CONCENTRATION INEQUALITIES FOR LOG-CONCAVE DISTRIBUTIONS
WITH APPLICATIONS TO RANDOM SURFACE FLUCTUATIONS

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Abstract. We derive two concentration inequalities for linear functions of log-concave distributions: an enhanced version of the classical Brascamp–Lieb concentration inequality, and an inequality quantifying log-concavity of marginals in a manner suitable for obtaining variance and tail probability bounds.

These inequalities are applied to the statistical mechanics problem of estimating the fluctuations of random surfaces of the \( \nabla \phi \) type. The classical Brascamp–Lieb inequality bounds the fluctuations whenever the interaction potential is uniformly convex. We extend these bounds to the case of convex potentials whose second derivative vanishes only on a zero measure set, when the underlying graph is a \( d \)-dimensional discrete torus. The result applies, in particular, to potentials of the form \( U(x) = |x|^p \) with \( p > 1 \) and answers a question discussed by Brascamp–Lieb–Lebowitz (1975). Additionally, new tail probability bounds are obtained for the family of potentials \( U(x) = |x|^p + x^2 \), \( p > 2 \). This result answers a question mentioned by Deuschel and Giacomin (2000).

1. Introduction

The goal of this paper is two-fold: to present new concentration inequalities for log-concave distributions and to apply these inequalities to the problem of bounding the fluctuations of random surfaces of \( \nabla \phi \) type arising in statistical mechanics.

1.1. Concentration inequalities for log-concave distributions. A log-concave distribution is a probability distribution on \( \mathbb{R}^n \) with a density \( \exp(-f) \) with respect to Lebesgue measure where \( f : \mathbb{R}^n \to (-\infty, \infty] \) is convex (so that \( \exp(-f) \) is log-concave). Without restricting generality, we impose the convention that \( \{x : f(x) < \infty\} \) is open for convex \( f \).

Let \( X \) be a random vector in \( \mathbb{R}^n \) with a log-concave distribution. We wish to study the concentration properties of linear functions \( \langle \eta, X \rangle \), for vectors \( \eta \in \mathbb{R}^n \), providing both variance and tail probability estimates. A seminal result in this context is the Brascamp–Lieb concentration inequality [12, 13], which states that

\[
\text{Var}(\langle \eta, X \rangle) \leq \mathbb{E}(\langle \eta, (\text{Hess } f)(X)^{-1}\eta \rangle),
\]

where \( \text{Hess } f \) is the Hessian matrix of \( f \). In this paper, the matrix \( (\text{Hess } f)(x) \) is defined as the unique symmetric matrix for which the second-order Taylor expansion

\[
f(x + y) = f(x) + \langle y, (\nabla f)(x) \rangle + \frac{1}{2} \langle y, (\text{Hess } f)(x)y \rangle + o(\|y\|^2)
\]

is valid at \( x \) for some vector \( (\nabla f)(x) \). The convexity of \( f \) implies, by Aleksandrov’s theorem [3] (see also [24, Theorem 6.9]), that \( \text{Hess } f \) exists and is positive semidefinite almost everywhere on \( \{x : f(x) < \infty\} \). Rockafellar [47] discusses the relation of this definition with other possible definitions of second derivatives of \( f \). For the one-dimensional case see also Section 2.1. We remark that the expression \( \langle \eta, (\text{Hess } f)(X)^{-1}\eta \rangle \) appearing in (1) may have a well-defined finite value even if \( \text{Hess } f \) is not invertible. This is the case exactly when the kernel of \( \text{Hess } f \) is orthogonal to \( \eta \); see Lemma 2.3.

The theorem of Brascamp–Lieb is in fact more general than (1), allowing to bound also the variance of non-linear functions of \( X \) but our focus here is on the linear case. Our first theorem provides an enhancement of the concentration inequality (1) which is useful in cases when the
expectation of \( \langle \eta, (\text{Hess } f)(X)^{-1} \eta \rangle \) is much larger than its typical value (including cases with infinite expectation).

**Theorem 1.1** (Quantile Brascamp–Lieb type inequality). There exists a constant \( C > 0 \) so that the following holds. Let \( X \) be a random vector with a log-concave density \( \exp(-f) \) and let \( \eta \) be a non-zero vector in \( \mathbb{R}^n \). For each \( t > 0 \),
\[
\text{Var}(\langle \eta, X \rangle) \leq \frac{Ct}{\mathbb{P}(\langle \eta, (\text{Hess } f)(X)^{-1} \eta \rangle \leq t)^3}.
\]
In particular,
\[
\text{Var}(\langle \eta, X \rangle) \leq 8C\text{Med}(\langle \eta, (\text{Hess } f)(X)^{-1} \eta \rangle)
\]
where Med(\( Y \)) is any median of the random variable \( Y \), i.e., a number \( t \) satisfying \( \mathbb{P}(Y \geq t) \geq \frac{1}{2} \) and \( \mathbb{P}(Y \leq t) \geq \frac{1}{2} \).

For the follow-up, recall that the classical Prékopa-Leindler inequality (see below) implies that the density function of \( \langle \eta, X \rangle \) is itself log-concave. Our second result asserts a quantitative version of this log-concavity.

**Theorem 1.2** (Quantitative log-concavity). Let \( X \) be a random vector with a log-concave density \( \exp(-f) \) and let \( \eta \) be a non-zero vector in \( \mathbb{R}^n \). Denote by \( \alpha_{\eta} : \mathbb{R} \to [0, \infty) \) the (log-concave) density function of \( \langle \eta, X \rangle \). Define, for each \( t > 0 \) and each \( x \in \mathbb{R}^n \) satisfying \( f(x) < \infty \),
\[
D_{\eta,x}(t) := \inf_{\frac{x^+ + x^-}{2} = 2t} \frac{f(x^+) + f(x^-) - 2f(x)}{2}.
\]
Define further, for each \( D \geq 0 \), \( t > 0 \) and \( s \in \mathbb{R} \) satisfying that \( \alpha_{\eta}(s) > 0 \)
\[
\gamma_{\eta}(D, s, t) := \mathbb{P}(D_{\eta,X}(t) \geq D \mid \langle \eta, X \rangle = s).
\]
Then the inequality
\[
\sqrt{\alpha_{\eta}(s - t)\alpha_{\eta}(s + t)} \leq \left( 1 - \gamma_{\eta}(D, s, t)(1 - e^{-D}) \right) \cdot \alpha_{\eta}(s)
\]
holds for every \( D, t \) and \( s \) as above. In particular, if \( \mathbb{P}(D_{\eta,X}(t) \geq D) \geq \frac{3}{4} \) then
\[
\mathbb{P}\left( \sqrt{\alpha(\langle \eta, X \rangle + t)\alpha(\langle \eta, X \rangle - t)} \leq \left( 1 - \frac{1}{2} (1 - e^{-D}) \right) \alpha(\langle \eta, X \rangle) \right) \geq \frac{1}{2}.
\]
We remark that although \( \gamma_{\eta}(D, s, t) \) is defined in (6) as a conditional expectation and hence it a priori only makes sense for almost every \( s \in \mathbb{R} \) with \( \alpha_{\eta}(s) > 0 \), we may in fact define it for all such \( s \); see (41) below.

The main motivation for Theorem 1.2 is Lemma 1.3 below. It is worthwhile to emphasize that the assumption (9) is also the conclusion of Theorem 1.2. Taken together, the theorem and lemma provide a second route (an alternative to Theorem 1.1) to obtaining variance bounds for linear functions of random vectors with log-concave densities.

**Lemma 1.3.** There exists a constant \( C > 0 \) so that the following holds. Let \( \xi \) be a random variable in \( \mathbb{R} \) with log-concave density \( \alpha : \mathbb{R} \to [0, \infty) \). Let \( t > 0 \) and \( 0 < \delta < 1 \). If
\[
\mathbb{P}\left( \sqrt{\alpha(\xi + t)\alpha(\xi - t)} \leq (1 - \delta)\alpha(\xi) \right) \geq \frac{1}{2}
\]
then
\[
\text{Var} \xi \leq \left( \frac{Ct}{\delta} \right)^2.
\]
In addition to its use in bounding the variance (via Lemma 1.3), Theorem 1.2 can also provide tail probability bounds for the distribution of \((\eta, X)\). This is enabled by taking the parameter \(t\) large and finding values of \(D\) for which \(\gamma_0(D, s, t)(1 - e^{-D})\) is close to one for a suitable range of \(s\). In our application we do not explore this direction to its fullest and rather demonstrate it in a simpler situation where one has \(\gamma_0(D, s, t) = 1\) for suitable \(D, s, t\) (see the proof of Theorem 1.5). Still, new results are obtained even in this simplified setup.

The proofs of both Theorem 1.1 and Theorem 1.2 make use of the Prékopa-Leindler inequality, introduced in [45, 36, 46], from convexity theory.

**Theorem** (Prékopa-Leindler inequality). Let \(m \geq 1\) be an integer, \(0 < \lambda < 1\) be real and \(F_1, F_2, F\) be non-negative, Lebesgue integrable real functions on \(\mathbb{R}^m\) satisfying
\[
F((1 - \lambda)x + \lambda y) \geq F_1(x)^{1-\lambda}F_2(y)^\lambda, \quad x, y \in \mathbb{R}^m.
\]
Then
\[
\int F(x) dx \geq \left( \int F_1(x) dx \right)^{1-\lambda} \left( \int F_2(x) dx \right)^\lambda.
\]

A proof may be found, for instance, in Schneider's book [48, Theorem 7.1.2] (proving the \(m = 1\) case directly with a change of variable and proceeding by induction on \(m\)). In proving Theorem 1.2 we apply the Prékopa-Leindler inequality with \(F_1, F_2\) being the restrictions of the log-concave density \(\exp(-f)\) to the hyperplanes \((\eta, x) = s+t\) and \((\eta, x) = s-t\), respectively (so that \(m = n-1\)) and set \(F(x) = \exp(-f(x) - D_{\eta,x}(t))\) restricted to \((\eta, x) = s\). The function \(D_{\eta,x}(t)\) is natural in this context as it makes \(F\) the smallest function for which the assumption (11) is still satisfied with our choices of \(F_1, F_2\) and \(\lambda = \frac{1}{2}\). The proof of Theorem 1.1 makes use of the Prékopa-Leindler inequality in a similar manner but focuses on an infinitesimal analog of the quantity \(D_{\eta,x}(t)\) as \(t\) tends to 0.

### 1.2. Fluctuations of random surfaces.

As an application of the above concentration inequalities, we provide new upper bounds for the fluctuations of random surface models of the \(\nabla \varphi\) type (see Section 1.3 for background). We consider the following family of models: Let \(G = (V(G), E(G))\) be a finite, connected graph with a distinguished set \(\emptyset \neq V_0 \subsetneq V(G)\), let \(\varphi_0 : V_0 \to \mathbb{R}\) be a function and let \(U\) be a potential, i.e., a measurable function \(U : \mathbb{R} \to (-\infty, \infty]\) satisfying \(U(x) = U(-x)\) for all \(x\). The random surface model on \(G\) with potential \(U\) and boundary conditions \((V_0, \varphi_0)\) is the probability measure \(\mu_{G, V_0, \varphi_0, U}\) on functions \(\varphi : G \to \mathbb{R}\) defined by
\[
d\mu_{G, V_0, \varphi_0, U}(\varphi) := \frac{1}{Z_{G, V_0, \varphi_0, U}} \exp\left( - \sum_{e \in E(G)} U(\nabla_e \varphi) \right) \prod_{v \in V_0} \delta(\varphi(v) - \varphi_0(v)) \prod_{v \in V(G) \setminus V_0} d\varphi(v),
\]
where \(d\varphi(v)\) denotes Lebesgue measure on \(\varphi(v)\) and \(\delta(\cdot)\) is the Dirac delta symbol. Thus \(\mu_{G, V_0, \varphi_0, U}\) is supported on
\[
\Omega_{G, V_0, \varphi_0} := \{ \varphi : V(G) \to \mathbb{R} \text{ such that } \varphi|_{V_0} \equiv \varphi_0 \},
\]
and \(\prod_{v \in V_0} \delta(\varphi(v) - \varphi_0(v)) \prod_{v \in V(G) \setminus V_0} d\varphi(v)\) is Lebesgue measure on this set. The normalization constant (partition function)
\[
Z_{G, V_0, \varphi_0, U} := \int_{\mathbb{R}^{|V(G)|}} \exp\left( - \sum_{e \in E(G)} U(\nabla_e \varphi) \right) \prod_{v \in V_0} \delta(\varphi(v) - \varphi_0(v)) \prod_{v \in V(G) \setminus V_0} d\varphi(v),
\]
makes \(\mu_{G, V_0, \varphi_0, U}\) a probability measure. We require that \(0 < Z_{G, V_0, \varphi_0, U} < \infty\), a requirement which is satisfied under the assumptions of our main results. It is convenient to assume that the edges \(E(G)\) are endowed with some fixed orientation, so that \(\nabla_e \varphi\) can be unambiguously defined as \(\varphi(u) - \varphi(v)\) for an edge \(e = (u, v)\), but it is important to note that the choice of orientation is immaterial for expressions such as \(U(\nabla_e \varphi)\) as \(U\) is an even function.
Our main concern is with the fluctuations of the random surface model on \( d \)-dimensional lattice graphs with zero boundary values. Specifically, we work with the following two families of graphs:

**Setting 1.** The \( d \)-dimensional bipartite torus \( \mathbb{T}_{2L}^d \): Here \( V(\mathbb{T}_{2L}^d) := \{-L+1, -L+2, \ldots, L-1, L\}^d \) with \( v \) adjacent to \( w \) when \( v \) and \( w \) are equal in all but one coordinate and differ by exactly one modulo \( 2L \) in that coordinate. We set \( V_0 = \{0\} \) (where \( 0 = (0, \ldots, 0) \)) and \( \varphi_0 \equiv 0 \). For brevity, we write \( \mu_{\mathbb{T}_{2L}^d,U} \) for \( \mu_{\mathbb{T}_{2L}^d,V_0,\varphi_0,U} \). As usual, we write \( \|v\|_1 \) for the \( \ell^1 \)-norm of \( v \).

