We introduce a new family of random compact metric spaces $S_\alpha$ for $\alpha \in (1, 2)$, which we call stable shredded spheres. They are constructed from excursions of $\alpha$-stable Lévy processes on $[0,1]$ possessing no negative jumps. Informally, viewing the graph of the Lévy excursion in the plane, each jump of the process is "cut open" and replaced by a circle and then all points on the graph at equal height which are not separated by a jump are identified. We show that the shredded spheres arise as scaling limits of models of causal random planar maps with large faces introduced by Di Francesco and Guitter. We also establish that their Hausdorff dimension is almost surely equal to $\alpha$. Point identification in the shredded spheres is intimately connected to the presence of decrease points in stable spectrally positive Lévy processes as studied by Bertoin in the 90’s.

1. Introduction.

1.1. Random planar geometries. In recent years, there has been considerable progress in the study of random two dimensional surfaces, in particular random planar maps and their scaling limits. Planar maps are finite connected graphs properly drawn on the two-sphere.

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viewed up to continuous deformations. A natural way to choose a random planar map is to pick uniformly from the set of triangulations of the sphere with a fixed number of triangles. Due to bijections of Bouttier, Di Francesco and Guitter [12] (building on [16, 44]) the random triangulations may be viewed as random labelled trees which are further in bijection with random processes. These correspondences have been used to prove many deep results about random planar maps. Most notable are the independent proofs of Miermont [40] (uniform quadrangulations) and Le Gall [36] (uniform triangulations and $2p$-angulations) that properly rescaled random maps converge, in the Gromov–Hausdorff sense, towards a compact metric space called the Brownian map (this type of convergence is often referred to as a scaling limit). Subsequently, it has been shown that the Brownian map is the scaling limit for a large class of different discrete models, see e.g. [1, 10, 39]. It has also been constructed within the Gaussian Free Field by Miller and Sheffield [41].

Another family of random compact metric spaces was introduced in [37] and referred to as stable maps. We mention these here since they are relevant to the model which we introduce in the current paper. The stable maps arise as scaling limits, at least along subsequences, of random planar maps defined by assigning weights to the faces such that the degree of a typical face is in the domain of attraction of a stable law with index $\alpha \in (1, 2)$. See [17, Section 5.2] for more details. One of the motivation for studying stable maps is that they appear in statistical mechanical models on planar maps [13, 37, 42].

1.2. Causal maps. Models of random planar maps became popular in high energy physics in the 1980’s and 1990’s due to their connections to random matrix models and Liouville quantum gravity (see e.g. [2]). In the latter, one quantizes gravity via a path integral which is formally defined as an integral over a suitable set of space-time geometries with a weight which is given in terms of the action of the classical gravity theory. In order to make sense of this integral, one discretizes the geometries and the problem essentially boils down to performing the sum over a suitable set of planar maps having a fixed size and an appropriate weight which is derived from the gravity action. The weights may be viewed as a probability measure on this set of planar maps by normalizing. When restricted to triangulations of the sphere, we recover the model of uniform triangulation of the sphere described above whose scaling limit is the Brownian map. This model is referred to by physicists as dynamical triangulations (DT), see Figure 2.

An important question in the model of DT is what constitutes a suitable set of space-time geometries in the path integral. Ambjørn and Loll [3] argued that causality should be taken into account and imposed a ‘causal’ condition on the planar maps appearing in the discretized path integral. This was initially formulated for triangulations roughly as follows. The sphere is divided into lines of constant latitudes (including the poles) and each adjacent pair of latitudes is triangulated with a single layer of triangles, see Figure 2. This model is referred to as causal dynamical triangulations (CDT). Although a large uniformly random CDT is a non-trivial object, it turns out that its scaling limit is simply a line segment [18, Theorem 1]. The reason is that the latitudes collapse to points in the limit since a pair of typical points on a given latitude are a factor of a logarithm closer to each other than to (say) the south pole. The Gromov-Hausdorff limit is thus a useless concept to study properties of large CDT.

In this paper, building on the work of Di Franscesco and Guitter [21], we study a model of generalized CDT where the faces are allowed to have arbitrary degrees and they are assigned weights such that the degree of a typical face is in the domain of attraction of an $\alpha$-stable law, $\alpha \in (1, 2)$ (see Figure 2 for an illustration and the next section for a precise statement). This is analogous to the definition of the stable maps which was mentioned above. As we will see these random maps will have a non-trivial scaling limit.
1.3. The hard multimer model and its scaling limits. Let us define our model of hard multimers, which is closely related to one originally defined by Di Francesco and Guitter [21]. We will use it to define the model of causal maps through a bijection.

Consider the semi-infinite cylinder represented by the set $C = [0,1] \times \mathbb{R}_+$ with the vertical boundaries identified. We draw equally spaced lines $[0,1] \times i$, for $i \in \{0,1,2,\ldots\}$ in $C$ which we will sometimes refer to as time-slices. A hard multimer configuration is defined by drawing vertical lines of integer length (zero or larger) in $C$ so that their endpoints lie on time-slices and so that no two such lines intersect, see Figure 3. A vertical line is referred to as a multimer. We will assume that a multimer configuration is connected in the sense that the projection of all the multimers onto $0 \times \mathbb{R}_+$ is an interval of the form $[0,k]$, $k \geq 0$. One of the endpoints which lie on the first time-slice is singled out and called the root. We consider a multimer configuration as a planar map whose vertex set is the set of intersection points of horizontal and vertical lines and the edges are the line segments connecting these points. Let $\mathcal{M}$ denote the set of connected multimer configurations and $\mathcal{M}_n$ the subset of those with exactly $n$ vertices.

To remove any arbitrariness in the drawing of a multimer configuration we will use the following convention referred to as a left staircase boundary condition in [21]: The root is drawn furthest to the left and the leftmost multimer which intersects layer $[0,1] \times [i,i+1]$, $i > 0$, is to the right of the leftmost multimer which intersects the layer immediately below. See Figure 3 for an illustration of these concepts.

Given a multimer configuration $M \in \mathcal{M}$ we define its vertical dual $G$ by placing a vertex in each face of $M$, as well as a vertex below the lowest time-slice and a vertex above the
highest time-slice, and an edge between vertices such that the corresponding faces share a horizontal boundary. The graph distance on $G$ is thus the minimal number of horizontal time-slices to cross to go from one point to the other by avoiding all the multimers (such a path can go around the cylinder if needed). The vertical dual will be called a causal map. Its faces are in one to one correspondence with the multimers and the degree of a face equals 2 times the number of vertices in the corresponding multimer. Note that a causal triangulation is recovered by letting all multimers have length 1 and by including the horizontal edges in the dual.

Let $\mu$ be a probability measure on $\{-1,0,1,2,\ldots\}$. We define a probability distribution $P_{\mu}$ on the set of multimer configurations $M_n$ by assigning to each $M \in M_n$ the probability

$$
P_{\mu}^n(M) = \frac{1}{Z_n} \prod_{m \text{ multimer in } M} \mu(|m|) (\mu - 1)^{|m|}
$$

where $|m|$ denotes the length (number of edges) of a multimer and $Z_n$ is the appropriate normalization. The presence of the factor $(\mu - 1)^{|m|}$ is also a convention of normalization and is included here to make the link with random walk clearer. This model is related to that of Di Francesco and Guitter [21], see Section 2.1.1. Denote a random multimer configuration distributed by $P_{\mu}$ by $M_{\mu}$. The associated random vertical dual graph will be denoted by $G_{\mu}$ and we call it a random causal map. We regard $G_{\mu}$ as a metric space with metric given by the graph distance.

When $\mu$ has mean zero and has finite variance, the result of [18] should hold and we believe that $G_{\mu}$ falls in the universality class of so-called generic causal dynamical triangulations, see Section 4.4. In this work we suppose that $\mu$ is centered but has infinite variance. Specifically we suppose that

$$
\mu([k, \infty)) \sim |\Gamma(1 - \alpha)|^{-1} k^{-\alpha} \quad \text{with} \quad \alpha \in (1, 2).
$$

So that $\mu$ is in the strict domain of attraction of the spectrally positive $\alpha$-stable random variable. Then an interesting random compact metric space appears as the scaling limit of $G_{\mu}$ for each $\alpha$:

**Theorem 1.1 (Shredded spheres as scaling limits).** There is a family of random compact metric spaces $(S_\alpha : 1 < \alpha < 2)$, called the stable shredded spheres such that if $\mu$ is centered and satisfies (1.2) then we have

$$
n^{-1/\alpha} \cdot G_{\mu} \xrightarrow{d} S_\alpha,
$$

in distribution for the Gromov–Hausdorff topology.

Here $c \cdot G$ means that graph distances in $G$ are multiplied by the constant $c$. For background on the Gromov–Hausdorff topology, we refer to [14]. A simulation of a large $G_{\mu}$ is shown in Figure 1.

1.4. Shredded spheres as a metric gluing of two trees. A precise definition of $S_\alpha$ is given in Section 3, but roughly speaking we construct $S_\alpha$ as follows. Start with an excursion $\mathcal{E} = (\mathcal{E}(t))_{t \in [0,1]}$ of an $\alpha$-stable Lévy process with no negative jumps. As it is well-known, one may construct a real tree by “gluing the underside” of $\mathcal{E}$ (identifying points at the same height which are not separated by a local minimum), see [26]. In the same way, another real tree may be obtained by applying the same construction to the reflected excursion $(-\mathcal{E}(-t) + \max \mathcal{E})_{t \in [0,1]}$ (with cyclically adjusted time-paramterization). We obtain the shredded sphere $S_\alpha$ by applying both identifications simultaneously; that is, we “glue” both the underside and
overside of $E$. In a more precise sense, we endow $[0, 1]$ with a random pseudo-metric $D^*$ obtained as

$$D^*(s, t) = \inf \left\{ \sum_{i=1}^{n} D^u(a_i, b_i) + D^d(b_i, a_{i+1}) : s = a_1, b_1, ..., a_n, b_n, a_{n+1} = t \right\},$$

where $D^u$ and $D^d$ are the pseudo-metrics on $[0, 1]$ coding for the tree ‘below’ and ‘above’ $E$. This type of constructions is classic in the theory of random maps, e.g. the Brownian map [36] or in the mating of trees theory [24]. But it is usually challenging to show that $D^*$ appears as the limiting metric of rescaled discrete models: in the case of the Brownian map, this corresponds to the breakthroughs of Le Gall [36] and Miermont [40]. Also, contrary to the case of random planar maps, our models do not possess re-rooting invariance which was crucial in the aforementioned results. Our salvation will come from noting that the large multimers create impassable barriers in the scaling limit. More precisely, we can define another pseudo metric $V(s, t)$ on $[0, 1]$ obtained by the minimal vertical total variation to go from $s$ to $t$ by avoiding the jumps of $E$. Our main result is that $V = D^*$ almost surely for $E$, see Theorem 3.1 (this is not a deterministic statement, see Remark 3.2 for a counterexample). With these ingredients at hands, the proof of Theorem 1.1 is rather easy. We also show (Theorem 4.3) that

$$V(s, t) = 0 \iff D^u(s, t) = 0 \text{ or } D^d(s, t) = 0,$$

thus characterizing the point identifications in $S_\alpha$. Our methodology, based on the use of large faces to control the metric, will be applied to the study of stable maps [37] in a forthcoming work [20]. A key step is a powerful bootstrapping argument introduced in a different context by Gwynne and Miller [29].

We also study various properties of the $\alpha$-stable shredded sphere which are connected to fine properties of the $\alpha$-stable Lévy process. For example, the local behavior of such processes entails that $S_\alpha$ is almost surely of Hausdorff dimension $\alpha$ (Proposition 4.1). Our proof of this closely mimics the case of the $\alpha$-stable looptrees [19]. Furthermore, our shredded spheres $S_\alpha$ possess large faces which are the image of the jumps of $E$ in our construction. We show using the work of Bertoin [7] on increase points of Lévy processes that there are large faces which touch each other. Unfortunately, as in the case of the mating-of-trees theory of Duplantier, Miller and Sheffield [24, Question 11.2] we were unable to decide whether the graph of faces in $S_\alpha$ is connected (two faces are adjacent if they touch each other). More generally we leave the following question open:

**Question 1.2.** Is it possible to characterize the topology of $S_\alpha$?

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2. Discrete encodings and bounds on distances. In this section we recall the coding of (random) multimer configurations using (random) walks. This will motivate our definition and construction of the shredded sphere $S_\alpha$ using the normalized excursion $E$ of an $\alpha$-stable spectrally positive Lévy process. We refer the reader to [8, 15] for the construction and basic properties of (excursions of) stable Lévy processes. We shall also prove lower and upper bounds for asymptotic distances in $G_n^\alpha$ reducing the proof of Theorem 1.1 to properties of $E$.

