

VARIABLE SELECTION WITH HAMMING LOSS

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Abstract We derive non-asymptotic bounds for the minimax risk of variable selection under expected Hamming loss in the Gaussian mean model in \mathbb{R}^d for classes of s -sparse vectors separated from 0 by a constant $a > 0$. In some cases, we get exact expressions for the non-asymptotic minimax risk as a function of d, s, a and find explicitly the minimax selectors. These results are extended to dependent or non-Gaussian observations and to the problem of crowdsourcing. Analogous conclusions are obtained for the probability of wrong recovery of the sparsity pattern. As corollaries, we derive necessary and sufficient conditions for such asymptotic properties as almost full recovery and exact recovery. Moreover, we propose data-driven selectors that provide almost full and exact recovery adaptively to the parameters of the classes.

1. Introduction. In recent years, the problem of variable selection in high-dimensional regression models has been extensively studied from the theoretical and computational viewpoints. In making effective high-dimensional inference, sparsity plays a key role. With regard to variable selection in sparse high-dimensional regression, the Lasso, Dantzig selector, other penalized techniques as well as marginal regression were analyzed in detail; see, for example, [20, 27, 24, 19, 23, 25, 21, 12, 15] and the references cited therein. Several other recent papers deal with sparse variable selection in nonparametric regression; see, for example, [17, 5, 10, 14, 8].

In this paper, we study the problem of variable selection in the Gaussian sequence model

$$(1) \quad X_j = \theta_j + \sigma \xi_j, \quad j = 1, \dots, d,$$

where ξ_1, \dots, ξ_d are i.i.d. standard Gaussian random variables, $\sigma > 0$ is the noise level, and $\theta = (\theta_1, \dots, \theta_d)$ is an unknown vector of parameters to be estimated. We assume that θ is (s, a) -sparse, which is understood in the sense that θ belongs to one of the following sets:

$$\Theta_d(s, a) = \left\{ \theta \in \mathbb{R}^d : \text{there exists a set } S \subseteq \{1, \dots, d\} \text{ with } s \text{ elements} \right. \\ \left. \text{such that } |\theta_j| \geq a \text{ for all } j \in S, \text{ and } \theta_j = 0 \text{ for all } j \notin S \right\}$$

or

$$\Theta_d^+(s, a) = \left\{ \theta \in \mathbb{R}^d : \text{there exists a set } S \subseteq \{1, \dots, d\} \text{ with } s \text{ elements} \right. \\ \left. \text{such that } \theta_j \geq a \text{ for all } j \in S, \text{ and } \theta_j = 0 \text{ for all } j \notin S \right\}.$$

Keywords and phrases: adaptive variable selection, almost full recovery, exact recovery, Hamming loss, minimax selectors, nonasymptotic minimax selection bounds, phase transitions

Here, $a > 0$ and $s \in \{1, \dots, d\}$ are given constants.

We study the problem of selecting the relevant components of θ , that is, of estimating the vector

$$\eta = \eta(\theta) = (I(\theta_j \neq 0))_{j=1, \dots, d},$$

where $I(\cdot)$ is the indicator function. As estimators of η , we consider any measurable functions $\hat{\eta} = \hat{\eta}(X_1, \dots, X_n)$ of (X_1, \dots, X_n) taking values in $\{0, 1\}^d$. Such estimators will be called *selectors*. We characterize the loss of a selector $\hat{\eta}$ as an estimator of η by the Hamming distance between $\hat{\eta}$ and η , that is, by the number of positions at which $\hat{\eta}$ and η differ:

$$|\hat{\eta} - \eta| \triangleq \sum_{j=1}^d |\hat{\eta}_j - \eta_j| = \sum_{j=1}^d I(\hat{\eta}_j \neq \eta_j).$$

Here, $\hat{\eta}_j$ and $\eta_j = \eta_j(\theta)$ are the j th components of $\hat{\eta}$ and $\eta = \eta(\theta)$, respectively. The expected Hamming loss of a selector $\hat{\eta}$ is defined as $\mathbf{E}_\theta |\hat{\eta} - \eta|$, where \mathbf{E}_θ denotes the expectation with respect to the distribution \mathbf{P}_θ of (X_1, \dots, X_n) satisfying (1). Another well-known risk measure is the probability of wrong recovery $\mathbf{P}_\theta(\hat{S} \neq S(\theta))$, where $\hat{S} = \{j : \hat{\eta}_j = 1\}$ and $S(\theta) = \{j : \eta_j(\theta) = 1\}$. It can be viewed as the Hamming distance with an indicator loss and is related to the expected Hamming loss as follows:

$$(2) \quad \mathbf{P}_\theta(\hat{S} \neq S(\theta)) = \mathbf{P}_\theta(|\hat{\eta} - \eta| \geq 1) \leq \mathbf{E}_\theta |\hat{\eta} - \eta|.$$

In view of the last inequality, bounding the expected Hamming loss provides a stronger result than bounding the probability of wrong recovery.

Most of the literature on variable selection in high dimensions focuses on the recovery of the sparsity pattern, that is, on constructing selectors such that the probability $\mathbf{P}_\theta(\hat{S} \neq S(\theta))$ is close to 0 in some asymptotic sense (see, for example, [20, 27, 24, 19, 23, 25, 21]). These papers consider high-dimensional linear regression settings with deterministic or random covariates. In particular, for the sequence model (1), one gets that if $a > C\sigma\sqrt{\log d}$ for some $C > 0$ large enough, then there exist selectors such that $\mathbf{P}_\theta(\hat{S} \neq S(\theta))$ tends to 0, while this is not the case if $a < c\sigma\sqrt{\log d}$ for some $c > 0$ small enough. More insight into variable selection was provided in [12, 15] by considering a Hamming risk close to the one we have defined above. Assuming that $s \sim d^{1-\beta}$ for some $\beta \in (0, 1)$, the papers [12, 15] establish an asymptotic in d “phase diagram” that partitions the parameter space into three regions called the exact recovery, almost full recovery, and no recovery regions. This is done in a Bayesian setup for the linear regression model with i.i.d. Gaussian covariates and random θ . Note also that in [12, 15] the knowledge of β is required to construct the selectors, so that in this sense the methods are not adaptive. The selectors are of the form $\hat{\eta}_j = I(|X_j| \geq t)$ with threshold $t = \tau(\beta)\sigma\sqrt{\log d}$ for some function $\tau(\cdot) > 0$. More recently, these asymptotic results were extended to a combined minimax - Bayes Hamming risk on a certain class of vectors θ in [16].

The present paper makes further steps in the analysis of variable selection with a Hamming loss initiated in [12, 15]. Unlike [12, 15], we study the sequence model (1) rather than Gaussian regression and analyze the behavior of the minimax risk rather than that of the Bayes risk with a specific prior. Furthermore, we consider not only $s \sim d^{1-\beta}$ but general s and derive non-asymptotic results that are valid for any sample size. Remarkably, we get an exact expression for the non-asymptotic minimax risk and find explicitly the minimax selectors. Finally, we construct data-driven selectors that are simultaneously adaptive to the parameters a and s .

Specifically, we consider the minimax risk

$$(3) \quad \inf_{\tilde{\eta}} \sup_{\theta \in \Theta} \frac{1}{s} \mathbf{E}_{\theta} |\tilde{\eta} - \eta|$$

for $\Theta = \Theta_d(s, a)$ and $\Theta = \Theta_d^+(s, a)$, where $\inf_{\tilde{\eta}}$ denotes the infimum over all selectors $\tilde{\eta}$. In Section 2, for both classes $\Theta = \Theta_d^+(s, a)$ and $\Theta = \Theta_d(s, a)$ we find the exact values of the minimax risks and derive minimax selectors for any fixed $d, s, a > 0$ such that $s < d$. For $\Theta = \Theta_d(s, a)$ we also propose another selector attaining the minimax risk up to the factor 2. Interestingly, the thresholds that correspond to the minimax optimal selectors do not have the classical form $A\sigma\sqrt{\log d}$ for some $A > 0$; the optimal threshold is a function of a and s . Analogous minimax results are obtained for the risk measured by the probability of wrong recovery $\mathbf{P}_{\theta}(\hat{S} \neq S(\theta))$. Section 3 considers extensions of the non-asymptotic exact minimax theorems of Section 2 to settings with non-Gaussian or dependent observations. In Section 4, as asymptotic corollaries of these results, we establish sharp conditions under which exact and almost full recovery are achievable. Section 5 is devoted to the construction of adaptive selectors that achieve almost full and exact recovery without the knowledge of the parameters a and s . Most of the proofs are given in the Appendix.

Finally, note that quite recently several papers have studied the expected Hamming loss in other problems of variable selection. Asymptotic behavior of the minimax risk analogous to (3) for classes Θ different from the sparsity classes that we consider here was analyzed in [8] and without the normalizing factor $1/s$ in [14]. Oracle inequalities for Hamming risks in the problem of multiple classification under sparsity constraints are established in [22]. The paper [26] introduces an asymptotically minimax approach based on the Hamming loss in the problem of community detection in networks.

2. Non-asymptotic minimax selectors. In what follows, we assume that $s < d$. We first consider minimax variable selection for the class $\Theta_d^+(s, a)$. For this class, we will use a selector $\hat{\eta}^+$ with the components

$$(4) \quad \hat{\eta}_j^+ = I(X_j \geq t), \quad j = 1, \dots, d,$$

where the threshold is defined by

$$(5) \quad t = \frac{a}{2} + \frac{\sigma^2}{a} \log \left(\frac{d}{s} - 1 \right).$$

Set

$$\Psi_+(d, s, a) = \left(\frac{d}{s} - 1 \right) \Phi \left(-\frac{a}{2\sigma} - \frac{\sigma}{a} \log \left(\frac{d}{s} - 1 \right) \right) + \Phi \left(-\frac{a}{2\sigma} + \frac{\sigma}{a} \log \left(\frac{d}{s} - 1 \right) \right),$$

where $\Phi(\cdot)$ denotes the standard Gaussian cumulative distribution function.

THEOREM 2.1. *For any $a > 0$ and $s < d$ the selector $\hat{\eta}^+$ in (4) with the threshold t defined in (5) satisfies*

$$(6) \quad \sup_{\theta \in \Theta_d^+(s, a)} \frac{1}{s} \mathbf{E}_{\theta} |\hat{\eta}^+ - \eta| \leq \Psi_+(d, s, a).$$

The proof is given in the Appendix.

The next theorem gives a lower bound on the minimax risk showing that the upper bound in Theorem 2.1 is tight.

THEOREM 2.2. *For any $a > 0$ and $s < d$ we have*

$$\inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d^+(s,a)} \frac{1}{s} \mathbf{E}_{\theta} |\tilde{\eta} - \eta| \geq \Psi_+(d, s, a),$$

where $\inf_{\tilde{\eta}}$ denotes the infimum over all selectors $\tilde{\eta}$.

The proof is given in the Appendix.

As a straightforward corollary of Theorems 2.1 and 2.2, we obtain that the estimator $\hat{\eta}^+$ is minimax in the exact sense for the class $\Theta_d^+(s, a)$ and the minimax risk satisfies

$$(7) \quad \inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d^+(s,a)} \frac{1}{s} \mathbf{E}_{\theta} |\tilde{\eta} - \eta| = \sup_{\theta \in \Theta_d^+(s,a)} \frac{1}{s} \mathbf{E}_{\theta} |\hat{\eta}^+ - \eta| = \Psi_+(d, s, a).$$

Remarkably, this holds under no assumptions on d, s, a except for, of course, some minimal conditions under which the problem ever makes sense: $a > 0$ and $s < d$. Analogous non-asymptotic minimax result is valid for the class

$$\Theta_d^-(s, a) = \left\{ \theta \in \mathbb{R}^d : \text{there exists a set } S \subseteq \{1, \dots, d\} \text{ with } s \text{ elements} \right. \\ \left. \text{such that } \theta_j \leq -a \text{ for all } j \in S, \text{ and } \theta_j = 0 \text{ for all } j \notin S \right\}.$$

We omit details here.

Next, consider the class $\Theta_d(s, a)$. A direct analog of $\hat{\eta}^+$ for $\Theta_d(s, a)$ is a selector $\hat{\eta}$ with the components

$$(8) \quad \hat{\eta}_j = I(|X_j| \geq t), \quad j = 1, \dots, d,$$

where the threshold t is defined in (5). Set

$$\Psi(d, s, a) = \left(\frac{d}{s} - 1 \right) \Phi \left(-\frac{a}{2\sigma} - \frac{\sigma}{a} \log \left(\frac{d}{s} - 1 \right) \right) + \Phi \left(-\left(\frac{a}{2\sigma} - \frac{\sigma}{a} \log \left(\frac{d}{s} - 1 \right) \right)_+ \right),$$

where $x_+ = \max(x, 0)$. Note that

$$(9) \quad \Psi(d, s, a) \leq \Psi_+(d, s, a).$$

We have the following bound.

THEOREM 2.3. *For any $a > 0$ and $s < d$ the selector $\hat{\eta}$ in (8) with the threshold t defined in (5) satisfies*

$$(10) \quad \sup_{\theta \in \Theta_d(s,a)} \frac{1}{s} \mathbf{E}_{\theta} |\hat{\eta} - \eta| \leq 2\Psi(d, s, a).$$

The proof is given in the Appendix.

For the minimax risk on the class $\Theta_d(s, a)$, we have the following corollary, which is an immediate consequence of Theorems 2.2, 2.3, and inequality (9).

COROLLARY 2.1. *For any $a > 0$ and $s < d$ the selector $\hat{\eta}$ in (8) with the threshold t defined in (5) satisfies*

$$(11) \quad \sup_{\theta \in \Theta_d(s,a)} \mathbf{E}_\theta |\hat{\eta} - \eta| \leq 2 \inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d(s,a)} \mathbf{E}_\theta |\tilde{\eta} - \eta|.$$

Thus, the risk of the thresholding estimator (8) cannot be greater than the minimax risk over the class $\Theta_d(s, a)$ multiplied by 2.

We turn now to exact minimax variable selection over the class $\Theta_d(s, a)$. Consider a selector $\bar{\eta} = (\bar{\eta}_1, \dots, \bar{\eta}_d)$ with the components

$$(12) \quad \bar{\eta}_j = I \left(\log \left(\cosh \left(\frac{aX_j}{\sigma^2} \right) \right) \geq t \right), \quad j = 1, \dots, d,$$

where the threshold is defined by

$$(13) \quad t = \frac{a^2}{2\sigma^2} + \log \left(\frac{d}{s} - 1 \right).$$

Set

$$\begin{aligned} \bar{\Psi}(d, s, a) &= \left(\frac{d}{s} - 1 \right) \mathbf{P} \left(e^{-\frac{a^2}{2\sigma^2}} \cosh \left(\frac{a\xi}{\sigma} \right) \geq \frac{d}{s} - 1 \right) \\ &\quad + \mathbf{P} \left(e^{-\frac{a^2}{2\sigma^2}} \cosh \left(\frac{a\xi}{\sigma} + \frac{a^2}{\sigma^2} \right) < \frac{d}{s} - 1 \right), \end{aligned}$$

where ξ denotes a standard Gaussian random variable. Our aim is to show that $\bar{\Psi}(d, s, a)$ is the minimax risk of variable selection under the Hamming loss over the class $\Theta_d(s, a)$ and that it is achieved by the selector in (12). We first prove that $\bar{\Psi}(d, s, a)$ is an upper bound on the maximum risk of the selector (12).

