OPTIMAL MAXIMIN $L_1$-DISTANCE LATIN HYPERCUBE DESIGNS BASED ON GOOD LATTICE POINT DESIGNS

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Maximin distance Latin hypercube designs are commonly used for computer experiments, but the construction of such designs is challenging. We construct a series of maximin Latin hypercube designs via Williams transformations of good lattice point designs. Some constructed designs are optimal under the maximin $L_1$-distance criterion, while others are asymptotically optimal. Moreover, these designs are also shown to have small pairwise correlations between columns.

1. Introduction. Computer experiments are increasingly being used to investigate complex systems (Sacks et al., 1989; Santner et al., 2003; Fang et al., 2006; Morris and Moore, 2015). A general design approach to planning computer experiments is to seek design points that fill a design region as uniformly as possible (Lin and Tang 2015). Representative designs include Latin hypercube designs (LHDs) and their modifications, maximin distance designs (Johnson et al., 1990) and uniform designs (Fang and Wang, 1994). LHDs have uniform one-dimensional projections and orthogonal-array based LHDs (Tang, 1993; He and Tang, 2013, 2014; He et al., 2017) have improved two- or three-dimensional projections. Many researchers have constructed orthogonal or nearly orthogonal LHDs; see, among others, Ye (1998), Steinberg and Lin (2006), Cioppa and Lucas (2007), Lin et al. (2009), Sun et al. (2009), Yang and Liu (2012), Georgiou and Efthimiou (2014), Lin and Tang (2015), and Sun and Tang (2017). However, these LHDs are often not space-filling in high dimensions (Joseph and Hung, 2008; Xiao and Xu, 2018).

A maximin distance design spreads design points over the design space in such a way that the separation distance, i.e., the minimal distance between pairs of points, is maximized. Computer experiments are often modeled as Gaussian processes. When the correlations between observations rapidly decrease as the distances between design points increase, maximin distance designs are asymptotically $D$-optimal in the sense that they maximize the

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determinant of the correlation matrix (Johnson et al. 1990). The choice of distances is application dependent. Some researchers worked on the $L_2$-distance and proposed algorithms such as simulated annealing (Morris and Mitchell, 1995; Joseph and Hung, 2008; Ba et al., 2015) and swarm optimization algorithms (Moon et al., 2011; Chen et al., 2013) to construct maximin distance LHDs. However, such methods are not efficient for constructing large designs due to their computational complexity. Nevertheless, large designs are needed for computer experiments; for example, Morris (1991) considered many simulation models involving hundreds of factors. Zhou and Xu (2015) studied both $L_1$- and $L_2$-distances of good lattice point (GLP) designs. The GLP method was introduced by Korobov (1959) for numerical evaluation of multivariate integrals and has been widely used in quasi-Monte Carlo method, uniform designs and computer experiments (Fang and Wang, 1994). Zhou and Xu (2015) showed that permuting levels can increase the separation distances of GLP designs. It is infeasible to conduct all level permutations, so they considered only linear permutations, which limits the ability of generating good designs. Xiao and Xu (2017) proposed construction methods via Costas arrays and obtained some LHDs with large minimal $L_1$-distance.

In this paper, we propose a series of systematic methods to construct maximin $L_1$-distance LHDs. The $L_1$-distance provides a lower bound for the $L_2$-distance by the Cauchy-Schwarz inequality so that the constructed designs also perform well regarding the $L_2$-distance. The proposed method is based on the Williams transformation and its modification. The Williams transformation was first used by Williams (1949) to construct Latin square designs that are balanced for nearest neighbors. Bailey (1982) and Edmondson (1993) used the transformation to construct designs orthogonal to polynomial trends. Butler (2001) used the transformation to construct optimal and orthogonal LHDs under a second-order cosine model. Our purpose is different from theirs. We apply the Williams transformation to GLP designs and construct a class of asymptotically optimal maximin LHDs. Applying the leave-one-out method we obtain another class of asymptotically optimal maximin LHDs. By modifying the Williams transformation, we obtain a class of exactly optimal maximin LHDs. Moreover, all resulting designs have small pairwise correlations between columns and the average correlations converge to zero as the design sizes increase. This near orthogonality is desirable for estimating potential linear trend efficiently in a Gaussian process.

This paper is organized as follows. Section 2 provides the construction methods. Sections 3 and 4 give theoretical results on separation distances
and correlations of some special constructed designs. Section 5 extends the theoretical results to a general situation. Concluding remarks are given in Section 6. Proofs are deferred to the Appendix.

2. Construction methods. An $N \times n$ LHD is an $N \times n$ matrix where each column is a permutation of $N$ equally spaced levels, denoted by $0, \ldots, N-1$ or $1, \ldots, N$. The $L_1$-distance between two vectors $x_1 = (x_{11}, \ldots, x_{1n})$ and $x_2 = (x_{21}, \ldots, x_{2n})$ is $d(x_1, x_2) = \sum_{j=1}^{n} |x_{1j} - x_{2j}|$. For an $N \times n$ design matrix $D$, let $x_i$ be the $i$th row, $i = 1, \ldots, N$, and $d_{ik}(D)$ be the $L_1$-distance between the $i$th and $k$th rows of $D$, i.e., $d_{ik}(D) = d(x_i, x_k)$. The $L_1$-distance of $D$, denoted by $d(D) = \min\{d_{ik}(D) : i \neq k, i, k = 1, \ldots, N\}$, is the minimum $L_1$-distance between any two distinct rows in $D$. The maximin distance criterion (Johnson et al. 1990) is to maximize $d(D)$ among all possible designs. For an $N \times n$ LHD, the average pairwise $L_1$-distance between rows is $(N^2+1)n/3$ (Zhou and Xu, 2015). Because the minimum pairwise $L_1$-distance cannot exceed the integer part of the average, we have the following result.

Lemma 1. For any $N \times n$ LHD $D$, $d(D) \leq d_{\text{upper}} = \lfloor (N+1)n/3 \rfloor$, where $[x]$ is the integer part of $x$.

Let $h = (h_1, \ldots, h_n)$ be a set of positive integers smaller than and coprime to $N$. An $N \times n$ GLP design $D = (x_{ij})$ is defined by $x_{ij} = i \times h_j \pmod{N}$ for $i = 1, \ldots, N$ and $j = 1, \ldots, n$. The last row of $D$ is a vector of zeros. Each column of $D$ is a permutation of the set $\{0, \ldots, N-1\}$. Thus a GLP design is an LHD. We can construct an $N \times n$ GLP design for any $n \leq \phi(N)$, where $\phi(N)$ is the Euler function, i.e., the number of positive integers smaller than and coprime to $N$. Let $D_b = D + b \pmod{N}$ for $b = 0, \ldots, N-1$, that is, $D_b$ is a linearly permuted GLP design. Then $D_b$ is still an LHD. Zhou and Xu (2015) showed that $d(D_b) \geq d(D)$ for any $b$ and proposed to search $b$ that maximizes $d(D_b)$.

2.1. Williams transformation. Given an integer $N$, for $x = 0, \ldots, N-1$, the Williams transformation is defined by

\begin{equation}
W(x) = \begin{cases} 
2x, & \text{for } 0 \leq x < N/2; \\
2(N-x) - 1, & \text{for } N/2 \leq x < N.
\end{cases}
\end{equation}

The Williams transformation is a permutation of $\{0, \ldots, N-1\}$. Hence, for an LHD $D = (x_{ij})$, $W(D) = (W(x_{ij}))$ is also an LHD. The following example shows that the Williams transformation can further increase the $L_1$-distance of linearly permuted GLP designs.
Table 1

The $L_1$-distances of $D_b$ and $E_b$ in Example 1

<table>
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<tr>
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<tr>
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<td>$d(E_b)$</td>
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</table>

Example 1. Consider $N = 11$ and $h = (1, \ldots, 10)$. The GLP design $D = (x_{ij})$ is an $11 \times 10$ LHD with $x_{ij} = i \times j \pmod{11}$ and $d(D) = 30$. For each $b = 0, \ldots, 10$, we obtain two designs via linear permutation and Williams transformation, namely, $D_b = D + b \pmod{11}$ and $E_b = W(D_b)$. Table 1 shows the $L_1$-distances of $D_b$ and $E_b$. The linearly permuted designs $D_b$’s have distances ranging from 30 to 34, while the distances for $E_b$’s vary from 10 to 39. The upper bound from Lemma 1 is 40. The best design from $D_b$’s is $D_1$ or $D_9$ with $d(D_1) = d(D_9) = 34$, while the best design from $E_b$’s is $E_1$ or $E_4$ with $d(E_1) = d(E_4) = 39$.

