Logarithmic heat kernel estimates without curvature restrictions

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Abstract

The main results of the article are short time estimates and asymptotic estimates for the first two order derivatives of the logarithmic heat kernel of a complete Riemannian manifold. We remove all curvature restrictions and also develop several techniques.

A basic tool developed here is intrinsic stochastic variations with prescribed second order covariant differentials, allowing to obtain a path integration representation for the second order derivatives of the heat semigroup $P_t$ on a complete Riemannian manifold, again without any assumptions on the curvature. The novelty is the introduction of an $\epsilon^2$ term in the variation allowing greater control. We also construct a family of cut-off stochastic processes adapted to an exhaustion by compact subsets with smooth boundaries, each process is constructed path by path and differentiable in time, furthermore the differentials have locally uniformly bounded moments with respect to the Brownian motion measures, allowing to by-pass the lack of continuity of the exit time of the Brownian motions on its initial position.

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1 Introduction

Let \((M, g)\) be an \(n\)-dimensional connected and complete Riemannian manifold endowed with the Levi-Civita connection \(\nabla\). Let \(\Delta\) denote the Laplace-Beltrami operator, and let \(p(t, x, y)\) denote its heat kernel, by which we mean the minimal positive fundamental solution to the equation \(\frac{\partial}{\partial t} = \frac{1}{2} \Delta\). The objective of this article is to provide estimates on the first and the second order gradients of \(\log p(t, x, \cdot)\), without imposing any curvature conditions on \(M\). For a fixed \(x \in M\), we use the abbreviation \(\log p\) for the logarithmic heat kernel \(\log p(t, x, \cdot)\) and use \(\nabla \log p\) and \(\nabla^2 \log p\) for its first and second order derivatives respectively.

We begin with explaining some of the motivations and potential applications. Let \(o \in M\) be fixed, we denote \(P_o(M) := \{\gamma \in C([0, 1]; M) : \gamma(0) = o\}\) the based path space over \(M\). Likewise, let \(L_o(M)\) denote the based loop space over \(M\), \(L_o(M) := \{\gamma \in P_o(M) : \gamma(0) = \gamma(1) = o\}\). A classical problem is to seek a suitable probability measure on \(P_o(M)\) or \(L_o(M)\), with which analysis on these infinite dimensional non-linear spaces can be made and understanding of the path spaces can be furthered. If \(M\) is compact or more generally with bounded geometry, a natural candidate for the probability measure on \(L_o(M)\) is the probability distribution of the diffusion process with the infinitesimal operator \(L := \frac{1}{2} \Delta + \nabla \log p(1 - t, \cdot, o)\) and the initial value \(o\). This is the Brownian bridge measure. Since there is no analogue of a Lebesque measure, translation invariant, on \(L_o(M)\), the Brownian bridge measure is essentially the canonical measure to use. Indeed, for \(M = \mathbb{R}^n\) the Brownian bridge measure is a Gaussian measure and it is quasi-invariant under translations of Cameron-Martin vectors. To construct such a diffusion process, which is usually called the Ornstein-Uhlenbeck process, we define a pre-Dirichlet form. This form will be called the Ornstein-Uhlenbeck (O-U) Dirichlet form. To verify that the pre-Dirichlet form yields a Markov process, it is necessary to show it is closed – a property following readily once we have an integration by parts (IBP) formula. The key ingredient for such an IBP formula is suitable short time estimates on \(\nabla \log p\) and \(\nabla^2 \log p\). We refer the reader to Aida [1, 2], Airault and Malliavin [4], Driver [22], Hsu [44] and Li [53] for more detail.

Another interesting problem is to establish functional inequalities for the O-U Dirichlet form. This includes the Poincaré inequality and logarithmic Sobolev inequality. They describe the long time behaviours of the associated diffusion process. The logarithmic Sobolev inequality for Gaussian measures was obtained by Gross in the celebrated paper [39]. However, this is not known to hold for loop space over a general manifold \(M\). When \(M\) was the hyperbolic space, Poincaré inequality was shown to hold on \(L_o(M)\) by the authors of the article [16] and Aida [3]. If \(M\) was compact simply connected with strictly positive Ricci curvature, a weak Poincaré inequality with explicit rate function was also established by the authors of the article [17]. It was shown in Gross [40] that the Poincaré inequality for O-U Dirichlet form did not hold on \(L_o(M)\) when \(M\) was not simply connected. Soon after, Eberle [24] constructed a simply connected compact manifold for which the Poincaré inequality for O-U Dirichlet form did not hold on \(L_o(M)\). When the based manifold \(M\) was compact, Aida [1], Eberle [23], Gong and Ma [35], Gong, Röckner and Wu [36] and Gross [40] have obtained weighted log-Sobolev inequalities or other different versions of modified log-Sobolev
inequalities on $L_0(M)$. In all the results mentioned above, the crucial ingredient was again the asymptotic estimates for $\nabla \log p$ and $\nabla^2 \log p$.

We want to stress that all the results mentioned above have been established for the base manifold $M$ compact or with some bounded geometry conditions, since the short time or asymptotic estimates for $\nabla \log p$ and $\nabla^2 \log p$ were only known for manifolds with such restrictions. Our immediate concern is to study the construction of diffusion processes and functional inequalities on $L_0(M)$ without any bounded geometry conditions on $M$. We will obtain short time or asymptotic estimates for $\nabla \log p$ and $\nabla^2 \log p$ in this paper. These estimates will be applied to study several problems on $L_0(M)$ in a forthcoming paper [15].

It is intriguing that estimates for $\nabla \log p$ and $\nabla^2 \log p$ are also main tools for proving the continuous counterpart of Talagrand’s conjecture for the hypercube $\Omega_n = \{-1, 1\}^n$ which we explain below. Let $\sigma'$ denote the configuration with the $i$th coordinate of $\sigma$ flipped and let $\sigma_i$ denote the $i$-th component of $\sigma \in \Omega_n$. Let $\mu_n \equiv 2^{-n}$ be the uniform measure on $\Omega_n$ which is reversible associated with the generator $Lf(\sigma) := \frac{1}{2} \sum_{i=1}^{n} (f(\sigma') - f(\sigma))$ where $\sigma \in \Omega_n$. Setting $T_s f(\sigma) := \int_{\Omega_n} f(\eta) \prod_{i=1}^{n} (1 + e^{-\sigma_i \eta_i}) d\mu_n(\eta)$, then Talagrand’s conjecture states that for any $s > 0$ there exists a constant $c_s$ independent of the dimension $n$ such that $\mu_n(\left\{ \sigma : T_s f(\sigma) \geq t \right\}) \leq c_s \frac{1}{t \sqrt{\log t}}$ for $t > 1$. The value $c_s$ is uniformly in the function $f$ with $\|f\|_{L^1(\mu_n)} = 1$ and in the dimension. The continuous counter-part of the conjecture is for the Ornstein-Uhlenbeck semi-group $T_t$ with generator $\Delta - x \cdot \nabla$

\[
\sup_{f \geq 0, \|f\|_{L^1(\gamma_n)} = 1} \gamma_n \left( \left\{ \sigma : T_s f(\sigma) \geq t \right\} \right) \leq c_s \frac{1}{t \sqrt{\log t}}, \quad t \geq 2,
\]

where $\gamma_n \sim N(0, I_{n \times n})$ is the standard $n$-dimensional normal distribution. This was proven to be affirmative in Ball, Barthe, Bednorz, Oleszkiewicz and Wolff [9]. The dimension free best constants were given in Eldan and Lee [26] and Lehec [47] where the key ingredients are:

1. For any $g \in L^1(\gamma_n)$ and any $s > 0$, $\nabla^2 (\log g) \geq -c_s^2 \text{Id}$.

2. For any $g \in L^1(\gamma_n)$ non-negative and with $\nabla^2 (\log g) \geq -\beta \text{Id}$ with a $\beta > 0$, one has $\gamma_n(g \geq t) \leq \frac{C_s}{t \sqrt{\log t}}$ for any $t > 1$.

Here $\text{Id}$ is the identity operator. Such estimates for non-Gaussian measures and also for the $M/M/\infty$ queue on $\mathbb{N}$ were obtained by Gozlan, Li, Madiman, Roberto and Samson [37].

## 2 Main Results

The short time and asymptotic estimates are presented in (2.1)–(2.5) below. To the best of our knowledge, such estimates were obtained only for a Riemannian manifold with bounded geometry including a compact Riemannian manifold. Gradient and Hessian estimates of the form (2.1-2.2) were proved by Sheu [64] for $\mathbb{R}^n$ with a non-trivial Riemannian metrics where the objective was a non-degenerate parabolic PDEs with bounded derivatives up to order three, and (2.1) for a compact Riemannian manifold can be found in Driver [22], obtained using a result of Hamilton [41], Corollary 1.3 and the Gaussian bounds on heat kernels, see e.g. Li and Yau [48], Cheeger and Yau [14], Davies [20], Setti [63], and Varopoulos [73, 72]. The estimate (2.2) was shown in Hsu [43] again for the compact case. For a non-compact Riemannian manifold with non-negative Ricci curvature, (2.1) was obtained by Kotschwar [46]. Under a bounded geometry condition together with a volume non-collapsing condition, similar estimates were obtained by Souplet and Zhang [65] and Engoulavit [33]. For the
heat kernel associated with the Witten Laplacian operator, these estimates were proved by X.D. Li [49] under a bounded geometry condition on the Bakry–Emery Ricci curvature. In addition, in all the references mentioned above, suitable bounded geometry conditions were required. Likewise, the bounded geometry restrictions are used to derive differential Harnack inequalities and global heat kernel estimates, by Cheeger, Gromov and Taylor [13], Cheng, P. Li and Yau [19], Hamilton [41], P. Li and Yau [48], they provide an important step toward (2.1)–(2.2). Meanwhile, the asymptotic gradient estimate (2.4) was first shown in Bismut [12] for a compact Riemannian manifold. It was extended to the hypo-elliptic heat kernel and the heat kernel on a vector bundle, for \( M \) with bounded geometry, respectively by Ben Arous [10], Ben Arous and Léandre [11] and Norris [62], c.f. also Azencott [8].

The asymptotic second order gradient estimate (2.5) was established by Malliavin and Stroock [59] for a compact Riemannian manifold. For ‘asymptotically flat’ Riemannian manifolds with poles and bounded geometry this can be found in Aida [2]. On cut-locus estimates was studied by Neel [61].

A natural question is then whether the estimates (2.1)–(2.5) still hold for a general non-compact Riemannian manifold? Note that in Azencott [7], it was illustrated that Gaussian type heat kernel estimates could not be automatically extended to an arbitrary manifold and may fail if the completeness of the Riemannian metric was removed.

We state the main estimate. For any \( y \in M \), let \( \text{Cut}(y) \) be the cut-locus of \( y \).

**Theorem 2.1** (Theorems 6.7 and 6.10). Suppose that \( M \) is a complete Riemannian manifold with Riemannian distance \( d \).

1. For every compact subset \( K \) of \( M \), the following statements hold.
   
   \[
   \left| \nabla_x \log p(t, x, y) \right|_{T_x M} \leq C(K) \left( \frac{1}{\sqrt{t}} + \frac{d(x, y)}{t} \right), \quad (2.1)
   \]
   \[
   \left| \nabla^2_x \log p(t, x, y) \right|_{T_x M \otimes T_x M} \leq C(K) \left( \frac{d^2(x, y)}{t^2} + \frac{1}{t} \right) \quad (2.2)
   \]
   
   for any \( x, y \in K \) and for any \( t \in (0, 1] \).

   (b) There exist positive constants \( t_0(y, K) \) and \( C_1(y, K) \) such that
   
   \[
   \left| t \nabla^2_x \log p(t, x, y) \right|_{T_x M} \leq C_1(y, K) \left( d(x, y) + \sqrt{t} \right), \quad x \in K, \; t \in (0, t_0(y, K)] \quad (2.3)
   \]
   
   where \( I_{T_x M} \) is the identical map on \( T_x M \).

2. Let \( y \in M \) and assume that \( \tilde{K} \subset M \setminus \text{Cut}(y) \) is a compact set. Then
   
   \[
   \lim_{t \downarrow 0} \sup_{x \in \tilde{K}} \left| t \nabla_x \log p(t, x, y) + \nabla_x \left( \frac{d^2(x, y)}{2} \right) \right|_{T_x M} = 0, \quad (2.4)
   \]
   \[
   \lim_{t \downarrow 0} \sup_{x \in \tilde{K}} \left| t \nabla^2_x \log p(t, x, y) + \nabla^2_x \left( \frac{d^2(x, y)}{2} \right) \right|_{T_x M \otimes T_x M} = 0. \quad (2.5)
   \]

**Remarks on the main theorem.** As explained in Section 1, these estimates are crucial for the stochastic analysis of the loop space \( L_0(M) \). Despite of the collective efforts, so far,
these type of results have been largely proved only for based manifolds with bounded geometry. While in this paper, we only need to assume that the based manifold $M$ is complete and stochastically complete. For analysis on the path space $P_o(M)$ over a general complete Riemannian manifold without curvature conditions, some work have already been done by Chen and Wu [18] and Hsu and Ouyang [45]. For $P_o(M)$, the content of Theorem 2.1 is not essential. In a forthcoming paper [15], we shall apply these to obtain integration by parts formula and construct of O-U Dirichlet form on $L_o(M)$, and to prove several functional inequalities on $L_o(M)$.

Our main idea is to obtain localised asymptotic comparison theorems for the first and the second order gradients of logarithmic heat kernel (see Proposition 6.6 and 6.9 below). One novelty is a new second order derivative formula via a new type of (second order) stochastic variation for Brownian paths on the orthonormal frame bundles, which is in particular different from that used by Bismut [12] or Stroock [66]. The idea of stochastic variation was initiated in [12] for obtaining an integration by part formula. While the choice of the variation in [66] will produce a term with (the time reverse of) a non-random vector field on $L^{2,1}(\Omega; \mathbb{R}^n)$, see also Malliavin and Stroock [59, (1.5)], it seems not possible to replace the non-random vector field in their paper by a random one (otherwise the time reversed field is not adapted, hence Itô’s integral is not well defined), which prevents the extension of the formula in [59] to a general non-compact $M$ by a suitable localisation argument. We shall choose a variation (see Section 4 below) with desired properties, which in particular ensures that the formula for the second order gradient of heat semigroup can take a random vector fields. This is the key step for us to extend the new formula to a general complete $M$ (see e.g. Theorem 3.1 below). The expression we obtain for the second order gradient of heat semigroup is different from that by Elworthy and Li [32], Li [52, 54], or from that in Arnaudon, Plank and Thalmaier [6] or that in Thompson [69]. We prove the formula by combining the second order stochastic variation (shown to hold for a compact manifold) and approximation arguments (for a non-compact manifold), which is totally different from that in [6, 69]. This new method is adapted for both the proof of Proposition 6.9 here and the integration by parts formula in our forthcoming paper [15].

3 Expression for the second order gradient of heat semigroup

Throughout the paper, $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ denotes a filtered probability space satisfying the standard assumptions, and $B_t = (B_1^t, B_2^t, \ldots, B_n^t)$ is a standard $\mathbb{R}^n$-valued Brownian motion. Let $L(\Omega; \mathbb{R}^n)$ denote the collection of all stochastic processes $h : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ which are $\mathcal{F}_t$-adapted. Let $h'(\cdot, \omega)$ denote the time derivative of $h(\cdot, \omega)$. We define the Cameron-Martin space on the Wiener space as follows

$$L^{2,1}(\Omega; \mathbb{R}^n) := \left\{ h \in L(\Omega; \mathbb{R}^n) : h(\cdot, \omega) \text{ is absolutely continuous for a.s. } \omega \in \Omega, \right.$$ \left.$$ \text{and } \mathbb{E}\left[ \int_0^1 |h'(s, \omega)|^2 \, ds \right] < \infty \right\}.$$  

Elements of $L^{2,1}(\Omega; \mathbb{R}^n)$ are usually called (random) Cameron-Martin vectors. Let $C_b(M)$ and $C_c(M)$ denote the collection of all real valued bounded and continuous functions on $M$ and continuous functions with compact supports in $M$ respectively. Let $\mathfrak{so}(n)$ denote the set of of anti-symmetric $n \times n$ matrices and let $SO(n)$ denote the collection of orthonormal $n \times n$ matrices.
The curvature. Let $R_x$ denote the sectional curvature tensor and let $\text{Ric}_x$ denote the Ricci curvature tensor at $x \in M$ respectively. Thus both $R_x : T_xM \times T_xM \to T_xM \times T_xM$ and $\text{Ric}^x : T_xM \to T_xM$ are linear map, the latter is given by the duality:

$$\langle \text{Ric}^x_i(v_1), v_2 \rangle_{T_xM} = \text{Ric}_x(v_1, v_2), \quad \forall v_1, v_2 \in T_xM.$$ 

The horizontal Brownian motion. Given a point $x \in M$, let $O_xM$ denote the space of linear isometries from $\mathbb{R}^n$ to $T_xM$. Let $OM := \cup_{x \in M} O_xM$, which is the orthonormal frame bundle over $M$, and let $\pi : OM \to M$ denote the canonical projection which takes a frame $u \in O_xM$ to its base point $x$. For every $u \in OM$, we define $R_u : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ and $\text{ric}_u : \mathbb{R}^n \to \mathbb{R}^n$ by

$$R_u(e_1, e_2) := u^{-1}(R_{\pi(u)}(ue_1, ue_2)),$$

$$\text{ric}_u(e_1) := u^{-1}(\text{Ric}^x_{\pi(u)}(ue_1))$$

for every $e_1, e_2 \in \mathbb{R}^n$.

