CONVERGENCE IN LAW FOR COMPLEX GAUSSIAN MULTIPLICATIVE CHAOS IN PHASE III

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Abstract. Gaussian Multiplicative Chaos (GMC) is informally defined as a random measure \( e^{\gamma X} dx \) where \( X \) is Gaussian field on \( \mathbb{R}^d \) (or an open subset of it) whose correlation function is of the form \( K(x, y) = \log \frac{1}{|x-y|} + L(x, y) \), where \( L \) is a continuous function of \( x \) and \( y \) and \( \gamma = \alpha + i\beta \) is a complex parameter. In the present paper, we consider the case \( \gamma \in \mathcal{P}_{III} \) where
\[
\mathcal{P}_{III} := \{ \alpha + i\beta : \alpha, \gamma \in \mathbb{R}, |\alpha| < \sqrt{d/2}, \alpha^2 + \beta^2 \geq d \}.
\]
We prove that if \( X \) is replaced by an approximation \( X_\varepsilon \) obtained by convolution with a smooth kernel, then the random distribution \( e^{\gamma X_\varepsilon} dx \), when properly rescaled, has an explicit nontrivial limit in law when \( \varepsilon \) goes to zero. This limit does not depend on the specific convolution kernel which is used to define \( X_\varepsilon \) and can be described as a complex Gaussian white noise with a random intensity given by a real GMC associated with parameter \( 2\alpha \).

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1. Model and results

1.1. The exponential of a log-correlated field via convolution approximation.
Given an open domain \( \mathcal{D} \subset \mathbb{R}^d \), we consider \( K : \mathcal{D}^2 \to \mathbb{R} \) to be a positive definite kernel on \( \mathcal{D} \) which admits a decomposition in the following form
\[
K(x, y) := L(x, y) + \log \frac{1}{|x-y|},
\]
where \( L \) is a continuous function. A kernel \( K \) is positive definite if for any bounded continuous function \( \rho \) with compact support on \( \mathcal{D} \)
\[
\int_{\mathcal{D}^2} K(x, y)\rho(x)\rho(y)dx dy \geq 0.
\]
We want to consider a Gaussian field \( X \) with covariance \( K \) and find a way to make sense of the distribution \( e^{\gamma X_\varepsilon} dx \) when \( \gamma = \alpha + i\beta \) is a complex parameter. Such an exponentiation a log-correlated field is what is called Gaussian Multiplicative Chaos (GMC) and has been the focus of a large amount of mathematical work first in the case \( \gamma \in \mathbb{R} \) (see [21] for a review and references) and more recently \( \gamma \in \mathbb{C} \) (see [10, 11, 12, 15, 16, 17] and references therein). The standard way to define GMC is to define the field \( X \) as a random distribution and then approximate \( X \) by random functions \( X_\varepsilon \) and consider the limit of \( e^{\gamma X_\varepsilon(x)} dx \) as \( \varepsilon \) tends to zero to define the exponential of \( X \).

Let us review this process in more details. Since \( K \) is infinite on the diagonal, it is not possible to define directly a Gaussian field indexed by \( \mathcal{D} \) with covariance function \( K \). We consider thus a distributional Gaussian field, that is, a field indexed by a set of signed
measure. Like in [1], we define $\mathcal{M}_K^+$ to be the set of positive Borel measures on $\mathcal{D}$ such that
\[
\int_{\mathcal{D}^2} |K(x,y)|\mu(dx)\mu(dy) < \infty
\] (1.3)
and let $\mathcal{M}_K$ be the space of signed measure spanned by $\mathcal{M}_K^+$
\[
\mathcal{M}_K := \{\mu_+ - \mu_- : \mu_+, \mu_- \in \mathcal{M}_K^+\}.
\] (1.4)
We define $\mathcal{K}$ as the following quadratic form on $\mathcal{M}_K$
\[
\mathcal{K}(\mu, \mu') = \int_{\mathcal{D}^2} K(x,y)\mu(dx)\mu'(dy),
\] (1.5)
and finally define $X = (\langle X, \mu \rangle)_{\mu \in \mathcal{M}_K}$ as the random field indexed by $\mathcal{M}_K$ with covariance kernel given by $\mathcal{K}$.

The distributional field $X$ can be approximated by a sequence of functional fields - that is, fields indexed by a subset of $\mathcal{D}$ by the mean of convolution with smooth kernels (we have to consider a strict subset of $\mathcal{D}$ to avoid boundary effects). Consider $\theta$ a nonnegative $C^\infty$ function whose compact support is included in $B(0,1)$ the $d$-dimensional Euclidean ball of radius one, $\int_{B(0,1)} \theta(x)dx = 1$. We define for $\varepsilon \in (0,1)$, $\theta_\varepsilon := \varepsilon^{-d}\theta(\varepsilon^{-1} \cdot)$ and consider the convoluted version of $X$ on the set
\[
\mathcal{D}_\varepsilon := \{x \in \mathcal{D} : \min_{y \in \mathbb{R}^d \setminus \mathcal{D}} |x - y| \geq 2\varepsilon\}
\] (1.6)
(or $\mathcal{D}_\varepsilon = \mathbb{R}^d$ convention if $\mathcal{D} = \mathbb{R}^d$). It is defined by
\[
X_\varepsilon(x) := \langle X, \theta_\varepsilon(x - \cdot) \rangle
\] (1.7)
where the function $\theta_\varepsilon(x - \cdot)$ is identified with the measure $\theta_\varepsilon(x - y)dy$ on $\mathcal{D}$. With this definition one can check that $X_\varepsilon(x)$ has covariance
\[
K_\varepsilon(x,y) := \mathbb{E}[X_\varepsilon(x)X_\varepsilon(y)] = \int_{\mathbb{R}^{2d}} \theta_\varepsilon(x - z_1)\theta_\varepsilon(y - z_2)K(z_1, z_2)dz_1dz_2.
\] (1.8)
We simply write $K_\varepsilon(x)$ when $x = y$. Note that since $K_\varepsilon$ is infinitely differentiable, by Kolmogorov’s criterion (see e.g. [18, Theorem 2.9]), there exists a version of $X_\varepsilon$ which is continuous in $x$. Considering this version of the field, we can make sense of integrals of measurable functionals of $X_\varepsilon(\cdot)$. This allows to define, for any $f \in C^\infty_c(\mathcal{D})$ (the set of real valued infinitely differentiable functions with compact support), the quantity
\[
M_\varepsilon^{(\gamma)}(f) := \int_{\mathbb{R}^d} f(x)e^{\gamma X_\varepsilon(x)} - \frac{\gamma^2}{2}K_\varepsilon(x)1_{\mathcal{D}_\varepsilon}(x)dx.
\] (1.9)
The restriction to $\mathcal{D}_\varepsilon$ ensures that $K_\varepsilon$ is well defined and uniformly bounded, which is convenient. It disappears when $\varepsilon$ goes to zero, since for $\varepsilon$ sufficiently small the support of $f$ is included in $\mathcal{D}_\varepsilon$. The question of focus in the present paper is the existence of a nontrivial limit of the distribution $M_\varepsilon^{(\gamma)}(\cdot)$ when $\varepsilon$ tend to zero (possibly with a rescaling by a factor depending on $\varepsilon$). Such a limit gives natural interpretation for the formal expression $e^{\gamma X}dx$.

**Remark 1.1.** Note that, even if we have chosen to omit the dependence in $\theta$ in the notation for better readability, $X_\varepsilon$ and hence $M_\varepsilon^{(\gamma)}$ depend on the choice of the convolution kernel.
1.2. The case of real GMC. The case when the parameter in the exponentiation is real (in that case we write it as $\alpha$ instead of $\gamma$) has been extensively studied, starting with the work of Kahane [14] (see for instance [1] [5] [22], we refer to the introduction in [1] for a detailed chronological account). These works established that when $\alpha \in (-\sqrt{2d}, \sqrt{2d})$ then $M^\alpha_\varepsilon$ converges to a non trivial limit. As this result, together its variant Theorem B presented in the next section, play a pivotal role in our proof so we state it in full details.

**Theorem A.** [1] Theorem 1.1] For any $\alpha \in \mathbb{R}$ with $|\alpha| < \sqrt{2d}$, then there exists a random distribution $M_0^{\alpha}$ such that for every $\theta$ and every $f \in C^\infty_c(\mathcal{D})$, we have the following convergence in $L^1$

$$\lim_{\varepsilon \to 0} M^\alpha_\varepsilon(f) = M^{\alpha}(f).$$

The distribution $M_0^{\alpha}$ is a locally finite measure whose support is dense in $\mathcal{D}$. The limit does not depend on the convolution kernel $\theta$ used in the definition of $M^\alpha_\varepsilon$.

The condition $|\alpha| < \sqrt{2d}$ is optimal : when $|\alpha| \geq \sqrt{2d}$, then $\lim_{\varepsilon \to 0} M^{\alpha}_\varepsilon(f) = 0$ for all $f$. When $\alpha = \pm \sqrt{2d}$, one can still obtain a nontrivial limit by adding a scaling factor: the measure $\sqrt{\log(1/\varepsilon)}M^\alpha_\varepsilon(\pm \sqrt{2d})$ converges in probability to a positive measure referred to as the critical multiplicative chaos (see [4] for a first derivation [9] for uniqueness of the limit and [20] for an up-to-date review). When $|\alpha| > \sqrt{2d}$, one should still obtain after an adequate rescaling a convergence a different nature, but one that only holds in law (see [19] for such a result with $X_\varepsilon$ replaced by a martingale sequence of approximation similar to the one considered in Section 2 of the present paper).

1.3. Complex white noise with random intensity given by a Real GMC. In order to describe our limit, let us introduce the notion of the complex white noise with a random intensity.

For $\gamma \in \mathcal{P}_1$, we define $\mathfrak{W}(\gamma)$ to be a complex white noise with intensity measure given by $M_0^{2\alpha}(e^{\gamma|L|^2} \cdot)$ where $\alpha \in (-\sqrt{2d}, \sqrt{2d})$. It is a random linear form which is constructed jointly with $X$, on an extended space (we let $\mathcal{P}$ denote the corresponding probability). Conditionally to $X$, for $f \in C^\infty_c(\mathcal{D})$, $\mathfrak{W}(\gamma)(f)$ is a complex Gaussian random variable, with independent real and imaginary parts whose variances equal

$$M_0^{2\alpha}(e^{\gamma|L|^2} f^2) = \int_{\mathcal{D}} e^{\gamma|L|^2(x,x)} f(x)^2 M_0^{2\alpha}(dx).$$

Formally, $\mathfrak{W}(\gamma)(\cdot)$ is a random process indexed by $C^\infty_c(\mathcal{D})$ whose joint law with $X$, which we denote by $\mathcal{P}$, satisfies for any $n, m \geq 1$, $\mu_1, \ldots, \mu_m \in \mathcal{M}_K, f_1, \ldots, f_n \in C^\infty_c(\mathbb{R}^d)$ and any bounded measurable function $F$ on $\mathbb{R}^m \times \mathbb{C}^n$

$$\mathbb{E} \left[ F \left( (\langle X, \mu_i \rangle)_{i=1}^m, (\mathfrak{W}(\gamma)(f_j))_{j=1}^n \right) \right] = \mathbb{E} \otimes \mathbb{E} \left[ F \left( (\langle X, \mu_i \rangle)_{i=1}^m, \Sigma[\gamma, X, (f_j)_{j=1}^n] \cdot \mathcal{N}_n \right) \right]$$

(1.10)

where $\mathcal{N}_n$ is an $n$ dimensional vector (with probability distribution $\mathcal{P}$) whose coordinate are IID standard complex Gaussian variables, and $\Sigma[\gamma, X, (f_j)_{j=1}^n]$ is the positive definite square root of the matrix

$$\left( M^{2\alpha}(e^{\gamma|L|^2} f_i f_j) \right)_{i,j=1}^n.$$
The process $\mathcal{M}^{(\gamma)}$ can be interpreted as a random distribution. More precisely, there exists a version of the process $\mathcal{M}^{(\gamma)}$ taking values in the local Sobolev space $H^{-u}_{\text{loc}}(D)$ with $u > d/2$ (see definition (1.13) below). This regularity for $\mathcal{M}^{(\gamma)}$ can for instance be obtained by the combining Proposition 1.4 and Theorem 1.5 proved below.

1.4. Our main result. Our main theorem concerns the convergence of $M_{\varepsilon}^{(\gamma)}$ for complex values $\gamma$. More precisely we consider $\gamma$ in the following range of parameters

$$P_{\text{III}}' := \{ \alpha + i\beta : \alpha, \beta \in \mathbb{R}, |\alpha| < \sqrt{d/2}, \alpha^2 + \beta^2 > d \}. \quad (1.11)$$

We require an assumption on the regularity of the function $L$ present in (1.1) (a condition which is also present for papers investigating the subcritical complex chaos [11, 15] for a similar reasons). Let us recall the definition for the Sobolev space with index $s \in \mathbb{R}$ on $\mathbb{R}^k$ which is the Hilbert space of complex valued function associated with the norm

$$\|\varphi\|_{H^s(\mathbb{R}^k)} := \left( \int_{\mathbb{R}^k} (1 + |\xi|^2)^s |\hat{\varphi}(\xi)|^2 d\xi \right)^{1/2}, \quad (1.12)$$

where $\hat{\varphi}(\xi)$ is the Fourier transform of $\varphi$ defined for $\varphi \in C^\infty_c(\mathbb{R}^k)$ by

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^k} e^{i\xi x} \varphi(x) dx.$$

Now for $U \subset \mathbb{R}^k$ open, the local Sobolev space $H^s_{\text{loc}}(U)$ denotes the function which belongs to $H^s(U)$ after multiplication by an arbitrary smooth function with compact support

$$H^s_{\text{loc}}(U) := \left\{ \varphi : U \to \mathbb{R} \mid \rho \varphi \in H^s(\mathbb{R}^d) \text{ for all } \rho \in C^\infty_c(U) \right\}, \quad (1.13)$$

where above $\rho \varphi$ is identified with its extension by zero on $\mathbb{R}^k$. We are going to assume that the covariance kernel $K$ is of the form (1.1) with $L \in H^s_{\text{loc}}(D^2)$ for some exponent $s > d$. Before stating the result we need to introduce some notation. Let us define the function $\ell_\theta$ on $\mathbb{R}^d$, obtained by convoluting $z \mapsto \log 1/|z|$ twice with $\theta$, that is

$$\ell_\theta(z) := \int_{\mathbb{R}^d} \log \left( \frac{1}{|z + z_1 - z_2|} \right) \theta(z_1)\theta(z_2) dz_1dz_2. \quad (1.14)$$

and set

$$v(\varepsilon, \theta, \gamma) := \begin{cases} \varepsilon^{\frac{|\gamma|^2-d}{2}} \left( \frac{1}{2} \int_{\mathbb{R}^d} \varepsilon^{\frac{1}{2}|\gamma|^2} \ell_\theta(z) dz \right)^{-1/2} & \text{if } |\gamma| > \sqrt{d}, \\ \left( \frac{\pi^{d/2}}{\Gamma(d/2)} \log \frac{1}{\varepsilon} \right)^{-1/2} & \text{if } |\gamma| = \sqrt{d}, \end{cases} \quad (1.15)$$

(the quantity $\frac{\pi^{d/2}}{\Gamma(d/2)}$ corresponds to half of the volume of the $(d-1)$-dimensional sphere). Our result establishes that for $\gamma \in P_{\text{III}}'$, in the limit when $\varepsilon$ goes to zero $v(\varepsilon, \theta, \gamma)M_{\varepsilon}^{(\gamma)}$ converges in law towards a complex Gaussian white noise, whose random intensity is $M_0^{(2\alpha)} \left( e^{i\gamma^2 L} \right)$.

**Theorem 1.2.** Let $X$ be a Gaussian random field on $D$ whose covariance kernel is of the form (1.1) with $L \in H^s_{\text{loc}}(D^2)$ for some $s > d$. For $\gamma \in P_{\text{III}}'$, $u > d/2$, the distribution $v(\varepsilon, \theta, \gamma)M_{\varepsilon}^{(\gamma)}$ converges in law, for the $H^{-u}_{\text{loc}}(D)$ topology, towards $\mathcal{M}^{(\gamma)}$ defined in (1.10). More precisely we have the following joint convergence in law

$$(X, v(\varepsilon, \theta, \gamma)M_{\varepsilon}^{(\gamma)}) \xrightarrow{\varepsilon \to 0} (X, \mathcal{M}^{(\gamma)}). \quad (1.16)$$
Remark 1.3. Note that, as it was the case in Theorem 2.1 or its extension to the complex case [15 Theorem 2.1], the limit $M^{(\gamma)}$ does not depend on the convolution kernel used to define $X_\varepsilon$ (recall Remark 1.1). On the other hand the rescaling factor $v(\varepsilon, \theta, \gamma)$ depends on $\theta$ when $|\gamma| > \sqrt{d}$ ($\gamma \in \mathcal{P}_\text{II}$ in the terminology of (1.21)). This is quite natural, in that case the scaling includes a power of $\varepsilon$, and the change $\varepsilon \to c\varepsilon$ is equivalent to a change of convolution Kernel $\theta \to \frac{1}{c}\theta(c)$. On the other hand, in the boundary case when $|\gamma| = \sqrt{d}$ the rescaling is logarithmic in $\varepsilon$ and in that case $v(\varepsilon, \theta, \gamma)$ does not depend on $\theta$.

The convergence (1.16) indicates that the randomness of the limit of $\mathcal{M}^{(\gamma)}$ splits into two parts: the intensity of the noise which is determined by the realization of the field $X$ and an additional Gaussian randomness which is independent from $X$. In particular (1.16) implies that $v(\varepsilon, \theta, \gamma)M^{(\gamma)}$ does not converge to a limit in probability. It is a particular case of stable convergence (see [8 VIII-Section 5c]).