Example 1 shows that the Williams transformation can generate designs with larger distances than the linear permutation. Inspired by this, we propose a new construction for maximin LHDs:

Algorithm 1 (Williams transformation of linearly permuted GLP designs).

Step 1. Given a pair of integers $N$ and $n \leq \phi(N)$, generate an $N \times n$ GLP design $D$.

Step 2. For $b = 0, \ldots, N - 1$, generate $D_b = D + b \pmod{N}$ and $E_b = W(D_b)$.

Step 3. Find the best $D_b$ and $E_b$ which maximize $d(D_b)$ and $d(E_b)$, respectively.

As an illustration, we apply Algorithm 1 for $N = 7, \ldots, 30$ and $n = \phi(N)$. Table 2 compares LHDs generated by the linear permutation, the Williams transformation, R package SLHD provided by Ba, Myers and Brenneman (2015), and the Gilbert and Golomb methods proposed by Xiao and Xu (2017). The SLHD package adopts the $L_2$-distance measure, so we ran the command `maximinSLHD` with option $t = 1$ and default settings for 100 times, and chose the design with the largest $L_1$-distance. The Williams transformation always offers better designs than the linear permutation except for $N = 13$, and consistently outperforms the Gilbert and Golomb methods, which only work for prime $N$. Compared to the SLHD package, the Williams transformation performs better for designs with moderate to large sizes. The Williams transformation performs specially well when $N$ is a prime.
2.2. Leave-one-out method. Since the last row of a GLP design $D$ is $(0, \ldots, 0)$, then the last rows of $D_b$ and $E_b$ are $(b, \ldots, b)$ and $(W(b), \ldots, W(b))$, respectively. The leave-one-out method is to delete the constant row of a design and rearrange the levels so that the resulting design is still an LHD. Specifically, starting from $D_b$, we delete the last row and reduce the levels $b+1, \ldots, N-1$ by one, which gives us an $(N-1) \times n$ LHD, denoted by $D_b^*$. Similarly, from $E_b$, we obtain another $(N-1) \times n$ LHD, denoted by $E_b^*$. Table 3 compares the $L_1$-distances of $D_b^*$ and $E_b^*$ for $N = 7, \ldots, 30$, as well as the $(N-1) \times n$ designs generated by R package SLHD and the Gilbert and Golomb methods. From Table 3, the leave-one-out Williams transformation generates designs with larger $L_1$-distance than other methods in most cases. It performs specially well when $N$ is a prime.

2.3. Modified Williams transformation. To construct other maximin LHDs, we propose a modified Williams transformation. For $x = 0, \ldots, N-1$, define

$$w(x) = \begin{cases} 2x, & \text{for } 0 \leq x < N/2; \\ 2(N-x), & \text{for } N/2 \leq x < N. \end{cases}$$

The following lemma shows an important connection between the original and modified Williams transformations.

**Lemma 2.** Let $N$ be an odd prime, $D$ be an $N \times (N-1)$ GLP design, and $D_b = D + b \pmod{N}$ for $b = 0, \ldots, N-1$. Then $d_{ik}(w(D_b)) = d_{ik}(W(D_b))$ for $i+k \neq N$ and $i, k = 1, \ldots, N-1$. 

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<th>SLHD</th>
<th>Gil</th>
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<td>Note: LP, linear permutation; WT, Williams transformation; SLHD, R package SLHD; Gil, Gilbert method; Gol, Golomb method.</td>
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Table 3
Comparison of $L_1$-distances of $(N - 1) \times n$ LHDs

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Note: LP-1, leave-one-out linear permutation; WT-1, leave-one-out Williams transformation.

Table 4
The design matrices of $D$ and $w(D)/2$ in Example 2

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<th>$w(D)/2$</th>
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The $w(x)$ is always an even number, so $w(D_b)$ is not an LHD. We can construct LHDs by selecting some submatrices of $w(D)/2$. Let us see an illustrating example.

Example 2. Consider $N = 11$ and the $11 \times 10$ GLP design $D$. The design matrices of $D$ and $w(D)/2$ are shown in Table 4. If we divide the design matrix of $w(D)/2$ into four blocks as shown in Table 4, then each block is an LHD. Denote $H_1$ and $H_2$ as the top two blocks, and $H_3$ and $H_4$ as the bottom two blocks, respectively. It can be verified that $H_1$ and $H_2$ are $5 \times 5$ LHDs with $d(H_1) = d(H_2) = 10$, which attains the upper bound of $L_1$-distance in Lemma 1. In fact, $H_1$ and $H_2$ are the same design up to column permutations; in addition, $H_3$ and $H_4$ can be obtained by adding a
**Table 5**

*Comparison of $L_1$-distances of $m \times m$ LHDs*

<table>
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Note: MWT, modified Williams transformation; Wel, Welch.

row of zeros to $H_1$ and $H_2$, respectively.

Generally, suppose that $N$ is an odd prime with $N = 2m+1$ and $D = (x_{ij})$ is the $N \times (N-1)$ GLP design. Since $x_{ij} + x_{(N-i)j} = N$ and $x_{ij} + x_{i(N-j)} = N$ for any $i, j = 1, \ldots, N-1$, then

$$
D = \begin{pmatrix}
A_1 & N - A_2 \\
N - A_3 & A_4 \\
0_m & 0_m
\end{pmatrix}
$$

and

$$
w(D) = \begin{pmatrix}
w(A_1) & w(A_2) \\
w(A_3) & w(A_4) \\
0_m & 0_m
\end{pmatrix},
$$

where $A_1$ is the $m \times m$ leading principal submatrix of $D$, and $A_2, A_3$, and $A_4$ can be obtained from $A_1$ by reversing the order of columns, rows, and both, respectively. In fact, $w(A_1), \ldots, w(A_4)$ are the same design up to row and column permutations, each column of which is a permutation of $\{2, 4, \ldots, 2m\}$. Let

$$
H = w(A_1)/2
$$

be an $m \times m$ LHD from the modified Williams transformation. Table 5 compares LHDs generated by the modified Williams transformation, the R package SLHD, and the Welch, Gilbert and Golomb methods from Xiao and Xu (2017). The modified Williams transformation always provides better designs than any other methods. In fact, the $L_1$-distance of each design generated by the modified Williams transformation in Table 5 attains the upper bound given in Lemma 1.

**3. Theoretical results.** The Williams transformation leads to a remarkably simple design structure in terms of the $L_1$-distance when $N$ is an odd prime.
To maximize $d(E_b)$, we need to maximize $\min\{f(b), -2f(b)\}$. Let $c_0 = \lfloor(\sqrt{(N^2 - 1)/12})\rfloor$,

$$c = \begin{cases} 
c_0, & \text{if } c_0^2 + 2(c_0 + 1)^2 \geq (N^2 - 1)/4; \\
c_0 + 1, & \text{otherwise},
\end{cases}$$

and

$$(3.1) \quad b = W^{-1} \left( \frac{N - 1}{2} \pm c \right).$$

It can be verified that either choice of $b$ defined in (3.1) maximizes $\min\{f(b), -2f(b)\}$ and leads to the best $E_b$.

**Example 3.** Consider $N = 11$. Then $c_0 = \lfloor(\sqrt{11^2 - 1})/12\rfloor = 3$. Since $c_0^2 + 2(c_0 + 1)^2 \geq (N^2 - 1)/4$, set $c = 3$. By (3.1), $b = 1$ or 4. For either $b = 1$
or \( b = 4 \), by Theorem 1, for \( i \neq k \),

\[
d_{ik}(E_b) = \begin{cases} 
39, & \text{for } i = 11 \text{ or } k = 11, \\
42, & \text{for } i = 11 - k, \\
40, & \text{otherwise.}
\end{cases}
\]

Hence, \( d(E_1) = d(E_4) = 39 \).

