]$. On this set we obtain the following bound

$$\begin{aligned}&n\sum_{l=1,2}\mathbb{E}\Big[\sum_{i,j:t^{(l)}_{i,n}\wedge t^{(3-l)}_{j,n}\le T}\int_{t^{(l)}_{i-1,n}}^{t^{(l)}_{i,n}}\|\sigma_s-\sigma(r)_s\|^2ds\big|\mathcal{I}^{(3-l)}_{j,n}\big|\mathbb{1}_{\{\mathcal{I}^{(l)}_{i,n}\cap\mathcal{I}^{(3-l)}_{i,n}\neq\emptyset\}}\mathbb{1}_{\Omega(\delta,N,K',r)}\Big|\mathcal{S}\Big] \\ &\quad\le 4\delta^2H_n(T)+2N(K'+\delta)^2\sup_{u,s\in[0,T],|u-s|\le T2^{-r}+4|\pi_n|_T}\big|H_n(u)-H_n(s)\big|.\end{aligned}\tag{6.13}$$

which converges to $4\delta^2H(T)$ as $n\to\infty$. Further it holds

$$\begin{aligned}&n\sum_{l=1,2}\mathbb{E}\Big[\sum_{i,j:t^{(l)}_{i,n}\wedge t^{(3-l)}_{j,n}\le T}\int_{t^{(l)}_{i-1,n}}^{t^{(l)}_{i,n}}\|\sigma_s-\sigma(r)_s\|^2ds\big|\mathcal{I}^{(3-l)}_{j,n}\big|\mathbb{1}_{\{\mathcal{I}^{(l)}_{i,n}\cap\mathcal{I}^{(3-l)}_{i,n}\neq\emptyset\}}\mathbb{1}_{\Omega(\delta,N,K',r)^C}\Big|\mathcal{S}\Big] \\ &\quad\le KH_n(T)\mathbb{P}(\Omega(\delta,N,K',r)^C|\mathcal{S})=KH_n(T)\mathbb{P}(\Omega(\delta,N,K',r)^C)\end{aligned}$$

which vanishes as $r, n \to \infty$ for any $\delta > 0$ because of $\mathbb{P}(\Omega(\delta, N, K', r)) \to 1$ for $N, K', r \to \infty$ and any $\delta > 0$ since σ is càdlàg. Combining this with (6.13) then yields that the last sum in (6.12) vanishes as $r, n \to \infty$ because δ can be chosen arbitrarily small. $\square$

Proposition 6.3. *If Condition 3.1 is fulfilled, we have on $\Omega_T^{(d)}$ the $\mathcal{X}$-stable convergence*

$$n \sum_{t_{i,n}^{(1)} \wedge t_{j,n}^{(2)} \leq T} \left(\Delta_{i,n}^{(1)} N(q)\right)^2 \left(\Delta_{j,n}^{(2)} C\right)^2 + \left(\left(\Delta_{i,n}^{(1)} C\right)^2 \left(\Delta_{j,n}^{(2)} N(q)\right)^2\right) \mathbb{1}_{\{\mathcal{I}_{i,n}^{(1)} \cap \mathcal{I}_{j,n}^{(2)} \neq \emptyset\}}$$
$$\xrightarrow{\mathcal{L}-s} \sum_{p: S_p \leq T} \left(\left(\Delta X_{S_p}^{(1)}\right)^2 R^{(1)}(S_p) + \left(\Delta X_{S_p}^{(2)}\right)^2 R^{(2)}(S_p)\right)$$

as $n \to \infty$ and then $q \to \infty$. Here, the $R^{(l)}(S_p)$, $l = 1, 2$, are as defined in (3.5).

Proof. Step 1. Denote by $P_T^{(l)}(q)$ the number of jumps of $N^{(l)}(q)$ in $[0, T]$, by $(S_{q,p}^{(l)})_{p \leq P_T^{(l)}(q)}$ the jump times of $N^{(l)}(q)$ in $[0, T]$ ordered by the size of $\| \int_{\mathbb{R}^2} z \mu(S_{q,p}^{(l)}, dz) \|$ and set

$$Y_n^{(l)}(s) = \left((Z_n^{(l)}(s))^*, U_{n,-}^{(l)}(s), U_{n,+}^{(l)}(s)\right)^*, \; Y^{(l)}(s) = \left((Z^{(l)}(s))^*, U_-^{(l)}(s), U_+^{(l)}(s)\right)^*,$$

with $U_{n,-}^{(l)}(s) = (\overline{W}_s^{(l)} - \overline{W}_{\tau_{n,-}^{(l)}}^{(l)})/(\delta_{n,-}^{(l)}(s))^{1/2}$, $U_{n,+}^{(l)}(s) = (\overline{W}_{\tau_{n,-}^{(l)}}^{(l)} - \overline{W}_s^{(l)})/(\delta_{n,+}^{(l)}(s))^{1/2}$.

We begin by showing that Condition 3.1(iii) yields the $\mathcal{X}$-stable convergence of all the $Y_n^{(l)}(S_{q,p}^{(3-l)})$ to the respective $Y^{(l)}(S_{q,p}^{(3-l)})$ on $\Omega_T^{(d)}$, i.e. we have to show

$$\mathbb{E}\left[\Lambda f\left(\left(Y_n^{(2)}(S_{q,p}^{(1)})\right)_{p \leq P_T^{(1)}(q)}, \left(Y_n^{(1)}(S_{q,p}^{(2)})\right)_{p \leq P_T^{(2)}(q)}\right) \mathbb{1}_{\Omega_T^{(d)}}\right]$$
$$\to \widetilde{\mathbb{E}}\left[\Lambda f\left(\left(Y^{(2)}(S_{q,p}^{(1)})\right)_{p \leq P_T^{(1)}(q)}, \left(Y^{(1)}(S_{q,p}^{(2)})\right)_{p \leq P_T^{(2)}(q)}\right) \mathbb{1}_{\Omega_T^{(d)}}\right] \quad (6.14)$$

for all $\mathcal{X}$-measurable bounded random variables Λ and all bounded Lipschitz functions f. We will use the same techniques to prove this as were used in Bibinger and Vetter (2015) e.g. in the proof of Proposition 3. Similar arguments can also be found in Jacod and Protter (1998) (Lemma 6.2) and Jacod (2008) (Lemma 5.8).

Denote by $\Omega(q, m, n)$ the subset of $\Omega_T^{(d)}$ on which $2|\pi_n|_T < 1/m$ and where two different jumps $S_{q,p_1}^{(l_1)}, S_{q,p_2}^{(l_2)} \leq T$ are further apart than $2|\pi_n|_T$. As $\Omega(q, m, n) \to \Omega_T^{(d)}$ for $n \to \infty$ it suffices to prove (6.14) with the indicator $\mathbb{1}_{\Omega(q,m,n)}$ added in both expectations. Further we set (see the paragraph above (3.1) for the definition of $\overline{W}$)

$$B^{(l)}(m) = \bigcup_{S_{q,p}^{(l)} \leq T} \left(\max\{S_{q,p}^{(l)} - 1/m, 0\}, \min\{S_{q,p}^{(l)} + 1/m, T\}\right],$$
$$\overline{W}^{(l)}(m)_t = \int_0^t \mathbb{1}_{B^{(l)}(m)}(s) d\overline{W}^{(l)}(s).$$

Let $\mathcal{G}(m)$ denote the σ-algebra generated by $\overline{W}(m)$ and the jump times $S_{q,p}^{(l)} \leq T$. By conditioning on $\sigma(\mathcal{G}(m) \cup \mathcal{S})$ we see that for proving (6.14) with the indicator $\mathbb{1}_{\Omega(q,m,n)}$ added in both expectations it is sufficient to consider only $\mathcal{G}(m)$-measurable

Λ', as restricted to $\Omega(q,m,n)$ the $Y_n^{(l)}(S_p)$ are $\sigma(\mathcal{G}(m)\cup\mathcal{S})$-measurable. By Lemma 2.1 in Jacod and Protter (1998) we may in particular choose Λ' of the form

$$\Lambda' = \gamma(\overline{W}(m))\kappa\big((S_{q,p}^{(1)})_{p\le P_T^{(1)}(q)},(S_{q,p}^{(2)})_{p\le P_T^{(2)}(q)}\big). \tag{6.15}$$

As $\overline{W}^{(l)}(m)$ converges to 0 in L^1 as $m\to\infty$ and because γ,κ,f are bounded we obtain

$$\lim_{m\to\infty}\limsup_{n\to\infty}\Big|\mathbb{E}\big[\gamma(0)\kappa\big((S_{q,p}^{(1)})_{p\le P_T^{(1)}(q)},(S_{q,p}^{(2)})_{p\le P_T^{(2)}(q)}\big)f\big(\big((Y_n^{(l)}(S_{q,p}^{(3-l)}))_{p\le P_T^{(3-l)}(q)}\big)_{l=1,2}\big)\mathbb{1}_{\Omega_T^{(d)}}\big]$$
$$-\mathbb{E}\big[\gamma(\overline{W}(m))\kappa\big((S_{q,p}^{(1)})_{\le P_T^{(1)}(q)},(S_{q,p}^{(2)})_{\le P_T^{(2)}(q)}\big)f\big(\big((Y_n^{(l)}(S_{q,p}^{(3-l)}))_{\le P_T^{(3-l)}(q)}\big)_{l=1,2}\big)\mathbb{1}_{\Omega_T^{(d)}}\mathbb{1}_{\Omega(q,m,n)}\big]\Big|=0$$

and the analogous result for $Y_n^{(l)}(S_{q,p}^{(3-l)})$ replaced with $Y^{(l)}(S_{q,p}^{(3-l)})$. Hence it remains to prove

$$\mathbb{E}\big[\kappa\big((S_{q,p}^{(1)})_{p\le P_T^{(1)}(q)},(S_{q,p}^{(2)})_{p\le P_T^{(2)}(q)}\big)f\big(\big((Y_n^{(l)}(S_{q,p}^{(3-l)}))_{p\le P_T^{(3-l)}(q)}\big)_{l=1,2}\big)\mathbb{1}_{\Omega_T^{(d)}}\big]$$
$$\to\widetilde{\mathbb{E}}\big[\kappa\big((S_{q,p}^{(1)})_{p\le P_T^{(1)}(q)},(S_{q,p}^{(2)})_{p\le P_T^{(2)}(q)}\big)f\big(\big((Y^{(l)}(S_{q,p}^{(3-l)}))_{p\le P_T^{(3-l)}(q)}\big)_{l=1,2}\big)\mathbb{1}_{\Omega_T^{(d)}}\big] \tag{6.16}$$

for all bounded Lipschitz functions κ,f.