To prove Theorem 1.2 we prove separately the tightness of $v(\varepsilon, \theta, \gamma)M^{(\gamma)}$ in $H^{-u}(\mathcal{D})$ if $u > d/2$ and the convergence of the finite dimensional marginals. The proof of the tightness result, while a bit technical, follows a standard approach and for this reason is given in Appendix [3].

Proposition 1.4. Under the assumptions of Theorem 1.2, given $\rho \in C^{\infty}_c(\mathcal{D})$. The random sequence $(v(\varepsilon, \theta, \gamma)M^{(\gamma)}(\rho \cdot))_{\varepsilon \in (0, 1)}$ is tight in $H^{-u}(\mathbb{R}^d)$ for any $u > d/2$.

To prove the convergence of finite dimensional marginals of $(X, v(\varepsilon, \theta, \gamma)M^{(\gamma)})$, using Lévy’s Theorem, it is sufficient to prove the pointwise convergence of the Fourier transform of the $\mathbb{R}^{m+2m}$ valued random vectors of the type

\[
\left(\langle \mu_j, X \rangle\right)_{j=1}^m, \left(v(\varepsilon, \theta, \gamma)\mathcal{R}(M^{(\gamma)}(f_k))\right)_{k=1}^n, \left(v(\varepsilon, \theta, \gamma)\mathcal{I}(M^{(\gamma)}(f_k))\right)_{k=1}^n
\]

for $\mu_1, \ldots, \mu_m \in \mathcal{M}_K, f_1, \ldots, f_n \in C^{\infty}_c(\mathcal{D})$, where here and in the remainder of the paper $\mathcal{R}$ and $\mathcal{I}$ are used to denote the real part of a complex number. This amounts to checking the convergence of $\mathbb{E} \left[ e^{i\langle X, \mu \rangle + i\varepsilon(\theta, \gamma)M^{(\gamma)}(f, \omega)} \right]$ for every $\mu \in \mathcal{M}_K, f \in C^{\infty}_c(\mathcal{D})$ and $\omega \in [0, 2\pi)$, where

\[
M^{(\gamma)}(f, \omega) := \mathcal{R} \left(e^{-i\omega M^{(\gamma)}(f)}\right)
\]

\[
= \int_{\mathcal{D}_\varepsilon} f(x)e^{ix\psi(x)} + \xi(x)^2 K(x) \cos(\beta(x) - 2\alpha^2 K(x) - \omega) dx.
\]

We let $\mathcal{F}_X$ denote the $\sigma$-algebra generated by the process $X$

\[
\mathcal{F}_X := \sigma(\langle X, \mu \rangle, \mu \in \mathcal{M}_K).
\]

Note that from the definition in (1.10) we have for every $f \in C^{\infty}_c(\mathcal{D})$ and $\omega \in [0, 2\pi)$

\[
\mathbb{E} \left[ e^{i\theta(\omega) - \omega} \right] = e^{-\frac{i}{2}M^{(2\alpha)}(\omega^2 f^2)}.
\]

The following result is the main technical achievement of the paper.

Theorem 1.5. Under the assumption of Theorem 1.2, given $f \in C^{\infty}_c(\mathcal{D}), \omega \in [0, 2\pi)$, and $\mu \in \mathcal{M}_K$ we have

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ e^{i\langle X, \mu \rangle + i\varepsilon(\theta, \gamma)M^{(\gamma)}(f, \omega)} \right] = \mathbb{E} \left[ e^{i\langle X, \mu \rangle + i\mathcal{R}(e^{-i\omega} \mathcal{I}(\mu))} \right]
\]

\[
= \mathbb{E} \left[ e^{i\langle X, \mu \rangle - \frac{i}{2}M^{(2\alpha)}(\omega^2 f^2)} \right].
\]
1.5. A review of results on complex GMC. Let us now try to give some perspectives on our result by showing how it relates to the existing literature on complex GMC. The set \( P_{\text{III}}' \) in (1.11) corresponds in fact - up to boundary - to one of the phases in a diagram which contains three. Let us introduce these three phases for the needs of the discussion

\[
P_{\text{sub}} := \{ \alpha + i \beta : \alpha^2 + \beta^2 < d \} \cup \{ \alpha + i \beta : \alpha \in (\sqrt{d/2}, \sqrt{2d}) \ ; |\alpha| + |\beta| < \sqrt{2d} \},
\]

\[
P_{\text{II}} := \{ \alpha + i \beta : |\alpha| + |\beta| > \sqrt{2d} \ ; |\alpha| > \sqrt{d/2} \},
\]

\[
P_{\text{III}} := \{ \alpha + i \beta : \alpha^2 + \beta^2 > d \ ; |\alpha| < \sqrt{d/2} \}.
\]

(1.21)

This phase diagram appears in [17] in the context of the study of complex GMC is defined by considering

\[
M^{(\alpha,\beta)}(\epsilon)(dx) := e^{\alpha X_\epsilon + i\beta Y_\epsilon - \frac{\alpha^2 + \beta^2}{2} K_\epsilon(x)} dx
\]

(1.22)

where \( X \) and \( Y \) are independent log-correlated fields of covariance \( K \) [4] and also for related models such as complex Multiplicative Cascades [3] complex Random Energy Model [13] or complex branching Brownian Motion [6, 7]. Each of the phases in (1.21) is conjectured to correspond to a different scaling regime for \( M^{(\gamma)}(\epsilon) \).

The subcritical phase \( P_{\text{sub}} \). When \( \gamma \in P_{\text{sub}} \) is has been proved that \( M^{(\gamma)} \) converges in probability to a random distribution. More precisely is has been proved in [11] (with the assumption \( L \in H^s_{\text{loc}}(D^2) \), for \( s > d \) that the random distribution \( M^{(\alpha)} \) of Theorem A has a unique analytic continuation on the domain \( P_{\text{sub}} \). In [15], it was proved (under the same assumption) that this analytic continuation is the limit of \( M^{(\gamma)} \) for any choice of convolution kernel \( \theta \), extending Theorem A to the full region \( P_{\text{sub}} \). Convergence of \( M^{(\alpha,\beta)} \), recall (1.22), for \( (\alpha, \beta) \in P_{\text{sub}} \) (identified with a subset of \( \mathbb{R}^2 \)) is also established in [15] (and earlier in [17]) for the martingale approximation. The convergence should be in law.

The glassy phase \( P_{\text{II}} \). When \( \gamma \in P_{\text{II}} \), it is conjectured that in the limit when \( \epsilon \to 0 \) the distribution of \( M^{(\gamma)} \) is supported by small neighborhood of the points where \( X_\epsilon \) is close to be maximized, yielding an atomic distribution (a countable sum of weighted Dirac masses) in the limit. In this case the right scaling should be \((\log 1/\epsilon)^{\frac{4}{\ell_2}} \epsilon^{\frac{\alpha}{\ell_2} - d} M^{(\gamma)}(dx)\). This phenomenon is called freezing and has been proved in [6, 19] for the complex exponential of Branching Brownian Motion, but it remains a challenging conjecture for complex GMC (both for \( M^{(\alpha,\beta)} \) and \( M^{(\gamma)} \)).

The third phase \( P_{\text{III}} \). When \( \gamma \in P_{\text{III}} \) the fluctuations of \( X_\epsilon \) makes the phases of \( e^{i\beta X_\epsilon(x)} \) decorrelate even on small scale, and this accounts for the appearance of a white noise appear in the limit. The intensity of the corresponding white noise has to be given by the limit the square of the modulus of the local variations, that is \( e^{2\gamma X_\epsilon} dx \). The convergence of \( \epsilon^{(\gamma)^2 - d} M^{(\alpha,\beta)} \) (recall (1.22)) was established in [17]. The proof relied on the computation of all the conditional moments of \( M^{(\alpha,\beta)} \) when conditioning w.r.t. to the field \( X \). This

\[1\] To be completely accurate, the field \( X_\epsilon \) considered in [17] is not a convolution but rather a martingale approximation of the type considered in Section [2]. This difference is not relevant for the present discussion.
approach is heavily relying on the independence of $X$ and $Y$ and cannot be adapted to the present context.

In this paper, partly inspired the techniques used in [16] to study the Sine-Gordon model (which corresponds to the case $\alpha = 0$) we take a completely different approach which relies on convergence of martingale brackets after using a martingale decomposition.

1.6. Open questions. Note that our result Theorem 1.2 does not only compute the scaling limit of $M_\epsilon^{(\gamma)}$ for $\gamma \in \mathcal{P}_{III}$ but also when $\gamma$ is on the frontier between $\mathcal{P}_{III}$ and $\mathcal{P}_{sub}$ (recall the definition of $\mathcal{P}_{III} \ (1.11)$). Let us discuss here shortly what should occur on the rest of the frontier between $\mathcal{P}_{III}$ and other phases.

To formulate this conjecture, let us introduce the critical real multiplicative chaos, which corresponds to the point $|\alpha| = \sqrt{2d}$ and $\beta = 0$. It has been proved [4, 9] that while $M_\epsilon^{(\pm \sqrt{2d})}(dx)$ converges to zero, we obtain a non trivial limit in probability after rescaling by an appropriate factor. More precisely we have

$$\lim_{\epsilon \to 0} (\log 1/\epsilon)^{1/2} M_\epsilon^{(\pm \sqrt{2d})}(dx) =: M^{(\pm \sqrt{2d})}(dx).$$

The measure $M^{(\pm \sqrt{2d})}$ is referred to as the critical multiplicative chaos.

The frontier $\mathcal{P}_{II}/\mathcal{P}_{III}$. When $|\alpha| = \sqrt{d/2}$, $|\beta| > \sqrt{d/2}$, it is natural to conjecture that $M_\epsilon^{(\gamma)}$ properly renormalized should converges to a white noise whose intensity is given by $M^{(2\alpha)}$. Such a result has been proved in [17] for the chaos given in (1.22). The proper renormalization to consider in that case should be $(\log 1/\epsilon)^{1/2} |\gamma|^2 - d M_\epsilon^{(\gamma)}$.

The triple point $|\alpha| = |\beta| = \sqrt{d/2}$. This should be similar to the $\mathcal{P}_{II}/\mathcal{P}_{III}$ frontier though possibly more technical to handle. In that case $(\log 1/\epsilon)^{-1/2} M_\epsilon^{(\gamma)}$ should converge to a complex white noise with intensity given by $M^{(2\alpha)}$.

First hints on the organization of the paper. Proposition 1.4 is proved in Appendix B. The proof of Theorem 1.5 which is the main technical achievement of the paper rely on a martingale decomposition of $M_\epsilon^{(\gamma)}$ which is introduced in Section 2. Additional details on this decomposition are needed in order to explain how the remaining sections are organized, a more detailed picture is given in Section 2.6.

Notation. Let us list here a few convention adopted in the paper. If $G$ is a generic function of two variables we will write $G(x)$ for $G(x, x)$. If $(J_s)_{s \geq 0}$ is a continuous function or a random process indexed by $s$ we set

$$J_{[a, b]} := E_b - J_a \quad (1.23)$$

The letter $C$ is used to denote generic positive constants used in the computation. The value of $C$ is allowed to change from one equation to another within the same proof.

2. Decompositing the proof of Theorem 1.5

2.1. Star-scale invariant kernels. We say that the kernel $K$ has a star-scale invariant part (with kernel $\kappa$) if it can be written in the form

$$K(x, y) = K_0(x, y) + \int_0^\infty \kappa(e^t|x - y|)dt \quad (2.1)$$
where \(K_0(x,y)\) is a bounded H{"o}lder continuous positive definite kernel on \(D\) (recall (1.2)) and the function \(\kappa : \mathbb{R}_+ \to \mathbb{R}\), satisfies the following assumptions:

(i) \(\kappa\) is Lipschitz-continuous and non-negative,
(ii) \(\kappa(0) = 1\), \(\kappa(r) = 0\) for \(r \geq 1\).
(iii) \((x,y) \mapsto \kappa(|x-y|)\) defines a positive definite function on \(\mathbb{R}^d \times \mathbb{R}^d\).

Note that if \(\kappa\) satisfies (2.1) then

\[
L(x,y) := K(x,y) + \log |x-y|,
\]

(2.2)
can be extended to a continuous function on \(D^2\), so that \(K\) having a star-scale invariant part implies that it can be written in the form (1.1).

The converse is not true and there are positive definite kernel \(K\) of the type given in Equation (1.1) that cannot be decomposed as in (2.1) for any choice of \(\kappa\). However if \(L \in H^s_{\text{loc}}(D^2)\) for some \(s > d\), then \(K\) can be approximated very well by a kernel of the type (2.1). This is the content of the following result (proved in Appendix A).

**Lemma 2.1.** Given \(K\) a covariance kernel on \(D\) of the form (1.1) with \(L \in H^s_{\text{loc}}(D^2)\) for \(s > d\), \(D'\) a bounded open set whose closure satisfies \(\overline{D'} \subset D\) and \(\delta > 0\) there exists a kernel \(K^{(\delta)}\) of the form (2.1) on \(D'\) such that

(A) For all \(x, y \in D'\), \(|K^{(\delta)}(x,y) - K(x,y)| \leq \delta\).
(B) \(\Delta^{(\delta)}(x,y) = K^{(\delta)}(x,y) - K(x,y)\) is a positive definite kernel on \(D'\).

In order to prove Theorem 1.5, the most important step is to prove the convergence of \(M_{\varepsilon}^{(\gamma)}(f, \omega)\) for a field whose covariance kernel satisfies the assumption in (2.1) and use our approximation Lemma 2.1 to conclude.

**Proposition 2.2.** Given a Gaussian field \(X\) defined on \(D\) whose covariance kernel satisfies (2.1), a convolution kernel \(\theta\), and \(\gamma \in \mathcal{P}_{\text{III}'}\), we have for every \(f \in C_c^\infty(D)\), \(\omega \in [0,2\pi]\), and \(\mu \in \mathcal{M}_K\) we have

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ e^{i(X,\mu) + iv(\varepsilon,\beta)M_{\varepsilon}^{(\gamma)}(f,\omega)} \right] = \mathbb{E} \left[ e^{-i(X,\mu) - \frac{1}{2}M_0^{(2\alpha)}(v|\gamma|^2L^2f^2)} \right].
\]

(2.3)
The proof of Proposition 2.2 is technical and requires several steps. We provide a detailed road map at the end of this section. We explain first how our main result follows from it.

2.2. **Proving Theorem 1.5 from Proposition 2.2** Let us fix \(f \in C_c^\infty(D)\) and \(\omega \in (0,2\pi]\) and \(\mu \in \mathcal{M}_K\). Given \(\eta > 0\) we consider a bounded open set \(D'\) which includes the support of \(f\) and which is such that

\[
\int_{(D',D')^2} K(x,y) \mu(dx)\mu(dy) \leq \eta
\]

(2.4)
and whose topological closure satisfies \(\overline{D'} \subset D\). We let \(K^{(\delta)}\) be a kernel satisfying the assumptions (A) and (B) of Lemma 2.1. Given \(\delta > 0\), we can construct two Gaussian fields \(X\) and \(X^{(\delta)}\) indexed by \(D\) and \(D'\) on the same probability space in such a way that the field \(Z^{(\delta)} := X^{(\delta)} - X\) defined on \(D'\) is independent of \(X\) and has covariance \(\Delta^{(\delta)}(x,y)\). Letting \(M_{\varepsilon}^{(\gamma,\delta)}\) denote the exponential of the convoluted field \(X_{\varepsilon}^{(\delta)}\), and \(\mu'\) denote the restriction of \(\mu\) to \(D'\) we have from Proposition 2.2

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ e^{i(X^{(\delta)},\mu') + iv(\varepsilon,\beta)M_{\varepsilon}^{(\gamma,\delta)}(f,\omega)} \right] = \mathbb{E} \left[ e^{-i(X^{(\delta)},\mu') - \frac{1}{2}M_0^{(2\alpha,\delta)}(v|\gamma|^2L^2f^2)} \right].
\]

(2.5)
Now in order to obtain a conclusion for the exponential of the original field $X$ we have to replace $X^{(δ)}$ by $X$ and $μ'$ by $μ$ in both sides of the above convergence. Using Jensen’s inequality and the triangle inequality we have

$$
\left| E \left[ e^{i⟨X, μ⟩ + iv(ε, θ, γ)M^{(γ)}_ε(f, ω)} - e^{i⟨X^{(δ)}, μ'⟩ + iv(ε, θ, γ)M^{(γ, δ)}_ε(f, ω)} \right] \right| 
\leq E \left[ e^{iv(ε, θ, γ)M^{(γ)}_ε(f, ω)} - e^{iv(ε, θ, γ)M^{(γ)}_ε(f, ω)} \right] + E \left[ |e^{i⟨X^{(δ)}, μ'⟩} - e^{i⟨X, μ⟩}| \right],
\tag{2.6}
$$

and in the same manner

$$
\left| E \left[ e^{i⟨X, μ⟩ - \frac{1}{2}M^{(2α)}_0(ε|γ|^2 L f^2) - e^{i⟨X^{(δ)}, μ'⟩ - \frac{1}{2}M^{(2α, δ)}_0(ε|γ|^2 L f^2) \right]} \right| 
\leq E \left[ e^{-\frac{1}{2}M^{(2α)}_0(ε|γ|^2 L f^2)} - e^{-\frac{1}{2}M^{(2α, δ)}_0(ε|γ|^2 L f^2)} \right] + E \left[ |e^{i⟨X^{(δ)}, μ'⟩} - e^{i⟨X, μ⟩}| \right].
\tag{2.7}
$$

In both r.h.s. of (2.6)-(2.7) the second term in the sum is easier to control. As $u \mapsto e^{iz}$ is Lipshitz we have

$$
E \left[ |e^{i⟨X^{(δ)}, μ'⟩} - e^{i⟨X, μ⟩}| \right] \leq E \left[ |⟨X^{(δ)}, μ'⟩ - ⟨X, μ⟩| \right] \leq E \left[ (⟨X^{(δ)}, μ'⟩ - ⟨X, μ⟩)^2 \right]^{1/2}.
\tag{2.8}
$$

The variance of can be computed explicitly, we have (recall (2.4))

$$
E \left[ (⟨X^{(δ)}, μ'⟩ - ⟨X, μ⟩)^2 \right] = \int_{(D')^2} \Delta^{(δ)}(x,y)μ(dx)μ(dy) + \int_{(D\backslash D')^2} K(x,y)μ(dx)μ(dy)
\leq δ |μ|(D')^2 + η.
\tag{2.9}
$$

Note that the assumption $μ ∈ M_K$ implies that $|μ|(D') < ∞$. Indeed since $L$ is bounded from below on $D^2$, there exists $a > 0$ such that

$$
∀ x, y ∈ D', |x - y| < a \rightarrow K(x, y) ≥ 1.
$$

Hence if $A ⊂ D'$ has diameter smaller than $a$, $μ ∈ M_K$ implies that

$$
||μ|(A)||^2 ≤ \int_{A×A} |K(x, y)||μ|(dx)||μ|(dy) < ∞,
$$

and we can conclude by considering a covering of $D'$ by finitely many such sets. Hence we have $μ' ∈ M_K^{(δ)}$ which we have implicitly to obtain (2.5).