**Setting 2.** The \( d \)-dimensional box \( \Lambda_L^d \): Here \( V(\Lambda_L^d) := \{1, \ldots, L\}^d \subset \mathbb{Z}^d \) with the usual nearest-neighbor adjacency of \( \mathbb{Z}^d \). We set \( V_0 \) to be the vertices of \( V(\Lambda_L^d) \) which are adjacent in \( \mathbb{Z}^d \) to a vertex outside \( V(\Lambda_L^d) \) and \( \varphi_0 \equiv 0 \). Again, we write \( \mu_{\Lambda_L^d,U} \) for \( \mu_{\Lambda_L^d,V_0,\varphi_0,U} \).

The properties of random surfaces with general interaction potentials first received rigorous consideration in the seminal 1975 work of Brascamp–Lieb–Lebowitz \([14]\). They conjectured that the fluctuations of such random surfaces in dimension \( d = 3 \) are uniformly bounded in the system size for potentials \( U \) satisfying that \( \int \exp(-pU(x))\,dx < \infty \) for all \( p > 0 \). Among the main results of their work is a proof, using the Brascamp–Lieb concentration inequality \((1)\), that the conjecture holds for uniformly convex potentials \( U \) (i.e., potentials satisfying \( \inf_x U''(x) > 0 \)). To date, this remains the main case for which the conjecture is verified (Section 1.3 details additional progress) with the result missing even for the potential \( U(x) = x^4 \) which was emphasized in \([14]\) and in the more recent survey of Velenik \([50, \text{Problem 1}]\). Our first result verifies the conjecture for a large class of potentials, including the case \( U(x) = x^4 \) and more generally the family \( U(x) = |x|^p \), \( 1 < p < \infty \). Our result further applies in dimension \( d = 2 \) where it shows that fluctuations are of order at most the square root of the logarithm of the system size, matching lower bounds of \([14]\) up to a constant prefactor.

**Theorem 1.4.** Suppose that \( U : \mathbb{R} \to (-\infty, \infty] \) is such that \( U(x) = U(-x) \) for all \( x \), and, in addition, the following assumption is satisfied:

\[ U \text{ is convex and } U''(x) > 0 \text{ Lebesgue almost-everywhere (a.e.) on } \{x: U(x) < \infty\}. \tag{16} \]

Let \( d \geq 2 \) and \( L \geq 2 \) be integers and let \( \varphi \) be randomly sampled from \( \mu_{\mathbb{T}_{2L}^d,U} \) (Setting 1). Then there exists \( C > 0 \), depending on \( U \) and \( d \) but not on \( L \), such that for any \( v \in V(\mathbb{T}_{2L}^d) \setminus \{0\} \) we have

\[
\begin{align*}
\text{if } d = 2 & : \quad \text{Var}(\varphi(v)) \leq C \log(1 + \|v\|_1), \\
\text{if } d \geq 3 & : \quad \text{Var}(\varphi(v)) \leq C.
\end{align*}
\tag{17}
\]

As discussed above, convexity of \( U \) implies the existence of its second derivative almost everywhere on \( \{x: U(x) < \infty\} \). Theorem 1.4 may be proved using either Theorem 1.1 or Theorem 1.2, and relying on additional facts specific to the random surface model. We demonstrate both approaches in Section 4.2.

As mentioned earlier, linear functions of log-concave distributions have themselves a log-concave distribution. Since log-concave distributions have tail probabilities decaying at least exponentially fast on the scale of their standard deviation, we may deduce such tail probability bounds for the variables \( \varphi(v) \) from Theorem 1.4. For uniformly convex potentials, exponential decay may be upgraded to sub-Gaussian decay as a consequence of the Brascamp–Lieb concentration inequality (see, e.g., \([26, \text{Theorem 4.9}]\)). Deuschel–Giacomin \([21, \text{Remark 2.11}]\) discuss the question of whether in dimensions \( d \geq 3 \) the tail probabilities of \( \varphi(v) \) exhibit faster than sub-Gaussian decay (on the scale of the standard deviation of \( \varphi(v) \)) when the potential grows faster than quadratically at infinity. Our second theorem shows that this is the case for potentials of the form \( U(x) = |x|^p + x^2 \), \( p > 2 \). The obtained upper bounds on the tail probabilities match, for \( d \neq p \), the lower bounds obtained by \([21]\) up to a constant multiple in the exponent (the lower bounds apply to vertices sufficiently separated from the boundary set \( V_0 \)).
Theorem 1.5. Let \( d \geq 3 \), \( L \geq 2 \) be integers and \( p > 2 \) real. Set \( U(x) = |x|^p + x^2 \). Suppose that either

(i) \( \varphi \) is randomly sampled from \( \mu_{T_{2L}^d,U} \), \( V_0 \subset V(T_{2L}^d) \) is as in Setting 1 and \( v \in V(T_{2L}^d) \setminus V_0 \); or

(ii) \( \varphi \) is randomly sampled from \( \mu_{\Lambda_{2L}^d,U} \), \( V_0 \subset V(\Lambda_{2L}^d) \) is as in Setting 2 and \( v \in V(\Lambda_{2L}^d) \setminus V_0 \).

Then there exist \( C, c > 0 \), depending on \( p \) and \( d \) but not on \( L \) or \( v \), such that for all \( t > 2 \),

\[
P(\varphi(v) > t) \leq \begin{cases} 
C \exp(-c t^d) & d < p \\
C \exp \left( -c \frac{t^d}{(\log t)^{d-1}} \right) & d = p \\
C \exp(-c t^p) & d > p 
\end{cases} \tag{18}
\]

Theorem 1.5 is derived from Theorem 1.2, via additional facts specific to the random surface model, in Section 4.1. An extension to non-zero boundary conditions is discussed at the end of Section 4.1.

1.3. Discussion and Background. Concentration inequalities for spaces with uniform convexity estimates (possibility of non-quadratic nature) were established in various contexts; see Gromov–Milman [29], Bobkov–Ledoux [8] and Milman–Sodin [40]. Such concentration inequalities imply, for instance, tail bounds for Lipschitz functions [8, Corollary 4.1], reminiscent of the tail bounds of Theorem 1.5. E. Milman [39] has informed us that in an unpublished work (circa 2008) with S. Sodin they have been able to use uniform convexity estimates to prove tail probability bounds of the form \( C \exp(-c t^p) \) (similarly to (18)) for potentials of the form \( |x|^p \) in dimensions \( d > p > 2 \). In this context we emphasize that the concentration inequalities provided by Theorem 1.1 and Theorem 1.2 do not require uniform convexity but rather apply when a quantitative convexity assumption (in the direction of the functional \( \eta \)) is given on a subset of the full space. Some improvements of the Brascamp–Lieb inequality, which may be used in the absence of uniform convexity, are known in the literature such as the inequality of Bobkov–Ledoux [9, Theorem 2.4] (further extended by Nguyen [43] and Kolesnikov–Milman [33, 34]) and the inequality of Veyssière [51] and its extension [33, Theorem 4.1].

The random surface models introduced above, sometimes called \( \nabla \varphi \) interface models, constitute natural examples of statistical mechanics models with non-compact state space and also serve as effective models for various interfaces arising in spin systems [27, 26, 50]. We consider them in the standard setting of the lattice graphs \( T_{2L}^d \) or \( \Lambda_{2L}^d \) and focus on dimensions \( d \geq 2 \) as one-dimensional random surfaces are equivalent to random walks and are well understood.

The most well-known example of a random surface is the lattice Gaussian free field (LGFF), which has \( U(x) = x^2 \). In this case, the Gaussian distribution of the surface considerably simplifies its analysis. A long-standing prediction is that many properties of the LGFF are universal, holding for a general class of potentials. We briefly review here the progress made on several aspects of this phenomenon.

The scaling limit of the LGFF is the continuum Gaussian free field (CGFF), the higher-dimensional analogue of Brownian motion. Analogously to the invariance principle for random walks, it is expected that random surfaces satisfying mild integrability conditions on \( U \) converge to the CGFF. The state-of-the-art is, however, very far from this goal. A general convergence result [17, 42, 28, 38] has been proved only when \( U \) is twice continuously differentiable and satisfies

\[
0 < \inf_x U''(x) \leq \sup_x U''(x) < \infty. \tag{19}
\]

Additionally convergence is proved [6, 52] when \( \exp(-U) \) is an average of Gaussian functions, and for non-convex potentials arising as small perturbations of potentials satisfying (19) [19, 2, 30, 1].

Our concern in this paper is rather with the thermodynamic limit of the surface — the study of individual (or multiple) heights of the surface in the infinite-volume limit (i.e., as \( L \to \infty \)). The
most basic aspect of this study is to decide whether the surface localizes or delocalizes. This is quantified, for instance, by the variance at a vertex $v$ with $\|v\| \approx L$ for a surface sampled in the torus $\mathbb{T}^d_{2L}$ (setting 1 above), or by the variance at the origin for a surface sampled in $\Lambda^d_L$ (setting 2) above. In the seminal 1975 work of Brascamp–Lieb–Lebowitz [14] it is conjectured that for every $d$-dimensional torus $T$ quantified, for instance, by the variance at a vertex $\|v\| \approx L$.

In this section we prove the results of Section 1.1: Theorem 1.1, Theorem 1.2 and Lemma 1.3.

2. Proofs of concentration inequalities

In this section we prove the results of Section 1.1: Theorem 1.1, Theorem 1.2 and Lemma 1.3.
2.1. One-dimensional distributions with log-concave density. We begin by assembling a collection of useful facts about one-dimensional log-concave distributions, including a proof of Lemma 1.3.

As discussed in Section 1.1, the second derivative of a convex function may be defined using the validity of a Taylor expansion, which in the one-dimensional case takes the form

\[ f(x + y) = f(x) + y f'(x) + \frac{1}{2} f''(x) y^2 + o(y^2). \]  

(20)

In fact, in the one-dimensional case the usual definitions of \( f' \) and \( f'' \) may also be used, in the following way: Right and left derivatives, \( f'_{\text{left}}(t) \) and \( f'_{\text{right}}(t) \), exist everywhere on \( I := \{ x \in \mathbb{R} : f(x) < \infty \} \), are non-decreasing and coincide at all but countably many points of \( I \). Consequently, \( f'' \), defined as the derivative of any non-decreasing extension of \( f' \) to \( I \) (say, \( f'_{\text{right}}(t) \)), exists almost everywhere on \( I \) by Lebesgue’s theorem on increasing functions [32, Theorem 1 of Section 3.2.2]. It is straightforward that the expansion (20) then holds for almost every \( x \in I \). In addition, the inequality

\[ f'_{\text{right}}(t_+) - f'_{\text{left}}(t_-) \geq \int_{t_-}^{t_+} f''(t) \, dt \]  

(21)

holds for all \( t_- < t_+ \) in \( I \); see [32, Theorem 1 of Section 7.2.1].

The following proposition relates the concentration properties of one-dimensional log-concave distributions to the maximum of their density.

**Proposition 2.1.** Let \( \xi \) be a real-valued random variable with log-concave distribution. Denote its density by \( \alpha : \mathbb{R} \to [0, \infty) \). Then

1. for some absolute constants \( 0 < c_1 < 1 < C_1 \) it holds that
   \[ \frac{c_1}{\sqrt{\text{Var} \xi}} \leq \sup_{s \in \mathbb{R}} \alpha(s) \leq \frac{C_1}{\sqrt{\text{Var} \xi}} ; \]
2. for an absolute constant \( C_2 > 1 \) it holds that
   \[ \sup_{s \in \mathbb{R}} \alpha(s) \leq C_2 \cdot \inf \left\{ a \in \mathbb{R} : \mathbb{P}(\alpha(\xi) < a) > \frac{1}{4} \right\} ; \]
3. for each \( p > 0 \),
   \[ \mathbb{P}(\alpha(\xi) \leq p) \leq \frac{p}{\sup_{s \in \mathbb{R}} \alpha(s)} . \]

These properties are standard. Proofs may be found, for instance, in [7, Proposition 4.1] (Item 1), [35, Lemma 5.2] (Item 2) and [37, Lemma 5.6] (Item 3).