2.1. Coding by a walk. Let $\mathcal{M}_n^\mu$ be our random multimer configuration with $n$ vertices and $G_n^\mu$ be its vertical dual. We associate with $\mathcal{M}_n^\mu$ a random walk $E_n$ as follows. For each multimer $m \in \mathcal{M}_n^\mu$ except for the root, draw a horizontal segment from its bottom point going to the left and stop at the first vertex encountered (blue lines in Figure 4). The bottom point of the root is instead connected vertically to $\delta_n$, an additional point below the lowest time slice. This construction results in a tree. As in [21] we now perform the left-to-right contour walk around this tree, starting at the bottom of the root, going up the multimers in single steps (of sizes $k \in \{0, 1, 2, \ldots\}$) and going down by steps of $-1$ when needed (horizontal blue edges are traversed without taking a step in the corresponding walk). The resulting contour function, with time indexed by $\{0, 1, \ldots, n + 1\}$, is denoted by $E_n$; it starts at level 0 and stops at level $-1$ at time $n + 1$. We extend the domain of $E_n$ to the interval $[0, n + 1]$ by linear interpolation between the integer points. The construction, which is a bijection between multimer configurations and excursions of walks whose jumps are in $\{-1, 0, 1, 2, 3, \ldots\}$, should be clear on Figure 4. Note that $E_n$ records the vertical distances to the lowest point of $\mathcal{M}_n^\mu$.

The underlying multimer configuration can be identified with the closed set formed by

\[
\text{Slits}_n = \bigcup_{0 \leq k \leq n-1} \{k + 1\} \times [E_n(k), E_n(k + 1)] \subset [0, n] \times \mathbb{R}_+.
\]

(Only multimers contribute to the union since $[j, j - 1] = \emptyset$.) Notice that we consider here the cylinder of width $n + 1$ whereas we considered before a cylinder of width 1, but this horizontal dilation is irrelevant in what follows. In this representation, the vertices of $G_n^\mu$ except for the top and bottom vertices can be identified with $(t, E_n(t))$ for those times $t \in \mathcal{R}_n$ where

\[
\mathcal{R}_n = \{k + \frac{1}{2} : E_n(k + 1) = E_n(k) - 1\} \cap [0, n + 1].
\]
It is also more convenient to start our cylinder at height $-1$ to include the vertex $\emptyset^u$, and we shall always imagine that the vertices $\emptyset^u$ and $\emptyset^d$ are respectively placed at height $-1/2$ and $\sup E_n + 1/2$. See Figure 5 for an illustration.

It will be convenient to associate an element of $\mathcal{R}_n$ to each $t \in [0, n]$, hence we define

$$t^\circ = \begin{cases} \max \{ r \in \mathcal{R}_n : r \leq t \}, & \text{if } t \geq \min \mathcal{R}_n, \\ \max \mathcal{R}_n, & \text{otherwise}. \end{cases}$$

Note that if $t \in \mathcal{R}_n$ then $E_n(t) + 1/2$ is the graph distance between $\emptyset^u$ and the corresponding vertex in $\mathcal{G}_n$ via (2.2). We also consider the piecewise constant excursion $\overline{E}_n(t) := E_n(t^\circ)$. Note that $\overline{E}_n(\cdot)$ is càdlàg and agrees with $E_n(\cdot)$ on the set $\mathcal{R}_n$, and that $\overline{E}_n(t) = 1/2$ for $0 \leq t < \min \mathcal{R}_n$.

2.1.1. Law of the walk under Boltzmann measures. Let us compute the law of $\overline{E}_n$ in the case when the multimer configuration has a “Boltzmann” law with a generic weight sequence. We do so in order to shed light on the normalization conditions used in the introduction. If
\( q = (q_k : k \geq 1) \) is a sequence of non-negative weights (not necessarily of sum 1) we can define a probability distribution \( P_n^q \) on the set of multimer configurations of size \( n \) by putting

\[
(2.4) \quad P_n^q(M) \propto \prod_{m \text{ multimer in } M} q_{|m|}, \quad \text{for each } M \in \mathcal{M}_n,
\]

where \(|m|\) denotes the length (number of edges) of a multimer. Notice that since we restrict to configurations with \( n \) vertices in total, for any \( \lambda, \xi > 0 \) we have

\[
\prod_{m \text{ multimer in } M} q_{|m|} = (\lambda \xi)^{-n} \prod_{m \text{ multimer in } M} \left( \xi q_{|m|} \lambda^{|m|+1} \right) : \xi^{|m|}.
\]

We shall assume that \( \lambda, \xi > 0 \) are chosen so that \((\mu_k : k \geq -1)\) defined by

\[
\mu_k = \xi q_k \lambda^{k+1}, \quad \text{for } k \geq 0, \quad \text{and } \mu_{-1} = \xi,
\]

is a probability measure. By doing so, we are back to the setting (1.1) used in the introduction. The additional freedom on \((\lambda, \xi)\) allows us, in generic situations, to further fix the mean of \( \mu \) to be 1 (see [32, Section 4] for further details in the equivalent context of simply generated trees). Now it is easy to see that the push forward of this distribution on the set of excursion paths by the above bijection gives the law of a \( \alpha \)-random walk conditioned to first hit \(-1\) at time \( n + 1 \). Our standing assumptions (1.2) is then to ask that the step distribution \( \mu \) of that random walk is critical and in the domain of attraction of the \( \alpha \)-stable law.

Our model is related to that of Di Francesco and Guitter [21] as follows: setting \( \mu_0 = 0 \) and \( t_i = \mu_i(\mu_{-1})^i \) for \( i \geq 1 \), we have

\[
(2.5) \quad \sum_{n \geq 0} Z_n = \lim_{T \to \infty} \sum_{\{t_i\}} Z_T(\{t_i\}),
\]

where the right-hand-side uses the notation of [21, Section 2.2] for the partition-function of multimer-configurations on \( T \) time-slices. Thus, we restrict the number of vertices \( n \) rather than the number of time-slices \( T \).

2.2. A first scaling limit. The above coding of multimer configurations by walks enables us to define right away the scaling limit for the coding excursion which will be our entry door to the continuous world.

By \( \alpha \)-stable Lévy process we mean a stable spectrally positive Lévy process \( X \) of index \( \alpha \), normalized so that for every \( \lambda > 0 \),

\[
\mathbb{E}[\exp(-\lambda X_t)] = \exp(t\lambda^\alpha).
\]

Equivalently, the Lévy measure \( \Pi \) of \( X \) is supported by \( \mathbb{R}_+ \) and we have

\[
(2.6) \quad \Pi([r, \infty)) = |\Gamma(1-\alpha)|^{-1} r^{-\alpha}, \quad \text{for } r > 0.
\]

The trajectories of \( X \) a.s. belong to the Skorokhod space \( D(\mathbb{R}_+, \mathbb{R}) \) of right-continuous with left limits (càdlàg) functions, endowed with the Skorokhod topology (see [11, Chap. 3]). The dependence of \( X \) in \( \alpha \) will be implicit in the whole paper. We consider (see [15]) the normalized excursion \( \mathcal{E} \) of \( X \) above its infimum. Notice that \( \mathcal{E} \) is a.s. a random càdlàg function on \([0, 1]\) such that \( \mathcal{E}(0) = \mathcal{E}(1) = 0 \) and \( \mathcal{E}(s) > 0 \) for every \( s \in (0, 1) \).

**Proposition 2.1** (Scaling limit of the coding excursion). If \( \mu \) is centered and satisfies (1.2) then the random càdlàg excursion \( \mathcal{E}_n \) associated to \( \mathcal{M}^\alpha_n \) by the above construction satisfies

\[
(2.7) \quad (n^{-1/\alpha} \cdot \mathcal{E}_n(nt))_{t \in [0, 1]} \to (\mathcal{E}(t))_{t \in [0, 1]}
\]

in the sense of weak convergence with respect to the Skorokhod metric on \( \mathbb{D} [0, 1] \).
PROOF. By the discussion at the end of the last section, the law of $E_n$ is that of a random walk with i.i.d. increments of distribution $\mu$ conditioned on reaching $-1$ for the first time at $n + 1$. The result then follows from an adaptation of the arguments in [25, Lemmas 4.5-6] (to deal with the interpolation between integer-points using the random function $t^\circ$ in (2.3)).

Proposition 2.1 already gives the order of the diameter of $\mathcal{G}_n^\mu$, since $\overline{E}_n + 1/2$ records the distances to $\sigma^u$ in $\mathcal{G}_n^\mu$. Using Skorokhod’s embedding theorem, we can suppose that our multimer configurations $\mathcal{M}_n^\mu$ are coupled in such a way that the convergence of Proposition 2.1 is almost sure. This then means that there are (random) continuous increasing bijective functions $\psi_n : [0, 1] \to [0, n]$ so that

$$\lim_{n \to \infty} (n^{-1/\alpha} \cdot \overline{E}_n \circ \psi_n(t) : t \in [0, 1]) = (E(t) : t \in [0, 1])$$

almost surely for the supremum norm on $[0, 1]$ and so that $\|\frac{1}{n} \psi_n - \text{Id}\|_{\infty} \to 0$. In the following we shall write $\overline{E}_n = n^{-1/\alpha} \overline{E}_n \circ \psi_n$.

Our goal is to use (2.8) to show almost sure convergence of the rescaled graphs $\mathcal{G}_n^\mu$, thereby establishing our main result Theorem 1.1. The road will be quite long and we shall start with discrete upper and lower bounds on the distances in $\mathcal{G}_n^\mu$ which pass to the limit.

2.3. Two trees and an upper bound. Recall the encoding of $\mathcal{M}_n^\mu$ by $E_n$ and let us denote by $\sigma^d$ and $\sigma^u$ the top and bottom vertices of $\mathcal{G}_n^\mu$ respectively. We define two trees, which we call the up-tree (rooted at $\sigma^u$) and down-tree (rooted at $\sigma^d$), as follows, see Figure 6. The up-tree is obtained by connecting each vertex of $\mathcal{G}_n^\mu$ (except $\sigma^d$) to its leftmost neighbor in the time-slice below, and the down-tree is obtained by connecting each vertex of $\mathcal{G}_n^\mu$ (except $\sigma^u$) to its rightmost neighbor in the time-slice above. Both trees are subgraphs of $\mathcal{G}_n^\mu$ but their union does not yield $\mathcal{G}_n^\mu$ in general.

![Figure 6](image_url)

Fig 6. Illustration of the two trees lying inside $\mathcal{G}_n^\mu$, the up-tree is in red and is rooted at the bottom vertex $\sigma^u$, the down-tree is in blue and is rooted at the top vertex $\sigma^d$. Both trees are subgraphs of $\mathcal{G}_n^\mu$ and the distances in those trees are easily expressed using $E_n$. On the right it is shown how to recover the trees from the walk which corresponds to the multimer configuration.