Next we bound the respective first summands in the r.h.s. of (2.6) and (2.7). We prove that there exists a constant $C$ (allowed to depend on $f$, $γ$, and on the covariance kernel $K$) such that for all $ε$ and $δ$ sufficiently small

$$
E \left[ e^{iv(ε, θ, γ)M^{(γ)}_ε(f, ω)} - e^{iv(ε, θ, γ)M^{(γ)}_ε(f, ω)} \right] ≤ C\sqrt{δ},
\tag{2.10}
$$

and

$$
E \left[ e^{-\frac{1}{2}M^{(2α)}_0(ε|γ|^2 L f^2)} - e^{-\frac{1}{2}M^{(2α, δ)}_0(ε|γ|^2 L f^2)} \right] ≤ C\sqrt{δ}.
$$

Before proving (2.10) let us show how it can be used to conclude. The combination of (2.5)-(2.10) yields

$$
\limsup_{ε \to 0} \left| E \left[ e^{i⟨X, μ⟩ + iv(ε, θ, γ)M^{(γ)}_ε(f, ω)} \right] - E \left[ e^{-i⟨X, μ⟩ - \frac{1}{2}M^{(2α)}_0(ε|γ|^2 L f^2)} \right] \right| ≤ 2C\sqrt{δ} + 2\sqrt{δ |μ|(D')^2 + η}.
\tag{2.11}
$$
The r.h.s. above can be made arbitrarily small by taking first \( \eta \) small (which fixes \( D' \)) and then \( \delta \) small, and thus the limit in the r.h.s. is equal to zero.

Let us now prove (2.10). Using the fact that \( u \rightarrow e^{iu} \) and \( u \rightarrow e^{-u} \) are Lipschitz functions on \( \mathbb{R} \) and \( \mathbb{R}_+ \) respectively (for the first line we also use that \( |\text{Re}(e^{-iwz})| \leq |z| \)) we have

\[
\begin{align*}
\left| \mathbb{E} \left[ e^{i\nu(x,\theta,\gamma)\frac{M_{\epsilon,\nu}(\gamma,\delta)}{M_{\epsilon}^{(\gamma)}(f,\omega)} - e^{i\nu(x,\theta,\gamma)\frac{M_{\epsilon}(\gamma,\delta)}{M_{\epsilon}^{(\gamma)}(f,\omega)}} \right] \right| & \leq v(\epsilon, \theta, \gamma) \mathbb{E} \left[ \left| M_{\epsilon}^{(\gamma,\delta)}(f) - M_{\epsilon}^{(\gamma)}(f) \right| \right], \\
\left| \mathbb{E} \left[ e^{-M_{\epsilon}^{(2\alpha,\delta)}(\epsilon f^2)} - e^{-M_{\epsilon}^{(2\alpha)}(\epsilon f^2)} \right] \right| & \leq \mathbb{E} \left[ \left| M_{\epsilon}^{(2\alpha,\delta)}(\epsilon f^2) - M_{\epsilon}^{(2\alpha)}(\epsilon f^2) \right| \right].
\end{align*}
\]

(2.12)

To bound the r.h.s. of the first line in (2.12), we rely on Cauchy-Schwarz and evaluate the second moment. Using the independence of \( X \) and \( Z(\delta) \), the assumption \( B \) on \( \Delta(\delta) \) and setting

\[
\Delta_{\epsilon}(x,y) := \int_{(D')^2} \Delta(\delta)(z_1, z_2) \theta_{\epsilon}(x - z_1) \theta_{\epsilon}(y - z_2) dz_1 dz_2.
\]

(2.13)

we obtain that - here we assume that \( \epsilon \) is sufficiently small so that the support of \( f \) is included in \( D' \) (recall (1.6))

\[
\mathbb{E} \left[ \left| M_{\epsilon}^{(\gamma,\delta)}(f) - M_{\epsilon}^{(\gamma)}(f) \right|^2 \right] = \int_{(D')^2} f(x)f(y) \left( e^{\gamma^2 \Delta_{\epsilon}(x,y)} - 1 \right) e^{\gamma^2 K_{\epsilon}(x,y)} dx dy 
\leq \left( e^{\delta} - 1 \right) \int_{(D')^2} f(x)f(y) e^{\gamma^2 K_{\epsilon}(x,y)} dx dy. \tag{2.14}
\]

In the second line above we simply used that \( \Delta_{\epsilon}(x,y) \leq \delta \) which follows immediately from the assumption \( A \) in Lemma 2.1. Using the estimate (3.4) for \( K_{\epsilon} \) we obtain that

\[
\int_{D^2} f(x)f(y) e^{\gamma^2 K_{\epsilon}(x,y)} dx dy \leq C \int_{\mathbb{R}^d} f(x)f(y) (|x - y| + \epsilon)^{-\gamma^2} dx dy 
\leq e^{C \gamma^2 \|f\|_2^2} \int_{\mathbb{R}^d} (|z| + \epsilon)^{-\gamma^2} dz. \tag{2.15}
\]

The right-hand side is of order \( \epsilon^{d-\gamma^2} \) if \( |\gamma| > \sqrt{d} \) and of order \( \log(1/\epsilon) \) if \( |\gamma| = \sqrt{d} \). This allows to conclude from (2.14) that for a constant \( C \) which may depend on all parameters but \( \delta \) and \( \epsilon \), we have

\[
v(\epsilon, \theta, \gamma)^2 \mathbb{E} \left[ \left| M_{\epsilon}^{(\gamma,\delta)}(f) - M_{\epsilon}^{(\gamma)}(f) \right|^2 \right] \leq C \delta. \tag{2.16}
\]

Let us now evaluate the r.h.s. of the second line in (2.12). Since, by Theorem A \( M_{\epsilon}^{(2\alpha,\delta)}(\epsilon f^2) \) and \( M_{\epsilon}^{(2\alpha)}(\epsilon f^2) \) both converge in \( L^1 \), it is sufficient to prove a bound on

\[
\mathbb{E} \left[ \left| M_{\epsilon}^{(2\alpha,\delta)}(\epsilon |f|^2 f^2) - M_{\epsilon}^{(2\alpha)}(\epsilon |f|^2 f^2) \right| \right]
\]

which is uniform in \( \epsilon \). Assuming that the support of \( f \) is included in \( D' \) (recall (1.6)), letting \( Z_{\epsilon} \) denote the field \( Z \) convoluted with \( \theta_{\epsilon} \) we have

\[
\mathbb{E} \left[ \left| M_{\epsilon}^{(2\alpha,\delta)}(\epsilon |f|^2 f^2) - M_{\epsilon}^{(2\alpha)}(\epsilon |f|^2 f^2) \right| \right] 
\leq \mathbb{E} \left[ \int_{D'} e^{2\alpha Z_{\epsilon}(x) - 2\alpha^2 \Delta_{\epsilon}(x)} - 1 \left| e^{\gamma^2 L(x)} f^2 (x) M_{\epsilon}^{(2\alpha)}(dx) \right| \right]. \tag{2.17}
\]
Averaging with respect to $X$, and using the fact that $X$ and $Z$ are independent, we obtain that the l.h.s. in (2.17) is equal to

\[
\int_{\mathcal{D}} \mathbb{E} \left[ e^{2\alpha Z_t(x)-2\alpha^2 \Delta^{(\delta)}(x)} - 1 \right] e^{\gamma^2 L(x)} f^2(x)dx \leq \sqrt{e^{4\alpha^2 \delta} - 1} \int_{\mathcal{D}} e^{\gamma^2 L(x)} f^2(x)dx \tag{2.18}
\]

where for the inequality we only used Cauchy-Schwartz together with

\[
\mathbb{E} \left[ \left( e^{2\alpha Z_t(x)-2\alpha^2 \Delta^{(\delta)}(x)} - 1 \right)^2 \right] = e^{4\alpha^2 \Delta^{(\delta)}(x)} - 1 \leq e^{4\alpha^2 \delta} - 1. \tag{2.19}
\]

\[\square\]

2.3. The martingale approximation of the GMC. Using the assumption (2.1), we can introduce another functional approximation of the field $X$ besides the convolution approximation. Introducing the notation $Q_t(x, y) := \kappa(\epsilon^2 (x-y))$, we define $(X_t(x))_{x \in \mathbb{R}^d, t \geq 0}$ as the Gaussian field parameterized by $\mathcal{D} \times \mathbb{R}_+$ with covariance function

\[
\mathbb{E}[X_s(x)X_t(y)] = K_0(x, y) + \int_0^{s \wedge t} Q_u(x, y)du =: K_{s \wedge t}(x, y). \tag{2.20}
\]

By construction $X_t(\cdot)$ is a martingale for the canonical filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$ defined by

\[
\mathcal{F}_t := \sigma(X_s, s \in [0, t)). \tag{2.21}
\]

Note that $\mathcal{F}_X \subset \mathcal{F}_X$, and that the inclusion is strict. Since, from our assumptions on $K_0$ and $\kappa$, $K_0(x, y)$ is Hölder continuous (in space and time) then by Kolmogorov-Chensov Theorem $(X_t(x))_{x \in \mathcal{D}, t \geq 0}$ admits a modification which is space time continuous. On the same probability space, one can define a distributional field $X$ indexed by $\mathcal{M}_K$ by setting

\[
\langle X, \mu \rangle := \lim_{t \to \infty} \int X_t(x)d\mu(x), \tag{2.22}
\]

(the convergence holds in $L^2$). The field $X$ has covariance $K$ and $(X_t)_{t \geq 0}$ is thus a sequence of approximation for $X$. We then consider the following distribution valued martingale by setting for $f \in C_c^\infty(\mathcal{D})$

\[
M^{(\gamma)}_t(f) := \int_{\mathbb{R}^d} f(x)e^{\gamma X_t(x)-\frac{\gamma^2 K_t(x)}{2}} dx. \tag{2.23}
\]

The notation $K_t$, $X_t$ and $M^{(\gamma)}_t$ conflicts with $K_\varepsilon$, $X_\varepsilon$ and $M^{(\gamma)}_\varepsilon$ introduced earlier, but we believe this abuse of notation to be harmless for all our purposes. In our computations, the variable $\varepsilon$ is always used for convolution approximation while $t$ and a other latin letters is used for martingale approximations.

The following result illustrates the fact that the martingale approximation has an effect which is similar to the convolution approximation. It can be deduced from the proof in [1] and is a particular case of [13 Remark 2.4].

\textbf{Theorem B.} For any $\alpha \in \mathbb{R}$ with $|\alpha| < \sqrt{2d}$, then there exists a random distribution $M^{(\alpha)}_0$ such that for every $\theta$ and every continuous $f \in C_c^\infty(\mathcal{D})$, we have the following convergence in $L^1$

\[
\lim_{t \to \infty} M^{(\alpha)}_t(f) = M^{(\alpha)}_0(f), \tag{2.24}
\]

where the distribution $M^{(\alpha)}_0$ is the same as in Theorem A.
Before proving Proposition 2.2, we are going to show that $M_t^{(\gamma)}$, once renormalized also converges to a white noise with intensity $M_0^{(2\alpha)}$. We prove only the convergence of finite dimensional distributions. Tightness follows from the proof of Proposition 1.4 displayed in Appendix B which directly adapts to this case. While this result is not necessary to prove Proposition 2.2, it provides a step of intermediate difficulty and thus serves a didactical purpose. Furthermore Theorem 2.3 presents an interest in itself since it provides further indication that the scaling limit is universal since it is the same for the convolution approximation and for martingale approximation.

As for the convolution case, we define for $\omega \in [0, 2\pi)$

$$M_t^{(\gamma)}(f, \omega) := \Re(e^{-i\omega M_t^{(\gamma)}(f)}). \quad (2.24)$$

and set

$$\nu(t, \kappa, \gamma) := \begin{cases} \frac{d-|\gamma|^2}{2} e^{-1/2} \int_{\mathbb{R}^d} e^{-|\gamma|^2 \ell_\kappa(|z|)} dz, & \text{if } |\gamma| > \sqrt{d}, \\ \left(\frac{d}{\Gamma(d/2)}\right)^{-1/2}, & \text{if } |\gamma| = \sqrt{d}, \end{cases} \quad (2.25)$$

where

$$\ell_\kappa(r) := \int_0^\infty \kappa(e^{-\ell} r) - \kappa(e^{-\ell} t) dt. \quad (2.26)$$

**Theorem 2.3.** Let $X$ a Gaussian field on $\mathcal{D}$ whose covariance kernel satisfies (2.1), and $\gamma \in \mathcal{P}^0_{\mathbb{H}}$. We have for every $f \in C_c^\infty(\mathcal{D})$ $\omega \in [0, 2\pi)$, and $\mu \in \mathcal{M}_K$

$$\lim_{t \to \infty} E \left[ e^{i\langle \mu, X \rangle + i\nu(t, \kappa, \gamma) M_t^{(\gamma)}(f, \omega)} \right] = E \left[ e^{i\langle \mu, X \rangle - \frac{1}{2} M_0^{(2\alpha)}(e|\gamma|^2 L f^2)} \right]. \quad (2.27)$$

The function $L$ appearing in the theorem above is simply defined by (1.1). We prefer to use $L$ instead of $K_0$ in the result since the latter depends on the specific choice which is made for $\kappa$ while $L$ is fully determined by the kernel $K$. We have

$$L(x, y) := K_0(x, y) + \int_0^\infty \kappa(e^t |x - y|) dt - \log \frac{1}{|x - y|}. \quad (2.28)$$

Examining the value of the difference of the two last summands near the diagonal, we obtain that $L(x)$ and $K_0(x)$ only differ by a constant, we have

$$L(x, x) = K_0(x, x) - j_\kappa. \quad (2.29)$$

where

$$j_\kappa := \int_0^\infty (1 - \kappa(e^{-\ell} t)) dt. \quad (2.30)$$

The proof of Theorem 2.3 relies on a very simple strategy. We need to prove that the quadratic variation of $(M_t^{(\gamma)}(f, \omega))_{t \geq 0}$ (recall that this a continuous martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$) satisfies the law of large number given in the following proposition, and then use a martingale Central Limit Theorem which we introduce in the next section.

**Proposition 2.4.** Under the assumption of Theorem 2.3 setting $N_t := M_t^{(\gamma)}(f, \omega)$ we have

$$\lim_{t \to \infty} \nu(t, \kappa, \gamma)^2 \langle N_t \rangle_\kappa = M_0^{(2\alpha)}(e^{-|\gamma|^2 L f^2}). \quad (2.31)$$
2.4. A martingale CLT with a random variance. One of the key ideas our proof is the use of the following Central Limit Theorem with a random variance. We consider \((\mathcal{F}_t)_{t \geq 0}\) to be a continuous filtration, \((N_t)_{t \geq 0}\) be a continuous local martingale for the filtration \((\mathcal{F}_t)_{t \geq 0}\), \(Z\) a non-negative random variable and \(v : \mathbb{R}_+ \to \mathbb{R}\) be a non-increasing positive continuous function such that \(\lim_{t \to \infty} v(t) = 0\). The following result is a particular case of [8] Theorem 5.50, Chap. VIII-Section 5c (the reference only treats the case \(v(t) = t^{-1/2}\) but this is just a matter of time rescaling).

**Theorem 2.5.** If the quadratic variation of \(N\) satisfies the following convergence in probability

\[
\lim_{t \to \infty} v(t)^2 \langle N \rangle_t = Z
\]  

then we have for every \(\xi, \xi' \in \mathbb{R}\) and \(Y\) bounded real valued \(\mathcal{F}_\infty\)-measurable random variable

\[
\lim_{t \to \infty} \mathbb{E} \left[ e^{i\xi v(t)N_t + i\xi'Y} \right] = \mathbb{E} \left[ e^{-\frac{\xi^2}{2} Z + i\xi'Y} \right].
\]  

In other words we have the following joint convergence in distribution

\[
(Y, v(t)N_t) \overset{\mathcal{D}}{\longrightarrow} (Y, \sqrt{Z} \times \mathcal{N})
\]  

where \(\mathcal{N}\) is a standard normal variable independent of \(Z\).