Given a vector $e \in \mathbb{R}^n$, we denote by $H_e$ the associated canonical horizontal vector field on $OM$ with the property that $(T\pi)_u(H_e) = ue \in T_{\pi(u)}M$. Thus the solution of the ODE

$$u'(t) = H_e(u(t))$$

projects to the geodesic on $M$ with the initial position $x$ and the initial speed $u(0)(e)$.

We choose an orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$. Suppose $\{U_t\}_{t \geq 0}$ is the solution of following $OM$-valued Stratonovich stochastic differential equation

$$dU_t = \sum_{i=1}^n H_{e_i}(U_t) \circ dB_t^i, \quad (3.1)$$

where the initial value $U_0$ is a fixed orthonormal basis of $T_xM$. We usually call $\{U_t\}_{0 \leq t < \zeta}$ the canonical horizontal Brownian motion, where $\zeta : \Omega \to \mathbb{R}_+$ is the life time for $U_t$. Let $X^x_t := \pi(U_t)$, $0 \leq t < \zeta(x)$, then $X^x_t$ is a Brownian motion on $M$ with initial value $x$ and life time $\zeta(x)$. This is the celebrated intrinsic construction of $M$-valued Brownian motion by Eells and Elworthy [25] and Elworthy [28], see also Malliavin [58]. It is well known that the Brownian motion on $M$ does not explode if and only if the horizontal Brownian motion $U_t$ on $OM$ does not explode. In particular, it does not rely on the choice of an isometrically embedding from $M$ to an ambient Euclidean space. Let

$$P_tf(x) := \mathbb{E}\left[ f(X^x_{\zeta}) \mathbf{1}_{\{\zeta < \zeta(x)\}} \right]$$

be the heat semigroup associated to Brownian motion $X$.

The superscript $x$ may be omitted if there is no risk of confusion.

3.1 Second order gradient of the heat semigroup

Let $\{U_t\}_{0 \leq t < \zeta(x)}$ denote the horizontal Brownian motion on $M$ and $\{X^x_t = \pi(U_t)\}_{0 \leq t < \zeta(x)}$ is the Brownian motion on $M$ with initial value $x$ and life time $\zeta(x)$. For any $h \in L^2(\Omega; \mathbb{R}^n)$, we set

$$\Gamma^h_t := \int_0^t R_{U_s}(\circ dB_s, h(s)), \quad \Theta^h_t := h'(t) + \frac{1}{2} \text{ric}_{U_t}(h(t)),$$

(3.2)

It is easy to see that $\Gamma^h_t$ is an $\text{so}(n)$-valued process. For $t \geq 0$, we define

$$A^h_t := \Gamma^h_t h'(t) + \frac{1}{2} U_t^{-1} \nabla \text{Ric}^x_{X^x_t}(U_t h(t), U_t h(t)) - \frac{1}{2} \Gamma^h_t \text{ric}_{U_t}(h(t)) + \frac{1}{2} \text{ric}_{U_t} \left( \Gamma^h_t h(t) \right).$$

(3.3)
We are now ready to state one of our main tools, the second order gradient formula on a general complete $M$.

**Theorem 3.1.** Suppose that $M$ is a complete Riemannian manifold. Let $\{D_m\}_{m=1}^{\infty}$ denote the increasing family of exhaustive relatively compact open sets of $M$ and let $\{l_m\}_{m=1}^{\infty}$ denote the cut-off vector fields as constructed in Lemma 5.1. Let $x \in m$, and there exists $m_0 \in \mathbb{N}$ such that $x \in D_{m_0+1}$.

For every $m > m_0$, $v \in T_x M$, and $t \in (0,1]$, we define

$$h(s) := \left( \frac{t - 2s}{t} \right)^+ \cdot l_m (s, X^x_t) \cdot U_0^{-1} v, \quad s \geq 0.$$  

Then $h \in L^{2,1}(\Omega; \mathbb{R}^n)$. Furthermore, for any $f \in C_b(M)$ we have

$$\langle \nabla^2 P_t f(x), v \otimes v \rangle_{T_x M \otimes T_x M} = E_x \left[ \left( \int_0^t \langle \Theta^h_s, dB_s \rangle \right)^2 - \int_0^t \langle \Lambda^h_s, dB_s \rangle - \int_0^t \left| \Theta^h_s \right|^2 ds \right] f(X^x_t) 1_{\{t < \zeta(x)\}}. \quad (3.4)$$

In particular, the processes $l_m(t, \gamma)$ equals to 1 at any time before $\gamma$ exits $D_{m-1}$ and equals to zero after it exits $D_m$ for the first time. So it is obvious to see that $h(t,\gamma) = U_0^{-1} v$ at $t = 0$ and vanishes after the first exit time of $\gamma$ from $D_m$.

### 3.2 Comments

The main idea for proving the second order gradient of the heat semigroup $P_t$ is to approximate the formula on $M$ by those for a family of specific compact manifolds. We first use a result of Greene and Wu [38] to construct a family of relatively compact exhausting open subsets $\{D_m\}_{m=1}^{\infty}$, which is valid for a complete Riemannian manifold $M$. This allows to construct a series of random cut-off vector fields $l_m \in L^{2,1}(\Omega; \mathbb{R}^n)$ vanishing, as soon as the sample path exits $D_m$ for the first time, with the necessary quantitative estimates needed for the localisation. See Lemma 5.1 below for details. The lemma is partly inspired by the work of Thalmaier [67] and Thalmaier and Wang [68], where geodesic balls are used. For the purpose of embedding into compact manifolds, we make sure that each $D_m$ having a smooth boundary which, because of the cut locus, cannot be taken as granted of geodesic balls on arbitrary Riemannian manifolds.

We want to remark that this offers a more powerful (and also a more reliable) alternative to localisation with stopping times, the latter has been commonly used in stochastic calculus and occasionally incorrectly used. The stopping time argument relies on a continuity assumption on the Brownian motion with respect to the initial value. Such continuity condition seems not easy to verify (for stopping times), and ought not be applied casually, see e.g. Elworthy [27], Li and Sheutzow [55], and Li [51] for more details. Note, however, that exit times from regular domains do have good regularity properties in the sense of Malliavin calculus, we refer the reader to the work of Airault, Malliavin, and Ren [5] for more details.

Cut-off vector fields have been previously applied by Arnaudon, Plank and Thalmaier [6], Thompson [69], Thalmaier [67], and Thalmaier and Wang [68] to provide a localised differential formula for heat semigroups. As explained earlier, we use a new type of (second order) stochastic variation argument to construct the global second order gradient formula given below. In particular, the expression here is different from that of Elworthy and Li [32], Arnaudon, Plank, and Thalmaier [6], Li [52, 54] and Thompson [69] and particularly we do not use the doubly parallel translation operators used in [52, 54].
3.3 Comparison theorems

The outline of the proof is as follows. We first show that the formula holds for a compact Riemannian manifold, this proof is given in Section 4 using a new stochastic variation. To pass from a compact manifold to a non-compact manifold, we use a suitable isometric embedding from $D_m$ into a compact Riemannian manifold $\tilde{M}_m$, as well as the quantitative cut-off process $l_m$, constructed by Lemma 7.1 and Lemma 5.1 respectively.

Denote by $p_{\tilde{M}_m}(t,x,y)$ the heat kernel on $\tilde{M}_m$. Although the heat kernel of a Riemannian manifold is determined in a global manner by the Riemannian metric, we obtain, below, short time comparison theorems between $\nabla \log p_{\tilde{M}_m}$, $\nabla^2 \log p_{\tilde{M}_m}$ and $\nabla \log p$, $\nabla^2 \log p$. These are used for proving (2.1)–(2.5).

The comparison theorem below allows us to obtain estimates for $\nabla^2 \log p$, with the successive applications of first order and second order gradient formula as well as comparison estimates for functionals of the Brownian motions on $M$ and that on $\tilde{M}_m$.

**Proposition 3.2.** (Propositions 6.6 and 6.9) Suppose $K$ is a compact subset of $M$. For any constant $L > 1$, there exists a $m_0 = m_0(K,L) \in \mathbb{N}$, which may depend on $K$ and $L$, such that for all $m \geq m_0$ we could find a positive time $t_0 = t_0(K,L,m)$ such that

$$\sup_{x,y \in K} \left| \nabla_x \log p(t,x,y) - \nabla_x \log p_{\tilde{M}_m}(t,x,y) \right|_{T_xM} \leq C(m)e^{-\frac{t}{L}}, \quad \forall t \in (0,t_0],$$

$$\sup_{x,y \in K} e^{\frac{t}{L}} \left| \nabla^2_x \log p(t,x,y) - \nabla^2_x \log p_{\tilde{M}_m}(t,x,y) \right|_{T_xM \otimes T_xM} \leq C(m)e^{-\frac{t}{L}}, \quad \forall t \in (0,t_0],$$

where $C(m)$ is a positive constant depending on $m$.

4 Second Order Variation on a Compact Manifold

Throughout this section, $M$ is an $n$-dimensional compact Riemannian manifold. In Proposition 4.4 below, we shall establish (3.4) for a compact manifold, which is a fundamental step toward Theorem 3.1.

The first second order differential formula for the heat semigroup $P_t$ was obtained by Elworthy and Li [31] for a non-compact manifold, however with restrictions on their curvature. Another disadvantage of the formula was its involvement of a non-intrinsic curvature which was due to the application of the derivative flow of gradient stochastic differential equations, as well as a martingale approach developed in Li [50]. An intrinsic formula for $\nabla^2 P_t f$ was given by Stroock [66] for a compact Riemannian manifold, while a localised intrinsic formula was obtained by Arnaudon, Plank and Thalmaier [6] with the martingale approach. The study of the second order gradient of the Feynman-Kac semigroup of an operator $\Delta + V$, with a potential function, was pioneered by Li [54, 52], where a path integration formula was obtained with the help of doubly damped stochastic parallel transport equation. (The first order gradient formula was previously obtained in Li and Thompson [56], c.f. [31, 30].) A localised version of the Hessian formula (still with doubly stochastic damped parallel translations) for the Feynman-Kac semigroup was derived by Thompson [69].

However, all the expressions mentioned earlier do not seem to lead to our application, such as the proof of Proposition 3.2. To overcome this problem we introduce a quantitative localisation procedure and obtain a second order gradient formula to which this localisation method can be applied.

One of our main tools is to extend Bismut’s idea to perturb the $M$-valued Brownian motion with initial value $\xi(\varepsilon)$ (where $\xi(\varepsilon)$ is a smooth curve in $M$), they will be constructed as
solutions of a family of SDEs with the driving Brownian motion \( \{ B_t \}_{t \geq 0} \) rotated and translated appropriately. The rotation and translation exerted on \( \{ B_t \}_{t \geq 0} \) transmits the variation in the initial value of the Brownian motion on the manifold to variations, in the same parameter, of the Radon Nikodym derivatives of a family of probability measures, with respect to which the solutions are Brownian motions on \( M \). This simple and elegant idea was applied in Bismut [12] for deducing an integration by parts formula. Incidentally, such integration by parts formula and the first order gradient formula of the heat semigroup were proved to be equivalent on a compact manifold by Elworthy and Li [32]. In Stroock [66], by calculating the concrete form of the second variation introduced by Bismut, this idea was adapted for obtaining the second order derivative formula for the heat semigroup on a compact manifold. As explained earlier, the choice of stochastic variation in [66] (see also Malliavin and Stroock [59, (1.5)]) will produce a term coming from the time reverse of a non-random vector field on \( L^{2,1}(\Omega; \mathbb{R}^n) \), and it seems not possible to replace the non-random vector field by a random one (otherwise the time reversed field is not adapted, hence Itô’s integral is not well defined). Therefore the formula obtained in Stroock [66] may not be extended to the one with a random vector field and so is not suitable for extension to non-compact manifolds with the localisation technique we introduce shortly.

One crucial ingredient for our choice of the stochastic variation is that it ensures (4.10), which implies that the second variation vanishes at time \( t \) when we choose a vector field \( h \) in the translated part satisfying \( h(t) \equiv 0 \). This allows us to derive a second order gradient formula with localised vector fields and to extend it to a general (non-compact) complete Riemannian manifold.

### 4.1 A novel stochastic variation with a second order term

As before, \( \{ U_t \}_{0 \leq t \leq \zeta(x)} \) is the solution of equation (3.1) with initial point \( U_0 = x \). In Bismut [12] the following classical perturbation for the driving force \( B_t \) was used:

\[
\hat{B}_t^\varepsilon = \int_0^t e^{-\varepsilon \Gamma_{t}^h} dB_s + \varepsilon \int_0^t \left( h'(s) + \frac{1}{2} \text{ric}_{U_s} h(s) \right) ds.
\]

where \( h \in L^{2,1}(\Omega; \mathbb{R}^n) \) is a chosen Cameron-Martin vector and \( \Gamma_{t}^h := \int_0^t R_{U_s} (\circ dB_s, h(s)) \). This perturbation of the noise works well with the first variation for which one needs to ensure that \( \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon = 0} \pi(U^\varepsilon_t) \neq 0 \) and has been the popular and standard perturbation, as used also in Driver [21], Fang and Malliavin [34]. Other variation of the noise are also of first order perturbations.

However, with the above mentioned variation, \( \frac{\partial^2}{\partial \varepsilon^2} \bigg|_{\varepsilon = 0} \pi(U^\varepsilon_t) \neq 0 \) as long as \( h(t) \neq 0 \). To solve this problem, we will introduce a second order variation (such perturbation is not unique and we may find a slightly different choice). Unlike the case with the classical perturbation, this time we cannot avoid differentiating the structure equation so have to choose a connection on the frame bundle. Our approach is inspired by the theory of linear connections induced by a SDE developed by Elworthy, LeJan and Li [29]. We believe that the same method can also be used for higher order variations.

For any \( h \in L^{2,1}(\Omega; \mathbb{R}^n) \), we have defined an \( \mathfrak{so}(n) \)-valued process \( \Gamma_{t}^h \) and \( \mathbb{R}^n \)-valued process \( \Theta_{t}^h, A_{t}^h \) by (3.2) and (3.3) respectively. We first introduce the translation and define the \( \mathbb{R}^n \)-valued process \( B_{t}^{\varepsilon, h} \) as follows

\[
B_{t}^{\varepsilon, h} := B_t + \varepsilon \int_0^t h'(s) ds + \frac{\varepsilon^2}{2} \int_0^t \Phi_s^h ds, \quad (4.1)
\]
where \( \Phi_t^h := \Gamma^h_t h'(t) \). We then introduce a rotation for \( \mathbb{R}^n \)-valued Brownian motion. Let us first set

\[
\Gamma^{(2),h}_t := \int_0^t \nabla_{\nabla_{\pi(U_s)}(U_s h(s), U_s \circ dB_s, U_s h(s))} - \int_0^t \Gamma^h_s R_{U_s}(\circ dB_s, h(s))
+ \int_0^t R_{U_s}(h'(s), h(s)) ds + \int_0^t R_{U_s}(\circ dB_s, \Gamma^h_s h(s)).
\] (4.2)

It is easy to see that \( \Gamma^{(2),h}_t \) is an \( \mathfrak{so}(n) \)-valued process. Then for every \( \varepsilon > 0 \), we define \( SO(n) \)-valued process \( G^{\varepsilon,h}_t \) as follows

\[
G^{\varepsilon,h}_t := \exp\left(-\varepsilon \Gamma^{(2),h}_t - \frac{\varepsilon^2}{2} \Gamma^{(2),h}_t\right),
\]

where \( \exp : \mathfrak{so}(n) \to SO(n) \) is the exponential map in the Lie algebra \( \mathfrak{so}(n) \) of \( SO(n) \).

We can now introduce \( B^{\varepsilon,h}_t \), the variation of \( B_t \), as well as the corresponding equation on \( OM \).

**Definition 4.1.** Let \( \xi(\varepsilon), \varepsilon \in (-1, 1) \), be a geodesic with \( \xi(0) = x \). Let \( \{U^{\varepsilon,h}_t : \varepsilon \in (-1, 1)\} \) be a parallel orthonormal frame along \( \xi(\varepsilon) \) with \( \pi(U^{\varepsilon,h}_t) = \xi(\varepsilon) \). Let \( U^{\varepsilon}_t \) denote the solution of the following equation with initial condition \( U^{\varepsilon,0}_0 \),

\[
dU^{\varepsilon,h}_t = \sum_{i=1}^n H_{\varepsilon i}(U^{\varepsilon,h}_t) \circ dB^{\varepsilon,h}_t, \quad d\tilde{B}^{\varepsilon,h}_t = G^{\varepsilon,h}_t \circ dB^{\varepsilon,h}_t.
\] (4.3)

We define \( X^{\varepsilon,\xi(\varepsilon),h}_t = \pi(U^{\varepsilon,h}_t) \). If \( \varepsilon = 0 \), then \( X^{\varepsilon,0,h}_t = X^\varepsilon_t \) with \( X^\varepsilon_t = \pi(U_t^\varepsilon) \).

We remark that the perturbation in \( U^{\varepsilon,h}_t \) has a translation part \( B^{\varepsilon,h}_t \), and a rotation part \( G^{\varepsilon,h}_t \). The rotation \( G^{\varepsilon,h}_t \) is chosen to offset precisely the twisting effects induced by the second order stochastic variation.

For simplicity we omit the subscript \( h \), in \( \Theta^h_t, \Lambda^{h}_t, X^{\varepsilon,h}_t, \Gamma^{h}_t, \Gamma^{(2),h}_t, G^{\varepsilon,h}_t, B^{\varepsilon,h}_t \) and \( U^{\varepsilon,h}_t \), from time to time.

Let \( \varpi \) and \( \theta \) denote respectively the \( \mathfrak{so}(n) \)-valued connection 1-form and the \( \mathbb{R}^n \)-valued solder 1-form respectively. Set

\[
\varpi^\varepsilon_t := \varpi \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right), \quad \theta^\varepsilon_t := \theta \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right).
\]

Through this paper, we use \( D_t, d_t \) to denote the stochastic covariant differential for vector fields and stochastic differential on \( M \) along a semi-martingale respectively and \( \frac{\partial}{\partial \varepsilon} \) denotes the covariant derivative for vector fields on \( M \) with respect to the variable \( \varepsilon \).