Recall from the last section that the vertices of $\mathcal{G}_n^\mu$, except for $\sigma^u$ and $\sigma^d$, are in correspondence with $\mathcal{R}_n$. Distances within these trees can be expressed using the excursion $E_n$: 

\[ (n^{-1/\alpha} \cdot \overline{E}_n \circ \psi_n(t) : t \in [0, 1]) \]
if \( u, v \in \mathcal{R}_n \) the distances \( D^u(u, v) \) (resp. \( D^d(u, v) \)) in the up-tree (resp. down-tree) between the vertices associated with \( u \) and \( v \) in \( \mathcal{G}_n^u \) are given by

\[
D^u_n(u, v) = \mathcal{E}_n(u) + \mathcal{E}_n(v) - 2 \min_{w \in [u \wedge v, u \vee v]} \mathcal{E}_n(w) + 1 - \delta_{u,v},
\]

\[
D^d_n(u, v) = 1 - \delta_{u,v} + 2 \min \left\{ \max_{w \in [u \wedge v, u \vee v]} \mathcal{E}_n(w), \max_{w \in [0, u \vee v] \cup [u \wedge v, n]} \mathcal{E}_n(w) \right\} - \mathcal{E}_n(u) - \mathcal{E}_n(v).
\]

(The definition of \( D^d \) can be interpreted in terms of the “mirror” excursion of \( \mathcal{E}_n \), we explain this in the continuous setting below, see (2.10).)

Let us write \( D_n(u, v) \) for the distance in \( \mathcal{G}_n^u \) between (the vertices associated with) \( u, v \in \mathcal{R}_n \). Clearly we have both \( D_n(u, v) \leq D^u_n(u, v) \) and \( D_n(u, v) \leq D^d_n(u, v) \). In order to see \( D_n, D^u_n \) and \( D^d_n \) as pseudo-distances on \([0, 1]\), similar to what we did for \( \mathcal{E}_n \) just after Proposition 2.1, we shall introduce for \( s, t \in [0, 1] \)

\[
\tilde{D}_n(s, t) = D_n(\psi_n(s), \psi_n(t),)^{\circ}, \quad \tilde{D}^u_n(s, t) = D^u_n(\psi_n(s), \psi_n(t),)^{\circ}, \quad \tilde{D}^d_n(s, t) = D^d_n(\psi_n(s), \psi_n(t),)^{\circ},
\]

where \( \psi_n \) is the time-change in (2.8). With these notations, the distance metric in \( \mathcal{G}_n^u \) between vertices (excepting \( \emptyset^u \) and \( \emptyset^d \)) is given by the quotient of \([0, 1]\) by \( (D_n = 0) \) endowed with the projection of \( \tilde{D}_n \).

We now move on to the continuous setting and define accordingly for \( s, t \in [0, 1] \)

\[
D^u(s, t) = \mathcal{E}(s) + \mathcal{E}(t) - 2 \inf_{u \in [s \wedge t, s \vee t]} \mathcal{E}(u).
\]

To help the reader understand the next definition notice that we always have \( \inf_{u \in [s \wedge t, s \vee t]} \mathcal{E}(u) = 0 \) and so we could have replaced \( \inf_{u \in [s \wedge t, s \vee t]} \mathcal{E}(u) \) by \( \max \{ \inf_{u \in [s \wedge t, s \vee t]} \mathcal{E}(u), \inf_{u \in [0, s \wedge t] \cup [s \vee t, 1]} \mathcal{E}(u) \} \) in the last definition. The second pseudo distance is the same as the first but for the “mirror” excursion \( -\mathcal{E}(-\cdot) + \max \mathcal{E} \) and is given for \( s, t \in [0, 1] \) by

\[
D^d(s, t) = 2 \min \left\{ \sup_{u \in [s \wedge t, s \vee t]} \mathcal{E}(u), \sup_{u \in [0, s \wedge t] \cup [s \vee t, 1]} \mathcal{E}(u) \right\} - \mathcal{E}(s) - \mathcal{E}(t).
\]

It is easy to check using [26, eq. (3)] that \( D^u \) and \( D^d \) are pseudo distances. Furthermore the quotient metric spaces \(([0, 1]/\{D^i = 0\}, D^i)\) for \( i \in \{u, d\} \) are both real trees, which we refer to as the trees coded by \( \mathcal{E} \) and by \( -\mathcal{E}(-\cdot) + \max \mathcal{E} \), respectively. In a different context, such trees were considered by Lambert [34] as special cases of random real trees which belong to the family of so-called splitting trees. In our case we “glue these trees together”, that is we consider \( D^* \) the largest pseudo distance on \([0, 1]\) that is both smaller than \( D^u \) and \( D^d \) viz

\[
D^*(s, t) = \inf \left\{ \sum_{i=1}^{k} D^u(a_i, b_i) + D^d(b_i, a_{i+1}) : s = a_1, b_1, \ldots, a_k, b_k, a_{k+1} = t \right\}.
\]

Our asymptotic upper bound on the distances in \( \mathcal{G}_n^u \) (similar to the bound \( D \leq D^* \) in the case of the Brownian map, see [35, Proposition 3.3]) reads now as follows:

**Proposition 2.2 (Upper bound on distances).** *Almost surely, under the coupling in (2.8)*

\[
\limsup_{n \to \infty} \sup_{s,t \in [0,1]} \left( n^{-1/\alpha} \tilde{D}_n(s, t) - D^*(s, t) \right) \leq 0.
\]
PROOF. We start by noting from (2.8) and by definition of $\tilde{D}_n^{u/d}$ and $D^{u/d}$ that we have
\[ n^{-1/\alpha} \tilde{D}_n^{u/d} \xrightarrow{\text{a.s.}} D^{u/d}, \]
uniformly over $[0,1]^2$. Using (2.8) and properties of càdlàg functions, it is easy to see that for any given $\epsilon > 0$ there is an $N$ such that for all large enough $n$, for any $s \in [0,1]$, we can find $s' \in D(N) = \{k2^{-N} : k = 0, 1, \ldots, 2^N\}$ such that
\[ n^{-1/\alpha} \tilde{D}_n(s,s') \leq \epsilon/4 \quad \text{and} \quad D^{u/d}_n(s,s') \leq \epsilon/4. \]
Using the bounds $\tilde{D}_n \leq \tilde{D}_n^{u}$ as well as $D^* \leq D^{u}$ and the triangle inequality, it follows that for all large enough $n$ we have
\[ (2.12)\sup_{s,t \in [0,1]} \left( n^{-1/\alpha} \tilde{D}_n(s,t) - D^*(s,t) \right) \leq \epsilon + \max_{s,t \in D(N)} \left( n^{-1/\alpha} \tilde{D}_n(s,t) - D^*(s,t) \right), \]
Now, for each fixed $s_0, t_0$ we claim that $\limsup_{n \to \infty} n^{-1/\alpha} \tilde{D}_n(s_0, t_0) \leq D^*(s_0, t_0)$. To see this, for a given $\eta > 0$, fix a finite sequence $a_i, b_i$ such that
\[ D^*(s_0, t_0) \geq \sum_{i=1}^{k} D^u(a_i, b_i) + D^d(b_i, a_{i+1}) - \eta. \]
Using that
\[ \tilde{D}_n(s_0, t_0) \leq \sum_{i=1}^{k} \tilde{D}_n^u(a_i, b_i) + \tilde{D}_n^d(b_i, a_{i+1}) \]
and the fact that $n^{-1/\alpha} \tilde{D}_n^{u/d}(s,t) \to D^{u/d}(s,t)$ we deduce the claim by letting $\eta \to 0$. Since there is a finite number of pairs of points $(s,t) \in D(N)^2$, we can take the lim sup and $n \to \infty$ in (2.12) and get the statement of the proposition by letting $\epsilon \to 0$. \qed

2.4. Lower bound on distances. We now turn our attention to a lower bound on distances in $G_n^\mu$ for which the role of the large multimers becomes crucial. The rough idea is that large multimers form obstacles when travelling horizontally, allowing us to bound distances from below by a vertical height variation.

Recall the setup of the last section and let us imagine that $G_n^\mu$ is drawn on the cylinder $[0,n+1] \times [-1,\infty)$ on top of the coding walk $E_n$ as in Figure 5. More precisely, the vertices of $G_n^\mu$ are the points $(t, E(t))$ for $t \in \mathcal{R}_n$, together with the additional vertices $\varnothing^u$ and $\varnothing^d$ which are respectively at height $-1/2$ and $H+1/2$ where $H = \max E_n$. Recall the definition of Slits$_n$ from (2.1). Finally, the edges of the graph $G_n^\mu$ can also be drawn on the cylinder, such that the edges do not intersect Slits$_n$ and such that the vertical coordinate is monotone along each edge, see Figure 5.

Consider two times in $\mathcal{R}_n$ corresponding to two vertices $a$ and $b$ of $G_n^\mu$, and consider a geodesic path in $G_n^\mu$ going from $a$ to $b$. In the above embedding, such a path may be seen as a continuous curve $\gamma$ from $[0,1]$ to the cylinder $[0,n+1] \times [-1,\infty)$ avoiding Slits$_n$. Since two adjacent vertices in $G_n^\mu$ necessarily are at height difference 1 and since edges have monotone vertical variation, we can write
\[ D_n(i + \frac{1}{2}, j + \frac{1}{2}) = \text{VarHt}(\gamma), \]
where $\text{VarHt}(\gamma)$ is the total variation of the vertical coordinate of the path $\gamma = ((\gamma_x(t), \gamma_y(t)) : 0 \leq t \leq 1)$, that is
\[ \text{VarHt}(\gamma) = \sup_{0=t_0<t_1<

\cdots<n-1} \sum_{i=0}^{n-1} |\gamma_y(t_{i+1}) - \gamma_y(t_i)|. \]
Let us mimic the above construction in the continuous world using the excursion $\mathcal{E}$. For $s,t \in [0,1]$ let

\begin{equation}
V(s,t) = \inf_{\gamma : (s,\mathcal{E}(s)) \to (t,\mathcal{E}(t))} \text{VarHt}(\gamma),
\end{equation}

where the infimum is taken over all continuous paths $\gamma$ on the cylinder $[0,1] \times \mathbb{R}_+$ with the vertical boundaries identified, going from $(s, \mathcal{E}(s))$ to $(t, \mathcal{E}(t))$ and which does not cross any segments of

$$\text{Slits}(\mathcal{E}) = \bigcup_{i \geq 0} \{t_i\} \times \{\mathcal{E}(t_i), \mathcal{E}(t_i)\}$$

where $(t_i : i \geq 0)$ is an enumeration of the jumps of $\mathcal{E}$. Above we say that $\gamma$ crosses a vertical segment $V = \{\tau\} \times [a,b]$ if there exist times $g,d \in [0,1]$ so that $\gamma(t) \in \{\tau\} \times (a,b)$ for all $t \in [g \land d, g \lor d]$ and so that any neighbourhood of $g$ contains a time where $\gamma$ is to the left of $V$, and any neighbourhood of $d$ contains a time where $\gamma$ is to the right of $V$. Notice that $V(s,t)$ is almost surely finite and that $V(s,t) \geq |\mathcal{E}(s) - \mathcal{E}(t)|$.

We state a useful lemma which uses only basic analysis:

**Lemma 2.3.** Almost surely, for any $s,t \in [0,1]$ there exists a continuous path $\gamma : (s, \mathcal{E}(s)) \to (t, \mathcal{E}(t))$ which does not cross any segment of $\text{Slits}(\mathcal{E})$ and such that $\text{VarHt}(\gamma) = V(s,t)$ and where the $x$-component of $\gamma$ is monotone (on the cylinder).

**Proof.** The lemma is deterministic and works if we replace $\mathcal{E}$ by any càdlàg function. Let us first see why we can restrict ourselves to paths $\gamma = (\gamma_x, \gamma_y)$ whose $x$-coordinate $\gamma_x$ is monotone (recall that we are on the cylinder).