**Proof of Theorem 2.5.** It follows directly from the combination of Proposition 2.4 and Theorem 2.5.

\[\square\]

2.5. Roadmap to prove Proposition 2.2. To prove Proposition 2.2 we use some of the ideas used the proof of Theorem 2.3. We consider a martingale indexed by \(t\) whose limit is given by \(M_t^{(\gamma)}\). For this we consider \((X_{t,\varepsilon})_{t \in D_\varepsilon}\) (recall (1.6)) defined by

\[
X_{t,\varepsilon}(x) = \int_{\mathbb{R}^d} \theta_\varepsilon(x - z)X_t(z)dz.
\]  

We let \(K_{t,\varepsilon}\) denote the covariance of \(X_{t,\varepsilon}\)

\[
K_{t,\varepsilon}(x, y) := \int_{(\mathbb{R}^d)^2} \theta_\varepsilon(x - z_1)\theta_\varepsilon(y - z_2)K_t(z_1, z_2)dz_1dz_2
\]  

Note that if \(X\) is the field defined by (2.22) then we have

\[
\lim_{t \to \infty} X_{t,\varepsilon}(x) = X_\varepsilon(x) \quad \text{and} \quad \mathbb{E}[X_\varepsilon(x) \mid \mathcal{F}_t] = X_\varepsilon(x).
\]

Given \(\gamma \in \mathcal{P}^\prime_{III}, f \in C_\varepsilon^\infty(D)\) and \(\omega \in [0, 2\pi]\), we define

\[
N_t^{(\varepsilon)} = \int_{\mathbb{R}^d} f(x)e^{\alpha X_{t,\varepsilon}(x) - \frac{(\alpha^2 - \beta^2)K_{t,\varepsilon}(x)}{2}} \cos(\beta X_{t,\varepsilon}(x) - \alpha\beta K_{t,\varepsilon}(x) - \omega)dx.
\]  

Note that

\[
N_t^{(\varepsilon)} = \mathbb{E}[M_\varepsilon^{(\gamma)}(f, \omega) \mid \mathcal{F}_t] \quad \text{and} \quad \lim_{t \to \infty} N_t^{(\varepsilon)} = M_\varepsilon^{(\gamma)}(f, \omega).
\]  

Instead of proving an asymptotic statement concerning the quadratic variation when \(t\) tends to infinity like in Proposition 2.4 we must identify the behavior of \(\langle N^{(\varepsilon)} \rangle_\infty\) when \(\varepsilon\) tends to 0.

**Proposition 2.6.** Under the assumption of Theorem 2.3 and with \(N_t^{(\varepsilon)}\)

\[
\lim_{\varepsilon \to 0} v(\varepsilon, \theta, \gamma)^2 \langle N^{(\varepsilon)} \rangle_\infty = M_0^{(2\alpha)}(e^{-|\gamma|^2/L}f^2).
\]  

\[\square\]
We cannot deduce Theorem 2.3 from Proposition 2.6 using Theorem 2.5 but are going to use similar ideas to conclude.

2.6. Organization of the paper.

- In Section 3 we introduce a couple of classical tools and technical estimates that are used throughout the paper.
- Sections 4 is devoted to the proof of Theorem 2.3
- Sections 5 and 6 are devoted to the proof of Proposition 2.2. The proof are a bit more technical than those concerning Theorem 2.3 but follow the same main ideas. In Section 5, we show how Proposition 2.2 can be deduced from Proposition 2.6 while in Section 6 we prove Proposition 2.6.
- Two technical results are proved in appendices. Lemma 2.1 is proved in Appendix A, while Proposition 1.4 is proved in Appendix B.

3. Technical preliminaries

3.1. Gaussian tools. Before starting the proof, let us mention one classical inequality and one classical identity that we use repeatedly in the proof. First, for any $\sigma > 0$ and $t \geq 0$, we have

$$\frac{1}{\sqrt{2\pi \sigma}} \int_{t}^{\infty} e^{-u^2 / (2\sigma^2)} \, du \leq e^{-t^2 / (2\sigma^2)}.$$  \hfill (3.1)

We refer to this inequality as the Gaussian tail bound. Our second tool is the Cameron-Martin formula. Let $(Y(z))_{z \in \mathcal{Z}}$ be an arbitrary centered Gaussian field indexed by an arbitrary set $\mathcal{Z}$. We let $H$ denote its covariance and $\mathbb{P}$ denote its law. For any $z \in \mathcal{Z}$ let us define $\mathbb{P}_z$ the measure tilted by $Y(z)$.

$$\frac{d\mathbb{P}_z}{d\mathbb{P}} := e^{Y(z) - \frac{1}{2} H(z,z)} \hfill (3.2)$$

**Proposition 3.1.** Under the probability law $\mathbb{P}_z$, $Y$ is a Gaussian field with covariance $H$, with mean value equal to

$$\mathbb{E}_z[Y(z')] = H(z,z').$$  \hfill (3.3)

3.2. Estimates for the covariance kernels. We list in this section a collection of useful estimates for the kernel $K_z, K_t, K_{t,\varepsilon}$ defined in (1.8), (2.20) and (2.36) (under the most general assumption (1.1) for $K_z$ and (2.1) for the other kernels). First note that if we fix a compact in $\mathcal{K} \subset \mathcal{D}$, then there exists a constant $C$ (depending on $K, \mathcal{K}, \theta$ and $\kappa$) such that for every $t \geq 0$ and every $\varepsilon$ such that $\mathcal{D}_\varepsilon \subset \mathcal{K}$, every $x$ and $y$ in $\mathcal{K}$

$$|K_t(x,y) - \log \left( \frac{1}{|x-y| \vee e^{-t}} \right)| \leq C,$$

$$|K_z(x,y) - \log \left( \frac{1}{|x-y| \vee \varepsilon} \right)| \leq C,$$

$$|K_{t,\varepsilon}(x,y) - \log \left( \frac{1}{|x-y| \vee \varepsilon \vee e^{-t}} \right)| \leq C.$$  \hfill (3.4)

The estimates above can be proved by hand using the definitions and are left to the reader. Secondly, let us give some estimate concerning the local regularity of the Kernel near the diagonal.
Lemma 3.2. There exists a function \( \eta : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \lim_{a \to 0} \eta(a) = 0 \) and a constant \( C \) which is such that for every \( x, y \in \mathcal{K} \), \( t \geq 0 \) and every \( \varepsilon \in (0, 1) \) such that \( \mathcal{K} \subset \mathcal{D}_\varepsilon 
abla \),

\[
|K_t(x, y) - K_t(x, x)| \leq \eta(|x - y|) + Ce^t|x - y|,
\]

\[
|K_{t, \varepsilon}(x, y) - K_{t, \varepsilon}(x, x)| \leq \eta(|x - y|) + Ce^t|x - y|.
\]

Proof. We let \( d_0 := \inf_{x \in \mathcal{K}, y \in \mathbb{R}^d, t} |x - y| \in (0, \infty) \) and set

\[
\mathcal{K}' := \{ x + z : x \in \mathcal{K}, \ |z| \leq (d_0/2) \wedge 1 \}.
\]

The set \( \mathcal{K}' \) is a compact subset of \( \mathcal{D} \). Letting \( \eta \) denote the modulus of continuity of \( K_0 \) restricted to \( \mathcal{K}' \). Then we have by definition for \( x, y \) in \( \mathcal{K}' \)

\[
|K_0(x, y) - K_0(x, x)| \leq \eta(|x - y|)
\]

The identity also holds for \( K_{0, \varepsilon} \) and \( x, y \in \mathcal{K} \). Indeed the assumption \( \mathcal{K} \subset \mathcal{D}_\varepsilon \) implies that the integral \( (2.36) \) defining \( K_{t, \varepsilon} \) only runs over a subset of \( \mathcal{K}' \) and convolution reduces the modulus of continuity. We can thus assume that \( K_0 \equiv 0 \). In that case the inequality

\[
|K_t(x, y) - K_t(x, x)| \leq Ce^t|x - y|
\]

follows from the fact that \( \kappa \) is Lipshitz. The identity holds also for \( K_{t, \varepsilon} \) since convolution reduces the value of the Lipshitz constant. \( \square \)

4. Proof of Proposition 2.4

Our proof is simply based on an explicit computation of the quadratic variation using Itô calculus. This computation is more convenient (for notation) when considering the complex valued martingale

\[
W_t := M_t^{(\gamma)}(f) = \int_\mathcal{D} e^{\gamma X_t(x) - \frac{\gamma^2}{2} K_t(x)} f(x) dx.
\]

The bracket of \( M_t^{(\gamma)}(f, \omega) \) can then easily be expressed in terms of \( \langle W, \overline{W} \rangle_t \) and \( \langle W, W \rangle_t \). Informally we are going to prove that for large \( t \)

\[
d\langle W, \overline{W} \rangle_t \approx C e^{|\gamma|^2 - d} M_0^{(2a)} \left( e^{|\gamma|^2 t} f^2 \right) dt
\]

and that \( d\langle W, W \rangle_t \) is of a much smaller order. We then simply integrate that inequality in \( t \) to obtain the required asymptotics.

4.1. Computing the quadratic variation. We fix \( \gamma \in \mathcal{P}_{\text{III}} \) and \( f \in C^2_\text{c}(\mathbb{R}^d) \) be fixed. We define the complex martingale \( W_t \) by

\[
W_t := M_t^{(\gamma)}(f) = \int_\mathcal{D} e^{\gamma X_t(x) - \frac{\gamma^2}{2} K_t(x)} f(x) dx
\]

Now note that for \( x \in \mathcal{D} \), \( (X_t(x))_{t \geq 0} \) is a continuous time martingale. Using Itô calculus we have

\[
dW_t := \gamma \int_\mathcal{D} e^{\gamma X_t(x) - \frac{\gamma^2}{2} K_t(x)} f(x) dX_t(x) dx.
\]

Now by construction we have \( d\langle X(x), X(y) \rangle_t = Q_t(x, y) \) and we obtain the following expressions for the brackets \( \langle W, \overline{W} \rangle_t \) and \( \langle W, W \rangle_t \)

\[
\langle W, \overline{W} \rangle_t = |\gamma|^2 \int_0^t A_s ds \quad \text{and} \quad \langle W, W \rangle_t = \gamma^2 \int_0^t B_s ds,
\]
where $A_s$ and $B_s$ are defined by
\begin{align}
A_s &:= \int_{\mathbb{D}^2} f(x)f(y)Q_s(x,y)e^{\gamma X_s(x)+\tau X_s(y)+(\beta^2-\alpha^2)K_s(x)} \, dx \, dy, \\
B_s &:= \int_{\mathbb{D}^2} f(x)f(y)Q_s(x,y)e^{\gamma (X_s(x)+X_s(y))-\gamma^2 K_s(x)} \, dx \, dy.
\end{align}
(4.3)

The core of our proof is to show that $A_s$ and $B_s$ properly rescaled converge to $M_0^{(2\alpha)}$ and 0 respectively. Let us define $\hat{K}_s = K_s - K_0$ or
\begin{equation}
\hat{K}_s(x,y) := \int_0^s \kappa(e^t|x-y|) \, dt
\end{equation}
and set
\begin{equation}
a(s, \kappa, \gamma) := \int_{\mathbb{R}^2} Q_s(0,z)e^{\gamma^2 \hat{K}_s(0,z)} \, dz.
\end{equation}

**Proposition 4.1.** We have
\begin{align}
\lim_{t \to \infty} a(t, \kappa, \gamma)^{-1} A_t &= M_0^{(2\alpha)}(e^{\gamma^2 |\kappa|^2}) , \\
\lim_{t \to \infty} a(t, \kappa, \gamma)^{-1} B_t &= 0.
\end{align}
(4.5)

The proof of Proposition 4.1 is detailed in the next subsection and requires several technical steps. Let us show first that it implies Proposition 2.4. For this we require the to check that the following estimate holds (meaning that $\overline{\tau}(t, \kappa, \gamma)$ is the right quantity to normalize with).

**Lemma 4.2.** We have
\begin{equation}
\lim_{t \to \infty} \overline{\tau}(t, \kappa, \gamma)^2 |\gamma|^2 \int_0^t a(s, \kappa, \gamma) \, ds = 2e^{-|\gamma|^2|\kappa|}.
\end{equation}
(4.6)

**Proof.** We have
\begin{align}
\int_0^t |\gamma|^2 a(s, \kappa, \gamma) \, ds &= \int_{|z| \leq 1} \left( e^{\gamma^2 \hat{K}_s(0,z)} - 1 \right) \, dz \\
&= e^{(|\gamma|^2-d)t} \int_{|y| \leq e^t} \left( e^{-|\gamma|^2 \int_0^t (1-\kappa(e^{-s}y)) \, ds} - e^{-|\gamma|^2 t} \right) \, dy,
\end{align}
(4.7)

where the last inequality is obtained using the change of variable $y = e^t z$. When $|\gamma| > \sqrt{d}$ the $e^{-|\gamma|^2t}$ term, once integrated tends to zero. Now we have
\begin{equation}
\lim_{t \to \infty} \int_0^t (1-\kappa(e^{-s}y)) \, ds = \ell_\kappa(y) + \ell_\kappa,
\end{equation}
(4.8)
and when $|y| \leq e^t$ we have
\begin{equation}
\int_0^t (1-\kappa(e^{-s}y)) \, ds \geq (\log |y| - C)_+
\end{equation}
(4.9)

With (4.8) and (4.9) we use dominated convergence and obtain
\begin{equation}
\lim_{t \to \infty} \int_{\mathbb{R}^d} e^{-|\gamma|^2 \int_0^t (1-\kappa(e^{-s}|y|)) \, ds} 1_{|y| \leq e^t} \, dy = e^{-|\gamma|^2 |\kappa|} \int_{\mathbb{R}^d} e^{-|\gamma|^2 \ell_\kappa(y)} \, dy,
\end{equation}
(4.10)
so that (4.6) holds. When $|\gamma| = \sqrt{d}$, considering the first equality in (4.7), we disregard the $-1$ in the integral which only yields a contribution of constant order. Observe that we have $\hat{K}_t(0,z) \leq t$ for all $z$, and that whenever $|z| \in [e^{-t}, 1]$

$$\hat{K}_t(0,z) = \log \frac{1}{|z|} + \int_0^{\log(1/|z|)} (\kappa(e^t|z|) - 1)dz = \log \frac{1}{|z|} - j\kappa. \quad (4.11)$$

This yields

$$\int_{|z| \leq 1} e^{\hat{K}_t(0,z)}dz = e^{-j\kappa} \int_{|z| \in [e^{-t}, 1]} |z|^{-d}dz + \int_{|z| \leq e^{-t}} e^{\hat{K}_t(0,z)}dz. \quad (4.12)$$

The first term is exactly $e^{-j\kappa} t^{2\pi/d/2} (1/d^2)$ (the last factor being the volume of the $d - 1$ dimensional sphere) while the other term is of order one which yield (4.6) in that case too. □

Proof of Proposition 2.4 using Proposition 4.1. The quadratic variation of $N_t = M_t^{(\gamma)}(f, \omega)$ is a linear combination of $\langle W, W \rangle_t$ and $\langle W, \bar{W} \rangle_t$. We have

$$\langle N \rangle_t = \frac{1}{2} \langle W, W \rangle_t + \frac{1}{2} \Im(e^{-2i\omega} \langle W, \bar{W} \rangle_t) \quad (4.13)$$

In view of Lemma 4.2 (4.13) and (2.29), we only need to prove the following convergence in $L^1$

$$\lim_{t \to \infty} \int_0^t |\gamma|^2 a(s, \kappa, \gamma)ds = M_0^{(2\alpha)}(e^{|\gamma|^2|K|K_0} f^2) \quad \text{and} \quad \lim_{t \to \infty} \int_0^t a(s, \kappa, \gamma)ds = 0, \quad (4.14)$$

As a direct consequence of (4.2) and (4.5) we have

$$\lim_{t \to \infty} \int_0^{\sqrt{t}} |\gamma|^2 a(s, \kappa, \gamma)ds = M_0^{(2\alpha)}(e^{|\gamma|^2|K|K_0} f^2) \quad \text{and} \quad \lim_{t \to \infty} \int_0^{\sqrt{t}} a(s, \kappa, \gamma)ds = 0. \quad (4.15)$$

To conclude we only need to check that both $\int_0^{\sqrt{t}} a(s, \kappa, \gamma)ds$ and $\langle W, \bar{W} \rangle_{\sqrt{t}}$ are negligible with respect to $\int_0^t |\gamma|^2 a(s, \kappa, \gamma)ds$. For the first quantity, it is sufficient to observe, using the definition and (3.3), that $a(s, \kappa, \gamma)$ is of order $e^{((|\gamma|^2 - d)s$. As for the second, we have

$$\mathbb{E} \left[ \langle W, \bar{W} \rangle_u \right] = \mathbb{E} \left[ |W_u - W_0|^2 \right] \leq \int_{\mathbb{D}^2} f(x)f(y)e^{|\gamma|^2 K_u(x,y)}dxdy 
\leq e^{|\gamma|^2|K_0|} \|f\|_2^2 \int_{\mathbb{R}^d} e^{(|\gamma|^2 - d)u}dz, \quad (4.16)$$

Now $\int_{\mathbb{R}^d} e^{(|\gamma|^2 - d)u}dz$ is either of order $u$ (if $|\gamma|^2 = d$) or $e^{(|\gamma|^2 - d)u}$ (if $|\gamma|^2 > d$) and with $u = \sqrt{t}$ this is negligible w.r.t. $\int_0^t |\gamma|^2 a(s, \kappa, \gamma)ds$ in all cases. □

4.2. Proof of Proposition 4.1. For the sake of simplicity we are going to assume (recall (2.1)) that $K_0 = 0$. This does not alter the proof at all but provides some welcome simplification in the notation (for instance we have $K_t(x) := K_t(x, x) = t$ for every $x$). Since in that case $K$ is translation invariant, we can extend it to a kernel $\mathbb{R}^d \times \mathbb{R}^d$, thus without loss of generality we are going to assume that $\mathcal{D} = \mathbb{R}^d$. In this case note that $L(x, x) = -j\kappa$ for all $x \in \mathbb{R}^d$. To alleviate further the notation we write $a(t)$ for $a(t, \kappa, \gamma)$.
We set for this proof $u := u_t = t - \log t$. We are going to show the result using intermediate steps. Recalling the notation \((1.23)\) we set
\[
A_t^{(1)} := \int_{\mathbb{R}^d} f^2(x)Q_t(x, y)e^{\gamma X_t(x)+\tau X_t(y)+(\beta^2-\alpha^2)t}dx dy,
\]
\[
A_t^{(2)} := \int_{\mathbb{R}^d} f^2(x)Q_t(x, y)e^{\gamma X_u(x)+\tau X_u(y)+|\gamma|^2 K_{u, t}(x,y)+(\beta^2-\alpha^2)u}dx dy
\]  \tag{4.17}
We set for this proof
\[
18 \quad \text{HUBERT LACOIN}
\]
Using the triangle inequality we have
\[
|A_t - a(t)M_0^{(2\alpha)}(f^2)| \leq |A_t - A_t^{(1)}|
\]
\[+ |A_t^{(1)} - A_t^{(2)}| + |A_t^{(2)} - a(t)M_0^{(2\alpha)}(f^2)| + a(t)|M_0^{(2\alpha)}(f^2) - M_0^{(2\alpha)}(f^2)|. \tag{4.18}
\]
We are going to show that the expectation of each of the summands in the right-hand side of \((4.18)\) are $o(e^{(\gamma^2 - d)t})$ (which is the same as $o(a(t))$). The fourth one is controlled by applying Theorem \(\mathcal{E}\) The first three require more work.