**Lemma 4.1.** If we choose \( h \in L^{2,1}(\Omega; \mathbb{R}^n) \) such that \( h(0) = U^{-1}_0 \left( \frac{\partial}{\partial \varepsilon} \right)_{\varepsilon=0} \xi(\varepsilon) \), then

\[
\varpi^\varepsilon_t = \int_0^t R_{U_s}(G^\varepsilon_t \circ dB^\varepsilon_t, \theta^\varepsilon_t).
\] (4.4)

And \( \theta^\varepsilon_t \) satisfy the following equation,

\[
\begin{aligned}
\frac{d\theta^\varepsilon_t}{dt} &= -\left( \Gamma_t + \varepsilon \Gamma^{(2)}_t^\varepsilon \right) G_t^\varepsilon_t \circ dB_t^\varepsilon + \varpi^\varepsilon_t G_t^\varepsilon_t \circ dB_t^\varepsilon + G_t^\varepsilon_t \left( h'(t) + \varepsilon \Phi_t \right) dt,
\end{aligned}
\]

\[
\theta^\varepsilon_0 = (U^\varepsilon_0)^{-1} \frac{d\xi(\varepsilon)}{d\varepsilon}.
\] (4.5)
In particular, we have
\[
\left\{
\begin{align*}
\theta^0_t :&= \theta \left( \frac{\partial}{\partial \varepsilon} \right)_{\varepsilon=0} U^\varepsilon_t = h(t), \\
\varpi^0_t &= \Gamma_t \\
\frac{D}{\partial \varepsilon} \bigg|_{\varepsilon=0} (U^\varepsilon_t G^\varepsilon_t) &= 0, \quad \forall \ v \in \mathbb{R}^d, \\
\frac{\partial X^\varepsilon_t}{\partial \varepsilon} \bigg|_{\varepsilon=0} &= U_t h(t).
\end{align*}
\right.
\] (4.6)

Proof. We first use the structure equation
\[
\mathrm{d}\varpi \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t, \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right) = -\varpi \wedge \varpi \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t, \frac{\partial}{\partial \varepsilon} U^\varepsilon_t + R_{U^\varepsilon_t} \left( \theta \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right), \theta \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right) \right) \right)
\]
\[
= -\sum_{i=1}^n \varpi \wedge \varpi \left( H_{e_i}(U^\varepsilon_t) \circ dB^\varepsilon_{t,i}, \frac{\partial}{\partial \varepsilon} U^\varepsilon_t + R_{U^\varepsilon_t} \left( \theta \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right), \theta \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right) \right) \right)
\]
\[
= R_{U^\varepsilon_t} \left( \theta \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right), \theta \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right) \right)
\]
to obtain
\[
\mathrm{d}\varpi^\varepsilon_t = \mathrm{d}\varpi \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t, \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right) = R_{U^\varepsilon_t} \left( \theta \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right), \theta \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right) \right) = R_{U^\varepsilon_t} \left( G^\varepsilon_t \circ dB^\varepsilon_t, \theta \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right) \right).
\]
Since at time 0, the variation \{U^\varepsilon_0; \varepsilon \in (-1, 1)\} is parallel along the geodesic \xi, \varpi^0_0 = 0. Then (4.4) follows immediately.

Here we have used the Transfer Principle: on the compact manifold \(M\) we could treat the Stratonovich integral as the ordinary derivative (with respect to time variable) in the computation. Crucially we could exchange the order of differentiations and integrations. The transfer principle is well known for compact manifolds, see e.g. [34] or [57], but not automatically apply to non-compact manifolds nor automatically to the less smooth case nor to the derivative processes. This is used in similar computations later in the article without further comment.

Due to the torsion free property, the time derivative and the derivative for \(\varepsilon\) could commute:
\[
D_t \frac{\partial}{\partial \varepsilon} = \frac{\partial}{\partial \varepsilon} D_t. \quad \text{Also note that} \quad \theta^r_t = (U^\varepsilon_t)^{-1} T \pi \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_t \right), \quad \text{so we have,}
\]
\[
\mathrm{d}\theta^r_t = (U^\varepsilon_t)^{-1} \left( D_t \left( \frac{\partial}{\partial \varepsilon} X^\varepsilon_t \right) \right) = (U^\varepsilon_t)^{-1} \left( \frac{D}{\partial \varepsilon} \mathrm{d} X^\varepsilon_t \right)
\]
\[
= (U^\varepsilon_t)^{-1} \left( \frac{D}{\partial \varepsilon} (U^\varepsilon_t G^\varepsilon_t \circ dB^\varepsilon_t) \right)
\]
\[
= \varpi^r_t G^\varepsilon_t \circ dB^\varepsilon_t + \frac{\partial G^\varepsilon_t}{\partial \varepsilon} \circ dB^\varepsilon_t + G^\varepsilon_t \circ 
\]
\[
= \varpi^r_t G^\varepsilon_t \circ dB^\varepsilon_t - \left( \Gamma_t + \varepsilon \Gamma_t^{(2)} \right) G^\varepsilon_t \circ dB^\varepsilon_t + G^\varepsilon_t \left( h'(t) + \varepsilon \Phi_t \right) \mathrm{d}t,
\]
where the fourth equality is due to
\[
\frac{D}{\partial \varepsilon} (U^\varepsilon_t G^\varepsilon_t) = U^\varepsilon_t \left( \varpi^r_t G^\varepsilon_t + \frac{\partial}{\partial \varepsilon} G^\varepsilon_t \right).
\] (4.8)
So we have obtained the first equation in (4.5). The initial condition in (4.5) follows trivially from the fact \(\theta^r_0 = (U^\varepsilon_0)^{-1} \pi \left( \frac{\partial}{\partial \varepsilon} U^\varepsilon_0 \right)\). \{U^\varepsilon_0; \varepsilon \in (-1, 1)\} is a parallel orthonormal frame bundle along \(\xi(\cdot)\) and \(X^\varepsilon_t = \xi(\varepsilon)\).
Based on the fact that
\[ ς^0_t = \int_0^t R_{t,s}(\circ dB_s, θ^0_s), \quad Γ_t = \int_0^t R_{t,s}(\circ dB_s, h(s)), \]
and taking \( ε = 0 \) in (4.5) we arrive at
\[ dθ^0_t = \left( \int_0^t R_{t,s}(\circ dB_s, θ^0_s) - \int_0^t R_{t,s}(\circ dB_s, h(s)) \right) \circ dB_t + h'(t) dt, \quad θ^0_0 = h(0). \]

It is easy to verify that \( θ^0_t = h(t) \) is the unique solution to above equation, proving the first line of (4.6). Then plugging in \( θ^0_t = h(t) \) into (4.4) to see that \( ς^0_t = Γ_t \), so we have
\[ \frac{D}{dε} \bigg|_{ε=0} (U^ε_t G^ε_t e) = U_t \left( ς^0_t e + \frac{∂}{∂ε} \bigg|_{ε=0} G^ε_t e \right) = U_t (Γ_t e - Γ(e)) = 0, \]
which is the third line of (4.6). Finally, \( D_t \left( \frac{∂X^ε_t}{∂ε} \big|_{ε=0} \right) = U_t dh^0_t = U_t h'(t) dt \), giving \( \frac{∂X^ε_t}{∂ε} \big|_{ε=0} = U_t h(t) \). This completes the proof.

In particular, we obtain the following lemma:

**Lemma 4.2.** For every \( h ∈ L^{2,1}(Ω; \mathbb{R}^n) \) with \( h(0) \equiv v = U_0^{-1} \left( \frac{∂}{∂ε} \bigg|_{ε=0} ξ(ε) \right) \), we have
\[ \frac{∂}{∂ε} \bigg|_{ε=0} ς^ε_t = \int_0^t R_{t,s}(\circ dB_s, η_s) + Γ^{(2)}_t, \quad (4.9) \]
where \( η_s := \frac{∂η_s}{∂ε} \bigg|_{ε=0} \) and \( Γ^{(2)}_t \) is defined by (4.2).

**Proof.** By the first line of (4.6) we have \( θ^0_t = h(t) \). We differentiate the integral expression (4.4) for \( ς^ε_t \) and apply the third line of (4.6) to obtain
\[
\frac{∂}{∂ε} \bigg|_{ε=0} ς^ε_t = \frac{∂}{∂ε} \bigg|_{ε=0} \int_0^t \left( U^ε_s G^ε_s e \circ dB_s^ε, U^ε_s θ^ε_s \right) \\
= \int_0^t \frac{∂}{∂ε} \bigg|_{ε=0} \left( U^ε_s G^ε_s e \circ dB_s^ε, U^ε_s θ^ε_s \right) \\
= \int_0^t \left( \frac{∂G^ε_s}{∂ε} \bigg|_{ε=0} \right) R_{t,s}(\circ dB_s, θ^0_s) + \int_0^t U^{-1}_s \nabla R_{t,s}(\circ dB_s, θ^0_s) \\
+ \int_0^t R_{t,s}(\circ dB_s, θ^0_s) + \int_0^t R_{t,s}(\circ dB_s, \left( \frac{∂G^ε_s}{∂ε} \bigg|_{ε=0} \right) θ^0_s) \\
+ \int_0^t R_{t,s}(\circ dB_s, \left( \frac{∂θ^ε_s}{∂ε} \bigg|_{ε=0} \right) θ^0_s),
\]
Here the last term is \( \int_0^t R_{t,s}(\circ dB_s, η_s) \), while the sum of the rest is \( Γ^{(2)}_t \), so we have completed the proof.

We observe that \( η_s = \frac{∂η_s}{∂ε} \bigg|_{ε=0} \) is essentially the second variation of \( π(U^ε_s) \).

**Lemma 4.3.** For every \( h ∈ L^{2,1}(Ω; \mathbb{R}^n) \) with \( h(0) \equiv v = U_0^{-1} \left( \frac{∂}{∂ε} \bigg|_{ε=0} ξ(ε) \right) \), we have \( η_t \equiv 0 \) for all \( t ∈ [0, 1] \) and
\[ \frac{D}{dε} \bigg|_{ε=0} \left( \frac{∂X^ε_t}{∂ε} \right) = U_t Γ_t h(t). \]
Proof. We recall the first equation of (4.5)

\[ d\eta_t^\varepsilon = -\left( \Gamma_t + \varepsilon \Gamma_t^{(2)} \right) G_t^\varepsilon \circ dB_t^\varepsilon + \varpi_t^\varepsilon G_t^\varepsilon \circ dB_t^\varepsilon + G_t^\varepsilon (h'(t) + \varepsilon \Gamma_t h'(t)) dt. \]

Differentiating it at \( \varepsilon = 0 \), using (4.9) and the following fact

\[ \varpi_t^0 = \Gamma_t, \quad \frac{\partial B_t^\varepsilon}{\partial \varepsilon} \big|_{\varepsilon=0} = h'(t), \quad \frac{\partial G_t^\varepsilon}{\partial \varepsilon} \big|_{\varepsilon=0} = -\Gamma_t, \quad \Phi_t = \Gamma_t h'(t), \]

we could obtain

\[
d\eta_t = -\left( \Gamma_t^{(2)} + \Gamma_t \frac{\partial G_t^\varepsilon}{\partial \varepsilon} \right) \frac{\partial B_t}{\partial \varepsilon} \big|_{\varepsilon=0} \circ dB_t - \Gamma_t \circ \left( \frac{\partial B_t^\varepsilon}{\partial \varepsilon} \big|_{\varepsilon=0} \right) + \left( \varpi_t^0 \frac{\partial G_t^\varepsilon}{\partial \varepsilon} \big|_{\varepsilon=0} + \varpi_t^0 \frac{\partial \varpi_t^\varepsilon}{\partial \varepsilon} \big|_{\varepsilon=0} \right) \circ dB_t
\]

\[
\quad + \varpi_t^0 \left( \frac{\partial B_t^\varepsilon}{\partial \varepsilon} \big|_{\varepsilon=0} \right) + \frac{\partial G_t^\varepsilon}{\partial \varepsilon} \big|_{\varepsilon=0} h'(t) dt + \Gamma_t h'(t) dt
\]

\[
= \left( \int_0^t R_{U_s} (\circ dB_s, \eta_s) \right) \circ dB_t.
\]

At the same time, since \( X_0^\varepsilon = \xi(\varepsilon) \), \( \xi(\cdot) \) is a geodesic, and also \( \{ U_0^\varepsilon, \varepsilon \in (-1, 1) \} \) is a parallel orthonormal frame bundle along \( \xi(\cdot) \), we could verify that

\[
\eta_0 = \frac{\partial \theta_0^\varepsilon}{\partial \varepsilon} \big|_{\varepsilon=0} = U_0^{-1} \left( \frac{D}{\partial \varepsilon} \big|_{\varepsilon=0} \left( \frac{\partial \xi(\varepsilon)}{\partial \varepsilon} \right) \right) = 0.
\]

Observe that the unique solution to following equation is \( v_t \equiv 0 \)

\[
dv_t = \left( \int_0^t R_{U_s} (\circ dB_s, v_s) \right) \circ dB_t, \quad v_0 = 0.
\]

Then we derive that \( \eta_t \equiv 0 \) for all \( t \in [0, 1] \).

Moreover, note that by definition we have \( \frac{\partial X_t^\varepsilon}{\partial \varepsilon} = U_t^\varepsilon \theta_t^\varepsilon \), due to the fact \( \eta_t = \frac{\partial \theta_t^\varepsilon}{\partial \varepsilon} \big|_{\varepsilon=0} \equiv 0 \) we obtain

\[
\frac{D}{\partial \varepsilon} \big|_{\varepsilon=0} \left( \frac{\partial X_t^\varepsilon}{\partial \varepsilon} \right) = \frac{D}{\partial \varepsilon} \big|_{\varepsilon=0} \left( U_t \left( \varpi_t^0 \theta_t^0 + \frac{\partial \theta_t^\varepsilon}{\partial \varepsilon} \big|_{\varepsilon=0} \right) \right) = U_t \Gamma_t h(t).
\]

Now we have obtained (4.10). \( \square \)

### 4.2 Proof for the 2nd order gradient formula on a compact manifold

**Proposition 4.4.** Let \( t > 0 \), \( x \in M \) and \( v \in T_x M \). Then for any \( f \in C_0(M) \) and \( h \in L^{2,1}(\Omega, \mathbb{R}^n) \) satisfying that \( h(0) = U_0^{-1} v \) and \( h(t) = 0 \) a.s., we have

\[
\langle \nabla P_t f(x), v \rangle_{T_x M} = -\mathbb{E} \left[ f(X_t^x) \int_0^t \langle \Theta_s^h, dB_s \rangle \right], \tag{4.11}
\]

where \( \Theta_t^h := h'(t) + \frac{1}{2} \text{ric}_{U_t}(h(t)) \). Furthermore,

\[
\langle \nabla^2 P_t f(x), v \otimes v \rangle_{T_x M \otimes T_x M} = \mathbb{E} \left[ f(X_t^x) \left( \int_0^t \langle \Theta_s^h, dB_s \rangle \right)^2 - \int_0^t \langle \Lambda_s^h, dB_s \rangle - \int_0^t \left| \Theta_s^h \right|^2 ds \right]. \tag{4.12}
\]
Proof. We take $\xi(\cdot)$ to be a geodesic with initial value $\xi(0) = x$ and initial velocity $\frac{\partial \xi(\cdot)}{\partial x}\bigg|_{x=0} = v$. Let $\{U_0^\varepsilon \in (-1, 1)\}$ denote the parallel orthonormal frame bundle along $\xi(\cdot)$ with $\frac{\partial U_0^\varepsilon}{\partial x}\bigg|_{x=0} = U_0$. In particular, it holds that $\pi(U_0^\varepsilon) = \xi(\varepsilon)$. Recall that $U_t^\varepsilon$ is the solution to (4.3) with initial value $U_0^\varepsilon$ chosen above. It holds that

$$
\int_0^t G_s^\varepsilon \circ dB_s^\varepsilon = \int_0^t G_s^\varepsilon \circ dB_s + \int_0^t \frac{1}{2} d\langle G_s^\varepsilon, B_s \rangle_s + \int_0^t G_s^\varepsilon \bigg( \varepsilon h'(s) + \frac{\varepsilon^2}{2} \Gamma_s h'(s) \bigg) ds
$$

Here we have used that

$$
d\langle G_s^\varepsilon, B_t \rangle_t = -\varepsilon G_t^\varepsilon d \langle \Gamma_s, B_t \rangle_t - \frac{\varepsilon^2}{2} G_t^\varepsilon d \langle \Gamma_s^2, B_t \rangle_t
$$

Let $W_t^\varepsilon := \int_0^t G_s^\varepsilon dB_s$ be still an $\mathbb{R}^n$-valued Brownian motion, so we have

$$
dU_t^\varepsilon = H(U_t^\varepsilon) \circ \left( dW_t^\varepsilon + G_s^\varepsilon \left( \varepsilon \Theta_t + \frac{\varepsilon^2}{2} \Lambda_t \right) dt \right).
$$

Let

$$
M_t^\varepsilon := \exp \left( -\int_0^t \left( \varepsilon \Theta_t + \frac{\varepsilon^2}{2} \Lambda_t \right) dB_s \right) - \int_0^t \left( \frac{\varepsilon^2}{2} \left| \Theta_t + \frac{\varepsilon^2}{2} \Lambda_t \right|^2 \right) ds.
$$

Then by the Girsanov theorem, the distribution of $\{U_s^\varepsilon; s \in [0, t]\}$ under $dQ^\varepsilon := M_t^\varepsilon d\mathbb{P}$ is the same as that of $\{U_s^0, x; s \in [0, t]\}$, where $U_s^0, x$ is the solution to equation (3.1) with initial value $U_0^0 = U_0^\varepsilon$. Therefore we obtain

$$
P_t f(\xi(\varepsilon)) = \mathbb{E} \left[ f(X_t^\varepsilon(\xi)) \right] = \mathbb{E} \left[ f(X_t^\varepsilon(\xi)) M_t^\varepsilon \right],
$$

where $X_t^\varepsilon(\xi) = \pi(U_t^\varepsilon)$, $X_t^\varepsilon = \pi(U_t^{0, x})$.