Based on the upper bound in Lemma 1, we define the distance efficiency as

\[
d_{\text{eff}}(D) = \frac{d(D)}{d_{\text{upper}}} = \frac{d(D)}{\lfloor (N + 1)n/3 \rfloor}.
\]

When \( N \) is a prime, \( n = \phi(N) = N - 1 \) and \((N + 1)n/3 = (N^2 - 1)/3\) is an integer. In this case, \( d_{\text{eff}}(E_b) = d(D)/((N + 1)n/3)\). For example, for the designs \( E_1 \) and \( E_4 \) in Example 3, \( d_{\text{eff}}(E_1) = d_{\text{eff}}(E_4) = 39/40 = 0.975 \).

Generally, we have the following result.

**Theorem 2.** For an odd prime \( N \) and \( b \) defined in (3.1),

\[
d(E_b) \geq \frac{N^2 - 1}{3} - \frac{2}{3} \sqrt[3]{\frac{N^2 - 1}{3}} \quad \text{and} \quad d_{\text{eff}}(E_b) \geq 1 - \frac{2}{\sqrt[3]{3(N^2 - 1)}}.
\]

As \( N \to \infty \), \( d_{\text{eff}}(E_b) \to 1 \); so \( E_b \) is asymptotically optimal under the maximin distance criterion. For the leave-one-out design \( E_b^* \) defined in Section 2.2, we have the following result.

**Theorem 3.** For an odd prime \( N \) and \( b \) defined in (3.1),

\[
d(E_b^*) \geq \frac{N^2 - 7}{3} + \frac{1}{3} \sqrt[3]{\frac{N^2 - 1}{3}} - (N - 1) \quad \text{when } N \geq 7.
\]

When \( N \geq 7 \), \( d_{\text{eff}}(E_b^*) \geq 1 - (3 - 1/\sqrt[3]{3})/N > 1 - 2.43/N \).

For an odd prime \( N = 2m + 1 \) and the \( m \times m \) design \( H \) constructed in (2.4), we have even better results. By Lemma 2 and Theorem 1, \( d_{ik}(w(D)) = (N^2 - 1)/3 \) for \( i \neq k, i, k = 1, \ldots, m \). By the structure of \( w(D) \) shown in (2.3), \( d_{ik}(w(A_1)) = d_{ik}(w(D))/2 = (N^2 - 1)/6 \); so \( H \) is an equidistant LHD and \( d(H) = (N^2 - 1)/12 = (m + 1)m/3 \).

**Theorem 4.** Let \( N = 2m + 1 \) be an odd prime, \( D = (x_{ij}) \) be an \( N \times (N - 1) \) GLP design, and \( A_1 \) be the \( m \times m \) leading principal submatrix of \( D \), that is, \( A_1 = (x_{ij}) \) with \( i, j = 1, \ldots, m \). Then \( H = w(A_1)/2 \) is a maximin distance LHD with \( d(H) = (m + 1)m/3 \).
The modified Williams transformation generates exact maximin LHDs when $N$ is an odd prime. The constructed $H$ is a cyclic Latin square, with each level occurring once in each row and once in each column. We can add a row of zeros to $H$ to obtain an $(m+1) \times m$ LHD, denoted by $H^*$. It is easy to see that $d(H^*) = d(H) = (m+1)m/3$ and $d_{\text{eff}}(H^*) = (m+1)/(m+2) \to 1$ as $m \to \infty$.

The proposed methods are also useful in the construction of maximin $L_2$-distance designs. An upper bound for the $L_2$-distance of an $N \times n$ LHD is $d^{(2)}_{\text{upper}} = \sqrt{N(N+1)n/6}$ (Zhou and Xu, 2015). By the Cauchy-Schwarz inequality, we have $\|x\|_2 \geq \|x\|_1/\sqrt{n}$ for any $n$-vector $x$, so $d^{(2)}_{\text{eff}} > \sqrt{2/3} d_{\text{eff}}$, where $d^{(2)}_{\text{eff}}$ is the $L_2$-distance efficiency. Therefore, for an (asymptotically) optimal design under the maximin $L_1$-distance criterion, its $L_2$-distance efficiency will tend to be greater than $\sqrt{2/3} > 0.816$. This is a loose lower bound, and yet it illustrates the good performance of our constructed designs regarding the $L_2$-distance. Numerical calculation shows that our proposed methods are able to produce designs with $L_2$-distance efficiencies greater than 0.95 for large $N$.

4. Additional results on correlations. We now consider the pairwise correlation between columns for the constructed designs. For any $N \times n$ design $D = (x_{ij})$, define

\begin{equation}
\rho_{\text{ave}}(D) = \frac{\sum_{j \neq k} |\rho_{jk}|}{n(n-1)},
\end{equation}

where $\rho_{jk}$ is the correlation between columns $j$ and $k$ of $D$. The $\rho_{\text{ave}}$ in (4.1) is a performance measure on the overall pairwise column correlations for design $D$. A good design should have a low $\rho_{\text{ave}}$ value to reduce correlations between factors and reduce the variance of coefficients estimates.

Consider the $\rho_{\text{ave}}$ values for the designs from the Williams transformation. For each prime $N$, Table 6 compares the $\rho_{\text{ave}}$ values of designs from the linear permutation, Williams transformation (with $b$ chosen by (3.1)), Gilbert, and Golomb methods. The Williams transformation always generates designs with the smallest $\rho_{\text{ave}}$ values. In fact, we have a general result on the average correlation $\rho_{\text{ave}}(E_b)$ for any $b = 0, \ldots, N - 1$, not restricted to the $b$ defined in (3.1).

**Theorem 5.** Let $N$ be an odd prime and $D$ be an $N \times (N-1)$ GLP design, $D_b = D + b \pmod{N}$, and $E_b = W(D_b)$ for $b = 0, \ldots, N - 1$. Then $\rho_{\text{ave}}(E_b) < 2/(N-2)$. 

OPTIMAL MAXIMIN $L_1$-DISTANCE LATIN HYPERCUBE DESIGNS

**Table 6**

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For a prime $N$, $\rho_{\text{ave}}(E_b) \to 0$ as $N \to \infty$ for any $b = 0, \ldots, N - 1$. This property makes it possible to generate large LHDs with tiny pairwise column correlations without any computer search. For the leave-one-out Williams transformation, we have the following result.

**Theorem 6.** Let $N$ be an odd prime, $D$ be an $N \times (N - 1)$ GLP design, $D_b = D + b \pmod N$, $E_b = W(D_b)$, and $E_b^*$ be the leave-one-out design obtained from $E_b$ for $b = 0, \ldots, N - 1$. Then $\rho_{\text{ave}}(E_b^*) < 5(N + 1)/(N - 2)^2$ for any $b = 0, \ldots, N - 1$.

Table 7 compares designs obtained from the leave-one-out linear permutation, leave-one-out Williams transformation, Gilbert, and Golomb methods. The leave-one-out Williams transformation generates designs with the smallest $\rho_{\text{ave}}$ values except for $N = 13$. 
Table 8

Comparison of the $\rho_{\text{ave}}$ values for $m \times m$ LHDs

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For the modified Williams transformation, we have the following result.

**Theorem 7.** Let $N = 2m + 1$ be an odd prime, $D = (x_{ij})$ be an $N \times (N - 1)$ GLP design, $A_1$ be the $m \times m$ leading principal submatrix of $D$, that is, $A_1 = (x_{ij})$ with $i, j = 1, \ldots, m$, and $H = w(A_1)/2$. Then $\rho_{\text{ave}}(H) < 2/(m - 1)$.

Table 8 compares the $\rho_{\text{ave}}$ values of designs generated by the modified Williams transformation and some other available methods. The modified Williams transformation always provides designs with the smallest $\rho_{\text{ave}}$ values.

**5. Extension.** We consider extending the results to a general case where $N = kp$ with $k$ and $p$ being prime numbers. Let

$$b = \lfloor N(1 + 1/\sqrt{3})/4 \rfloor,$$

and $E_b$ be the $N \times \phi(N)$ design constructed by the Williams transformation. Figure 2 (top) shows the values of $d_{\text{eff}}(E_b)$ for $N = 2p, 3p, 5p$ and $7p$ and $p \leq 200$. The $d_{\text{eff}}(E_b)$ increases quickly as $N$ increases and reaches 0.9 when $N$ is around 30. When $N > 100$, the $d_{\text{eff}}(E_b)$ values are typically greater than 0.95 and converge to 1 for $N = 2p$ and $N = 7p$. The $d_{\text{eff}}(E_b)$ values do not converge to 1 for $N = 3p$ and $N = 5p$, possibly due to the looseness of the upper bound $d_{\text{upper}}$. In addition, Figure 2 (bottom) shows that $\rho_{\text{ave}}(E_b)$ goes to 0 quickly as $N$ increases.