**Step 1: Bounding** $\mathbb{E}[|A_t - A_t^{(1)}|]$. We let $w_f(u) := \max_{|x_1 - x_2| \leq u} f(x_1 - x_2)$ denote the modulus of continuity of $f$. We have
\[
|A_t - A_t^{(1)}| \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |f(x)||f(y) - f(x)|Q_t(x, y)e^{\alpha(X_t(x)+X_t(y)+(\beta^2-\alpha^2)t}dx dy \tag{4.19}
\]
and hence
\[
\mathbb{E} \left[ |A_t - A_t^{(1)}| \right] \leq w_f(e^{-t}) \int_{\mathbb{R}^d} |f(x)| 1_{\{|y - x| \leq e^{-t}\}} e^{(\gamma^2)t} dx dy
\]
\[
= \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)} e^{(\gamma^2 - d)t} w_f(e^{-t}) \int_{\mathbb{R}^d} |f(x)| dx. \tag{4.20}
\]
This implies that
\[
\lim_{t \to \infty} e^{(d-\gamma^2)t} \mathbb{E} \left[ |A_t - A_t^{(1)}| \right] = 0. \tag{4.21}
\]

**Step 1: Bounding** $\mathbb{E}[|A_t^{(1)} - A_t^{(2)}|]$. Now let us consider the second term $A_t^{(2)} - A_t^{(1)}$. This is the most delicate step. Let us set
\[
\xi_t(x, y) := e^{\gamma X_t(x)+\tau X_t(y)+(\beta^2-\alpha^2)t} - \mathbb{E}[e^{\gamma X_t(x)+\tau X_t(y)+(\beta^2-\alpha^2)t} | \mathcal{F}_u],
\]
\[
\zeta_t(x, y) := Q_t(x, y)f(x)^2 \xi_t(x, y)
\]  \tag{4.22}
we have
\[
A_t^{(1)} - A_t^{(2)} = \int_{\mathbb{R}^d} \zeta_t(x, y) dx dy
\]  \tag{4.23}
Our method to bound $A_t^{(1)} - A_t^{(2)}$ depends on whether $\alpha \in [0, \sqrt{d}/2]$ or $\alpha \in [\sqrt{d}/2, \sqrt{d}/2]$ (by symmetry we can assume without loss of generality that $\alpha \geq 0$). When $\alpha < \sqrt{d}/2$ it is sufficient to compute the second moment of $A_t^{(2)} - A_t^{(1)}$. We have
\[
\mathbb{E} \left[ |A_t^{(1)} - A_t^{(2)}|^2 \right] = \int_{\mathbb{R}^{2d}} \mathbb{E} \left[ \zeta_t(x_1, y_1)\zeta_t(x_2, y_2) \right] dx_1 dx_2 dy_1 dy_2. \tag{4.24}
\]
We have for any $x_1, x_2, y_1, y_2$
\[
|\mathbb{E} \left[ \zeta_t(x_1, y_1)\zeta_t(x_2, y_2) \right]| \leq \frac{1}{2} \left( \mathbb{E} \left[ \zeta_t(x_1, y_1)^2 \right] + \mathbb{E} \left[ \zeta_t(x_2, y_2)^2 \right] \right) \leq e^{(6\alpha^2+2\beta^2)t}. \tag{4.25}
\]
We have
\[ |x_1 - x_2| \geq e^{-u} + 2e^{-t} \Rightarrow \begin{cases} |x_1 - y_1| \vee |x_2 - y_2| \geq e^{-t} \quad \text{or} \quad \min_{z_1 \in \{x_1,y_1\}, \ z_2 \in \{x_2,y_2\}} |z_1 - z_2| \geq e^u \end{cases}. \]

If the first condition on the r.h.s. holds we have \( Q_t(x_1, y_1)Q_t(x_2, y_2) = 0 \). If the second condition holds, since at distance \( e^{-u} \) the correlation of the field \( X_{[u,t]} \) vanish, we have \( E[\zeta_t(x_1, y_1)\zeta_t(x_2, y_2) \mid \mathcal{F}_u] = 0 \). Hence (since \( u < t \)) we have
\[ |x_1 - x_2| \geq 3e^{-u} \Rightarrow E[\zeta_t(x_1, y_1)\zeta_t(x_2, y_2) \mid \mathcal{F}_u] = 0, \tag{4.26} \]
and (4.25) and (4.26) combined imply
\[
E \left[ |A_t^{(2)} - A_t^{(1)}|^2 \right] \\
\leq \int_{\mathbb{R}^d} e^{(\alpha^2+2\beta^2)t} f(x_1)^2 f(x_2)^2 Q_t(x_1, y_1)Q_t(x_2, y_2)1_{|x_1-x_2| \leq 3e^{-u}} dx_1 dx_2 dy_1 dy_2 \\
\leq Ce^{2(\gamma^2-d)t}e^{4\alpha^2t-du} \|A\|_1^2. \tag{4.27} \]

If \( 4\alpha^2 < d \) then the second exponential factor goes to zero (recall that \( u = t - \log t \)) and we can deduce that
\[
\lim_{t \to \infty} e^{(d-|\gamma|^2)t} E \left[ |A_t^{(2)} - A_t^{(1)}|^2 \right] = \lim_{t \to \infty} e^{(d-|\gamma|^2)t} E \left[ |A_t^{(2)} - A_t^{(1)}|^2 \right]^{1/2} = 0. \tag{4.28} \]

When \( \alpha \in [\sqrt{\alpha}/2, \sqrt{d}/2] \) we need to combine the above argument with a truncation procedure. Let us fix some parameter \( \lambda \) that satisfies the following assumptions
\[ \lambda \in (2 \alpha, 4 \alpha) \quad \text{and} \quad \frac{1}{2}(4 \alpha - \lambda)^2 > 4 \alpha^2 - d. \tag{4.29} \]

Such a value of \( \lambda \) exists when \( \alpha \in [\sqrt{\alpha}/2, \sqrt{d}/2] \). We introduce the event
\[ A_t(x) := \{ X_u(x) \leq \lambda u \} \]
(4.30)

Now recalling (4.23) we have
\[
A_t^{(2)} - A_t^{(1)} = \int_{\mathbb{R}^{2d}} \zeta_t(x,y)1_{A_t^c(x)} dx dy + \int_{\mathbb{R}^{2d}} \zeta_t(x,y)1_{A_t(x)} dx dy \\
=: W_{t}^{(1)} + W_{t}^{(2)}. \tag{4.31} \]

We compute the first moment of \( W_{t}^{(1)} \) and the second moment of \( W_{t}^{(2)} \). We have
\[
E[|\zeta_t(x,y)|1_{A_t^c(x)}] \leq Q_t(x,y)|f(x)|^2 \left[ e^{(\beta^2-\alpha^2)t} E \left[ e^{\alpha(X_t(x)+X_t(y))}1_{A_t^c(x)} \right] \\
+ e^{(\beta^2-\alpha^2)u+|\alpha|^2K_{[u,t]}(x,y)} E \left[ e^{\alpha(X_u(x)+X_u(y))}1_{A_t^c(x)} \right] \right]. \tag{4.32} \]

Using the Cameron-Martin formula (Proposition 3.1) to compute the expectations we find
\[
E \left[ e^{\alpha(X_t(x)+X_t(y))}1_{A_t^c(x)} \right] = e^{\alpha^2(t+K_t(x,y))} P \left[ X_u(x) > \lambda u - \alpha(u + K_u(x,y)) \right], \tag{4.33} \]
\[
E \left[ e^{\alpha(X_u(x)+X_u(y))}1_{A_t^c(x)} \right] = e^{\alpha^2(u+K_u(x,y))} P \left[ X_u(x) > \lambda u - \alpha(u + K_u(x,y)) \right]. \tag{4.34} \]
Using \( u \) and \( t \) as upper bounds for \( K_t \) and \( K_u \), we obtain that
\[
\mathbb{E} [ |\tilde{\zeta}_t(x,y)1_{A_t(x)}(x)| ] \leq 2Q_t(x,y)|f(x)|^2 e^{(\gamma)^2 t} \mathbb{P} [ X_u(x) \geq (\lambda - 2\alpha)u ] .
\] (4.34)

Recalling that \( \lambda > 2\alpha \), using Gaussian tail estimates (3.1), we bound the above probability by \( e^{-(\lambda-2\alpha)^2 t} \), and obtain
\[
\mathbb{E} [ |W_t^{(1)}(x)| ] \leq 2\|f\|^2 e^{(\gamma)^2 - \frac{(\lambda-2\alpha)^2 t}{2}} \int_{\mathbb{R}^d} K(e^{\gamma} |z|) \, dz \leq C\|f\|^2 e^{(\gamma)^2 - \frac{(\lambda-2\alpha)^2 t}{2}} .
\] (4.35)

Let us now control the second moment of \( W_t^{(2)}(x) \). We have
\[
\mathbb{E} \left[ |W_t^{(2)}(x)|^2 \right] = \int_{\mathbb{R}^{4d}} \mathbb{E} \left[ \tilde{\zeta}_t(x,y)1_{A_t(x) \cap A_t(y)}1_{A_t(x)} \right] \, dx_1 \, dx_2 \, dy_1 \, dy_2 .
\] (4.36)

From (4.26), as \( A_t(x) \cap A_t(y) \) is \( F_u \) measurable, (4.26) implies that
\[
|x_1 - x_2| \geq 3e^{-u} \Rightarrow \mathbb{E} \left[ \tilde{\zeta}_t(x_1,y_1)\tilde{\zeta}_t(x_2,y_2)1_{A_t(x_1) \cap A_t(x_2)} \right] = 0 .
\] (4.37)

When \( |x_1 - x_2| < 3e^{-u} \) we have
\[
\left| \mathbb{E} \left[ \tilde{\zeta}_t(x_1,y_1)\tilde{\zeta}_t(x_2,y_2)1_{A_t(x_1) \cap A_t(x_2)} \right] \right| \leq \|f\|^2 Q_t(x_1,y_1)Q_t(x_2,y_2) \mathbb{E} \left[ |\tilde{\zeta}_t(x_1,y_1)|^2 1_{A_t(x_1)} + |\tilde{\zeta}_t(x_2,y_2)|^2 1_{A_t(x_2)} \right] ,
\] (4.38)

so that
\[
\mathbb{E} \left[ |W_t^{(2)}(x)|^2 \right] \leq \int_{\mathbb{R}^{4d}} |f(x_1)f(x_2)|^2 Q_t(x_1,y_1)Q_t(x_2,y_2) \mathbb{E} \left[ |\tilde{\zeta}_t(x_1,y_1)|^2 1_{A_t(x_1)} + |\tilde{\zeta}_t(x_2,y_2)|^2 1_{A_t(x_2)} \right] 
\times \sup_{|x-y| \leq 3e^{-u}} \mathbb{E} \left[ |\xi_t(x,y)|^2 1_{A_t(x)} \right] .
\] (4.39)

The integral in the first line is smaller than \( C\|f\|^4 e^{-\alpha(2t+u)} \). Finally, using the Cameron-Martin formula (Proposition 3.1), we have
\[
\mathbb{E} \left[ |\tilde{\zeta}_t(x,y)|^2 1_{A_t(x)} \right] \leq \mathbb{E} \left[ e^{\gamma X_t(x) + \tau X_t(y) + (\beta^2 - \alpha^2) t} 1_{A_t(x)} \right] 
= e^{4\alpha^2 K_t(x,y) + \gamma^2 t} \mathbb{P} [ X_u(x) \leq \lambda u - 2\alpha(u + K_u(x,y)) ] .
\] (4.40)

Using the fact that \( K_u(x,y) \geq u - 1 \) if \( |x-y| \leq e^{-t} \), and the fact that \( \lambda < 4\alpha \) (4.29), we have from Gaussian tails estimates (3.1)
\[
\mathbb{P} [ X_u(x) \leq \lambda u - 2\alpha(u + K_u(x,y)) ] \leq C e^{-\frac{(4\alpha - \lambda)^2 u}{2}} .
\] (4.41)

The combination (4.39), (4.40) and (4.41) yields
\[
\mathbb{E} \left[ |W_t^{(2)}(x)|^2 \right] \leq C\|f\|^4 e^{4\alpha^2 + 2(\gamma)^2 t} e^{-\frac{(4\alpha - \lambda)^2 u}{2}} \leq \|f\|^4 e^{2(\gamma)^2 - \delta t} .
\] (4.42)

where, using the fact that \( |t - u| \ll t \), the last line is valid for \( t \) sufficiently large with (cf. (4.29))
\[
\delta := \frac{1}{4} [(4\alpha - \lambda)^2 - 4\alpha^2 + d] > 0 .
\] (4.43)

Recalling (4.31), combining (4.35) and (4.42) we obtain that
\[
\lim_{t \to \infty} e^{(d-\gamma)^2 t} \mathbb{E} \left[ |A_t^{(2)} - A_t^{(1)}| \right] = 0 .
\] (4.44)
Step 3: Bounding $\mathbb{E}[(A^{(2)} - a(t) M_0^{(2\alpha)}(f^2))]$. This is easier and just comes down to a simple first moment estimate. We have
\[
a(t) M_0^{(2\alpha)}(f^2) := \int_{\mathbb{R}^2} f(x)^2 Q_t(x, y) e^{\gamma |y|^2 K_t(x, y)} e^{2\alpha X_u(x) - 2\alpha^2 u} \, dx \, dy
\]
so that from Jensen’s inequality
\[
\mathbb{E} \left[ |a(t) M_0^{2\alpha}(f^2) - A_t^{(2)}| \right] \leq \int_{\mathbb{R}^2} f(x)^2 Q_t(x, y) e^{\gamma |y|^2 K_t(x, y)} \times \mathbb{E} \left[ |e^{2\alpha X_u(x) - 2\alpha^2 u} - e^{\gamma X_u(x) + \beta X_u(y) - |\gamma|^2 K_u(x, y) + (\beta^2 - \alpha^2) u}| \right] \, dx \, dy.
\] (4.46)
Using the Cameron-Martin formula (Proposition 3.1) the expectation in the integral is equal to
\[
\mathbb{E} \left[ e^{2\alpha X_u(x) - 2\alpha^2 u} \left| 1 - e^{\gamma (X_u(y) - X_u(x)) + |\gamma|^2 (u - K_u(x, y))} \right| \right]
= \mathbb{E} \left[ \left| 1 - e^{\gamma (X_u(y) - X_u(x)) - \gamma^2 (u - K_u(x, y))} \right| \right].
\] (4.47)
We can explicitly compute the second moment in the last line. We obtain
\[
\mathbb{E} \left[ \left| 1 - e^{\gamma (X_u(y) - X_u(x)) - \gamma^2 (u - K_u(x, y))} \right|^2 \right] = e^{2|\gamma|^2 (u - K_u(x, y))} - 1.
\] (4.48)
When $|x - y| \leq e^{-t}$, using Lemma 3.2 (note that $K_u(x, x) = u$)
\[
e^{2|\gamma|^2 (u - K_u(x, y))} - 1 \leq e^{2|\gamma|^2 C e^{u-t} + \eta(e^{-t})} - 1 \leq \delta(t).
\] (4.49)
where $\lim_{t \to \infty} \delta(t) = 0$. Going back to (4.46), combining (4.47)-(4.49) yields
\[
\mathbb{E} \left[ |a(t) M_0^{2\alpha}(f^2) - A_t^{(2)}| \right] \leq C \sqrt{\delta(t)} \int_{\mathbb{R}^2} f(x)^2 Q_t(x, y) e^{\gamma |y|^2 K_t(x, y)} \, dx \, dy = C \sqrt{\delta(t)} \|f\|_{L^2}^2 a(t),
\] (4.50)
and finishes the proof for the convergence of $A_t$. We are left with that of $B_t$.