We first assume $f \in C^2_b(M)$, differentiating (4.13) with respect to $\varepsilon$ yields that

$$
\langle \nabla P_t f(x), v \rangle_{T_s M \otimes T_x M} = \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} P_t f(\xi(\varepsilon))
$$

$$
= \mathbb{E} \left[ \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} f(\xi(\varepsilon)) \right] + \mathbb{E} \left[ f(X_t) \left. \left( \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} M_t^\varepsilon \right) \right],
$$

Another round of differentiation gives:

$$
\langle \nabla^2 P_t f(x), v \otimes v \rangle_{T_s M \otimes T_x M} = \left. \frac{\partial^2}{\partial \varepsilon^2} \right|_{\varepsilon=0} P_t f(\xi(\varepsilon))
$$

$$
= \mathbb{E} \left[ \left. \frac{\partial^2}{\partial \varepsilon^2} \right|_{\varepsilon=0} f(\xi(\varepsilon)) \right] + 2 \mathbb{E} \left[ \left. \left( \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} f(\xi(\varepsilon)) \right) \left( \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} M_t^\varepsilon \right) \right] + \mathbb{E} \left[ f(X_t) \left. \left( \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} M_t^\varepsilon \right) \right].
$$
Furthermore, \[ \frac{\partial}{\partial \varepsilon} \left|_{\varepsilon=0} \right. f \left( X_t^{\varepsilon, \xi(e)} \right) = \langle \nabla f (X_t^\varepsilon), U_t h(t) \rangle_{T_{X_t^\varepsilon} M} = 0 \]

and also,

\[ \frac{\partial}{\partial \varepsilon} \left|_{\varepsilon=0} \right. M_t^\varepsilon = - \int_0^t \langle \Theta_s, dB_s \rangle. \]

Furthermore,

\[ \frac{\partial^2}{\partial \varepsilon^2} \left|_{\varepsilon=0} \right. f \left( X_t^{\varepsilon, \xi(e)} \right) = \left\langle \nabla^2 f (X_t^\varepsilon), \left. \frac{\partial X_t^{\varepsilon, \xi(e)}}{\partial \varepsilon} \right|_{\varepsilon=0}, \left. \frac{\partial X_t^{\varepsilon, \xi(e)}}{\partial \varepsilon} \right|_{\varepsilon=0} \right\rangle_{T_{X_t^\varepsilon} M \otimes T_{X_t^\varepsilon} M} + \left\langle \nabla f (X_t^\varepsilon), D_{\varepsilon} \left( \frac{\partial X_t^{\varepsilon, \xi(e)}}{\partial \varepsilon} \right) \right\rangle_{T_{X_t^\varepsilon} M} = 0, \]

\[ \frac{\partial^2}{\partial \varepsilon^2} \left|_{\varepsilon=0} \right. M_t^\varepsilon = \left( \int_0^t \langle \Theta_s, dB_s \rangle \right)^2 - \int_0^t \langle \Lambda_s, dB_s \rangle - \int_0^t |\Theta_s|^2 \, ds. \]

Crucially this special choice of variation ensures that \[ \frac{\partial^2}{\partial \varepsilon^2} \left|_{\varepsilon=0} \right. f \left( X_t^{\varepsilon, \xi(e)} \right) \] depends only on \( h(t) \), not on the history of the process \( h \).

Putting these back to (4.14) and (4.15) yields (4.11), (4.12) for \( f \in C_b^2(M) \). By standard approximation procedure and the compact property of \( M \) we see that these equalities still hold for any \( f \in C_b(M) \).

5 Quantitative Cut-off Processes

From now on, we assume that \( M \) is an \( n \)-dimensional general complete Riemannian manifold, not necessarily compact.

In this section we introduce a class of cut-off processes satisfying estimates crucial for the localisation procedures, which we shall apply later to (4.12) and to obtain the asymptotic gradient estimates for the logarithmic heat kernel.

Since geodesic balls have typically non-regular boundary, we firstly construct a family of relatively compact open sets \( \{D_m\}_{m=1}^\infty \) with smooth boundary which plays the roles of geodesic balls and such that \( \bigcup_{m=1}^\infty D_m = M \). Our localisation procedure crucially relies on \( D_m \) has smooth boundaries, see Lemma 7.1. We first use a result in Greene and Wu [38] on the existence of a smooth approximate distance function, which is valid for complete manifold, and then construct a family of cut off vector fields adapted to \( \{D_m\}_{m=1}^\infty \). Fixing an \( o \in M \), denote by \( d \) the Riemannian distance function on \( M \) from \( o \). Since \( M \) is complete, according to [38] there exists a non-negative smooth function \( \hat{d} : M \to \mathbb{R}_+ \) with the property that \( 0 \leq |\nabla \hat{d}| \leq 1 \) and

\[ \hat{d}(x) - \frac{1}{2} \hat{d}(x) < 1, \quad \forall x \in M. \]

For every non-negative \( m \), define \( D_m := \hat{d}^{-1}((-\infty, m)) := \{ z \in M : \hat{d}(z) < m \} \), then it is easy to verify \( B_o(2m - 2) \subset D_m \subset B_o(2m + 2) \), where \( B_o(r) := \{ z \in M : \hat{d}(z) < r \} \) is the
geodesic ball centred at \( o \) with radius \( r \). Let \( \phi : \mathbb{R} \to [0, 1] \) be a smooth function such that

\[
\phi(r) = \begin{cases} 
1, & r \leq 1 \\
(0, 1), & r \in (1, 2) \\
0, & r \geq 2.
\end{cases}
\] (5.1)

Setting

\[
f_m(z) := \phi\left( \hat{d}(z) - m + 2 \right), \quad z \in M,
\] (5.2)

then it is easy to see that

\[
f_m(z) = \begin{cases} 
1, & \text{if } z \in \overline{D}_{m-1} \\
0, & \text{if } z \in D_m \\
\in (0, 1), & \text{otherwise}
\end{cases}
\]

and \( D_m = \{ z \in M; f_m(z) > 0 \} \). Without loss of generality we can assume that \( D_m \) is a bounded connected open set (otherwise we could take the connected component of \( D_m \) containing \( B_o(2m - 2) \)). Moreover, since \( \partial D_m = \{ z \in M; \hat{d}(z) = m \} \) and \( |\nabla \hat{d}(z)| \neq 0 \) for all \( z \in M \), we know \( \partial D_m \) is a smooth \( n - 1 \) dimensional submanifold of \( M \).

As before we suppose that \( \{ U_t \}_{0 \leq t \leq \zeta(x)} \) is the solution to the canonical horizontal equation (3.1) with \( \zeta(x) \) denoting its explosion time, and \( \{ X^x_t := \pi(U_t) \}_{0 \leq t < \zeta(x)} \) is a Brownian motion on \( M \) with initial value \( x := \pi(U_0) \).

Let \( \partial \) denote the cemetery state for \( M \) and set \( \bar{M} = M \cup \{ \partial \} \). Given a \( x \in M \) we let

\[
P_x(\bar{M}) := \{ \gamma \in C([0, 1]; \bar{M}) : \gamma(0) = x \}
\]

denote the collection of all \( \bar{M} \)-valued continuous paths with initial value \( x \). Let \( \mu_x \) denote the Brownian motion measure on \( P_x(\bar{M}) \). We also refer the natural filtration of the canonical process \( \gamma(\cdot) \) as the canonical filtration on \( P_x(\bar{M}) \), which is augmented to be complete and right continuous as usual.

It is well known that the distribution of \( \{ X^x_t \}_{0 \leq t < \zeta(x)} \) and \( \{ U_t \}_{0 \leq t < \zeta(x)} \) under \( P \) is the same as that of the canonical process \( \{ \gamma(t) \}_{0 \leq t < \zeta(\gamma)} \) and its horizontal lift \( \{ U_t(\gamma) \}_{0 \leq t < \zeta(\gamma)} \) under \( \mu_x \), where \( \zeta(\gamma) \) denotes the explosion time of \( \gamma(\cdot) \). Set

\[
\tau_m(\gamma) = \tau_{D_m}(\gamma) := \inf \{ s \geq 0 : \gamma(s) \notin D_m \}.
\]

**Lemma 5.1.** For any \( m \in \mathbb{N} \) there exists a stochastic process (vector field) \( l_m : [0, 1] \times P_x(\bar{M}) \to [0, 1] \), such that

1. \( l_m(t, \gamma) = \begin{cases} 
1, & t \leq \tau_{m-1}(\gamma) \wedge 1 \\
0, & t > \tau_m(\gamma)
\end{cases} \).

2. **Absolute continuity:** \( l_m(t, \cdot) \) is adapted to the canonical filtration and \( l_m(\cdot, \gamma) \) is absolutely continuous for \( \mu_x \)-a.s. \( \gamma \in P_x(\bar{M}) \).

3. **Local uniform moment estimates:** For every positive integer \( k \in \mathbb{N} \), we have

\[
\sup_{x \in D_{m-1}} \int_{P_x(\bar{M})} \int_0^1 |l_m'(s, \gamma)|^k ds \, \mu_x(d\gamma) \leq C_1(m, k)
\] (5.3)

for some positive constant \( C_1(m, k) \) (which may depends on \( m \) and \( k \)).
Proof. In the proof, the constant $C$ (which may depend on $m$) will change in different lines. The main idea of the proof is inspired by the article of Thalmaier [67] and Thalmaier and Wang [68].

(1) Since for any $m \geq 1$, $D_m \subset D_{m+1} \uparrow M$, there exists a $m_0 \in \mathbb{N}$ such that

$$
\begin{cases}
  x \in D_m, & \text{when } m \geq m_0 \\
  x \notin D_m, & \text{when } 1 \leq m < m_0
\end{cases}
$$

When $x \notin D_m$, let $l_m(t, \gamma) \equiv 0$. In the following, we will consider the case of $x \in D_m$ (which implies that $\tau_m(\gamma) > 0$) without loss of generality. Let $f_m : M \to \mathbb{R}_+$ be the function given by (5.2), we define a sequence of functions:

$$
T_m(t, \gamma) := \begin{cases}
  \int_0^t \frac{ds}{[f_m(\gamma(s))]^2}, & t < \tau_m(\gamma) \\
  \infty, & t \geq \tau_m(\gamma).
\end{cases}
$$

Then each $T_m(\cdot, \gamma)$ is an increasing right continuous function of $t$. For any $t \geq 0$, set

$$
A_m(t, \gamma) := \inf \{s \geq 0 : T_m(s, \gamma) \geq t\}.
$$

We may omit the parameter $\gamma$ in the notation of $T_m(t, \gamma), A_m(t, \gamma)$ for simplicity in the proof.

Since $\inf_{s \in [0, t]} f_m(\gamma(t)) > 0$ for $t < \tau_m(\gamma)$, then $T_m(t) < \infty$ for every $t < \tau_m$ and $T_m(\cdot)$ is strictly increasing and continuous in $[0, \tau_m]$ (with respect to the variable $t$). Therefore $A_m(\cdot)$ is continuous on $[0, T_m(\tau_m)]$ and $T_m(A_m(t)) = t$ for every $0 \leq \tau_m < T_m(\tau_m)$. Furthermore we have $T_m(\tau_m) = \infty$. To see this we only need to observe that

$$
f_m(\gamma(s)) = f_m(\gamma(s)) - f_m(\gamma(\tau_m)) \leq \frac{1}{2} \sup_{x \in D_m} |\nabla^2 f_m(x)| d(\gamma(s), \gamma(\tau_m))^2
\leq C_m(\gamma)\sqrt{|s - \tau_m|}, \quad \forall s < \tau_m,$$

where $C_m(\gamma)$ is a constant, and we applied the property that $d(\gamma(s), \gamma(\tau_m)) \leq C_m(\gamma)|s - \tau_m|^{1/4}$ which is easy to prove by the Kolmogorov criterion. Combining the fact $T_m(\tau_m) = \infty$ with $T_m(t) < \infty$ for all $0 \leq t < \tau_m$ immediately yields that $A_m(T_m(t)) = t$ for every $0 \leq t \leq \tau_m$ and $\tau_m > A_m(t)$ for every $0 \leq t < \infty$.

Next, we use the truncation function $\phi : \mathbb{R} \to \mathbb{R}$ in (5.1) to define

$$
l_m(t, \gamma) = \phi \left( \int_0^t \frac{\phi(T_m(s) - 2)}{f_m^2(\gamma(s))} ds \right), \tag{5.4}
$$

which is clearly adapted to the canonical filtration. Suppose that $t \geq \tau_m > A_m(3)$, then

$$
\int_0^t \frac{\phi(T_m(s) - 2)}{f_m^2(\gamma(s))} ds \geq \int_0^t 1_{\{T_m(s) \leq 3\}} f_m^{-2}(\gamma(s)) ds
= \int_0^t 1_{\{s \leq A_m(3)\}} f_m^{-2}(\gamma(s)) ds
= T_m(A_m(3)) = 3,
$$

which implies $l_m(t, \gamma) = 0$ for $t \geq \tau_m$ by the definition of $\phi$.

If $s \leq \tau_{m-1}(\gamma)$ then $f_m(\gamma(s)) = 1$ and so $T_m(s) = s$. Consequently, $\phi(T_m(s) - 2) = 1$ for every $s \leq \tau_{m-1} \land 1$. Hence we obtain

$$
l_m(t, \gamma) = \phi(t \land 1) = 1, \quad \forall t \leq \tau_{m-1} \land 1,
$$
concluding the proof of part (1).

(2) Still by the expression of (5.4) we know the conclusion of part (2) holds.

(3) Now it only remains to verify the estimates (5.3). Firstly,

\[ |\phi'(t)\left(\int_0^t \phi(\frac{T_m(s) - 2}{f_m^2(\gamma(s))}) \, ds\right)| |\phi(\frac{T_m(t) - 2}{f_m^2(\gamma(t))})| \leq \|\phi'\|_\infty f_m^{-2}(\gamma(t)) 1_{\{\phi(T_m(t) - 2) \neq 0\}} \leq C f_m^{-2}(\gamma(t)) 1_{\{T_m(t) \leq 4\}}. \]

Then for every \( k \in \mathbb{N} \),

\[
\int_0^1 |\phi'(s)|^k \, ds \leq C \int_0^1 f_m^{-2k}(\gamma(s)) 1_{\{T_m(s) \leq 4\}} \, ds \\
\leq C \int_0^1 f_m^{-2k+2}(\gamma(s)) 1_{\{s \leq A_m(4)\}} \, ds \\
= C \int_0^{A_m(4)} f_m^{-2k+2}(\gamma(A_m(r))) \, dr \\
\leq C \int_0^4 f_m^{-2k+2}(\gamma(A_m(r))) \, dr. \tag{5.5}
\]

Observe that the distribution of \( X^x \) under \( \mathbb{P} \) is the same as that of \( \gamma(\cdot) \) under \( \mu_x \),

\[
\sup_{x \in D_{m-1}} \mathbb{E} \left[ \int_0^4 f_m^{-2k+2}(\gamma(A_m(s))) \, d\mu_x(\gamma) \right] = \sup_{x \in D_{m-1}} \mathbb{E} \left[ \int_0^4 f_m^{-2k+2}(X^x_{A_m(s) \wedge S_{j,m}}) \, ds \right]. \tag{5.6}
\]

Let \( S_{j,m}(\gamma) := \inf\left\{ t > 0; f_m(\gamma(t)) \leq \frac{1}{2} \right\} \). According to Itô’s formula we obtain for all \( j, k \in \mathbb{N} \) and \( x \in D_{m-1} \),

\[
\mathbb{E} \left[ f_m^{-k}(X^x_{A_m(t) \wedge S_{j,m}}) \right] = f_m^{-k}(x) + \frac{1}{2} \mathbb{E} \left[ \int_0^{A_m(t) \wedge S_{j,m}} f_m^{-2k}(\Delta f_m)(X_s) \, ds \right] \\
= 1 + \frac{1}{2} \mathbb{E} \left[ \int_0^{A_m(t) \wedge S_{j,m}} \left( f_m^2 \Delta f_m \right)(X^x_{A_m(T_m(s))}) \, dT_m(s) \right], \tag{5.7}
\]

we have applied the fact that \( A_m(T_m(s)) = s \) for every \( 0 \leq s < S_{j,m} \) and \( f_m(x) = 1 \) for all \( x \in D_{m-1} \). Meanwhile we have

\[
f_m^2 \Delta (f_m^{-k}) = k(k + 1) f_m^{-k} \left| \nabla f_m \right|^2 - k f_m^{-k+1} \Delta f_m \\
= k(k + 1) f_m^{-k} \left| \phi'(\hat{d} - m + 2) \right|^2 \left| \nabla \hat{d} \right|^2 \\
- k f_m^{-k} \left( f_m \phi''(\hat{d} - m + 2) \left| \nabla \hat{d} \right|^2 + \phi'(\hat{d} - m + 2) f_m \Delta \hat{d} \right) \\
\leq k(k + 1) f_m^{-k} \left( \left| \phi' \right|_\infty + \left| \phi'' \right|_\infty + \left| \phi' \right|_\infty \sup_{z \in D_m} |\Delta \hat{d}(z)| \right) \leq C f_m^{-k}. \]
Putting this into (5.7) we arrive at
\[
\mathbb{E} \left[ f_m^{-k} (X_{A_m(t)} \wedge S_{j,m}) \right] \leq 1 + C \mathbb{E} \left[ \int_0^{A_m(t) \wedge S_{j,m}} f_m^{-k} (X_{A_m(s)}) dT_m(s) \right] \\
\leq 1 + C \int_0^t \mathbb{E} \left[ f_m^{-k} (X_{A_m(r) \wedge S_{j,m}}) \right] dr,
\]
where the last step follows from the procedure of change of variable \( u = T_m(s) \) and the fact \( A_m(t) \leq t \).