The next lemma controls the quantiles of the second logarithmic derivative of a log-concave density.

**Lemma 2.2.** Let \( \xi \) be a real-valued random variable with log-concave density \( \alpha = \exp(-f) \). Let \( C \geq 4 \) and let \( M = \max_{t \in \mathbb{R}} \alpha(t) \). Then

\[ \mathbb{P} \left( f''(\xi) > (CM)^2 \right) \leq 4C^{-1} . \]

Proof. Choose \( t_+ \in \mathbb{R} \) so that \( \mathbb{P}(\xi > t_+) = C^{-1} \). We claim that \( f'_{\text{right}}(t_+) \leq CM \). Indeed, assume the converse. In that case, \( f'(x) > CM \) almost everywhere on the set \( \{ x \in (t_+, \infty) : f(x) < \infty \} \) and therefore it holds that \( f(t_+ + y) > f(t_+) + CMy \) for every \( y > 0 \). Given that \( \exp(-f(t_+)) \leq M \), we have

\[ \mathbb{P}(\xi > t_+) = \int_{0}^{\infty} \exp(-f(t_+ + y)) \, dy \leq M \int_{0}^{\infty} e^{-CMy} \, dy = \frac{M}{CM} = C^{-1} , \]
a contradiction.

Similarly, choose $t_-$ so that $\mathbb{P}(\xi < t_-) = C^{-1}$. Just as above, we conclude that $f'_{\text{left}}(t_-) \geq -CM$. Now we have

$$2CM \overset{(a)}{=} CM - (-CM) \overset{(b)}{=} f'_{\text{right}}(t_+) - f'_{\text{left}}(t_-) \overset{(c)}{=} \int_{t_-}^{t_+} f''(t) \, dt \overset{(d)}{=} \int_{t_-}^{t_+} \mathbb{1} \{ f''(t) \geq (CM)^2 \} \, dt$$

where the inequality (c) is (21) above. Therefore

$$\mathbb{P}(t_- \leq \xi < t_+ \text{ and } f''(\xi) > (CM)^2) = \int_{t_-}^{t_+} \exp(-f(t)) \cdot \mathbb{1} \{ f''(t) \geq (CM)^2 \} \, dt \leq M \cdot \frac{2CM}{(CM)^2} = 2C^{-1}.$$

In total, we get

$$\mathbb{P}(f''(\xi) > (CM)^2) \leq \mathbb{P}(\xi < t_-) + \mathbb{P}(\xi > t_+) + \mathbb{P}(t_- \leq \xi \leq t_+ \text{ and } f''(\xi) > (CM)^2) \leq C^{-1} + C^{-1} + 2C^{-1} = 4C^{-1},$$

as desired. \hfill \square

Finally, we proceed with the proof of Lemma 1.3.

**Proof of Lemma 1.3.** Our goal will be to show that the choice

$$C := 8C_1 \max(1, \ln C_2)$$

suffices, where the values of $C_1$ and $C_2$ are those from Item 1 and Item 2 of Proposition 2.1. We argue by contradiction, assuming that (10) does not hold with this choice of $C$.

Denote

$$a_0 := \inf \left\{ a \in \mathbb{R} : \mathbb{P}(\alpha(\xi) < a) > \frac{1}{4} \right\}.$$

Due to log-concavity of $\alpha$, there is a unique pair of values $r_1 < r_2$ such that $\alpha(s) < a_0$ whenever $s < r_1$ or $s > r_2$, and $\alpha(s) > a_0$ whenever $r_1 < s < r_2$. By definition of $a_0$, we have

$$\int_{r_1}^{r_2} \alpha(s) \, ds = \frac{3}{4},$$

and, consequently,

$$r_2 - r_1 > \frac{1}{\sup_{s \in \mathbb{R}} \alpha(s)} \int_{r_1}^{r_2} \alpha(s) \, ds = \frac{3}{4} \cdot \sup_{s \in \mathbb{R}} \alpha(s) > \frac{3\sqrt{\text{Var}\xi}}{4C_1}, \quad (22)$$

where the last inequality is an application of Item 1 of Proposition 2.1. By our assumption, the inequality (10) does not hold, i.e., $\sqrt{\text{Var}\xi} > \frac{Ct}{\delta}$, which, combined with (22), yields $r_2 - r_1 > \frac{3C_2}{4C_1\delta}$.

Denote $k := \frac{C}{8C_1\delta}$. Then $r_1 + kt < r_2 - kt$, and, moreover,

$$\mathbb{P}(\xi \in [r_1 + kt, r_2 - kt]) \geq \frac{3}{4} - 2kt \cdot \sup_{s \in \mathbb{R}} \alpha(s) \geq \frac{3}{4} - \frac{2C_1kt}{\sqrt{\text{Var}\xi}} = \frac{3}{4} - \frac{Ct/\delta}{4\sqrt{\text{Var}\xi}} > \frac{1}{2}.$$

Combined with (9), this means that the set

$$S = \left\{ s \in (r_1 + kt, r_2 - kt) : \sqrt{\alpha(s + t)} \alpha(s - t) \leq (1 - \delta)\alpha(s) \right\}$$
is non-empty. Let \( s_0 \in S \). Then either \( \alpha(s_0 + t) \leq (1 - \delta)\alpha(s_0) \), or \( \alpha(s_0 - t) \leq (1 - \delta)\alpha(s_0) \). We consider only the first case, since the other one is similar. By the choice of \( C \), we have \( k > 1 \) and therefore \( s_0 < s_0 + t < s_0 + kt < r_2 \). Consequently, the log-concavity of \( \alpha \) implies that

\[
a_0 \leq \lim_{r \nearrow r_2} \alpha(r) < (1 - \delta)^k \alpha(s_0).
\]

But

\[
(1 - \delta)^k = \exp(k \ln(1 - \delta)) < \exp(-k\delta) = \exp\left(-\frac{C}{8C_1}\right) \leq \frac{1}{C_2},
\]

which implies that

\[
a_0 < \frac{\alpha(s_0)}{C_2} \leq \frac{\sup_{s \in \mathbb{R}} \alpha(s)}{C_2}.
\]

This contradicts Item 2 of Proposition 2.1 and thereby finishes the proof. \( \square \)

2.2. Proof of the quantile Brascamp–Lieb type inequality. In this section we prove Theorem 1.1.

The next lemma connects the quadratic form \( \langle \mathbf{n}, (\text{Hess} f)(\mathbf{x})^{-1}\mathbf{n} \rangle \) to a quantitative measure of the convexity of \( f \) at \( \mathbf{x} \).

**Lemma 2.3.** Let \( f : \mathbb{R}^n \to (-\infty, \infty] \) be a convex function and \( \mathbf{x} \in \mathbb{R}^n \) be a point such that the Taylor expansion

\[
f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{v} \rangle + \frac{1}{2} \langle \mathbf{y}, \mathbf{H}\mathbf{y} \rangle + o(\|\mathbf{y}\|^2)
\]

exists for some vector \( \mathbf{v} \) and positive definite matrix \( \mathbf{H} \). Let \( \mathbf{n} \in \mathbb{R}^n \) be a unit vector and set \( s := \langle \mathbf{n}, \mathbf{x} \rangle \). Then the following holds as \( \gamma \searrow 0 \):

\[
\inf_{\mathbf{x}^+, \mathbf{x}^- \in \mathbb{R}^n : \mathbf{x}^+ + \mathbf{x}^- = 2\mathbf{x}, \langle \mathbf{n}, \mathbf{x}^\pm \rangle = 0} \left( f(\mathbf{x}^+) + f(\mathbf{x}^-) - 2f(\mathbf{x}) \right) = \frac{\gamma^2}{\langle \mathbf{n}, \mathbf{H}^{-1}\mathbf{n} \rangle} + o(\gamma^2). \tag{24}
\]

**Proof.** Replacing \( f \) by \( \tilde{f} \), where \( \tilde{f}(\mathbf{z}) := f(\mathbf{z}) - \langle \mathbf{z} - \mathbf{x}, \mathbf{v} \rangle - f(\mathbf{x}) \) does not change (24). Hence we do not lose any generality assuming that \( f(\mathbf{x}) = 0 \) and \( \mathbf{v} = \mathbf{0} \). The rest of the argument is provided under these assumptions.

We claim that the following approximation holds if \( \gamma > 0 \) is sufficiently small:

\[
\inf_{\mathbf{x}^\pm \in \mathbb{R}^n : \langle \mathbf{n}, \mathbf{x}^\pm \rangle = s \pm \gamma} f(\mathbf{x}^\pm) = \frac{1}{2} \left( \gamma \mathbf{y}_0, \mathbf{H}\mathbf{y}_0 \right) + o(\gamma^2) = \frac{\gamma^2}{2\langle \mathbf{n}, \mathbf{H}^{-1}\mathbf{n} \rangle} + o(\gamma^2) \quad \text{where} \quad \mathbf{y}_0 := \frac{\mathbf{H}^{-1}\mathbf{n}}{\langle \mathbf{n}, \mathbf{H}^{-1}\mathbf{n} \rangle}. \tag{25}
\]

We use the standard fact that if \( \mathbf{H} \) is an \( n \times n \) positive definite matrix and \( \mathbf{n} \in \mathbb{R}^n \setminus \{\mathbf{0}\} \) then

\[
\inf_{\mathbf{y} \in \mathbb{R}^n : \langle \mathbf{n}, \mathbf{y} \rangle = 1} \langle \mathbf{y}, \mathbf{H}\mathbf{y} \rangle = \frac{1}{\langle \mathbf{n}, \mathbf{H}^{-1}\mathbf{n} \rangle} \quad \text{and} \quad \arg\min_{\mathbf{y} \in \mathbb{R}^n : \langle \mathbf{n}, \mathbf{y} \rangle = 1} \langle \mathbf{y}, \mathbf{H}\mathbf{y} \rangle = \frac{\mathbf{H}^{-1}\mathbf{n}}{\langle \mathbf{n}, \mathbf{H}^{-1}\mathbf{n} \rangle}. \tag{26}
\]

(These extend to the case that \( \mathbf{H} \) is positive semidefinite, with a suitable interpretation depending on whether \( \mathbf{n} \) is orthogonal to the kernel of \( \mathbf{H} \) or not.)

We will prove (25) with the plus sign, as the other case is identical. Let \( R := \|\mathbf{y}_0\| \). The strict convexity of the quadratic form \( \mathbf{y} \mapsto \langle \mathbf{y}, \mathbf{H}\mathbf{y} \rangle \) implies that for all sufficiently small \( \gamma > 0 \) one has

\[
\inf_{\mathbf{x}^+ \in \mathbb{R}^n : \langle \mathbf{n}, \mathbf{x}^+ \rangle = s + \gamma, \|\mathbf{x}^+ - \mathbf{x}\| = 2R\gamma} f(\mathbf{x}^+) > f(\mathbf{x} + \gamma \mathbf{y}_0),
\]
and, consequently, by the convexity of $f$,
\[
\inf \frac{f(x^+)}{f(x)} > f(x + \gamma y_0) = \frac{1}{2} \langle \gamma y_0, H \gamma y_0 \rangle + o(\gamma^2). \tag{27}
\]

On the other hand,
\[
\inf \frac{f(x^+)}{f(x)} = \frac{1}{2} \inf \frac{\langle y, H y \rangle + o(\gamma^2)}{\langle y, \gamma y \rangle} = \frac{1}{2} \langle \gamma y_0, H \gamma y_0 \rangle + o(\gamma^2). \tag{28}
\]

Combining (27) and (28) indeed yields (25) with the plus sign.

Now the lower bound for the left-hand side of (24) is obtained as follows:
\[
\inf_{x^+, x^- :x=x^+ + x^-} \left( f(x^+) + f(x^-) - 2f(x) \right) = \inf_{x^+, x^- :x=x^+ + x^-} \left( f(x^+) + f(x^-) \right) \geq \inf_{x^+ :x^+ = x^+ + x^-} f(x^+) + \inf_{x^- :x^- = s - x^+} f(x^-) = \frac{\gamma^2}{\langle n, H^{-1} n \rangle} + o(\gamma^2).
\]

The matching upper bound is achieved by setting $x^\pm := x \pm \gamma y_0$.

Let now $\exp(-f)$ be a log-concave probability density on $\mathbb{R}^n$, and let $X$ be a random vector in $\mathbb{R}^n$ sampled according to the probability measure $\exp(-f(x))\,dx$. As mentioned above, it is a simple corollary of the Prékopa–Leindler inequality that the distribution of a one-dimensional marginal $\langle n, X \rangle$, where $n$ is a unit vector, is log-concave. Our next lemma quantifies this log-concavity by making use of Lemma 2.3.

From now on, we will write $m_{n-1}$ for the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$.