THEOREM 2.4. *For any $a > 0$ and $s < d$ the selector $\bar{\eta}$ in (12) with the threshold t defined in (13) satisfies*

$$\sup_{\theta \in \Theta_d(s,a)} \frac{1}{s} \mathbf{E}_\theta |\bar{\eta} - \eta| \leq \bar{\Psi}(d, s, a).$$

The next theorem establishes the lower bound on the minimax risk showing that the upper bound in Theorem 2.4 cannot be improved.

THEOREM 2.5. *For any $a > 0$ and $s < d$ we have*

$$\inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d(s,a)} \frac{1}{s} \mathbf{E}_\theta |\tilde{\eta} - \eta| \geq \bar{\Psi}(d, s, a),$$

where $\inf_{\tilde{\eta}}$ denotes the infimum over all selectors $\tilde{\eta}$.

Next, we discuss a connection to the Bayesian setting. It is not hard to check that, for each j , the minimax optimal selector $\hat{\eta}_j^+$ coincides with the Bayes test of the null hypothesis $H_0 : \theta_j = 0$ against the alternative $H_0 : \theta_j = a$ with prior probabilities $1 - s/d$ and s/d , respectively. This can be also seen from the proof of Theorem 2.2. Furthermore, the minimax risk on $\Theta_d^+(s, a)$ is exactly equal to the risk of the corresponding Bayes selector as shown in the next proposition. Analogous result holds for the class $\Theta_d(s, a)$ but we do not state it here for the sake of brevity.

PROPOSITION 2.1. *Let μ be the uniform distribution on the set Θ' of all θ in $\Theta_d^+(s, a)$ such that s components θ_j of θ are equal to a and the remaining $d - s$ components are 0. Then,*

$$(14) \quad \inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d^+(s, a)} \mathbf{E}_{\theta} |\tilde{\eta} - \eta| = \inf_{\tilde{\eta}} \int \mathbf{E}_{\theta} |\tilde{\eta} - \eta| \mu(d\theta).$$

The proof is given in the Appendix.

Finally, we show how the above non-asymptotic minimax results can be extended to the probability of wrong recovery. For any selector $\tilde{\eta}$, we denote by $S_{\tilde{\eta}}$ the selected set of indices: $S_{\tilde{\eta}} = \{j : \tilde{\eta}_j = 1\}$. A selector $\tilde{\eta} = (\tilde{\eta}_1, \dots, \tilde{\eta}_d)$ will be called *separable* if its j th component $\tilde{\eta}_j$ depends only on X_j for all $j = 1, \dots, d$. We denote by \mathcal{T} the set of all separable selectors.

THEOREM 2.6. *For any $a > 0$ and $s < d$ the selectors $\hat{\eta}$ in (8) and $\hat{\eta}^+$ in (4) with the threshold t defined in (5), and the selector $\bar{\eta}$ in (12) with the threshold t defined in (13) satisfy*

$$(15) \quad \sup_{\theta \in \Theta_d^+(s, a)} \mathbf{P}_{\theta}(S_{\hat{\eta}^+} \neq S(\theta)) \leq s\Psi_+(d, s, a),$$

$$(16) \quad \sup_{\theta \in \Theta_d(s, a)} \mathbf{P}_{\theta}(S_{\bar{\eta}} \neq S(\theta)) \leq s\bar{\Psi}(d, s, a)$$

and

$$(17) \quad \sup_{\theta \in \Theta_d(s, a)} \mathbf{P}_{\theta}(S_{\hat{\eta}} \neq S(\theta)) \leq 2s\Psi(d, s, a).$$

Furthermore,

$$(18) \quad \inf_{\tilde{\eta} \in \mathcal{T}} \sup_{\theta \in \Theta_d^+(s, a)} \mathbf{P}_{\theta}(S_{\tilde{\eta}} \neq S(\theta)) \geq \frac{s\Psi_+(d, s, a)}{1 + s\Psi_+(d, s, a)},$$

and

$$(19) \quad \inf_{\tilde{\eta} \in \mathcal{T}} \sup_{\theta \in \Theta_d(s, a)} \mathbf{P}_{\theta}(S_{\tilde{\eta}} \neq S(\theta)) \geq \frac{s\bar{\Psi}(d, s, a)}{1 + s\bar{\Psi}(d, s, a)}.$$

The proof is given in the Appendix.

Although Theorem 2.6 does not provide the exact minimax solution, it implies sharp minimaxity in asymptotic sense. Indeed, an interesting case is when the minimax risk in Theorem 2.6 goes to 0 as $d \rightarrow \infty$. Assuming that s and a are functions of d , this corresponds to $s\Psi_+(d, s, a) \rightarrow 0$ as $d \rightarrow \infty$. In this natural asymptotic setup, the upper and lower bounds of Theorem 2.6 for both classes $\Theta_d^+(s, a)$ and $\Theta_d(s, a)$ are sharp. We discuss this issue in Section 4, cf. Theorem 4.5.

REMARK 2.1. *Papers [12, 15, 16] use a different Hamming loss defined in terms of vectors of signs. In our setting, this would mean considering not $|\hat{\eta} - \eta|$ but the following loss: $\sum_{j=1}^d I(\text{sign}(\hat{\theta}_j) \neq \text{sign}(\theta_j))$, where $\hat{\theta}_j$ is an estimator of θ_j and $\text{sign}(x) = I(x > 0) - I(x < 0)$. Theorems of this section are easily adapted to such a loss, but in this case the corresponding expressions for the non-asymptotic risk contain additional terms and we do not obtain exact minimax solutions as above. On the other hand, these additional terms are smaller than $\Psi(d, s, a)$ and $\Psi_+(d, s, a)$, and in the asymptotic analysis, such as the one performed in Sections 4 and 5, can often be neglected. Thus, in many cases, one gets the same asymptotic results for both losses. We do not discuss this issue in more detail here.*

3. Generalizations and extensions. Before proceeding to asymptotic corollaries, we discuss some generalizations and extensions of the non-asymptotic results of Section 2.

3.1. *Dependent observations.* It is easy to see that Theorems 2.1 and 2.3 do not use any information on the dependence between the observations and thus remain valid for dependent X_j . Furthermore, a minimax optimality property holds under dependence as well. To be specific, denote by $\mathcal{N}_d(\theta, \Sigma)$ the d -dimensional Gaussian distribution with mean θ and covariance matrix Σ . Assume that the distribution \mathbf{P} of (X_1, \dots, X_d) belongs to the class

$$\mathcal{P}_d^+(s, a, \sigma^2) = \{\mathcal{N}_d(\theta, \Sigma) : \theta \in \Theta_d^+(s, a), \sigma_{ii} = \sigma^2, \text{ for all } i = 1, \dots, d\}$$

where we denote by σ_{ii} the diagonal entries of Σ . Note that, for distributions in this class, Σ can be any covariance matrix with constant diagonal elements.

THEOREM 3.1. *For any $a > 0$ and $s < d$, and for the selector $\hat{\eta}^+$ in (4) with the threshold t defined in (5) we have*

$$\inf_{\tilde{\eta}} \sup_{\mathbf{P} \in \mathcal{P}_d^+(s, a, \sigma^2)} \mathbf{E}_{\mathbf{P}} |\tilde{\eta} - \eta| = \sup_{\mathbf{P} \in \mathcal{P}_d^+(s, a, \sigma^2)} \mathbf{E}_{\mathbf{P}} |\hat{\eta}^+ - \eta| = s\Psi_+(d, s, a),$$

where $\inf_{\tilde{\eta}}$ denotes the infimum over all selectors $\tilde{\eta}$, and $\mathbf{E}_{\mathbf{P}}$ denotes the expectation with respect to \mathbf{P} .

PROOF. The upper bound $\Psi_+(d, s, a)$ on the minimax risk follows from the fact that the proofs of Theorems 2.1 and 2.3 are not affected by the dependence. Indeed, both the selector and the Hamming loss proceed coordinate-wisely. The lower bound $\Psi_+(d, s, a)$ on the minimax risk follows from Theorem 2.2 after observing that the maximum over $\mathcal{P}_d^+(s, a, \sigma^2)$ is greater than the maximum over the subfamily of Gaussian vectors with independent entries $\{\mathcal{N}_d(\theta, \sigma^2 I_d) : \theta \in \Theta_d^+(s, a)\}$, where I_d is the $d \times d$ identity matrix. \square

An interesting consequence of Theorem 3.1 and of (7) is that the model with independent X_j is the least favorable model, in the exact non-asymptotic sense, for the problem of variable selection with Hamming loss on the class of vectors $\Theta_d^+(s, a)$.

This fact was also noticed and discussed in [13] for the detection problem. That paper considers the Gaussian model with covariance matrix Σ that is not necessarily a diagonal matrix. It is shown that faster detection rates are achieved in the case of dependent observations (under some assumptions) than in the case of independent data. It would be interesting to extend these results to the variable selection problem in hand.

3.2. *Non-Gaussian models.* As a building block for extension to non-Gaussian observations, we first consider the following simple model. We observe independent random variables X_1, \dots, X_d with values in a measurable space $(\mathcal{X}, \mathcal{U})$ such that s among them are distributed according to the probability distribution \mathbf{P}_1 and the other $d - s$ are distributed according to the probability distribution \mathbf{P}_0 . We assume that $\mathbf{P}_0 \neq \mathbf{P}_1$. Let f_0 and f_1 be densities of \mathbf{P}_0 and \mathbf{P}_1 with respect to some dominating measure. Denote by $\eta = (\eta_1, \dots, \eta_d)$ the vector such that $\eta_j = 1$ if the distribution of X_j is \mathbf{P}_1 and $\eta_j = 0$ if it is \mathbf{P}_0 . Let $\Theta_d(s, f_0, f_1)$ be the set of all such vectors η . Consider the selector $\hat{\eta} = (\hat{\eta}_1, \dots, \hat{\eta}_d)$ where

$$(20) \quad \hat{\eta}_j = I \left(\log \frac{f_1}{f_0}(X_j) \geq \log \left(\frac{d}{s} - 1 \right) \right), \quad j = 1, \dots, d.$$

THEOREM 3.2. *For any $s < d$, the selector $\hat{\eta}$ in (20) satisfies*

$$(21) \quad \sup_{\eta \in \Theta_d(s, f_0, f_1)} \mathbf{E}|\hat{\eta} - \eta| = \inf_{\tilde{\eta}} \sup_{\eta \in \Theta_d(s, f_0, f_1)} \mathbf{E}|\tilde{\eta} - \eta| = s\Psi,$$

where $\inf_{\tilde{\eta}}$ denotes the infimum over all selectors, and

$$(22) \quad \Psi = \mathbf{P}_1\left(\log \frac{f_1}{f_0}(X) < \log\left(\frac{d}{s} - 1\right)\right) + \left(\frac{d}{s} - 1\right)\mathbf{P}_0\left(\log \frac{f_1}{f_0}(X) \geq \log\left(\frac{d}{s} - 1\right)\right).$$

PROOF. The proof of the upper bound $\sup_{\eta \in \Theta_d(s, f_0, f_1)} \mathbf{E}|\hat{\eta} - \eta| \leq s\Psi$ is obvious. The proof of the lower bound $\inf_{\tilde{\eta}} \sup_{\eta \in \Theta_d(s, f_0, f_1)} \mathbf{E}|\tilde{\eta} - \eta| \geq s\Psi$ follows the same lines as the proof of Theorem 2.2 with the only difference in the definition of probability measures. We replace the Gaussian distributions centered at 0 and a by the distributions \mathbf{P}_0 and \mathbf{P}_1 , respectively. With this change, the Bayesian version of the Neyman-Pearson lemma leads to the optimal test statistic T^* of the form

$$T^*(X) = I\left(\frac{(s/d)f_1(X)}{(1-s/d)f_0(X)} > 1\right),$$

and, respectively, to the lower bound (22) on the minimax risk. \square

Suppose now that instead of two measures \mathbf{P}_0 and \mathbf{P}_1 we have a parametric family of probability measures $\{\mathbf{P}_a, a \in \mathcal{U}\}$ where $\mathcal{U} \subseteq \mathbb{R}$. Let \mathbf{f}_a be a density of \mathbf{P}_a with respect to some dominating measure. Recall that the family $\{\mathbf{f}_a, a \in \mathcal{U}\}$ is said to have the Monotone Likelihood Ratio (MLR) property if, for all a_0, a_1 in \mathcal{U} such that $a_0 < a_1$, the log-likelihood ratio $\log(\mathbf{f}_{a_1}(X)/\mathbf{f}_{a_0}(X))$ is an increasing function of X . In particular, this implies, cf. [18, Lemma 3.4.2] that $\{\mathbf{f}_a, a \in \mathcal{U}\}$ is a stochastically ordered family, i.e.,

$$(23) \quad F_a(x) \geq F_{a'}(x) \quad \text{for all } x \quad \text{if } a < a'$$

where F_a is the cumulative distribution function corresponding to \mathbf{f}_a . Using these facts, we generalize the non-asymptotic results of the previous section in two ways. First, we allow for not necessarily Gaussian distributions and second, instead of the set of parameters $\Theta_d^+(s, a)$, we consider the following set with two restrictions:

$$\Theta_d^+(s, a_0, a_1) = \left\{ \theta \in \mathbb{R}^d : \text{there exists a set } S \subseteq \{1, \dots, d\} \text{ with } s \text{ elements} \right. \\ \left. \text{such that } \theta_j \geq a_1 \text{ for all } j \in S, \text{ and } \theta_j \leq a_0 \text{ for all } j \notin S \right\}$$

where $a_0 < a_1$. In what follows, we use the notation $f_j = \mathbf{f}_{a_j}, j = 0, 1$.

THEOREM 3.3. *Let $\{\mathbf{f}_a, a \in \mathcal{U}\}$ be a family with the MLR property, and let $a_0, a_1 \in \mathcal{U}$ be such that $a_0 < a_1$. Then, for any $s < d$, the selector $\hat{\eta}$ in (20) with $f_0 = \mathbf{f}_{a_0}$ and $f_1 = \mathbf{f}_{a_1}$ satisfies*

$$\sup_{\theta \in \Theta_d^+(s, a_0, a_1)} \mathbf{E}_\theta|\hat{\eta} - \eta| = \inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d^+(s, a_0, a_1)} \mathbf{E}_\theta|\tilde{\eta} - \eta| = s\Psi,$$

where $\inf_{\tilde{\eta}}$ denotes the infimum over all selectors and Ψ is given in (22).