Hence by Grownwall’s inequality we arrive at for all \( k, j \in \mathbb{N} \),
\[
\mathbb{E} \left[ f_m^{-k} (X_{A_m(t)} \wedge S_{j,m}) \right] \leq C e^{Ct}.
\]
Then letting \( j \to \infty \) and observing that \( A_m(t) \leq \tau_m = \lim_{j \to \infty} S_{j,m} \) we obtain for all \( k \in \mathbb{N} \),
\[
\mathbb{E} \left[ f_m^{-k} (X_{A_m(t)}) \right] \leq C e^{Ct},
\]
combing this with (5.5) yields (5.3). This completes the proof for Lemma 5.1. \( \square \)

6 Proof of the Main Estimates

In this section, we shall apply the cut-off procedures, using the quantitative localised vector fields introduced in Section 5, to obtain short time as well as asymptotic first and second order gradient estimates for the logarithmic heat kernel of a complete Riemannian manifold without imposing on it any curvature bounds.

Let \( \{ D_m \}_{m=1}^{\infty} \) and \( \{ f_m \}_{m=1}^{\infty} \) be the sequences of domains and functions constructed in Section 5. Recall that for every \( m, D_m = \{ x \in M : f_m(x) > 0 \} \) is a bounded connected open set. By Lemma 7.1 from the Appendix there exists a compact Riemannian manifold \( \tilde{M}_m \) such that \( D_m \) is isometrically embedded into \( \tilde{M}_m \) as an open set. We could and will view \( D_m \subset \tilde{M}_m \) as an open subset of \( \tilde{M}_m \). In particular, we have
\[
d_{\tilde{M}_m}(x, y) = d(x, y), \quad \forall x, y \in B_o(2m - 2), \tag{6.1}
\]
where \( d \) and \( d_{\tilde{M}_m} \) are the Riemannian distance function on \( M \) and \( \tilde{M}_m \). We denote the heat kernel on \( M \) and \( \tilde{M}_m \) by \( p(t, x, y) \) and \( p_{\tilde{M}_m}(t, x, y) \) respectively. For every \( e \in \mathbb{R}^n \) we also let \( H_e^m \) denote the horizontal lift of \( u e \) on \( TO(\tilde{M}_m) \).

Let us fix a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Let \( \{ B_t \}_{t \geq 0} \) be the standard \( \mathbb{R}^n \)-valued Brownian motion with \( B_t = (B^1_t, \ldots, B^n_t) \), and we denote by \( \mathcal{F}_t \) the filtration generated by it. Now we fix an orthonormal basis \( \{ e_i \}_{i=1}^n \) of \( \mathbb{R}^n \).

For \( x \in D_m \subset \tilde{M}_m \) and \( U_0 \) a frame at \( x \) so that \( U_0 \in O_x M = O_x \tilde{M}_m \), let \( U_t^m \) denote the solution to the following \( O(\tilde{M}_m) \)-valued stochastic differential equation
\[
dU_t^m = \sum_{i=1}^n H_{e_i}^m (U_t^m) \circ dB_t^i, \quad U_0^m = U_0. \tag{6.2}
\]
Set \( X_t^{m,x} := \pi(U_t^m) \). This is a \( \tilde{M}_m \)-valued Brownian motion. Recall that \( X_t^x := \pi(U_t) \), where \( U_t \) is the solution to (3.1), with the same driving Brownian motion \( B_t \) and the same initial value \( U_0 \) as in (6.2).
Throughout this section, for every $m, k \in \mathbb{N}$ with $k \geq m$, we define
\[
\tau_m := \inf \{ t > 0; X^k_t \notin D_m \}, \quad \tau^k_m := \inf \{ t > 0; X^k_t \notin D_m \}.
\]
Note that for every $k > m$, $H^k_{\tau^k_m} = H^m_{\tau_m}$ on $\pi^{-1}(D_m)$. It is easy to verify that
\[
\tau_m = \tau^k_m, \quad X^k_t = X_t^{k,x}, \quad \forall \ k \geq m > 1, \ 0 \leq t \leq \tau_m.
\] (6.3)
As before, the superscript $x$ may be omitted from time to time when there is no risk of confusion. The probability and the expectation for the functional generated by $X^x$ or $X^{m,x}$ (with respect to $\mathbb{P}$) are denoted by $\mathbb{P}_x$ and $\mathbb{E}_x$ respectively in this section.

If $M$ is compact, then when $m$ is large enough we have $D_m = M$ and we can take $\tilde{M}_m = M$ (we do not have to apply Lemma 7.1 when $M$ is compact), then all the conclusions in this section automatically hold. Hence in this section we always assume that $M$ is non-compact.

We shall use the following estimates which are crucial for our proof.

**Lemma 6.1** ([7, 60, 70, 71]). For any $x, y \in M$,
\[
\lim_{t \downarrow 0} t \log p(t, x, y) = -\frac{d(x, y)^2}{2}.
\] (6.4)
and the convergence is uniformly in $(x, y)$ on $K \times K$ for any compact subset $K$.
Moreover, for every connected bounded open set $D \supseteq K$ with smooth boundary,
\[
\lim_{t \downarrow 0} t \log \mathbb{P}_x (\tau_D < t) = -\frac{d(x, \partial D)^2}{2}, \quad \forall \ x \in K.
\] (6.5)
Here $\tau_D := \inf \{ t > 0; X_t \notin D \}$ is the first exit time from $D$ and $d(x, \partial D) := \inf_{z \in \partial D} d(x, z)$. And the convergence is also uniform in $x$ on $K$.

The asymptotic estimates (6.4) and (6.5) were firstly shown to hold for $\mathbb{R}^n$ in Varadhan [70, 71], extension to a complete Riemannian manifold was given in Molchanov [60]. In addition, Azencott [8] and [42] indicated that these statements may fail for an incomplete Riemannian manifold. We shall also use the following statement, which follows readily from the small time asymptotics and the Gaussian heat kernel upper bounds.

**Lemma 6.2.** ([8], [42, Lemma 2.2]) For any compact subset $K$ of $M$ and any positive number $r$, Then there exists a positive number $t_0$ such that
\[
\sup_{t \in (0, t_0]} \sup_{d(z,y) \geq r, y \in K} p(t, z, y) \leq 1.
\] (6.6)

### 6.1 Comparison theorem for functional integrals involving approximate heat kernels

Let $D_m$ denote the relatively compact subset and let $t_m : [0, 1] \times P_x(M) \to \mathbb{R}$ be the cutoff processes adapted to $D_m$, as constructed by Lemma 5.1. Let $p^{D_m}(t, x, y)$ denote the Dirichlet heat kernel on $D_m$. Let $K$ be a compact set and $x, y \in K$ be such that $d(x, y) < d(x, \partial D^m) \vee d(y, \partial D^m)$, then $p(t, x, y)$ and $p^{D_m}(t, x, y)$ are asymptotically the same for small $t$. See [8], Lemma 2.3 on page 156.

Below we give a quantitative estimate on $p$ and $p^{D_m}$ on a compact set $K \times K$, for sufficiently large $m$. By sufficiently large, we mean that $m \geq m_0$ for a natural number $m_0$ and $m_0$ may depend on other data. In all the results below, it depends on the compact set $K$ and the prescribed exponential factor $L > 0$. 
Lemma 6.3. Suppose that $K$ is a compact subset of $M$ and $L > 1$ is a positive number. Then for sufficiently large $m$, there exists a positive number $t_0 = t_0(K, L, m)$ such that for every $t \in (0, t_0]$,

$$
sup_{x,y \in K} |p(t, x, y) - p^{D_m}(t, x, y)| \leq e^{-\frac{2L}{t}}, \tag{6.7}
$$

$$
sup_{x,y \in K} |p_{\tilde{M}_m}(t, x, y) - p^{D_m}(t, x, y)| \leq e^{-\frac{2L}{t}}. \tag{6.8}
$$

In particular, for every $t \in (0, t_0]$, 

$$
sup_{x,y \in K} |p(t, x, y) - p_{\tilde{M}_m}(t, x, y)| \leq e^{-\frac{L}{7}}. \tag{6.9}
$$

Proof. The estimates in (6.7) could be found in Azencott [8, section 4.2], also in Bismut [12, Section III.a] and Hsu [44, The proof of Theorem 5.1.1]. Here we include a proof for the convenience of the reader. The technique and the intermediate estimates will be used later.

By the strong Markovian property,

$$
P_t f(x) = E_x \left[ f \left( X_t \right) 1_{\{t \leq \tau_m\}} \right] + E_x \left[ E_{X_{\tau_m}} \left[ f \left( X_{t-\tau_m} \right) \right] 1_{\{\tau_m < t < \zeta\}} \right]
$$

and so for any $x, y \in K$ and $t > 0$,

$$
p(t, x, y) = p^{D_m}(t, x, y) + E_x \left[ p \left( t - \tau_m, X_{\tau_m}, y \right) 1_{\{\tau_m < t < \zeta\}} \right]. \tag{6.9}
$$

Since $M$ is non-compact, given any number $L > 1$, there exists a natural number $m_0$ such that

$$
K \subset B_o(2m_0 - 2), \quad d(K, \partial D_{m_0}) \geq d(K, \partial B_o(2m_0 - 2)) > 4L.
$$

Then, according to (6.5) and (6.6), for every $m \geq m_0$, we could find a positive number $t_0(K, L, m)$ such that for any $t \in (0, t_0]$,

$$
P_x \left( t > \tau_m \right) \leq \exp \left( -\frac{d(x, \partial D_m)^2 - 1}{2t} \right) \leq e^{-\frac{2L}{t}}, \quad \forall x \in K, \quad p(t, z, y) \leq 1, \quad \text{for all} \quad z \in \partial D_m, \quad \text{and} \quad y \in K.
$$

By these estimates we obtain that, for all $m \geq m_0$ and all $t \in (0, t_0]$,

$$
E_x \left[ p \left( t - \tau_m, X_{\tau_m}, y \right) 1_{\{t > \tau_{D_m}\}} \right] \leq \sup_{t \in (0, t_0]} \sup_{z \in \partial D_m, y \in K} \sup_{x \in K} p(t, z, y), \quad E_x \left( t > \tau_m \right) \leq e^{-\frac{2L}{t}}.
$$

Putting this into (6.9) we arrive at that for all $m \geq m_0$, all $x, y \in K$, and for all $t \in (0, t_0]$,

$$
|p(t, x, y) - p^{D_m}(t, x, y)| \leq e^{-\frac{2L}{t}}. \tag{6.10}
$$

Note that for every $m \geq m_0$, $D_m \subset \tilde{M}_m$ and $x \in K$,

$$
d_{\tilde{M}_m}(x, \partial D_m) \geq d_{\tilde{M}_m}(x, \partial B_o(2m - 2)) = d(x, \partial B_o(2m - 2))
$$

which is due to (6.1). By the same argument for (6.10) and changing the constant $t_0$ if necessary we could find a $t_0(K, L, m)$ such that for all $m \geq m_0$,

$$
|p_{\tilde{M}_m}(t, x, y) - p^{D_m}(t, x, y)| \leq e^{-\frac{2L}{t}}, \quad x, y \in K, \quad t \in (0, t_0].
$$

This, together with (6.10), yields (6.7) and (6.8). \qed
Lemma 6.4. Suppose that $K$ is a compact subset of $M$ and $L > 1$ is a positive number.

(1) For $m_0$ sufficiently large and any $m > m_0$, there exists a $t_0(K, L, m)$ such that for every $0 < s \leq \frac{t}{2}$ and $0 < t \leq t_0$, we have

$$
\sup_{x,y \in K} \sup_{z \in D_{m_0}} \left| \frac{p(t-s, x, z)}{p(t, x, y)} - \frac{p_{M_m}(t-s, x, z)}{p_{M_m}(t, x, y)} \right| \leq 2e^{-\frac{4(L+\tau_0+1)}{t}}, \quad (6.11)
$$

(2) Suppose $\Upsilon_t$ is an $F_t$ adapted process, and for any $q > 0$ and $m \geq 1$ we set

$$
F^q_m(\Upsilon, X) = \left( \int_0^s \Upsilon t'_m(r, X) \, dB_r \right)^q, \quad F^q_m(\Upsilon, X^m) = \left( \int_0^s \Upsilon t'_{m}(r, X^m) \, dB_r \right)^q.
$$

We also assume that

$$
\sup_{x \in K} \mathbb{E}_x \left[ \int_0^{1+\tau_m} |\Upsilon_s|^q \, ds \right] < \infty, \quad \forall m \geq 1 \quad (6.12)
$$

for some $q \in \mathbb{N}$. Then for every sufficiently large $m$ (any $m$ greater than some number $m_0(K, L)$), we can find a positive number $t_0(K, L, m)$ with the property that

$$
\sup_{x,y \in K} \mathbb{E}_x \left[ F^q_m(\Upsilon, X) \frac{p(t-s, X, x, y)}{p(t, x, y)} \right] - \mathbb{E}_x \left[ F^q_m(\Upsilon, X^m) \frac{p_{M_m}(t-s, X^m, y)}{p_{M_m}(t, x, y)} \right] 
\leq C(m) e^{-\frac{4}{t}} \quad (6.13)
$$

for any $0 < t \leq t_0$, $0 < s \leq \frac{t}{2}$. Here the positive constant $C(m)$ may depend on $m$ and on $\alpha_m := \sup_{x \in K} \mathbb{E}_x \left[ \int_0^1 |\Upsilon t'_m(r, X)|^q \, dr \right]$. (Note that $l'_m(r, X) \neq 0$ only for $r < \tau_m = \tau_m$ so the quantity is well defined.)

Proof. In the proof, the constant $C(m)$ may represent different constants in different lines. Let $r_0 := \sup_{x,y \in K} d_M(x, y)$ denote the diameter of $K$. Since $M$ is non-compact, we can choose a natural number $\tilde{m}_0$ (which may depend on $K$ and $L$) such that

$$
K \subset B_o(2\tilde{m}_0 - 2) \subset D_{\tilde{m}_0}
$$

and for all $m > \tilde{m}_0$,

$$
d(K, \partial B_o(2\tilde{m}_0 - 2)) = d_{M_m}(K, \partial B_o(2\tilde{m}_0 - 2)) > 4(L + r_0 + 1).
$$

Also, by the heat kernel comparison (6.7) and (6.8), we can find a $m_0 > \tilde{m}_0$ so that for all $m > m_0$, there exists a constant $t_2(K, L, m) > 0$ such that,

$$
\left| p(t, z, y) - p_{M_m}(t, z, y) \right| \leq e^{-\frac{4(L+\tau_0+1)}{t}}, \quad \forall t \in (0, t_2], \ z, y \in D_{m_0}. \quad (6.14)
$$

According to the asymptotic relations (6.4) and (6.5), for every $m > m_0$ (taking $m_0$ larger as is necessary) we could find a constant $0 < t_1(K, L, m) \leq t_2$ such that for all $t \in (0, t_1]$,

$$
p(t, z, y) \leq e^\frac{t}{4}, \quad p_{M_m}(t, z, y) \leq e^\frac{t}{4}, \quad \forall z, y \in D_{m_0}, \quad (6.15)
$$

$$
p(t, z, y) \geq e^{-\frac{t}{4(L+\tau_0+1)}}, \quad p_{M_m}(t, z, y) \geq e^{-\frac{t}{4(L+\tau_0+1)}}, \quad \forall z, y \in K, \quad (6.16)
$$

$$
\mathbb{P}_z(t_{m_0} < t) \leq e^{-\frac{4(L+\tau_0+1)}{t}}, \quad \forall z \in K. \quad (6.17)
$$
By the small time locally uniform heat kernel bound (6.6), for every \( m > m_0 \) there exists a number \( 0 < t_0(K, L, m) \leq t_1 \) such that for all \( t \in (0, t_0) \),
\[
p(t, z_1, y) \vee p_{\tilde{M}_m}(t, z_2, y) \leq 1, \quad \forall z_1 \in M \cap D_{m_0}^c, \ z_2 \in \tilde{M}_m \cap D_{m_0}^c, \ y \in K. \quad (6.18)
\]

Therefore for every \( m > m_0 \) and for every \( 0 < \frac{t}{2} \), every \( 0 < t \leq t_0 \), and for all \( x, y \in K \) and \( z \in D_{m_0} \), we have
\[
\left| \frac{p(t-s, x, z)}{p(t,x,y)} - \frac{p_{\tilde{M}_m}(t-s, x, z)}{p_{\tilde{M}_m}(t, x, y)} \right| \leq \frac{p(t-s, x, z) \left| p_{\tilde{M}_m}(t, x, y) - p(t, x, y) \right| + p(t, x, y) \left| p_{\tilde{M}_m}(t-s, x, z) - p(t-s, x, z) \right|}{p(t, x, y) p_{\tilde{M}_m}(t, x, y)} 
\leq 2e^{-\frac{2(1+\alpha_1)}{4} \frac{t}{t}\frac{2}{|z|^{q+1}} \frac{t}{t} \frac{1}{2}} \leq 2e^{-\frac{4t}{t}}.
\]

(6.19)

Here the second step above is due to (6.14)--(6.16). Thus, we finish the proof of (1).