We present the asymptotic optimality of $E_b$ for $N = 2p$ based on the theoretical results in Section 3. It is possible to establish similar results for other cases with more elaborate arguments, which we do not pursue here.
Fig 2. The values of $d_{\text{eff}}(E_b)$ (top) and $\rho_{\text{ave}}(E_b)$ (bottom) with $b$ defined in (5.1).
Theorem 8. Let $p$ be an odd prime, $N = 2p$, $D$ be an $N \times \phi(N)$ GLP design, $D_b = D + b \pmod{N}$, and $E_b = W(D_b)$. For $b$ defined in (5.1), $d_{\text{eff}}(E_b) = 1 - O(1/N)$. As $N \to \infty$, $d_{\text{eff}}(E_b) \to 1$.

Now we consider an extension of the leave-one-out procedure. We can generate many asymptotically optimal LHDs by applying the following leave-out-one procedure for rows or columns. When we delete any row from an $N \times n$ LHD $D$ and rearrange the levels as in the leave-one-out method in Section 2.2, the distance of the resulting design will reduce at most by $n$. When we delete any column from an $N \times n$ LHD $D$, the distance will reduce at most by $N - 1$. Deleting multiple columns and rows together is equivalent to repeating the leave-one-out procedure for multiple times. The following result can be derived.

Theorem 9. Let $D$ be an $N \times n$ LHD. Deleting any $k_r$ rows and $k_c$ columns and rearranging the levels yields an $(N - k_r) \times (n - k_c)$ LHD, denoted by $D^*$. Then $d_{\text{eff}}(D^*) \geq d_{\text{eff}}(D) - 3k_r/(N - k_r) - 3k_c/(n - k_c)$.

For $N = kp$ and $n = \phi(N)$, $n \to \infty$ as $N \to \infty$. If $k_r$ and $k_c$ are fixed constants not increasing with $N$, $d_{\text{eff}}(D^*) \to 1$ as $N \to \infty$. This multiple leave-one-out procedure yields many asymptotically optimal LHDs with different sizes. For example, let $k = 3$ and $p = 41$, we obtain a $123 \times 80$ LHD with $d_{\text{eff}} = 0.956$. Delete the last 22 rows and rearrange the levels; we obtain a $101 \times 80$ LHD with $d_{\text{eff}} = 0.948$. Let $k = 2$ and $p = 61$, we obtain a $122 \times 60$ LHD with $d_{\text{eff}} = 0.980$. Delete the last 21 rows and rearrange the levels; we obtain a $101 \times 60$ LHD with $d_{\text{eff}} = 0.961$. Let $k = 5$ and $p = 103$, we obtain a $515 \times 408$ LHD with $d_{\text{eff}} = 0.962$. Delete the last 3 rows and the last 8 columns, and rearrange the levels, we obtain a $512 \times 400$ LHD with $d_{\text{eff}} = 0.953$. A distinctive feature of our method is the excellent performance for moderate and large designs. Many other methods slow down quickly as the design size increases and usually give designs with poor distance efficiencies. In contrast, our method generates moderate and large designs with guaranteed high distance efficiencies without search, as long as the ratios of $k_r/N$ and $k_c/\phi(N)$ are small. When the ratios are relatively large, this simple procedure may not work well and further research is needed.

6. Concluding remarks. We have proposed a series of systematic methods for the construction of maximin LHDs via the Williams transformation and its modification. The Williams transformation and leave-one-out method produce asymptotically optimal LHDs under the maximin distance criterion, and the modified Williams transformation generates equidistant
LHDs under the $L_1$-distance. Xu (1999) showed that equidistant LHDs are universally optimal for computer experiments. The average correlations between columns of the constructed designs converge to zero as the design sizes increase. Moreover, the constructed designs often have larger $L_1$-distance and smaller average correlation than existing designs even for designs with small sizes.

The Williams transformation can be applied to other designs as well. We have explored the Williams transformation on regular fractional factorial designs and found that it can substantially improve design efficiencies for estimating polynomial models. We will report the results in a separate paper.

Acknowledgement. The authors thank an editor, an associate editor and two reviewers for their helpful comments.

APPENDIX: PROOFS

We need to distinguish two addition operations. For clarify, let $\oplus$ be the addition operation over the Galois field $\{0, \ldots, N-1\}$. Let $D = (x_{ij})$ be the $N \times \phi(N)$ GLP design and $D_b = (x_{ij} \oplus b)$. When $N$ is a prime, $x_i = (x_{i1}, \ldots, x_{i(N-1)})$ and $x_i \oplus b = (x_{i1} \oplus b, \ldots, x_{i(N-1)} \oplus b)$ are the $i$th row of $D$ and $D_b$, respectively, $x_i$ is a permutation of $\{1, \ldots, N-1\}$ for $i = 2, \ldots, N-1$; and $x_1 = (1, \ldots, N-1)$. The designs $D$ and $D_b$ have some important properties which are crucial for the proofs of all theoretical results. We first summarize these properties in the following lemma.

**Lemma 3.** Let $N$ be an odd prime.

(i) For $i \neq k$ and $i, k = 1, \ldots, N-1$, there exists a unique $q \in \{2, \ldots, N-1\}$ such that $k = iq \pmod{N}$. For any given $b$, the two matrices

\[
\begin{pmatrix}
    x_i \oplus b \\
    x_k \oplus b
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
    x_1 \oplus b \\
    x_q \oplus b
\end{pmatrix}
\]

are the same up to column permutations. In addition, $q = N-1$ if and only if $i + k = N$.

(ii) For any $b = 0, \ldots, N-1$ and $i = 2, \ldots, N-2$, denote $a = (1-i)b \pmod{N}$. The two matrices

\[
\begin{pmatrix}
    x_1 \oplus b & b \\
    x_i \oplus b & b
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
    x_1 & 0 \\
    x_i \oplus a & a
\end{pmatrix}
\]

are the same up to column permutations.
Proof. Part (i) is obvious from the definition of $D$ and $D_b$. For (ii), denote $\tilde{x}_i = (x_i, 0)$ for $i = 1, \ldots, N$. Then $\tilde{x}_1 \oplus b = i(\tilde{x}_1 \oplus b) \oplus a$. The result follows by noting that $\tilde{x}_1 \oplus b$ is a permutation of $\tilde{x}_1$ and $i\tilde{x}_1 \oplus a = \tilde{x}_i \oplus a = (x_i \oplus a, a)$.

Proof of Lemma 2. We divide the proof in four steps.

Step 1. For $i + k \neq N$, $i \neq k$, and $i, k = 1, \ldots, N - 1$, by Lemma 3(i), there exists a unique $q \in \{2, \ldots, N - 2\}$ such that $d_{ik}(W(D_b)) = d_{1q}(w(D_b))$ and $d_{ik}(w(D_b)) = d_{1q}(w(D_b))$. Therefore, it suffices to show that $d_{1i}(W(D_b)) = d_{11}(w(D_b))$ for any $b = 0, \ldots, N - 1$ and $i = 2, \ldots, N - 2$.