**Lemma 2.4.** Let $\exp(-f)$ be a log-concave probability density on $\mathbb{R}^n$. Let $n \in \mathbb{R}^n$ be a unit vector. Define $\alpha : \mathbb{R} \to [0, \infty)$ (the log-concave marginal density) by
\[
\alpha(r) := \int_{\{x \in \mathbb{R}^n : \langle n, x \rangle = r\}} \exp(-f(x)) \, dm_{n-1}(x).
\]

Let $s \in \mathbb{R}$ be such that $\alpha(s) > 0$. Suppose $\text{Hess} \, f$ is a positive definite matrix $m_{n-1}$-almost-everywhere on $\{x \in \mathbb{R}^n : \langle n, x \rangle = s, f(x) < \infty\}$. Suppose further that $(\ln \alpha)^\prime\prime$ is defined at $s$ (in the sense of Section 2.1). Then
\[
-(\ln \alpha)^\prime\prime(s) \geq \frac{\int_{\{x \in \mathbb{R}^n : \langle n, x \rangle = s, f(x) < \infty\}} \frac{1}{\langle n, \text{Hess} \, f(x)^{-1} n \rangle} \cdot e^{-f(x)} \, dm_{n-1}(x)}{\alpha(s)}. \tag{29}
\]

**Proof.** Define
\[
\Pi := \{x \in \mathbb{R}^n : \langle n, x \rangle = s, f(x) < \infty, (\text{Hess} \, f)(x) \text{ is positive definite}\},
\]
\[
m(x, \gamma) := \inf_{x^+, x^- \in \mathbb{R}^n : x^+ + x^- = 2x} \left( f(x^+) + f(x^-) - 2f(x) \right),
\]
\[
X(\gamma, \varepsilon) := \left\{ x \in \Pi : m(x, \gamma) \geq \frac{(1 - \varepsilon)\gamma^2}{\langle n, (\text{Hess} \, f(x)^{-1} n \rangle} \text{ for all } 0 < \gamma_1 < \gamma \right\}.
\]

By Lemma 2.3, for every $\varepsilon > 0$ it holds that $X(\gamma, \varepsilon)$ increases to $\Pi$ as $\gamma$ decreases to 0.
Fix $\gamma, \varepsilon > 0$. For $\gamma_1 \in (0, \gamma)$, we may apply the Prékopa–Leindler inequality to the following functions on $\{x \in \mathbb{R}^n : \langle n, x \rangle = s\}$:

$$
F(x) := \exp(-f(x)) \cdot p(x), \quad F_1(x) := \exp(-f(x + \gamma_1 n)), \quad F_2(x) := \exp(-f(x - \gamma_1 n)),
$$

where

$$
p(x) := \begin{cases} 
\exp\left(-\frac{(1-\varepsilon)\gamma_1^2}{2(n, \text{Hess } f)(x)^{-1} n}\right) & \text{if } x \in X(\gamma, \varepsilon), \\
1 & \text{if } x \in \Pi \setminus X(\gamma, \varepsilon)
\end{cases}
$$

to obtain

$$
\alpha(s+\gamma_1)^{\frac{1}{2}} \alpha(s-\gamma_1)^{\frac{1}{2}} \leq \alpha(s) \left(1 - \frac{\int_{X(\gamma, \varepsilon)} \left(1 - \exp\left(-\frac{(1-\varepsilon)\gamma_1^2}{2(n, \text{Hess } f)(x)^{-1} n}\right)\right) \cdot e^{-f(x)} dm_{n-1}(x)}{\alpha(s)}\right). \tag{30}
$$

Now set

$$
X(\gamma, \varepsilon, \delta) := \{x \in X(\gamma, \varepsilon) : \langle n, (\text{Hess } f)(x)^{-1} n \rangle \geq \delta\}
$$

and note that $X(\gamma, \varepsilon, \delta)$ increases to $X(\gamma, \varepsilon)$ as $\delta$ decreases to 0 as $\text{Hess } f$ is positive definite on $\Pi$. It follows that (30) continues to hold when $X(\gamma, \varepsilon)$ is replaced by $X(\gamma, \varepsilon, \delta)$ for any $\delta > 0$. Taking logarithms and using the Taylor expansion (of the logarithm and the exponential) we obtain

$$
\frac{1}{2} \ln \alpha(s + \gamma_1) + \frac{1}{2} \ln \alpha(s - \gamma_1) - \ln \alpha(s) \leq 
\ln \left(1 - \frac{\int_{X(\gamma, \varepsilon, \delta)} \left(1 - \exp\left(-\frac{(1-\varepsilon)\gamma_1^2}{2(n, \text{Hess } f)(x)^{-1} n}\right)\right) \cdot e^{-f(x)} dm_{n-1}(x)}{\alpha(s)}\right) =
- \gamma_1^2 \cdot \frac{\int_{X(\gamma, \varepsilon, \delta)} \frac{1-\varepsilon}{2(n, \text{Hess } f)(x)^{-1} n} \cdot e^{-f(x)} dm_{n-1}(x)}{\alpha(s)} + o(\gamma_1^2)
$$
as $\gamma_1 \searrow 0$. Thus, since $(\ln \alpha)''(s)$ is well defined,

$$
-(\ln \alpha)''(s) \geq \frac{\int_{X(\gamma, \varepsilon, \delta)} \frac{1-\varepsilon}{2(n, \text{Hess } f)(x)^{-1} n} \cdot e^{-f(x)} dm_{n-1}(x)}{\alpha(s)}. 
$$

The lemma follows by taking $\delta \searrow 0$ followed by $\gamma \searrow 0$ and lastly $\varepsilon \searrow 0$. \hfill \Box

The next lemma combined with part 1 of Proposition 2.1 implies Theorem 1.1.

**Lemma 2.5.** Let $X$ be a random vector with a log-concave density $\exp(-f)$. Let $n$ be a unit vector in $\mathbb{R}^n$. Let $t > 0$ and set

$$
p := \mathbb{P}\left(\langle n, (\text{Hess } f)(X)^{-1} n \rangle \leq t\right). \tag{31}
$$

Let $\alpha$ be the (log-concave) density of $\langle n, X \rangle$. Then

$$
\max_{s \in \mathbb{R}} \alpha(s) \geq \frac{p^{3/2}}{(8 - 4p)^{\sqrt{2t}}}. \tag{32}
$$

**Proof.** We first prove the statement under the additional assumption that $(\text{Hess } f)(X)$ is positive definite almost surely. With this assumption we may apply Lemma 2.4 to conclude that

$$
-(\ln \alpha)''(\langle n, X \rangle) \geq \mathbb{E}\left(\frac{1}{\langle n, (\text{Hess } f)(X)^{-1} n \rangle | \langle n, X \rangle}\right). \tag{33}
$$
almost surely. In particular,

\[-(\ln \alpha)''(\langle n, X \rangle) \geq \frac{1}{t} \cdot \mathbb{P}(\langle n, (\text{Hess } f)(X)^{-1}n \rangle \leq t | \langle n, X \rangle). \tag{34}\]

Markov’s inequality shows that for any random variable \(0 \leq \theta \leq 1\) one has \(\mathbb{P}\left(\theta > \frac{E(\theta)}{2}\right) > \frac{E(\theta)}{2 - E(\theta)}\). Applying this to the random variable \(\mathbb{P}(\langle n, (\text{Hess } f)(X)^{-1}n \rangle \leq t | \langle n, X \rangle)\), whose expectation is \(p\) by (31), we may continue (34) to obtain

\[\mathbb{P}\left(-(\ln \alpha)''(\langle n, X \rangle) > \frac{p}{2t}\right) > \frac{p}{2 - p}. \tag{35}\]

This is to be compared with the conclusion of Lemma 2.2, which states that for any \(C \geq 4\),

\[\mathbb{P}\left(-(\ln \alpha)''(\langle n, X \rangle) > \left(C \max_{s \in \mathbb{R}} \alpha(s)\right)^2\right) \leq 4C^{-1}. \tag{36}\]

Substituting \(C = \frac{8 - 4p}{p}\), the two inequalities show that \(\left(\frac{8 - 4p}{p} \max_{s \in \mathbb{R}} \alpha(s)\right)^2 > \frac{p}{2t}\), proving (32), under our additional assumption on the positivity of \(\text{Hess } f\).

To treat the case of general \(f\), define, for \(\varepsilon > 0\),

\[f_{\varepsilon}(x) := f(x) + \frac{1}{2} \varepsilon \|x\|^2 - \ln Z_{\varepsilon}\]

where \(Z_{\varepsilon} = \int \exp(-f(x) - \frac{1}{2} \varepsilon \|x\|^2) \, dx\) is chosen so that \(\int f_{\varepsilon}(x) \, dx = 1\). Note that \(Z_{\varepsilon} \to 1\) as \(\varepsilon \searrow 0\) by the dominated convergence theorem. Let \(X_{\varepsilon}\) be a random vector with density \(f_{\varepsilon}\). Note that \(\text{Hess } f_{\varepsilon}(x) = \text{Hess } f(x) + \varepsilon \text{Id}\) for almost every \(x\). In particular, \(\langle n, (\text{Hess } f_{\varepsilon})(x)^{-1}n \rangle \geq \langle n, (\text{Hess } f)(x)^{-1}n \rangle\) almost everywhere. Thus

\[p = \mathbb{P}\left(\langle n, (\text{Hess } f)(X)^{-1}n \rangle \leq t\right) \leq \lim_{\varepsilon \searrow 0} \mathbb{P}\left(\langle n, (\text{Hess } f_{\varepsilon})(X_{\varepsilon})^{-1}n \rangle \leq t\right)\]

by a second application of the dominated convergence theorem. Now set \(\alpha_{\varepsilon}\) to be the log-concave density of \(\langle n, X_{\varepsilon}\rangle\). It is straightforward that for every \(s \in \mathbb{R}\),

\[\alpha(s) \geq \frac{1}{Z_{\varepsilon}} \alpha_{\varepsilon}(s).\]

Combining the above facts we obtain the conclusion (32) as a consequence of the theorem applied to \(X_{\varepsilon}\), by taking the limit \(\varepsilon \searrow 0\).

\[\square\]

**Proof of Theorem 1.1.** Define the unit vector \(n := \frac{\eta}{\|\eta\|}\) and let \(\alpha\) be the log-concave density of \(\langle n, X \rangle\). Let \(r > 0\) and set

\[p := \mathbb{P}\left(\langle n, \text{Hess } f(X)^{-1}n \rangle \leq r\right). \tag{37}\]

Lemma 2.5 shows that

\[\max_{s \in \mathbb{R}} \alpha(s) \geq \frac{p^{3/2}}{(8 - 4p)^{3/2}} \geq \frac{p^{3/2}}{8\sqrt{2}r}. \tag{38}\]

Part 1 of Proposition 2.1 then implies that

\[\text{Var}(\langle n, X \rangle) \leq \frac{Cr}{p^3}, \tag{39}\]

for a universal constant \(C > 0\). The conclusion (3) now follows by setting \(r = \frac{1}{\|\eta\|^2} . \tag{39}\]

\[\square\]
2.3. Proof of the quantitative log-concavity theorem. In this section we prove Theorem 1.2. As in the theorem, let $X$ be a random vector with a log-concave density $\exp(-f)$ and $\eta \in \mathbb{R}^n$. Denote by $\alpha_\eta : \mathbb{R} \to [0, \infty)$ the (log-concave) density function of $\langle \eta, X \rangle$. Fix $D \geq 0$, $t > 0$ and $s \in \mathbb{R}$ satisfying that $\alpha_\eta(s) > 0$.

Denote by $m_{n-1}$ the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$. We fix a representative of the marginal density $\alpha_\eta$ by setting

$$\alpha_\eta(r) = \int_{\{x \in \mathbb{R}^n : \langle \eta, x \rangle = r\}} \exp(-f(x)) \, dm_{n-1}(x), \quad r \in \mathbb{R}. \quad (40)$$

Similarly, we fix a representative of $\gamma_\eta(D, r, t)$ by setting

$$\gamma_\eta(D, r, t) \alpha_\eta(r) = \int_{\{x \in \mathbb{R}^n : \langle \eta, x \rangle = r, D_{\eta, x}(t) \geq D\}} \exp(-f(x)) \, dm_{n-1}(x) \quad (41)$$

for $r$ in the open interval $\{r \in \mathbb{R} : \alpha_\eta(r) > 0\}$.

Recall the definition of $D_{\eta, x}$ from (5) and note that it takes values in $[0, \infty)$ by the convexity of $f$. Note further that $D_{\eta, x}(t)$ is upper semi-continuous and, in particular, measurable.

We first prove (7). Aiming to apply the Prékopa-Leindler inequality, define $F_1, F_2 : \mathbb{R}^{n-1} \to [0, \infty)$ to be the restrictions of the density $\exp(-f(x))$ to the hyperplanes $\langle \eta, x \rangle = s + t$ and $\langle \eta, x \rangle = s - t$, respectively (the hyperplanes are parameterized as $\mathbb{R}^{n-1}$ and are equipped with a standard Lebesgue measure - the projection of $m_{n-1}$). Set $g : \mathbb{R}^n \to (-\infty, \infty]$ to equal $f(x) + D_{\eta, x}(t)$ for $x$ satisfying $f(x) < \infty$ and to equal $\infty$ for other $x$. Lastly, define $F : \mathbb{R}^{n-1} \to [0, \infty)$ as the restriction of $\exp(-g(x))$ to the hyperplane $\langle \eta, x \rangle = s$.