Assume that $\gamma_x(0) \leq \gamma_x(1)$ and let $t' = \sup_{t \in [0,1]} \{\gamma_x(t) = \gamma_x(0)\}$. Let us assume for concreteness that $\gamma_x(t) \geq \gamma_x(0)$ for all $t \in [t',1]$. This means that the path $\gamma$ does not wrap around the cylinder after time $t'$ and we will show that we may modify it such that its $x$-coordinate is monotonically increasing. Define a new path $\gamma^+ = (\gamma^+_x, \gamma_y)$ where

$$\gamma^+_x(t) = \min \left\{ \sup_{s \in [0,t]} \gamma_x(s), \gamma_x(1) \right\}.$$

The path $\gamma^+$ is by construction continuous, its $x$-coordinate is monotonically increasing and it has the same endpoints and the same height variation as $\gamma$. It remains to argue that it does not cross any slits. Assume the opposite, then there is a slit $\{(\tau) \times [a,b]\}$ and times $g \leq d$ such that $\gamma^+(t) \in (\tau) \times (a,b)$ for all $t \in [g,d]$ and there is a $\delta > 0$ such that $\gamma^+_x(t) < \tau$ for $t \in (g - \delta, g]$ and $\gamma^+_x(t) > \tau$ for $t \in (d, d + \delta)$. Let $g' = \inf\{t \leq d : \gamma_x(t) = \{\tau\}\}$. Since $d$ is a point of increase of $\gamma^+_x$ it holds that $\gamma_x(d) = \gamma^+_x(d) = \tau$ and thus the infimum is over a non-empty set. Since $\gamma_x(t) \leq \gamma^+_x(t) < \tau$ for $t \in (g - \delta, g]$ it holds that $g' \geq g$. Therefore, $\gamma(t) \in (\tau) \times (a,b)$ for all $t \in [g',d]$. Now, for any $\varepsilon > 0$ there is a $t \in (g' - \varepsilon, g')$ such that $\gamma_x(t) < \tau$ (otherwise, $g'$ could be made smaller) and a $t' \in (d, d + \varepsilon)$ such that $\gamma_x(t') > \tau$ (since $d$ is a point of increase of $\gamma^+_x$). We have thus shown that $\gamma$ crosses a slit which is a contradiction.

We now restrict ourselves to paths for which the $x$-coordinate is monotone. Consider a sequence of such paths $\gamma_n = (\gamma_{x,n}, \gamma_{y,n})$ such that $\text{VarHt}(\gamma_n) \to V(s,t)$ as $n \to \infty$. Since the height variation converges, $\gamma_{n,y}$ are of a uniformly bounded variation. Since $\gamma_{x,n}$ are monotone and uniformly bounded, they are also of a uniformly bounded variation. Therefore we may assume, using continuity and by reparameterizing, that the sequence is $C$-Lipschitz.
for some $c > 0$. By Helly’s selection theorem (see e.g. [33], Theorems 4 and 5 in Section 36) $\gamma_n$ has a subsequential limit which we denote by $\gamma$, and

$$\text{Var}H_t(\gamma) \leq \lim_{n \to \infty} \text{Var}H(\gamma_n) = V(s,t).$$

Moreover, the Lipschitz condition guarantees that $\gamma$ is continuous. Finally, it is easy to see that $\gamma$ does not cross any slits since none of the $\gamma_n$ did.

**Proposition 2.4** (Lower bound on distances). *Almost surely, under the coupling in (2.8)*

$$\liminf_{n \to \infty} \inf_{s,t \in [0,1]} \left( n^{-\alpha} \hat{D}_n(s,t) - V(s,t) \right) \geq 0.$$

**Proof.** As in Proposition 2.2 we use that, given $\varepsilon > 0$, there is $N$ such that for all large enough $n$ we have

$$\inf_{s,t \in [0,1]} \left( n^{-\alpha} \hat{D}_n(s,t) - V(s,t) \right) \geq -\varepsilon + \min_{s,t \in D(N)} \left( n^{-\alpha} \hat{D}_n(s,t) - V(s,t) \right).$$

Moreover, for each such $s$, $t$ and $n$ there is a continuous path $\gamma_{n,\varepsilon}$ from $(s, \tilde{\gamma}_n(s))$ to $(t, \tilde{\gamma}_n(t))$ which does not cross slits $\tilde{\text{Slits}}(\tilde{\gamma}_n)$ such that $n^{-\alpha} \hat{D}_n(s,t) \geq -\varepsilon + \text{Var}H(\gamma_{n,\varepsilon})$. We also define for $\varepsilon > 0$ and $0 \leq \delta \leq \varepsilon$, the set

$$\text{Slits}^\delta(\varepsilon) = \bigcup_{i \geq 0} \{ \{ t_i \} \times [E(t_i-) + \delta/2, E(t_i) - \delta/2] : E(t_i) - E(t_i^-) > \varepsilon \}$$

which consists of the slits of length larger than $\varepsilon$ which are furthermore truncated from the top and bottom by $\delta/2$. Denote the $x$-coordinates of the slits in $\text{Slits}^\delta(\tilde{\gamma}_n)$, by $t_{i,n}$, $i = 1, \ldots, m_{\varepsilon,n}$ and of the slits in $\text{Slits}^\delta(\varepsilon)$ by $t_i$, $i = 1, \ldots, m_{\varepsilon}$. Since $\tilde{\gamma}_n$ converges uniformly to $\varepsilon$ we may assume, by choosing $n$ large enough, that $m_{\varepsilon,n} = m_{\varepsilon}$ and (by permuting the jump-times) that $t_{i,n} = t_i$ and

$$|\tilde{\gamma}_n(t_i) - E(t_i)| < \varepsilon/2, \quad |\tilde{\gamma}_n(t_i^-) - E(t_i^-)| < \varepsilon/2$$

for all $1 \leq i \leq m_{\varepsilon}$. Arguing as in Lemma 2.3, we may find a subsequence along which the $\gamma_{n,\varepsilon}$ converge towards a continuous curve $\gamma_{\varepsilon}$ which does not cross $\text{Slits}^\delta(\varepsilon)$ and letting $\varepsilon \to 0$ we may find a sequence among the $\gamma_{\varepsilon}$ which converges towards a continuous curve $\gamma$ which does not cross $\text{Slits}(\varepsilon)$ and such that

$$\liminf_{\varepsilon \to 0} \liminf_{n \to \infty} \text{Var}H(\gamma_{n,\varepsilon}) \geq \liminf_{\varepsilon \to 0} \text{Var}H(\gamma_{\varepsilon}) \geq \text{Var}H(\gamma) \geq V(s,t)$$

which concludes the proof.

**3. The $\alpha$-stable shredded spheres and $V = D^\ast$.** Recall the definition of the two (random) pseudo-distances $D^\ast$ given in (2.11) and $V$ given in (2.13). In this section we establish our main result:

**Theorem 3.1.** *For any $\alpha \in (1,2)$, almost surely for $\varepsilon$ we have $V = D^\ast$.***

Given the above theorem we can equivalently define the shredded sphere $S_\alpha$ as the quotient metric space $[0,1]/\{D^\ast = 0\}$ equipped with the (projection of) $D^\ast$ or similarly using $V$ instead of $D^\ast$. Let us show why this, combined with our asymptotic upper and lower bounds for the distances in $\mathcal{G}_m^\ast$, will easily imply Theorem 1.1.
Proof of Theorem 1.1. Recall from the discussion after Proposition 2.1 that we have assumed that \( \tilde{E}_n \to E \) uniformly as \( n \to \infty \) almost surely. Recall also that \( \mathcal{G}_n^\mu \setminus \{\varnothing^u, \varnothing^d\} \) (considered as a metric space) is the quotient of \([0, 1]\) by \( \tilde{D}_n = 0 \) endowed with the projection of \( \bar{D}_n \). Similarly, \( S_n \) is the quotient of \([0, 1]\) by the pseudo-distance \( V \) or equivalently by \( D^\ast \). Those projections from \([0, 1]\) induce a natural correspondence between \( n^{-1/\alpha} \cdot \mathcal{G}_n^\mu \setminus \{\varnothing^u, \varnothing^d\} \) and \( S_\alpha \) whose distortion converges to 0 by Propositions 2.2 and 2.4 combined with Theorem 3.1. Since \( \{\varnothing^u, \varnothing^d\} \) are at distance \( n^{-1/\alpha} \) from another vertex in \( n^{-1/\alpha} \cdot \mathcal{G}_n^\mu \) the result is granted.

Let us give a rough idea of the proof of Theorem 3.1. First, it is straightforward to see that the inequality \( V \leq D^\ast \) is always valid for any càdlàg function \( E \). Indeed, for any \( s = a_1, b_1, \ldots, a_k, b_k, a_{k+1} = t \) one can easily construct a path \( \gamma \) going from \((s, E(s))\) to \((t, E(t))\) and not crossing \( \text{Slits}(E) \) whose height variation is as close as we wish to \( \sum_{i=1}^k D^u(a_i, b_i) + D^d(b_i, a_{i+1}) \). Hence
\[
(3.1) \quad D^\ast(s, t) \geq V(s, t), \quad \forall s, t \in [0, 1].
\]

However the inequality \( D^\ast \leq V \) is much harder. We give an example below which shows that this is not a deterministic result (indeed we shall rely on probabilistic properties of \( E \)). We shall first prove a version of the inequality \( D^\ast \leq V \) for an unconditioned Lévy process \( X \). To do that, we introduce an exploration method of a path not crossing \( \text{Slits}(X) \) for a given scale \( \varepsilon > 0 \). This encodes the path into a word on two letters \( a \) and \( b \). For each word obtained, this decomposes \( X \) into i.i.d. pieces for which we can estimate the height variation needed to cross it. The proof then goes in two stages. We first prove that \( D^\ast \leq C \cdot V \) for some constant \( C > 1 \); then we bootstrap the argument to show that \( C \) must be equal to 1. This last scheme of proof was used earlier by Gwynne and Miller [29] in a very different and independent context, also dealing with random metrics.

Remark 3.2 (A deterministic counter-example). Introduce the functions
\[
f_1(t) = \begin{cases} 
0, & \text{if } t \in [0, 1/3), \\
3(1 - 2t), & \text{if } t \in [1/3, 2/3), \text{ and for } k \geq 1, \\
0, & \text{if } t \in [2/3, 1],
\end{cases}
\]
where \( \frac{1}{2} \leq \beta \leq 1 \). Let (see Figure 7)
\[
F(t) = \sum_{k \geq 1} f_k(t).
\]
For the function \( F \) we have \( V(0, 1) = 0 \) (for \( \gamma \) one can choose a straight horizontal line) and it is not hard to show that \( D^\ast(0, 1) = 2 \).

3.1. Exploration of a path. We now turn to the proof of Theorem 3.1. We shall first prove the result for the (unconditioned) \( \alpha \)-stable Lévy process \( X = (X_t)_{t \geq 0} \), the result for the excursion \( E \) will follow by local absolute continuity. The main idea is an encoding of a path \( \gamma \) at a given scale by a word which induces a Markovian exploration of the Lévy process \( X \).

We define the corresponding objects \( V, D^\ast, \text{Slits,} \ldots \) for \( X \) as follows. Firstly, for \( 0 \leq s < t \),
\[
D^u_X(s, t) = X(s) + X(t) - 2 \inf_{u \in [s,t]} X(u), \quad D^d_X(s, t) = 2 \sup_{u \in [s,t]} X(u) - X(s) - X(t)
\]
and

\[ D^*_X(s,t) = \inf \left\{ \sum_{i=1}^{k} D^b_X(a_i, b_i) + D^d_X(b_i, a_{i+1}) : s = a_1, b_1, \ldots, a_k, b_k, a_{k+1} = t \right\}. \]

Next, \( \text{Slits}(X) = \bigcup_{i \geq 0} \{ t_i \} \times [X(t_i -), X(t_i)] \) where \( (t_i : i \geq 0) \) is an enumeration of the jumps of \( X \), and \( V(s,t) = \inf \text{VarHt}(\gamma) \), where the infimum is taken over all continuous paths \( \gamma \) on \( [0, \infty) \times \mathbb{R} \) going from \( (s, X(s)) \) to \( (t, X(t)) \) and which do not cross any segments of \( \text{Slits}(X) \). Note that we do not use periodic time for these definitions. By abuse of notation we still write \( V, D^*, \text{Slits}, \ldots \) for \( V_X, D^*_X, \text{Slits}(X), \ldots \) and trust that the choice will be clear from context.