Final step: Bounding $\mathbb{E} [|B_t|]$. For $B_t$ we use the same ideas as for $A_t$, but require only one intermediate step. We define
\[
B_t^{(1)} = \mathbb{E}[B_t \mid \mathcal{F}_u] = \int_{\mathbb{R}^2} f(x) \, f(y) Q_s(x, y) e^{\gamma (X_u(x) + X_u(y)) - \gamma^2 (u - K_{[u,t]}(x,y))} \, dx \, dy
\] (4.51)
and provide a separate bound for $\mathbb{E} \left[ |B_t^{(1)}| \right]$ and $\mathbb{E} \left[ |B_t^{(1)}| - B_t | \right]$. We have
\[
\mathbb{E} \left[ |B_t^{(1)}| \right] \leq \int_{\mathbb{R}^2} f(x) \, f(y) Q_s(x, y) \mathbb{E} \left[ e^{a(X_u(x) + X_u(y)) + (\beta^2 - \alpha^2)(u - K_{[u,t]}(x,y))} \right] \, dx \, dy
\] (4.52)
\[
\leq \int_{\mathbb{R}^2} f(x) \, f(y) Q_s(x, y) e^{a^2 K_t(x,y) + \beta^2 (u - K_{[u,t]}(x,y))} \, dx \, dy
\]
\[
\leq C \|f\|_{L^2}^2 e^{(\gamma^2 - \alpha^2) t - 2\beta^2 (t - u)}.
\]
Hence $\lim_{t \to \infty} e^{\gamma^2} \mathbb{E} \left[ |B_t^{(1)}| \right] = 0$. We can then bound $\mathbb{E}[|B_t - B_t^{(1)}|]$. The reasoning is identical to that used for $A^{(1)} - A_t^{(2)}$ so we provide details only the more delicate case
 Then we decompose the expectation we want to bound, recalling (1.23) and in analogy with (4.22) set
\[ ξ_t(x, y) = e^{γ(X_t(x) + X_t(y)) - γ^2 t} - \mathbb{E}[e^{γX_t(x) + γX_t(y) - γ^2 t} \mid \mathcal{F}_u], \]
\[ ζ_t(x, y) := Q_τ(x, y)f(x)f(y)ξ_t(x, y). \]
We have
\[ |B_t^{(1)} - B_t| \leq \left| \int_{\mathbb{R}^d} ζ_t(x, y)1_{A_t^c}(x)dx dy \right| + \left| \int_{\mathbb{R}^d} ζ_t(x, y)1_{A_t}(x)dx dy \right| \tag{4.54} \]
We prove similarly to (4.32) that
\[ \mathbb{E}\left[|ξ_t(x, y)|1_{A_t^c}(x)\right] \leq Q_τ(x, y)f(x)f(y)e^{γ^2 t - \frac{(λ - 2α)^2}{2} u}, \tag{4.55} \]
which implies that
\[ \mathbb{E}\left[\int_{\mathbb{R}^d} ζ_t(x, y)1_{A_t}(x)dx dy\right] \leq C\|f\|^2 e^{(λ - 2α)^2 u}. \tag{4.56} \]
Finally we control the second moment of the first term in (4.54) like that of \( W_t^{(2)} \) in (4.39).
Repeating the same steps we prove that
\[ \mathbb{E}\left[\left| \int_{\mathbb{R}^d} ζ_t(x, y)1_{A_t}(x)dx dy\right|^2 \right] \leq C\|f\|^4 e^{4α^2 + 2(γ - d) |t| - \frac{(4α - λ)^2}{2} u} \tag{4.57} \]
and conclude in the same manner.

\[ \square \]

5. Deducing Proposition 2.2 from Proposition 2.6

The present section and the next are dedicated to the proof of Proposition 2.2 on which the main result of the paper - Theorem 1.5 - relies. While with the combination of martingale filtration and convolution, the notation become more cumbersome and the computations a bit more delicate, the proof rely almost exclusively on the ideas developed in Section 4 and on an adaptation of Theorem 2.5.

In this section, we show that Proposition 2.6 implies Proposition 2.2 Recall that \( N_t^{(ε)} = \mathbb{E}[M^{(γ)}_ε(f, ω) \mid \mathcal{F}_τ] \). We set for the proof \( Z := M^{(2α)}_ε(e^Lf^2) \) and \( Z^A := \mathbb{E}[Z \wedge A \mid \mathcal{F}_τ] \).
We write simply \( v(ε) \) for \( v(ε, θ, γ) \). Since \( N_{ε}^{(γ)} = M^{(γ)}_ε(f, ω) \) we need to show that
\[ \lim_{ε \to 0} \mathbb{E}\left[ e^{i(X, μ) + iv(ε)N_{ε}^{(γ)}} - e^{i(X, μ) - \frac{1}{2} Z} \right] = 0. \tag{5.1} \]
We set \( t = t_ε := \sqrt{\log 1/ε} \), and (with the convention \( \inf \emptyset = \infty \))
\[ T(A, ε) := \inf\{s \geq 0 : \langle N(ε)x \rangle_s \geq v(ε)^{-2}A\}. \tag{5.2} \]
We also set
\[ Y := e^{i(X, μ)} \quad \text{and} \quad Y_t := \mathbb{E}[Y \mid \mathcal{F}_t]. \tag{5.3} \]
Then we decompose the expectation we want to bound, recalling (1.23)
\[ \left| \mathbb{E}\left[ Y(e^{iv(ε)N_{ε}^{(γ)}} - e^{-\frac{1}{2} Z}) \right] \right| \leq \left| \mathbb{E}\left[ Ye^{iv(ε)N_{ε}^{(γ)}} - Y_t e^{iv(ε)N_{t,T}^{(ε)}} \right] \right| + \left| \mathbb{E}\left[ Y_t \left( e^{iv(ε)N_{t,T}^{(ε)}} - e^{-\frac{1}{2} Z^A} \right) \right] \right| + \left| \mathbb{E}\left[ Y_t e^{-\frac{1}{2} Z^A} - Ye^{-\frac{1}{2} Z} \right] \right|. \tag{5.4} \]
Denoting the three summands appearing in the r.h.s. by \( E_1(\varepsilon, A) \), \( E_2(\varepsilon, A) \) and \( E_3(\varepsilon, A) \), we are going to prove (5.1) by showing that
\[
\forall j \in \{1, 2, 3\}, \quad \lim_{A \to \infty} \limsup_{\varepsilon \to 0} E_j(\varepsilon, A) = 0. \tag{5.5}
\]

For \( E_3 \) we can use dominated convergence twice
\[
\lim_{A \to \infty} \lim_{\varepsilon \to 0} E_3(\varepsilon, A) = \lim_{A \to \infty} \mathbb{E} \left[ Y \left( e^{-\frac{1}{2} Z \wedge A} - e^{-\frac{1}{2} Z} \right) \right] = 0.
\]

For \( E_1 \) we have (recall that \( |Y|, |Y| \leq 1 \))
\[
\mathbb{E} \left[ \left| Y e^{iv(\varepsilon) N_t(\varepsilon)} - Y e^{iv(\varepsilon) N_t(\varepsilon)} \right| \right] \leq 2\mathbb{P}[T < \infty] + \mathbb{E} \left[ \left| Y - Y e^{-iv(\varepsilon) N_t(\varepsilon)} \right| \right]. \tag{5.6}
\]

Combining Proposition 2.4 and Portmanteau Theorem we have
\[
\limsup_{\varepsilon \to 0} \mathbb{P}[T, A] < \infty \leq \mathbb{P}[Z \geq A]. \tag{5.7}
\]

To bound the second term in (5.6), since \( Y_t \) converges to \( Y \) in \( L^1 \), we only need to show that \( v(\varepsilon) N_t(\varepsilon) \) tends to zero in probability. Using (3.4) we have
\[
\mathbb{E} \left[ \left| N_t(\varepsilon) \right|^2 \right] = \int_{D^2} e^{\left| x \right|^2} dx dy \leq C \int_{D^2} (|x - y| \wedge e^{-t})^{\frac{1}{2}} dx dy. \tag{5.8}
\]

The integral on the r.h.s. in (5.8) is of order \( e^{\left| \gamma \right|^2 - |x - y|} \) if \( |\gamma| > \sqrt{d} \) and of order \( t \) if \( |\gamma| = \sqrt{d} \). In both cases, with our value of \( t_\varepsilon \) we obtain that
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ v(\varepsilon)^2 \left| N_t(\varepsilon) \right|^2 \right] = 0.
\]

Finally for \( j = 2 \), using the fact that \( (N_t(\varepsilon))_{t \geq t} \) is a martingale for the filtration \( \mathcal{F}_t \) with bounded quadratic variation, we obtain using martingale exponentiation (at \( s = \infty \)) that
\[
\mathbb{E} \left[ e^{iv(\varepsilon) N_t(\varepsilon)} + \frac{v(\varepsilon)^2}{2} \langle N^{(\varepsilon)} \rangle_{t \wedge T, T} \bigg| \mathcal{F}_t \right] = 1. \tag{5.9}
\]

As a consequence we have
\[
\mathbb{E} \left[ Y_t \left( e^{iv(\varepsilon) N_t(\varepsilon)} + \frac{v(\varepsilon)^2}{2} \langle N^{(\varepsilon)} \rangle_{t \wedge T, T} - e^{-\frac{1}{2} Z_t^A} - e^{-\frac{1}{2} Z_t^A} \right) \right] = 0 \tag{5.10}
\]
and thus
\[
E_2(\varepsilon, A) = \mathbb{E} \left[ Y_t \left( e^{iv(\varepsilon) N_t(\varepsilon)} - e^{iv(\varepsilon) N_t(\varepsilon)} - \frac{1}{2} \langle N^{(\varepsilon)} \rangle_{t \wedge T, T} - Z_t^A \right) \right] \leq \mathbb{E} \left[ \left| 1 - e^{-\frac{1}{2} \langle N^{(\varepsilon)} \rangle_{t \wedge T, T} - Z_t^A} \right| \right]. \tag{5.11}
\]

To conclude, we simply observe that both \( v(\varepsilon)^2 \langle N^{(\varepsilon)} \rangle_{t \wedge T, T} \) and \( Z_t^A \) are bounded and both converge in probability to \( Z \wedge A \) when \( \varepsilon \) goes to zero so that by dominated convergence
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \left| 1 - e^{-\frac{1}{2} \langle N^{(\varepsilon)} \rangle_{t \wedge T, T} - Z_t^A} \right| \right] = 0. \tag{5.12}
\]
This concludes the proof of (5.1). \( \square \)
6. PROOF OF PROPOSITION 2.6

As for Proposition 2.4, we are going to control the derivative of the brackets associated with a complex valued martingale. We set

\[ W_t^{(\varepsilon)} := \int_{\mathbb{R}^d} f(x)e^{\gamma X_{t,\varepsilon}(x)-\gamma^2 K_{t,\varepsilon}(x,y)}1_{D_\varepsilon}dx. \]

Assuming that \( \varepsilon \) is sufficiently small, the support of \( f \) is included in \( D_\varepsilon \) so we do not need to worry about the indicator function. Using that \( (X_{t,\varepsilon}(x))_{t \geq 0} \) is a continuous martingale, we have from Itô calculus

\[ dW_t^{(\varepsilon)} := \gamma \int_{\mathbb{R}^d} f(x)e^{\gamma X_{t,\varepsilon}(x)-\gamma^2 K_{t,\varepsilon}(x,y)}dX_{t,\varepsilon}(x)ds. \] (6.1)

Since by construction we have \( d\langle X_{t,\varepsilon}(x), X_{t,\varepsilon}(y) \rangle_t = Q_{t,\varepsilon}(x, y), \) where

\[ Q_{t,\varepsilon}(x, y) := \int_{\mathbb{R}^d} \theta_\varepsilon(x-z_1)\theta_\varepsilon(y-z_2)\kappa(t|z_1-z_2|)dz_1dz_2, \] (6.2)

this yields

\[ \langle W^{(\varepsilon)}, W^{(\varepsilon)} \rangle_\infty = |\gamma|^2 \int_0^\infty A_{t,\varepsilon}dt \quad \text{and} \quad \langle W^{(\varepsilon)}, W^{(\varepsilon)} \rangle_\infty = \gamma^2 \int_0^\infty B_{t,\varepsilon}dt, \] (6.3)

where we have set

\[ A_{t,\varepsilon} := \int_{\mathbb{R}^d} f(x)f(y)Q_{t,\varepsilon}(x, y)e^{\gamma X_{t,\varepsilon}(x)+\gamma X_{t,\varepsilon}(y)+\frac{\gamma^2-\lambda^2}{2}(K_{t,\varepsilon}(x)+K_{t,\varepsilon}(y))}dx dy, \]

\[ B_{t,\varepsilon} := \int_{\mathbb{R}^d} f(x)f(y)Q_{t,\varepsilon}(x, y)e^{\gamma (X_{t,\varepsilon}(x)+X_{t,\varepsilon}(y))+\frac{\gamma^2}{2}(K_{t,\varepsilon}(x)+K_{t,\varepsilon}(y))}dx dy. \] (6.4)

Recalling the definition of \( \hat{K}_s \) let us define

\[ \hat{K}_{s,\varepsilon}(x, y) := \int_{\mathbb{R}^d} \hat{K}(z_1, z_2)\theta_\varepsilon(x-z_1)\theta_\varepsilon(y-z_2)dz_1dz_2 = \int_0^s Q_{t,\varepsilon}(x, y)dt \]

\[ a(t, \kappa, \gamma, \varepsilon) := \int_{\mathbb{R}^d} Q_{t,\varepsilon}(z)e^{\gamma^2 \hat{K}_{t,\varepsilon}(0,z)}dz. \] (6.5)

We write simply \( \hat{K}_s \) when \( t = \infty \). The key estimate to prove Proposition 2.6 is the following

**Proposition 6.1.** We have the following convergences in \( L^1 \) (valid for the joint limit)

\[ \lim_{(t,\varepsilon) \rightarrow (\infty, 0)} a(t, \kappa, \gamma, \varepsilon)^{-1}A_{t,\varepsilon} = M_0^{(2\alpha)}(\varepsilon K_0 f^2), \]

\[ \lim_{(t,\varepsilon) \rightarrow (\infty, 0)} a(t, \kappa, \gamma, \varepsilon)^{-1}B_{t,\varepsilon} = 0. \] (6.6)

As for Proposition 2.4 in order to deduce Proposition 2.6 from Proposition 6.1 we must check that \( v(\varepsilon, \theta) \) is the correct renormalization.

**Lemma 6.2.** We have

\[ \lim_{\varepsilon \rightarrow 0} |\gamma|^2 v(\varepsilon, \theta)^2 \int_0^\infty a(t, \kappa, \gamma, \varepsilon)dt = 2e^{-|\gamma|^2 j_\kappa}. \] (6.7)
Proof. When $|\gamma|^2 > d$, using the fact that $\hat{K}_\epsilon(0, z) = 0$ when $|z| \geq 1 + 2\epsilon$, and the definition of $v(\epsilon, \theta)$ (1.15), we have

$$|\gamma|^2 v(\epsilon, \theta)^2 \int_0^\infty a(t, \kappa, \gamma, \epsilon) = v(\epsilon, \theta)^2 \int_{\mathbb{R}^d} \left( e|\gamma|^2\hat{K}_\epsilon(0, z) - 1 \right) dz$$

$$= \frac{2}{\int_{\mathbb{R}^d} e|\gamma|^2 d\gamma} \int_{|y| \leq \epsilon^{-1} + 2} \left( e|\gamma|^2(\hat{K}_\epsilon(0, \epsilon y) - \log(1/\epsilon)) - \epsilon|\gamma|^2 \right) dy. \quad (6.8)$$

Integrating $\epsilon|\gamma|^2$ yields something of order $\epsilon^{d-1}|\gamma|^2 = o(1)$. Hence we just need to compute the limit of $\int_{|y| \leq \epsilon^{-1} + 2} e|\gamma|^2(\hat{K}_\epsilon(0, \epsilon y) - \log(1/\epsilon)) dy$. We consider the following decomposition

$$\hat{K}(x, y) = \log \left( \frac{1}{|x-y| + 1} \right) + U(x, y) \quad (6.9)$$

where

$$U(x, y) := \int_0^\infty \kappa(e t|x-y|) dt - \log \left( \frac{1}{|x-y| + 1} \right).$$

The function $U$ is uniformly bounded and equal to $-j_\kappa$ on the diagonal (recall (2.30)). When $|y| \leq 2 + \epsilon^{-1}$ and $\epsilon \leq 1/4$ we have

$$\hat{K}_\epsilon(0, \epsilon y) = \int_{\mathbb{R}^{2d}} \theta_\epsilon(z_1)\theta_\epsilon(z_2) \left( \log \left( \frac{1}{|z_1 + z_2 - \epsilon y|} \right) + U(\epsilon y - z_1 + z_2) \right) dz_1 dz_2. \quad (6.10)$$

Performing a change of variable and recalling (1.14) we have

$$\int_{\mathbb{R}^{2d}} \theta_\epsilon(z_1)\theta_\epsilon(z_2) \log \left( \frac{1}{|z_1 + z_2 - \epsilon y|} \right) dz_1 dz_2 = \ell_\theta(y) + \log(1/\epsilon), \quad (6.11)$$

while

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^{2d}} \theta_\epsilon(z_1)\theta_\epsilon(z_2) U(\epsilon y - z_1 + z_2) = -j_\kappa. \quad (6.12)$$

As a conclusion we obtain that

$$\lim_{\epsilon \to 0} \left( \hat{K}_\epsilon(0, \epsilon y) - \log(1/\epsilon) \right) = \ell_\theta(y), \quad (6.13)$$

$$\left| \left( \hat{K}_\epsilon(0, \epsilon y) - \log(1/\epsilon) \right) - \ell_\theta(y) \right| \leq C.$$  

By dominated convergence, this proves that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^d} e|\gamma|^2(\hat{K}_\epsilon(0, \epsilon y) - \log(1/\epsilon)) 1_{\{|y| \geq 2\epsilon^{-1} + 2\}} dy = e^{-|\gamma|^2 j_\kappa} \int_{\mathbb{R}^d} e|\gamma|^2 \ell_\theta(y) dy, \quad (6.14)$$

and hence, recalling (6.8), that (6.7) holds. Now when $|\gamma| = \sqrt{d}$ we split the integral

$$\int_{\mathbb{R}^d} e^{d\hat{K}_\epsilon(0, z)} - 1 1_{\{|z| \leq 1 + 2 \epsilon\}} dz$$

in three parts (again we can throw the $-1$ out since it does not yield a significant contribution) controlled in (6.15), (6.16) and (6.17) respectively. Since $\hat{K}_\epsilon(0, z) \leq \log(1/\epsilon) + C$ for all $|z|$ we have

$$\int_{|z| \leq \epsilon \log \log(1/\epsilon)} e^{d\hat{K}_\epsilon(0, z)} dz \leq C \log \log(1/\epsilon)^d. \quad (6.15)$$

Using (6.10) (replacing $\epsilon y$ by $z$) we see that $\hat{K}_\epsilon(0, z) \leq \log(3)(1/\epsilon) + C$ when $|z| \geq \log \log(1/\epsilon)$. This yields

$$\int_{\log \log(1/\epsilon) \leq |z| \leq 1 + 2 \epsilon} e^{d\hat{K}_\epsilon(0, z)} dz \leq C \log \log(1/\epsilon)^d. \quad (6.16)$$
Finally, using (6.10) ($U$ is a Lipschitz function) we obtain that
\[ \{ |z| \in (\varepsilon \log (1/\varepsilon), \log (1/\varepsilon)^{-1}) \} \Rightarrow \{ |\hat{K}_\varepsilon(0, z) - \log |z| + j_n| \leq C \log (1/\varepsilon)^{-1} \} \]
so that
\[ \lim_{\varepsilon \to 0} \frac{e^{d_{\varepsilon}}}{|z|} |e^{d_{\varepsilon}} \hat{K}_\varepsilon(0, z) \, dz = 1. \]  
(6.17)

The denominator is asymptotically equivalent to $\frac{2\pi d \varepsilon^2}{(d/2)} \log (1/\varepsilon)$ (the prefactor is simply the volume of the $d - 1$ dimensional sphere of radius one). Combined with (6.15) and (6.16), this yields (6.7) also in that case.  