PROOF. We have

$$\begin{aligned} \sup_{\theta \in \Theta_d^+(s, a_0, a_1)} \frac{1}{s} \mathbf{E}_\theta |\hat{\eta} - \eta| &= \sup_{a \geq a_1} \mathbf{P}_a \left(\log \frac{f_1}{f_0}(X) < \log \left(\frac{d}{s} - 1 \right) \right) \\ &\quad + \sup_{a \leq a_0} \left(\frac{d}{s} - 1 \right) \mathbf{P}_a \left(\log \frac{f_1}{f_0}(X) \geq \log \left(\frac{d}{s} - 1 \right) \right) = \Psi, \end{aligned}$$

where the last equality is due to the monotonicity of $\log \frac{f_1}{f_0}(X)$ and to the stochastic order property (23). The proof of the lower bound

$$\inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d^+(s, a_0, a_1)} \frac{1}{s} \mathbf{E}_\theta |\tilde{\eta} - \eta| \geq \Psi$$

follows from the fact $\sup_{\theta \in \Theta_d^+(s, a_0, a_1)} \mathbf{E}_\theta |\tilde{\eta} - \eta| \geq \sup_{\eta \in \Theta_d(s, f_0, f_1)} \mathbf{E} |\tilde{\eta} - \eta|$ and Theorem 3.2. \square

Example 1. Let \mathbf{f}_a be the Gaussian $\mathcal{N}(a, \sigma^2)$ density with some $\sigma^2 > 0$, and let $a_0 < a_1$. For $f_1 = \mathbf{f}_{a_1}$ and $f_0 = \mathbf{f}_{a_0}$, the log-likelihood ratio

$$\log \frac{f_1}{f_0}(X) = X \frac{a_1 - a_0}{\sigma^2} - \frac{a_1^2 - a_0^2}{2\sigma^2}$$

is increasing in X . By Theorem 3.3, the minimax optimal selector $\hat{\eta}$ on the class $\Theta_d^+(s, a_0, a_1)$ is a vector with components

$$(24) \quad \hat{\eta}_j = I(X_j \geq t(a_0, a_1)), \quad j = 1, \dots, d,$$

where

$$t(a_0, a_1) = \frac{a_1 + a_0}{2} + \frac{\sigma^2 \log(d/s - 1)}{a_1 - a_0}.$$

Note that for $a_0 = 0$ it coincides with the selector in (4) with $a = a_1$, which is minimax optimal on $\Theta_d^+(s, a_1)$. Moreover, the minimax risk only depends on a_0 and a_1 through the difference $\delta = a_1 - a_0$:

$$\Psi = \Phi \left(-\frac{\delta}{2} + \frac{\sigma^2 \log(d/s - 1)}{\delta} \right) + \left(\frac{d}{s} - 1 \right) \Phi \left(-\frac{\delta}{2} + \frac{\sigma^2 \log(d/s - 1)}{\delta} \right).$$

Example 2 Let \mathbf{P}_a be the Bernoulli distribution $B(a)$ with parameter $a \in (0, 1)$, and $0 < a_0 < a_1 < 1$. Denoting by \mathbf{f}_a the density of \mathbf{P}_a with respect to the counting measure we have, for $f_1 = \mathbf{f}_{a_1}$ and $f_0 = \mathbf{f}_{a_0}$,

$$\log \frac{f_1}{f_0}(X) = X \log \left(\frac{a_1}{1 - a_1} \frac{1 - a_0}{a_0} \right) + \log \frac{1 - a_1}{1 - a_0}$$

which is increasing in X for $0 < a_0 < a_1 < 1$. The minimax optimal selector $\hat{\eta}$ on the class $\Theta_d^+(s, a_0, a_1)$ is a vector with components $\hat{\eta}_j$ in (24) where the threshold $t(a_0, a_1)$ is given by

$$t(a_0, a_1) = \frac{\log \left(\frac{d}{s} - 1 \right) - \log \frac{1 - a_1}{1 - a_0}}{\log \left(\frac{a_1}{1 - a_1} \frac{1 - a_0}{a_0} \right)}.$$

Note that the minimax selector $\hat{\eta}_j$ differs from the naive selector $\hat{\eta}_j^n = X_j$. Indeed since $X_j \in \{0, 1\}$ we have $\hat{\eta}_j = 1$ if either $X_j = 1$ or $t(a_0, a_1) \leq 0$, and $\hat{\eta}_j = 0$ if either $X_j = 0$ or $t(a_0, a_1) > 1$. The value Ψ in the minimax risk has the form

$$\begin{aligned} \Psi &= \mathbf{P}_{a_1}(X < t(a_0, a_1)) + \left(\frac{d}{s} - 1\right) \mathbf{P}_{a_0}(X \geq t(a_0, a_1)) \\ &= \begin{cases} d/s - 1, & t(a_0, a_1) \leq 0, \\ 1 - a_1 + a_0(d/s - 1), & 0 < t(a_0, a_1) < 1, \\ 1, & t(a_0, a_1) \geq 1. \end{cases} \end{aligned}$$

In the asymptotic regime when $d \rightarrow \infty$ and $s \rightarrow \infty$, the minimax risk $s\Psi$ can converge to 0 only when the parameters d, s, a_0, a_1 are kept such that $0 < t(a_0, a_1) < 1$, and in addition $(1 - a_1)s \rightarrow 0$, $a_0(d - s) \rightarrow 0$. Thus, the risk can converge to 0 only when the Bernoulli probabilities a_1 and a_0 tend sufficiently fast to 1 and to 0, respectively.

Example 3. Let \mathbf{P}_a be the Poisson distribution with parameter $a > 0$, and let $a_1 > a_0 > 0$. Denoting by f_a the density of \mathbf{P}_a with respect to the counting measure we have

$$\log \frac{f_1}{f_0}(X) = X \log \left(\frac{a_1}{a_0} \right) - a_1 + a_0,$$

which is increasing in X . The components of the minimax optimal selector $\hat{\eta}$ are given by (24) with

$$t(a_0, a_1) = \frac{\log(d/s - 1) + a_1 - a_0}{\log(a_1/a_0)}.$$

Note that $t(a_0, a_1) > 0$ as soon as $d/s \geq 2$ and $a_1 > a_0 > 0$. The minimax risk has the form $\Psi = \mathbf{P}_{a_1}(X < t(a_0, a_1)) + (d/s - 1)\mathbf{P}_{a_0}(X \geq t(a_0, a_1))$.

3.3. Crowdsourcing with sparsity constraint. The problem of crowdsourcing with two classes is a clustering problem that can be formalized as follows, cf. [11]. Assume that m workers provide class assignments for d items. The class assignment X_{ij} of the i th worker for the j th item is assumed to have a Bernoulli distribution $B(a_{i0})$ if the j th item belongs to class 0, and a Bernoulli distribution $B(a_{i1})$ if it belongs to class 1. Here, $a_{i0}, a_{i1} \in (0, 1)$ and $a_{i0} \neq a_{i1}$ for $i = 1, \dots, m$. The observations $(X_{ij}, i = 1, \dots, m, j = 1, \dots, d)$ are assumed to be jointly independent. Thus, each vector $X_j = (X_{1j}, \dots, X_{mj})$ is distributed according to \mathbf{P}_0 or to \mathbf{P}_1 where each of these two measures is a product of Bernoulli measures, and $\mathbf{P}_0 \neq \mathbf{P}_1$. We assume that there are s vectors X_j with distribution \mathbf{P}_1 , and $d - s$ vectors X_j with distribution \mathbf{P}_0 . The aim is to recover the binary vector of class labels $\eta = (\eta_1, \dots, \eta_d)$ based on the observations $\mathbf{X} = (X_1, \dots, X_d)$. Here, $\eta_j \in \{0, 1\}$ satisfies $\eta_j = k$ if the j th item belongs to class $k \in \{0, 1\}$. Thus, we are in the framework of Theorem 3.2 with a particular form of the log-likelihood ratio

$$(25) \quad \log \frac{f_1}{f_0}(X_j) = \sum_{i=1}^m \left(X_{ij} \log \left(\frac{a_{i1}}{1 - a_{i1}} \frac{1 - a_{i0}}{a_{i0}} \right) + \log \frac{1 - a_{i1}}{1 - a_{i0}} \right),$$

where f_k is the density of \mathbf{P}_k , $k \in \{0, 1\}$. The following corollary is an immediate consequence of Theorem 3.2.

COROLLARY 3.1. *Let $s < d$, $a_{i0}, a_{i1} \in (0, 1)$ and $a_{i0} \neq a_{i1}$ for $i = 1, \dots, m$. Then, the selector $\hat{\eta}$ in (20) with $\log \frac{f_1}{f_0}(X_j)$ defined in (25) is minimax optimal on the class $\Theta_d(s, f_0, f_1)$. The corresponding minimax risk is given in (22).*

Thus, a selector which is minimax optimal in the exact non-asymptotic sense is explicitly given by formula (20). For suitable combinations of parameters d, s, a_{i0}, a_{i1} , the exact value of the minimax risk Ψ can be further analyzed to obtain asymptotics of interest. Gao et al. [11] have studied a setting of crowdsourcing problem which is different from the one we consider here. They did not assume sparsity s , and instead of the class $\Theta_d(s, f_0, f_1)$ of s -sparse binary sequences, they considered the class of all possible binary sequences $\{0, 1\}^d$. For this class, Gao et al. [11] did not derive the exact minimax solution but rather analyzed specific asymptotics of the minimax risk $\inf_{\tilde{\eta}} \sup_{\eta \in \{0,1\}^d} d^{-1} \mathbf{E}|\tilde{\eta} - \eta|$ in large deviations perspective.

4. Asymptotic analysis. Phase transitions. In this section, we conduct the asymptotic analysis of the problem of variable selection. The results are derived as corollaries of the minimax bounds of Section 2. We will assume that $d \rightarrow \infty$ and that parameters $a = a_d$ and $s = s_d$ depend on d .

The first two asymptotic properties we study here are *exact recovery* and *almost full recovery*. We use this terminology following [12, 15] but we define these properties in a different way, as asymptotic minimax properties for classes of vectors θ . The papers [12, 15] considered a Bayesian setup with random θ and studied a linear regression model with i.i.d. Gaussian regressors rather than the sequence model (1).

The study of *exact recovery* and *almost full recovery* will be done here only for the classes $\Theta_d(s_d, a_d)$. The corresponding results for the classes $\Theta_d^+(s_d, a_d)$ or $\Theta_d^-(s_d, a_d)$ are completely analogous. We do not state them here for the sake of brevity.

DEFINITION 4.1. *Let $(\Theta_d(s_d, a_d))_{d \geq 1}$ be a sequence of classes of sparse vectors.*

- *We say that **exact recovery is possible** for $(\Theta_d(s_d, a_d))_{d \geq 1}$ if there exists a selector $\hat{\eta}$ such that*

$$(26) \quad \lim_{d \rightarrow \infty} \sup_{\theta \in \Theta_d(s_d, a_d)} \mathbf{E}_\theta |\hat{\eta} - \eta| = 0.$$

In this case, we say that $\hat{\eta}$ achieves exact recovery.

- *We say that **almost full recovery is possible** for $(\Theta_d(s_d, a_d))_{d \geq 1}$ if there exists a selector $\hat{\eta}$ such that*

$$(27) \quad \lim_{d \rightarrow \infty} \sup_{\theta \in \Theta_d(s_d, a_d)} \frac{1}{s_d} \mathbf{E}_\theta |\hat{\eta} - \eta| = 0.$$

In this case, we say that $\hat{\eta}$ achieves almost full recovery.

It is of interest to characterize the sequences $(s_d, a_d)_{d \geq 1}$, for which exact recovery and almost full recovery are possible. To describe the impossibility of exact or almost full recovery, we need the following definition.

DEFINITION 4.2. *Let $(\Theta_d(s_d, a_d))_{d \geq 1}$ be a sequence of classes of sparse vectors.*

- *We say that **exact recovery is impossible** for $(\Theta_d(s_d, a_d))_{d \geq 1}$ if*

$$(28) \quad \liminf_{d \rightarrow \infty} \inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d(s_d, a_d)} \mathbf{E}_\theta |\tilde{\eta} - \eta| > 0,$$

- We say that **almost full recovery is impossible** for $(\Theta_d(s_d, a_d))_{d \geq 1}$ if

$$(29) \quad \liminf_{d \rightarrow \infty} \inf_{\hat{\eta}} \sup_{\theta \in \Theta_d(s_d, a_d)} \frac{1}{s_d} \mathbf{E}_\theta |\hat{\eta} - \eta| > 0,$$

where $\inf_{\hat{\eta}}$ denotes the infimum over all selectors.

The following general characterization theorem is a straightforward corollary of the results of Section 2.

THEOREM 4.1. (i) *Almost full recovery is possible for $(\Theta_d(s_d, a_d))_{d \geq 1}$ if and only if*

$$(30) \quad \Psi_+(d, s_d, a_d) \rightarrow 0 \quad \text{as } d \rightarrow \infty.$$

In this case, the selector $\hat{\eta}$ defined in (8) with threshold (5) achieves almost full recovery.

(ii) *Exact recovery is possible for $(\Theta_d(s_d, a_d))_{d \geq 1}$ if and only if*

$$(31) \quad s_d \Psi_+(d, s_d, a_d) \rightarrow 0 \quad \text{as } d \rightarrow \infty.$$

In this case, the selector $\hat{\eta}$ defined in (8) with threshold (5) achieves exact recovery.

Although this theorem gives a complete solution to the problem, conditions (30) and (31) are not quite explicit. Intuitively, we would like to get a ‘‘phase transition’’ values a_d^* such that exact (or almost full) recovery is possible for a_d greater than a_d^* and is impossible for a_d smaller than a_d^* . Our aim now is to find such ‘‘phase transition’’ values. We first do it in the almost full recovery framework.

The following bounds for the tails of Gaussian distribution will be useful:

$$(32) \quad \sqrt{\frac{2}{\pi}} \frac{e^{-y^2/2}}{y + \sqrt{y^2 + 4}} < \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-u^2/2} du \leq \sqrt{\frac{2}{\pi}} \frac{e^{-y^2/2}}{y + \sqrt{y^2 + 8/\pi}},$$

for all $y \geq 0$. These bounds are an immediate consequence of formula 7.1.13. in [3] with $x = y/\sqrt{2}$.

Furthermore, we will need some non-asymptotic bounds for the expected Hamming loss that will play a key role in the subsequent asymptotic analysis. They are given in the next theorem.

THEOREM 4.2. *Assume that $s < d/2$.*

(i) *If*

$$(33) \quad a^2 \geq \sigma^2 \left(2 \log((d-s)/s) + W \right) \text{ for some } W > 0,$$

then the selector $\hat{\eta}$ defined in (8) with threshold (5) satisfies

$$(34) \quad \sup_{\theta \in \Theta_d(s, a)} \mathbf{E}_\theta |\hat{\eta} - \eta| \leq (2 + \sqrt{2\pi})s \Phi(-\Delta),$$

where Δ is defined by

$$(35) \quad \Delta = \frac{W}{2\sqrt{2 \log((d-s)/s) + W}}.$$

(ii) If $a > 0$ is such that

$$(36) \quad a^2 \leq \sigma^2 \left(2 \log((d-s)/s) + W \right) \text{ for some } W > 0,$$

then

$$(37) \quad \inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d(s,a)} \mathbf{E}_\theta |\tilde{\eta} - \eta| \geq s \Phi(-\Delta),$$

where the infimum is taken over all selectors $\tilde{\eta}$ and $\Delta > 0$ is defined in (35).

The proof is given in the Appendix.