Step 2. By Lemma 3(ii), to prove $d_{1i}(W(D_b)) = d_{1i}(w(D_b))$, we only need to show that $d(W(x_1), W(x_i \oplus a) + W(a) = d(w(x_1), w(x_i \oplus a) + w(a))$ for any $a = 0, \ldots, N - 1$. Note that $W(a) = w(a)$ if $a < N/2$, and $W(a) = w(a) - 1$ if $a > N/2$. It suffices to show that

\[(A.1) \quad d(W(x_1), W(x_i \oplus a)) = \begin{cases} d(w(x_1), w(x_i \oplus a)), & \text{if } a < N/2; \\ d(w(x_1), w(x_i \oplus a)) + 1, & \text{if } a > N/2. \end{cases} \]

Step 3. Recall that $x_1 = (1, \ldots, N - 1)$ and $x_i \oplus a = (x_i \oplus a, \ldots, x_i(N - 1) \oplus a)$. Then $d(W(x_1), W(x_i \oplus a)) = \sum_{j=1}^{N-1} \left|W(j) - W(x_{ij} \oplus a)\right|$ and $d(w(x_1), w(x_i \oplus a)) = \sum_{j=1}^{N-1} \left|w(j) - w(x_{ij} \oplus a)\right|$. It can be shown that

\[|W(j) - W(x_{ij} \oplus a)| = \begin{cases} |w(j) - w(x_{ij} \oplus a)|, & \text{for } j \in I \cup J; \\ |w(j) - w(x_{ij} \oplus a)| - 1, & \text{for } j \in U \setminus I; \\ |w(j) - w(x_{ij} \oplus a)| + 1, & \text{for } j \in V \setminus J, \end{cases} \]

where

\[I = \{j : j < N/2, (x_{ij} \oplus a) < N/2\}, \quad J = \{j : j > N/2, (x_{ij} \oplus a) > N/2\}, \quad U = \{j : j + (x_{ij} \oplus a) < N\}, \quad \text{and} \quad V = \{j : j + (x_{ij} \oplus a) \geq N\}.\]

Therefore, to prove (A.1), we need to show that if $a < N/2$, $U \setminus I$ and $V \setminus J$ contain the same number of elements; and if $a > N/2$, $U \setminus I$ contains one less element than $V \setminus J$.

Step 4. Denote $\#S$ as the number of elements in a set $S$. Since $\#(U \setminus I) = \#U - \#I$ and $\#(V \setminus J) = \#V - \#J$, we want to show that

\[\#U = \#V \quad \text{and} \quad \begin{cases} \#I = \#J, & \text{if } a < N/2; \\ \#I = \#J + 1, & \text{if } a > N/2. \end{cases} \]

Since

\[x_{(i+1)j} \oplus a = \begin{cases} j + (x_{ij} \oplus a), & \text{for } j \in U; \\ j + (x_{ij} \oplus a) - N, & \text{for } j \in V, \end{cases} \]
then $\sum_{j=1}^{N-1} (x_{i(j+1)} \oplus a) = \sum_{j=1}^{N-1} (x_{ij} \oplus a) + \sum_{j=1}^{N-1} j - (\#V)N$. Because both $x_i$ and $x_{i+1}$ are permutations of $\{1, \ldots, N-1\}$, $\sum_{j=1}^{N-1} (x_{i(j+1)} \oplus a) = \sum_{j=1}^{N-1} (x_{ij} \oplus a)$, which leads to $\#V = \sum_{j=1}^{N-1} j/N = (N-1)/2$. Because $\#U + \#V = N - 1$, $\#U = \#V = (N-1)/2$. Denote $I_1 = \{ j : j > N/2, x_{ij} \oplus a < N/2 \}$. If $a < N/2$, $\#I + \#I_1 = \#J + \#I_1 = (N-1)/2$ so $\#I = \#J$. If $a > N/2$, $\#I + \#I_1 = (N+1)/2$ and $\#J + \#I_1 = (N-1)/2$ so $\#I = \#J + 1$. This completes the proof. 

To prove Theorem 1, we need the following lemma.

**Lemma 4.** For all $i = 2, \ldots, N-2$ and $b = 0, \ldots, N-1$, $d(x_1 \oplus b, x_i \oplus b) + d(N - (x_1 \oplus b), x_i \oplus b) = (2N^2 + 1)/3 - |N - 2b|$.

**Proof.** We divide the proofs in three steps.

*Step 1.* By Lemma 3(ii),

\[
\begin{align*}
d(x_1 \oplus b, x_i \oplus b) &= d(x_1, x_i \oplus a) + a, \quad \text{and} \\
d(N - (x_1 \oplus b), x_i \oplus b) + |N - 2b| &= d(N - x_1, x_i \oplus a) + N - a,
\end{align*}
\]

where $a = (1 - i)b \mod N$. Then,

\[
d(x_1 \oplus b, x_i \oplus b) + d(N - (x_1 \oplus b), x_i \oplus b) = d(x_1, x_i \oplus a) + d(N - x_1, x_i \oplus a) + N - |N - 2b|.
\]

Hence, it suffices to show that $d(x_1, x_i \oplus a) + d(N - x_1, x_i \oplus a) = (2N^2 + 1)/3 - N = (N-1)(2N-1)/3$ for any $a = 0, \ldots, N-1$.

*Step 2.* Let $g_i(a) = d(x_1, x_i \oplus a) + d(N - x_1, x_i \oplus a)$. If we can prove $g_i(0) = g_i(1) = \cdots = g_i(N - 1)$, we will have

\[
g_i(a) = \frac{1}{N} \sum_{c=0}^{N-1} g_i(c) = \frac{1}{N} \sum_{c=0}^{N-1} (d(x_1, x_i \oplus c) + d(N - x_1, x_i \oplus c)).
\]

Because $\sum_{c=0}^{N-1} d(N - x_1, x_i \oplus c) = \sum_{c=0}^{N-1} d(x_1, x_i \oplus c)$, then

\[
g_i(a) = \frac{2}{N} \sum_{c=0}^{N-1} d(x_1, x_i \oplus c) = \frac{2}{N} \sum_{c=0}^{N-1} \sum_{j=1}^{N-1} |j - (x_{ij} \oplus c)|
\]

\[
= \frac{2}{N} \sum_{j=1}^{N-1} \sum_{k=0}^{N-1} |j - k| = (N-1)(2N-1)/3.
\]

*Step 3.* Now we prove that $g_i(0) = g_i(1) = \cdots = g_i(N - 1)$. It suffices to show that $g_i(a + 1) = g_i(a)$ for any $a = 0, \ldots, N - 2$. Recall that $g_i(a) =$
Then we only need to show that
\[
|j - (x_{ij} \oplus (a + 1))| + |N - j - (x_{ij} \oplus (a + 1))|
\]
\[
= \begin{cases}
|j - (x_{ij} \oplus a)| + |N - j - (x_{ij} \oplus a)|, & \text{for } j \in S_1 \cup S_2; \\
|j - (x_{ij} \oplus a)| + |N - j - (x_{ij} \oplus a)| + 2, & \text{for } j \in S_3; \\
|j - (x_{ij} \oplus a)| + |N - j - (x_{ij} \oplus a)| - 2, & \text{for } j \in S_4,
\end{cases}
\]
where
\[
S_1 = \{j : j \leq x_{ij} \oplus a < N - j\}, \quad S_2 = \{j : N - j \leq x_{ij} \oplus a < j\}, \\
S_3 = \{j : x_{ij} \oplus a \geq j, x_{ij} \oplus a \geq N - j\}, \quad S_4 = \{j : x_{ij} \oplus a < j, x_{ij} \oplus a < N - j\},
\]
we only need to show that \#S_3 = \#S_4. Note that
\[
\begin{align*}
x_{(i-1)j} \oplus a &= x_{ij} \oplus a - j \quad \text{and} \quad x_{(i+1)j} \oplus a = x_{ij} \oplus a + j, & \text{for } j \in S_1; \\
x_{(i-1)j} \oplus a &= x_{ij} \oplus a - j + N \quad \text{and} \quad x_{(i+1)j} \oplus a = x_{ij} \oplus a + j - N, & \text{for } j \in S_2; \\
x_{(i-1)j} \oplus a &= x_{ij} \oplus a - j \quad \text{and} \quad x_{(i+1)j} \oplus a = x_{ij} \oplus a + j - N, & \text{for } j \in S_3; \\
x_{(i-1)j} \oplus a &= x_{ij} \oplus a - j + N \quad \text{and} \quad x_{(i+1)j} \oplus a = x_{ij} \oplus a + j, & \text{for } j \in S_4.
\end{align*}
\]
Then
\[
(A.2) \quad \sum_{j=1}^{N-1} ((x_{(i-1)j} \oplus a) + (x_{(i+1)j} \oplus a)) = 2 \sum_{j=1}^{N-1} (x_{ij} \oplus a) - N(\#S_3 - \#S_4).
\]
Because \(x_i \oplus a\) is a permutation of \(\{0, \ldots, a - 1, a + 1, \ldots, N - 1\}\) for any \(i < N\), \(\sum_{j=1}^{N-1} (x_{ij} \oplus a) = \sum_{j=1}^{N-1} (x_{ij} \oplus a) = \sum_{j=1}^{N-1} (x_{(i+1)j} \oplus a)\). By (A.2), \(N(\#S_3 - \#S_4) = 0\) so \#S_3 = \#S_4. This completes the proof. \(\square\)

**Proof of Theorem 1.** For the first case, note that \(W(x_i \oplus b)\) is a permutation of \(\{0, \ldots, W(b) - 1, W(b) + 1, \ldots, N - 1\}\), and \(W(x_N \oplus b)\) is a constant vector with each component equal to \(W(b)\), so \(d_i(N)(E_b) = d_{N_i}(E_b) = \sum_{j=0}^{N-1} |j - W(b)| = (N^2 - 1)/3 + f(b)\).