The above definitions imply that the assumption (11) of the Prékopa-Leindler inequality with $\lambda = \frac{1}{2}$ is satisfied, i.e.,

$$F\left(\frac{1}{2}(x + y)\right) \geq \sqrt{F_1(x)F_2(y)}, \quad x, y \in \mathbb{R}^{n-1}. \quad (42)$$

Consequently,

$$\int F(x) \, dx \geq \sqrt{\int F_1(x) \, dx \int F_2(x) \, dx} = \sqrt{\alpha_\eta(s + t) \alpha_\eta(s - t)}. \quad (43)$$

To prove (7), it remains to note that

$$\int F(x) \, dx \leq (1 - \gamma_\eta(D, s, t)(1 - e^{-D})) \cdot \alpha_\eta(s). \quad (44)$$

Indeed, by the definition of $g$ and (41),

$$\int F(x) \, dx = \int_{\{x \in \mathbb{R}^n : \langle \eta, x \rangle = s\}} \exp(-g(x)) \, dm_{n-1}(x) \leq ((1 - \gamma_\eta(D, s, t)) + \exp(-D)\gamma_\eta(D, s, t)) \alpha_\eta(s).$$

We proceed to prove (8). Define the random variable $\Gamma := \mathbb{P}(D_{\eta, X}(t) \geq D \mid \langle \eta, X \rangle)$. Then, by (7),

$$\sqrt{\alpha_\eta(\langle \eta, X \rangle - t) \alpha_\eta(\langle \eta, X \rangle + t)} \leq (1 - \Gamma \cdot (1 - e^{-D})) \cdot \alpha_\eta(\langle \eta, X \rangle). \quad (45)$$

Hence it suffices to prove that $\mathbb{P}(\Gamma \geq \frac{1}{2}) \geq \frac{1}{2}$. This follows from the fact that $\mathbb{E}(\Gamma) \geq \frac{2}{4}$ (by Markov’s inequality applied to $1 - \Gamma$).
3. Random surface preliminaries

In this section we develop the technical tools necessary to prove our main results on random surfaces, Theorem 1.4 and Theorem 1.5. These tools will be put together in the following Section 4 to enable the use of the concentration results of Section 1.1 in the random surface context.

3.1. Isoperimetric properties of $\Lambda^d_L$. In this section we provide isoperimetric estimates for two graphs: the box $\Lambda^d_L$ and the graph obtained by performing a bond percolation process on $\Lambda^d_L$. The bond percolation process we will need for our application to random surfaces is not the usual independent percolation, but is still super-critical in a suitable sense. We postpone the formal definition; it is given later by (46).

For a graph $G = (V(G), E(G))$ and a subset $X \subseteq V(G)$ write
\[ E(G)|_X := \{ \{x, y\} \in E(G) : \{x, y\} \subseteq X \} \]
\[ \partial_G X := \{ \{x, y\} \in E(G) : \#(\{x, y\} \cap X) = 1 \}. \]
If $X \neq \emptyset$, let $G|_X := (X, E(G)|_X)$ be the induced subgraph of $G$ on $X$. For connected $G$, we will further use the class of connected subsets whose complement is also connected,
\[ C(G) := \{ X \subseteq V(G) : X \notin \{ \emptyset, V(G) \}, \text{both } G|_X \text{ and } G|_{V(G) \setminus X} \text{ are connected} \}. \]
We write $|S|$ for the cardinality of a finite set $S$.

3.1.1. The box $\Lambda^d_L$. The following is the isoperimetric inequality we need.

**Lemma 3.1.** Let $d \geq 2$ and $L \geq 1$. If $X \subseteq V(\Lambda^d_L)$ and $|X| \leq 3L^d/4$, then
\[ |\partial_{\Lambda^d_L} X| \geq |X|^\frac{d-1}{d}. \]

The inequality follows as a corollary of the following result.

**Lemma 3.2.** (Bollobás and Leader [11]) Let $d \geq 2$ and $L \geq 1$. Given $X \subseteq V(\Lambda^d_L)$, it holds that
\[ \begin{align*}
& (i) \quad |\partial_{\Lambda^d_L} X| \geq \min_{r \in \{1, \ldots, d\}} |X|^{1-\frac{1}{d}} r L^{\frac{d}{d-1}} - 1 \quad \text{if } |X| \leq L^d/2; \\
& (ii) \quad |\partial_{\Lambda^d_L} X| \geq L^{d-1} \quad \text{if } L^d/4 \leq |X| \leq 3L^d/4.
\end{align*} \]

These results appear in [11] as Theorem 3 (item (i)) and Corollary 4 (item (ii)).

**Proof of Lemma 3.1.** If $|X| \leq L^d/2$, then assertion (i) of Lemma 3.2 implies
\[ |\partial_{\Lambda^d_L} X| \geq \frac{|X|}{L} \min_{r \in \{1, \ldots, d\}} r \left( \frac{L^d}{|X|} \right)^\frac{1}{d} \geq \frac{|X|}{L} \left( \min_{r \in \{1, \ldots, d\}} \left( \frac{L^d}{|X|} \right) \right)^\frac{1}{d} = |X|^\frac{d-1}{d}. \]

In the remaining case, $L^d/2 < |X| \leq 3L^d/4$, assertion (ii) of Lemma 3.2 yields
\[ |\partial_{\Lambda^d_L} X| \geq L^{d-1} > |X|^\frac{d-1}{d}. \]

3.1.2. Bond percolation on $\Lambda^d_L$. In this section we study isoperimetry for random spanning subgraphs of $\Lambda^d_L$ (i.e., random subgraphs whose vertex set is $V(\Lambda^d_L)$). Specifically, for $0 < p < 1$ we consider random spanning subgraphs $\Lambda^d_{L,p} \subseteq \Lambda^d_L$ satisfying the property
\[ \mathbb{P} \left( E' \cap E(\Lambda^d_{L,p}) = \emptyset \right) \leq (1 - p)^{|E'|} \quad \text{for each } E' \subseteq E(\Lambda^d_L). \]

Independent bond percolation with parameter $p$ certainly satisfies (46) but our application to random surfaces will make use of a more general percolation process, for which (46) still holds true. The next lemma studies the (anchored) isoperimetric properties of $\Lambda^d_{L,p}$ for $p$ sufficiently close to 1 (see also [44] for related statements).
Lemma 3.3. Let $d \geq 2$. There is a function $q : (0,1) \to [0,1]$ satisfying $\lim_{p \to 1} q(p) = 1$ such that the following holds. Let $L \geq 1$ and let $a, b \in V(\Lambda^d_L)$. Let $0 < p < 1$. Suppose $\Lambda^d_{L,p}$ is a random spanning subgraph of $\Lambda^d_L$ satisfying (46). Let $G_a$ denote the connected component of $a$ in $\Lambda^d_{L,p}$ (considered as a connected graph). Then each of the following events holds with probability at least $q(p)$:

1. $b \in V(G_a)$.
2. For each set $X \in C(G_a)$ satisfying $a \in X$ and $|X| \leq \frac{1}{2} |V(G_a)|$ it holds that

$$|\partial_{G_a} X| \geq \frac{1}{2} |X|^\frac{d-1}{2}.$$  

The proof of the lemma makes use of the following standard estimate for the number of connected sets with connected complement. For $a \in V(\Lambda^d_L)$ and integer $m \geq 1$ define

$$C_{a,m}(\Lambda^d_L) := \left\{ X \in C(\Lambda^d_L) : a \in X, |X| \leq 3L^d/4, |\partial_{\Lambda^d_L} X| = m \right\}.$$  

Lemma 3.4. Let $d \geq 2$ and $L \geq 1$. There exists $C > 1$, depending only on $d$, such that the inequality

$$\left| C_{a,m}(\Lambda^d_L) \right| \leq C^m$$  

holds for each $a \in V(\Lambda^d_L)$ and every integer $m \geq 1$.

As we could not find a reference for this exact statement, we provide a proof at the end of the section.

Proof of Lemma 3.3. For each integer $m \geq 1$ and $v \in \Lambda^d_L$, define the event

$$\mathcal{E}_{v,m} := \left\{ \exists X \in C_{v,m}(\Lambda^d_L) \text{ satisfying } |\partial_{\Lambda^d_{L,p}} X| < \frac{1}{2} m \right\}.$$  

Define also $\mathcal{E}_v := \bigcup_{m=1}^{\infty} \mathcal{E}_{v,m}$. We first prove the convergence

$$\lim_{p \to 1} P(\mathcal{E}_v) = 0 \quad \text{uniformly in } L, v,$$  

where here and later we also implicitly require the uniformity of the statements in the choice of processes $(\Lambda^d_{L,p})$ satisfying (46). To see this, first consider a specific $X \in C_{v,m}$. The inequality $|\partial_{\Lambda^d_{L,p}} X| < \frac{1}{2} m$ implies that some $[m/2]$-edge subset of $\partial_{\Lambda^d_L} X$ is completely removed when passing from $\Lambda^d_L$ to $\partial_{\Lambda^d_{L,p}} X$. Taking a union bound over such subsets, we obtain

$$P\left( \left| \partial_{\Lambda^d_{L,p}} X \right| < \frac{1}{2} m \right) \leq m \left( \frac{m}{[m/2]} \right)^2 (1-p)^{[m/2]} \leq (4(1-p))^{m/2}.$$  

Now, if $C$ is the constant from Lemma 3.4, and if $p$ is sufficiently close to 1, then (47) is implied by the following inequalities:

$$P(\mathcal{E}_v) \leq \sum_{m=1}^{\infty} P(\mathcal{E}_{v,m}) \leq \sum_{m=1}^{\infty} (4C^2(1-p))^{m/2} \leq 3C \sqrt{1-p}.$$  

We next prove that

$$\lim_{p \to 1} P(v \notin V(G_a)) = 0 \quad \text{uniformly in } L, a \text{ and } v \in V(\Lambda^d_L).$$  

To prove (48), we assume that for some $v \in V(\Lambda^d_L)$ the property $v \notin V(G_a)$ holds. Then there exists a set $X \in C(\Lambda^d_L)$ with $a \in X$, $v \notin X$ and $\partial_{\Lambda^d_{L,p}} X = \emptyset$. Further, min $\{ |X|, |V(\Lambda^d_L) \setminus X| \} \leq L^d/2 \leq 3L^d/4$, whence the event $\mathcal{E}_a \cup \mathcal{E}_v$ holds. (Indeed, $\mathcal{E}_a$ holds if $|X| \leq 3L^d/4$ and $\mathcal{E}_v$ holds
if $|V(\Lambda^d_L) \setminus X| \leq 3L^d/4$. Therefore $P(v \notin V(G_a)) \leq P(\mathcal{E}_a) + P(\mathcal{E}_a)$, and (48) follows from (47). Taking $v = 0$ proves the assertion of the lemma regarding item (1).

In order to prove the assertion of the lemma regarding item (2), define the event

$$\mathcal{E} := \left\{ |V(G_a)| < L^d/2 \right\}.$$  

We note that

$$\lim_{p \to 1} P(\mathcal{E}) = 0 \quad \text{uniformly in } L, a \quad (49)$$

as a consequence of (48) and the following chain of inequalities:

$$P(\mathcal{E}) \leq \frac{2}{L^d} \mathbb{E} \left| V(\Lambda^d_L) \setminus V(G_a) \right| = \frac{2}{L^d} \sum_{v \in V(\Lambda^d_L)} P(v \notin V(G_a)) \leq 2 \cdot \max_{v \in V(\Lambda^d_L)} P(v \notin V(G_a)).$$

It thus suffices to show that the event in item (2) holds when neither $\mathcal{E}_a$ nor $\mathcal{E}$ hold. Let $X \in C(G_a)$ for which $a \in X$ and $|X| \geq \frac{1}{2} |V(G_a)|$. Let us extend $X$ to a set $X' = X \cup Y$, where

$$Y := \left\{ w \in V(\Lambda^d_L) \setminus V(G_a) : \exists \text{ path in } \Lambda^d_L \setminus X \text{ between } w \text{ and } V(G_a) \setminus X \right\}$$

(informally, $X'$ is obtained from $X$ by “filling holes”). From the definition of $Y$ it is evident that $X' \in C(\Lambda^d_L)$, $V(G_a) \cap X' = X$ and $\partial_{\Lambda^d_L, p} X' = \partial_{\Lambda^d_L} X$. Additionally, when $\mathcal{E}$ does not hold, we have

$$|X'| \leq L^d - |V(G_a) \setminus X| \leq L^d - \frac{1}{2} |V(G_a)| \leq 3L^d/4.$$  

Lastly, using Lemma 3.1 and the fact that $\mathcal{E}_a$ does not hold, we conclude that

$$|\partial_{\Lambda^d_L} X'| = \left| \partial_{\Lambda^d_L, p} X' \right| \geq \frac{1}{2} \left| \partial_{\Lambda^d_L} X' \right| \geq \frac{1}{2} |X'|^{\frac{d+1}{d}} \geq \frac{1}{2} |X|^{\frac{d+1}{d}},$$

so the event in item (2) indeed occurs, provided that neither $\mathcal{E}$ nor $\mathcal{E}_a$ holds. $\square$

We now give the proof of Lemma 3.4.