The next theorem is an easy consequence of Theorem 4.2. It describes a ‘‘phase transition’’ for a_d in the problem of almost full recovery.

THEOREM 4.3. *Assume that $\limsup_{d \rightarrow \infty} s_d/d < 1/2$.*

(i) *If, for all d large enough,*

$$a_d^2 \geq \sigma^2 \left(2 \log((d-s_d)/s_d) + A_d \sqrt{2 \log((d-s_d)/s_d)} \right)$$

for an arbitrary sequence $A_d \rightarrow \infty$, as $d \rightarrow \infty$, then the selector $\hat{\eta}$ defined by (8) and (5) achieves almost full recovery:

$$\lim_{d \rightarrow \infty} \sup_{\theta \in \Theta_d(s_d, a_d)} \frac{1}{s_d} \mathbf{E}_\theta |\hat{\eta} - \eta| = 0.$$

(ii) *Moreover, if there exists $A > 0$ such that for all d large enough the reverse inequality holds:*

$$a_d^2 \leq \sigma^2 \left(2 \log((d-s_d)/s_d) + A \sqrt{2 \log((d-s_d)/s_d)} \right)$$

then almost full recovery is impossible:

$$\liminf_{d \rightarrow \infty} \inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d(s_d, a_d)} \frac{1}{s_d} \mathbf{E}_\theta |\tilde{\eta} - \eta| \geq \Phi\left(-\frac{A}{2}\right) > 0.$$

Here, $\inf_{\tilde{\eta}}$ is the infimum over all selectors $\tilde{\eta}$.

The proof is given in the Appendix.

Under the natural sparsity assumption that

$$(38) \quad d/s_d \rightarrow \infty \quad \text{as} \quad d \rightarrow \infty,$$

Theorem 4.3 shows that the ‘‘phase transition’’ for almost full recovery occurs at the value $a_d = a_d^*$, where

$$(39) \quad a_d^* = \sigma \sqrt{2 \log((d-s_d)/s_d)} \left(1 + o(1) \right).$$

Furthermore, Theorem 4.3 details the behavior of the $o(1)$ term here.

We now state a corollary of Theorem 4.3 under simplified assumptions.

COROLLARY 4.1. *Assume that (38) holds and set*

$$a_d = \sigma \sqrt{2(1 + \delta) \log(d/s_d)}, \quad \text{for some } \delta > 0.$$

Then the selector $\hat{\eta}$ defined by (8) with threshold $t = \sigma \sqrt{2(1 + \varepsilon(\delta)) \log(d/s_d)}$ where $\varepsilon(\delta) > 0$ depends only on δ , achieves almost full recovery.

In the particular case of $s_d = d^{1-\beta}(1 + o(1))$ for some $\beta \in (0, 1)$, condition (38) is satisfied. Then $\log(d/s_d) = \beta(1 + o(1)) \log d$ and it follows from Corollary 4.1 that for $a_d = \sigma \sqrt{2\beta(1 + \delta) \log d}$ the selector with components $\hat{\eta}_j = I(|X_j| > \sigma \sqrt{2\beta(1 + \varepsilon) \log d})$ achieves almost full recovery. This is in agreement with the findings of [12, 15] where an analogous particular case of s_d was considered for a different model and the Bayesian definition of almost full recovery.

We now turn to the problem of exact recovery. First, notice that if

$$\limsup_{d \rightarrow \infty} s_d < \infty$$

the properties of exact recovery and almost full recovery are equivalent. Therefore, it suffices to consider exact recovery only when $s_d \rightarrow \infty$ as $d \rightarrow \infty$. Under this assumption, the "phase transition" for a_d in the problem of exact recovery is described in the next theorem.

THEOREM 4.4. *Assume that $s_d \rightarrow \infty$ as $d \rightarrow \infty$, and $\limsup_{d \rightarrow \infty} s_d/d < 1/2$.*

(i) *If*

$$a_d^2 \geq \sigma^2 \left(2 \log((d - s_d)/s_d) + W_d \right)$$

for all d large enough, where the sequence W_d is such that

$$(40) \quad \liminf_{d \rightarrow \infty} \frac{W_d}{4 \left(\log(s_d) + \sqrt{\log(s_d) \log(d - s_d)} \right)} \geq 1,$$

then the selector $\hat{\eta}$ defined by (8) and (5) achieves exact recovery:

$$(41) \quad \lim_{d \rightarrow \infty} \sup_{\theta \in \Theta_d(s_d, a_d)} \mathbf{E}_\theta |\hat{\eta} - \eta| = 0.$$

(ii) *If the complementary condition holds:*

$$a_d^2 \leq \sigma^2 \left(2 \log((d - s_d)/s_d) + W_d \right)$$

for all d large enough, where the sequence W_d is such that

$$(42) \quad \limsup_{d \rightarrow \infty} \frac{W_d}{4 \left(\log(s_d) + \sqrt{\log(s_d) \log(d - s_d)} \right)} < 1,$$

then exact recovery is impossible, and moreover we have

$$\lim_{d \rightarrow \infty} \inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d(s_d, a_d)} \mathbf{E}_\theta |\tilde{\eta} - \eta| = \infty.$$

Here, $\inf_{\tilde{\eta}}$ is the infimum over all selectors $\tilde{\eta}$.

The proof is given in the Appendix.

Some remarks are in order here. First of all, Theorem 4.4 shows that the ‘‘phase transition’’ for exact recovery occurs at $W_d = 4\left(\log(s_d) + \sqrt{\log(s_d)\log(d-s_d)}\right)$, which corresponds to the critical value $a_d = a_d^*$ of the form

$$(43) \quad a_d^* = \sigma\left(\sqrt{2\log(d-s_d)} + \sqrt{2\log s_d}\right).$$

This value is greater than the critical value a_d^* for almost full recovery, cf. (39), which is intuitively quite clear. The optimal threshold (5) corresponding to (43) has a simple form:

$$t_d^* = \frac{a_d^*}{2} + \frac{\sigma^2}{a_d^*} \log\left(\frac{d}{s_d} - 1\right) = \sigma\sqrt{2\log(d-s_d)}.$$

For example, if $s_d = d^{1-\beta}(1 + o(1))$ for some $\beta \in (0, 1)$, then $a_d^* \sim \sigma(1 + \sqrt{1-\beta})\sqrt{2\log d}$. In this particular case, Theorem 4.4 implies that if $a_d = \sigma(1 + \sqrt{1-\beta})\sqrt{2(1+\delta)\log d}$ for some $\delta > 0$, then exact recovery is possible and the selector with threshold $t = \sigma\sqrt{2(1+\varepsilon)\log d}$ for some $\varepsilon > 0$ achieves exact recovery. This is in agreement with the results of [12, 15] where an analogous particular case of s_d was considered for a different model and the Bayesian definition of exact recovery. For our model, even a sharper result is true; namely, a simple universal threshold $t = \sigma\sqrt{2\log d}$ guarantees exact recovery adaptively in the parameters a and s . Intuitively, this is suggested by the form of t_d^* . The precise statement is given in Theorem 5.1 below.

Finally, we state an asymptotic corollary of Theorem 2.6 showing that the selector $\hat{\eta}$ considered above is sharp in the asymptotically minimax sense with respect to the risk defined as the probability of wrong recovery.

THEOREM 4.5. *Assume that exact recovery is possible for the classes $(\Theta_d(s_d, a_d))_{d \geq 1}$ and $(\Theta_d^+(s_d, a_d))_{d \geq 1}$, that is, condition (31) holds. Then, for the selectors $\hat{\eta}$ and $\hat{\eta}^+$ defined by (8), (4) and (5), and for the selector $\bar{\eta}$ defined by (12) and (13), we have*

$$\lim_{d \rightarrow \infty} \sup_{\theta \in \Theta_d^+(s_d, a_d)} \frac{\mathbf{P}_\theta(S_{\hat{\eta}^+} \neq S(\theta))}{s_d \Psi_+(d, s_d, a_d)} = \lim_{d \rightarrow \infty} \inf_{\bar{\eta} \in \mathcal{T}} \sup_{\theta \in \Theta_d^+(s_d, a_d)} \frac{\mathbf{P}_\theta(S_{\bar{\eta}} \neq S(\theta))}{s_d \Psi_+(d, s_d, a_d)} = 1,$$

$$\lim_{d \rightarrow \infty} \sup_{\theta \in \Theta_d(s_d, a_d)} \frac{\mathbf{P}_\theta(S_{\bar{\eta}} \neq S(\theta))}{s_d \bar{\Psi}(d, s_d, a_d)} = \lim_{d \rightarrow \infty} \inf_{\bar{\eta} \in \mathcal{T}} \sup_{\theta \in \Theta_d(s_d, a_d)} \frac{\mathbf{P}_\theta(S_{\bar{\eta}} \neq S(\theta))}{s_d \bar{\Psi}(d, s_d, a_d)} = 1,$$

and

$$\limsup_{d \rightarrow \infty} \sup_{\theta \in \Theta_d(s_d, a_d)} \frac{\mathbf{P}_\theta(S_{\hat{\eta}} \neq S(\theta))}{s_d \Psi_+(d, s_d, a_d)} \leq 2.$$

Note that the threshold (5) depends on the parameters s and a , so that the selectors considered in all the results above are not adaptive. In the next section, we propose adaptive selectors that achieve almost full recovery and exact recovery without the knowledge of s and a .

REMARK 4.1. *Another procedure of variable selection is the exhaustive search estimator of the support $S(\theta)$ defined as*

$$\tilde{S} = \operatorname{argmax}_{C \subseteq \{1, \dots, d\}: |C|=s} \sum_{j \in C} X_j.$$

This estimator was studied by Butucea et al. [7]. The selection procedure can be equivalently stated as choosing the indices j corresponding to s largest order statistics of the sample (X_1, \dots, X_d) . In [7, Theorem 2.5], it was shown that, on the class $\Theta_d^+(s_d, a_d)$, the probability of wrong recovery $\mathbf{P}_\theta(\tilde{S} \neq S(\theta))$ tends to 0 as $d \rightarrow \infty$ under a stronger condition on (s_d, a_d) than (31). The rate of this convergence was not analyzed there. If we denote by $\eta_{\tilde{S}}$ the selector with components $I(j \in \tilde{S})$ for j from 1 to d , it can be proved that $\mathbf{E}|\eta_{\tilde{S}} - \eta| \leq 2\mathbf{E}|\hat{\eta}^+ - \eta|$ and thus the risk of $\eta_{\tilde{S}}$ is within the factor 2 of the minimax risk over the class $\Theta_d^+(s, a)$. Thus, it does not enjoy the non-asymptotic sharp optimality that we have established for the selector defined by (4) and (5) over the class $\Theta_d^+(s, a)$ and for the selector defined by (12) and (13) over the class $\Theta_d(s, a)$.

5. Adaptive selectors. In this section, we consider the asymptotic setup as in Section 4 and construct the selectors that provide almost full and exact recovery adaptively, that is, without the knowledge of a and s .

As discussed in Section 4, the issue of adaptation for exact recovery is almost trivial. Indeed, the expressions for minimal value a_d^* , for which exact recovery is possible (cf. (43)), and for the corresponding optimal threshold t_d^* suggest that taking a selector with the universal threshold $t = \sigma\sqrt{2\log d}$ is enough to achieve exact recovery simultaneously for all values (a_d, s_d) , for which the exact recovery is possible. This point is formalized in the next theorem.

THEOREM 5.1. *Assume that $s_d \rightarrow \infty$ as $d \rightarrow \infty$ and that $\limsup_{d \rightarrow \infty} s_d/d < 1/2$. Let the sequence $(a_d)_{d \geq 1}$ be above the phase transition level for exact recovery, that is, $a_d \geq a_d^*$ for all d , where a_d^* is defined in (43). Then the selector $\hat{\eta}$ defined by (8) with threshold $t = \sigma\sqrt{2\log d}$ achieves exact recovery.*

The proof of this theorem is given in the Appendix.

We now turn to the problem of adaption for almost full recovery. Ideally, we would like to construct a selector that achieves almost full recovery for all sequences $(s_d, a_d)_{d \geq 1}$ for which almost full recovery is possible. We have seen in Section 4 that this includes a much broader range of values than in case of exact recovery. Thus, using the adaptive selector of Theorem 5.1 for almost full recovery does not give a satisfactory result, and we have to take a different approach.

Following Section 4, we will use the notation

$$a_0(s, A) \triangleq \sigma \left(2\log((d-s)/s) + A\sqrt{\log((d-s)/s)} \right)^{1/2}.$$

As shown in Section 4, it makes sense to consider the classes $\Theta_d(s, a)$ only when $a \geq a_0(s, A)$ with some $A > 0$, since for other values of a almost full recovery is impossible. Only such classes will be studied below.

In the asymptotic setup of Section 4 we have used the assumption that $d/s_d \rightarrow \infty$ (the sparsity assumption), which is now transformed into the condition

$$(44) \quad s_d \in \mathcal{S}_d \triangleq \{1, 2, \dots, s_d^*\} \text{ where } s_d^* \text{ is an integer such that } \frac{d}{s_d^*} \rightarrow \infty \text{ as } d \rightarrow \infty.$$

Assuming s_d to be known, we have shown in Section 4 that almost full recovery is achievable for all $a \geq a_0(s_d, A_d)$, where A_d tends to infinity as $d \rightarrow \infty$. The rate of growth of A_d was allowed

to be arbitrarily slow there, cf. Theorem 4.3. However, for adaptive estimation considered in this section we will need the following mild assumption on the growth of A_d :

$$(45) \quad A_d \geq c_0 \left(\log \log \left(\frac{d}{s_d^*} - 1 \right) \right)^{1/2},$$

where $c_0 > 0$ is an absolute constant. In what follows, we will assume that $s_d^* \leq d/4$, so that the right-hand side of (45) is well-defined.

Consider a grid of points $\{g_1, \dots, g_M\}$ on \mathcal{S}_d , where $g_j = 2^{j-1}$ and M is the maximal integer such that $g_M \leq s_d^*$. For each g_m , $m = 1, \dots, M$, we define a selector

$$\hat{\eta}(g_m) = (\hat{\eta}_j(g_m))_{j=1, \dots, d} \triangleq (I(|X_j| \geq w(g_m)))_{j=1, \dots, d},$$

where

$$w(s) = \sigma \sqrt{2 \log \left(\frac{d}{s} - 1 \right)}.$$

Note that $w(s)$ is monotonically decreasing. We now choose the “best” index m , for which g_m is near the true (but unknown) value of s , by the following data-driven procedure:

$$(46) \quad \hat{m} = \min \left\{ m \in \{2, \dots, M\} : \sum_{j=1}^d I(w(g_k) \leq |X_j| < w(g_{k-1})) \leq \tau g_k \text{ for all } k \geq m \right\},$$

where

$$\tau = \left(\log(d/s_d^* - 1) \right)^{-\frac{1}{7}},$$

and we set $\hat{m} = M$ if the set in (46) is empty. Finally, we define an adaptive selector as

$$\hat{\eta}^{\text{ad}} = \hat{\eta}(g_{\hat{m}}).$$

This adaptive procedure is quite natural in the sense that it can be related to the Lepski method or to wavelet thresholding that are widely used for adaptive estimation. Indeed, as in wavelet methods, we consider dyadic blocks determined by the grid points g_j . The value $\sum_{j=1}^d I(w(g_k) \leq |X_j| < w(g_{k-1}))$ is the number of observations within the k th block. If this number is too small (below a suitably chosen threshold) we decide that the block corresponds to pure noise and it is rejected; in other words, this k is not considered as a good candidate for \hat{m} . This argument is analogous to wavelet thresholding. We start from the largest k (equivalently, smallest $w(g_k)$) and perform this procedure until we find the first block, which is not rejected. The corresponding value k determines our choice of \hat{m} as defined in (46).