3.1.1. Exploration along a fixed word at scale \( \varepsilon > 0 \). For each positive integer \( \ell \) we will consider ‘words’ \( w \in \{b, h\}^\ell \) of length \( \ell \) on the two letters \( b \) and \( h \). We write \( \#h(w) \) and \( \#b(w) \) for the number of entries \( h \) and \( b \) in a word \( w \), respectively, and \( \#w = \#h(w) + \#b(w) = \ell \) for its total length. For \( \varepsilon > 0 \) and a word of length \( \ell \) define a sequences of stopping times \( (S_k, T_k)_{k=1}^\ell \), as follows. First \( S_0 = T_0 = 0 \), and then for \( 1 \leq k \leq \ell \),

\[ S_k = \inf \{ t \geq T_{k-1} : X_t \geq X_{T_{k-1}} + \varepsilon \} \]  
\[ T_k = \begin{cases} S_k, & \text{if } w_k = h, \\ \inf \{ t \geq S_k : X_t = X_{S_k} \}, & \text{otherwise}. \end{cases} \]

See Figure 8. Let us describe by a few sentences the definition of these stopping times. Initially, we sit on the point \( (0,0) \) of \( X \). We first wait for the first time \( S_1 \) where the process \( X \) exceeds level \( \varepsilon > 0 \). This will happen in finite time and almost surely the process \( X \) makes a jump at that time. We then look at the first letter of our word \( w \); if it is an \( h \), then we decide to go “above” that jump and we move on to the point \( (S_1, X_{S_1}) \). If it is a \( b \), then we decide to go “below” that jump: we then first wait until \( T_1 \) for the Lévy process to go down and reach again the value \( X_{S_1} \) and then we move on to the point \( (T_1, X_{T_1}) \). Notice that \( X \) is
continuous at time $T_1$ since it has no negative jumps, in particular $X_{T_1} = X_{T_1-} = X_{S_1-}$. We then iterate that construction and by induction all those stopping times are almost surely finite and $X$ makes a jump at each time $S_k$.

![Figure 8. Example of a curve $\gamma$ (green) with corresponding word $W^\varepsilon(\gamma) = hbbh$.](image)

To each $k$ such that $w_k = b$, we associate a random variable

$$\chi_k = \frac{1}{\varepsilon} (X_{T_{k-1}} - X_{S_k-}),$$

which measures how far ‘below’ $X_{T_{k-1}}$ we find the bottom of the jump at time $S_k$. The values of $\chi_k$ are in the range $[-1, \infty)$ and the distribution of $\chi_k$ does not depend on $\varepsilon$. Note that for a fixed word $w$, by the strong Markov property the processes $(X_{t+T_k} - X_{T_k})_{0 \leq t \leq S_{k+1} - T_k}$ for $k = 0, \ldots, \ell - 1$ are i.i.d. In particular, the $\chi_k$ are i.i.d. of law $\chi$.

In the next subsection we shall derive estimates valid for a fixed word $w$ but we shall later apply our results to random words obtained by exploring a path $\gamma$ at scale $\varepsilon > 0$. To get some intuition for the coming lemmas let us describe this construction right now.

### 3.1.2. Word associated to a path at scale $\varepsilon > 0$.

Fix $t \geq 0$. We consider a continuous path $\gamma = (\gamma_x, \gamma_y)$ going from $(0, X_0)$ to $(t, X_t)$, with $\gamma_x$ non-decreasing, which does not cross Slits$(X)$. For any $\varepsilon > 0$, we associate with this path and the scale $\varepsilon > 0$, a word

$$W^\varepsilon(\gamma)$$

on the letters $\{b, h\}$ as follows. Build the word $W^\varepsilon(\gamma)$ letter-by-letter: we first wait for time $S_1$ when $X$ exceeds level $\varepsilon > 0$. Since the path $\gamma$ cannot cross the slit caused by the jump of $X$ at time $S_1$ it must get around either above or below it. If it goes above, then the first letter of $W^\varepsilon(\gamma)$ is set to $h$; otherwise it is set to $b$. In the second case, we wait for the Lévy process $X$ to come back to the point $X_{S_1-}$ and we iterate the construction. The words stop after $\ell$ letters where $\ell = \inf\{k \geq 0 : S_{k+1} \geq t\}$. Note that the random word $W^\varepsilon(\gamma)$ then depends on the whole process $(X_s)_{0 \leq s \leq t}$ via the path $\gamma$. See Figure 8. The word $W^\varepsilon(\gamma)$ encodes partial information about $D^*(0,t)$ and $\text{VarHt}(\gamma)$:
**Lemma 3.3.** With the above notation we have

\[ \text{D}_x^*(0, t) \leq \text{VarHt}(\gamma) + 2\varepsilon \#b(W^\varepsilon(\gamma)) + 2\varepsilon, \]

\[ \text{VarHt}(\gamma) \geq \varepsilon \#h(W^\varepsilon(\gamma)) + \varepsilon \sum_{k : W_k^\varepsilon(\gamma) = b} \chi_k. \]

In particular, when \( \gamma \) is a path minimizing \( V(0, t) \) as in Lemma 2.3 we can replace \( \text{VarHt}(\gamma) \) by \( V(0, t) \) in the above displays.

**Proof.** We decompose \( W^\varepsilon(\gamma) \) into consecutive blocks of \( h \)'s and \( b \)'s:

\[ W^\varepsilon(\gamma) = h^j b^j h^i \cdots h^{i_{m-1}} b^{i_m}. \]

We set \( k_0 = 0, T_0^* = 0 \), and for \( 1 \leq j \leq m - 1 \) we let \( k_j = i_1 + \cdots + i_j \) and \( T_j^* = T_{k_j} \), and also set \( T_m^* = t \). Thus the \( T_j^* \) delimit the times (for \( X \) when \( W^\varepsilon(\gamma) \) switches between \( h \) and \( b \) or vice versa. Note that \( \gamma \) starts at \((0, 0) = (T_0^*, X(T_0^*))\), then goes above \((T_1^*, X(T_1^*))\), below \((S_1^*, X(S_1^*) = (S_2^*, X(T_2^*)), \text{above} (T_3^*, X(T_3^*)), \text{and so on, ending at} (T_m^*, X(T_m^*)) = (t, X(t)). Using continuity, this means that there are times \( 0 = t_0^* \leq t_1^* \leq t_2^* \leq \cdots \leq t_m^* = 1 \) such that \( \gamma_y(t_j^*) = X(T_j^*) \) for all \( j \). We will define another path \( \gamma^* \) which satisfies \( \gamma^*(t_j^*) = (T_j^*, X(T_j^*)) \) for all \( 0 \leq j \leq m \). For both bounds (3.5) and (3.6) we will then use the simple estimate

\[ \text{VarHt}(\gamma) \geq \sum_{j=1}^m |\gamma_y(t_j^*) - \gamma_y(t_{j-1}^*)| = \sum_{j=1}^m |\gamma_y^*(t_j^*) - \gamma_y^*(t_{j-1}^*)|. \]

Now we define \( \gamma^*(s) \) for \( t_{j-1}^* \leq s \leq t_j^* \). The construction is best understood with a picture, see Figure 9.

- For odd \( j \), meaning that we are in a block of \( h \)'s, we let \( \gamma^* \) iteratively go first up by \( \varepsilon \) from \((T_k, X(T_k))\), then across to the next slit at time \( S_{k+1} = T_{k+1} \), then up along that slit to \((T_{k+1}, X(T_{k+1}))\), and so on, as on the left in Figure 9. If the block of \( h \)'s is at the end of \( W^\varepsilon(\gamma) \) (i.e. \( j = m - 1 \) and \( i_m = 0 \)) then we finish \( \gamma^* \) from \((T_t, X(T_t))\) in a similar way, going down from \((t, X(T_t) + \varepsilon)\) at the very end.

- For even \( j \), meaning that we are in a block of \( b \)'s, we let \( \gamma^* \) iteratively go up by \( \varepsilon \) from \((T_k, X(T_k))\), then across until time \( S_{k+1} \), then down along the slit to height \( X(S_{k+1}) = X(T_{k+1}) \), then across to \((T_{k+1}, X(T_{k+1}))\), and so on, as on the right in Figure 9. If the block of \( b \)'s is at the end of \( W^\varepsilon(\gamma) \) (i.e. \( j = m \)) then we finish \( \gamma^* \) from \((S_{t}, X(S_{t}))\) in a similar way, either straight across and up if \( T_t > t \), otherwise up by \( \varepsilon \) from \((T_t, X(T_t))\)), across, and finally down.

We now note the following about the height-variations \( \text{VarHt}(\gamma^*(s) : t_{j-1}^* \leq s \leq t_j^*) \). For simplicity, let us assume that \( T_j^* < t \) (similar considerations apply to the last bit of path).

- For odd \( j \), \( \gamma_y^*(s) \) is non-decreasing in the interval, and we have

\[ \text{VarHt}(\gamma^*(s) : t_{j-1}^* \leq s \leq t_j^*) = \gamma_y^*(t_j^*) - \gamma_y^*(t_{j-1}^*) \geq \varepsilon i_j. \]

Actually, the height-variation of \( \gamma^* \) realises the \( D^d \)-distance we between the endpoints:

\[ \text{VarHt}(\gamma^*(s) : t_{j-1}^* \leq s \leq t_j^*) = D^d(T_{j-1}^*, T_j^*). \]

- For even \( j \), by summing the sizes of the vertical steps we get

\[ \text{VarHt}(\gamma^*(s) : t_{j-1}^* \leq s \leq t_j^*) = \sum_{k = k_{j-1}}^{k_j-1} (2\varepsilon + \varepsilon \chi_k). \]
Summing instead the displacements (with sign), we get

\[ (3.11) \quad \gamma^*_y(t^*_j) - \gamma^*_y(t^*_j-1) = - \sum_{k=k_{j-1}}^{k_j-1} \varepsilon \chi_k. \]

Combined with (3.10), this gives

\[ (3.12) \quad \text{VarHt}(\gamma^*(s) : t^*_j-1 \leq s \leq t^*_j) \leq |\gamma^*_y(t^*_j) - \gamma^*_y(t^*_j-1)| + 2\varepsilon \varepsilon_j. \]

The way we chose \( \gamma^* \) means that we also have

\[ (3.13) \quad \text{VarHt}(\gamma^*(s) : t^*_j-1 \leq s \leq t^*_j) \geq \sum_{k=k_{j-1}}^{k_j-1} \left[ D^d(T_k, S_{k+1}) + D^u(S_{k+1}, T_{k+1}) \right]. \]

To prove (3.5), we use the decomposition of \([0, t]\) suggested by (3.9) and (3.13), the identity in (3.8), and the bound in (3.12) to get

\[ D^*(0, t) \leq \sum_{j=1}^{m} \left[ |\gamma^*_y(t^*_j) - \gamma^*_y(t^*_j-1)| + 2\varepsilon \varepsilon_j 1\{j \text{ even}\} \right] + 2\varepsilon, \]

where the last \( 2\varepsilon \) comes from the last bit of \( \gamma^* \). Now (3.5) follows from (3.7). Finally, (3.6) follows by using (3.7) together with the bound in (3.8) and the identity in (3.11). \( \square \)

### 3.2. Probabilistic estimates for fixed words

In this section we fix a (long) word \( w \) and estimate its height difference \( \sum \chi_k \). Our goal is to get large deviations estimates in order to prove a statement valid for all words simultaneously (in particular to words \( W^\varepsilon(\gamma) \) coming from the exploration of any path at scale \( \varepsilon > 0 \)). Recall that \( \alpha \in (1, 2) \).

We start by an estimate on the ‘underjump’ of law \( \chi = \chi_k \), which by a standard result in renewal theory is related to the size-biasing of the Lévy measure. In particular \( \mathbb{P}(\chi \geq r) \) decays as \( r^{-\alpha+1} \) as \( r \to \infty \). For our purposes we shall need:

**Proposition 3.4.** Write \( \beta = \alpha - 1 \in (0, 1) \) and let \( \mathcal{S} \) be a strictly positive stable random variable of index \( \beta \), i.e. \( \mathbb{E}[e^{-\lambda \mathcal{S}}] = e^{-\lambda^\beta} \) for all \( \lambda > 0 \). Then there are constants \( a, b > 0 \) depending only on \( \alpha \) such that we have the stochastic domination

\[-a + b \mathcal{S} \leq \chi.\]
PROOF. The density of $\chi$ is explicitly known, see [23, Example 7] ($\chi$ has the distribution of $-X_{r+1}^\tau$ in their notation). In particular, there is some constant $c$ such that for all $r \geq 0$ we have $\mathbb{P}(\chi > r) = e^{(r + 1)^{-\beta}}$. Also, there is another constant $C > 0$ such that $\mathbb{P}(\mathcal{S} > r) \leq Cr^{-\beta}$ for all $r \geq 0$, see [43, Property 1.2.15]. Thus we can pick $a, b > 0$ as claimed.