To prove the result for the second case, \(i = N - k\), it suffices to prove the result for the third case. This is because the total pairwise \(L_1\)-distance between distinct rows of \(W(D_b)\) is \(t = (N - 1)\sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1} |j_1 - j_2| = N(N - 1)^2(N + 1)/6\). Out of all the pairs of distinct rows, \(N - 1\) pairs belong to the first case with a total distance \(t_1 = (N - 1)[(N^2 - 1)/3 + f(b)]\), \((N - 1)(N - 3)/2\) pairs belong to the third case with a total distance \(t_2 = (N^2 - 1)(N - 1)(N - 3)/6\), and \((N - 1)/2\) pairs belong to the second case. By Lemma 3(i), \(d_i(N-i)(E_b) = d_{1(N-i)}(E_b)\) for any \(i\). Therefore, \(d_i(N-i)(E_b) = (t - t_1 - t_2)/[(N - 1)/2] = (N^2 - 1)/3 - 2f(b)\).
Now we prove the result for the last case where $i \neq N - k$, $i \neq N$, and $k \neq N$. By Lemmas 2 and 3(i), it suffices to consider $d_{i}(E_b) = d(W(x_i \oplus b), W(x_i \oplus b)) = d(w(x_i \oplus b), w(x_i \oplus b))$ for $i = 2, \ldots, N - 2$. Denote

$$
B = \begin{pmatrix}
B_1 & B_2 & B_3 & B_4 \\
2(x_1 \oplus b) & 2(x_1 \oplus b) & 2N - 2(x_1 \oplus b) & 2N - 2(x_1 \oplus b) \\
2(x_i \oplus b) & 2N - 2(x_i \oplus b) & 2N - 2(x_i \oplus b) & 2N - 2(x_i \oplus b)
\end{pmatrix},
$$

then $d_{i}(E_b) = d(B_1)$. By column permutations, $B$ can be rearranged as

$$
C = \begin{pmatrix}
B_1 & B_2 & B_3 & B_4 \\
2(x_1 \oplus b) & 2(x_1 \oplus b) & 2N - 2(x_1 \oplus b) & 2N - 2(x_1 \oplus b) \\
2(x_i \oplus b) & 2N - 2(x_i \oplus b) & 2N - 2(x_i \oplus b) & 2N - 2(x_i \oplus b)
\end{pmatrix}.
$$

By Lemma 4, $d(B) = d(C) = 4((2N^2 + 1)/3 - |N - 2b|)$. Note that $d(B_1) = d(B_4)$ and $d(B_2) = d(B_3)$. For $B_2$, in both $w(x_1 \oplus b)$ and $w(x_i \oplus b)$, 0 and $w(b)$ appear once and all other even numbers smaller than $N$ appear twice. Then $d(B_2) = \sum_{j=1}^{N-1} (N - w(x_j \oplus b) - w(x_j \oplus b)) = (N^2 + 1) - 2|N - 2b|$. Therefore, $d_{i}(E_b) = d(B_1) = (d(B) - 2d(B_2))/2 = (N^2 - 1)/3.

**Proof of Theorem 2.** If $c_0^2 + 2(c_0 + 1)^2 \geq (N^2 - 1)/4$, then $c_0 = 2\sqrt{(N^2 - 1)/12} - 2/9 - 2/3$ and $c_0^2 \geq (N^2 - 1)/12 - (4/3)(\sqrt{(N^2 - 1)/12}$. Hence, $d(E_b) = (N^2 - 1)/4 + c_0^2 \geq (N^2 - 1)/3 - (4/3)(\sqrt{(N^2 - 1)/12}$. Similarly, if $c_0^2 + 2(c_0 + 1)^2 < (N^2 - 1)/4$, $c_0 + 1 \leq \sqrt{(N^2 - 1)/12} - 2/9 + 1/3$, and $(c_0 + 1)^2 \leq (N^2 - 1)/12 + (2/3)(\sqrt{(N^2 - 1)/12}$. Then $d(E_b) = (N^2 - 1)/2 - 2(c_0 + 1)^2 \geq (N^2 - 1)/3 - (4/3)(\sqrt{(N^2 - 1)/12}$. Therefore,

$$
d(E_b) \geq \frac{N^2 - 1}{3} - \frac{4}{3}\sqrt{\frac{N^2 - 1}{12}} = \frac{N^2 - 1}{3} - 2\sqrt{\frac{N^2 - 1}{3}}.
$$

By the definition in (3.2), $d_{\text{eff}}(E_b) = d(E_b)/((N^2 - 1)/3) \geq 1/2\sqrt{3(N^2 - 1)}$.

**Proof of Theorem 3.** Let $e_i = (e_{i1}, \ldots, e_{i(N-1)})$ and $e_k = (e_{k1}, \ldots, e_{k(N-1)})$ be two distinct rows of $E_b$ for $i, k = 1, \ldots, N - 1$, and $e_i^* = (e_{i1}^*, \ldots, e_{i(N-1)}^*)$ and $e_k^* = (e_{k1}^*, \ldots, e_{k(N-1)}^*)$ be the corresponding rows of $E_b^*$. For $j = 1, \ldots, N - 1$, if $e_{ij} > W(b) > e_{kj}$ or $e_{kj} > W(b) > e_{ij}$, $|e_{ij} - e_{kj}| = |e_{ij} - e_{kj}| - 1$; otherwise, $|e_{ij} - e_{kj}| = |e_{ij} - e_{kj}|$. Since the number of $j$’s such that $e_{ij} > W(b) > e_{kj}$ (or $e_{kj} > W(b) > e_{ij}$) cannot exceed min{$W(b), N - 1 - W(b)$}, then $d(E_b) \geq d(E_b^*) - 2\min\{W(b), N - 1 - W(b)\}$.

For the $b$ defined in (3.1), $\min\{W(b), N - 1 - W(b)\} = (N - 1)/2 - c$. Then $d(E_b) \geq d(E_b) - (N - 1) + 2c = d(E_b) - (N - 1) + 2(\sqrt{(N^2 - 1)/12} - 1)$. 


By Theorem 2, \( d(E_b^*) \geq (N^2 - 7)/3 + \sqrt{(N^2 - 1)/3} - (N - 1) \). When \( N \geq 7 \), we have \( d_{\text{eff}}(E_b^*) = d(E_b^*)/[N(N - 1)/3] \geq d(E_b^*)/(N(N - 1)/3) \geq 1 + 1/(\sqrt{3N}) - 3/N > 1 - 2.43/N \). \( \Box \)

**Proof of Theorem 5.** Let \( \rho_{jk} \) be the correlation between the \( j \)th and \( k \)th columns of \( E_b \). Denote the \( j \)th column of \( D_b \) as \( \tilde{z}_j \oplus b \) for \( j = 1, \ldots, N - 1 \), then \( \tilde{z}_j \oplus b = (x_j \oplus b, b)^T \). By Lemma 3(i), there exists a unique \( q \in \{2, \ldots, N - 1\} \) such that \( \rho_{jk} = \rho_{1q} \). Thus,

\[
\rho_{\text{ave}}(E_b) = \sum_{j=2}^{N-1} |\rho_{1j}| \tag{A.3}
\]

where

\[
\rho_{1j} = \text{cor}(W(\tilde{z}_1 \oplus b), W(\tilde{z}_j \oplus b)) = \frac{\sum_{i=1}^{N} (W(x_{i1} \oplus b) - \frac{N-1}{2}) (W(x_{ij} \oplus b) - \frac{N-1}{2})}{(N^3 - N)/12} \tag{A.4}
\]