Further note that by another density argument it suffices to consider functions f of the form

$$f\big(\big((Y_n^{(l)}(S_{q,p}^{(3-l)}))_{p\le P_T^{(3-l)}(q)}\big)_{l=1,2}\big)$$
$$=\prod_{l=1,2}\prod_{p:S_{q,p}^{(3-l)}\le T}f_p^{(l)}(Z_n^{(l)}(S_{q,p}^{(3-l)}))\tilde f_p^{(l)}\big(U_{n,-}^{(l)}(S_{q,p}^{(3-l)}),U_{n,+}^{(l)}(S_{q,p}^{(3-l)})\big).$$

Then because the $U_{n,-}^{(l)}(S_{q,p}^{(3-l)})$, $U_{n,+}^{(l)}(S_{q,p}^{(3-l)})$, $U_{-}^{(l)}(S_{q,p}^{(3-l)})$, $U_{+}^{(l)}(S_{q,p}^{(3-l)})$ are i.i.d. $\mathcal{N}(0,1)$ distributed and independent of μ and $Z_n^{(l)}(S_{q,p}^{(3-l)})$ respectively $Z^{(l)}(S_{q,p}^{(3-l)})$, (6.16) becomes

$$\mathbb{E}\big[\kappa\big((S_{q,p}^{(1)})_{p\le P_T^{(1)}(q)},(S_{q,p}^{(2)})_{p\le P_T^{(2)}(q)}\big)\prod_{l=1,2}\prod_{p\le P_T^{(3-l)}(q)}f_p^{(l)}(Z_n^{(l)}(S_{q,p}^{(3-l)}))\mathbb{1}_{\Omega_T^{(d)}}\big]$$
$$\to\widetilde{\mathbb{E}}\big[\kappa\big((S_{q,p}^{(1)})_{p\le P_T^{(1)}(q)},(S_{q,p}^{(2)})_{p\le P_T^{(3-l)}(q)}\big)\prod_{l=1,2}\prod_{p:S_{q,p}^{(3-l)}\le T}f_p^{(l)}(Z^{(l)}(S_{q,p}^{(3-l)}))\mathbb{1}_{\Omega_T^{(d)}}\big].$$

This is exactly Condition 3.1(iii) as conditional on the event that there are k_l jumps of $N^{(l)}(q)$, $l=1,2$, in $[0,T]$ all the $S_{q,p}^{(l)}$ are independent uniformly distributed on $[0,T]$. Note that the second expectation can be written in the form (3.4) as the $Z^{(l)}(s)$ are independent of the $S_{q,p}^{(l)}$ and of each other. Hence we have shown

$$\big(\big((Y_n^{(l)}(S_{q,p}^{(3-l)}))_{p\le P_T^{(3-l)}(q)}\big)_{l=1,2}\big)\xrightarrow{\mathcal{L}-s}\big(\big((Y^{(l)}(S_{q,p}^{(3-l)}))_{p\le P_T^{(3-l)}(q)}\big)_{l=1,2}\big). \tag{6.17}$$

Step 2. We reconsider the discretized volatility process $\sigma(r)$ from the proof of Proposition 6.2 and set

$$\tilde\sigma(r)_s=\begin{cases}\sigma_{S_{q,p}^{(l)}} & \text{if } s\in[S_{q,p}^{(l)},\lceil S_{q,p}^{(l)}/2^r\rceil/2^r)\\ \sigma(r)_s & \text{otherwise}\end{cases},\quad \widetilde C(r)_t=\int_0^t\tilde\sigma(r)_s ds.$$

Denote by $\Omega(q,r,n)$ the subset of $\Omega_T^{(d)}$ where two different jumps $S_{q,p_1}^{(l_1)} \neq S_{q,p_2}^{(l_2)}$ are further apart than $2|\pi_n|_T$ and the jump times $S_{q,p}^{(l)}$ are further away than $2|\pi_n|_T$ from the discontinuities $k/2^r$ of $\sigma(r)$. On this set we get

$$n \sum_{l=1,2} \sum_{i,j: t_{i,n}^{(l)} \wedge t_{j,n}^{(3-l)} \leq T} \big(\Delta_{i,n}^{(l)} N(q)\big)^2 \big(\Delta_{j,n}^{(3-l)} \widetilde{C}(r)\big)^2 \mathbb{1}_{\{\mathcal{I}_{i,n}^{(l)} \cap \mathcal{I}_{i,n}^{(3-l)} \neq \emptyset\}} \mathbb{1}_{\Omega(q,r,n)}$$
$$= \sum_{l=1,2} \sum_{p \leq P_T^{(l)}(q)} \big(\Delta N^{(l)}(q)_{S_{q,p}^{(l)}}\big)^2 \widetilde{R}_n^{(3-l)}(S_{q,p}^{(l)}, r) \mathbb{1}_{\Omega(q,r,n)} \tag{6.18}$$

where

$$\widetilde{R}_n^{(l)}(s,r) = (\tilde{\sigma}_{s-}^{(l)}(r))^2 \eta_{n,-}^{(l)}(s) + \Big(\tilde{\sigma}_{s-}^{(l)}(r)(\delta_{n,-}^{(l)}(s))^{1/2} U_{n,-}^{(l)}(s)$$
$$+ \tilde{\sigma}_s^{(l)}(r)(\delta_{n,+}^{(l)}(s))^{1/2} U_{n,+}^{(l)}(s)\Big)^2 + (\tilde{\sigma}_s^{(l)}(r))^2 \eta_{n,+}^{(l)}(s), \quad s \in [0,T], \quad l = 1,2.$$

From (6.17) and Proposition 2.2 in Podolskij and Vetter (2010) we get

$$\Big(N(q), \tilde{\sigma}(r), \big((S_{q,p}^{(l)})_{p \leq P_T^{(l)}(q)}\big)_{l=1,2}, \big((Y_n^{(l)}(S_{q,p}^{(3-l)}))_{p \leq P_T^{(3-l)}(q)}\big)_{l=1,2}\Big)$$
$$\xrightarrow{\mathcal{L}-s} \Big(N(q), \tilde{\sigma}(r), \big((S_{q,p}^{(l)})_{p \leq P_T^{(l)}(q)}\big)_{l=1,2}, \big((Y^{(l)}(S_{q,p}^{(3-l)}))_{p \leq P_T^{(3-l)}(q)}\big)_{l=1,2}\Big)$$

which yields, using the continuous mapping theorem,

$$\sum_{l=1,2} \sum_{p \leq P_T^{(l)}(q)} \big(\Delta N^{(l)}(q)_{S_{q,p}^{(l)}}\big)^2 \widetilde{R}_n^{(3-l)}(S_{q,p}^{(l)}, r) \mathbb{1}_{\Omega_T^{(d)}}$$
$$\xrightarrow{\mathcal{L}-s} \sum_{l=1,2} \sum_{p \leq P_T^{(l)}(q)} \big(\Delta N^{(l)}(q)_{S_{q,p}^{(l)}}\big)^2 \widetilde{R}^{(3-l)}(S_{q,p}^{(l)}, r) \mathbb{1}_{\Omega_T^{(d)}} \tag{6.19}$$

where $\widetilde{R}^{(l)}(s,r)$ is defined as $R^{(l)}(s)$ (see (3.5)) with σ replaced by $\tilde{\sigma}(r)$. Note that we may replace the left hand side of (6.19) by (6.18), since $\Omega(q,r,n) \to \Omega_T^{(d)}$ as $n \to \infty$.

But the convergence in (6.19) is even preserved if we replace $\tilde{\sigma}(r)$ by σ, because we get convergence in probability for both sides as $r \to \infty$: For the left hand side of (6.18) we use that the number of jumps of $N(q)$ and their size is bounded in probability and a similar argument as for the last step in the proof of Proposition 6.2. For the right hand side of (6.19) we use in addition that the first moments of the $Z^{(l)}(s)$ are uniformly bounded.

Step 3. We have

$$\sum_{p: S_p \leq T} \sum_{l=1,2} \big(\Delta X_{S_p}^{(3-l)}\big)^2 R^{(l)}(S_p) - \sum_{l=1,2} \sum_{p: S_{q,p}^{(l)} \leq T} \big(\Delta N^{(3-l)}(q)_{S_{q,p}^{(l)}}\big)^2 R^{(l)}(S_{q,p}^{(l)})$$
$$= \sum_{p: S_p \leq T} \sum_{l=1,2} \big(\Delta M^{(3-l)}(q)_{S_p}\big)^2 R^{(l)}(S_p). \tag{6.20}$$

Computing the $\mathcal{X}$-conditional expectation first and applying dominated convergence afterwards, it is easy to see that the right hand side of (6.20) converges to zero in probability as $q \to \infty$. This finishes the proof of Proposition 6.3. □

The following lemma is needed for the proof of Proposition 6.5.

Lemma 6.4. *Let Condition 2.1 be satisfied. Then there exists a constant K which is independent of (i,j) such that*

$$\mathbb{E}\big[\big(\Delta_{i,n}^{(l)}C\big)^2\big(\Delta_{j,n}^{(3-l)}M(q)\big)^2\big|\mathcal{S}\big] \leq Ke_q|\mathcal{I}_{i,n}^{(l)}||\mathcal{I}_{j,n}^{(3-l)}|, \quad l=1,2.$$

On the set $\Omega_T^{(d)}$ we further have

$$\mathbb{E}\big[\big(\Delta_{i,n}^{(l)}M(q)\big)^2\big(\Delta_{j,n}^{(3-l)}M(q')\big)^2\mathbb{1}_{\Omega_T^{(d)}}\big|\mathcal{S}\big] \leq Ke_qe_{q'}|\mathcal{I}_{i,n}^{(1)}||\mathcal{I}_{j,n}^{(2)}|. \tag{6.21}$$

Proof. If $\mathcal{I}_{i,n}^{(l)} \cap \mathcal{I}_{j,n}^{(3-l)} = \emptyset$ we use iterated expectations and Lemma 6.1. If the intervals do overlap, we use iterated expectations for the non-overlapping parts to obtain

$$\mathbb{E}\big[\big(\Delta_{i,n}^{(l)}C\big)^2\big(\Delta_{j,n}^{(3-l)}M(q)\big)^2\big|\mathcal{S}\big] \leq Ke_q\big(|\mathcal{I}_{i,n}^{(l)}||\mathcal{I}_{j,n}^{(3-l)}| - |\mathcal{I}_{i,n}^{(l)} \cap \mathcal{I}_{j,n}^{(3-l)}|^2\big)$$
$$+ \mathbb{E}\big[\big(C^{(l)}_{t_{i,n}^{(l)}\wedge t_{j,n}^{(3-l)}} - C^{(l)}_{t_{i-1,n}^{(l)}\vee t_{j-1,n}^{(3-l)}}\big)^2\big(M^{(3-l)}(q)_{t_{i,n}^{(l)}\wedge t_{j,n}^{(3-l)}} - M^{(3-l)}(q)_{t_{i-1,n}^{(l)}\vee t_{j-1,n}^{(3-l)}}\big)^2\big|\mathcal{S}\big]$$

and an analogous result for (6.21). The claim now follows from Lemma 8.2 in Jacod and Todorov (2009) which is basically Lemma 6.4 for $\mathcal{I}_{i,n}^{(l)} = \mathcal{I}_{j,n}^{(3-l)}$. The generalization to $q \neq q'$ here does not complicate the proof. □