**Proposition 3.5.** There is a constant $c_\alpha$ depending only on $\alpha$ such that the following holds. Let $w \in \{b, h\}^\ell$ be a fixed word with $\#b = m$. Then for any $q > 0$, writing $\kappa = \frac{q-1}{2-\alpha} > 0$, we have that

$$\mathbb{P}\left( \sum_{k: w_k = b} \chi_k \leq qm \right) \leq \exp(-c_\alpha q^{-\kappa} m).$$

**Proof.** Writing $\mathcal{S}(t)$ for a $\beta = (\alpha - 1)$-stable subordinator viewed at time $t$, and $\mathcal{S}_k$ for independent copies of $\mathcal{S}(1)$ and $\mathcal{S}$, we have using Proposition 3.4 and the fact that the $(\chi_k : w_k = b)$ are i.i.d. for any given word $w$ that

$$\mathbb{P}\left( \sum_{k: w_k = b} \chi_k \leq qm \right) \leq \mathbb{P}\left( \sum_{k: w_k = b} \mathcal{S}_k \leq \frac{q+a}{b} m \right) = \mathbb{P}(\mathcal{S}(m) \leq \frac{q+a}{b} m)$$

$$= \mathbb{P}(\mathcal{S}(1) \leq \frac{q+a}{b} m^{1-1/\beta}).$$

For $\delta \in (0, 1)$ we have [8, p. 221] that

$$\mathbb{P}(\mathcal{S}(1) \leq \delta) \leq \exp(-c\delta^{\beta/(\beta-1)})$$

for some constant $c$ depending only on $\alpha$. We get

$$\mathbb{P}(\mathcal{S}(1) \leq \frac{q+a}{b} m^{1-1/\beta}) \leq \exp(-c(\frac{q+a}{b})^{\beta/(\beta-1)} m)$$

as claimed.

3.3. A first bound. In this section we use the previous estimates to show a first bound: There exists some $\delta > 0$ such that almost surely, in the Lévy process $X$ we have

$$\forall t \geq 0, \quad V(0, t) \leq D^*(0, t) \leq (1 + \delta)V(0, t).$$

We will eventually see that $\delta$ can be made arbitrarily small, however in this first step we will need to think about $\delta$ as a large constant. Recall that the inequality $V \leq D^*$ is deterministically true by extending (3.1) from the case of $E$ to the case of $X$. Towards (3.18), fix $t > 0$ and consider a continuous path $\gamma$ starting from $(0, 0)$ and ending at $(t, X_t)$ minimizing the height variation as in Lemma 2.3. We shall consider the associated word $W^\varepsilon(\gamma)$ as $\varepsilon \to 0$. Let us give first the rough idea of the proof to help the reader follow our steps. First, if the word is short, in the sense that $\#W^\varepsilon(\gamma) = o(1/\varepsilon)$ then (3.5) automatically gives $D^*(0, t) = V(0, t)$. The problem might come from long words $W^\varepsilon(\gamma)$ containing more that $\varepsilon^{-1}$ letters, i.e. of paths $\gamma$ oscillating frenetically around our Lévy process $X$. We may seem in bad shape since there are $2\varepsilon^{-1}$ words of length $\varepsilon^{-1}$. Our salvation will come from the large deviations bound in Proposition 3.5 which roughly says that if the word has too many letters $b$, then the path $\gamma$ has a large height variation. While the exponential control in Proposition 3.5 does not readily “kill” the entropic term $2\varepsilon^{-1}$, it does so if we restrict to words with a small proportion of letters $h$.

Let us proceed. Using the deterministic bounds of Lemma 3.3 we can prove:
Proposition 3.6. Let \( t > 0 \) be such that \( D^*(0, t) > (1 + \delta) V(0, t) \). Then eventually as \( \varepsilon \downarrow 0 \) we have that \( \#b(W^\varepsilon(\gamma)) \geq \varepsilon^{-1/2} \) and that \( W^\varepsilon(\gamma) \) belongs to the set

\[
\text{Bad} = \left\{ \text{words } w : \#h(w) \leq \frac{3}{\delta} \#w, \sum_{k : w_k = b} \chi_k \leq \frac{3}{\delta} \#b(w) \right\}.
\]

Proof. The first assertion is easy to prove. Indeed if the number of \( b \) letters in \( W^\varepsilon(\gamma) \) is \( \leq \varepsilon^{-1/2} \) infinitely often as \( \varepsilon \to 0 \) then (3.5) gives \( D^*(0, t) \leq V(0, t) + 4\varepsilon^{1/2} \) meaning that \( D^*(0, t) = V(0, t) \). For the second assertion, if \( t > 0 \) is such that \( D^*(0, t) > (1 + \delta) V(0, t) \) then (3.5) gives for all \( \varepsilon > 0 \) with \( W = W^\varepsilon(\gamma) \)

\[
(3.19) \quad V(0, t) + 2\varepsilon \# b(W) + 2\varepsilon \geq D^*(0, t) - (1 + \delta) V(0, t).
\]

Consequently \( \#b(W) > \frac{\delta}{2} \varepsilon^{-1} V(0, t) - 1 \). At the same time, (3.6) gives \( \#h(W) \leq \varepsilon^{-1} V(0, t) \). Putting these together we conclude that eventually we have \( \#h(W) \leq \frac{3}{\delta} \#b(W) \leq \frac{3}{\delta} \#W \). Moreover, using (3.6) again, we also have

\[
\varepsilon \sum_{k : W_k = b} \chi_k \leq V(0, t) < \frac{2\varepsilon}{\delta} (\#b(W) + 1),
\]

and thus eventually we have \( \sum_{k : w_k = b} \chi_k \leq \frac{3}{\delta} \#b(W) \), using the fact that \( \#b(W) \to \infty \) by the first part of the proof. \( \square \)

Given the previous lemma, the next result finishes the proof of (3.18).

Lemma 3.7. For \( \delta > 0 \) large enough, with probability one, the set of words of length \( \ell \) belonging to \( \text{Bad} \) is eventually empty as \( \ell \to \infty \).

Proof. For \( \ell \geq 1, p \in (0, \frac{1}{2}) \) and \( q > 0 \) define

\[
\mathcal{B}(\ell, p, q) = \left\{ \text{words } w : \#w = \ell, \#h(w) \leq p \#w, \sum_{k : w_k = b} \chi_k \leq q \#b(w) \right\}.
\]

This is a random set of words because of the variables \( \chi_k \) which depend on \( X \). It is well-known that the number of words \( w \) satisfying \( \#w = \ell \) and \( \#h(w) \leq p \#w \) is at most \( \exp(\ell[p \log(\frac{1}{p}) + (1 - p) \log(\frac{1}{1 - p})]) \), see e.g. [28, Theorem 3.1]. Summing over all such words \( w \), we can use Proposition 3.5 to bound

\[
\mathbb{E}[\#\mathcal{B}(\ell, p, q)] \leq \sum_{w : \#h(w) \leq p \ell, \#w = \ell} \mathbb{P}\left( \sum_{k : w_k = b} \chi_k \leq q \#b(w) \right)
\]

\[
\leq \sum_{w : \#h(w) \leq p \ell, \#w = \ell} \exp\left( - c_\alpha \ell^{-\kappa} \#b(w) \right)
\]

\[
\leq \exp\left( - \ell [c_\alpha \ell^{-\kappa} (1 - p) - p \log(\frac{1}{p}) - (1 - p) \log(\frac{1}{1 - p})] \right)
\]

where \( c_\alpha, \kappa > 0 \) are from Proposition 3.5. Now, putting \( p = q = \frac{3}{\varepsilon} \) in the last display, we can choose \( \delta \) large enough so that \( c_\alpha \ell^{-\kappa} (1 - p) - p \log(\frac{1}{p}) - (1 - p) \log(\frac{1}{1 - p}) > 0 \). Then the series \( \sum_\ell \mathbb{E}[\#\mathcal{B}(\ell, \frac{3}{\varepsilon}, \frac{3}{\varepsilon})] \) is convergent, and thus \( \sum_\ell \#\mathcal{B}(\ell, \frac{3}{\varepsilon}, \frac{3}{\varepsilon}) \) is convergent almost surely from which it follows that \( \mathcal{B}(\ell, \frac{3}{\varepsilon}, \frac{3}{\varepsilon}) \) is eventually empty almost surely. \( \square \)
3.4. Bootstrapping. In this section we finally prove $D^* = V$ for the Lévy process $X$. Our strategy is first to extend (3.18) to all pairs of time $s,t \geq 0$. We then use these bounds to sharpen Lemma 3.3, and by using the same kind of arguments as in the last section this gives $\delta = 0$. Let us proceed.

We first claim that in the Lévy process $X$, we have

$$\forall s,t \geq 0, \quad V(s,t) \leq D^*(s,t) \leq (1 + \delta)V(s,t). \quad (3.22)$$

This is true with probability one for any fixed $s = s_0 > 0$ and any $t > s_0$ by (3.18) and invariance by time translation. By countable intersection, the above display is true with probability one for any $0 \leq s \leq t$ with $s \in \mathbb{Q}$. The bound is then extended to all pairs $0 \leq s \leq t$ using the fact that both $s \mapsto V(s,t_0)$ and $s \mapsto D^*(s,t_0)$ are (almost surely) right-continuous in $s \leq t_0$.

We now use (3.22) as a ‘deterministic input’ in order to sharpen the geometric estimates of Lemma 3.3. Thus we suppose that we are given a right-continuous path $X$ satisfying (3.22) and we aim at controlling the height variation and the distance $D^*$ of a path $\gamma$ starting from $(0,0)$ and ending at $(t,X_t)$. We assume that $\gamma$ is optimal, i.e. $\text{Var} H_t(\gamma) = V(0,t)$, as in Lemma 2.3. As in Lemma 3.3 we consider the word $W^\varepsilon(\gamma)$ and examine the contributions to the height variation of $\gamma$ along sequences of the same letter $b$ or $h$. We will use several definitions from the proof of that lemma. Recall in particular the times $0 = T_0^* < T^*_1 < \ldots < T_{m-1}^* < T_m^* = t$ which delimit the consecutive sequences of $h$’s or $b$’s, the times $T^*_j$ such that $\gamma_y(t_j^*) = X(T_j^*)$, and the path $\gamma^*$ satisfying $\gamma^*(t_j^*) = (T_j^*,X(T_j^*))$. We define

$$\text{Var} H_t(\gamma,\varepsilon) = \sum_{1 \leq j \leq m-1} \text{Var} H_t(\gamma(s) : t_{j-1}^* \leq s \leq t_j^*), \quad \text{Var} H_t(\gamma,\varepsilon) = \sum_{1 \leq j \leq m} \text{Var} H_t(\gamma(s) : t_{j-1}^* \leq s \leq t_j^*),$$

noting that the latter also includes the “last bit of path”. We have

$$V(0,t) = \text{Var} H_t(\gamma) = \text{Var} H_t(\gamma,\varepsilon) + \text{Var} H_t(\gamma,\varepsilon). \quad (3.23)$$

**Lemma 3.8.** We have

$$\text{Var} H_t(\gamma,\varepsilon) \geq \varepsilon \# h(W^\varepsilon(\gamma)) \quad (3.24)$$

$$D^*(0,t) \leq \text{Var} H_t(\gamma,\varepsilon) + (1 + \delta)\text{Var} H_t(\gamma,\varepsilon) \quad (3.25)$$

**Proof.** For (3.24), note that by (3.8) we have

$$\text{Var} H_t(\gamma,\varepsilon) \geq \sum_{1 \leq j \leq m-1} \| \gamma_y(t_j^*) - \gamma_y(t_{j-1}^*) \| = \sum_{1 \leq j \leq m-1} \| \gamma_y(t_j^*) - \gamma_y(t_{j-1}^*) \| \geq \varepsilon \# h(W^\varepsilon(\gamma)).$$

For (3.25), by using subadditivity, (3.8), (3.9), as well as (3.22) we get

$$D^*(0,t) \leq \sum_{1 \leq j \leq m-1} D^*(T_{j-1}^*,T_j^*) + \sum_{1 \leq j \leq m} D^*(T_{j-1}^*,T_j^*) \leq \text{Var} H_t(\gamma,\varepsilon) + (1 + \delta) \sum_{1 \leq j \leq m} V(T_{j-1}^*,T_j^*).$$

Clearly $V(T_{j-1}^*,T_j^*) \leq \text{Var} H_t(\gamma(s) : t_{j-1}^* \leq s \leq t_j^*)$, which finishes the proof.