THEOREM 5.2. *Let $c_0 \geq 16$. Then the selector $\hat{\eta}^{\text{ad}}$ adaptively achieves almost full recovery in the following sense:*

$$(47) \quad \lim_{d \rightarrow \infty} \sup_{\theta \in \Theta_d(s_d, a_d)} \frac{1}{s_d} \mathbf{E}_\theta |\hat{\eta}^{\text{ad}} - \eta| = 0$$

for all sequences $(s_d, a_d)_{d \geq 1}$ such that (44) holds and $a_d \geq a_0(s_d, A_d)$, where A_d satisfies (45).

REMARK 5.1. *Another family of variable selection methods originates from the theory of multiple testing. These are, for example, the Benjamini-Hochberg, Benjamini-Yekutieli or SLOPE procedures. We refer to [6] for a recent overview and comparison of these techniques. They have the same structure as the exhaustive search procedure in that they keep only the largest order statistics. The difference is that the value s (which is usually not known in practice) is replaced by an estimator \hat{s} obtained from comparing the i th order statistic of $(|X_1|, \dots, |X_d|)$ with a suitable normal quantile depending on i . The analysis of these methods in the literature is focused on the evaluation of false discovery rate (FDR). Asymptotic power calculations for the Benjamini-Hochberg procedure are given in [4]. To the best of our knowledge, the behavior of the risk $\mathbf{P}_\theta(\tilde{S} \neq S(\theta))$ and of the Hamming risk, even in a simple consistency perspective, was not studied.*

REMARK 5.2. *In this paper, the variance σ was supposed to be known. Extension to the case of unknown σ can be treated as described, for example, in [9]. Namely, we replace σ in the definition of the threshold $w(s)$ by a statistic $\hat{\sigma}$ defined in [9, Section 3]. As shown in [9, Proposition 1], this statistic is such that $\sigma \leq \hat{\sigma} \leq C'\sigma$ with high probability provided that $s \leq d/2$, and $d \geq d_0$ for some absolute constants $C' > 1$, $d_0 \geq 1$. Then, replacing σ by $\hat{\sigma}$ in the expression for $w(s)$, one can show that Theorem 5.2 remains valid with this choice of $w(s)$ independent of σ , up to a change in numerical constants in the definition of the adaptive procedure. With this modification, we obtain a procedure which is completely data-driven and enjoys the property of almost full recovery under the mild conditions given in Theorem 5.2. The same modification can be done in Theorem 5.1. Namely, under the assumptions of Theorem 5.1 and $a_d \geq c'a_d^*$, where $c' \geq 1$ is a numerical constant, the selector $\hat{\eta}$ defined by (8) with threshold $t = \hat{\sigma}\sqrt{2\log d}$ achieves exact recovery when σ is unknown.*

REMARK 5.3. *In this section, the problem of adaptive variable selection was considered only for the classes $\Theta_d(s_d, a_d)$. The corresponding results for classes $\Theta_d^+(s_d, a_d)$ and $\Theta_d^-(s_d, a_d)$ are completely analogous. We do not state them here for the sake of brevity.*

6. Appendix.

PROOF OF THEOREM 2.3. We have, for any $t > 0$,

$$\begin{aligned} |\hat{\eta} - \eta| &= \sum_{j:\eta_j=0} \hat{\eta}_j + \sum_{j:\eta_j=1} (1 - \hat{\eta}_j) \\ &= \sum_{j:\eta_j=0} I(|\sigma\xi_j| \geq t) + \sum_{j:\eta_j=1} I(|\sigma\xi_j + \theta_j| < t). \end{aligned}$$

Now, for any $\theta \in \Theta_d(s, a)$ and any $t > 0$,

$$\begin{aligned} \mathbf{E}(I(|\sigma\xi_j + \theta_j| < t)) &\leq \mathbf{P}(|\theta_j| - |\sigma\xi_j| < t) \leq \mathbf{P}(|\xi| > (a - t)/\sigma) \\ &= \mathbf{P}(|\xi| > (a - t)_+/\sigma), \end{aligned}$$

where ξ denotes a standard Gaussian random variable. Thus, for any $\theta \in \Theta_d(s, a)$,

$$(48) \quad \frac{1}{s}\mathbf{E}_\theta|\hat{\eta} - \eta| \leq \left(\frac{d}{s} - 1\right) \mathbf{P}\left(|\xi| \geq \frac{t}{\sigma}\right) + \mathbf{P}\left(|\xi| > \frac{(a - t)_+}{\sigma}\right) = 2\Psi(d, s, a).$$

Note that the inequality here is valid for any $t > 0$, not necessarily for t defined in (5). \square

PROOF OF THEOREM 2.1. Arguing as in the proof of Theorem 2.3, we obtain

$$|\hat{\eta}^+ - \eta| = \sum_{j:\eta_j=0} I(\xi_j \geq t) + \sum_{j:\eta_j=1} I(\sigma\xi_j + \theta_j < t),$$

and $\mathbf{E}(I(\sigma\xi_j + \theta_j < t)) \leq \mathbf{P}(\xi < (t - a)/\sigma)$. Thus, for any $\theta \in \Theta_d^+(s, a)$,

$$\frac{1}{s} \mathbf{E}_\theta |\hat{\eta}^+ - \eta| \leq \left(\frac{d}{s} - 1\right) \mathbf{P}(\xi \geq t/\sigma) + \mathbf{P}(\xi < (t - a)/\sigma) = \Psi_+(d, s, a).$$

□

PROOF OF THEOREM 2.2. An estimator $\bar{\eta} = (\bar{\eta}_1, \dots, \bar{\eta}_d)$ of η (not necessarily a selector) will be called *separable* if $\bar{\eta}_j$ depends only on X_j for all $j = 1, \dots, d$. First note that instead of considering all selectors, it suffices to prove the lower bound for the class of separable estimators $\bar{\eta}$ with components $\bar{\eta}_j \in [0, 1]$. Indeed, for any selector $\tilde{\eta}$, using Jensen's inequality, we obtain

$$\mathbf{E}_\theta |\tilde{\eta} - \eta| = \sum_{j=1}^d \mathbf{E}_\theta |\tilde{\eta}_j - \eta_j| = \sum_{j=1}^d \mathbf{E}_{j, \theta_j} \mathbf{E}_{\{\theta_i, i \neq j\}} |\tilde{\eta}_j - \eta_j| \geq \sum_{j=1}^d \mathbf{E}_{j, \theta_j} |\bar{\eta}_j - \eta_j|$$

where $\bar{\eta}_j = \mathbf{E}_{\{\theta_i, i \neq j\}}(\tilde{\eta}_j)$, and the symbols \mathbf{E}_{j, θ_j} and $\mathbf{E}_{\{\theta_i, i \neq j\}}$ stand for the expectations over the distributions of X_j and $(X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_d)$, respectively. Clearly, $\bar{\eta}_j$ depends only on X_j and takes on values in $[0, 1]$. Thus,

$$(49) \quad \inf_{\bar{\eta}} \sup_{\theta \in \Theta_d^+(s, a)} \frac{1}{s} \mathbf{E}_\theta |\tilde{\eta} - \eta| \geq \inf_{\bar{\eta} \in \mathcal{T}_{[0,1]}} \sup_{\theta \in \Theta_d^+(s, a)} \frac{1}{s} \sum_{j=1}^d \mathbf{E}_{j, \theta_j} |\bar{\eta}_j - \eta_j|$$

where $\mathcal{T}_{[0,1]}$ is the class of all separable estimators $\bar{\eta}$ with components $\bar{\eta}_j \in [0, 1]$.

Let Θ' be the set of all θ in $\Theta_d^+(s, a)$ such that s components θ_j of θ are equal to a and the remaining $d - s$ components are 0. Denote by $|\Theta'| = \binom{d}{s}$ the cardinality of Θ' . Then, for any $\bar{\eta} \in \mathcal{T}_{[0,1]}$ we have

$$(50) \quad \begin{aligned} & \sup_{\theta \in \Theta_d^+(s, a)} \frac{1}{s} \sum_{j=1}^d \mathbf{E}_{j, \theta_j} |\bar{\eta}_j - \eta_j| \geq \frac{1}{s|\Theta'|} \sum_{\theta \in \Theta'} \sum_{j=1}^d \mathbf{E}_{j, \theta_j} |\bar{\eta}_j - \eta_j| \\ &= \frac{1}{s|\Theta'|} \sum_{j=1}^d \left(\sum_{\theta \in \Theta': \theta_j=0} \mathbf{E}_{j,0}(\bar{\eta}_j) + \sum_{\theta \in \Theta': \theta_j=a} \mathbf{E}_{j,a}(1 - \bar{\eta}_j) \right) \\ &= \frac{1}{s} \sum_{j=1}^d \left(\left(1 - \frac{s}{d}\right) \mathbf{E}_{j,0}(\bar{\eta}_j) + \frac{s}{d} \mathbf{E}_{j,a}(1 - \bar{\eta}_j) \right) \\ &\geq \frac{d}{s} \inf_{T \in [0,1]} \left(\left(1 - \frac{s}{d}\right) \mathbb{E}_0(T) + \frac{s}{d} \mathbb{E}_a(1 - T) \right), \end{aligned}$$

where we have used that $|\{\theta \in \Theta' : \theta_j = a\}| = \binom{d-1}{s-1} = s|\Theta'|/d$. In the last line of display (50), \mathbb{E}_a is understood as the expectation with respect to the distribution of $X = u + \sigma\xi$, where $\xi \sim \mathcal{N}(0, 1)$ and $\inf_{T \in [0,1]}$ denotes the infimum over all $[0, 1]$ -valued statistics $T(X)$. Set

$$L^* = \inf_{T \in [0,1]} \left(\left(1 - \frac{s}{d}\right) \mathbb{E}_0(T) + \frac{s}{d} \mathbb{E}_a(1 - T) \right)$$

By the Bayesian version of the Neyman-Pearson lemma, the infimum here is attained for $T = T^*$ given by

$$T^*(X) = I \left(\frac{(s/d)\varphi_\sigma(X-a)}{(1-s/d)\varphi_\sigma(X)} > 1 \right)$$

where $\varphi_\sigma(\cdot)$ is the density of an $\mathcal{N}(0, \sigma^2)$ distribution. Thus,

$$L^* = \left(1 - \frac{s}{d}\right) \mathbf{P} \left(\frac{\varphi_\sigma(\sigma\xi - a)}{\varphi_\sigma(\sigma\xi)} > \frac{d}{s} - 1 \right) + \frac{s}{d} \mathbf{P} \left(\frac{\varphi_\sigma(\sigma\xi)}{\varphi_\sigma(\sigma\xi + a)} \leq \frac{d}{s} - 1 \right).$$

Combining this with (49) and (50), we get

$$\begin{aligned} & \inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d^+(s,a)} \frac{1}{s} \mathbf{E}_\theta |\tilde{\eta} - \eta| \\ & \geq \left(\frac{d}{s} - 1\right) \mathbf{P} \left(\exp \left(\frac{a\xi}{\sigma} - \frac{a^2}{2\sigma^2} \right) > \frac{d}{s} - 1 \right) + \mathbf{P} \left(\exp \left(\frac{a\xi}{\sigma} + \frac{a^2}{2\sigma^2} \right) \leq \frac{d}{s} - 1 \right) \\ & = \left(\frac{d}{s} - 1\right) \mathbf{P} \left(\xi > \frac{a}{2\sigma} + \frac{\sigma}{a} \log \left(\frac{d}{s} - 1 \right) \right) + \mathbf{P} \left(\xi \leq -\frac{a}{2\sigma} + \frac{\sigma}{a} \log \left(\frac{d}{s} - 1 \right) \right) \\ & = \Psi_+(d, s, a). \end{aligned}$$

□

PROOF OF PROPOSITION 2.1. Using (7) it suffices to show that the right hand side of (14) is bounded from above and from below by $s\Psi_+(d, s, a)$. The upper bound is obvious in view of Theorem 2.1. To prove lower bound, we follow the same lines as in the proof of Theorem 2.2. The only difference is that, instead of (49), we now use the inequality

$$\inf_{\tilde{\eta}} \frac{1}{s|\Theta'|} \sum_{\theta \in \Theta'} \mathbf{E}_\theta |\tilde{\eta} - \eta| \geq \inf_{\tilde{\eta} \in \mathcal{T}_{[0,1]}} \frac{1}{s|\Theta'|} \sum_{\theta \in \Theta'} \sum_{j=1}^d \mathbf{E}_{j,\theta_j} |\tilde{\eta}_j - \eta_j|,$$

and we do not need the first inequality in (50). □

PROOF OF THEOREM 2.4. For any $\theta \in \Theta_d(s, a)$, we have

$$\begin{aligned} (51) \quad \mathbf{E}_\theta |\tilde{\eta} - \eta| &= \sum_{j:\theta_j=0} \mathbf{P}_{j,0}(\tilde{\eta}_j = 1) + \sum_{j:\theta_j \geq a} \mathbf{P}_{j,\theta_j}(\tilde{\eta}_j = 0) + \sum_{j:\theta_j \leq -a} \mathbf{P}_{j,\theta_j}(\tilde{\eta}_j = 0) \\ &= (d-s) \mathbf{P} \left(e^{-\frac{a^2}{2\sigma^2}} \cosh \left(\frac{a\xi}{\sigma} \right) > \frac{d}{s} - 1 \right) \\ &\quad + \sum_{j:\theta_j \geq a} \mathbf{P}_{j,\theta_j}(\tilde{\eta}_j = 0) + \sum_{j:\theta_j \leq -a} \mathbf{P}_{j,\theta_j}(\tilde{\eta}_j = 0), \end{aligned}$$

where \mathbf{P}_{j,θ_j} denotes the distribution of X_j , and ξ is a standard Gaussian random variable. We now bound from above the probabilities $\mathbf{P}_{j,\theta_j}(\tilde{\eta}_j = 0)$. Introduce the notation

$$g(x) = \cosh \left(\frac{(x + \sigma\xi)a}{\sigma^2} \right), \quad \forall x \in \mathbb{R},$$

and

$$u = \exp \left(\frac{a^2}{2\sigma^2} + \log \left(\frac{d}{s} - 1 \right) \right).$$