Proposition 6.5. *If Condition 3.1 is fulfilled, we have*

$$\lim_{q\to\infty}\limsup_{n\to\infty}\mathbb{P}\big(\big\{\big|nV(f,\pi_n)_T - R(n,q)_T\big| > \varepsilon\big\} \cap \Omega_T^{(d)}\big) = 0 \quad \forall \varepsilon > 0$$

with

$$R(n,q)_T = n\sum_{i,j:t_{i,n}^{(1)}\wedge t_{j,n}^{(2)}\leq T}\big(\big(\Delta_{i,n}^{(1)}C\big)^2\big(\Delta_{j,n}^{(2)}C\big)^2 + \big(\Delta_{i,n}^{(1)}N(q)\big)^2\big(\Delta_{j,n}^{(2)}C\big)^2$$
$$+ \big(\Delta_{i,n}^{(1)}C\big)^2\big(\Delta_{j,n}^{(2)}N(q)\big)^2\big)\mathbb{1}_{\{\mathcal{I}_{i,n}^{(1)}\cap\mathcal{I}_{i,n}^{(2)}\neq\emptyset\}}.$$

Proof. Since γ is bounded by Condition 2.1 we can write

$$X = X_0 + B(q') + C + M(q')$$

on $[0,T]$ for some positive number q' (not necessarily an integer) which yields

$$N(q) = B(q') - B(q) + M(q') - M(q). \tag{6.22}$$

We apply inequality (6.8) with

$$a_l = 0, \quad b_l = \Delta_{i,n}^{(l)}B(q), \quad c_l = \Delta_{i,n}^{(l)}M(q), \quad d_l = \Delta_{i,n}^{(l)}C + \Delta_{i,n}^{(l)}N(q).$$

Then, we have

$$\rho n\sum_{i,j:t_{i,n}^{(1)}\wedge t_{j,n}^{(2)}\leq T}\big(\Delta_{i,n}^{(1)}C + \Delta_{i,n}^{(1)}N(q)\big)^2\big(\Delta_{j,n}^{(2)}C + \Delta_{j,n}^{(2)}N(q)\big)\big)^2\mathbb{1}_{\{\mathcal{I}_{i,n}^{(1)}\cap\mathcal{I}_{i,n}^{(2)}\neq\emptyset\}} \tag{6.23}$$
$$\leq \rho n\sum_{i,j:t_{i,n}^{(1)}\wedge t_{j,n}^{(2)}\leq T}4\big((\Delta_{i,n}^{(1)}C)^2 + (\Delta_{i,n}^{(1)}N(q))^2\big)\big((\Delta_{j,n}^{(2)}C)^2 + (\Delta_{j,n}^{(2)}N(q))^2\big)$$
$$\times \mathbb{1}_{\{\mathcal{I}_{i,n}^{(1)}\cap\mathcal{I}_{i,n}^{(2)}\neq\emptyset\}},$$

and the latter term is bounded in probability by Propositions 6.2 and 6.3. Hence, it converges to zero for $\rho \to 0$.

We also get for $l = 1, 2$ using (6.22), Lemma 6.1 and Lemma 6.4,

$$
\begin{aligned}
&\mathbb{E}\Big[c_\rho n \sum_{i,j: t^{(3-l)}_{i,n} \wedge t^{(l)}_{j,n} \leq T} \big((\Delta^{(3-l)}_{i,n} B(q))^2 + (\Delta^{(3-l)}_{i,n} M(q))^2 \big) \\
&\quad \times \big((\Delta^{(l)}_{j,n} B(q))^2 + (\Delta^{(l)}_{j,n} M(q))^2 + (\Delta^{(l)}_{j,n} C + \Delta^{(l)}_{j,n} N(q))^2 \big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{1}_{\{\mathcal{I}^{(3-l)}_{i,n} \cap \mathcal{I}^{(l)}_{i,n} \neq \emptyset\}} \mathbb{1}_{\Omega^{(d)}_T} \big| \mathcal{S}\Big] \\
&\leq c_\rho n \sum_{i,j: t^{(3-l)}_{i,n} \wedge t^{(l)}_{j,n} \leq T} \big(K_q |\mathcal{I}^{(3-l)}_{i,n}| + K e_q \big) |\mathcal{I}^{(3-l)}_{i,n}| \\
&\quad \times \big(K_q |\mathcal{I}^{(l)}_{j,n}| + K e_q + 2K + 8(K_q + K_{q'}) |\mathcal{I}^{(l)}_{j,n}| + 8K(e_q + e_{q'}) \big) |\mathcal{I}^{(l)}_{j,n}| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{1}_{\{\mathcal{I}^{(3-l)}_{i,n} \cap \mathcal{I}^{(l)}_{i,n} \neq \emptyset\}} \\
&\leq c_\rho \big(K_q |\pi_n|_T + K e_q \big) \big(H_n(T) + O_{\mathbb{P}}(n (|\pi_n|_T)^2) \big),
\end{aligned}
$$

where the latter bound converges to zero for $n \to \infty$ and then $q \to \infty$. Therefore, inequality (6.8) shows that only the terms as in (6.23) remain in the limit. On $\Omega^{(d)}_T$, the terms that occur in (6.23) but not in $R(n,q)_T$ are of the form

$$
\begin{aligned}
n &\sum_{i,j: t^{(l)}_{i,n} \wedge t^{(3-l)}_{j,n} \leq T} \big((\Delta^{(l)}_{i,n} C)^2 + 2 (\Delta^{(l)}_{i,n} C)(\Delta^{(l)}_{i,n} N(q)) + (\Delta^{(l)}_{i,n} N(q))^2 \big) \\
&\qquad \times \big((\Delta^{(3-l)}_{j,n} C)(\Delta^{(3-l)}_{j,n} N(q)) \big) \mathbb{1}_{\{\mathcal{I}^{(l)}_{i,n} \cap \mathcal{I}^{(3-l)}_{j,n} \neq \emptyset\}}, \quad l = 1, 2.
\end{aligned}
\tag{6.24}
$$

From similar arguments as before, we obtain that the sum over terms containing the product $(\Delta^{(l)}_{i,n} N(q))(\Delta^{(3-l)}_{j,n} N(q))$ converges to zero because we are on $\Omega^{(d)}_T$.

For the remaining terms we obtain

$$
\begin{aligned}
n &\sum_{i,j: t^{(l)}_{i,n} \wedge t^{(3-l)}_{j,n} \leq T} (\Delta^{(l)}_{i,n} C)^2 (\Delta^{(3-l)}_{j,n} C)(\Delta^{(3-l)}_{j,n} N(q)) \mathbb{1}_{\{\mathcal{I}^{(l)}_{i,n} \cap \mathcal{I}^{(3-l)}_{j,n} \neq \emptyset\}} \\
&\leq \Big(\sup_{j: t^{(3-l)}_{j,n} \leq T} \Delta^{(3-l)}_{j,n} C \Big) n \sum_{i,j: t^{(l)}_{i,n} \wedge t^{(3-l)}_{j,n} \leq T} (\Delta^{(l)}_{i,n} C)^2 (\Delta^{(3-l)}_{j,n} N(q)) \mathbb{1}_{\{\mathcal{I}^{(l)}_{i,n} \cap \mathcal{I}^{(3-l)}_{j,n} \neq \emptyset\}}
\end{aligned}
$$

where the right hand side tends to zero as $n \to \infty$ for all $q > 0$ because the supremum vanishes as C is continuous and because the sum converges stably in law on $\Omega^{(d)}_T$ to

$$
\sum_{S^{(3-l)}_{p,q} \leq T} \Delta N^{(3-l)}(q)_{S^{(3-l)}_{p,q}} R^{(l)}(S^{(3-l)}_{p,q}),
$$

where $(S^{(3-l)}_{p,q})_{p \in \mathbb{N}}$ denotes an enumeration of the jump times of $N^{(3-l)}(q)$. The stable convergence can be proven similarly as Proposition 6.5 and follows from Condition 3.1(iii). □

Proof of Theorem 3.2. This is a direct consequence of Propositions 6.2, 6.3 and 6.5 as well as (2.3). □

6.4 Proof for the testing procedure

Proof of Theorem 4.2. For proving (4.4) we will show

$$\widetilde{\mathbb{P}}\big(nV(f,\pi_n)_T > A_{n,T} + \widehat{Q}_{n,T}(1-\alpha)\big|F^{(d)}\big) \to \alpha, \tag{6.25}$$

for all $F^{(d)} \subset \Omega_T^{(d)}$ with $\mathbb{P}(F^{(d)}) > 0$. To this end, we will prove in the sequel that Condition 3.1 ensures

$$A_{n,T} \xrightarrow{\mathbb{P}} \widetilde{C}_T, \tag{6.26}$$

as well as

$$\widehat{Q}_{n,T}(\alpha)\mathbb{1}_{\Omega_T^{(d)}} \xrightarrow{\widetilde{\mathbb{P}}} Q(\alpha)\mathbb{1}_{\Omega_T^{(d)}} \tag{6.27}$$

for each $\alpha \in [0,1]$, where $Q(\alpha)$ denotes the $\mathcal{X}$-conditional α quantile of $\widetilde{D}_T$ on $\Omega_T^{(d)}$ defined in (4.2).

Then, Theorem 3.2 and (6.26) yield the $\mathcal{X}$-stable convergence

$$nV(f,\pi_n)_T - A_{n,T} \xrightarrow{\mathcal{L}-s} \widetilde{D}_T$$

on $\Omega_T^{(d)}$ from which we obtain together with (6.27)

$$(nV(f,\pi_n)_T - A_{n,T}, \widehat{Q}_{n,T}(1-\alpha))\mathbb{1}_{\Omega_T^{(d)}} \xrightarrow{\mathcal{L}-s} (\widetilde{D}_T, Q(1-\alpha))\mathbb{1}_{\Omega_T^{(d)}}$$

by Proposition 2.5(i) in Podolskij and Vetter (2010). Then, finally,

$$\begin{aligned} &\widetilde{\mathbb{P}}\big(\{nV(f,\pi_n)_T > A_{n,T} + \widehat{Q}_{n,T}(1-\alpha)\} \cap F^{(d)}\big) \\ &\quad \to \widetilde{\mathbb{P}}\big(\{\widetilde{D}_T > Q(1-\alpha)\} \cap F^{(d)}\big) = \alpha\mathbb{P}\big(F^{(d)}\big) \end{aligned}$$

where the last equality follows from the definition of $Q(\alpha)$. This implies (6.25) and hence (4.4).