If $X \in C(\Lambda^d_L)$, then, in a certain sense, the boundary of $X$ is connected. This is established in the next lemma which is proved by Deuschel and Pisztora [22, part (ii) of Lemma 2.1] (see also the related paper of Timár [49]).

**Lemma 3.5.** Let $\Pi^d_L$ be the graph defined by

$$V(\Pi^d_L) := E(\Lambda^d_L), \quad \{e_1, e_2\} \in E(\Pi^d_L) \text{ if the midpoints of } e_1 \text{ and } e_2 \text{ are at } \ell^\infty \text{ distance in } (0, 1].$$

Let $X \in C(\Lambda^d_L)$. Then the induced subgraph $\Pi^d_L|_{\partial X}$ is connected, where $\partial X$ stands for $\partial_{\Lambda^d_L} X$.

The following standard lemma gives a bound on the number of connected subsets of a graph containing a given vertex. A proof may be found in [10, Chapter 45].

**Lemma 3.6.** Let $G = (V(G), E(G))$ be a graph with maximal degree $\Delta \geq 3$. Let $v \in V(G)$ and integer $m \geq 1$. Then

$$|\{S \subset V(G) : S \text{ is connected, } v \in S \text{ and } |S| = m\}| \leq (\Delta - 1)^{m-1}. \quad \text{(50)}$$

**Proof of Lemma 3.4.** Since $X \in C_{a,m}(\Lambda^d_L)$ is uniquely determined by $\partial X := \partial_{\Lambda^d_L} X$, it is sufficient to enumerate all possibilities for $\partial X$. Given an edge $e_1 \in \partial X$, the connectivity property of Lemma 3.5 and the counting estimate of Lemma 3.6 show that there are at most $C^m$ options for $\partial X$, where $C$ depends only on $d$. To find the starting edge $e_1$ the following argument may be used. Choose $X_0 \subseteq V(\Lambda^d_L)$ so that $a \in X_0$, $\Lambda^d_L|_{X_0}$ is connected and $|X_0| = \left\lfloor \min \left\{ \frac{d}{d+1}, 3L^d/4 \right\} \right\rfloor + 1$. Lemma 3.1 implies that $|X| < |X_0|$, from which we conclude that $\partial X \cap E(\Lambda^d_L|_{X_0}) \neq \emptyset$. Consequently, $e_1$ may
be chosen as one of at most \(|E(\Lambda^d,|x_0|)\) \(\leq d m \frac{d}{d - 1} + 1 \leq C^m \) options for a constant \(C\) depending only on \(d\).

\[\square\]

### 3.2. Energy estimate via isoperimetry

The following lemma is the main result of this section. It is a close relative of the result by Benjamini and Kozma [4, Theorem 2.1], with the difference that our result is applicable for non-quadratic potentials.

**Lemma 3.7.** Let \(G\) be a finite connected graph. Let \(l\) be an integer such that \(2^l \leq |V(G)| < 2^{l+1}\). For each \(i \in \{1, 2, \ldots, l\}\) and for each vertex \(v \in V(G)\) write

\[
M_i(v) := \max \left\{ |E(G[X])| + |\partial_X X| : X \in \mathcal{C}(G), \ v \in X, \ \left| \frac{|V(G)|}{2^{i+1}} \right| < |X| \leq \left| \frac{|V(G)|}{2^i} \right| \right\},
\]

\[
m_i(v) := \min \left\{ |\partial_X X| : X \in \mathcal{C}(G), \ v \in X, \ \left| \frac{|V(G)|}{2^{i+1}} \right| < |X| \leq \left| \frac{|V(G)|}{2^i} \right| \right\},
\]

with the convention that a maximum, or minimum, over an empty collection remains undefined.

Let, finally, \(U : \mathbb{R} \rightarrow \mathbb{R}\) be a convex function with \(U(0) = 0\) and \(U(-x) \equiv U(x)\). Then the inequality

\[
\inf_{\varphi : V(G) \rightarrow \mathbb{R}} \sum_{e \in E(G)} U(\nabla_e \varphi) \geq \inf_{p_1, \ldots, p_l \geq 0 \atop q_1, q_{l+1}, q_{l+2} \geq 0} \left( \sum_{i=1}^l M_i(a) \cdot U \left( \frac{p_i m_i(a)}{M_i(a)} \right) + \sum_{i=1}^l M_i(b) \cdot U \left( \frac{q_i m_i(b)}{M_i(b)} \right) \right).
\]

holds for every two distinct vertices \(a, b \in V(G)\), when all quantities \(M_i(a), m_i(a), M_i(b), m_i(b)\) are defined. In the presence of undefined terms the inequality continues to hold with the following modification: Whenever one of \(M_i(a), m_i(a)\) (respectively, \(M_i(b), m_i(b)\)) is undefined the corresponding summand is set to 0 and the additional restriction \(p_i = 0\) (respectively, \(q_i = 0\)) is added to the infimum.

We remark that while some of the terms \(M_i(v), m_i(v)\) may indeed be undefined for some graphs \(G\) and vertices \(v \in V(G)\), it is always the case that for every distinct \(a, b \in V(G)\) there is an \(X \in \mathcal{C}(G)\) with \(a \in X\) and \(b \notin X\), implying that for at least one \(i\) either both \(M_i(a), m_i(a)\) or both \(M_i(b), m_i(b)\) are defined.

Before we proceed with the proof, let us show that a well-known energy bound for the quadratic potential on graphs with the isoperimetry of a \(\mathbb{Z}^d\) lattice follows as an immediate corollary.

**Corollary 3.8.** In the setup of Lemma 3.7, suppose that the graph \(G\) and vertices \(a, b \in V(G)\) are such that the quantities \(M_i(a), m_i(a), M_i(b), m_i(b)\) which are defined satisfy the inequalities

\[
M_{l-i}(v) \leq C 2^i \quad \text{and} \quad m_{l-i}(v) \geq c 2^{\frac{d}{d-1}i} \quad \text{for} \ 1 \leq i \leq l \ \text{and} \ v \in \{a, b\},
\]

where \(d \geq 2\) is an integer and \(C, c > 0\). Then there exists \(c' = c'(d, C, c) > 0\) such that

\[
\inf_{\varphi : V(G) \rightarrow \mathbb{R}} \sum_{e \in E(G)} (\nabla_e \varphi)^2 \geq \inf_{p_1, \ldots, p_l \geq 0 \atop q_1, q_{l+1}, q_{l+2} \geq 0} \left( \sum_{i=0}^{l-1} 2^{i(1-2/d)} (p_{l-i}^2 + q_{l-i}^2) \right).
\]

**Proof.** Lemma 3.7 implies that for some \(c_0 = c_0(d, C, c)\) we have

\[
\inf_{\varphi : V(G) \rightarrow \mathbb{R}} \sum_{e \in E(G)} (\nabla_e \varphi)^2 \geq c_0 \inf_{p_1, \ldots, p_l \geq 0 \atop q_1, q_{l+1}, q_{l+2} \geq 0} \left( \sum_{i=0}^{l-1} 2^{i(1-2/d)} (p_{l-i}^2 + q_{l-i}^2) \right).
\]

The corollary follows by using the Cauchy–Schwarz inequality in the form \((\sum r_i)^2 \leq (\sum a_i r_i^2)(\sum 1/a_i)\), where \(r_{2i-1} = p_{l-i}, r_{2i} = q_{l-i}, a_{2i-1} = a_{2i} = 2^{i(1-2/d)}\) for \(i = 1, 2, \ldots, l\).

\[\square\]
Proof of Lemma 3.7. The proof is given for the case that all the quantities \( M_i(a), m_i(a), M_i(b), m_i(b) \) are defined. The required modifications when some of the terms are undefined are straightforward.

Note that \( U \) has to be continuous on \( \mathbb{R} \), since it is convex and attains only finite values.

No generality is lost if we assume \( \varphi(a) = 1 \) and \( \varphi(b) = 0 \). But then the inequality

\[
\sum_{e \in E(G)} U(\nabla_e \varphi) \geq \sum_{e \in E(G)} U(\nabla_e \bar{\varphi})
\]

holds, where

\[
\bar{\varphi}(v) := \begin{cases} 
0, & \text{if } \varphi(v) < 0 \\
\varphi(v), & \text{if } 0 \leq \varphi(v) \leq 1 \\
1, & \text{if } \varphi(v) > 1
\end{cases}
\]

for all \( v \in V(G) \).

Thus it is sufficient to consider \( \varphi \in [0, 1]^{V(G)} \). For the reasons of compactness, the infimum

\[
\inf_{\varphi : V(G) \to \mathbb{R}} \sum_{e \in E(G)} U(\nabla_e \varphi)
\]

is attained on a non-empty compact set \( \Omega_{\min} \subseteq [0, 1]^{V(G)} \). Let

\[
\varphi_{\min} = \arg \min_{\varphi \in \Omega_{\min}} \sum_{e \in E(G)} (\nabla_e \varphi)^2
\]

(the quadratic function is used here for convenience and can be replaced by other strictly convex, even functions on \( \mathbb{R} \)). Then for every \( s \in [0, 1) \) it holds that \( X(s) \in C(G) \), where

\[
X(s) := \{ v \in V(G) : \varphi_{\min}(v) \leq s \}.
\]

Indeed, assume for the contradiction that \( G|_{X(s)} \) is disconnected. If \( X_0 \subseteq X(s) \) is such that \( b \notin X_0 \) and \( G|_{X_0} \) is a connected component of \( G|_{X(s)} \), define

\[
\varphi_\varepsilon(v) := \begin{cases} 
\varphi_{\min}(v), & \text{if } v \notin X_0 \\
\varphi_{\min}(v) + \varepsilon, & \text{if } v \in X_0
\end{cases}
\]

for all \( v \in V(G) \).

Then, for all sufficiently small \( \varepsilon > 0 \),

\[
\sum_{e \in E(G)} U(\nabla_e \varphi_\varepsilon) \leq \sum_{e \in E(G)} U(\nabla_e \varphi_{\min}) \quad \text{and} \quad \sum_{e \in E(G)} (\nabla_e \varphi_\varepsilon)^2 \leq \sum_{e \in E(G)} (\nabla_e \varphi_{\min})^2,
\]

a contradiction to the choice of \( \varphi_{\min} \). The induced graph \( G|_{V(G) \setminus X(s)} \) is connected for similar reasons. From now on we will write \( \varphi \) instead of \( \varphi_{\min} \), since this will not cause any confusion.

Given a number \( \nu \in \mathbb{N} \) and an edge \( e = \{u, v\} \in E(G) \), denote

\[
T_\nu(e) = \{0, 1, 2, \ldots, \nu - 1\} \cap [\min \nu \cdot \varphi(e), \max \nu \cdot \varphi(e)],
\]

with \( \nu \cdot \varphi(e) := \{\nu \cdot \varphi(u), \nu \cdot \varphi(v)\} \), so that

\[
\tau \in T_\nu(e) \quad \text{if and only if} \quad e \in \partial_G X(\tau/\nu).
\]

Then

\[
\sum_{e \in E(G)} U(\nabla_e \varphi) = \lim_{\nu \to \infty} \sum_{e \in E(G)} U\left(\frac{|T_\nu(e)|}{\nu}\right).
\]

Let us define a partition

\[
\{0, 1, 2, \ldots, \nu - 1\} = T_a^1 \cup \ldots \cup T_a^l \cup T_b^1 \cup \ldots \cup T_b^l.
\]
according to the following equivalence relations:

\[
\tau \in T^b_{i,\nu} \iff \left| \frac{\tau(G)}{2} \right| < \left| X \left( \frac{\tau}{\nu} \right) \right| \leq \left| \frac{\tau(G)}{2} \right|, \quad i = 1, 2, \ldots, l,
\]

\[
\tau \in T^a_{i,\nu} \iff \left| \frac{\tau(G)}{4} \right| < \left| V(G) \setminus X \left( \frac{\tau}{\nu} \right) \right| < \left| \frac{\tau(G)}{2} \right|,
\]

\[
\tau \in T^a_{i,\nu} \iff \left| \frac{\tau(G)}{2\nu} \right| < \left| V(G) \setminus X \left( \frac{\tau}{\nu} \right) \right| \leq \left| \frac{\tau(G)}{2} \right|, \quad i = 2, 3, \ldots, l.
\] (52)

It is not hard to check that indeed, according to the definition (52), each number \( \tau \) becomes assigned to exactly one set \( T^a_{i,\nu} \) or \( T^b_{i,\nu} \).