We have

$$\mathbf{P}_{j,\theta_j}(\bar{\eta}_j = 0) = \mathbf{P}(g(\theta_j) < u) = \mathbf{P}(-b - \theta_j < \sigma\xi < b - \theta_j).$$

where $b = (\sigma^2/a)\operatorname{arccosh}(u) > 0$. It is easy to check that the function $x \mapsto \mathbf{P}(-b - x < \sigma\xi < b - x)$ is monotonically decreasing on $[0, \infty)$. Therefore, the maximum of $\mathbf{P}(-b - \theta_j < \sigma\xi < b - \theta_j)$ over $\theta_j \geq a$ is attained at $\theta_j = a$. Thus, for any $\theta_j \geq a$ we have

$$(52) \quad \mathbf{P}_{j,\theta_j}(\bar{\eta}_j = 0) \leq \mathbf{P}(g(a) < u) = \mathbf{P}\left(e^{-\frac{a^2}{2\sigma^2}} \cosh\left(\frac{(a + \sigma\xi)a}{\sigma^2}\right) < \frac{d}{s} - 1\right).$$

Analogously, for any $\theta_j \leq -a$,

$$(53) \quad \begin{aligned} \mathbf{P}_{j,\theta_j}(\bar{\eta}_j = 0) &\leq \mathbf{P}\left(e^{-\frac{a^2}{2\sigma^2}} \cosh\left(\frac{(-a + \sigma\xi)a}{\sigma^2}\right) < \frac{d}{s} - 1\right) \\ &= \mathbf{P}\left(e^{-\frac{a^2}{2\sigma^2}} \cosh\left(\frac{(a + \sigma\xi)a}{\sigma^2}\right) < \frac{d}{s} - 1\right) \end{aligned}$$

where the last equality follows from the fact that ξ has the same distribution as $-\xi$ and \cosh is an even function. Combining (51) – (53) proves the theorem. \square

PROOF OF THEOREM 2.5. We follow the lines of the proof of Theorem 2.2 with suitable modifications. The same argument shows that instead of considering all selectors, it suffices to prove the lower bound for the class of separable estimators $\bar{\eta}$ with components $\bar{\eta}_j \in [0, 1]$. Thus,

$$(54) \quad \inf_{\bar{\eta}} \sup_{\theta \in \Theta_d(s,a)} \mathbf{E}_\theta |\bar{\eta} - \eta| \geq \inf_{\bar{\eta} \in \mathcal{T}_{[0,1]}} \sup_{\theta \in \Theta_d(s,a)} \sum_{j=1}^d \mathbf{E}_{j,\theta_j} |\bar{\eta}_j - \eta_j|$$

where $\mathcal{T}_{[0,1]}$ is the class of all separable estimators $\bar{\eta}$ with components $\bar{\eta}_j \in [0, 1]$ and \mathbf{E}_{j,θ_j} denotes the expectation with respect to \mathbf{P}_{j,θ_j} .

Let Θ^+ and Θ^- be the sets of all θ in $\Theta_d(s, a)$ such that $d - s$ components θ_j of θ are equal to 0 and the remaining s components are equal to a (for $\theta \in \Theta^+$) or to $-a$ (for $\theta \in \Theta^-$). For any $\bar{\eta} \in \mathcal{T}_{[0,1]}$ we have

$$\sup_{\theta \in \Theta_d(s,a)} \sum_{j=1}^d \mathbf{E}_{j,\theta_j} |\bar{\eta}_j - \eta_j| \geq \frac{1}{2} \left\{ \sup_{\theta \in \Theta^+} \sum_{j=1}^d \mathbf{E}_{j,\theta_j} |\bar{\eta}_j - \eta_j| + \sup_{\theta \in \Theta^-} \sum_{j=1}^d \mathbf{E}_{j,\theta_j} |\bar{\eta}_j - \eta_j| \right\}.$$

As shown in the proof of Theorem 2.2, for any $\bar{\eta} \in \mathcal{T}_{[0,1]}$,

$$\sup_{\theta \in \Theta^+} \sum_{j=1}^d \mathbf{E}_{j,\theta_j} |\bar{\eta}_j - \eta_j| \geq \sum_{j=1}^d \left(\left(1 - \frac{s}{d}\right) \mathbf{E}_{j,0}(\bar{\eta}_j) + \frac{s}{d} \mathbf{E}_{j,a}(1 - \bar{\eta}_j) \right).$$

Analogously,

$$\sup_{\theta \in \Theta^-} \sum_{j=1}^d \mathbf{E}_{j,\theta_j} |\bar{\eta}_j - \eta_j| \geq \sum_{j=1}^d \left(\left(1 - \frac{s}{d}\right) \mathbf{E}_{j,0}(\bar{\eta}_j) + \frac{s}{d} \mathbf{E}_{j,-a}(1 - \bar{\eta}_j) \right).$$

From the last three displays we obtain

$$\sup_{\theta \in \Theta_d(s,a)} \sum_{j=1}^d \mathbf{E}_{j,\theta_j} |\bar{\eta}_j - \eta_j| \geq \sum_{j=1}^d \left(\left(1 - \frac{s}{d}\right) \mathbf{E}_{j,0}(\bar{\eta}_j) + \frac{s}{d} \bar{\mathbf{E}}_j(1 - \bar{\eta}_j) \right).$$

where $\bar{\mathbf{E}}_j$ is the expectation with respect to the measure $\bar{\mathbf{P}}_j = (\mathbf{P}_{j,a} + \mathbf{P}_{j,-a})/2$. It follows that

$$(55) \quad \sup_{\theta \in \Theta_d(s,a)} \sum_{j=1}^d \mathbf{E}_{j,\theta_j} |\bar{\eta}_j - \eta_j| \geq \inf_{T \in [0,1]} \left((d-s) \mathbb{E}_0(T) + s \bar{\mathbb{E}}(1-T) \right).$$

Here, \mathbb{E}_0 denotes the expectation with respect to the distribution of X with density $\varphi_\sigma(\cdot)$, $\bar{\mathbb{E}}$ is the expectation with respect to the distribution of X with mixture density $\bar{\varphi}_\sigma(\cdot) = (\varphi_\sigma(\cdot + a) + \varphi_\sigma(\cdot - a))/2$, and $\inf_{T \in [0,1]}$ denotes the infimum over all $[0,1]$ -valued statistics $T(X)$. Recall that we denote by $\varphi_\sigma(\cdot)$ is the density of $\mathcal{N}(0, \sigma^2)$ distribution. Set

$$\tilde{L} = \inf_{T \in [0,1]} \left(\left(1 - \frac{s}{d}\right) \mathbb{E}_0(T) + \frac{s}{d} \bar{\mathbb{E}}(1-T) \right).$$

By the Bayesian version of the Neyman-Pearson lemma, the infimum here is attained for $T = \tilde{T}$ given by

$$\tilde{T}(X) = I \left(\frac{(s/d) \bar{\varphi}_\sigma(X)}{(1-s/d) \varphi_\sigma(X)} > 1 \right).$$

Thus,

$$(56) \quad \begin{aligned} \tilde{L} &= \left(1 - \frac{s}{d}\right) \mathbf{P} \left(\frac{\bar{\varphi}_\sigma(\sigma\xi)}{\varphi_\sigma(\sigma\xi)} > \frac{d}{s} - 1 \right) + \frac{s}{2d} \mathbb{P}_a \left(\frac{\bar{\varphi}_\sigma(X)}{\varphi_\sigma(X)} \leq \frac{d}{s} - 1 \right) \\ &\quad + \frac{s}{2d} \mathbb{P}_{-a} \left(\frac{\bar{\varphi}_\sigma(X)}{\varphi_\sigma(X)} \leq \frac{d}{s} - 1 \right) \\ &= \left(1 - \frac{s}{d}\right) \mathbf{P} \left(e^{-\frac{a^2}{2\sigma^2}} \cosh \left(\frac{a\xi}{\sigma} \right) > \frac{d}{s} - 1 \right) \\ &\quad + \frac{s}{2d} \mathbb{P}_a \left(\frac{\bar{\varphi}_\sigma(X)}{\varphi_\sigma(X)} \leq \frac{d}{s} - 1 \right) + \frac{s}{2d} \mathbb{P}_{-a} \left(\frac{\bar{\varphi}_\sigma(X)}{\varphi_\sigma(X)} \leq \frac{d}{s} - 1 \right) \end{aligned}$$

where \mathbb{P}_u denotes the probability distribution of X with density $\varphi_\sigma(\cdot - u)$. Note that, for all $x \in \mathbb{R}$,

$$\frac{\bar{\varphi}_\sigma(x)}{\varphi_\sigma(x)} = e^{-\frac{a^2}{2\sigma^2}} \cosh \left(\frac{ax}{\sigma^2} \right).$$

Using this formula with $x = \sigma\xi + a$ and $x = \sigma\xi - a$, and the facts that $\cosh(\cdot)$ is an even function and ξ coincides with $-\xi$ in distribution, we obtain

$$\begin{aligned} \mathbb{P}_a \left(\frac{\bar{\varphi}_\sigma(X)}{\varphi_\sigma(X)} \leq \frac{d}{s} - 1 \right) &= \mathbb{P}_{-a} \left(\frac{\bar{\varphi}_\sigma(X)}{\varphi_\sigma(X)} \leq \frac{d}{s} - 1 \right) \\ &= \mathbf{P} \left(e^{-\frac{a^2}{2\sigma^2}} \cosh \left(\frac{a\xi}{\sigma} + \frac{a^2}{\sigma^2} \right) \leq \frac{d}{s} - 1 \right). \end{aligned}$$

Thus, $\tilde{L} = (s/d) \bar{\Psi}(d, s, a)$. Combining this equality with (54) and (55) proves the theorem. \square

PROOF OF THEOREM 2.6. The upper bounds (15), (16) and (17) follow immediately from (2) and Theorems 2.1, 2.4 and 2.3, respectively. We now prove the lower bound (18). To this end, first note that for any $\theta \in \Theta_d^+(s, a)$ and any $\tilde{\eta} \in \mathcal{T}$ we have

$$\mathbf{P}_\theta(S_{\tilde{\eta}} \neq S(\theta)) = \mathbf{P}_\theta(\cup_{j=1}^d \{\tilde{\eta}_j \neq \eta_j\}) = 1 - \prod_{j=1}^d p_j(\theta),$$

where $p_j(\theta) \triangleq \mathbf{P}_\theta(\tilde{\eta}_j = \eta_j)$. Hence, for any $\tilde{\eta} \in \mathcal{T}$,

$$(57) \quad \sup_{\theta \in \Theta_d^+(s, a)} \mathbf{P}_\theta(S_{\tilde{\eta}} \neq S(\theta)) \geq \max_{\theta \in \Theta'} \mathbf{P}_\theta(S_{\tilde{\eta}} \neq S(\theta)) = 1 - p_*,$$

where Θ' is the subset of $\Theta_d^+(s, a)$ defined in the proof of Theorem 2.2, and $p_* = \min_{\theta \in \Theta'} \prod_{j=1}^d p_j(\theta)$.

Next, for any selector $\tilde{\eta}$ we have $\mathbf{P}_\theta(S_{\tilde{\eta}} \neq S(\theta)) \geq \mathbf{P}_\theta(|\tilde{\eta} - \eta| = 1)$. Therefore,

$$(58) \quad \sup_{\theta \in \Theta_d^+(s, a)} \mathbf{P}_\theta(S_{\tilde{\eta}} \neq S(\theta)) \geq \frac{1}{|\Theta'|} \sum_{\theta \in \Theta'} \mathbf{P}_\theta(|\tilde{\eta} - \eta| = 1).$$

Here, $\mathbf{P}_\theta(|\tilde{\eta} - \eta| = 1) = \mathbf{P}_\theta(\cup_{j=1}^d B_j)$ with the random events $B_j = \{|\tilde{\eta}_j - \eta_j| = 1, \text{ and } \tilde{\eta}_i = \eta_i, \forall i \neq j\}$. Since the events B_j are disjoint, for any $\tilde{\eta} \in \mathcal{T}$ we get

$$\begin{aligned} & \frac{1}{|\Theta'|} \sum_{\theta \in \Theta'} \mathbf{P}_\theta(|\tilde{\eta} - \eta| = 1) = \frac{1}{|\Theta'|} \sum_{\theta \in \Theta'} \sum_{j=1}^d \mathbf{P}_\theta(B_j) \\ &= \frac{1}{|\Theta'|} \sum_{j=1}^d \left(\sum_{\theta \in \Theta': \theta_j=0} \mathbf{P}_{j,0}(\tilde{\eta}_j = 1) \prod_{i \neq j} p_i(\theta) + \sum_{\theta \in \Theta': \theta_j=a} \mathbf{P}_{j,a}(\tilde{\eta}_j = 0) \prod_{i \neq j} p_i(\theta) \right) \\ &\geq \frac{p_*}{|\Theta'|} \sum_{j=1}^d \left(\sum_{\theta \in \Theta': \theta_j=0} \mathbf{P}_{j,0}(\tilde{\eta}_j = 1) + \sum_{\theta \in \Theta': \theta_j=a} \mathbf{P}_{j,a}(\tilde{\eta}_j = 0) \right) \\ (59) \quad &= \frac{p_*}{|\Theta'|} \sum_{j=1}^d \left(\sum_{\theta \in \Theta': \theta_j=0} \mathbf{E}_{j,0}(\tilde{\eta}_j) + \sum_{\theta \in \Theta': \theta_j=a} \mathbf{E}_{j,a}(1 - \tilde{\eta}_j) \right) \end{aligned}$$

where $\mathbf{P}_{j,u}$ denotes the distribution of X_j when $\theta_j = u$. We now bound the right-hand side of (59) by following the argument from the last three lines of (50) to the end of the proof of Theorem 2.2. Applying this argument yields that, for any $\tilde{\eta} \in \mathcal{T}$,

$$(60) \quad \frac{1}{|\Theta'|} \sum_{\theta \in \Theta'} \mathbf{P}_\theta(|\tilde{\eta} - \eta| = 1) \geq p^* d \tilde{L} \geq p^* s \Psi_+(d, s, a).$$

Combining (57), (58), and (60), we find that, for any $\tilde{\eta} \in \mathcal{T}$,

$$\sup_{\theta \in \Theta_d^+(s, a)} \mathbf{P}_\theta(S_{\tilde{\eta}} \neq S(\theta)) \geq \min_{0 \leq p^* \leq 1} \max\{1 - p^*, p^* s \Psi_+(d, s, a)\} = \frac{s \Psi_+(d, s, a)}{1 + s \Psi_+(d, s, a)}.$$

We now prove the lower bound (19). Let the sets Θ^+ and Θ^- and the constants $p_j(\theta)$ be the same as in the proof of Theorem 2.5. Then

$$\sup_{\theta \in \Theta_d(s, a)} \mathbf{P}_\theta(S_{\tilde{\eta}} \neq S(\theta)) \geq \max_{\theta \in \Theta^+ \cup \Theta^-} \mathbf{P}_\theta(S_{\tilde{\eta}} \neq S(\theta)) = 1 - \bar{p},$$

where $\bar{p} = \min_{\theta \in \Theta^+ \cup \Theta^-} \prod_{j=1}^d p_j(\theta)$.