For all \( m > m_0 \), let us split the terms as follows,
\[
E_x \left[ F_m^q(Y_m, x_m) \frac{p(t-s, X_s, y)}{p(t, x, y)} \right] = E_x \left[ F_m^q(Y_m, x_m) \frac{p(t-s, X_s, y)}{p(t, x, y)} 1_{\{t \leq \tau_m\}} \right] + E_x \left[ F_m^q(Y_m, x_m) \frac{p(t-s, X_s, y)}{p(t, x, y)} 1_{\{t > \tau_m\}} \right]
\]
\[
= : J_1^m(s, t) + J_2^m(s, t).
\]

Since \( l_m^r(r, X_s) \neq 0 \) if only if \( t < \tau_m \), then \( l_m^r(r, X_s) = l_m^r(r, X) \) and we have
\[
F_m^q(Y_m, x_m) = \left( \int_{0}^{s} \left[ Y_m l_m^r(r, X_s) dB_r \right]^q \right) = \left( \int_{0}^{s} \left[ Y_m l_m^r(r, X) dB_r \right]^q \right) = F_m^q(Y_m, x_m).
\]

Note also, \( X_s^m = X_s \) for every \( s \leq \frac{t}{2} \). It holds that
\[
E_x \left[ F_m^q(Y_m, x_m) \frac{p_{\tilde{M}_m}(t-s, X_s, y)}{p_{\tilde{M}_m}(t, x, y)} \right] = E_x \left[ F_m^q(Y_m, x_m) \frac{p_{\tilde{M}_m}(t-s, X_s, y)}{p_{\tilde{M}_m}(t, x, y)} 1_{\{t \leq \tau_m\}} \right] + E_x \left[ F_m^q(Y_m, x_m) \frac{p_{\tilde{M}_m}(t-s, X_s, y)}{p_{\tilde{M}_m}(t, x, y)} 1_{\{t > \tau_m\}} \right]
\]
\[
= : J_1^m(s, t) + J_2^m(s, t), \quad 0 < s < \frac{t}{2}.
\]

Note that
\[
\alpha_m = \sup_{x \in K} E_x \left[ \int_{0}^{\frac{t}{2}} \left| Y_m l_m^r(r, X) \right|^q dr \right] < \infty.
\]

(6.20)

This follows from the moment estimates on \( l_m^r \), (5.3), the assumption (6.12), and also
\[
\alpha_m \leq \sup_{x \in K} E_x \left[ \int_{0}^{1/2} \left| Y_r \right|^{2q} dr \right]^{1/2} \sup_{x \in K} E_x \left[ \int_{0}^{1/2} \left| l_m^r(r, X) \right|^{2q} dr \right]^{1/2}.
\]

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For all $m > m_0$, $x, y \in K$, $0 < s \leq \frac{1}{2}$, and $0 < t \leq t_0$, we may assume that $t_0 \leq 2$,

$$|I_1^m(s, t) - J_1^m(s, t)| \leq \sup_{z \in D_{m_0}} \frac{p(t - s, z, y)}{p(t, x, y)} \cdot \frac{p_{M_m}(t - s, z, y)}{p_{M_m}(t, x, y)} \cdot \mathbb{E}_x \left[ \left| \int_0^s \mathbb{Y}_r l''_m(r, X.) \, dB_r \right|^q \right]$$

$$\leq C e^{-\frac{4L}{r}} \sup_{z \in K} \mathbb{E}_x \left[ \int_0^{1+\tau_m} |\mathbb{Y}_r l''_m(r, X.)|^q \, dr \right] = C \alpha_m e^{-\frac{4L}{r}}.$$

In the penultimate step, we have applied Burkholder-Davies-Gundy inequality and (6.11). According to (6.15) and (6.18) we also have

$$\sup_{z \in M, y \in K} p(t, z, y) \leq e^{\frac{1}{r}}, \quad \forall 0 < t \leq t_0.$$

Combining this with (6.16)–(6.17), Cauchy-Schwartz inequality, and Burkholder-Davies-Gundy inequality, we obtain that for every $m > m_0$, $x, y \in K$, $0 < s \leq \frac{1}{2}$, and $0 < t \leq t_0$,

$$|I_2^m(s, t)| \leq C e^{-\frac{2L}{r} + 1} \sup_{r \in [\frac{1}{r}, t], z \in M, y \in K} p(r, z, y) \mathbb{E}_x \left[ \left| \int_0^s \mathbb{Y}_r l''_m(r, X.) \, dB_r \right|^2 \right] \leq C \alpha_m e^{-\frac{2L}{r}}.$$

Here in the last step we used (6.20). Similarly, we obtain that for every $m > m_0$, $x, y \in K$,

$$|I_2^m(s, t)| \leq C \alpha_m e^{-\frac{2L}{r}}, \quad \text{for all } 0 < s \leq \frac{1}{2} \text{ and } 0 < t \leq t_0.$$

Combining the above estimates for $I_1^m, I_2^m, J_1^m, J_2^m$ we see that, for every $m > m_0$, $x, y \in K$, $0 < s \leq \frac{1}{2}$ and $0 < t \leq t_0$,

$$\left| \mathbb{E}_x \left[ F_m^q(\mathbb{Y}, X.) \frac{p(t - s, X_s, y)}{p(t, x, y)} \right] - \mathbb{E}_x \left[ F_m^q(\mathbb{Y}, X_s^m) \frac{p_{M_m}(t - s, X_s^m, y)}{p_{M_m}(t, x, y)} \right] \right| \leq |I_1^m(s, t) - J_1^m(s, t)| + |J_1^m(s, t)| + |J_2^m(s, t)| \leq C \alpha_m e^{-\frac{2L}{r}},$$

which is (6.13) and we have finished the proof. □

**Remark 6.1.** By the same arguments in the proof for (6.13) we could obtain the following under the conditions of Lemma 6.4: For sufficiently large $m$ we can find a positive number $t_0(K, L, m)$ so that for every $x, y \in K$, $0 < s \leq \frac{1}{2}$, and $0 < t \leq t_0$ the following estimates hold, replacing $l_m$ by $l_m'$, or $dB_r$ by $dr$.

$$\left\| \mathbb{E}_x \left[ \left( \int_0^s \mathbb{Y}_r l_m(r, X.) \, dB_r \right)^q \left( \frac{p(t - s, X_s, y)}{p(t, x, y)} - \frac{p_{M_m}(t - s, X_s^m, y)}{p_{M_m}(t, x, y)} \right) \right] \right\| \leq C(m) e^{-\frac{L}{2}},$$

(6.21)

$$\left\| \mathbb{E}_x \left[ \left( \int_0^s \mathbb{Y}_r l''_m(r, X.) \, dr \right)^q \left( \frac{p(t - s, X_s, y)}{p(t, x, y)} - \frac{p_{M_m}(t - s, X_s^m, y)}{p_{M_m}(t, x, y)} \right) \right] \right\| \leq C(m) e^{-\frac{L}{2}},$$

(6.22)
For each $k > m$ non-compact complete $M$

Proof. When $\Theta$ defined in (6.2) with $\pi$ given by (3.2), with the manifold $M$ is compact, (6.24) is just (4.11) established in Proposition 4.4. For general $k > m$, let $\{U^k_t\}_{t \geq 0}$ be the horizontal Brownian motion on compact manifold $\tilde{M}_k$ as defined in (6.2) with $\pi(U^k_t) = x \in D_m$. Set $X^k_t = \pi(U^k_t)$ and $P^k_t f(x) = E_x \left[ f(X^k_t) \right]$. Let

$$h(s) = \left( \frac{t - 2s}{t} \right)^+ l_m(s, X) \cdot U_0^{-1} v.$$ (6.25)

According to (6.3), precisely $\tau_m = \tau^k_m$ and $h(s) \neq 0$ if and only if when $s \leq \frac{t}{2} \wedge \tau_m$. So furthermore

$$h(s) = \left( \frac{t - 2s}{t} \right)^+ l_m(s, X^k) \mathbf{1}_{\{s < \frac{t}{2} \wedge \tau_m\}} \cdot U_0^{-1} v, \forall k > m,$$

$$h'(s) = \left( -\frac{2}{t} l_m(s, X^k) + l'_m(s, X^k) \left( \frac{t - 2s}{t} \right)^+ \right) \mathbf{1}_{\{s < \frac{t}{2} \wedge \tau_m\}} \cdot U_0^{-1} v, \forall k > m,$$

which means that we can replace $X$ by $X^k$ in the expression of $h(s)$ and $h'(s)$. Let $\Theta^{h,k}_s$ be given by (3.2), with the manifold $M$ replaced by $\tilde{M}_k$ (associated with $X^k$). Therefore by (3.2) we have the following expression

$$\Theta^{h,k}_s = h'(s) + \text{Ric}_{\tilde{M}_k}(h(s)) = h'(s) + \text{Ric}_M(h(s)) = \Theta^h_s, \forall k > m.$$ (6.26)

Here both sides of (6.26) vanish for $s > \tau_m$, meanwhile we have used the fact that $U_s = U^k_s$ when $s < \tau_m$ and $\text{Ric}_{\tilde{M}_k} = \text{Ric}_z$ for every $z \in D_m$ and

$$\text{Ric}_{\tilde{M}_k}(h(s)) = \text{Ric}_{U^k_z}(h(s)) \mathbf{1}_{\{s < \tau_m\}} = \text{Ric}_{U_s}(h(s)), \forall k > m.$$
Moreover, we observe that for any compact set $K \subset D_m$ and $q > 0$,
\[
\sup_{x \in K} \mathbb{E}_x \left[ \int_0^{1 \wedge \tau_m} |\text{ric}_U(U_0^{-1} v)|^q \, ds \right] \leq |v|^q \sup_{x \in K} \mathbb{E}_x \left[ \int_0^{1 \wedge \tau_m} |\text{ric}_U \mathbf{1}_{\{s < \tau_m\}}|^q \, ds \right] \leq |v|^q \sup_{z \in D_m} \|\text{ric}(z)\|^q < \infty.
\]
(6.27)

Combining this with (5.3) and the fact that $h(s) \neq 0$ only if $s \leq t \wedge \tau_m = t \wedge \tau_m^k$ yields immediately that
\[
\sup_{x \in D_m, v \in T_x M, |v| = 1} \mathbb{E}_x \left[ \int_0^t |\Theta^h_s|^2 \, ds \right] < \infty, \quad \forall \ t > 0.
\]
(6.28)

Thus, applying Proposition 4.4 to $P_t^k f$ (note that $\mathcal{M}_k$ is compact) and using (6.26) we obtain that for all $v \in T_x M$,
\[
\langle \nabla P_t^k f(x), v \rangle_{T_x M} = -\mathbb{E}_x \left[ f(X^k_t) \int_0^t \langle \Theta^h_s, dB_s \rangle \right] = -\mathbb{E}_x \left[ f(X^k_t) \int_0^t \langle \Theta^h_s, dB_s \rangle \right].
\]
(6.29)

For any function $\psi \in C^\infty_c(M)$ and vector field $V \in C^\infty_c(M; TM)$ with supports in $D_m$ satisfying that $|V(x)| \leq 1$ for all $x \in D_m$, we can use $\nabla$ and $dx$ for the the gradient operator and the Riemannian volume measure on both manifolds $M$ and $\mathcal{M}_k$, so we have
\[
\int_M \mathbb{E}_x \left[ f(X^k_t) \int_0^t \langle \Theta^h_s, dB_s \rangle \right] \psi(x) \, dx = \int_M \langle \nabla P_t^k f(x), V(x) \rangle_{T_x M} \psi(x) \, dx = -\int_M \mathbb{E}_x \left[ f(X^k_t) \right] \text{div}(V \psi)(x) \, dx.
\]
(6.30)

Here $h(x)$ is defined by (6.25) with $v = V(x)$.

Meanwhile note that $X_t = X^k_t$ if $t < \tau_k$, for every $x \in D_m$ it holds
\[
\lim_{k \to \infty} \mathbb{E}_x \left[ f(X^k_t) \int_0^t \langle \Theta^h_s, dB_s \rangle \right] = -\mathbb{E}_x \left[ f(X_t) \int_0^t \langle \Theta^h_s, dB_s \rangle \mathbf{1}_{\{t < \zeta\}} \right] \leq \lim_{k \to \infty} \mathbb{E}_x \left[ \left| f(X^k_t) - f(X_t) \right| \mathbf{1}_{\{t < \zeta\}} \right] \int_0^t \langle \Theta^h_s, dB_s \rangle \right] \leq \lim_{k \to \infty} \sqrt{\mathbb{E}_x \left[ |f(X^k_t) - f(X_t)\mathbf{1}_{\{t < \zeta\}}|^2 \right]} \sqrt{\mathbb{E}_x \left[ \int_0^t \langle \Theta^h_s, dB_s \rangle \right]^2} \leq \lim_{k \to \infty} \sqrt{2C} \|f\|_\infty \sqrt{\mathbb{E}_x (\tau_k \leq t < \zeta)} = 0,
\]
where
\[
C := \sup_{x \in D_m, v \in T_x M, |v| = 1} \mathbb{E}_x \left[ \int_0^t |\Theta^h_s|^2 \, ds \right].
\]

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is finite for every $t > 0$, which is due to (6.28). With this we may take $k \to \infty$ in (6.30), then
\[
\mathbb{E}_x \left[ f \left( X^t \right) \int_0^t \langle \Theta^h_s(x), dB_s \rangle \right] \to \mathbb{E}_x \left[ f \left( X^t \right) \int_0^t \langle \Theta^h_s(x), dB_s \rangle 1_{\{t < \zeta\}} \right]
\]
and consequently
\[
\int_{D_m} \mathbb{E}_x \left[ f \left( X^t \right) \int_0^t \langle \Theta^h_s(x), dB_s \rangle 1_{\{t < \zeta\}} \right] \psi(x) dx = \int_{D_m} \mathbb{E}_x \left[ f \left( X^t \right) 1_{\{t < \zeta\}} \right] \text{div}(V\psi)(x) dx.
\]
Since $m$ is arbitrary, so it follows that for all test vector fields $V \in C_c^\infty(M; TM)$ and test functions $\psi \in C_c^\infty(M)$,
\[
\int_M \mathbb{E}_x \left[ f \left( X^t \right) \int_0^t \langle \Theta^h_s(x), dB_s \rangle 1_{\{t < \zeta\}} \right] \psi(x) dx = \int_M P_tf(x) \text{div}(V\psi)(x) dx,
\]
which means that the weak (distributional) gradient $\nabla P_tf$ exists
\[
\langle \nabla P_tf(x), V(x) \rangle_{T_x M} = \mathbb{E}_x \left[ f \left( X^t \right) \int_0^t \langle \Theta^h_s(x), dB_s \rangle 1_{\{t < \zeta\}} \right], \quad x \in M.
\]
According to the same arguments in the proof of Lemma 7.2 in the Appendix, the functional $x \mapsto \mathbb{E}_x \left[ f \left( X^t \right) \int_0^t \langle \Theta^h_s(x), dB_s \rangle 1_{\{t < \zeta\}} \right]$ is continuous. So we have verified that the distributional derivative $\nabla P_tf$ exists and is continuous, then $\nabla P_tf$ is the classical gradient and expression (6.24) holds.

Now we present an estimate for the difference between the gradients of logarithmic heat kernels. Note that for every $x, y \in K \subset B_o(2m - 2) \subset D_m$, we could view $\nabla_x \log p(t, x, y)$ and $\nabla_x \log p_{M_m}(t, x, y)$ as vectors in $T_x M$, so that
\[
\| \nabla_x \log p(t, x, y) - \nabla_x \log p_{M_m}(t, x, y) \|_{T_x M} \leq C(m) e^{-\frac{L}{t}}
\]
is well defined.

Let $\partial$ be the cemetery point. We make the convention that $p(t, \partial, y) = 0$ for all $t$.

**Proposition 6.6.** Suppose that $K$ is a compact subset of $M$ and $L > 1$ is a positive number. Then for every sufficiently large $m$, we could find a number $t_0(K, L, m)$, depending on $K, L, m$, such that for every $0 < t \leq t_0$,
\[
\sup_{x, y \in K} \left| \nabla_x \log p(t, x, y) - \nabla_x \log p_{M_m}(t, x, y) \right|_{T_x M} \leq C(m) e^{-\frac{L}{t}}
\]
where $C(m)$ is a positive constant, which may depend on $m$.

**Proof.** Let us fix points $x, y \in K$ and a unit vector $v \in T_x M$. Let $m$ be a natural number such that $B_o(2m - 2) \subset K$. Let $t > 0$ be fixed.