With these improved geometric controls at hands we can now prove that we may take $\delta = 0$ in (3.22). The idea is to show that if $\delta > 0$ then one can find $0 < \sigma < \delta$ so that (3.22) holds with this $\delta$ replaced by $\sigma$. By considering the smallest $\delta \geq 0$ so that (3.22) holds with probability 1 we deduce indeed that $\delta = 0$. 


Proposition 3.9. Suppose that (3.22) holds with probability 1 with some \( \delta > 0 \). Then we can find \( 0 < \sigma < \delta \) so that (3.22) holds with probability 1 with \( \delta \) replaced by \( \sigma \).

Proof. By the argument presented at the beginning of this subsection, it is sufficient to find \( 0 < \sigma < \delta \) so that (3.18) holds almost surely when \( \delta \) is replaced by \( \sigma \). To do this, let us set

\[
\sigma = \frac{\delta}{2} \vee (\delta(1 - p^2_2)) \in (0, \delta),
\]

where \( p \in (0, \frac{1}{2}) \) is to be chosen at the end of the proof. Suppose \( t > 0 \) is such that

\[
(3.26) \quad D^*(0, t) > (1 + \sigma)V(0, t),
\]

and let \( \gamma \) be an optimal path as in Lemma 2.3 going from \((0, 0)\) to \((t, X_t)\). We consider the word \( W^\varepsilon(\gamma) \) coming from the exploration of \( \gamma \) at scale \( \varepsilon \) and write \( W = W^\varepsilon(\gamma) \) to simplify notation. Let us make the following observations on the behavior of such words.

- We first note that \( \#W \geq b(W) \geq \varepsilon^{-1/2} \) eventually; otherwise we would have \( V(0, t) = D^*(0, t) \) as already observed in Proposition 3.6.
- Our second claim is that eventually \( \#\lln(W) \leq p\#W \). To see this, we begin by noting that if \( \#W < \frac{\delta}{4}\varepsilon^{-1}V(0, t) \) then from (3.5) we already have

\[
D^*(0, t) \leq V(0, t) + 2\varepsilon^2\frac{\delta}{4}\varepsilon^{-1}V(0, t) + 2\varepsilon \leq (1 + \frac{\delta}{2})V(0, t) + 2\varepsilon \leq (1 + \sigma)V(0, t),
\]

after letting \( \varepsilon \to 0 \). We therefore proceed under the assumption \( \#W \geq \frac{\delta}{4}\varepsilon^{-1}V(0, t) \), and assume by contradiction that \( \#\lln(W) \geq p\#W \). This gives \( \#\lln(W) \geq p\frac{\delta}{4}\varepsilon^{-1}V(0, t) \) and by (3.24) we have

\[
\text{VarH}_{\lln}(\gamma, \varepsilon) \geq p\frac{\delta}{4}V(0, t).
\]

We can thus write

\[
D^*(0, t) \leq (1 + \delta)\text{VarH}_{\lln}(\gamma, \varepsilon) + \text{VarH}_{\lln}(\gamma, \varepsilon) = V(0, t) + \delta\text{VarH}_{\lln}(\gamma, \varepsilon)
\]

\[
\leq (1 + \delta(1 - p^2_2))V(0, t),
\]

and this yields a contradiction given our definition of \( \sigma \).

- Our final claim is that if we set \( q = \frac{4}{5} \), then we must have \( \sum_{k: W_k = b} \chi_k \leq q\#b(W) \). Otherwise combining (3.5), (3.6) and letting \( \varepsilon \to 0 \) would give \( D^*(0, t) \leq (1 + \frac{2}{5})V(0, t) = (1 + \frac{\delta}{2})V(0, t) \leq (1 + \sigma)V(0, t) \) which is excluded by assumption.

Gathering-up our findings, the words \( W^\varepsilon(\gamma) \) corresponding to explorations of minimizing paths going to times \( t > 0 \) satisfying (3.26) eventually belong to the set \( \mathcal{B}(\ell, p, \frac{4}{5}) \) for some \( \ell \geq \varepsilon^{-1/2} \) where those sets were defined in the proof of Lemma 3.7. Arguing exactly as in that proof, for our fixed \( q = \frac{4}{5} \) we can find \( p \) small enough so that \( \mathcal{B}(\ell, p, \frac{4}{5}) \) is eventually empty almost surely. This implies that there is no such \( t > 0 \) satisfying (3.26). This means that (3.18) holds for \( \delta \) replaced by \( \sigma \) and we extend to all pairs of times \( s, t \geq 0 \) using right-continuity as in the beginning of this section.

We can now finish the proof of Theorem 3.1:

Proof of Theorem 3.1. Thanks to the previous sections the equality \( V(s, t) = D^*(s, t) \) is granted for \( s, t \geq 0 \) in the stable Lévy process \( X \). To extend it to the excursion \( E \) we use local absolute continuity. More precisely, if \( [x]_1 \in [0, 1) \) denotes the fractional
part for $x \in \mathbb{R}$, for $0 < s_0 < t_0 < 1$, the processes $(\mathcal{E}(s_0 + u) - \mathcal{E}(s_0) : 0 \leq u \leq t_0 - s_0)$ and $(\mathcal{E}([t_0 + u]_1) - \mathcal{E}(t_0) : 0 \leq u \leq 1 - t_0 + s_0)$ are absolutely continuous with respect to $(X_u : 0 \leq u \leq t_0 - s_0)$ and $(X_u : 0 \leq u \leq 1 - t_0 + s_0)$ respectively. This follows from the Vervaat transform relating $\mathcal{E}$ to the normalized bridge $X^{br}$ of $X$ and absolute continuity relation between $X^{br}$ and $X$ (see [8] Chapter VIII.3, Formula (8)). Lemma 2.3 shows that to compute $\mathcal{V}(s_0, t_0)$ it is sufficient to restrict to path whose $x$-coordinate is monotone, that is either go from $s_0$ to $t_0$ or from $s_0$ to $0$ and then from $1$ to $t_0$. In both cases, we can compare with one of the above pieces of the Lévy process $X$ and we deduce from the last section that $\mathcal{D}^*(s_0, t_0) = \mathcal{V}(s_0, t_0)$ almost surely in $\mathcal{E}$. The result is extended to all $s, t$ by right-continuity.

4. Properties of the shredded spheres. In this section we establish a few basic properties of the shredded spheres. We first show that the Hausdorff dimension of $S_\alpha$ is $\alpha \in (1, 2)$ and characterize the point identifications made by $\mathcal{V}$ or $\mathcal{D}^*$ in Theorem 4.3. Towards understanding the topology of $S_\alpha$, we study the graph formed by the faces of $S_\alpha$ which are the scaling limits of the large multimers in $\mathcal{M}'_n$. We show that adjacent faces exist using the decrease points of stable processes [7] but leave the question of the connectedness of the graph of faces open.

In this section we see $S_\alpha$ as the quotient of $[0, 1]$ by the equivalence relation $\mathcal{V} = 0$ endowed with the projection of the pseudo-distance $\mathcal{V}$ or equivalently of $\mathcal{D}^*$. The canonical projection $[0, 1] \to S_\alpha$ is denoted by $\pi$.

4.1. Hausdorff dimension.

**Proposition 4.1.** For $\alpha \in (1, 2)$, almost surely the Hausdorff dimension of $S_\alpha$ is $\alpha$.

**Proof.** Both the upper bound $\dim(S_\alpha) \leq \alpha$ and the lower bound $\dim(S_\alpha) \geq \alpha$ can be proved very similarly to the corresponding statements for random stable looptrees [19, Section 3.3], with only a small modification needed for the lower bound (see also [4]). We give a brief outline of the argument and indicate the necessary modifications.

For the upper bound we note that $S_\alpha$ can be covered by the sets $\pi([t_i^{(e)}, t_{i+1}^{(e)})]$ where the $t_i^{(e)}$ form an increasing enumeration of the times $t$ such that $\Delta \mathcal{E}_i > \varepsilon$. Due to the bound $\mathcal{D}^* \leq D^\alpha$, the diameter of $\pi([t_i^{(e)}, t_{i+1}^{(e)})]$ in $S_\alpha$ is at most $2 \sup \{|\mathcal{E}(s) - \mathcal{E}(t)| : s, t \in [t_i^{(e)}, t_{i+1}^{(e)}]\}$. The same calculations as in [19, Section 3.3.1] then give $\dim(S_\alpha) \leq \alpha$.

For the lower bound, as in [19, Section 3.3.2] it suffices to show that for any $\eta > 0$, almost surely $\nu(B_r(\pi(U))) \leq r^{\alpha - \eta}$ for all $r > 0$ sufficiently small, where

- $U \in [0, 1]$ is a uniform random variable independent of $S_\alpha$;
- $\nu(\cdot)$ is the push-forward of Lebesgue measure on $[0, 1]$ under $\pi$;
- $B_r(\cdot)$ is the ball of radius $r$ in $S_\alpha$.

In place of [19, Lemma 3.13] we use the following:

**Lemma 4.2.** Fix $\eta > 0$. Almost surely, for every $\varepsilon$ small enough, there are jump times $S_\varepsilon, T_\varepsilon$ of $\mathcal{E}$ satisfying:

(i) $S_\varepsilon \in (U - \varepsilon, U)$ and $T_\varepsilon \in (U, U + \varepsilon)$,

(ii) for $R = S$ or $T$,

$$\mathcal{E}(R_\varepsilon -) \leq \mathcal{E}(U) - \varepsilon^{1/\alpha + \eta}, \quad \mathcal{E}(R_\varepsilon) \geq \mathcal{E}(U) + \varepsilon^{1/\alpha + \eta}.$$
See Figure 10 for an illustration. In the conditions of the lemma, the two jumps at times $T_\epsilon$ and $S_\epsilon$ create vertical barriers so that $V(s, U) \geq \varepsilon^{1/\alpha + \eta}$ whenever $s \notin [U - \varepsilon, U + \varepsilon]$. This implies the result.

To establish Lemma 4.2 it suffices, as in [19], to consider the case when the excursion $E$ is replaced by an unconditioned Lévy process $(X_t : t \geq 0)$ and $U$ is replaced by 0. The lemma follows as in [19] from an application of Borel–Cantelli along $\varepsilon = 2^{-k}$ once we have proved that $\mathbb{P}(B_\varepsilon) \leq C\varepsilon^\gamma$ for some $C, \gamma > 0$, where

$$B_\varepsilon = \{ \exists s \in [0, \varepsilon] : X_{s-} \in [-2\varepsilon^{1/\alpha + \eta}, -\varepsilon^{1/\alpha + \eta}], \Delta X_s \geq 3\varepsilon^{1/\alpha + \eta} \}.$$  

To prove the bound on $\mathbb{P}(B_\varepsilon)$ we consider the excursions of $(X_t : t \geq 0)$ away from 0. (This differs from [19] where they consider excursions of $X - X$ where $X$ is the running supremum process.)

Let $L_t$ be a local time of $X$ at 0 and let $(g_j, d_j)$, $j \in J$, denote the excursion intervals. Then, since $X$ has only positive jumps, each $(g_j, d_j)$ contains a unique jump time $h_j$ such that $X_{h_j}^- < 0$ and $X_{h_j} > 0$. The random measure

$$\sum_{j \in J} \delta(L_{h_j}, \Delta X_{h_j}, X_{h_j}^-)$$

is a Poisson point process with intensity measure $dt \Pi(dx) 1_{[0,\varepsilon]}(r) dr$, see e.g. the remark after Corollary 1 of [5]. Also, $L^{-1}$ is a subordinator [8, Prop. V.4] which by the scaling property of $X$ is stable with index $1 - 1/\alpha$. Therefore we can apply the same estimates as in [19] to obtain the required bound on $\mathbb{P}(B_\varepsilon)$ and hence the result.