For any selector $\tilde{\eta}$, we use that $\mathbf{P}_\theta(S_{\tilde{\eta}} \neq S(\theta)) \geq \mathbf{P}_\theta(|\tilde{\eta} - \eta| = 1)$ and, therefore,

$$\sup_{\theta \in \Theta_d(s,a)} \mathbf{P}_\theta(S_{\tilde{\eta}} \neq S(\theta)) \geq \frac{1}{2|\Theta^+|} \sum_{\theta \in \Theta^+} \mathbf{P}_\theta(|\tilde{\eta} - \eta| = 1) + \frac{1}{2|\Theta^-|} \sum_{\theta \in \Theta^-} \mathbf{P}_\theta(|\tilde{\eta} - \eta| = 1).$$

We continue along the same lines as in the proof of (59) to get, for any separable selector $\tilde{\eta}$,

$$\begin{aligned} \sup_{\theta \in \Theta_d(s,a)} \mathbf{P}_\theta(S_{\tilde{\eta}} \neq S(\theta)) &\geq \frac{\bar{p}}{2|\Theta^+|} \sum_{j=1}^d \left(\sum_{\theta \in \Theta^+ : \theta_j=0} \mathbf{E}_{j,0}(\tilde{\eta}_j) + \sum_{\theta \in \Theta^+ : \theta_j=a} \mathbf{E}_{j,a}(1 - \tilde{\eta}_j) \right) \\ &\quad + \frac{\bar{p}}{2|\Theta^-|} \sum_{j=1}^d \left(\sum_{\theta \in \Theta^- : \theta_j=0} \mathbf{E}_{j,0}(\tilde{\eta}_j) + \sum_{\theta \in \Theta^- : \theta_j=-a} \mathbf{E}_{j,-a}(1 - \tilde{\eta}_j) \right) \\ &\geq \frac{\bar{p}}{2} \sum_{j=1}^d \left(\left(1 - \frac{s}{d}\right) \mathbf{E}_{j,0}(\tilde{\eta}_j) + \frac{s}{d} \mathbf{E}_{j,a}(1 - \tilde{\eta}_j) \right) \\ &\quad + \frac{\bar{p}}{2} \sum_{j=1}^d \left(\left(1 - \frac{s}{d}\right) \mathbf{E}_{j,0}(\tilde{\eta}_j) + \frac{s}{d} \mathbf{E}_{j,-a}(1 - \tilde{\eta}_j) \right) \\ &= \bar{p} \sum_{j=1}^d \left(\left(1 - \frac{s}{d}\right) \mathbf{E}_{j,0}(\tilde{\eta}_j) + \frac{s}{d} \bar{\mathbf{E}}_j(1 - \tilde{\eta}_j) \right), \end{aligned}$$

where again $\bar{\mathbf{E}}_j$ denotes the expected value with respect to $\bar{\mathbf{P}}_j = \frac{1}{2}(\mathbf{P}_{j,a} + \mathbf{P}_{j,-a})$. Analogously to the proof of Theorem 2.5, the expression in the last display can be further bounded from below by $\bar{p}d\bar{L} = \bar{p}s\bar{\Psi}(d, s, a)$. Thus,

$$\sup_{\theta \in \Theta_d(s,a)} \mathbf{P}_\theta(S_{\tilde{\eta}} \neq S(\theta)) \geq \min_{0 \leq \bar{p} \leq 1} \max\{1 - \bar{p}, \bar{p}s\bar{\Psi}(d, s, a)\} = \frac{s\bar{\Psi}(d, s, a)}{1 + s\bar{\Psi}(d, s, a)}.$$

□

PROOF OF THEOREM 4.2. (i) In the proof of Theorem 2.3, we have obtained that

$$(61) \quad \sup_{\theta \in \Theta_d(s,a)} \frac{1}{s} \mathbf{E}_\theta |\hat{\eta} - \eta| \leq 2 \left(\frac{d}{s} - 1 \right) \Phi(-t/\sigma) + 2\Phi(-(a-t)_+/\sigma),$$

where $t = \frac{a}{2} + \frac{\sigma^2}{a} \log\left(\frac{d}{s} - 1\right)$ is the threshold (5). Since $a^2 \geq 2\sigma^2 \log(d/s - 1)$ we get that $a \geq t$ and that $t > a/2$, which is equivalent to $t > a - t$. Furthermore, $\left(\frac{d}{s} - 1\right)e^{-t^2/(2\sigma^2)} = e^{-(a-t)^2/(2\sigma^2)}$. These remarks and (32) imply that

$$\begin{aligned} \left(\frac{d}{s} - 1\right) \Phi(-t/\sigma) &\leq \sqrt{\frac{2}{\pi}} \frac{\exp(-(a-t)^2/(2\sigma^2))}{(a-t)/\sigma + \sqrt{(a-t)^2/\sigma^2 + 8/\pi}} \\ &\leq \frac{\exp(-(a-t)^2/(2\sigma^2))}{(a-t)/\sigma + \sqrt{(a-t)^2/\sigma^2 + 4}} \\ &\leq \sqrt{\frac{\pi}{2}} \Phi\left(-\frac{a-t}{\sigma}\right). \end{aligned}$$

Combining this with (61) we get

$$\sup_{\theta \in \Theta_d(s,a)} \frac{1}{s} \mathbf{E}_\theta |\hat{\eta} - \eta| \leq (2 + \sqrt{2\pi}) \Phi \left(-\frac{a-t}{\sigma} \right).$$

Now, to prove (34) it remains to note that under assumption (33),

$$\frac{a-t}{\sigma} = \frac{a}{2\sigma} - \frac{\sigma}{a} \log \left(\frac{d}{s} - 1 \right) = \frac{a^2 - 2\sigma^2 \log((d-s)/s)}{2a\sigma} \geq \Delta.$$

Indeed, assumption (33) states that $a \geq a_0 \triangleq \sigma \left(2 \log((d-s)/s) + W \right)^{1/2}$, and the function $a \mapsto (a^2 - 2\sigma^2 \log((d-s)/s))/a$ is monotonically increasing in $a > 0$. On the other hand,

$$(62) \quad (a_0^2 - 2\sigma^2 \log((d-s)/s))/(2a_0\sigma) = \Delta.$$

(ii) We now prove (37). By Theorem 2.2,

$$\inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d(s,a)} \frac{1}{s} \mathbf{E}_\theta |\tilde{\eta} - \eta| \geq \Psi_+(d, s, a) \geq \Phi \left(-\frac{a}{2\sigma} + \frac{\sigma}{a} \log \left(\frac{d}{s} - 1 \right) \right).$$

Here,

$$-\frac{a}{2\sigma} + \frac{\sigma}{a} \log \left(\frac{d}{s} - 1 \right) = \frac{2\sigma^2 \log((d-s)/s) - a^2}{2\sigma a}.$$

Observe that the function $a \mapsto (2\sigma^2 \log((d-s)/s) - a^2)/a$ is monotonically decreasing in $a > 0$ and that assumption (36) states that $a \leq a_0$. In view of (62), the value of its minimum for $a \leq a_0$ is equal to $-\Delta$. The bound (37) now follows by the monotonicity of $\Phi(\cdot)$. \square

PROOF OF THEOREM 4.3. Assume without loss of generality that d is large enough to have $(d-s_d)/s_d > 1$. We apply Theorem 4.2 with $W = A\sqrt{2 \log((d-s_d)/s_d)}$. Then,

$$\Delta^2 = \frac{A^2 \sqrt{2 \log((d-s_d)/s_d)}}{4(\sqrt{2 \log((d-s_d)/s_d)} + A)}.$$

By assumption, there exists $\nu > 0$ such that $(2+\nu)s_d \leq d$ for all d large enough. Equivalently, $d/s_d - 1 \geq 1 + \nu$ and therefore, using the monotonicity argument, we find

$$\Delta^2 \geq \frac{A^2 \sqrt{2 \log(1+\nu)}}{\sqrt{2 \log(1+\nu)} + A} \rightarrow \infty \quad \text{as } A \rightarrow \infty.$$

This and (34) imply part (i) of the theorem. Part (ii) follows from (37) by noticing that $\Delta^2 \leq \sup_{x>0} \frac{A^2 x}{4(x+A)} = A^2/4$ for any fixed $A > 0$. \square

PROOF OF THEOREM 4.4. Throughout the proof, we assume without loss of generality that d is large enough to have $s_d \geq 2$, and $(d-s_d)/s_d > 1$. Set $W_*(s) \triangleq 4 \left(\log s + \sqrt{\log s \log(d-s)} \right)$, and notice that

$$(63) \quad \frac{W_*(s_d)}{2\sqrt{2 \log((d-s_d)/s_d)} + W_*(s_d)} = \sqrt{2 \log s_d},$$

$$(64) \quad 2 \log((d-s_d)/s_d) + W_*(s_d) = 2 \left(\sqrt{\log(d-s_d)} + \sqrt{\log s_d} \right)^2.$$

If (40) holds, we have $W_d \geq W_*(s_d)$ for all d large enough. By the monotonicity of the quantity Δ defined in (35) with respect to W , this implies

$$(65) \quad \begin{aligned} \Delta_d &\triangleq \frac{W_d}{2\sqrt{2\log((d-s_d)/s_d) + W_d}} \\ &\geq \frac{W_*(s_d)}{2\sqrt{2\log((d-s_d)/s_d) + W_*(s_d)}} = \sqrt{2\log s_d}. \end{aligned}$$

Now, by Theorem 4.2 and using (32) we may write

$$(66) \quad \begin{aligned} \sup_{\theta \in \Theta_d(s_d, a_d)} \mathbf{E}_\theta |\hat{\eta} - \eta| &\leq (2 + \sqrt{2\pi})s_d \Phi(-\Delta_d) \\ &\leq 3s_d \min \left\{ 1, \frac{1}{\Delta_d} \right\} \exp \left(-\frac{\Delta_d^2}{2} \right) \\ &= 3 \min \left\{ 1, \frac{1}{\Delta_d} \right\} \exp \left(-\frac{\Delta_d^2 - 2\log s_d}{2} \right). \end{aligned}$$

This and (65) imply that, for all d large enough,

$$\sup_{\theta \in \Theta_d(s_d, a_d)} \mathbf{E}_\theta |\hat{\eta} - \eta| \leq 3 \min \left\{ 1, \frac{1}{\sqrt{2\log s_d}} \right\}.$$

Since $s_d \rightarrow \infty$, part (i) of the theorem follows.

We now prove part (ii) of the theorem. It suffices to consider $W_d > 0$ for all d large enough since for non-positive W_d almost full recovery is impossible and the result follows from part (ii) of Theorem 4.3. If (42) holds, there exists $A < 1$ such that $W_d \leq AW_*(s_d)$ for all d large enough. By the monotonicity of the quantity Δ defined in (35) with respect to W and in view of equation (63), this implies

$$(67) \quad \begin{aligned} &\Delta_d^2 - 2\log s_d \\ &\leq \frac{A^2 W_*^2(s_d)}{4(2\log((d-s_d)/s_d) + AW_*(s_d))} - \frac{W_*^2(s_d)}{4(2\log((d-s_d)/s_d) + W_*(s_d))} \\ &= \frac{(A-1)W_*^2(s_d)(AW_*(s_d) + 2(A+1)\log((d-s_d)/s_d))}{4(2\log((d-s_d)/s_d) + AW_*(s_d))(2\log((d-s_d)/s_d) + W_*(s_d))} \\ &\leq \frac{(A-1)AW_*^2(s_d)}{4(2\log((d-s_d)/s_d) + W_*(s_d))} \\ &= \frac{2(A-1)A \left(\log s_d + \sqrt{\log s_d \log(d-s_d)} \right)^2}{\left(\sqrt{\log(d-s_d)} + \sqrt{\log s_d} \right)^2} = 2(A-1)A \log s_d, \end{aligned}$$

where we have used the fact that $A < 1$ and equations (63), (64). Next, by Theorem 4.2 and using (32), we have

$$\begin{aligned} \inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d(s_d, a_d)} \mathbf{E}_\theta |\tilde{\eta} - \eta| &\geq s_d \Phi(-\Delta_d) \geq \frac{s_d}{4} \min \left\{ \frac{1}{2}, \frac{1}{\Delta_d} \right\} \exp \left(-\frac{\Delta_d^2}{2} \right) \\ &= \frac{1}{4} \min \left\{ \frac{1}{2}, \frac{1}{\Delta_d} \right\} \exp \left(-\frac{\Delta_d^2 - 2\log s_d}{2} \right). \end{aligned}$$

Combining this inequality with (67), we find that, for all d large enough,

$$\inf_{\tilde{\eta}} \sup_{\theta \in \Theta_d(s_d, a_d)} \mathbf{E}_\theta |\tilde{\eta} - \eta| \geq \frac{1}{4} \min \left\{ \frac{1}{2}, \frac{1}{\Delta_d} \right\} \exp((1-A)A \log s_d).$$

Since $A < 1$ and $\Delta_d \leq A\sqrt{2 \log s_d}$ by (67), the last expression tends to ∞ as $s_d \rightarrow \infty$. This proves part (ii) of the theorem. \square

PROOF OF THEOREM 5.1. By (48), for any $\theta \in \Theta_d(s_d, a_d)$, and any $t > 0$ we have

$$\mathbf{E}_\theta |\hat{\eta} - \eta| \leq (d - s_d) \mathbf{P}(|\xi| \geq t/\sigma) + s_d \mathbf{P}(|\xi| > (a_d - t)_+/\sigma),$$

where ξ is a standard normal random variable. It follows that, for any $a_d \geq a_d^*$, any $\theta \in \Theta_d(s_d, a_d)$, and any $t > 0$,

$$\mathbf{E}_\theta |\hat{\eta} - \eta| \leq d \mathbf{P}(|\xi| \geq t/\sigma) + s_d \mathbf{P}(|\xi| > (a_d^* - t)_+/\sigma).$$

Without loss of generality assume that $d \geq 6$ and $2 \leq s_d \leq d/2$. Then, using the inequality $\sqrt{x} - \sqrt{y} \leq (x - y)/\sqrt{2y}$, $\forall x > y > 0$, we find that, for $t = \sigma\sqrt{2 \log d}$,

$$\begin{aligned} (a_d^* - t)_+/\sigma &\geq \sqrt{2} \left(\sqrt{\log(d - s_d)} - \sqrt{\log d} + \sqrt{\log(s_d)} \right) \\ &\geq \sqrt{2 \log(s_d)} - \log \left(\frac{d}{d - s_d} \right) / \sqrt{\log(d - s_d)} \\ &\geq \sqrt{2 \log(s_d)} - (\log 2) / \sqrt{\log(d/2)} > 0. \end{aligned}$$

From this we also easily deduce that, for $2 \leq s_d \leq d/2$, we have $((a_d^* - t)_+/\sigma)^2/2 \geq \log(s_d) - \sqrt{2} \log 2$. Combining these remarks with (32) and (43), we find

$$\sup_{\theta \in \Theta_d(s_d, a_d)} \mathbf{E}_\theta |\hat{\eta} - \eta| \leq \frac{1}{\sqrt{2 \log d}} + \frac{s_d \exp(-\log(s_d) + \sqrt{2} \log 2)}{\sqrt{2 \log(s_d)}},$$

which immediately implies the theorem by taking the limit as $d \rightarrow \infty$. \square

PROOF OF THEOREM 5.2. Throughout the proof, we will write for brevity $s_d = s, a_d = a, A_d = A$, and set $\sigma = 1$. Since $\Theta_d(s, a) \subseteq \Theta_d(s, a_0(s, A))$ for all $a \geq a_0(s, A)$, it suffices to prove that

$$(68) \quad \lim_{d \rightarrow \infty} \sup_{\theta \in \Theta_d(s, a_0(s, A))} \frac{1}{s} \mathbf{E}_\theta |\hat{\eta}^{\text{ad}} - \eta| = 0.$$

Here $s \leq s_d^*$ and recall that throughout this section we assume that $s_d^* \leq d/4$; since we deal with asymptotics as $d/s_d^* \rightarrow \infty$, the latter assumption is without loss of generality in the current proof.