Define

\[
P_{a,\nu} := \{ (e, \tau) : e \in E(G), \ \tau \in T^a_{i,\nu}, \ \tau \in \tau(e) \},
\]

\[
P_{b,\nu} := \{ (e, \tau) : e \in E(G), \ \tau \in T^b_{i,\nu}, \ \tau \in \tau(e) \}.
\]

For the next steps of the proof it will be useful to recall that since \( U \) is convex and satisfies \( U(0) = 0 \) then \( U(tx) \leq tU(x) \) for \( t \in [0, 1] \) and \( x \geq 0 \), and consequently \( U(x_1 + \ldots + x_n) \geq U(x_1) + \ldots + U(x_n) \) for non-negative \( x_1, \ldots, x_n \). Thus

\[
U \left( \frac{|\tau'(\nu)|}{\nu} \right) = U \left( \frac{1}{\nu} \sum_{i=1}^{l} \sum_{\tau=0}^{\nu-1} 1[(e, \tau) \in P^a_{i,\nu}] + \frac{1}{\nu} \sum_{i=1}^{l} \sum_{\tau=0}^{\nu-1} 1[(e, \tau) \in P^b_{i,\nu}] \right)
\]

\[
\geq \sum_{i=1}^{l} U \left( \frac{1}{\nu} \sum_{\tau=0}^{\nu-1} 1[(e, \tau) \in P^a_{i,\nu}] \right) + \sum_{i=1}^{l} \frac{U}{\nu} \sum_{\tau=0}^{\nu-1} 1[(e, \tau) \in P^b_{i,\nu}] \right).
\]

Assume that \( T^b_{i,\nu} \neq \emptyset \). Set \( \tau^b_{i,\nu} := \max T^b_{i,\nu} \) and \( E^b_{i,\nu} := E \left( G \setminus X \left( \tau^b_{i,\nu}/\nu \right) \right) \cup \partial_G \left( X \left( \tau^b_{i,\nu}/\nu \right) \right) \). By definition, \( (e, \tau) \in P^b_{i,\nu} \) if and only if \( \tau \in T^b_{i,\nu} \) and \( e \in \partial_G X (\tau/\nu) \). Therefore, if \( e \in E(G) \) is such that there exists \( \tau \) with \( (e, \tau) \in P^b_{i,\nu} \) then \( e \in E^b_{i,\nu} \). We conclude that

\[
\sum_{e \in E(G)} U \left( \frac{1}{\nu} \sum_{\tau=0}^{\nu-1} 1[(e, \tau) \in P^b_{i,\nu}] \right) = \sum_{e \in E^b_{i,\nu}} U \left( \frac{1}{\nu} \sum_{\tau \in T^b_{i,\nu}} 1[(e, \tau) \in P^b_{i,\nu}] \right) \geq |E^b_{i,\nu}| \cdot U \left( \frac{|T^b_{i,\nu}|}{\nu} \cdot m_i(b) \cdot \frac{1}{|E^b_{i,\nu}|} \right) \geq M_i(b) \cdot U \left( \frac{|T^b_{i,\nu}|}{\nu} \cdot m_i(b) \cdot \frac{1}{M_i(b)} \right),
\]

where the inequalities use Jensen's inequality and the inequalities

\[
|E^b_{i,\nu}| \leq M_i(b) \quad \text{and} \quad \sum_{e \in E^b_{i,\nu}} 1[(e, \tau) \in P^b_{i,\nu}] = |\partial_G X (\tau/\nu) | \geq m_i(b) \quad \text{for every} \ \tau \in T^b_{i,\nu}.
\]

If \( T^b_{i,\nu} = \emptyset \), then we still have

\[
\sum_{e \in E} U \left( \frac{1}{\nu} \sum_{\tau=0}^{\nu-1} 1[(e, \tau) \in P^b_{i,\nu}] \right) \geq M_i(b) \cdot U \left( \frac{|T^b_{i,\nu}|}{\nu} \cdot m_i(b) \cdot \frac{1}{M_i(b)} \right),
\]

since both sides of the inequality are zero.

Similarly, for each \( i = 1, 2, \ldots, l \) it holds that

\[
\sum_{e \in E} U \left( \frac{1}{\nu} \sum_{\tau=0}^{\nu-1} 1[(e, \tau) \in P^a_{i,\nu}] \right) \geq M_i(a) \cdot U \left( \frac{|T^a_{i,\nu}|}{\nu} \cdot m_i(a) \cdot \frac{1}{M_i(a)} \right),
\]

since the roles of \( a \) and \( b \) are interchangeable.
The argument above yields

\[
\sum_{e \in E(G)} U \left( \frac{|T_\nu(e)|}{\nu} \right) \\
\geq \sum_{i=1}^{l} M_i(a) \cdot U \left( \frac{|T_{i,\nu}|}{\nu} \cdot m_i(a) \cdot \frac{1}{M_i(a)} \right) + \sum_{i=1}^{l} M_i(b) \cdot U \left( \frac{|T_{i,\nu}|}{\nu} \cdot m_i(b) \cdot \frac{1}{M_i(b)} \right) \\
\geq \inf_{p_1,\ldots,p_l \geq 0 \atop q_1,\ldots,q_l \geq 0 \atop p_1 + \ldots + p_l + q_1 + \ldots + q_l = 1} \left( \sum_{i=1}^{l} M_i(a) \cdot U \left( \frac{p_i m_i(a)}{M_i(a)} \right) + \sum_{i=1}^{l} M_i(b) \cdot U \left( \frac{q_i m_i(b)}{M_i(b)} \right) \right),
\]

as the second inequality is implied by the identity \(|T_{i,\nu}^a| + \ldots + |T_{i,\nu}^a| + |T_{i,\nu}^b| + \ldots + |T_{i,\nu}^b| = \nu\). The inequality (51) follows by passing to the limit \(\nu \to \infty\). \(\square\)

3.3. Estimates on the joint distribution of gradients. In this section we consider a random surface model sampled from the distribution \(\mu_{T_{2L}^d, U}\) of Setting 1 (as introduced in Section 1.2; in particular, we work on the ‘even’ torus \(T_{2L}^d\)). We impose the restrictions \(\int e^{-U(x)} \, dx < \infty\) and \(\inf_{\mathbb{R}} U(x) > -\infty\). We note that every non-constant convex potential satisfies these restrictions, but \(U\) is not assumed to be convex in this section.

Let \(\varphi\) be sampled from the measure \(\mu_{T_{2L}^d, U}\) of Setting 1. Given a measurable set \(S \subseteq [0, \infty)\), for an arbitrary set of edges \(E_0 \subseteq E(T_{2L}^d)\) one can consider the following event: the gradients \(|\nabla e \varphi|\) for edges \(e \in E_0\) all belong to the set \(S\). Our goal is to obtain an upper bound for the probability of such an event in the form of \(p(S, d, U)^{|E_0|}\), where \(p(S, d, U)\) is a certain explicit expression. The proof uses reflection positivity in the form of the chessboard estimate (in a way similar to [41]). In the later application to proving Theorem 1.4, \(S\) is chosen to contain a neighborhood of infinity and a neighborhood of the points where \(U''\) vanishes. The neighborhoods are chosen as a function of \(U\) in a such a way that \(p(S, d, U)\) is small.

**Lemma 3.9.** Let \(d \geq 2\) and \(L \geq 2\) be integers. Let \(U : \mathbb{R} \to (-\infty, \infty]\) be a potential such that \(U(-x) = U(x)\) for all \(x \in \mathbb{R}\), \(0 < \int e^{-U(x)} \, dx < \infty\) and \(\inf_{\mathbb{R}} U(x) > -\infty\). Let, finally, \(\varphi\) be a random surface randomly sampled according to the measure \(\mu_{T_{2L}^d, U}\). There exist positive numbers \(C(d, U)\) and \(c(d)\) (both independent of \(L\) and the second also independent of \(U\)) such that the inequality

\[
\mathbb{P}\left( |\nabla e \varphi| \in S \text{ for all } e \in E_0 \right) \leq p(S, d, U)^{|E_0|}
\]

holds for all \(E_0 \subseteq E(T_{2L}^d)\) and all measurable \(S \subseteq [0, \infty)\) if we set, by definition,

\[
p(S, d, U) := \left( C(d, U) \cdot \int_S e^{-U(x)} \, dx \right)^{c(d)}.
\]

Our use of the chessboard estimate is the main reason for working with periodic boundary conditions (i.e., on \(T_{2L}^d\)). We do not know to obtain an analog of Lemma 3.9 without using the chessboard estimate although it seems reasonable that an analog in Setting 2 (i.e., on the box \(\Lambda_{2L}^d\)) should hold. For a step in this direction see [18].

The rest of this section is devoted to the proof of Lemma 3.9. We therefore extend the use of the notation introduced in Lemma 3.9 to the entire section.

Define

\[
E_S[\varphi] := \{ e \in E(T_{2L}^d) : |\nabla e \varphi| \in S \}.
\]
Next, for \(1 \leq j \leq d\) and \(\sigma \in \{0, 1\}^d\), define \(E_{j, \sigma}(T_{2L}) \subseteq E(T_{2L})\) as a collection of edges of the form \(\{(x_1, x_2, \ldots, x_d), (y_1, y_2, \ldots, y_d)\}\), where

\[ x_i \equiv y_i \equiv \sigma_i \pmod{2} \quad \text{for all } i \neq j. \]

In particular, each edge \(e \in E_{j, \sigma}(T_{2L})\) is aligned with the \(j\)th coordinate vector. Note that \(E_{j, \sigma}(T_{2L})\) and \(E_{j', \sigma'}(T_{2L})\) are either disjoint or equal, with equality occurring exactly when \(j = j'\) and \(\sigma\) equals \(\sigma'\) on all but the \(j\)th coordinate.

The next proposition is known in the literature.

**Proposition 3.10.** The inequality

\[
P\left(E_0 \subseteq E_S[\varphi]\right) \leq P\left(E_{j, \sigma}(T_{2L}) \subseteq E_S\right)^{|E_0|/|E_{j, \sigma}(T_{2L})|}.
\]

holds for each \(1 \leq j \leq d\), \(\sigma \in \{0, 1\}^d\), all \(E_0 \subseteq E_{j, \sigma}(T_{2L})\) and all measurable \(S \subseteq [0, \infty)\).

**Proof.** See [5, Theorem 5.8] or [41, Section 3]. The proof is derived from the chessboard estimate. (See the aforementioned references also for the details of that technique.) \(\square\)

For the next proposition the reader may benefit from recalling the notion of the partition function \(Z_{G, V_0, \varphi_0, U}\), see formula (15). In our case, i.e., with \(G = T_{2L}\) and with boundary conditions \((V_0, \varphi_0)\) as in Setting 1 in Section 1, we shorten the notation to \(Z_{T_{2L}, U}\).

**Proposition 3.11.** There exists a positive constant \(C_3(d, U)\), independent of \(L\), such that

\[
Z_{T_{2L}, U} \geq C_3(d, U)^{1-|V(T_{2L})|}.
\]

**Proof.** See [41, Lemma 3.1]. \(\square\)

**Proposition 3.12.** There exists a positive constant \(C(d, U)\) such that the inequality

\[
P\left(E_{j, \sigma}(T_{2L}) \subseteq E_S[\varphi]\right) \leq \left(C(d, U) \cdot \int_S e^{-U(t)} \, dt\right)^{\frac{1}{2}|E_{j, \sigma}(T_{2L})|} \tag{53}
\]

holds for each \(1 \leq j \leq d\), each \(\sigma \in \{0, 1\}^d\) and every measurable subset \(S \subseteq [0, \infty)\).

**Proof.** Let \(T\) be an arbitrary tree satisfying the conditions

(a) \(V(T) = V(T_{2L})\);
(b) \(E(T) \subseteq E(T_{2L})\);
(c) \(|(E(T) \cap E_{j, \sigma}(T_{2L}))| \geq \frac{1}{2}|E_{j, \sigma}(T_{2L})|\).
Indeed, the first inequality holds by inclusion of the respective events; the second inequality holds from that proposition); finally, the last inequality is a consequence of \( (56) \).

Then the following inequalities hold.

\[
P\left( E_{j,\sigma}(T_{2L}^d) \subseteq E_{S}[\varphi] \right) \leq \mathbb{P}\left( E(T) \cap E_{j,\sigma}(T_{2L}^d) \subseteq E_{S}[\varphi] \right)
\]

\[
= \frac{1}{Z_{T_{2L}^d,U}} \int_{\Omega_{T_{2L}^d,U}} \prod_{e \in E(T_{2L}^d)} \exp(-U(\nabla_e \varphi)) \prod_{e \in E(T) \setminus E_{j,\sigma}(T_{2L}^d)} \mathbb{1}(|\nabla_e(\varphi)| \in S) \prod_{v \in V(T_{2L}^d) \setminus \{0\}} d\varphi(v)
\]

\[
\leq \frac{1}{Z_{T_{2L}^d,U}} \cdot \left[ \exp\left( -\inf_{t \in \mathbb{R}} U(t) \right) \right]^{|E(T_{2L}^d)\setminus E(T)|}
\cdot \left( \int_{\mathbb{R}} e^{-U(t)} \, dt \right)^{|E(T)\setminus E_{j,\sigma}(T_{2L}^d)|}
\cdot \left( \frac{2}{S} \int_{S} e^{-U(t)} \, dt \right)^{|E(T)\cap E_{j,\sigma}(T_{2L}^d)|}.
\]

(54)

We also note that, by the choice of the tree \( T \), one has

\[
|E(T) \cap E_{j,\sigma}(T_{2L}^d)| \geq \frac{1}{2} |E_{j,\sigma}(T_{2L}^d)|.
\]

(55)

Plugging (55) and the result of Proposition 3.11 into (54) indeed yields the estimate (53). \( \square \)

We are ready to finish the proof of Lemma 3.9.