4.2. Point identification. Our main result Theorem 3.1 gives a pretty clear idea of the metric in $\mathcal{S}_\alpha$, but one could wonder whether the quotient $V = 0$ identifies more points than the trivial identifications $\{D_u = 0\}$ and $\{D_d = 0\}$. The answer will require the use of the points of decrease of Lévy processes studied by Bertoin in the 90’s.

Recall that time $t \in \mathbb{R}$ is a (local) decrease time ($f(t)$ is a decrease point) of a càdlàg function $f$ if for some $\varepsilon > 0$ we have

$$f(t - x) \geq f(t) \geq f(t + x), \quad \text{for all } x \in [0, \varepsilon].$$
Brownian motion almost surely has no decrease times (nor increase times) by a famous result of Dvoretzky, Erdős and Kakutani [27]. However, Bertoin [7] has proved that spectrally positive stable Lévy processes almost surely possess decrease times but no increase times. This was further studied in [6, 38]. The points of decrease a priori enable curves $\gamma$ to "cross" the excursion $E$ and possibly perform identifications for $V$ which were not permitted by $D^u$ or $D^d$ only. We will show that this is not the case. The following result can be seen as an analog to the point identification in the Brownian map [35].

**Theorem 4.3.** Almost surely, for all $s, t \in [0, 1]$, if $V(s, t) = 0$ then either $D^u(s, t) = 0$ or $D^d(s, t) = 0$.

**Proof.** If $0 \leq s \leq t \leq 1$ are such that $V(s, t) = 0$, by Lemma 2.3 this means that $h = E(s) = E(t)$ and that one of the straight segments going from $(s, h) \to (t, h)$ or from $(t, h) \to (s, h)$ around the cylinder, does not cross Slits$(E)$. To fix ideas, let us assume we are in the first case. By local absolute continuity (see the proof of Theorem 3.1), it is enough to argue with the Lévy process $X$.

If $s < t$ are such that $V(s, t) = 0$ in $X$ then in particular the segment $[s, t] \times \{h\} \subset \mathbb{R}^2$ does not cross Slits$(X)$. To begin with, let us show that

\begin{equation}
X - h \text{ changes strict sign at most once on } [s, t].
\end{equation}

If $X - h$ changes sign a finite number $n$ of times on $[s, t]$, then if $n \geq 2$ there must be an increase time of $X$ in $[s, t]$ which is excluded by [7]. We just have to exclude the possibility that $n = \infty$.

To do so, fix two rational times $q_1 < q_2$ and let us look at the possible levels $h \geq X_{q_1}$ so that the horizontal line at level $h$ does not intersect Slits$(X)$. Using Figure 11 (Right) it is easy to see that those possible heights are included in

\[ R_{q_1, q_2} = \{ \overline{X}_s : s \in [q_1, q_2] \} \text{ where } \overline{X}_s = \sup_{u \in [q_1, s]} X_u. \]

By standard results (see [8, Lemma VIII.1]) the set $R_{q_1, q_2}$ is a regenerative set, i.e. a part of the range of a subordinator. In our case, the subordinator is stable of index equal to $\alpha \rho = \alpha - 1$ where $\rho = \mathbb{P}(X_1 > 0)$ is the positivity parameter computed by Zolotarev’s formula, see [8, p. 218]. In particular this set has Hausdorff dimension $\alpha - 1 < 1$. In particular, for any $q_1 < q_2 < q_3 < \cdots < q_K$ the intersection

\[ \bigcap_{i=1}^{K-1} R_{q_{2i-1}, q_{2i}} \]
is an intersection of (part of) regenerative sets, which conditionally on their starting points \(X_{q_1}, X_{q_3}, \ldots, X_{q_{2K}}\) are independent. Since those starting points are almost surely distinct, it follows from [31, Example 1] or [9] that as soon as \(K \geq 1 + \left\lceil \frac{1}{2-\alpha} \right\rceil\) the intersection in the previous display is almost surely empty.

Performing the intersection over all countable choices of rationals \(q_1 < q_2 < q_3 < \cdots < q_{2K}\) we deduce from the above consideration that for any \(h \in \mathbb{R}\), one cannot find rationals \(q_1 < q_2 < q_3 < \cdots < q_{2K}\) so that \(X_{q_{2i-1}} < h < X_{q_{2i}}\) for \(1 \leq i \leq K\) and so that \([q_1, q_{2K}] \times \{h\}\) does not intersect \(\text{Slits}(X)\). It follows that for any \(s, t, h \in \mathbb{R}\), if the segment \([s, t] \times \{h\} \subset \mathbb{R}^2\) does not cross \(\text{Slits}(X)\), then \(X - h\) changes strict sign at most \(K\) times. Together with the discussion just after (4.1), this finally proves (4.1).

Coming back to our pair of identified points \(s < t\), we deduce thanks to (4.1) that either \(X - h\) does not change (strict) sign, in which case we have

\[
\mathcal{D}_u(s, t) = 0 \quad \text{or} \quad \mathcal{D}_d(s, t) = 0,
\]

otherwise it changes sign once and since there are no increase times in \(X\), the points must be identified through a decrease time as is depicted in Figure 12.

![Figure 12. The last situation to treat: s and t are identified through a decrease point.](image)

In particular, we can find rational \(q_1 < q_2 < q_3 < q_4 < q_5 < q_6\) such that:

- level \(h\) belongs to \(\mathcal{R}_{q_1, q_2} = \{X_u : u \in [q_1, q_2]\}\) where \(X_u = \inf_{v \in [u, q_2]} X_v\) (purple in Figure 12),
- level \(h\) belongs to \(\mathcal{R}_{q_5, q_6} = \{X_u : u \in [q_5, q_6]\}\) where \(X_u = \sup_{v \in [q_5, u]} X_v\) (light blue in Figure 12),
- level \(h\) belongs to \(\mathcal{P}_{q_3, q_4} = \{X_w : q_3 < w < q_4\}\), such that \(X_{q_3} \leq X_w \geq X_b, \forall q_3 \leq a \leq w \leq b \leq q_4\) (green in Figure 12).

As above, the random sets \(\mathcal{R}_{q_1, q_2} - X_{q_2}, \mathcal{R}_{q_5, q_6} - X_{q_5}\) and \(\mathcal{P}_{q_3, q_4} - X_{q_3}\) are independent and of Hausdorff dimension respectively

\[
\dim(\mathcal{R}_{q_1, q_2}) = \dim(\mathcal{R}_{q_5, q_6}) = \alpha - 1 \quad \text{and} \quad \dim(\mathcal{P}_{q_3, q_4}) = 2 - \alpha,
\]

where the last dimension is computed in [38]. The random sets \(\mathcal{R}_{q_1, q_2} - X_{q_2}\) and \(\mathcal{R}_{q_5, q_6} - X_{q_5}\) are regenerative sets, whereas \(\mathcal{P}_{q_3, q_4} - \sup \mathcal{P}_{q_3, q_4}\) is absolutely continuous with respect to a stable regenerative set, see [38]. Since the sum of their codimensions is larger than 1 we conclude as above that their intersections is almost surely empty. Performing an intersection over all possible choices of the rationals, we deduce that the above situation cannot occur and the theorem is proved.

\[\square\]
4.3. Faces of \( S_\alpha \). Recall that \( S_\alpha \) may be realized as a weak limit of rescaled causal maps, according to Theorem 1.1. The positive jumps of the random path which encodes the causal map correspond to faces in the map (which appear as vertical duals of hard multimers). Therefore, in the continuum picture, we define a face in \( S_\alpha \) by relating it to a jump of the excursion of the Lévy process \( E \) which encodes \( S_\alpha \). For each jump time \( t \) of \( E \) we associate a face in \( S_\alpha \) of perimeter \( 2\Delta E(t) \) as follows: For each \( s \in [E(t^-), E(t)] \) let \( \ell_t(s) \) (resp. \( r_t(s) \)) be the first instant on the left of \( t \) (resp. on the right of \( t \)) such that

\[ E(\ell_t(s)) = E(r_t(s)) = s. \]

To be more precise, we need to see \( E \) as indexed cyclically by time and it may be that \( \ell_t(s) > t \), but we always have the cyclic ordering \( \ell_t(s) \to t \to r_t(s) \).

**Definition 4.4.** The face \( \mathcal{F}_t \) in \( S_\alpha \) corresponding to a jump at \( t \) is defined by

\[ \mathcal{F}_t = \pi(\{\ell_t(s) : s \in [E(t^-), E(t)]\}) \cup \pi(\{r_t(s) : s \in [E(t^-), E(t)]\}) \]

where \( \pi : [0,1] \to S_\alpha \) denotes the projection to the quotient.

The space \( S_\alpha \) possesses a countable number of faces. Due to the presence of decrease points in \( E \), two faces may have a point in common as is indicated in Figure 13.

**Proposition 4.5.** Almost surely, there exists a pair of faces in \( S_\alpha \) which have points in common and the set of common points is a perfect set (every point in the set is a limit point of the set).

**Proof.** As before, we argue using the unconditioned Lévy process \( X \). Let \( t \) be a jump time of \( X \) and let \( \zeta \) be an independent exponential time with parameter 1. Define global decrease times of \( X \) on the interval \([t, t + \zeta]\) as those times \( s \in [t, t + \zeta] \) at which

\[ X_{s'} \geq X_s \geq X_{s''} \quad \text{for all} \quad s' \in [t, s] \quad \text{and} \quad s'' \in [s, t + \zeta] \]

and let \( I \) be the set of all global decrease times on \([t, t + \zeta]\). By [6], the set \( I \) is a perfect set and \( P(I \neq \emptyset) > 0 \). On the event \( I \neq \emptyset \), let \( h \in I \). Then, since \( t + \zeta \) is a stopping time, \( X \) almost surely jumps accross level \( h \) at some time \( r > t + \zeta \). The set

\[ \pi(\{s \in I : X_{t^-} < X_{r^-} < X_s < X_t \land X_r\}) \]

then belongs to both the face \( \mathcal{F}_t \) and \( \mathcal{F}_r \) and is a perfect set.

For the same reason, and due to cyclicity, it is even possible that a face \( \mathcal{F}_t \) is adjacent to itself in the sense that \( \ell_t(s) = r_t(s) \). Such points are global cut-points of the shredded sphere \( S_\alpha \).

Due to Proposition 4.5, it is natural to consider the graph \( \mathcal{G}_f \) formed by the faces of \( S_\alpha \), where two faces are adjacent if they share a common point. The connectedness of the \( \mathcal{G}_f \) is a question similar to [24, Question 11.2]. We have not been able to find a complete solution, however, in the regime \( \alpha \) close to 1, it is possible to adapt the method of [30] to show that the graph is connected with probability one. The main idea is, for a given face \( \mathcal{F}_f \), to find another face \( \mathcal{F}' \) in a ‘Markovian’ way and so that \( \mathcal{F}' \) is typically larger than \( \mathcal{F} \) (i.e. so that the expectation of the logarithm of the ratio of their lengths is positive). The interested reader may contact us for more details.
4.4. The case $\alpha = 2$. The case $\alpha = 2$ corresponds to when $\mu$ has finite variance and this falls in the universality class of the generic causal triangulations studied e.g. in [18]. Although [18] deals with an infinite model, the results should extend and $n^{-1/2} \cdot \mathcal{G}_n^\mu$ should converge towards a segment of height given by (a constant multiple of) the maximum of Brownian excursion $e$. In the case of the Brownian excursion $e$, the definition of $D^*$ and $V$ also make sense, but we trivially have
\[ V(s, t) = |e(s) - e(t)|, \]
since $\text{Slits}$ is empty. Moreover, Theorem 3.1 still holds since we can adapt the subadditive techniques of [18, Section 2.3] (using local time to measure horizontal distances) and show that $D^*(s, t) = |e(s) - e(t)|$ as well. We refrain from doing so to keep the paper short.

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