**Proof of Proposition 2.6.** As $N^{(e)} = \Re e^{i\omega} W^{(e)}e^{-i\omega}$ we have
\[ \langle N^{(e)} \rangle_\infty = \frac{1}{2} \langle W^{(e)}, W^{(e)} \rangle_\infty + \frac{1}{2} \Re e^{-2i\omega} \langle W^{(e)}, W^{(e)} \rangle_\infty. \]  
(6.18)

In view of (6.7), it is sufficient to prove that
\[ \lim_{\varepsilon \to 0} \frac{\langle W^{(e)}, W^{(e)} \rangle_\infty}{|| \gamma \|^2 \int_0^\infty a(t, \kappa, \gamma, \varepsilon) \, dt} = M_0(2\alpha)(e^{K_0 f^2}) \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_0^\infty a(t, \kappa, \gamma, \varepsilon) \, dt = 0. \]  
(6.19)

Let us set $t_\varepsilon := \sqrt{\log (1/\varepsilon)}$ (we need $1 < t_\varepsilon < \log (1/\varepsilon)$). From (6.3) and Proposition 6.1 we have
\[ \lim_{\varepsilon \to 0} \frac{\langle W^{(e)}, W^{(e)} \rangle_{t, \infty}}{\int_0^\infty a(s, \kappa, \gamma, \varepsilon) \, ds} = M_0(2\alpha)(e^{K_0 f^2}) \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_0^\infty a(s, \kappa, \gamma, \varepsilon) \, ds = 0. \]

To conclude we just need to show that
\[ \lim_{\varepsilon \to 0} \frac{\mathbb{E} \left[ \langle W^{(e)}, W^{(e)} \rangle_{t, \infty} \right]}{\int_0^\infty a(s, \kappa, \gamma, \varepsilon) \, ds} = \lim_{\varepsilon \to 0} \int_0^\infty a(s, \kappa, \gamma, \varepsilon) \, ds = 0. \]  
(6.20)

This is done like in the proof of Proposition 2.4 in Section 4. The numerators are of order $e^{(|\gamma|^2 - d) t_\varepsilon}$ (if $|\gamma| > \sqrt{d}$) or $t_\varepsilon$ (if $|\gamma| = \sqrt{d}$), while the denominator is of order $\varepsilon d - |\gamma|^2$ or $|\log \varepsilon|$, which is much larger in both cases (details are left to the reader).  

**6.1. Proof of Proposition 6.1.** We proceed as in the case without convolution (Proposition 4.1) in several steps. The case of $A_{t, \varepsilon}$ is the more delicate one so we treat it in full details and then sketch briefly the proof for $B_{t, \varepsilon}$. We assume that $K_0 \equiv 0$ for the sake of simplifying notation. For the need of the computation we need to fix $u < t$ such that $X_{u, \varepsilon}(x) - X_{u, \varepsilon}(y)$ is small whenever $Q_{t, \varepsilon}(x, y) > 0$. Keeping (3.4) and Lemma 3.2 in mind, a convenient choice is
\[ u := u(t, \varepsilon) = \left( t \wedge \log \frac{1}{\varepsilon} \right) - \log \left( t \wedge \log \frac{1}{\varepsilon} \right). \]  
(6.21)

We set
\[ k(t, \varepsilon) := K_{t, \varepsilon}(x, x) \]
(since $K_0 \equiv 0$, the field is translation invariant so it does not depend on $x$). Note that from (3.2) we have
\[ \left| k(t, \varepsilon) - \left( t \wedge \log \frac{1}{\varepsilon} \right) \right| \leq C, \]  
(6.22)
so that we have in particular in the $\varepsilon \to 0$ and $t \to \infty$ regime

$$1 \ll k(t, \varepsilon) - k(u, \varepsilon) \ll k(t, \varepsilon) \quad \text{and} \quad |k(u, \varepsilon) - u| \leq C$$ \hfill (6.23)

We define, using \hfill (1.23)

$$A_{t, \varepsilon}^{(1)} := \int_{\mathbb{R}^d} f(x)^2 Q_{t, \varepsilon}(x, y)e^{\gamma X_{t, \varepsilon}(x) + \gamma X_{t, \varepsilon}(y) + (\beta - \alpha^2)k(t, \varepsilon)} \, dx \, dy,$$

$$A_{t, \varepsilon}^{(2)} := \int_{\mathbb{R}^d} f(x)^2 Q_{t, \varepsilon}(x, y)e^{\gamma X_{t, \varepsilon}(x) + \gamma X_{t, \varepsilon}(y) + (\beta - \alpha^2)k(u, \varepsilon) + |\gamma|^2 K_{t, u, \varepsilon}(x, y)} \, dx \, dy,$$ \hfill (6.24)

and we need to show that each of the four summands in the r.h.s. divided by $a(t, \varepsilon)$ tend to zero. We define, using (1.23)

$$M_{t, \varepsilon}^{(2a)}(f^2) := \int_{\mathbb{R}^d} f(x)^2 e^{2\alpha X_{t, \varepsilon}(x) - 2\alpha^2 k(t, \varepsilon)} \, dx \, dy.$$ \hfill (6.25)

Writing $a(t, \varepsilon)$ for $a(t, \kappa, \gamma, \varepsilon)$ we have

$$|A_{t, \varepsilon} - a(t, \varepsilon) M_0^{(2a)}(f^2)| \leq |A_{t, \varepsilon} - A_{t, \varepsilon}^{(1)}| + |A_{t, \varepsilon}^{(1)} - A_{t, \varepsilon}^{(2)}|$$

$$+ |A_{t, \varepsilon}^{(2)} - a(t, \kappa, \gamma, \varepsilon) M_{t, \varepsilon}^{(2a)}| + a(t, \varepsilon)|M_{t, \varepsilon}^{(2a)}(f^2) - M_0^{(2a)}(f^2)|$$ \hfill (6.26)

and we need to show that each of the four summands in the r.h.s. divided by $a(t, \kappa, \gamma, \varepsilon)$ tend to zero in $L^1$ when $(t, \varepsilon) \to (x, 0)$. This is done in four separate steps. The argument for the first three are essentially the same as for the proof of Proposition 4.1 in Section 4.

**Step 1: Bounding $\mathbb{E}\left[|A_{t, \varepsilon} - A_{t, \varepsilon}^{(1)}|\right]$.** Let us prove first that

$$\lim_{\varepsilon \to 0} \frac{a(t, \varepsilon)}{t} \mathbb{E}\left[|A_{t, \varepsilon}^{(1)} - A_{t, \varepsilon}^{(2)}|\right] = 0.$$ \hfill (6.27)

Now since $Q_{t, \varepsilon}(x, y) = 0$ when $|x - y| > e^{-t} + 2\varepsilon$, letting $w_f$ denote the modulus of continuity of $f$ we have

$$\mathbb{E}\left[|A_{t} - A_{t, \varepsilon}^{(1)}|\right] \leq w_f(e^{-t} + 2\varepsilon) \int_{\mathbb{R}^d} |f(x)| Q_{t, \varepsilon}(x, y) e^{\gamma K_{t, \varepsilon}(x, y)} \, dx \, dy$$

$$= w_f(e^{-t} + 2\varepsilon) \|f\|_1 a(t, \varepsilon).$$ \hfill (6.28)

We conclude the proof by simply observing that $w_f(e^{-t} + 2\varepsilon)$ tends to zero.

**Step 2: Bounding $\mathbb{E}\left[|A_{t, \varepsilon}^{(1)} - A_{t, \varepsilon}^{(2)}|\right]$.** Let us now prove that

$$\lim_{(t, \varepsilon) \to (\infty, 0)} \frac{a(t, \varepsilon)}{t} \mathbb{E}\left[|A_{t, \varepsilon}^{(1)} - A_{t, \varepsilon}^{(2)}|\right] = 0$$ \hfill (6.29)

We provide details only for the more delicate case $\alpha \in [\sqrt{d}/2, \sqrt{d}/2)$ (the reader can repeat the procedure in Section 4 for the case $\alpha \in [0, \sqrt{d}/2]$). Let us set

$$\xi_{t, \varepsilon}(x, y) := e^{\gamma X_{t, \varepsilon}(x) + \gamma X_{t, \varepsilon}(y) + (\beta - \alpha^2)k(t, \varepsilon) - \mathbb{E}[e^{\gamma X_{t, \varepsilon}(x) + \gamma X_{t, \varepsilon}(y) + (\beta - \alpha^2)k(t, \varepsilon) | \mathcal{F}_u}],$$

$$\zeta_{t, \varepsilon}(x, y) := Q_{t, \varepsilon}(x, y) f(x)^2 \xi_{t, \varepsilon}(x, y).$$ \hfill (6.30)

We fix $\lambda$ satisfying (4.29) and set similarly to (4.30)

$$A_{t, \varepsilon}(x) = \{X_{t, \varepsilon}(x) \leq \lambda k(u, \varepsilon)\}$$ \hfill (6.31)
satisfying

Thus we obtain that

To prove (6.29) we are going to bound the \( L^1 \) norm of the first term and the \( L^2 \) norm of the second term. We have using the Cameron-Martin formula (Proposition 3.1)

Using the Gaussian tail bound (3.1), this entails that

For the second moment computation \( W_{t,\varepsilon}^{(2)} \), note that the range of correlation of \( X_{[u,t],\varepsilon} \) is smaller than \( e^{-u} + 2\varepsilon \) and that \( Q_{t,\varepsilon}(x, y) = 0 \) when \( |x - y| \geq e^{-t} + 2\varepsilon \) so that, the same argument used to prove (4.26) yields

Hence taking \( \varepsilon \) sufficiently small and \( t \) sufficiently large, since \( \mathcal{A}_{t,\varepsilon}(1) \cap \mathcal{A}_{t,\varepsilon}(2) \) is \( \mathcal{F}_u \)-measurable, we have,

When \( x_1 \) and \( x_2 \) are closer to each other, we use the variance to bound the covariance. We have

Our definition of \( u \) and Lemma 3.2 implies that there exists some positive constant \( C \) satisfying

Thus using Gaussian tail bounds (3.1) we have, whenever \( Q_{t,\varepsilon}(x, y) > 0 \)

Thus we obtain that

Now the integral above can be bounded by

\[
C\|f\|^2_2 e^{-du} \left( \int_{\mathbb{R}^d} Q_{t,\varepsilon}(0, z) dz \right)^2. \tag{6.40}
\]
Thus we obtain that
\[
E \left[ |W_{t,ε}^{(2)}|^2 \right] \leq C \|f\|^2_2 a(t, ε)^2 e^{α^2 k(t, ε) - du - (\lambda - 4α^2 k(u, ε))^2 / 2}. \tag{6.41}
\]
It only remains to prove that the exponential factor in the r.h.s. goes to zero. This is simply a consequence of (4.29) and (6.23) (which essentially allows to replace \( u \) and \( k(u, ε) \) by \( k(t, ε) \)).

**Step 3: Bounding** \( E \left[ |A_{t,ε}^{(2)} - M_{u,ε}^{(2n)}| \right] \). We prove now that
\[
\lim_{(t, ε) \to (x, 0)} a(t, ε)^{-1} E \left[ A_{t,ε}^{(1)} - A_{t,ε}^{(2)} \right] = 0. \tag{6.42}
\]
This much easier, we have from Jensen’s inequality
\[
E \left[ A_{t,ε}^{(2)} - a(t, ε) M_{u,ε}^{(2n)} \right] \leq \int_{\mathbb{R}^2} f(x)^2 Q_t(x, y) e^{\gamma^2 K_{t,ε}(x,y)} \times E \left[ e^{2α X_{u,ε}(x) - 2α k(u, ε)} e^{\gamma^2 (k(u, ε) - K_{u,ε}(x,y))} - 1 \right] \, dx \, dy. \tag{6.43}
\]
Using the Cameron-Martin formula (Proposition 3.1) we have
\[
E \left[ e^{2α X_{u,ε}(x) - 2α k(u, ε)} e^{\gamma^2 (k(u, ε) - K_{u,ε}(x,y))} - 1 \right] = E \left[ e^{\gamma^2 (k(u, ε) - K_{u,ε}(x,y))} - 1 \right] \tag{6.44}
\]
and finally
\[
E \left[ e^{\gamma^2 (k(u, ε) - K_{u,ε}(x,y))} - 1 \right]^2 \leq E \left[ e^{\gamma^2 (k(u, ε) - K_{u,ε}(x,y))} - 1 \right] = e^{2|\gamma^2 (k(u, ε) - K_{u,ε}(x,y))} - 1. \tag{6.45}
\]
Now, using Lemma 3.2 and our definition of \( u \), we have whenever \(|x - y| \leq ε^t + 2ε\)
\[
k(u, ε) - K_{u,ε}(x,y) \leq Ce^{-\eta(|x - y| + η(|x - y|)} \leq δ(t, ε) \tag{6.46}
\]
where \( δ(t, ε) \) when \( t \) goes to infinity and \( ε \) does to 0. Altogether we obtain that
\[
E \left[ A_{t,ε}^{(2)} - a(t, ε) M_{u,ε}^{(2n)} \right] \leq C \|f\|^2_2 a(t, ε) \sqrt{δ(t, ε)}, \tag{6.47}
\]
concluding the proof of (6.42).

**Step 4: Bounding** \( E \left[ |M_{t,ε}^{(2n)} - M_0^{(2n)}| \right] \). Let us finally discuss the fourth term. We want to prove the following
\[
\lim_{t \to -\infty} \lim_{t \to -\infty} E \left[ |M_{t,ε}^{(2n)}(f^2) - M_0^{(2n)}(f^2)| \right] = 0. \tag{6.48}
\]
Omitting \( f^2 \) from the notation for better readability we have
\[
E \left[ |M_{t,ε}^{(2n)} - M_0^{(2n)}| \right] = E \left[ |M_{t,ε}^{(2n)} - M_t^{(2n)}| \right] + E \left[ |M_t^{(2n)} - M_0^{(2n)}| \right]. \tag{6.49}
\]
The second term tends to zero when \( t \to \infty \) thanks to Theorem 3. As for the first one, note that we have
\[
M_{t,ε}^{(2n)} - M_t^{(2n)} = E \left[ M_{ε}^{(2n)} - M_0^{(2n)} \mid F_t \right]
\]
so that by Jensen inequality for conditional expectation we have
\[
\mathbb{E} \left[ |M_t^{(2\alpha)} - M_t^{(2\alpha)}| \right] \leq \mathbb{E}[|M_t^{(2\alpha)} - M_0^{(2\alpha)}|],
\]
and by Theorem [A], the right-hand side tends to zero when \( \varepsilon \) goes to zero, yielding (6.48).

**Final step: Bounding \( B_{t,\varepsilon} \).** Finally to bound \( B_{t,\varepsilon} \) we proceed as we did for \( B_t \). We show
\[
\lim_{(t,\varepsilon) \to (\infty,0)} a(t,\varepsilon)^{-1} \mathbb{E}[[E[B_{t,\varepsilon} | F_u]]] = 0,
\]
\[
\lim_{(t,\varepsilon) \to (\infty,0)} a(t,\varepsilon)^{-1} \mathbb{E}[[B_{t,\varepsilon} - \mathbb{E}[B_{t,\varepsilon} | F_u]]] = 0.
\]

For the first line we repeat the argument of **Step 3** above while for the second line we use the same proof as in **Step 2**.

□

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**Appendix A. Proof of Lemma 2.1**

Let \( D' \) and \( \delta \) be fixed. We consider \( \kappa_0 \) a \( C^\infty \) kernel satisfying the assumptions listed below (2.1). Setting \( \eta := (s - d)/2 \) we consider \( K^\delta \) defined by
\[
K^\delta(x,y) := K(x,y) + \eta \delta \int_0^\infty e^{-\eta t} \kappa_0(e^t|x-y|)dt.
\]
Note that \( K' \) satisfies assumptions (A)-(B) since by construction
\[
(K^\delta - K)(x,y) = \eta \delta \int_0^\infty e^{-\eta t} \kappa_0(e^t|x-y|)dt
\]
is a positive definite kernel and is equal to \( \delta \) on the diagonal. The fact that \( K^\delta \) can be written in the form (2.1) is a consequence of the following claim.