The consistency claim (4.5) follows from the fact that $\widetilde{\Phi}_{n,T}^{(d)}$ converges to a strictly positive limit on $\Omega_T^{(j)}$ by Theorem 2.2 while $c_n = O_{\widetilde{\mathbb{P}}}(n^{-1})$. □

Proof of (6.26). Looking at the proof of Proposition 6.5, it is enough to show that

$$\begin{aligned} n \sum_{i,j: t_{i,n}^{(1)} \wedge t_{j,n}^{(2)} \le T} &\big(\Delta_{i,n}^{(1)} C + \Delta_{i,n}^{(1)} N(q)\big)^2 \big(\Delta_{j,n}^{(2)} C + \Delta_{j,n}^{(2)} N(q)\big)^2 \\ &\times \mathbb{1}_{\{|\Delta_{i,n}^{(1)} X| \le \beta |\mathcal{I}_{i,n}^{(1)}|^{\varpi} \wedge |\Delta_{j,n}^{(2)} X| \le \beta |\mathcal{I}_{j,n}^{(2)}|^{\varpi}\}} \mathbb{1}_{\{\mathcal{I}_{i,n}^{(1)} \cap \mathcal{I}_{j,n}^{(2)} \neq \emptyset\}} \end{aligned} \tag{6.28}$$

converges to $\widetilde{C}_T$.

We first deal with the terms involving big jumps. Let $S_{q,p}^{(l)}$, $p = 1, \ldots, P_T^{(l)}(q)$, denote the (finitely many) jump times of $N^{(l)}(q)$ in $[0,T]$. It then holds

$$\Big| n \sum_{l=1,2} \sum_{i,j:t_{i,n}^{(l)} \wedge t_{j,n}^{(3-l)} \leq T} \Delta_{i,n}^{(l)} N(q) \big(\Delta_{i,n}^{(l)} C + \Delta_{i,n}^{(l)} N(q)\big) \big(\Delta_{j,n}^{(3-l)} C + \Delta_{j,n}^{(3-l)} N(q)\big)^2$$

$$\times \mathbb{1}_{\{|\Delta_{i,n}^{(l)} X| \leq \beta |\mathcal{I}_{i,n}^{(l)}|^{\varpi} \wedge |\Delta_{j,n}^{(3-l)} X| \leq \beta |\mathcal{I}_{j,n}^{(3-l)}|^{\varpi}\}} \mathbb{1}_{\{\mathcal{I}_{i,n}^{(l)} \cap \mathcal{I}_{j,n}^{(3-l)} \neq \emptyset\}} \Big| \mathbb{1}_{\Omega(n,q,T)}$$

$$\leq n \sum_{l=1,2} \sum_{p=1}^{P_T^{(l)}(q)} \big|\Delta N^{(l)}(q)_{S_{q,p}^{(l)}}\big| \mathbb{1}_{\{|\Delta^{(l)}_{i_n^{(l)}(S_{q,p}^{(l)}),n} X| \leq \beta |\mathcal{I}^{(l)}_{i_n^{(l)}(S_{q,p}^{(l)}),n}|^{\varpi}\}}$$

$$\times \big|\Delta^{(l)}_{i_n^{(l)}(S_{q,p}^{(l)}),n} (C + N(q))\big| \sum_{j:t_{j,n}^{(3-l)} \leq T} \big(\Delta_{j,n}^{(3-l)} C + \Delta_{j,n}^{(3-l)} N(q)\big)^2 \mathbb{1}_{\Omega(n,q,T)} \quad (6.29)$$

where $\Omega(n,q,T)$ denotes the set where two jumps of $N(q)$ in $[0,T]$ are further apart than $2|\pi_n|_T$. (6.29) converges in probability to zero as $n \to \infty$ because of

$$|\Delta^{(l)}_{i_n^{(l)}(S_{q,p}^{(l)}),n} X| \xrightarrow{\mathbb{P}_{\mathcal{X}}} |\Delta N^{(l)}(q)_{S_{q,p}^{(l)}}| > 0, \quad |\mathcal{I}^{(l)}_{i_n^{(l)}(S_{q,p}^{(l)}),n}| \xrightarrow{\mathbb{P}_{\mathcal{X}}} 0$$

where $\mathbb{P}_{\mathcal{X}}$ denotes convergence in $\mathcal{X}$-conditional probabilities. $\mathbb{P}(\Omega(n,q,T)) \to 1$ as $n \to \infty$ then yields that the terms involving big jumps in (6.28) vanish asymptotically. Hence only the terms involving squared increments of $C^{(1)}, C^{(2)}$ contribute in the limit.

Using Proposition 6.2 it remains to show

$$\widetilde{L}_T = n \sum_{i,j:t_{i,n}^{(1)} \wedge t_{j,n}^{(2)} \leq T} \big(\Delta_{i,n}^{(1)} C\big)^2 \big(\Delta_{j,n}^{(2)} C\big)^2$$

$$\times \mathbb{1}_{\{|\Delta_{i,n}^{(1)} X| > \beta |\mathcal{I}_{i,n}^{(1)}|^{\varpi} \vee |\Delta_{j,n}^{(2)} X| > \beta |\mathcal{I}_{j,n}^{(2)}|^{\varpi}\}} \mathbb{1}_{\{\mathcal{I}_{i,n}^{(1)} \cap \mathcal{I}_{j,n}^{(2)} \neq \emptyset\}} \xrightarrow{\mathbb{P}} 0. \quad (6.30)$$

The conditional Markov inequality plus an application of Lemma 6.1 give

$$\mathbb{P}\big(|\Delta_{i,n}^{(l)} X| > \beta |\mathcal{I}_{i,n}^{(l)}|^{\varpi} \big| \mathcal{S}\big) \leq K |\mathcal{I}_{i,n}^{(l)}|^{1-2\varpi}. \quad (6.31)$$

Using

$$\widetilde{L}_T \leq n \sum_{i_1,i_2:t_{i_1,n}^{(1)} \wedge t_{i_2,n}^{(2)} \leq T} \big(\Delta_{i_1,n}^{(1)} C\big)^2 \big(\Delta_{i_2,n}^{(2)} C\big)^2$$

$$\times \sum_{l=1,2} \mathbb{1}_{\{|\Delta_{i_l,n}^{(l)} X > \beta |\mathcal{I}_{i_l,n}^{(l)}|^{\varpi}\}} \mathbb{1}_{\{\mathcal{I}_{i_1,n}^{(1)} \cap \mathcal{I}_{i_2,n}^{(2)} \neq \emptyset\}}$$

and the generalized Hölder inequality, as well as Lemma 6.1 and (6.31), we get

$$\mathbb{E}\big[\widetilde{L}_T \big| \mathcal{S}\big] \leq K n \sum_{i_1,i_2:t_{i_1,n}^{(1)} \wedge t_{i_2,n}^{(2)} \leq T} |\mathcal{I}_{i_1,n}^{(1)}| |\mathcal{I}_{i_2,n}^{(2)}| \sum_{l=1,2} |\mathcal{I}_{i_l,n}^{(l)}|^{(1-2\varpi)/p'} \mathbb{1}_{\{\mathcal{I}_{i,n}^{(1)} \cap \mathcal{I}_{j,n}^{(2)} \neq \emptyset\}}$$

$$\leq K (|\pi_n|_T)^{(1-2\varpi)/p'} H_n(T) \quad (6.32)$$

for any $p' > 1$, which tends to zero by Condition 2.1 and Condition 3.1(ii). This yields (6.30). □

For the proof of (6.27) we need a few preliminary results which yield that the convergence of the empirical $\mathcal{X}$-conditional distribution on the $\widehat{D}_{T,n,m}$, $m=1,\ldots,M_n$ restricted to $\Omega_T^{(d)}$ to the $\mathcal{X}$-conditional distribution of $\widetilde{D}_T$ restricted to $\Omega_T^{(d)}$ follows from the convergence of the common empirical distribution of the $\widehat{Z}_{n,m}^{(l_j)}(s_j)$ to the common distribution of the $Z^{(l_j)}(s_j)$ provided in Condition 4.1. These results are proved in Lemma 6.6 and Proposition 6.7.

Lemma 6.6. *Suppose that* $A_{n,j} \xrightarrow{\widetilde{\mathbb{P}}} A_j$ *for* $\mathcal{F}$*-measurable* $A_{n,j}\in\mathbb{R}^d$, $\mathcal{X}$*-measurable* $A_j\in\mathbb{R}^d$, *and let* $S_j\in[0,T]$, $j=1,\ldots,J$, *be almost surely distinct* $\mathcal{X}$*-measurable random variables. Then, under Condition 4.1, it holds*

$$\widetilde{\mathbb{P}}\left(\left|\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{\varphi((A_{n,j},\widehat{Y}_{n,m}^{(l_j)}(S_j))_{j=1,\ldots,J})\le\Upsilon\}}-\widetilde{\mathbb{P}}\big(\varphi((A_j,Y^{(l_j)}(S_j))_{j=1,\ldots,J})\le\Upsilon\big|\mathcal{X}\big)\right|>\varepsilon\right)\to 0$$

for any $\mathcal{X}$*-measurable random variable* Υ, *any* $\varepsilon>0$ *and any continuous function* $\varphi:\mathbb{R}^{(d+6)\times J}\to\mathbb{R}$ *such that the* $\mathcal{X}$*-conditional distribution of* $\varphi((A_j,Y^{(l_j)}(S_j))_{j=1,\ldots,J})$ *is almost surely continuous. Here,*

$$\widehat{Y}_{n,m}^{(l_j)}(S_j)=\big((\widehat{Z}_{n,m}^{(l_j)}(S_j))^*,U_{n,m,-}^{(l_j)}(S_j),U_{n,m,+}^{(l_j)}(S_j)\big)^*,$$

and $Y^{(l_j)}(S_j)$ *is defined as in the proof of Proposition 6.3.*

Proof. First, note that Condition 4.1 implies that

$$\begin{aligned}&\widetilde{\mathbb{P}}\left(\left|\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{\widehat{Y}_{n,m}^{(l_j)}(s_j)\le x_j,\ j=1,\ldots,J\}}-\widetilde{\mathbb{P}}(Y^{(l_j)}(s_j)\le x_j,\ j=1,\ldots,J)\right|>\varepsilon\right) \qquad (6.33)\\ &\le\widetilde{\mathbb{P}}\Big(\Big|\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{\widehat{Y}_{n,m}^{(l_j)}(s_j)\le x_j,\ j=1,\ldots,J\}}-\widetilde{\mathbb{P}}\big(\widehat{Y}_{n,1}^{(l_j)}(s_j)\le x_j,\ j=1,\ldots,J\big|\mathcal{S}\big)\Big|>\frac{\varepsilon}{2}\Big)\\ &\quad+\widetilde{\mathbb{P}}\Big(\big|\widetilde{\mathbb{P}}\big(\widehat{Y}_{n,1}^{(l_j)}(s_j)\le x_j,\ j=1,\ldots,J\big|\mathcal{S}\big)-\widetilde{\mathbb{P}}\big(Y(s_j)\le x_j,\ j=1,\ldots,J\big)\big|>\frac{\varepsilon}{2}\Big)\end{aligned}$$

converges to zero as $n\to\infty$ for any s_j,x_j, $j=1,\ldots,J$. In fact, the $(\widehat{Y}_{n,m}^{(l_j)}(s_j))_{j=1,\ldots,J}$ are conditionally on $\mathcal{S}$ independent and identically distributed as m varies. Therefore, $M_n\to\infty$, the conditional Chebyshev inequality and dominated convergence ensure that the first term vanishes asymptotically. In the second term we may factorize probabilities as the U's are all independent of the Z's. Then the second term converges to zero by (4.3) and the fact that the U's are all normally distributed.