For \( x \in [0, N] \), the Fourier cosine expansion of \( x - N/2 \) is given by

\[
x - \frac{N}{2} = \sum_{u=1}^{\infty} a_u \cos \left( \frac{u\pi x}{N} \right), \tag{A.5}
\]

with

\[
a_u = \frac{2}{N} \int_0^N \left( x - \frac{N}{2} \right) \cos \left( \frac{u\pi x}{N} \right) dx = \begin{cases} 0, & \text{if } u \text{ is even}; \\ -4N/(u^2\pi^2), & \text{if } u \text{ is odd}. \end{cases}
\]

By (A.5), for any \( x + 0.5 \in [0, N] \),

\[
x - \frac{N - 1}{2} = (x + 0.5) - \frac{N}{2} = \sum_{u=1}^{\infty} a_u \cos \left( \frac{u\pi(x + 0.5)}{N} \right).
\]

Then the numerator of (A.4) is

\[
\sum_{i=1}^{N} (W(x_{i1} \oplus b) - \frac{N-1}{2}) (W(x_{ij} \oplus b) - \frac{N-1}{2}) = \sum_{u=1}^{\infty} \sum_{v=1}^{\infty} a_u a_v s(u, v) = \frac{16N^2}{\pi^4} \sum_{u \text{ odd}} \sum_{v \text{ odd}} \frac{1}{u^2v^2} s(u, v), \tag{A.6}
\]

where

\[
s(u, v) = \sum_{i=1}^{N} \cos \left( \frac{u\pi(W(x_{i1} \oplus b) + 0.5)}{N} \right) \cos \left( \frac{v\pi(W(x_{ij} \oplus b) + 0.5)}{N} \right).
\]
By (2.1), for any \( x = 0, \ldots, N - 1 \), \( \cos(u \pi (W(x) + 0.5)/N) = \cos(u \pi (2x + 0.5)/N) \). Then

\[
(A.7) \quad s(u, v) = \sum_{i=1}^{N} \cos \left( \frac{u \pi (2x_{i1} + 2b + 0.5)}{N} \right) \cos \left( \frac{v \pi (2x_{ij} + 2b + 0.5)}{N} \right)
\]

\[
= \frac{1}{2} \sum_{i=1}^{N} \cos \left( \frac{2 \pi ((jv + u)i + c_1)}{N} \right) + \frac{1}{2} \sum_{i=1}^{N} \cos \left( \frac{2 \pi ((jv - u)i + c_2)}{N} \right),
\]

where \( c_1 = (b + 0.25)(u + v) \) and \( c_2 = (b + 0.25)(v - u) \). For positive odd numbers \( u \) and \( v \), let \( I_1 = \{ (u, v) : u = jv \text{ or } -jv, v \neq 0 \text{ (mod } N) \} \) and \( I_2 = \{ (u, v) : u = 0 \text{ and } v = 0 \text{ (mod } N) \} \). For \( (u, v) \in I_1 \), \( |s(u, v)| \leq N/2 \) because only one of the two items in (A.7) can be nonzero. For \( (u, v) \in I_2 \), \( |s(u, v)| \leq N \); for \( (u, v) \notin I_1 \cup I_2 \), \( s(u, v) = 0 \). Then by (A.3), (A.4), and (A.6),

\[
\rho_{\text{ave}}(E_b) = \frac{\sum_{j=2}^{N-1} \left| \sum_{i=1}^{N} \left( W(x_{i1} \oplus b) - \frac{N-1}{2} \right) \left( W(x_{ij} \oplus b) - \frac{N-1}{2} \right) \right|}{(N - 2)(N^3 - N)/12}
\]

\[
\leq \frac{192N^2}{\pi^4(N^3 - N)(N - 2)} \sum_{j=2}^{N-1} \left( \sum_{I_1} \frac{N}{2} \frac{1}{u^2v^2} + \sum_{I_2} \frac{N}{2} \frac{1}{u^2v^2} \right),
\]

\[
(A.8) \quad = \frac{192N^2}{\pi^4(N^2 - 1)(N - 2)} \sum_{j=2}^{N-1} \left( \sum_{I_1} \frac{1}{2u^2v^2} + \sum_{I_2} \frac{1}{2u^2v^2} \right).
\]

Since

\[
\sum_{j=2}^{N-1} \left( \sum_{I_1} \frac{1}{2u^2v^2} + \sum_{I_2} \frac{1}{u^2v^2} \right)
\]

\[
\leq \frac{1}{2} \sum_{\text{odd } v} \frac{1}{v^2} \left( \frac{2}{\sum_{\text{odd } u} u^2} - \sum_{k=0}^{\infty} (v + 2kN)^2 \right) - \sum_{\text{odd } k} \frac{1}{k^2 N^2}
\]

\[
\leq \sum_{\text{odd } v} \frac{1}{v^2} \sum_{\text{odd } u} u^2 - \frac{1}{2} \sum_{\text{odd } v} v^4 - \frac{1}{N^2} \sum_{\text{odd } v} \frac{1}{v^2} \sum_{\text{odd } k} \frac{1}{k^2}
\]

\[
= \frac{N^2 - 1}{N^2} \left( \frac{\pi^4}{8^2} \right) - \frac{\pi^4}{192},
\]

where we used the fact that \( \sum_{\text{odd } v} 1/v^2 = \pi^2/8 \) and \( \sum_{\text{odd } v} 1/v^4 = \pi^4/96. \)
Then by (A.8),
\[
\rho_{\text{ave}}(E_b) \leq \frac{1}{N-2} \frac{192N^2}{\pi^4(N^2-1)} \left( \frac{N^2 - 1}{N^2} \left( \frac{\pi^4}{8^2} \right) - \frac{\pi^4}{192} \right) \\
= \frac{1}{N-2} \left( 3 - \frac{N^2}{N^2-1} \right) < \frac{2}{N-2}.
\]

**Proof of Theorem 6.** For any \(b = 0, \ldots, N-1\), let \(E_b = (e_{ij})\). Because 
\[
\sum_{i=1}^{N}(e_{ij} - (N-1)/2)^2 = N(N^2-1)/12 \text{ for any } j = 1, \ldots, N-1, \text{ by Theorem 5, we have}
\]
(A.9) 
\[
\sum_{j=2}^{N-1} \sum_{i=1}^{N} \left| e_{i1} - \frac{N-1}{2} \right| \left| e_{ij} - \frac{N-1}{2} \right| < \frac{N(N^2-1)}{6}.
\]

Let \(\rho_{jk}^*\) be the correlation between the \(j\)th and \(k\)th columns of \(E_b^*\). Similar to (A.3),
(A.10) 
\[
\rho_{\text{ave}}(E_b^*) = \frac{\sum_{j=2}^{N-1} |\rho_{ij}^*|}{N-2}.
\]

Note that
(A.11) 
\[
\rho_{ij}^* = \frac{12C_0}{N(N-1)(N-2)}
\]

with
\[
C_0 = \sum_{e_{i1}<W(b)} (e_{i1} - \mu) (e_{ij} - \mu) + \sum_{e_{i1}>W(b)} (e_{i1} - 1 - \mu) (e_{ij} - \mu) \\
+ \sum_{e_{i1}<W(b) \atop e_{ij}>W(b)} (e_{i1} - \mu) (e_{ij} - 1 - \mu) + \sum_{e_{i1}>W(b) \atop e_{ij}>W(b)} (e_{i1} - 1 - \mu) (e_{ij} - 1 - \mu) \\
= \sum_{i=1}^{N} \left( e_{i1} - \frac{N-1}{2} \right) \left( e_{ij} - \frac{N-1}{2} \right) + C_1 + C_2,
\]

where \(\mu = (N-2)/2\),
\[
C_1 = \frac{1}{2} \left( \sum_{e_{i1}<W(b)} e_{ij} - \sum_{e_{i1}>W(b)} e_{ij} + \sum_{e_{ij}<W(b)} e_{i1} - \sum_{e_{ij}>W(b)} e_{i1} \right) \\
+ \frac{(N-1)^2}{4} - (W(b))^2
\]
and
\[ C_2 = \frac{1}{4} \left( \sum_{e_{i1} < W(b)} 1 + \sum_{e_{ij} < W(b)} 1 - \sum_{e_{i1} > W(b)} 1 - \sum_{e_{ij} > W(b)} 1 \right). \]