By (6.24) wherein $\Theta^h = h'(t) + \frac{1}{2} \text{ric}_{U_t}(h(t))$, we have, for every $f \in C_c(M)$,
\[
\langle \nabla P_tf(x), v \rangle_{T_x M} = \frac{2}{t} \mathbb{E}_x \left[ \int_0^t \langle l_m(s)U_0^{-1}v, dB_s \rangle f \left( X^t \right) 1_{\{t < \zeta\}} \right] - \mathbb{E}_x \left[ \int_0^t \langle \left( \frac{t - 2s}{t} \right) l_m(s)U_0^{-1}v, dB_s \rangle f \left( X^t \right) 1_{\{t < \zeta\}} \right] - \frac{1}{2} \mathbb{E}_x \left[ \int_0^t \langle \text{ric}_{U_t} \left( \left( \frac{t - 2s}{t} \right) l_m(s)U_0^{-1}v \right), dB_s \rangle f \left( X^t \right) 1_{\{t < \zeta\}} \right]
\]

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Since \( f \) has compact support, the indicator function \( 1_{\{t<\zeta\}} \) can be removed. Taking the conditional expectation on \( \sigma(X_t) \), we obtain, for all \( x \in M \) and almost everywhere \( y \in M \) (with respect to volume measure on \( M \)),

\[
\begin{align*}
t\langle \nabla_x \log p(t, x, y), v \rangle_{T_x M} &= t \langle \nabla_x p(t, x, y), v \rangle_{T_x M} = 2Ex \left[ 1_{\{t<\zeta\}} \int_0^{\frac{t}{2}} \langle l_m(s)U_0^{-1}v, dB_s \rangle \bigg| X_t = y \right] \\
&\quad - tEx \left[ 1_{\{t<\zeta\}} \int_0^{\frac{t}{2}} \langle \left( \frac{t - 2s}{t} \right) l'_m(s)U_0^{-1}v, dB_s \rangle \bigg| X_t = y \right] \\
&\quad - \frac{t}{2}Ex \left[ 1_{\{t<\zeta\}} \int_0^{\frac{t}{2}} \langle \text{ric}_{U_t}(\frac{t - 2s}{t})l_m(s)U_0^{-1}v, dB_s \rangle \bigg| X_t = y \right] \\
&= Ex \left[ 1_{\{t<\zeta\}} \int_0^{\frac{t}{2}} g_m(s)\langle U_0^{-1}v, dB_s \rangle \bigg| X_t = y \right].
\end{align*}
\]

where

\[
g_m(s) := 2l_m(s) - t \left( \frac{t - 2s}{t} \right) l'_m(s) - \frac{t}{2} \text{ric}_{U_t}(\frac{t - 2s}{t})l_m(s).
\]

Thus,

\[
\begin{align*}
t\langle \nabla_x \log p(t, x, y), v \rangle_{T_x M} &= Ex \left[ 1_{\{t<\zeta\}} \int_0^{\frac{t}{2}} g_m(s)\langle U_0^{-1}v, dB_s \rangle \bigg| X_t = y \right] \\
&= Ex \left[ \int_0^{\frac{t}{2}} g_m(s)\langle U_0^{-1}v, dB_s \rangle \frac{p \left( \frac{t}{2}, X_{\frac{t}{2}}, y \right)}{p(t, x, y)} 1_{\{\frac{t}{2}<\zeta\}} \right] \\
&= Ex \left[ \int_0^{\frac{t}{2}} g_m(s)\langle U_0^{-1}v, dB_s \rangle \frac{p \left( \frac{t}{2}, X_{\frac{t}{2}}, y \right)}{p(t, x, y)} \right].
\end{align*}
\]

We have used the property that for \( p \left( \frac{t}{2}, X_{\frac{t}{2}}, y \right) = 0 \) whenever \( \frac{t}{2} \geq \zeta(x) \).

Based on the heat kernel estimates in the previous lemmas, by the proof of Lemma 7.2 we know immediately

\[
x \mapsto Ex \left[ \int_0^{\frac{t}{2}} g_m(s)\langle U_0^{-1}v, dB_s \rangle \frac{p \left( \frac{t}{2}, X_{\frac{t}{2}}, y \right)}{p(t, x, y)} \right]
\]

is continuous. So the expression above is true for all \( x, y \in M \).

Since \( l'_m(s, X^m) = l'_m(s, X) \) and \( l_m(s, X^m) = l_m(s, X) \) for \( s < \tau_m \) and \( X_s = X_s^m \), applying the same arguments above to \( M_m \) we have

\[
\langle t\nabla \log p_{M_m}(t, x, y), v \rangle_{T_x M} = Ex \left[ \int_0^{\frac{t}{2}} g_m(s)\langle U_0^{-1}v, dB_s \rangle \frac{p_{M_m} \left( \frac{t}{2}, X_{\frac{t}{2}}, y \right)}{p_{M_m}(t, x, y)} \right].
\]

To apply Lemma 6.4 it remains to make moment estimates for \( \int_0^{t} g(s)\langle v, U_0 dB_s \rangle \). For any \( m \in \mathbb{N} \) large enough and \( q > 0 \), (6.27) implies that condition (6.12) in Lemma 6.4 holds for the process \( T_t = \text{ric}_{U_t} \) and we could apply (6.21) and (6.13) to conclude the estimates.

\[\square\]
We are now in a position to proceed to prove the gradient estimates for $\log p(t, x, y)$.

**Theorem 6.7.** The following statements hold.

(1) Suppose $x, y \in M$ and $x \notin \text{Cut}_M(y)$, then

$$
\lim_{t \downarrow 0} t \nabla_x \log p(t, x, y) = -\nabla_x \left( \frac{d^2(x, y)}{2} \right).
$$

(6.35)

Here the convergence is uniformly in $x$ on any compact subset of $M \setminus \text{Cut}_M(y)$.

(2) Let $K$ be a compact subset of $M$. Then there exists a positive constant $C(K)$, which may depend on $K$, such that

$$
|\nabla_x \log p(t, x, y)|_{T_x M} \leq C(K) \left( \frac{d(x, y)}{t} + \frac{1}{\sqrt{t}} \right), \quad x, y \in K, \quad t \in (0, 1].
$$

(6.36)

**Proof.** In the proof, the constant $C$ (which depends on $K$ or $K$) may change from line to line. For every $m \in \mathbb{N}$ with $K \subset B_m(2m - 2) \subset D_m$, we have

$$
t \nabla_x \log p(t, x, y) = t \nabla_x \log p_{\tilde{M}_m}(t, x, y) + \left( t \nabla_x \log p(t, x, y) - t \nabla_x \log p_{\tilde{M}_m}(t, x, y) \right).
$$

(6.37)

For each compact set $\tilde{K} \subset M \setminus \text{Cut}_M(y)$, by (6.31) we could choose a $m_0 \in \mathbb{N}$ large enough such that $\tilde{K} \subset B_{m_0}(2m_0 - 2) \subset D_{m_0}$ and

$$
\lim_{t \downarrow 0} \sup_{x \in \tilde{K}} |t \nabla_x \log p(t, x, y) - t \nabla_x \log p_{m_0}(t, x, y)|_{T_x M} = 0.
$$

At the same time, since $\tilde{M}_{m_0}$ is compact and $\tilde{K}$ is outside of the cut locus $\text{Cut}_{\tilde{M}_m}(y)$, we have

$$
\lim_{t \downarrow 0} \sup_{x \in \tilde{K}} \left| t \nabla_x \log p_{\tilde{M}_{m_0}}(t, x, y) + \nabla_x \left( \frac{d^2(x, y)}{2} \right) \right|_{T_x M} = 0.
$$

(6.35)

In the first step, we used that $d_{\tilde{M}_{m_0}}(x, y) = d(x, y)$ for $x, y \in \tilde{K}$, while the second step is due to Corollary 2.29 from Malliavin and Stroock [59], (see also Bismut [12] and Norris [62]). Plugging this into (6.37) with $m = m_0$, then we have shown (6.35).

Given a compact set $K \subset M$ and a constant $L > 1$, based on (6.31) there exists a sufficiently large natural number $m_0$ such that $K \subset B_m(2m_0 - 2) \subset D_{m_0}$ and and $t_0 \in (0, 1)$ such that

$$
\sup_{x, y \in K} |I^{m_0}(t, x, y)|_{T_x M} \leq C e^{-\frac{t}{t_0}}, \quad \forall \ t \in (0, t_0],
$$

(6.38)

Since $\tilde{M}_{m_0}$ is compact, we can apply Hsu [44, Theorem 5.5.3] or Sheu [64] to show that for all $x, y \in K$ and $t \in (0, 1]$,

$$
|\nabla_x \log p_{m_0}(t, x, y)|_{T_x M} \leq C(K) \left( \frac{d_{m_0}(x, y)}{t} + \frac{1}{\sqrt{t}} \right).
$$

(6.39)
Furthermore it holds that Proposition 4.4, we have established a second order gradient formula for $P$.

Now we can prove the claim for the second order gradient of logarithmic heat kernel. In 6.3 Proof of Theorem 3.1 and the main theorem: Hessian estimates

By now we have completed the proof of (6.36).

Remark:

(1) The gradient estimate (6.36) was proved in [66, 64, 44] for a complete manifold with Ricci curvature bounded from below by a constant $C_0$. In that case, the constant $C(K)$ in (6.36) in uniform and only depends on $C_0$, see also [49] for the case of the estimates for heat kernel associated with the Witten Laplacian operator.

(2) By carefully tracking the proof, we know the constant $C(K)$ from (6.36) depends only on $C_1(m_0)$, $\inf_{x \in D_{m_0}} \|\text{Ric}_x\|$, and $\sup_{x \in D_{m_0}} \mathbb{E}_x \int_0^1 |\nu_{m_0}(s)|^2 \, ds$, where $C_1(m_0)$ is the positive constant such that

$$|\nabla_x \log p_{M_{m_0}}(t, x, y)|_{T_x M} \leq C_1(m_0) \left( \frac{d_{M_{m_0}}(x, y)}{t} + \frac{1}{\sqrt{t}} \right).$$

$$= C_1(m_0) \left( \frac{d(x, y)}{t} + \frac{1}{\sqrt{t}} \right).$$

6.3 Proof of Theorem 3.1 and the main theorem: Hessian estimates

Now we can prove the claim for the second order gradient of logarithmic heat kernel. In Proposition 4.4, we have established a second order gradient formula for $P_t f$ on a compact manifold. In its proof we exchanged the differential and the integral operators several times, which may not hold if $M$ is not compact. So it is not trivial to extend Proposition 4.4 to a non-compact manifold.

To prove Theorem 3.1, we begin with comparing the terms in $\nabla^2 P_t^k$ and $\nabla^2 P_t$.

Lemma 6.8. Given a $x \in M$ and $a \in T_x M$, suppose that $m$ is sufficiently large so $x \in D_m$ and $k > m$. Let $\{U_t^k\}_{t \geq 0}$ be the horizontal Brownian motion on $\tilde{M}_k$ as defined in (6.2). Set $X_t^k = \pi(U_t^k)$ and $P_t^k f(x) = \mathbb{E}_x [f(X_t^k)]$. Let $h(s) = (\frac{\|\nu_{m_0}(s)\|}{t})^\frac{1}{2} \cdot l_m(s, X) \cdot U_0 \cdot v$ and define

$$I \left( \frac{t}{2}, X, v \right) := \left( \int_0^{\frac{t}{2}} \langle \Theta_s^h, dB_s \rangle \right)^2 - \int_0^{\frac{t}{2}} \langle \Lambda_s^h, dB_s \rangle - \int_0^{\frac{t}{2}} \left| \Theta_s^h \right|^2 \, ds \quad (6.40)$$

Let $I \left( \frac{t}{2}, X^k, v \right)$ be defined with the corresponding terms in $\tilde{M}_k$. Then we have

$$I \left( \frac{t}{2}, X^k, v \right) = I \left( \frac{t}{2}, X, v \right) = \left( \int_0^{\frac{t}{2}} \langle \Theta_s^h, dB_s \rangle \right)^2 - \int_0^{\frac{t}{2}} \langle \Lambda_s^h, dB_s \rangle - \int_0^{\frac{t}{2}} \left| \Theta_s^h \right|^2 \, ds. \quad (6.41)$$

Furthermore it holds that

$$\sup_{x \in D_m, v \in T_x M, |v| = 1} \mathbb{E}_x \left[ I \left( \frac{t}{2}, X, v \right) \cdot |v| \right] < \infty, \quad \forall t > 0. \quad (6.42)$$
4.4 to the compact manifold

\[ l \]

Let \( \Theta_{s}^{h,k}, \Gamma_{s}^{h,k}, \Lambda_{s}^{h,k} \) be the corresponding terms of \( \Theta_{s}^{h}, \Gamma_{s}^{h}, \Lambda_{s}^{h} \) defined on \( \tilde{M}_k \). By (6.26) we have

\[ \Theta_{s}^{h,k} = h'(s) + \text{ric}_{U^h_k}(h(s)) = h'(s) + \text{ric}_{U^h}(h(s)) = \Theta_{s}^{h}, \forall \, k > m. \quad (6.43) \]

Still based on (3.2), (3.3) and the same arguments for (6.43) we can obtain that

\[ \Gamma_{s}^{h,k} = \Gamma_{s}^{h}, \Lambda_{s}^{h,k} = \Lambda_{s}^{h}, \forall \, k > m. \]

Therefore the term \( I \left( \frac{1}{2}, X^k, v \right) \) in (6.40) is independent of \( k \) and the required identity (6.41) holds. Finally, (6.42) immediately follows from the moment estimates (5.3) for \( l_m^* \) and the same arguments for (6.27). \qed

We can now begin the Proof of Theorem 3.1. The idea of the proof is similar to that of Lemma 6.5. For convenience of the reader, here we provide a detailed proof. Let \( m_0 \in \mathbb{N} \) satisfy that \( x \in D_{m_0+1} \), then for every \( k > m > m_0 \) it holds that \( B_0(2m - 2) \subset D_m \subset D_k \). Let \( h(s) = \left( \frac{1 - 2s}{t} \right)^2 \cdot l_m(s, X^k) \cdot U_0^{-1} v = \left( \frac{1 - 2s}{t} \right)^2 \cdot l_m(s, X^k) \cdot U_0^{-1} v \). We can apply (4.12) in Proposition 4.4 to the compact manifold \( \tilde{M}_k \) to obtain that for every \( k > m \),

\[
\langle \nabla^2 P^k f(x), v \otimes v \rangle_{T_x \tilde{M} \otimes T_x \tilde{M}} \\
= \mathbb{E}_x \left[ f(X^k) \left( \int_0^t (\Theta_{s}^{h,k}, dB_s) - \int_0^t (\Lambda_{s}^{h,k}, dB_s) - \int_0^t \left| \Theta_{s}^{h,k} \right|^2 ds \right) \right] \\
= \mathbb{E}_x \left[ f(X^k) I \left( \frac{t}{2}, X^k, v \right) \right] = \mathbb{E}_x \left[ f(X^k) I \left( \frac{t}{2}, X, v \right) \right],
\]

where the process \( \Theta_{s}^{h,k}, \Lambda_{s}^{h,k} \) are defined by (3.2), (3.3) on \( \tilde{M}_k \), and in the last step we have applied (6.41).

According to (6.40) and integration by parts formula (on compact manifold \( \tilde{M}_k \)), for any \( \psi \in C^\infty_c(\tilde{M}), V \in C^\infty_c(\tilde{M}; TM) \) with \( \text{supp} \psi \cup \text{supp} V \subset D_m \) we have

\[
\int_M \mathbb{E}_x \left[ f \left( X^k_t \right) I \left( \frac{t}{2}, X, V(x) \right) \right] \psi(x) dx \\
= \int_M \langle \nabla^2 P^k f(x), V(x) \otimes V(x) \rangle_{T_x \tilde{M} \otimes T_x \tilde{M}} \psi(x) dx \\
= \int_{\tilde{M}_k} \langle \nabla^2 P^k f(x), V(x) \otimes V(x) \rangle_{T_x \tilde{M} \otimes T_x \tilde{M}} \psi(x) dx \\
= \int_{\tilde{M}_k} \mathbb{E}_x \left[ f \left( X^k_t \right) \Psi(\psi, V)(x) \right] dx = \int_M \mathbb{E}_x \left[ f \left( X^k_t \right) \right] \Psi(\psi, V)(x) dx.
\]

Here we denote the gradient operator and Riemannian volume measure on both \( M \) and \( \tilde{M}_k \) by \( \nabla \) and \( dx \), and we set

\[
\Psi(\psi, V)(x) := \text{div}(\text{div}(V \psi) V)(x) + \text{div}(\psi \nabla V) V(x) \\
= \psi(x) \left( \nabla (\nabla V) + (\text{div} V)^2 + \langle V, \nabla \text{div} V \rangle_{T_x \tilde{M}} \right)(x) \\
+ 2 \langle \nabla \psi, \nabla V + (\text{div} V) V \rangle_{T_x \tilde{M}}(x) + \langle \nabla^2 \psi(x), V(x) \otimes V(x) \rangle_{T_x \tilde{M}}.
\]
The second and last step above follow from the properties that Riemannian volume measure $dx$ and the second order gradient operator $\nabla^2$ on $M$ are the same as that on $\tilde{M}_k$, when they are restricted on $D_m$, the third equality is due to the integration by parts formula. Meanwhile note that $X_t = X_t^k$ if $t < \tau_k$, for every $x \in D_m$, it holds

\[
\lim_{k \to \infty} \left| \mathbb{E}_x \left[ f(X_t^k) I \left( \frac{t}{2}, X, V(x) \right) \right] - \mathbb{E}_x \left[ f(X_t) I \left( \frac{t}{2}, X, V(x) \right) \mathbf{1}_{\{t < \zeta\}} \right] \right| \leq \lim_{k \to \infty} \mathbb{E}_x \left[ \left| f(X_t^k) - f(X_t) \mathbf{1}_{\{t < \zeta\}} \right| \left| I \left( \frac{t}{2}, X, V(x) \right) \right| \right] \leq \lim_{k \to \infty} \sqrt{\mathbb{E}_x \left[ \left| f(X_t^k) - f(X_t) \mathbf{1}_{\{t < \zeta\}} \right|^2 \right]} \sqrt{\mathbb{E}_x \left[ \left| I \left( \frac{t}{2}, X, V(x) \right) \right|^2 \right]} \leq \lim_{k \to \infty} \sqrt{2C\|f\|_\infty} \sqrt{\mathbb{P}_x (\tau_k \leq t < \zeta)} = 0,
\]

where the last inequality is due to (6.42).