To shorten notation we set

$$\begin{aligned}\zeta_n&=\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{\varphi((A_{n,j},\widehat{Y}_{n,m}^{(l_j)}(S_j))_{j=1,\ldots,J})\le\Upsilon\}},\\ \zeta&=\widetilde{\mathbb{P}}\big(\varphi((A_j,Y^{(l_j)}(S_j))_{j=1,\ldots,J})\le\Upsilon\big|\mathcal{X}\big).\end{aligned}$$

The idea for the following steps is to approximate the function φ by piecewise constant functions, use (6.33) to prove the claim for those piecewise constant functions and to show that the convergence is preserved if we take limits. To formalize this approach let $K>0$, set $\square_k(K,r)=\{x\in\mathbb{R}^{6\times J}|x_{i,j}\in((k_{i,j}-1)2^{-r}K,k_{i,j}2^{-r}K],i\le$

$6, j \leq J\}$, $k = (k_1, \dots, k_J) \in \mathbb{Z}^{6\times J}$, $r \in \mathbb{N}$, and define

$$\begin{aligned}\zeta_n(K,r) &= \frac{1}{M_n}\sum_{m=1}^{M_n} \mathbb{1}_{\{\sum_{k\in\{-2^r,\dots,2^r\}^{6\times J}} \varphi((A_j,k_j 2^{-r}K)_{j\leq J})\mathbb{1}_{\{(\widehat{Y}^{(l_j)}_{n,m}(S_j))_{j\leq J})\in\square_k(K,r)\}}\leq\Upsilon\}}\\ &= \sum_{k\in\{-2^r,\dots,2^r\}^{6\times J}} \mathbb{1}_{\{\varphi((A_j,k_j 2^{-r}K)_{j\leq J})\leq\Upsilon\}} \frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{(\widehat{Y}^{(l_j)}_{n,m}(S_j))_{j\leq J})\in\square_k(K,r)\}},\\ \zeta(K,r) &= \sum_{k\in\{-2^r,\dots,2^r\}^{6\times J}} \mathbb{1}_{\{\varphi((A_j,k_j 2^{-r}K)_{j\leq J})\leq\Upsilon\}}\widetilde{\mathbb{P}}\big((Y^{(l_j)}(S_j))_{j\leq J})\in\square_k(K,r)\big|\mathcal{X}\big),\end{aligned}$$

where $\varphi((A_j, k_j 2^{-r}K)_{j\leq J})$ equals $\varphi((A_j,\cdot)_{j=1,\dots,J})$ evaluated at the rightmost vertex of $\square_k(K,r)$.

Using this notation it remains to show

$$\lim_{K\to\infty}\limsup_{r\to\infty}\limsup_{n\to\infty}\widetilde{\mathbb{P}}(|\zeta_n-\zeta_n(K,r)|>\varepsilon)=0 \quad \forall\varepsilon>0, \tag{6.34}$$

$$\lim_{n\to\infty}\widetilde{\mathbb{P}}(|\zeta_n(K,r)-\zeta(K,r)|>\varepsilon)=0 \quad \forall K,\varepsilon>0\ \forall r\in\mathbb{N}, \tag{6.35}$$

$$\lim_{K\to\infty}\limsup_{r\to\infty}\widetilde{\mathbb{P}}(|\zeta(K,r)-\zeta|>\varepsilon)=0 \quad \forall\varepsilon>0. \tag{6.36}$$

Step 1. We start by showing (6.35). It holds

$$\begin{aligned}&\widetilde{\mathbb{P}}(|\zeta_n(K,r)-\zeta(K,r)|>\varepsilon)\\ &\quad\leq \sum_{k\in\{-2^r,\dots,2^r\}^{6\times J}}\widetilde{\mathbb{E}}\Big[\widetilde{\mathbb{P}}\Big(\Big|\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{(\widehat{Y}^{(l_j)}_{n,m}(S_j))_{j=1,\dots,J})\in\square_k(K,r)\}}\\ &\qquad\qquad -\widetilde{\mathbb{P}}\big((Y^{(l_j)}(S_j))_{j=1,\dots,J})\in\square_k(K,r)\big|\mathcal{X}\big)\Big|>\varepsilon/(2^{r+1}+1)^{6J}\Big|\mathcal{X}\Big)\Big]\end{aligned}$$

where each conditional probability vanishes almost surely as $n\to\infty$ by (6.33) because the events

$$\{(\widehat{Y}^{(l_j)}_{n,m}(S_j))_{j=1,\dots,J})\in\square_k(K,r)\},\ \{(Y^{(l_j)}(S_j))_{j=1,\dots,J})\in\square_k(K,r)\}$$

may be written as unions/differences of events of the form

$$\{\widehat{Y}^{(l_j)}_{n,m}(S_j))_{j=1,\dots,J})\leq v_{k,i}(K,r)\},\ \{Y^{(l_j)}(S_j))_{j=1,\dots,J})\leq v_{k,i}(K,r)\}$$

where $(v_{k,i}(K,r))_i$ denotes the vertices of the cuboid $\square_k(K,r)$. Note that conditioning on $\mathcal{X}$ here simply has the effect of fixing the S_j. (6.35) then follows by dominated convergence.

Step 2. Next we show (6.34). It holds

$$\begin{aligned}|\zeta_n-\zeta_n(K,r)| &\leq \frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{(\widehat{Y}^{(l_j)}_{n,m}(S_j))_{j=1,\dots,J}\notin[-K,K]^{6\times J}\vee(A_j)_{j=1,\dots,J}\notin[-K,K]^{d\times J}\}}\\ &\quad+\sum_{k\in\{-2^r,\dots,2^r\}^{6\times J}}\big|\mathbb{1}_{\{\varphi((A_{n,j},\widehat{Y}^{(l_j)}_{n,m}(S_j))_{j=1,\dots,J})\leq\Upsilon\}}-\mathbb{1}_{\{\varphi((A_j,k_j 2^{-r}K)_{j=1,\dots,J})\leq\Upsilon\}}\big|\\ &\quad\times\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{(\widehat{Y}^{(l_j)}_{n,m}(S_j))_{j=1,\dots,J})\in\square_k(K,r)\}}\mathbb{1}_{\{(A_j)_{j=1,\dots,J}\in[-K,K]^{d\times J}\}}.\end{aligned} \tag{6.37}$$

The first term in (6.37) becomes arbitrarily small, because for $n \to \infty$ we obtain from (6.33)

$$\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{(\widehat{Y}_{n,m}^{(l_j)}(S_j))_{j=1,\dots,J}\notin[-K,K]^{6\times J}\}} \xrightarrow{\widetilde{\mathbb{P}}} \widetilde{\mathbb{P}}\big((Y^{(l_j)}(S_j))_{j=1,\dots,J}\notin[-K,K]^{6\times J}\big|\mathcal{X}\big)$$

as in Step 1, where the right hand side afterwards vanishes as $K\to\infty$.

Denote the second term in (6.37) by $\zeta_n'(K,r)$. Then it holds for $\delta>0$

$$\begin{aligned}\zeta_n'(K,r) \le \mathbb{1}_{\{\|(A_{n,j}-A_j)_{j=1,\dots,J}\|\ge\delta\}} + \frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{(\widehat{Y}_{n,m}^{(l_j)}(S_j))_{j=1,\dots,J}\in[-K,K]^{6\times J}\}}\\ \times\mathbb{1}_{\{|\varphi((A_j,\widehat{Y}_{n,m}^{(l_j)}(S_j))_{j=1,\dots,J})-\Upsilon|\le\rho(K+\delta,\delta,2^{-r}K)\}}\mathbb{1}_{\{(A_j)_{j=1,\dots,J}\in[-K,K]^{d\times J}\}}\end{aligned}\tag{6.38}$$

where

$$\rho(K,a,b)=\sup_{(x,y),(x',y')\in[-K,K]^{(d+6)\times J}:\|x-x'\|<a,\|y-y'\|_\infty\le b}|\varphi((x_j,y_j)_{j=1,\dots,J})-\varphi((x_j',y_j')_{j=1,\dots,J})|.$$

The first summand in (6.38) vanishes as $n\to\infty$ for all $\delta>0$ since $A_{n,j}\xrightarrow{\mathbb{P}}A_j$. Denoting the second summand in (6.38) by $\zeta_n''(K,r,\delta)$ we obtain further

$$\begin{aligned}\zeta_n''(K,r,\delta) &\le \frac{1}{M_n}\sum_{m=1}^{M_n}\sum_{k\in\{-2^r,\dots,2^r\}^{6\times J}}\mathbb{1}_{\{\min_{x\in\square_k(K,r)}|\varphi((A_j,x_j)_{j=1,\dots,J})-\Upsilon|\le\rho(K+\delta,\delta,2^{-r}K)\}}\\ &\qquad\times\mathbb{1}_{\{(\widehat{Y}_{n,m}^{(l_j)}(S_j))_{j=1,\dots,J}\in\square_k(K,r)\}}\mathbb{1}_{\{(A_j)_{j=1,\dots,J}\in[-K,K]^{d\times J}\}}\\ &\le \sum_{k\in\{-2^r,\dots,2^r\}^{6\times J}}\Big|\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{(\widehat{Y}_{n,m}^{(l_j)}(S_j))_{j=1,\dots,J}\in\square_k(K,r)\}}\\ &\qquad -\widetilde{\mathbb{P}}\big((Y^{(l_j)}(S_j))_{j=1,\dots,J}\in\square_k(K,r)\big|\mathcal{X}\big)\Big|\\ &\quad+\sum_{k\in\{-2^r,\dots,2^r\}^{6\times J}}\mathbb{1}_{\{\min_{x\in\square_k(K,r)}|\varphi((A_j,x_j)_{j=1,\dots,J})-\Upsilon|\le\rho(K+\delta,\delta,2^{-r}K)\}}\\ &\qquad\times\widetilde{\mathbb{P}}\big((Y^{(l_j)}(S_j))_{j=1,\dots,J}\in\square_k(K,r)\big|\mathcal{X}\big)\mathbb{1}_{\{(A_j)_{j=1,\dots,J}\in[-K,K]^{d\times J}\}}\end{aligned}\tag{6.39}$$

where the first sum vanishes for $n\to\infty$ as shown in Step 1. Denote the second sum in (6.39) by $\zeta_n'''(K,r,\delta)$. Then we finally obtain

$$\zeta_n'''(K,r,\delta)\le\widetilde{\mathbb{P}}\big(|\varphi((A_j,Y^{(l_j)}(S_j))_{j=1,\dots,J})-\Upsilon|\le 2\rho(K+\delta,\delta,2^{-r}K)\big|\mathcal{X}\big)$$

which converges to zero because $\varphi((A_j,Y^{(l_j)}(S_j))_{j=1,\dots,J})$ possesses almost surely a continuous $\mathcal{X}$-conditional distribution by assumption and because of

$$\lim_{\delta\to0}\limsup_{r\to\infty}\rho(K+\delta,\delta,2^{-r}K)=0$$

for all $K>0$ as φ is continuous. Hence altogether we have shown

$$\lim_{K\to\infty}\limsup_{\delta\to0}\limsup_{r\to\infty}\limsup_{n\to\infty}\widetilde{\mathbb{P}}(|\zeta_n'(K,r)|>\varepsilon)$$

for all $\varepsilon>0$ which yields (6.34).