If $s < g_M$, let $m_0 \in \{2, \dots, M\}$ be the index such that g_{m_0} is the minimal element of the grid, which is greater than the true underlying s . Thus, $g_{m_0}/2 = g_{m_0-1} \leq s < g_{m_0}$. If $s \in [g_M, s_d^*]$, we set $m_0 = M$. In both cases,

$$(69) \quad s \geq g_{m_0}/2.$$

We decompose the risk as follows:

$$\frac{1}{s} \mathbf{E}_\theta |\hat{\eta}^{\text{ad}} - \eta| = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \frac{1}{s} \mathbf{E}_\theta (|\hat{\eta}(g_{\hat{m}}) - \eta| I(\hat{m} \leq m_0)), \\ I_2 &= \frac{1}{s} \mathbf{E}_\theta (|\hat{\eta}(g_{\hat{m}}) - \eta| I(\hat{m} \geq m_0 + 1)). \end{aligned}$$

We now evaluate I_1 . Using the fact that $\hat{\eta}_j(g_m)$ is monotonically increasing in m and the definition of \hat{m} , we obtain that, on the event $\{\hat{m} \leq m_0\}$,

$$\begin{aligned} |\hat{\eta}(g_{\hat{m}}) - \hat{\eta}(g_{m_0})| &\leq \sum_{m=\hat{m}+1}^{m_0} |\hat{\eta}(g_m) - \hat{\eta}(g_{m-1})| \\ &= \sum_{m=\hat{m}+1}^{m_0} \sum_{j=1}^d (\hat{\eta}_j(g_m) - \hat{\eta}_j(g_{m-1})) \\ &= \sum_{m=\hat{m}+1}^{m_0} \sum_{j=1}^d I(w(g_m) \leq |X_j| < w(g_{m-1})) \\ &\leq \tau \sum_{m=\hat{m}+1}^{m_0} g_m \leq \tau s \sum_{m=2}^{m_0} 2^{m-m_0+1} \leq 4\tau s, \end{aligned}$$

where we have used the equality $g_m = 2^m$ and (69). Thus,

$$\begin{aligned} (70) \quad I_1 &\leq \frac{1}{s} \mathbf{E}_\theta (|\hat{\eta}(g_{\hat{m}}) - \hat{\eta}(g_{m_0})| I(\hat{m} \leq m_0)) + \frac{1}{s} \mathbf{E}_\theta |\hat{\eta}(g_{m_0}) - \eta| \\ &\leq 4\tau + \frac{1}{s} \mathbf{E}_\theta |\hat{\eta}(g_{m_0}) - \eta|. \end{aligned}$$

By (48), for any $\theta \in \Theta_d(s, a_0(s, A))$ we have

$$(71) \quad \frac{1}{s} \mathbf{E}_\theta |\hat{\eta}(g_{m_0}) - \eta| \leq \left(\frac{d}{s} - 1 \right) \mathbf{P}(|\xi| \geq w(g_{m_0})) + \mathbf{P}(|\xi| > (a_0(s, A) - w(g_{m_0}))_+)$$

where ξ is a standard Gaussian random variable. Using the bound on the Gaussian tail probability and the fact that $s \geq g_{m_0}/2$, we get

$$\begin{aligned} (72) \quad \left(\frac{d}{s} - 1 \right) \mathbf{P}(|\xi| \geq w(g_{m_0})) &\leq \frac{d/s - 1}{d/g_{m_0} - 1} \frac{\pi^{-1/2}}{\sqrt{\log(d/g_{m_0} - 1)}} \\ &\leq \frac{d - s}{d - 2s} \frac{2\pi^{-1/2}}{\sqrt{\log(d/s - 1)}} \leq \frac{3\pi^{-1/2}}{\sqrt{\log(d/s_d^* - 1)}}. \end{aligned}$$

To bound the second probability on the right-hand side of (71), we use the following lemma.

LEMMA 6.1. *Under the assumptions of Theorem 5.2, for any $m \geq m_0$ we have*

$$(73) \quad \mathbf{P}(|\xi| > (a_0(s, A) - w(g_m))_+) \leq (\log(d/s_d^* - 1))^{-\frac{1}{2}}.$$

Combining (71), (72) and (73) with $m = m_0$, we find

$$(74) \quad \frac{1}{s} \mathbf{E}_\theta |\hat{\eta}(g_{m_0}) - \eta| \leq \frac{3\pi^{-1/2} + 1}{\sqrt{\log(d/s_d^* - 1)}},$$

which together with (70) leads to the bound

$$(75) \quad I_1 \leq 4\tau + \frac{3\pi^{-1/2} + 1}{\sqrt{\log(d/s_d^* - 1)}}.$$

We now turn to the evaluation of I_2 . It is enough to consider the case $m_0 \leq M - 1$ since $I_2 = 0$ when $m_0 = M$. We have

$$(76) \quad \begin{aligned} I_2 &= \frac{1}{s} \sum_{m=m_0+1}^M \mathbf{E}_\theta (|\hat{\eta}(g_{\hat{m}}) - \eta| I(\hat{m} = m)) \\ &\leq \frac{1}{s} \sum_{m=m_0+1}^M (\mathbf{E}_\theta |\hat{\eta}(g_m) - \eta|)^{1/2} (\mathbf{P}_\theta(\hat{m} = m))^{1/2}. \end{aligned}$$

By definition, the event $\{\hat{m} = m\}$ occurs if and only if there exists some $\ell \geq m$ such that $\sum_{j=1}^d I(w_\ell \leq |X_j| < w_{\ell-1}) > \tau g_\ell \triangleq v_\ell$, where we set for brevity $w_\ell = w(g_\ell)$. Thus,

$$(77) \quad \mathbf{P}_\theta(\hat{m} = m) \leq \sum_{\ell=m}^M \mathbf{P}_\theta \left(\sum_{j=1}^d I(w_\ell \leq |X_j| < w_{\ell-1}) > v_\ell \right).$$

By Bernstein's inequality, for any $t > 0$ we have

$$(78) \quad \begin{aligned} &\mathbf{P}_\theta \left(\sum_{j=1}^d I(w_\ell \leq |X_j| < w_{\ell-1}) - \mathbf{E}_\theta \left(\sum_{j=1}^d I(w_\ell \leq |X_j| < w_{\ell-1}) \right) > t \right) \\ &\leq \exp \left(- \frac{t^2/2}{\sum_{j=1}^d \mathbf{E}_\theta (I(w_\ell \leq |X_j| < w_{\ell-1})) + 2t/3} \right), \end{aligned}$$

where we have used that, for random variables with values in $\{0, 1\}$, the variance is smaller than the expectation.

Now, similar to (48), for any $\theta \in \Theta_d(s, a_0(s, A))$,

$$\begin{aligned} &\mathbf{E}_\theta \left(\sum_{j=1}^d I(w_\ell \leq |X_j| < w_{\ell-1}) \right) \\ &\leq (d-s) \mathbf{P}(w_\ell \leq |\xi| < w_{\ell-1}) + \sum_{j:\theta_j \neq 0} \mathbf{P}(|\theta_j + \xi| < w_{\ell-1}) \\ &\leq (d-s) \mathbf{P}(|\xi| \geq w_\ell) + s \mathbf{P}(|\xi| > -(a_0(s, A) - w_{\ell-1})_+), \end{aligned}$$

where ξ is a standard Gaussian random variable. Since $\ell \geq m_0 + 1$, from Lemma 6.1 we get

$$(79) \quad \mathbf{P}(|\xi| > (a_0(s, A) - w_{\ell-1})_+) \leq (\log(d/s_d^* - 1))^{-\frac{1}{2}}.$$

Next, using the bound on the Gaussian tail probability and the inequalities $g_\ell \leq s_d^* \leq d/4$, we find

$$(80) \quad (d-s)\mathbf{P}(|\xi| \geq w_\ell) \leq \frac{d-s}{d/g_\ell - 1} \frac{\pi^{-1/2}}{\sqrt{\log(d/g_\ell - 1)}} \leq \frac{(4/3)\pi^{-1/2}g_\ell}{\sqrt{\log(d/s_d^* - 1)}}.$$

We now deduce from (79) and (80), and the inequality $s \leq g_\ell$ for $\ell \geq m_0 + 1$, that

$$(81) \quad \mathbf{E}_\theta \left(\sum_{j=1}^d I(w_\ell \leq |X_j| < w_{\ell-1}) \right) \leq \frac{((4/3)\pi^{-1/2} + 1)g_\ell}{\sqrt{\log(d/s_d^* - 1)}} \leq 2\tau g_\ell.$$

Taking in (78) $t = 3\tau g_\ell = 3v_\ell$ and using (81), we find

$$\mathbf{P}_\theta \left(\sum_{j=1}^d I(w_\ell \leq |X_j| < w_{\ell-1}) > v_\ell \right) \leq \exp(-C_1 v_\ell) = \exp(-C_1 2^\ell \tau),$$

for all $\ell \geq m_0 + 1$ and some absolute constant $C_1 > 0$. This implies

$$(82) \quad \mathbf{P}_\theta(\hat{m} = m) \leq \sum_{\ell=m}^M \exp(-C_1 2^\ell \tau) \leq C_2 2^{-m} \tau^{-1} \exp(-C_1 2^m \tau)$$

for some absolute constant $C_2 > 0$.

On the other hand, notice that the bounds (71), and (72) are valid not only for g_{m_0} but also for any g_m with $m \geq m_0 + 1$. Using this observation and Lemma 6.1 we get that, for any $\theta \in \Theta_d(s, a_0(s, A))$ and any $m \geq m_0 + 1$,

$$(83) \quad \begin{aligned} \mathbf{E}_\theta |\hat{\eta}(g_m) - \eta| &\leq s \left[\frac{d/s - 1}{d/g_m - 1} \frac{\pi^{-1/2}}{\sqrt{\log(d/g_m - 1)}} + (\log(d/s_d^* - 1))^{-\frac{1}{2}} \right] \\ &\leq \frac{((4/3)\pi^{-1/2} + 1)g_m}{\sqrt{\log(d/s_d^* - 1)}} \triangleq \tau' g_m = \tau' 2^m, \end{aligned}$$

where the last inequality follows from the same argument as in (80).

Now, we plug (82) and (83) in (76) to obtain

$$(84) \quad \begin{aligned} I_2 &\leq \frac{C_2^{1/2}(\tau'/\tau)^{1/2}}{s} \sum_{m=m_0+1}^M \exp(-C_1 2^{m-1} \tau) \\ &\leq C_3(\tau')^{1/2} \tau^{-3/2} \exp(-C_1 2^{m_0} \tau) \leq C_3(\tau')^{1/2} \tau^{-3/2} \end{aligned}$$

for some absolute constant $C_3 > 0$. Notice that $(\tau')^{1/2} = O((\log(d/s_d^* - 1))^{-\frac{1}{4}})$ as $d/s_d^* \rightarrow \infty$ while $\tau^{-3/2} = O((\log(d/s_d^* - 1))^{\frac{3}{14}})$. Thus, $I_2 = o(1)$ as $d \rightarrow \infty$. Since from (75) we also get that $I_1 = o(1)$ as $d \rightarrow \infty$, the proof is complete. \square

PROOF OF LEMMA 6.1. Let first $s < g_M$. Then, by definition of m_0 , we have $s < g_{m_0}$. Therefore, $s < g_m$ for $m \geq m_0$, and we have $w(g_m) < w(s)$. It follows that

$$a_0(s, A) - w(g_m) \geq a_0(s, A) - w(s) \geq \frac{\sqrt{A}}{2\sqrt{2}} \min \left(\frac{\sqrt{A}}{\sqrt{2}}, \log^{1/4}(d/s - 1) \right),$$

where we have used the elementary inequalities

$$\sqrt{x+y} - \sqrt{y} \geq y/(2\sqrt{x+y}) \geq (2\sqrt{2})^{-1} \min(y/\sqrt{x}, \sqrt{y})$$

with $x = 2 \log(d/s - 1)$ and $y = A\sqrt{\log(d/s - 1)}$. By assumption, $A \geq 16\sqrt{\log \log(d/s_d^* - 1)}$, so that we get

$$(85) \quad a_0(s, A) - w(g_m) \geq a_0(s, A) - w(s) \geq 4 \left(\log \log \left(\frac{d}{s_d^*} - 1 \right) \right)^{1/2}.$$

This and the standard bound on the Gaussian tail probability imply

$$(86) \quad \mathbf{P}(|\xi| > (a_0(s, A) - w(g_m))_+) \leq \exp(-(a_0(s, A) - w(g_m))^2/2)$$

$$(87) \quad \leq (\log(d/s_d^* - 1))^{-\frac{1}{2}}.$$

Let now $s \in [g_M, s_d^*]$. Then $m_0 = M$ and we need to prove the result only for $m = M$. By definition of M we have $s_d^* \leq 2g_M$. This and (85) imply

$$a_0(s, A) - w(g_M) \geq a_0(s, A) - w(s) - (w(s_d^*/2) - w(s_d^*)) \geq 4 \left(\log \log \left(\frac{d}{s_d^*} - 1 \right) \right)^{1/2} - (w(s_d^*/2) - w(s_d^*)).$$

Now, using the elementary inequality $\sqrt{\log(x+y)} - \sqrt{\log(x)} \leq y/(2x\sqrt{\log(x)})$ with $x = d/s_d^* - 1$ and $y = d/s_d^*$, and the fact that $s_d^* \leq d/4$ we find

$$w(s_d^*/2) - w(s_d^*) \leq \frac{1}{\sqrt{2 \log(d/s_d^* - 1)}} \frac{d}{d - s_d^*} \leq \frac{2\sqrt{2}}{3\sqrt{\log(d/s_d^* - 1)}} \leq 3 \left(\log \log \left(\frac{d}{s_d^*} - 1 \right) \right)^{1/2}.$$

The last two displays yield $a_0(s, A) - w(g_M) \geq \left(\log \log \left(\frac{d}{s_d^*} - 1 \right) \right)^{1/2}$, and we conclude as in (86). □

Acknowledgements. We would like to thank Felix Abramovich for helpful discussion of the results. The work of N.A. Stepanova was supported by an NSERC grant. The work of A.B. Tsybakov was supported by GENES and by the French National Research Agency (ANR) under the grants IPANEMA (ANR-13-BSH1-0004-02), and Labex ECODEC (ANR - 11-LABEX-0047). It was also supported by the "Chaire Economie et Gestion des Nouvelles Données", under the auspices of Institut Louis Bachelier, Havas-Media and Paris-Dauphine.

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