**Lemma A.1.** If \( t_0 \) is sufficiently large then the kernel \( G(t_0) \) defined by
\[
G(t_0)(x,y) := K^\delta(x,y) - \int_{t_0}^\infty \kappa_0(e^t|x-y|)dt = K^\delta(x,y) - \int_{t_0}^\infty \kappa_0(e^{t_0+u}|x-y|)du
\]
is positive definite and Hölder continuous.

The lemma implies that (2.1) is satisfied for \( K^\delta(x,y) \) with \( \kappa(u) = \kappa_0(e^{t_0}u) \). It is a consequence of the two following estimates.

**Lemma A.2.** Given \( s > d \), there exists a constant \( C(s,\kappa_0,K) \) such that for any \( \varphi \in C^\infty(D') \) we have
\[
\int_{\mathbb{R}^d} \left( K(x,y) - \int_{t_0}^\infty \kappa_0(e^t|x-y|)dt \right) \varphi(x)\varphi(y)dxdy \geq -Ce^{-\frac{(d-s)t_0}{2}}\|\varphi\|^2_{H^{-s/2}(\mathbb{R}^d)}
\]
Lemma A.3. For any $\varphi \in C_c^\infty(\mathcal{D}')$ we have
\[
\int \left( \eta \delta \int_0^\infty e^{-\eta t} \kappa_0(e^t|x-y|) dt \right) \varphi(x)\varphi(y) dx dy \geq \delta c(\kappa_0, s) \|\varphi\|_{H^{-s/2}(\mathbb{R}^d)}^2 \tag{A.4}
\]

Let us first deduce Lemma A.3 from Lemma A.2 and Lemma A.3.

Proof of Lemma A.3. First note that
\[
K^\delta(x, y) - \int_{t_0}^\infty \kappa_0(e^t|x-y|) dt = L(x, y) + \eta \delta \int_0^\infty e^{-\eta t} \kappa_0(e^t|x-y|) dt + \left( \log \frac{1}{|x-y|} - \int_{t_0}^\infty \kappa_0(e^t|x-y|) dt \right) \tag{A.5}
\]
Each of the three summands in the r.h.s. are Hölder continuous on $\mathcal{D}' \times \mathcal{D}'$ (L because of its Sobolev regularity combined with Morrey’s inequality, the case of the other summands can be checked by hand). Now combining (A.3) and (A.4) we have for any $\varphi \in C_c^\infty(\mathcal{D}')$
\[
\int_{\mathbb{R}^d} \left( K^\delta(x, y) - \int_{t_0}^\infty \kappa_0(e^t|x-y|) dt \right) \varphi(x)\varphi(y) dx dy \geq \left( \delta c(\kappa_0, s) - C e^{-\frac{(d-s)}{2} t_0} \right) \|\varphi\|_{H^{-s/2}(\mathbb{R}^d)}^2 \tag{A.6}
\]
and the r.h.s. is positive if $t_0$ is chosen to be sufficiently large.

\[\square\]

Proof of Lemma A.2. First we notice that we have
\[
K(x, y) = \left( L(x, y) + \log \frac{1}{|x-y|} - \int_0^\infty \kappa_0(e^t|x-y|) dt \right) + \int_0^\infty \kappa_0(e^t|x-y|) dt
\tag{A.7}
\]
where $L \in H^s_{\text{loc}}(\mathcal{D}' \times \mathcal{D}')$ (this is simply because $\log \left( \frac{1}{|x-y|} \right) - \int_0^\infty \kappa_0(e^t|x-y|) dt$ is a $C^\infty$ function). Now we consider $\varepsilon > 0$ sufficiently small so that $\mathcal{D}' \subset \mathcal{D}_\varepsilon (1.6)$. Recalling (1.8), we are going to use the $\varepsilon$ subscript notation for convolution with $\theta_\varepsilon$ on both coordinates for $L$ also. Since $K_\varepsilon$ is definite positive, it is sufficient to show that the inequality (A.3) holds for the following kernel
\[
K(x, y) - K_\varepsilon(x, y) - \int_{t_0}^\infty \kappa_0(e^t|x-y|) dt
\tag{A.8}
\]
Now using [11] Lemma 4.6, we have for any $\varphi \in C_c^\infty(\mathcal{D}')$
\[
\left| \int \varphi(x)\varphi(y)(L - L_\varepsilon)(x, y) dx dy \right| \leq C_{\mathcal{D}'} \|L - L_\varepsilon\|_{H^s(\mathbb{R}^d)} \|\varphi\|_{H^{-s/2}(\mathbb{R}^d)}^2 \tag{A.9}
\]
To control the remaining part, let us set
\[
\psi(x) := \int \left( K_\varepsilon(x, y) - \int_{t_0}^\infty \kappa_0(e^t|x-y|) dt \right) \varphi(y) dy. \tag{A.10}
\]
We have using Plancherel Theorem

\[
\int_{\mathbb{R}^d} \varphi(x) \varphi(y) \left( \int_0^\infty \kappa_0(e^t|x-y|)dt - \hat{K}_\varepsilon(x,y) \right) dx dy = - \int_{\mathbb{R}^d} \varphi(x) \hat{\psi}(x) dx = - (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{\psi}(\xi) \hat{\phi}(\xi) d\xi. \quad \text{(A.11)}
\]

Now using the formula for Fourier transform of convolution and rescaled functions, we have

\[
\hat{\psi}(\xi) = \left( \int_0^\infty \left( |\hat{\theta}(\varepsilon \xi)|^2 - 1_{\{t \leq t_0\}} \right) e^{-dt} \kappa_0(e^{-t} \xi) dt \right) \hat{\phi}(\xi) =: T(\xi) \hat{\phi}(\xi). \quad \text{(A.12)}
\]

To conclude we only need an upper bound on \( T(\xi) \). Note that since \(|\hat{\theta}(\xi)| \leq 1\) for every \( \xi \) and, since by Bochner’s Theorem \( \kappa_0 \) is pointwise real and non-negative, we have

\[
T(\xi) \leq |\hat{\theta}(\varepsilon \xi)|^2 \int_0^\infty e^{-dt} \kappa_0(e^{-t} \xi) dt \leq \frac{1}{d} \hat{\theta}(\varepsilon \xi)^2 e^{-d t_0}. \quad \text{(A.13)}
\]

We can now set \( \varepsilon = e^{-\frac{(s+d) t_0}{2s}} \). Now since \( \theta \) is \( C^\infty \), \( \hat{\theta}(\xi) \) decays faster than any negative power of \( |\xi| \) at infinity, and hence we can find a constant \( C \) (depending on \( \theta \) and \( s \)) which is such that

\[
T(\xi) \leq C e^{-d t_0} (1 + |\xi|^2)^{-s/2} \leq C e^{-(d-s) t_0/2} (1 + \xi^2)^{-s}, \quad \text{(A.14)}
\]

which is sufficient to conclude.

**Proof of Lemma A.3.** Following the same computation as in (A.11)-(A.12) we obtain that

\[
\int \left( \eta \delta \int_0^\infty e^{-\eta t} \kappa_0(e^t|x-y|) dt \right) \varphi(x) \varphi(y) dx dy
\]

\[
= \eta \delta \int \left( \int_0^\infty e^{-(\eta + d) t} \kappa_0(e^{-t} \xi) dt \right) \hat{\phi}(\xi)^2 d\xi. \quad \text{(A.15)}
\]

Now as \( \kappa_0 \) is non-negative (and positive around 0) we have

\[
\eta \int_0^\infty e^{-(\eta + d) t} \kappa_0(e^{-t} \xi) dt \geq c_\eta \left( 1 + |\xi|^2 \right)^{-(d+\eta)/2}, \quad \text{(A.16)}
\]

which is sufficient to conclude.

**Appendix B. Proof of Proposition 1.4.**

In order to check the tightness we work with the Fourier transform \( \widehat{M}_\varepsilon^{(\gamma, \rho)}(\xi) \) of \( M_\varepsilon^{(\gamma)}(\rho) \) which is almost surely finite. Most of the time we will omit the dependence in \( \gamma, \rho \) for better readability, and simply write \( v(\varepsilon) \) for \( v(\varepsilon, \theta, \gamma) \).

We need to show that \( (v(\varepsilon)M_\varepsilon^{(\gamma)})(\rho) \cdot \) is tight in \( H^{-u}(\mathbb{R}^d) \), which, by isometry, is equivalent to showing that \( v(\varepsilon)M_\varepsilon \) is tight in the space \( \widehat{H}^{-u}(\mathbb{R}^d) := L^2(\mathbb{R}^d, (1 + |\xi|^2)^{-u} d\xi) \). To prove the later statement we are going to use the following variant of the Frechet-Kolmogorov compactness criterion (see e.g. [2, Theorem 4.26]).

**Proposition B.1.** A subset of \( K \) of \( \widehat{H}^{-s}(\mathbb{R}^d) \) is relatively compact if and only if it satisfies the following conditions

(i) \( \lim_{R \to \infty} \sup_{\varphi \in K} \int_{|\xi| > R} |\varphi(\xi)|^2 (1 + |\xi|^2)^{-s} d\xi = 0. \)
(ii) \( \lim_{a \to 0} \sup_{\varphi \in K} \int_{\mathbb{R}^d} |\varphi(\xi + a) - \varphi(\xi)|^2 (1 + |\xi|^2)^{-u} \, d\xi = 0. \)

Now the tightness of \( \widehat{M}_\varepsilon \) will be proved using the following simple estimates.

**Lemma B.2.** We have

\[
\mathbb{E}[v(\varepsilon)^2 | \widehat{M}_\varepsilon(\xi)|^2] \leq C(\rho) \tag{B.1}
\]

\[
\mathbb{E}[v(\varepsilon)^2 | \widehat{M}_\varepsilon(\xi + a) - \widehat{M}_\varepsilon(\xi)|^2] \leq |a|^2. \tag{B.2}
\]

**Proof.** The proof of both bounds follows from direct computation. We have

\[
\mathbb{E} \left[ |\widehat{M}_\varepsilon(\xi)|^2 \right] = \int_{\mathbb{R}^{2d}} \rho(x) \rho(y) e^{i\xi(x-y)} \mathbb{E} \left[ e^{|X_\varepsilon(x) + \gamma X_\varepsilon(y) - \frac{\gamma^2}{2} K_\varepsilon(x) - \frac{\gamma^2}{2} K_\varepsilon(y)} \right] \, dx \, dy
\]

\[
= \int_{\mathbb{R}^{2d}} \rho(x) \rho(y) e^{i\xi(x-y)} e^{\gamma^2 K_\varepsilon(x,y)} \, dx \, dy \leq \int_{\mathbb{R}^{2d}} \rho(x) \rho(y) e^{\gamma^2 K_\varepsilon(x,y)} \, dx \, dy. \quad \tag{B.3}
\]

The last integral is of order \( v(\varepsilon)^2 \). In the same fashion we have

\[
\mathbb{E}[|\widehat{M}_\varepsilon(\xi + a) - \widehat{M}_\varepsilon(\xi)|^2] = \int_{\mathbb{R}^{2d}} \rho(x) \rho(y) \left( e^{i\xi,x - e^{i(\xi + a),x}} \right) \left( e^{-i\xi,y - e^{-i(\xi + a),y}} \right) e^{\gamma^2 K_\varepsilon(x,y)} \, dx \, dy \leq |a|^2 \int_{\mathbb{R}^{2d}} \rho(x) \rho(y) |x||y| e^{\gamma^2 K_\varepsilon(x,y)} \, dx \, dy. \quad \tag{B.4}
\]

Since the support of \( \rho \) is compact and hence bounded, the last integral is also of order \( v(\varepsilon)^2 \).

Now given \( A \) and \( d/2 < u' < u \) we define \( K^1_A := K^{(1)}_A \cap K^{(2)}_A \) with

\[
K^{(1)}_A := \left\{ \varphi : \int_{\mathbb{R}^d} |\varphi(\xi)|^2 (1 + |\xi|^2)^{-u'} \, d\xi \leq A \right\},
\]

\[
K^{(2)}_A := \left\{ \varphi : \forall |a| \leq 1, \|\varphi(\cdot + a) - \varphi\|_{H^{-u}(\mathbb{R}^d)} \leq A \sqrt{|a|} \right\}. \quad \tag{B.5}
\]

It is immediate to check from Lemma B.1 that \( K^{(1)}_A \cap K^{(2)}_A \) is relatively compact. To conclude the proof of Proposition 1.4 we only have to check the following.

**Lemma B.3.** We have

\[
\lim_{A \to \infty} \sup_{\varepsilon \in (0,\varepsilon_0)} \mathbb{P} \left[ v(\varepsilon) \widehat{M}_\varepsilon \notin K_A \right] = 0. \quad \tag{B.6}
\]

**Proof.** To show that

\[
\lim_{A \to \infty} \sup_{\varepsilon \in (0,\varepsilon_0)} \mathbb{P} \left[ v(\varepsilon) \widehat{M}_\varepsilon \notin K^{(1)}_A \right] = 0, \tag{B.7}
\]

it is sufficient to observe that from Proposition B.2 and the fact that \( s' > d/2 \), we have

\[
\mathbb{E} \left[ \int_{\mathbb{R}^d} v(\varepsilon)^2 |\widehat{M}_\varepsilon(\xi)|^2 (1 + |\xi|^2)^{-u'} \, d\xi \right] < C'(\rho). \quad \tag{B.8}
\]

Then (B.6) simply follows from Markov inequality. Let us now prove

\[
\lim_{A \to \infty} \sup_{\varepsilon \in (0,\varepsilon_0)} \mathbb{P} \left[ v(\varepsilon) \widehat{M}_\varepsilon \notin K^{(2)}_A \right] = 0. \quad \tag{B.9}
\]
We introduce
\[ \Phi(u) := \max_{\xi \in \mathbb{R}^d} \left( \frac{1 + |\xi - a|^2}{1 + |\xi|^2} \right)^{u/2}. \] (B.9)

Using again Lemma [B.2] and Markov inequality we have given \( b \in \mathbb{R}^d \)
\[ \mathbb{P} \left[ \int_{\mathbb{R}^d} v(\xi)^2 |\hat{M}_\varepsilon(\xi + b) - \hat{M}_\varepsilon(\xi)|^2 (1 + |\xi|^2)^u d\xi \geq u \right] \leq \frac{C|b|^2}{u}. \] (B.10)

We apply it to \( b_{k,i} = 2^{-k}e_i \) for \( k \geq 1 \) and \( i \in [1,d] \) where \( e_i \) are the unit coordinate vectors. Hence setting
\[ K_A^{(3)} := \left\{ \varphi : \forall (k,i) \in \mathbb{N} \times [1,d], \|\varphi(\cdot + b_{k,i}) - \varphi\|_{\dot{H}^{-u}(\mathbb{R}^d)} \geq \frac{A2^{-k/2}(\sqrt{2} - 1)}{\sqrt{2}d\Phi(u)} \right\}. \] (B.11)

We obtain after a union bound over \( i \) and \( k \) that
\[ \mathbb{P} \left[ v(\xi) \hat{M}_\varepsilon \notin K_A^{(3)} \right] \leq \frac{C}{A^2} \] (B.12)

Then (B.8) follows from the inclusion \( K_A^{(3)} \subset K_A^{(2)} \), which we prove now. Consider \( \varphi \in K_A^{(3)} \)

Now note that for any \( \phi \)
\[ \max_{|a| \leq 1} \frac{\|\phi(\cdot + a)\|_{\dot{H}^{-u}(\mathbb{R}^d)}}{\|\phi\|_{\dot{H}^{-u}(\mathbb{R}^d)}} < \Phi(s). \] (B.13)

Applying this to \( \phi = \varphi(\cdot + b_{k,i}) - \varphi \), we obtain that for all \( (k,i) \in \mathbb{N} \times [1,d] \), and \( |a| \leq 1 \)
\[ \|\varphi(\cdot + a + b_{k,i}) - \varphi(\cdot + a)\|_{\dot{H}^{-u}(\mathbb{R}^d)} \leq \frac{A2^{-k/2}(\sqrt{2} - 1)}{\sqrt{2}d}. \] (B.14)

Given \( a \) with \( 2^{-k_0} \leq |a| \leq 2^{1-k_0} \), assuming without loss of generality that all coordinates are positive we can write \( a \) in the following form
\[ a := \sum_{i=1}^{d} \sum_{k \geq k_0} \chi(k,i,a)b_{k,i}, \]
with \( \chi(k,i,a) \in \{0,1\} \) (the decomposition is not necessarily unique). We write
\[ a_{k,i} := \sum_{j=1}^{d} \sum_{m \geq k_0} \chi(m,i,a)b_{m,j}(1_{\{m \leq k-1\}} + 1_{\{m = k,j \leq i\}}). \]

Then using (B.14) and the triangle inequality we obtain that for every \( \phi \in K_A^{(3)}, k \geq k_0 \)
and \( i \in [1,d] \)
\[ \|\varphi(\cdot + a_{i,k}) - \varphi\|_{\dot{H}^{-u}(\mathbb{R}^d)} \leq A\left(1 - \frac{1}{\sqrt{2}}\right) \sum_{m = k_0}^{k} 2^{-k/2} \leq A2^{-k_0/2} \leq A\sqrt{|a|}. \] (B.15)

Passing to the limit we obtain that \( \|\varphi(\cdot + a) - \varphi\|_{\dot{H}^{-u}(\mathbb{R}^d)} \leq A\sqrt{|a|} \), and thus that \( \varphi \in K_A^{(2)} \), which concludes the proof. \( \square \)
REFERENCES


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