Step 3. It holds

$$\zeta(K,r)=\widetilde{\mathbb{P}}\Big(\sum_{k\in\{-2^r,\dots,2^r\}^{6\times J}}\varphi((A_j,k_j2^{-r}K)_{j\leq J})\mathbb{1}_{\{(Y^{(l_j)}(S_j))_{j=1,\dots,J}\in\square_k(K,r)\}}\leq\Upsilon\Big|\mathcal{X}\Big).$$

Hence

$$\begin{aligned}|\zeta(K,r)-\zeta|\leq{}&\widetilde{\mathbb{P}}((Y^{(l_j)}(S_j))_{j=1,\dots,J})\notin[-K,K]^{6\times J}|\mathcal{X})\\&+\widetilde{\mathbb{P}}(|\varphi((A_j,x_j)_{j=1,\dots,J})-\Upsilon|\leq\tilde{\rho}(K,r,(A_j)_{j=1,\dots,J})|\mathcal{X})\end{aligned}\tag{6.40}$$

where

$$\tilde{\rho}(K,r,(A_j)_{j=1,\dots J})=\sup_{y,y'\in[-K,K]^{6\times J}:\|y-y'\|\leq2^{-r}K}|\varphi((A_j,y_j)_{j=1,\dots,J})-\varphi((A_j,y'_j)_{j=1,\dots,J})|.$$

The first term on the right hand side of (6.40) vanishes almost surely as $K\to\infty$. Further it holds

$$\lim_{r\to\infty}\tilde{\rho}(K,r,(A_j)_{j=1,\dots,J})=0\ \text{ almost surely}$$

because $y\mapsto\varphi((A_j,y_j)_{j=1,\dots,J})$ is uniformly continuous on $[-K,K]^{6\times J}$ for fixed ω. Using this result the second term in (6.40) vanishes almost surely as $r\to\infty$ for any $K>0$ because the $\mathcal{X}$-conditional distribution of $\varphi((A_j,Y^{(l_j)}(S_j))_{j=1,\dots,J})$ is almost surely continuous by assumption. (6.36) then follows by dominated convergence. □

Proposition 6.7. *Suppose that Condition 4.1 is satisfied. Then*

$$\widetilde{\mathbb{P}}\big(\big\{\big|\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{\widehat{D}_{T,n,m}\leq\Upsilon\}}-\widetilde{\mathbb{P}}\big(\widetilde{D}_T\leq\Upsilon|\mathcal{X}\big)\big|>\varepsilon\big\}\cap\Omega_T^{(d)}\big)\to0\tag{6.41}$$

for any $\mathcal{X}$-measurable random variable Υ and all $\varepsilon>0$.

Proof. Step 1. We use S_j, $j=1,\dots,J$, to denote the jump times of the J largest jumps of X in $[0,T]$ with respect to a fixed norm on $\mathbb{R}$. Recall that on $\Omega_T^{(d)}$ only one component of X jumps at S_j, and we use l_j as the index of the component involving the jth jump. Therefore, setting

$$\begin{aligned}A_{n,j}&=\big(\widehat{\Delta}_nX^{(l_j)}(S_j),\hat{\sigma}_n^{(3-l_j)}(S_j,-),\hat{\sigma}_n^{(3-l_j)}(S_j,+)\big)\\A_j&=\big(\Delta X_{S_j}^{(l_j)},\sigma_{S_j-}^{(3-l_j)},\sigma_{S_j}^{(3-l_j)}\big),\end{aligned}$$

where the (consistent) estimators have been defined in (4.1), and defining φ via

$$\varphi((A_j,Y^{(l_j)}(S_j))_{j=1,\dots,J})=\sum_{j=1}^{J}(\Delta X_{S_j}^{(l_j)})^2R^{(3-l_j)}(S_j),$$

Lemma 6.6 proves

$$\widetilde{\mathbb{P}}\big(\big\{\big|\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{Y(J,n,m)\leq\Upsilon\}}-\widetilde{\mathbb{P}}\big(Y(J)\leq\Upsilon|\mathcal{X}\big)\big|>\varepsilon\big\}\cap\Omega_T^{(d)}\big)\to0\tag{6.42}$$

as the $\mathcal{X}$-conditional distribution of $\sum_{j=1}^{J}(\Delta X_{S_j}^{(l_j)})^2 R^{(3-l_j)}(S_j)$ is continuous on $\Omega_T^{(d)}$. In (6.42) we have used the notation

$$Y(J,n,m)=\sum_{j=1}^{J}\big(\widehat{\Delta}_n X^{(l_j)}(S_j)\big)^2\widehat{R}_{n,m}^{(3-l_j)}(S_j),\quad Y(J)=\sum_{j=1}^{J}\big(\Delta X_{S_j}^{(l_j)}\big)^2 R^{(3-l_j)}(S_j).$$

Step 2. We prove

$$\lim_{J\to\infty}\limsup_{n\to\infty}\frac{1}{M_n}\sum_{m=1}^{M_n}\widetilde{\mathbb{P}}\big(\big|Y(J,n,m)-\widehat{D}_{T,n,m}\big|>\varepsilon\big)\to 0 \tag{6.43}$$

for all $\varepsilon>0$. Denote by $\Omega(q,J,n)$ the set on which the jumps of $N(q)$ are among the J largest jumps and two different jumps of $N(q)$ are further apart than $|\pi_n|_T$. Obviously, $\mathbb{P}(\Omega(q,J,n))\to 1$ for $J,n\to\infty$ and any $q>0$. On the set $\Omega(q,J,n)$ we have

$$\begin{aligned}
&\big|Y(J,n,m)-\widehat{D}_{T,n,m}\big|\\
&\leq\sum_{l=1,2}\sum_{t_{i,n}^{(l)}\leq T,\nexists j:S_j\in\mathcal{I}_{i,n}^{(l)}}\big(\Delta_{i,n}^{(l)}B(q)+\Delta_{i,n}^{(l)}C+\Delta_{i,n}^{(l)}M(q)\big)^2\mathbb{1}_{\{|\Delta_{i,n}^{(l)}X^{(l)}|>\beta|\mathcal{I}_{i,n}^{(l)}|^{\varpi}\}}\\
&\qquad\qquad\times\widehat{R}_{n,m}^{(3-l_j)}(t_{i,n}^{(l)})\\
&\leq 2\sum_{l=1,2}\sum_{t_{i,n}^{(l)}\leq T}\big(\Delta_{i,n}^{(l)}B(q)+\Delta_{i,n}^{(l)}C+\Delta_{i,n}^{(l)}M(q)\big)^2\mathbb{1}_{\{|\Delta_{i,n}^{(l)}X^{(l)}|>\beta|\mathcal{I}_{i,n}^{(l)}|^{\varpi}\}}\\
&\qquad\times\Big(\frac{1}{b_n}\sum_{j:|t_{j,n}^{(3-l)}-t_{i,n}^{(l)}|\leq b_n}\big(\Delta_{j,n}^{(3-l)}X\big)^2\Big)\hat{\eta}_{n,m}^{(3-l)}(t_{i,n}^{(l)}),
\end{aligned}\tag{6.44}$$

where

$$\hat{\eta}_{n,m}^{(3-l)}(t_{i,n}^{(l)})=\hat{\eta}_{n,m,-}^{(3-l)}(s)+\hat{\delta}_{n,m,-}^{(3-l)}(s)U_{n,m,-}^{(3-l)}(s)^2+\hat{\delta}_{n,m,+}^{(3-l)}(s)U_{n,m,+}^{(3-l)}(s)^2+\hat{\eta}_{n,m,+}^{(3-l)}(s).$$

We first consider the increments over the overlapping observation intervals in the right hand side of (6.44). The $\mathcal{F}$-conditional mean of their sum is bounded by

$$\begin{aligned}
\frac{3|\pi_n|_T}{b_n}n\sum_{l=1,2}\sum_{t_{i,n}^{(l)},t_{j,n}^{(3-l)}\leq T}&\big(\Delta_{i,n}^{(l)}B(q)+\Delta_{i,n}^{(l)}C+\Delta_{i,n}^{(l)}M(q)\big)^2\mathbb{1}_{\{|\Delta_{i,n}^{(l)}X^{(l)}|>\beta|\mathcal{I}_{i,n}^{(l)}|^{\varpi}\}}\\
&\times\big(\Delta_{j,n}^{(3-l)}X\big)^2\mathbb{1}_{\{\mathcal{I}_{i,n}^{(l)}\cap\mathcal{I}_{j,n}^{(3-l)}\neq\emptyset\}},
\end{aligned}\tag{6.45}$$

since with $\mathcal{M}_n^{(l)}(s)=\sum_{i:t_{i,n}^{(l)}\leq T}|\mathcal{I}_{i,n}^{(l)}|\mathbb{1}_{\{\mathcal{I}_{i,n}^{(l)}\cap\mathcal{I}_{i_n^{(3-l)}(s),n}^{(3-l)}\neq\emptyset\}}$ we get

$$\begin{aligned}
\mathbb{E}\big[\hat{\eta}_{n,m}^{(3-l)}(t_{i,n}^{(l)})\big|\mathcal{F}\big]&=n\sum_{k_1\in\mathbb{Z},|k_2|\leq K_n}\big|\mathcal{I}_{k_1}^{(3-l)}\cap\mathcal{I}_{i+k_2}^{(l)}\big|\\
&\qquad\times\Big(\sum_{j_1\in\mathbb{Z},|j_2|\leq K_n}\big|\mathcal{I}_{j_1,n}^{(3-l)}\cap\mathcal{I}_{i+j_2,n}^{(l)}\big|\Big)^{-1}\mathcal{M}_n^{(3-l)}(t_{i+k_2,n}^{(l)})\\
&=n\sum_{k_2=-K_n}^{K_n}\big|\mathcal{I}_{i+k_2}^{(l)}\big|\Big(\sum_{j_2=-K_n}^{K_n}\big|\mathcal{I}_{i+j_2,n}^{(l)}\big|\Big)^{-1}\mathcal{M}_n^{(3-l)}(t_{i+k_2,n}^{(l)}) \qquad (6.46)\\
&\leq\sup_{k=-K_n,\ldots,K_n}n\mathcal{M}_n^{(3-l)}(t_{i+k,n}^{(l)})\leq 3n|\pi_n|_T.
\end{aligned}$$

Because of Theorem 3.2 the sum in (6.45) is of order $1/n$, while $|\pi_n|_T/b_n \xrightarrow{\mathbb{P}} 0$ for $n \to \infty$ by Condition 4.1. Hence, (6.45) vanishes.