Putting this into (6.45), letting $k \to \infty$ we see that for every $\psi \in C_c^\infty(M)$ and $V \in C^\infty(M; TM)$ with $\text{supp} \psi \cup \text{supp} V \subset D_m$,

\[
\int_{D_m} \mathbb{E}_x \left[ f(X_t) I \left( \frac{t}{2}, X, V(x) \right) \mathbf{1}_{\{t < \zeta\}} \right] \psi(x) dx = \int_{D_m} \mathbb{E}_x \left[ f(X_t) \mathbf{1}_{\{t < \zeta\}} \right] \Psi(\psi, V(x)) dx,
\]

which implies the weak (distributional) second order gradient $\nabla^2 P_t f$ exists on $D_m$ and

\[
\left\langle \nabla^2 P_t f(x), v \otimes v \right\rangle_{T_x M \otimes T_x M} = \mathbb{E}_x \left[ f(X_t) I \left( \frac{t}{2}, X, v \right) \mathbf{1}_{\{t < \zeta\}} \right], \quad x \in D_m, v \in T_x M.
\]

(6.46)

As shown by Lemma 7.2 in Appendix, the functional $x \mapsto \mathbb{E}_x \left[ f(X_t) I \left( \frac{t}{2}, X, V(x) \right) \mathbf{1}_{\{t < \zeta\}} \right]$ is continuous. Now that the distributional derivative $\nabla^2 P_t f$ exists and is continuous, then $\nabla^2 P_t f$ is the classical second order gradient on $D_m$ and expression (3.4) holds.

**Proposition 6.9.** Suppose that $K$ is a compact subset of $M$ and $L > 1$ is a positive constant. Then, for any sufficiently large $m$, we could find a $t_0(K, L, m)$ such that for any $t \in (0, t_0]$,

\[
\sup_{x, y \in K} e^{\frac{t}{2}} \left| t \nabla^2 \log p(t, x, y) - t \nabla^2 \log p_{\tilde{M}_m}(t, x, y) \right|_{T_x M \otimes T_x M} \leq C(m)e^{-\frac{t}{2}} \quad (6.47)
\]

where $C(m)$ is a positive constant which may depend on $m$.

**Proof.** Let us fix $x, y \in K$ and a unit vector $v \in T_x M$ with $|v| = 1$. Suppose that $m \in \mathbb{N}$ such that $K \subset B_0(2m - 2) \subset D_m$. Then by (3.4) we have

\[
\left\langle \nabla^2 P_t f(x), v \otimes v \right\rangle_{T_x M \otimes T_x M} = \mathbb{E}_x \left[ f(X_t) I \left( \frac{t}{2}, X, v \right) \mathbf{1}_{\{t < \zeta\}} \right],
\]

where $I \left( \frac{t}{2}, X, v \right)$ is defined by (6.40) with $h(s) := \left( \frac{t-2s}{t} \right)^+ \cdot l_m(s, X) \cdot U_0^{-1} v$.

By this representation and following the same arguments of (6.33) and (6.34) we obtain

\[
\left\langle \nabla^2 P_t f(x), v \otimes v \right\rangle_{T_x M \otimes T_x M} = \mathbb{E}_x \left[ I \left( \frac{t}{2}, X, v \right) \frac{p \left( \frac{t}{2}, X, y \right)}{p(t, x, y)} \right],
\]

\[
\left\langle \nabla^2 P_t f(x), v \otimes v \right\rangle_{T_x M \otimes T_x M} = \mathbb{E}_x \left[ I \left( \frac{t}{2}, X, v \right) \frac{p_{\tilde{M}_m} \left( \frac{t}{2}, X, y \right)}{p_{\tilde{M}_m}(t, x, y)} \right].
\]

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Based on above expression and following the same arguments in the proof of Proposition 6.6 (especially applying (6.13) and (6.21)–(6.23)) we could find a $m_0(K, L) \in \mathbb{N}$ such that for all $m \geq m_0$, there exists a $t_0(K, L, m) > 0$ such that

$$\sup_{x, y \in K} \left| \frac{\nabla^2_p(t, x, y)}{p(t, x, y)} - \frac{\nabla^2_{p_{M_m}}(t, x, y)}{p_{M_m}(t, x, y)} \right|_{T_x M \otimes T_y M} \leq C(m)e^{-\frac{t}{4}}, \quad t \in (0, t_0]. \quad (6.48)$$

Meanwhile we have

$$\langle \nabla^2_x \log p(t, x, y), v \otimes v \rangle_{T_x M \otimes T_y M} = \frac{\langle \nabla^2_x p(t, x, y), v \otimes v \rangle_{T_x M \otimes T_y M}}{p(t, x, y)} - \left( \frac{\langle \nabla x p(t, x, y), v \rangle_{T_x M}}{p(t, x, y)} \right)^2,$$

and the similar expression holds for $\langle \nabla^2_x \log p_{M_m}(t, x, y), v \otimes v \rangle_{T_x M \otimes T_y M}$. Together with (6.31) and (6.48), this yields (6.47) and concludes the proof.

With (6.47) we are in the position to prove the second part of the main theorem on the short time and asymptotic second order gradient estimates.

**Theorem 6.10.** The following statements hold.

1. Suppose $y \in M$ and $\bar{K} \subset M \setminus \text{Cut}_M(y)$ is a compact set, then

$$\lim_{t \downarrow 0} \sup_{x \in \bar{K}} \left| t \nabla^2_x \log p(t, x, y) + \nabla^2_x \left( \frac{d^2(x, y)}{2} \right) \right|_{T_x M \otimes T_y M} = 0. \quad (6.49)$$

2. Let $K$ be a compact set of $M$. For every $y \in M$ there exist positive constants $t_0(y, K)$ and $C_1(y, K)$, such that

$$|t \nabla^2_x \log p(t, x, y) + I_{T_x M}|_{T_x M \otimes T_y M} \leq C_1 \left( d(x, y) + \sqrt{t} \right), \quad x \in K, \ t \in (0, t_0], \quad (6.50)$$

where $I_{T_x M}$ is the identical map on $T_x M$.

3. Suppose $K \subset M$ is a compact subset of $M$, then there exists a positive constant $C_2(K)$, such that

$$|\nabla^2_x \log p(t, x, y)|_{T_x M \otimes T_y M} \leq C_2 \left( \frac{d^2(x, y)}{t^2} + \frac{1}{t} \right), \quad x, y \in K, \ t \in (0, 1]. \quad (6.51)$$

**Proof.** By Malliavin and Stroock [59, Corollary 2.29], Gong and Ma [35, Theorem 3.1] and Stroock [66] (or Sheu [64]), we know (6.49)–(6.51) hold when $M$ is compact. Then using the estimates (6.47) and following the same procedure as in the proof of Theorem 6.7 we can verify that (6.49)-(6.51) hold for any complete Riemannian manifold.

### 7 Appendix

Let $(M, g)$ and $D_m \subset M$ be the same terms in Section 5.

**Lemma 7.1.** For every $m \in \mathbb{Z}_+$, there exists a (smooth) compact Riemannian manifold $(M_m, \tilde{g}_m)$, such that $(D_m, g)$ is isometrically embedded into $(M_m, \tilde{g}_m)$ as an open set. In particular, if $y, x \in D_m$ and $x \notin \text{cut}_y(M)$, then $x \notin \text{cut}_y(M_m)$.
Proof. Let $G_m = D_{m+1}$, recall that $\partial G_m$ is a connected smooth $n - 1$-dimensional submanifold of $M$. Hence $\overline{G_m}$ is an $n$-dimensional manifold with smooth boundary, then there exist an atlas of local charts $\{(V_i, \psi_i)\}_{i=1}^N$ of $\overline{G_m}$ such that

1. $\bigcup_{i=1}^N V_i = \overline{G_m}$;

2. For $i = 1, \ldots, N_1 \leq N$, these are charts for the interior. So $V_i \cap \partial G_m = \emptyset$ and $\psi_i : V_i \to \mathbb{B}^n := \{z \in \mathbb{R}^n ; |z| < 1\}$ is a smooth diffeomorphism for all $1 \leq i \leq N_1$;

3. For all $i > N_1$, $V_i \cap \partial G_m \neq \emptyset,$

$$\psi_i : V_i \to \mathbb{B}^{n,+} := \{z = (z_1, \ldots, z_n) \in \mathbb{R}^n ; |z| < 1, z_1 \geq 0\}$$

is a smooth diffeomorphism and $\psi_i(V_i \cap \partial G_m) = \partial \mathbb{B}^{n,+}$.

By the Whitney embedding theorem, we could embed $M$ into a (ambient) Euclidean space $\mathbb{R}^p$. Let $\tilde{G}_m$ be an identical copy of $G_m$ in $\mathbb{R}^p$ endowed with the local charts $\{(\tilde{V}_i, \tilde{\psi}_i)\}_{i=1}^N$ (which is also an identical copy of $\{(V_i, \psi_i)\}_{i=1}^N$). We define $h : \partial G_m \to \tilde{G}_m$ by $h(x) := \tilde{\psi}_i^{-1}(\psi_i(x))$, if $x \in V_i \cap \partial G_m$; $h$ is well defined and is a smooth diffeomorphism.

We glue the boundary of $G_m$ and $\tilde{G}_m$ together to get $\tilde{M}_m := (G_m \sqcup \tilde{G}_m)/\sim$, where $\sim$ is an equivalent relation such that $x \sim y$ if and only if $h(x) = y$, $x \in \partial G_m$, $y \in \partial G_m$. Then $\tilde{M}_m$ is a smooth compact manifold without boundary. In fact, $\{(U_i, \phi_i)\}_{i=1}^{N+N_1} = \{(V_i, \psi_i)\}_{i=1}^N \cup \{\tilde{V}_i, \tilde{\psi}_i)\}_{i=1}^N \cup \{\tilde{V}_i, \tilde{\psi}_i)\}_{i=1}^{N_1}$ is a local charts of $\tilde{M}_m$. Here $\tilde{V}_i = (V_i \sqcup \tilde{V}_i)/\sim$ for every $N_1 < i \leq N$ and

$$\tilde{\psi}_i(x) = \begin{cases} \psi_i(x), & \text{if } x \in V_i, \\ \phi_i(\tilde{\psi}_i(x)), & \text{if } x \in \tilde{V}_i, \end{cases}$$

where $S : \mathbb{R}^n \to \mathbb{R}^n$ is a map such that $Sx = (-x_1, x_2, \ldots, x_n)$ for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. It is easy to see $\tilde{\psi}_i : \tilde{V}_i \to \mathbb{B}^n, N_1 < i \leq N$ is a smooth diffeomorphism, and the transition map between different local charts on $\{(U_i, \phi_i)\}_{i=1}^{N+N_1}$ is smooth.

We construct a smooth Riemannian metric $\tilde{g}_m$ on $\tilde{M}_m$ to ensure that $\tilde{g}_m(z) = g(z)$ for every $z \in D_m$. For the open set $D_m \subset G_m \subset \tilde{M}_m$, by the standard procedure (via the finite local charts) we could construct a function $\chi_m : \tilde{M}_m \to [0, 1]$ such that $\text{supp} \chi_m \subset G_m$ and $\chi_m(x) = 1$ for every $x \in D_m$. Note that $G_m$ could also be viewed as an open subset of $\tilde{M}_m$, so $\tilde{g}_m(x) := g(x)\chi_m(x), x \in \tilde{M}_m$ is well defined on $\tilde{M}_m$. Fixing a smooth Riemannian metric $g^0_m$ on $\tilde{M}_m$, which exists, we set

$$\tilde{g}_m(x) := g(x)\chi_m(x) + g^0_m(x)(1 - \chi_m(x)), x \in \tilde{M}_m.$$

It is easy to see $\tilde{g}_m$ is a smooth Riemannian metric on $\tilde{M}_m$ and $\tilde{g}_m(x) = g(x)$ for each $x \in D_m$. By now we have completed the proof. \qed

Let $I(t, X_v, v)$ be as defined in (6.40).

**Lemma 7.2.** For every fixed $f \in C_c^\infty(M)$, $V \in C^\infty(M; TM)$ with compact supports and $t > 0$, the function

$$F(x) := \mathbb{E} \left[ f \left( X^x_t \right) \left( \frac{t}{2} X^x_t, V(x) \right) \mathbf{1}_{\{t < \zeta(x)\}} \right], \ x \in M,$$

is continuous.
Proof. Let $\zeta(x)$ denotes the explosion time of the solution $X_t^x$ to (3.1) with the initial value $x$. Let $U$ be a frame at $x$. Then the explosion time of the horizontal Brownian motion agree with $\xi(x)$ almost surely. So we use $\xi$ for the explosion time of both. Furthermore, by Elworthy [28], there exist a maximal solution flow $\{U_t(\cdot, \omega)\}_{0 \leq t \leq \zeta(\cdot, \omega)}$ to (3.1) such that $U_t(u, \omega)$ is the solution of (3.1) with initial value $u \in OM$, and there is a null set $\Delta$ such that for all $\omega \notin \Delta$,

1. For each $t > 0$, set $\Xi_t(\omega) := \{u \in OM : t < \zeta(\cdot, \omega)\}$, Then $\Xi_t$ is open in $OM$ (i.e. $\zeta(\cdot, \omega) : OM \to \mathbb{R}_+$ is lower semi-continuous) and $U_t(\cdot, \omega) : \Xi_t(\omega) \to OM$ is a $C^\infty$ diffeomorphism onto its image.

2. For each fixed $u \in OM$ with $\pi(u) = x$, there exists a null set $\Delta(u)$ depending on $u$, such that $\zeta(u, \omega) = \zeta(X^x)$ for each $\omega \notin \Delta(u) \cup \Delta$.

Fix a point $x_0 \in M$. For each sequence $\{x_k\}_{k=1}^\infty$ with $\lim_{k \to \infty} x_k = x_0$, we take a sequence $\{u_k\}_{k=1}^\infty$ and $U_0$ in $OM$, such that $\pi(u_k) = x_k$, $\pi(U_0) = x_0$ and $\lim_{k \to \infty} u_k = u_0$ in $OM$. Set $\tilde{\Delta} := (\bigcup_{k=0}^\infty \Delta(u_k)) \cup \Delta$. For each $k$ and $\omega \notin \tilde{\Delta}$, $\zeta(U_k, \omega) = \zeta(x_k, \omega)$. By the lower semi-continuity of $\zeta$, $\zeta(x_0) \leq \liminf_{k \to \infty} \zeta(x_k)$, hence $u_k \in \Xi_t(\omega)$ for each $t < \zeta(x_0)$ when $k$ is large enough. By the property (1) above, we have immediately

$$
\lim_{k \to \infty} U_t(u_k, \omega) 1_{\{t < \zeta(x_k)\}} = U_t(u_0, \omega) 1_{\{t < \zeta(x_0)\}}, \quad \omega \notin \tilde{\Delta}, \ t > 0.
$$

Combing this with the definition $\Theta(s, X, v)$ and the expression (5.4) of $l_m$, we see that

$$
\lim_{k \to \infty} \Theta(s, X^{x_k}, V(x)) = \Theta(s, X^{x_0}, V(x_0)), \quad s > 0. \tag{7.1}
$$

Let $h(s, X, V(x)) := \left(\frac{t}{s^2} \wedge 0\right) \cdot l_m(s, X) \cdot u_0(x)^{-1} V(x)$, where $u_0(\cdot)$ is a smooth section of $OM$ with $\pi(u_0(x)) = x$. We only need to demonstrate the proof for one of the term in $I(t, X^x, v)$, for this we set

$$
\begin{align*}
\omega(x) := & \mathbb{E} \left[ f(X_t^x) \left( \int_0^t \Theta(s, X^{x_k}, V(x_k)) \, dB_s \right) 1_{\{t < \zeta(x)\}} \right] \\
= & \mathbb{E} \left[ f(X_t^x) \left( \int_0^t \left( h(s) + \frac{1}{2} \text{ric}_U h(s) \right) \, dB_s \right) 1_{\{t < \zeta(x)\}} \right].
\end{align*}
$$

For simplicity, we only prove the continuity for the function $x \to \omega(x)$, the continuity property for the other terms in $F(x)$ could be verified similarly.

According to (5.3) we obtain

$$
\sup_{k > 0} \mathbb{E} \left[ \left| \int_0^t \Theta(s, X^{x_k}, V(x_k)) \, dB_s \right|^4 \right] < \infty.
$$

Based on this and (7.1) we have

$$
\lim_{k \to \infty} \mathbb{E} \left[ \left| \int_0^t \Theta(s, X^{x_k}, V(x_k)) \, dB_s - \int_0^t \Theta(s, X^{x_0}, V(x_0)) \, dB_s \right|^2 \right] = 0.
$$

Similarly from (7.1) we arrive at

$$
\lim_{k \to \infty} \mathbb{E} \left[ \left| f(X_t^{x_k}) 1_{\{t < \zeta(x_k)\}} - f(X_t^{x_0}) 1_{\{t < \zeta(x_0)\}} \right|^2 \right] = 0.
$$

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Therefore by Cauchy-Schwartz inequality

\[
\lim_{k \to \infty} \left| w(x_k) - w(x_0) \right|^2 \\
\leq 2 \left\| f \right\|_\infty^2 \lim_{k \to \infty} E \left[ \left| \int_0^\frac{t}{2} \langle \Theta(s, X^{x_k}, V(x_k)), dB_s \rangle - \int_0^\frac{t}{2} \langle \Theta(s, X^{x_0}, V(x_0)), dB_s \rangle \right|^2 \right] \\
+ 2 \sup_{k > 0} \left[ \left| \int_0^\frac{t}{2} \langle \Theta(s, X^{x_k}, V(x_k)), dB_s \rangle \right|^4 \right] \cdot \lim_{k \to \infty} E \left[ \left| f(X^{x_k}) \mathbf{1}_{\{t < \zeta(x_k)\}} - f(X^{x_0}) \mathbf{1}_{\{t < \zeta(x_0)\}} \right|^2 \right]
\]

\[
= 0
\]

Since \( \{x_k\}_k \) is arbitrarily chosen, \( w(\cdot) \) is continuous at \( x_0 \in M \). Again \( x_0 \) is arbitrary, so \( w(\cdot) \) is continuous on \( M \). This completes the proof for the lemma.

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