It is easy to see that \(|C_1| \leq (N^2 - 1)/4\) and \(|C_2| \leq (N - 1)/4\). Hence, by (A.9), (A.10), and (A.11),
\[
\rho_{\text{ave}}(E_0^*) < \frac{12}{N(N-1)(N-2)^2} \left( \frac{N(N^2 - 1)}{6} + \frac{(N-2)(N^2 - 1)}{4} + \frac{(N-2)(N-1)}{4} \right)
\leq \frac{5(N+1)}{(N-2)^2}.
\]

**Proof of Theorem 7.** The proof is similar to that of Theorem 5. By (A.5), for \(j = 1, \ldots, (N-1)/2\),
\[
\sum_{i=1}^{N} \left( w(x_{i1}) - \frac{N}{2} \right) \left( w(x_{ij}) - \frac{N}{2} \right) = \frac{16N^2}{\pi^4} \sum_{\text{odd } u,v} \frac{1}{u^2 v^2} s(u,v),
\]
where
\[
s(u,v) = \sum_{i=1}^{N} \cos \left( \frac{u \pi w(x_{i1})}{N} \right) \cos \left( \frac{v \pi w(x_{ij})}{N} \right).
\]

Similar to (A.8), we can prove that
\[
\sum_{j=2}^{(N-1)/2} \sum_{i=1}^{N} \left| w(x_{i1}) - \frac{N}{2} \right| \left( w(x_{ij}) - \frac{N}{2} \right) \leq \frac{N^3}{24}.
\]

Since
\[
\sum_{i=1}^{N-1} \left( w(x_{i1}) - \frac{N+1}{2} \right) \left( w(x_{ij}) - \frac{N+1}{2} \right) = \sum_{i=1}^{N} \left( w(x_{i1}) - \frac{N}{2} \right) \left( w(x_{ij}) - \frac{N}{2} \right) - (N-1) + \frac{(N+1)^2 + 1}{4},
\]

Proof
then
\[
\sum_{j=2}^{(N-1)/2} \left| \sum_{i=1}^{N-1} \left( w(x_{i1}) - \frac{N+1}{2} \right) \left( w(x_{ij}) - \frac{N+1}{2} \right) \right| 
\leq \frac{N^3}{24} + \left( \frac{N-1}{2} - 1 \right) \left( \frac{(N+1)^2 + 1}{4} - (N-1) \right)
\]
= \frac{N^3}{6} - \frac{5N^2 - 12N + 18}{8} 
\leq \frac{1}{6}(N+1)(N-1)(N-3).
\]

Hence,

\[
\rho_{\text{ave}}(H) = \rho_{\text{ave}}(w(A_1)) 
= \frac{\sum_{j=2}^{(N-1)/2} \left| \sum_{i=1}^{N-1} \left( w(x_{i1}) - \frac{N+1}{2} \right) \left( w(x_{ij}) - \frac{N+1}{2} \right) \right|}{(m-1)(N+1)(N-1)(N-3)/12}
\leq \frac{2}{m-1}.
\]

\[\square\]

**Proof of Theorem 8.** To save space, we sketch only the main steps.

**Step 1.** For \( N = 2p \), \( \phi(N) = p - 1 \) and \( D = (x_{ij}) \) with \( x_{ij} = i(2j - 1) \pmod{N} \) for \( i = 1, \ldots, 2p \) and \( j = 1, \ldots, p - 1 \). With proper row and column permutations, \( D \) is equivalent to

\[
\left( \begin{array}{c}
2C \\
2C + p
\end{array} \right) \pmod{N},
\]

where \( C = (y_{ij}) \) is an \( p \times (p - 1) \) GLP design with \( y_{ij} = i \cdot j \pmod{p} \) for \( i = 1, \ldots, p \) and \( j = 1, \ldots, p - 1 \). Then \( E_b = W(D_b) \) is equivalent to

\[
\tilde{E}_b = \left( \begin{array}{c}
W(2C \oplus b) \\
W(2C \oplus (b + p))
\end{array} \right).
\]

**Step 2.** Consider \( W(2C \oplus b) \). If \( b \) is even, \( 2C \oplus b = 2(C + b/2 \pmod{p}) \). Then \( w(2C \oplus b) = 2w_p(C + b/2 \pmod{p}) \) where \( w \) is the modified Williams transformation defined in (2.2) and \( w_p \) is the modified Williams with \( N \) replaced by \( p \). By Lemma 2 and Theorem 1, \( d_{ik}(w(2C \oplus b)) = 2[d_{ik}(w_p(C + b/2 \pmod{p}))] = 2(N^2 - 1)/3 \) for \( i \neq k, i \neq p, k \neq p, \) and \( i + k \neq p \). Following the lines of Lemma 2 will result \( d_{ik}(W(2C \oplus b)) = d_{ik}(w(2C \oplus b)) \). Then

\[
d_{ik}(W(2C \oplus b)) = (N^2 - 4)/6 \text{ for } i \neq k, i \neq p, k \neq p, \text{ and } i + k \neq p.
\]
If $b$ is odd, $W(2C \oplus b) = N - 1 - W(2C \oplus (b + p))$ and (A.13) also holds.

Step 3. If $b$ is even, the last row of $W(2C \oplus b)$ is $(2b, \ldots, 2b)$ and each other row is a permutation of $\{0, 3, 4, \ldots, 2(p - 1) - 1, 2(p - 1)\}\backslash \{2b\}$. Based on this structure, we get

\begin{align}
(A.14) \quad d_{ip}(W(2C \oplus b)) &= \frac{N^2}{6} - \frac{N + 2}{4} + \frac{W(b)}{2} + \frac{g(b)}{2}, \\
(A.15) \quad d_{i(p-i)}(W(2C \oplus b)) &= \frac{N^2}{6} + \frac{N}{2} - 1 - W(b) - g(b),
\end{align}

where

$$g(b) = \left(\frac{W(b)}{2} - \frac{1}{2} \left(1 + \frac{1}{\sqrt{3}}\right)N\right) \left(\frac{W(b)}{2} - \frac{1}{2} \left(1 - \frac{1}{\sqrt{3}}\right)N\right).$$

Similarly, if $b$ is odd, (A.14) and (A.15) also hold.

Step 4. Because $W(2C \oplus b) = N - 1 - W(2C \oplus (b + p))$, $W(2C \oplus (b + p))$ has the same distance structure as $W(2C \oplus b)$.

Step 5. By the structure of $W(2C \oplus (b + p))$ and $W(2C \oplus b)$, by computation, we can get

\begin{align}
(A.16) \quad d_{i(p+k)}(\bar{E}(b)) &= \begin{cases} 
N^2/4 - l_1(b), & \text{for } i = k \neq p; \\
(N/2 - 1)l_1(b), & \text{for } i = k = p; \\
N^2/6 - l_1(b) + 1/3, & \text{for } (i, k) \in I_1; \\
N^2/6 - (N - 2)/4 + l_2(b)/2 - l_1(b), & \text{for } (i, k) \in I_2; \\
-N^2/12 + (N/2 - 1)l_1(b) + N/2 - l_2(b), & \text{for } (i, k) \in I_3.
\end{cases}
\end{align}

where $l_1(b) = \lfloor N - 2W(b) - 1 \rfloor$, $l_2(b) = W(b) + g(b)$, $I_1 = \{(i, k) : i \neq p, k \neq p, i + k \neq p\}$, $I_2 = \{(i, k) : i \neq p, k = p, \text{ or } i = p, k \neq p\}$, and $I_3 = \{(i, k) : i \neq p, k \neq p, i + k = p\}$.

Step 6. For $b = \lfloor N(1 + 1/\sqrt{3})/4 \rfloor$, $W(b) = 2b = \lfloor N(1 + 1/\sqrt{3})/2 \rfloor$ or $\lfloor N(1 + 1/\sqrt{3})/2 \rfloor + 1$, so $-N/\sqrt{3} \leq g(b) \leq 0$. Then $l_1(b) = O(N)$ and $l_2(b) = O(N)$. Since for any $N \times (N/2 - 1)$ LHD, $d_{\text{upper}} = (N + 1)(N - 2)/6$, by (A.13)–(A.16), it can be verified that $d_{\text{eff}}(E_b) = d_{\text{eff}}(\bar{E}_b) = 1 - O(1/N)$.

\section*{REFERENCES}


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