Next we deal with the increments over non-overlapping observation intervals in the right hand side of (6.44). An upper bound is obtained by taking iterated $\mathcal{S}$-conditional expectations using Lemma 6.1, the Hölder inequality as in (6.32) and (6.46), and it is given by

$$\sum_{l=1,2}\sum_{t_{i,n}^{(l)}\leq T}\big(K_q|\mathcal{I}_{i,n}^{(l)}|^2+K|\mathcal{I}_{i,n}^{(l)}|^{(p'+1-2\varpi)/p'}+Ke_q|\mathcal{I}_{i,n}^{(l)}|\big)\frac{2K\big(b_n+|\pi_n|_T\big)}{b_n}$$
$$\times n\sum_{k_2=-K_n}^{K_n}|\mathcal{I}_{i+k_2}^{(l)}|\Big(\sum_{j_2=-K_n}^{K_n}|\mathcal{I}_{i+j_2,n}^{(l)}|\Big)^{-1}\mathcal{M}_n^{(3-l)}(t_{i+k_2,n}^{(l)})$$
$$\leq K\big(K_q|\pi_n|_T+(|\pi_n|_T)^{(1-2\varpi)/p'}+e_q\big)O_{\mathbb{P}}(1)\sum_{l=1,2}\sum_{t_{i,n}^{(l)}\leq T}|\mathcal{I}_{i,n}^{(l)}|$$
$$\times n\sum_{k_2=-K_n}^{K_n}|\mathcal{I}_{i+k_2}^{(l)}|\Big(\sum_{j_2=-K_n}^{K_n}|\mathcal{I}_{i+j_2,n}^{(l)}|\Big)^{-1}\mathcal{M}_n^{(3-l)}(t_{i+k_2,n}^{(l)}).$$

Now (6.43) follows from Condition 3.1(ii) because of

$$n\sum_{l=1,2}\sum_{t_{i,n}^{(l)}\leq T}|\mathcal{I}_{i,n}^{(l)}|\sum_{k=-K_n}^{K_n}|\mathcal{I}_{i+k}^{(l)}|\Big(\sum_{j=-K_n}^{K_n}|\mathcal{I}_{i+j,n}^{(l)}|\Big)^{-1}\mathcal{M}_n^{(3-l)}(t_{i+k,n}^{(l)})$$
$$=n\sum_{l=1,2}\sum_{i,j:t_{i,n}^{(l)},t_{j,n}^{(3-l)}\leq T}|\mathcal{I}_{i,n}^{(l)}||\mathcal{I}_{j,n}^{(3-l)}|\mathbb{1}_{\{\mathcal{I}_{i,n}^{(l)}\cap\mathcal{I}_{j,n}^{(3-l)}\neq 0\}}\sum_{k=-K_n}^{K_n}|\mathcal{I}_{i+k,n}^{(l)}|\Big(\sum_{m=-K_n}^{K_n}|\mathcal{I}_{i+k+m,n}^{(l)}|\Big)^{-1}$$
$$\leq n\sum_{l=1,2}\sum_{t_{i,n}^{(l)},t_{j,n}^{(3-l)}\leq T}|\mathcal{I}_{i,n}^{(l)}||\mathcal{I}_{j,n}^{(3-l)}|\mathbb{1}_{\{\mathcal{I}_{i,n}^{(l)}\cap\mathcal{I}_{j,n}^{(3-l)}\neq 0\}}$$
$$\times\Big(\sum_{k=-K_n}^{0}|\mathcal{I}_{i+k,n}^{(l)}|\Big(\sum_{m=-K_n}^{0}|\mathcal{I}_{i+m,n}^{(l)}|\Big)^{-1}+\sum_{k=0}^{K_n}|\mathcal{I}_{i+k,n}^{(l)}|\Big(\sum_{m=0}^{K_n}|\mathcal{I}_{i+m,n}^{(l)}|\Big)^{-1}\Big)$$
$$\leq 2H_n(T)$$

and $q \to \infty$ afterwards.

Step 3. Using dominated convergence, $Y(J) \xrightarrow{\widetilde{\mathbb{P}}} \widetilde{D}_T$ as $J \to \infty$. Also, as the $\mathcal{X}$-conditional distribution of $\widetilde{D}_T$ is continuous on $\Omega_T^{(d)}$ by Condition 3.1(iii) and Condition 4.1, for any choice of $\varepsilon, \eta > 0$ there exists $\delta > 0$ such that

$$\widetilde{\mathbb{P}}\big(\{|\widetilde{\mathbb{P}}(\widetilde{D}_T\leq\Upsilon|\mathcal{X})-\widetilde{\mathbb{P}}(\widetilde{D}_T\pm\delta\leq\Upsilon|\mathcal{X})|>\eta\}\cap\Omega_T^{(d)}\big)<\varepsilon.$$

Then it is easy to deduce that

$$\widetilde{\mathbb{P}}\big(Y(J)\leq\Upsilon|\mathcal{X}\big)\mathbb{1}_{\Omega_T^{(d)}}\xrightarrow{\widetilde{\mathbb{P}}}\widetilde{\mathbb{P}}\big(\widetilde{D}_T\leq\Upsilon|\mathcal{X}\big)\mathbb{1}_{\Omega_T^{(d)}}\tag{6.47}$$

holds for $J \to \infty$.

Step 4. For any $\varepsilon > 0$ we have

$$\begin{aligned}
&\widetilde{\mathbb{E}}\Big[\Big|\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{Y(J,n,m)\le\Upsilon\}} - \frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{\widehat{D}_{T,n,m}\le\Upsilon\}}\Big|\mathbb{1}_{\Omega_T^{(d)}}\Big] \\
&\quad\le \widetilde{\mathbb{E}}\Big[\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{|Y(J,n,m)-\widehat{D}_{T,n,m}|\ge|Y(J,n,m)-\Upsilon|\}}\mathbb{1}_{\Omega_T^{(d)}}\Big] \\
&\quad\le \widetilde{\mathbb{E}}\Big[\frac{1}{M_n}\sum_{m=1}^{M_n}\big(\mathbb{1}_{\{|Y(J,n,m)-\widehat{D}_{T,n,m}|>\varepsilon\}} + \mathbb{1}_{\{|Y(J,n,m)-\Upsilon|\le\varepsilon\}}\big)\mathbb{1}_{\Omega_T^{(d)}}\Big].
\end{aligned}$$

By (6.42) and dominated convergence we obtain

$$\widetilde{\mathbb{E}}\Big[\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{|Y(J,n,m)-\Upsilon|\le\varepsilon\}}\mathbb{1}_{\Omega_T^{(d)}}\Big] \to \widetilde{\mathbb{P}}(\{|Y(J)-\Upsilon|\le\varepsilon\}\cap\Omega_T^{(d)}), \tag{6.48}$$

where the right hand side tends to zero as $\varepsilon \to 0$ using dominated convergence again, because the $\mathcal{X}$-conditional distribution of $Y(J)$ is continuous while Υ is $\mathcal{X}$-measurable. By (6.43) we also have

$$\lim_{J\to\infty}\limsup_{n\to\infty}\widetilde{\mathbb{E}}\Big[\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{|Y(J,n,m)-\widehat{D}_{T,n,m}|>\varepsilon\}}\Big] \to 0 \tag{6.49}$$

for all $\varepsilon > 0$. Thus, using (6.48) and (6.49), we obtain

$$\lim_{J\to\infty}\limsup_{n\to\infty}\widetilde{\mathbb{P}}\Big(\Big\{\Big|\frac{1}{M_n}\sum_{m=1}^{M_n}\big(\mathbb{1}_{\{Y(J,n,m)\le\Upsilon\}} - \mathbb{1}_{\{\widehat{D}_{T,n,m}\le\Upsilon\}}\big)\Big| > \varepsilon\Big\}\cap\Omega_T^{(d)}\Big) = 0 \tag{6.50}$$

for all $\varepsilon > 0$.
Step 5. The claim follows from (6.42), (6.47) and (6.50). □

Proof of (6.27). We have for arbitrary $\varepsilon > 0$

$$\begin{aligned}
&\widetilde{\mathbb{P}}\big(\{\widehat{Q}_{n,T}(\alpha) > Q(\alpha)+\varepsilon\}\cap\Omega_T^{(d)}\big) \\
&\quad= \widetilde{\mathbb{P}}\Big(\Big\{\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{\widehat{D}_{T,n,m}>Q(\alpha)+\varepsilon\}} > \frac{M_n-(\lfloor\alpha M_n\rfloor-1)}{M_n}\Big\}\cap\Omega_T^{(d)}\Big) \\
&\quad\le \widetilde{\mathbb{P}}\Big(\Big\{\frac{1}{M_n}\sum_{m=1}^{M_n}\mathbb{1}_{\{\widehat{D}_{T,n,m}>Q(\alpha)+\varepsilon\}} - \Upsilon(\alpha,\varepsilon) > (1-\alpha)-\Upsilon(\alpha,\varepsilon)\Big\}\cap\Omega_T^{(d)}\Big)
\end{aligned}$$

with $\Upsilon(\alpha,\varepsilon) = \widetilde{\mathbb{P}}\big(\widetilde{D}_T > Q(\alpha)+\varepsilon\big|\mathcal{X}\big)$. Because the $\mathcal{X}$-conditional distribution of $\widetilde{D}_T$ is continuous on $\Omega_T^{(d)}$ with a strictly positive density on $[0,\infty)$ by Condition 3.1(iii) and Condition 4.1, we have $\Upsilon(\alpha,\varepsilon) < 1-\alpha$ almost surely on $\Omega_T^{(d)}$. Then it is easy to deduce

$$\widetilde{\mathbb{P}}\big(\{\widehat{Q}_{n,T}(\alpha) > Q(\alpha)+\varepsilon\}\cap\Omega_T^{(d)}\big) \to 0$$

using Proposition 6.7. Analogously, we get $\widetilde{\mathbb{P}}\big(\widehat{Q}_{n,T}(\alpha) < Q(\alpha)-\varepsilon\big) \to 0$. □

6.5 Proof of (4.3) for Example 4.4

First, note that $\widehat{Z}_{n,m}^{(l_i)}(s_i)$ and $\widehat{Z}_{n,m}^{(l_j)}(s_j)$ are $\mathcal{S}$-conditionally independent if we are on the set $\Omega(n, s_i, s_j)$ on which $\widehat{Z}_{n,m}^{(l_i)}(s_i)$ and $\widehat{Z}_{n,m}^{(l_j)}(s_j)$ contain no common observation intervals. Without loss of generality let $s_i < s_j$. Using the Markov inequality we get

$$\begin{aligned}\mathbb{P}(\Omega(n, s_i, s_j)^c) &\le \mathbb{P}\big(\tau_{n,+}^{(l_i)}(t^{(3-l_i)}_{i_n^{(3-l_i)}(s_i)+K_n,n}) \ge s_i + (s_j - s_i)/2\big) \\ &\quad + \mathbb{P}\big(\tau_{n,-}^{(l_j)}(t^{(3-l_j)}_{i_n^{(3-l_j)}(s_j)-K_n-1,n}) \le s_i + (s_j - s_i)/2\big) \\ &\le 2K_{\lambda_1,\lambda_2} \frac{K_n/n}{(s_j - s_i)/2} \end{aligned} \tag{6.51}$$

for a generic constant K_{λ_1,λ_2}. The latter tends to zero as $n \to \infty$ because of $|\pi_n|_T K_n \xrightarrow{\mathbb{P}} 0$ and since the stochastic order of $|\pi_n|_T$ dominates $1/n$ as $n \to \infty$. Hence, we may assume $\widehat{Z}_{n,m}^{(l_i)}(s_i)$ and $\widehat{Z}_{n,m}^{(l_j)}(s_j)$ to be $\mathcal{S}$-conditionally independent, and it remains to prove (4.3) for $J = 1$. Also, we have seen in Example 3.4 that $Z^{(l)}(s)$ follows a continuous distribution. If we establish weak convergence of the $\mathcal{S}$-conditional distribution of $\widehat{Z}_{n,1}^{(l)}(s)$ to the (unconditional) one of $Z^{(l)}(s)$, then (4.3) follows from the Portmanteau theorem and dominated convergence.

By construction it holds

$$\widehat{Z}_{n,1}^{(l)}(s) \overset{\mathcal{L}_\mathcal{S}}{=} Z_n^{(l)}(U_{n,K_n}^{(l)}(s)), \quad U_{n,m}^{(l)}(s) \sim \mathcal{U}[t^{(3-l)}_{i_n^{(3-l)}(s)-m-1,n}, t^{(3-l)}_{i_n^{(3-l)}(s)+m,n}],$$

where $\mathcal{L}_\mathcal{S}$ denotes equality of the $\mathcal{S}$-conditional distribution. Hence it remains to show $Z_n^{(l)}(U_{n,K_n}^{(l)}(s)) \xrightarrow{\mathcal{L}_\mathcal{S}} Z^{(l)}(s)$ which is equivalent to

$$Z_1^{(l)}(U_{1,K_n}^{(l)}(ns)) \xrightarrow{\mathcal{L}_\mathcal{S}} Z^{(l)}(s). \tag{6.52}$$

In Example 3.4 the law of $Z^{(l)}(s)$ was obtained as the limit of the laws of $Z_1^{(l)}(ns)$ for $n \to \infty$. Hence (6.52) can be interpreted in the following way: If we take a fixed realization of the Poisson processes and shift this realization according to an independent uniform random variable on an interval around ns whose diameter increases with n, then this shifted realization has asymptotically the same distribution as $Z_1^{(l)}(ns)$. Hence observing a fixed realization of these Poisson processes around a uniformly distributed random time is due to the stationarity of the Poisson processes asymptotically the same as observing the random Poisson processes at a fixed time.

Lets give a formal proof of (6.52). First define for $l = 1, 2$

$$\begin{aligned}\tilde{t}_{0,n}^{(l)}(s) &= U_{1,K_n}^{(l)}(ns), \\ \tilde{t}_{k,n}^{(l)}(s) &= \inf\{t_{i,1}^{(l)} | t_{i,1}^{(l)} > \tilde{t}_{k-1,n}^{(l)}(s)\}, \quad k \ge 1, \\ \tilde{t}_{k,n}^{(l)}(s) &= \sup\{t_{i,1}^{(l)} | t_{i,1}^{(l)} < \tilde{t}_{k+1,n}^{(l)}(s)\}, \quad k \le -1.\end{aligned}$$

Since the number of increments occuring in $Z_1^{(l)}(U_{1,K_n}^{(l)}(ns))$ is bounded in probability and the number of exponentially distributed random variables used in Example 3.4 for the construction of $Z^{(l)}(s)$ is also bounded in probability it suffices to prove

$$\begin{aligned}&\big((\tilde{t}_{k,n}^{(l)}(s) - \tilde{t}_{k-1,n}^{(l)}(s))_{k=-K+1,\ldots,K}, (\tilde{t}_{k,n}^{(3-l)}(s) - \tilde{t}_{k-1,n}^{(3-l)}(s))_{k=-K+1,\ldots,K}\big) \\ &\qquad \xrightarrow{\mathcal{L}_\mathcal{S}} \big((E_k^{(l)})_{k=-K+1,\ldots,K}, (E_k^{(3-l)})_{k=-K+1,\ldots,K}\big)\end{aligned} \tag{6.53}$$

for all $K \in \mathbb{N}$ where $E_k^{(1)}, E_k^{(2)}$ are i.i.d. exponentially distributed random variables with parameters λ_1, λ_2, respectively.

We first show

$$(\tilde{t}_{k,n}^{(l)}(s) - \tilde{t}_{k-1,n}^{(l)}(s))_{k=-K+1,\dots,K} \xrightarrow{\mathcal{L}_{\mathcal{S}}} (E_k^{(l)})_{k=-K+1,\dots,K}. \tag{6.54}$$

To prove this we consider the $\mathcal{S}$-conditional characteristic function

$$\begin{aligned}
&\mathbb{E}\big[\exp\big(i \sum_{k=-K+1}^{K} v_k(\tilde{t}_{k,n}^{(l)}(s) - \tilde{t}_{k-1,n}^{(l)}(s))\big)\big|\mathcal{S}\big] \qquad (6.55)\\
&= O_{\mathbb{P}}(K_n^{-1/2}) + \sum_{j=-K_n^*(l)}^{K_n^*(l)} \big|\mathcal{I}_{i_1^{(l)}(ns)+j,1}^{(l)}\big|\Big(\sum_{j'=-K_n^*(l)}^{K_n^*(l)} \big|\mathcal{I}_{i_1^{(l)}(ns)+j',1}^{(l)}\big|\Big)^{-1}\\
&\qquad \times \exp\big(i\sum_{k=1}^{K-1}\big(v_{-k}\big|\mathcal{I}_{i_1^{(l)}(ns)+j-k,1}^{(l)}\big| + v_{k+1}\big|\mathcal{I}_{i_1^{(l)}(ns)+j+k,1}^{(l)}\big|\big)\big)\\
&\qquad \times \mathbb{E}\big[\exp\big(iv_0(U_1(ns) - t_{i_1^{(l)}(ns)+j-1,1}^{(l)}(s))\\
&\qquad\qquad + iv_1(t_{i_1^{(l)}(ns)+j,1}^{(l)}(s) - U_1(ns))\big)\big|\mathcal{S}, U_1(ns) \in \mathcal{I}_{i_1^{(l)}(ns)+j,1}^{(l)}\big]
\end{aligned}$$

where the number of observations of $X^{(l)}$ in the interval $[t_{i_n^{(3-l)}(s)-K_n-1,n}^{(3-l)}, t_{i_n^{(3-l)}(s)+K_n,n}^{(3-l)}]$ equals $(2K_n^*(l)+1)+O_{\mathbb{P}}(K_n^{1/2})$ with $K_n^*(l) = \lfloor K_n\lambda_l/\lambda_{3-l}\rfloor$. With a random variable $\kappa \sim \mathcal{U}[0,1]$ independent of $\mathcal{S}$ the conditional expectation in the last line equals

$$\begin{aligned}
&\mathbb{E}\big[\exp\big(iv_0\kappa\big|\mathcal{I}_{i_1^{(l)}(ns)+j,1}^{(l)}\big| + iv_1(1-\kappa)\big|\mathcal{I}_{i_1^{(l)}(ns)+j,1}^{(l)}\big|\big)\big|\mathcal{S}\big]\\
&\qquad = \frac{\exp\big(iv_0\big|\mathcal{I}_{i_1^{(l)}(ns)+j,1}^{(l)}\big|\big) - \exp\big(iv_1\big|\mathcal{I}_{i_1^{(l)}(ns)+j,1}^{(l)}\big|\big)}{i(v_0-v_1)\big|\mathcal{I}_{i_1^{(l)}(ns)+j,1}^{(l)}\big|}.
\end{aligned}$$

Except for $j=0$ the length of each observation interval $\mathcal{I}_{i_1^{(l)}(ns)+j,1}^{(l)}$ is exponentially distributed, up to asymptotically negligible boundary effects, with parameter λ_l. It follows easily that (6.55) has asymptotically the same distribution as

$$\Big(\sum_{j=-K_n^*(l)}^{K_n^*(l)} \widetilde{E}_j\Big)^{-1} \sum_{j=-K_n^*(l)}^{K_n^*(l)} \widetilde{E}_j \exp\big(i\sum_{k=1}^{K-1}(v_{-k}\widetilde{E}_{j-k} + v_{k+1}\widetilde{E}_{j+k})\big)\frac{\exp(iv_0\widetilde{E}_j) - \exp(iv_1\widetilde{E}_j)}{i(v_0-v_1)\widetilde{E}_j}$$

for i.i.d. exponentials $\widetilde{E}_j$, $j \in \mathbb{N}$, with parameter λ_l. Expanding by $(2K_n^*+1)^{-1}$ and using the law of large numbers (note that the summands are independent for $|j-j'| > 2K+1$), this expression converges almost surely to

$$\begin{aligned}
&\mathbb{E}\big[\lambda_l\widetilde{E}_0\exp\big(i\sum_{k=1}^{K-1}(v_{-k}\widetilde{E}_{j-k} + v_{k+1}\widetilde{E}_{j+k})\big)(i(v_0-v_1)\widetilde{E}_0)^{-1}\big(\exp(iv_0\widetilde{E}_0) - \exp(iv_1\widetilde{E}_0)\big)\big]\\
&= \mathbb{E}\big[\exp\big(i\sum_{k=1}^{K-1}(v_{-k}\widetilde{E}_{j-k} + v_{k+1}\widetilde{E}_{j+k})\big)\big]\int_0^\infty \lambda_l x\frac{\exp(iv_0x) - \exp(iv_1x)}{i(v_0-v_1)x}\lambda_l e^{-\lambda_l x}dx\\
&= \mathbb{E}\big[\exp\big(i\sum_{k=1}^{K-1}(v_{-k}\widetilde{E}_{j-k} + v_{k+1}\widetilde{E}_{j+k})\big)\big]\frac{\lambda_l}{\lambda_l - iv_0}\frac{\lambda_l}{\lambda_l - iv_1} \qquad (6.56)
\end{aligned}$$

which is the characteristic function of a vector of $2K$ independent Exp(λ_l)-distributed random variables. This yields (6.54).

Analogously to (6.54) we obtain

$$(\tilde{t}^{(3-l)}_{k,n}(s) - \tilde{t}^{(3-l)}_{k-1,n}(s))_{k=-K+1,\ldots,K} \xrightarrow{\mathcal{L}_S} (E^{(3-l)}_k)_{k=-K+1,\ldots,K}, \qquad (6.57)$$

and finally (6.54) and (6.57) yield (6.53), because by the stationarity of the Poisson process and the independence of the two processes we have that $\tilde{t}^{(l)}_{k,n}(s) - \tilde{t}^{(l)}_{k-1,n}(s)$ and $\tilde{t}^{(3-l)}_{k',n}(s) - \tilde{t}^{(3-l)}_{k'-1,n}(s)$ are asymptotically independent, because dependency only occurs in the $O_{\mathbb{P}}(K_n^{-1/2})$-term of (6.55) which is asymptotically negligible. □

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