**Proof of Lemma 3.9.** Given \( E_0 \), choose

\[
(j_0,\sigma_0) := \text{arg max}_{(j,\sigma)} \left| E_0 \cap E_{j,\sigma}(T_{2L}^d) \right|.
\]

Then

\[
\left| E_0 \cap E_{j_0,\sigma_0}(T_{2L}^d) \right| \geq \frac{|E_0|}{C_1(d)},
\]

(56)

where, by definition, \( C_1(d) := d \cdot 2^{d-1} \) is the number of distinct subsets \( E_{j,\sigma}(T_{2L}^d) \).

Let us note that the following inequalities hold:

\[
P\left( E_0 \subseteq E_{S}[\varphi] \right) \leq P\left( E_0 \cap E_{j_0,\sigma_0}(T_{2L}^d) \subseteq E_{S}[\varphi] \right)
\]

\[
\leq P\left( E_{j_0,\sigma_0}(T_{2L}^d) \subseteq E_{S}[\varphi] \right)^{|E_0 \cap E_{j_0,\sigma_0}(T_{2L}^d)|/|E_{j_0,\sigma_0}(T_{2L}^d)|}
\]

\[
\leq \left( C_2(d, U) \cdot \int_{S} e^{-U(t)} \, dt \right)^{1/2 \left| E_0 \cap E_{j_0,\sigma_0}(T_{2L}^d) \right|} \leq \left( C_2(d, U) \cdot \int_{S} e^{-U(t)} \, dt \right)^{1/2 |E_0|/C_1(d)}.
\]

Indeed, the first inequality holds by inclusion of the respective events; the second inequality holds by Proposition 3.10; the third inequality follows from Proposition 3.12 (with \( C_2 \) being the constant from that proposition); finally, the last inequality is a consequence of (56).
We finish the proof of Lemma 3.9 by letting $C(d, U) := C_2(d, U)$ and $c(d) := \frac{1}{2e^d}$.

\[
\[

4. Proofs of the results on random surfaces

In this section we prove our main results on random surfaces, Theorem 1.4 and Theorem 1.5.

4.1. Tail estimate for the potential $U(x) = |x|^p + x^2$, $p > 2$. In this section we prove Theorem 1.5. We will focus on Setting 2 (the box $\Lambda^d_L$), as the proof for Setting 1 is literally the same.

The proof is an application of the quantitative log-concavity result, Theorem 1.2. Let us first state a corollary of that result in the context of random surfaces. Recall the definitions of the measure on $\Omega$ and Lemma 1.5. We will focus on Setting 2 (the box $\Lambda^d_L$), noting that this distribution is absolutely continuous with respect to the Lebesgue measure on $\Omega$.

**Lemma 4.1.** Let $G$ be a finite connected graph, $V_0$ a proper subset of $V(G)$, $\varphi_0 : V_0 \to \mathbb{R}$ and $U : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be an even convex function which is not everywhere constant. Let $\eta \in \mathbb{R}^{V(G)}$ satisfy $\eta|_{V(G) \setminus V_0} \neq 0$. Denote by $\alpha_\eta : \mathbb{R} \to [0, \infty)$ the (log-concave) density function of $(\eta, \varphi)$, when $\varphi$ is sampled from the random surface measure $\mu_{G, V_0, \varphi_0, U}$ on the set $\Omega_{G, V_0, \varphi_0}$ from (13) and (14).

**Proof.** In order to use Theorem 1.2 we identify $\Omega_{G, V_0, \varphi_0}$ with $\mathbb{R}^{V(G) \setminus V_0}$ in the canonical fashion (restricting the functions to $V(G) \setminus V_0$). We let the random vector $X$ of Theorem 1.2 be sampled from $\mu_{G, V_0, \varphi_0, U}$, noting that this distribution is absolutely continuous with respect to the Lebesgue measure on $\Omega_{G, V_0, \varphi_0}$, with the log-concave density $\exp(-f)$ satisfying

\[
f(\varphi) = -\ln(Z_{G, V_0, \varphi_0, U}) + \sum_{e \in E(G)} U(\nabla_e \varphi).
\]

We may thus define $D_{\eta, \varphi}(t)$ via the formula (5). It then holds that

\[
D_{\eta}(s, t) = \inf_{\varphi \in \Omega_{G, V_0, \varphi_0}} D_{\eta, \varphi}(t).
\]

Recalling also the definition of $\gamma_\eta(D, s, t)$ from (6), our definitions imply that $\gamma_\eta(D_{\eta}(s, t), s, t) = 1$. Thus the inequality

\[
\sqrt{\alpha_\eta(s-t)\alpha_\eta(s+t)} \leq e^{-D_\eta(s,t)}\alpha_\eta(s)
\]

holds for every $t > 0$ and $s \in \mathbb{R}$ satisfying $\alpha_\eta(s) > 0$ by Theorem 1.2, establishing the first inequality in (60).

To see that $D_{\eta}(s, t) \geq D_\eta(t)$, establishing the second inequality in (60), observe the following. For each $\varphi^+, \varphi^- \in \Omega_{G, V_0, \varphi_0}$ satisfying $\langle \eta, \varphi^+ \rangle = s+t$ and $\langle \eta, \varphi^- \rangle = s-t$ we may define $\varphi = \frac{1}{2}(\varphi^+ + \varphi^-)$
and \( \psi = \frac{1}{2}(\varphi^+ - \varphi^-) \) so that
\[
\frac{1}{2} \left[ U(\nabla e \varphi^+) + U(\nabla e \varphi^-) \right] - U\left( \nabla e \varphi \right) = \frac{1}{2} \left( U(\nabla e \varphi + \nabla e \psi) + U(\nabla e \varphi - \nabla e \psi) \right) - U(\nabla e \varphi) \geq W(\nabla e \psi)
\]
and \( \psi : V(G) \to \mathbb{R} \) satisfies \( \psi \equiv 0 \) on \( V_0 \) and \( \langle \eta, \psi \rangle = t \).
\[\square\]

We proceed to prove Theorem 1.5 in Setting 2. Fix integers \( d \geq 3, L \geq 2 \) and real \( p > 2, t > 1 \). Set \( U(x) = |x|^p + x^2 \). Let \( \varphi \) be sampled from the random surface measure \( \mu_{\Lambda^d, U} \). Let \( V_0 \subset V(\Lambda^d_L) \) be as in Setting 2 and fix \( v \in V(\Lambda^d_L) \setminus V_0 \).

Let \( \alpha_v \) be the marginal density of \( \varphi(v) \). The function \( \alpha_v(s) \) is even and log-concave and therefore \( \alpha_v(s) \) is non-strictly increasing for \( s \leq 0 \) and non-strictly decreasing for \( s \geq 0 \). Combining this observation with Item 3 of Proposition 2.1 gives
\[
\mathbb{P}(|\varphi(v)| > t) \leq \mathbb{P}(\alpha_v(\varphi(v)) \leq \alpha_v(t)) \leq \frac{\alpha_v(t)}{\alpha_v(0)}.
\]
Consequently, it will suffice to prove the inequality
\[
\frac{\alpha_v(t)}{\alpha_v(0)} \leq C \exp(-c t^{\min\{p,d\}}).
\]

(62)

For convenience, in this section we will use the comparison operators \( \ll, \gg \) and \( \approx \) meaning, respectively, that the (positive) expression on the left is smaller than, is greater than, or equals the (positive) expression on the right up to a positive factor which depends only on \( d \) and \( p \) and not on any other parameter.

Lemma 4.1, applied to \( \alpha_v \) (noting that \( \varphi(v) = \langle \eta_v, \varphi \rangle \), where \( \eta_v : V(\Lambda^d_L) \to \mathbb{R} \) is the indicator function of the vertex \( v \)), yields
\[
\alpha_v(t) = \sqrt{\alpha_v(-t)\alpha_v(t)} \leq \exp(-D(t)) \cdot \alpha_v(0)
\]
where, using the abbreviations \( \Omega \) for \( \Omega_{\Lambda^d_L, \nu_0, \varphi_0} \) and \( E \) for \( E(\Lambda^d_L) \),
\[
D(t) = \inf_{\psi \in \Omega} \sum_{v \in E} W(\nabla e \psi),
\]
\[
W(r) = \inf_{s \in \mathbb{R}} \frac{1}{2} (U(s + r) + U(s - r)) - U(s) \approx |r|^p + r^2.
\]

(63)

(64)

(65)

In the last expression, the approximate equality follows by observing that
\[
\frac{1}{2}((s+r)^2 + (s-r)^2) - s^2 = r^2
\]
for all \( s, r \), while \( \frac{1}{2}(|s + r|^p + |s - r|^p) - |s|^p \) is \( \approx |r|^p \) when \( |r| \geq |s| \) and, by considering the second derivative of \( |x|^p \), is \( \approx r^2|s|^{p-2} \) when \( |r| \leq |s| \). Thus, the required inequality (62) will follow from (63) by obtaining a lower bound for the expression
\[
D^*(t) := \min_{\psi \in \Omega} \sum_{v \in E} (|\nabla e \psi|^p + (\nabla e \psi)^2).
\]
This will be accomplished by means of the energy estimate of Lemma 3.7 for \( G = \Lambda^d_L \), \( a = v \) and an arbitrary vertex \( b \in V_0 \). We adopt the notation \( m_i(\cdot) \) and \( M_i(\cdot) \) of that lemma. The number of vertices of the graph is \( N := |V(\Lambda^d_L)| = L^d \) and we fix \( l \) to be an integer such that \( 2^l \leq N < 2^{l+1} \).

It is well-known that \( \Lambda^d_L \) has the isoperimetry of \( \mathbb{Z}^d \) in the sense of Corollary 3.8, i.e.,
\[
M_i(a), M_i(b) \ll \frac{N}{2^l}, \quad m_i(a), m_i(b) \gg \left( \frac{N}{2^l} \right)^{\frac{d-1}{2}}.
\]

as follows from Lemma 3.1.
Lemma 3.7 yields
\[
\min_{\psi \in \Omega} \sum_{\psi(e) = t} W(\nabla_e \psi) = \min_{\psi \in \Omega} \sum_{\psi(e) = 1} W(t \nabla_e \psi)
\]
\[
\geq \inf_{p_1, \ldots, p_l \geq 0 \atop p_1 + \cdots + p_l = 1} \left( \sum_{i=1}^l M_i(a) \cdot W \left( t p_i m_i(a) \right) + \sum_{i=1}^l \frac{M_i(b)}{M_i(b)} \right)
\]
\[
\gg \inf_{p_1, \ldots, p_l \geq 0 \atop p_1 + \cdots + p_l = 1} \sum_{i=0}^{l-1} \left(2^i \left(1 - \frac{2}{d} \right) t^2 p_{l-i}^2 + 2^i \left(1 - \frac{p}{d} \right) t^p p_{l-i}^p\right). \quad (66)
\]

For the rest of the proof we may assume that \( l \) is sufficiently large. Indeed, if \( \sigma_l \) denotes the infimum in the rightmost part of (66) and \( f_l(p_1, p_2, \ldots, p_l) \) denotes the expression under that infimum, we have
\[
\sigma_l = \inf_{p_1, \ldots, p_l \geq 0 \atop p_1 + \cdots + p_l = 1} f_l(p_1, p_2, \ldots, p_l) = \inf_{p_1, \ldots, p_l \geq 0 \atop p_1 + \cdots + p_l = 1} f_{l+1}(0, p_1, p_2, \ldots, p_l)
\]
\[
\geq \inf_{p_1, \ldots, p_{l+1} \geq 0 \atop p_1 + \cdots + p_{l+1} = 1} f_{l+1}(p_1, p_2, \ldots, p_{l+1}) = \sigma_{l+1}.
\]

Consequently, increasing \( l \) may only weaken our estimate.

In addition, the fact that the partial derivative of \( f_l(p_1, p_2, \ldots, p_l) \) with respect to each \( p_i \) equals zero at \( p_i = 0 \) and is strictly positive when \( p_i > 0 \) implies that the infimum is attained for a vector with strictly positive coordinates.

Now, set
\[
(p_1, p_2, \ldots, p_l) = \arg \min_{p_1, \ldots, p_l \geq 0 \atop p_1 + \cdots + p_l = 1} \sum_{i=0}^{l-1} \left(2^i \left(1 - \frac{2}{d} \right) t^2 p_{l-i}^2 + 2^i \left(1 - \frac{p}{d} \right) t^p p_{l-i}^p\right);
\]

As explained above we have \( p_i > 0 \) for all \( i \). Therefore, there exists a \( \beta \) independent of \( i \) (a Lagrange multiplier) such that
\[
2 \cdot 2^i \left(1 - \frac{2}{d} \right) t^2 p_{l-i}^2 + p \cdot 2^i \left(1 - \frac{p}{d} \right) t^p p_{l-i}^p = \beta \quad \forall i = 1, 2, \ldots, l.
\]

Consequently,
\[
p_{l-i} \approx \min \left(2^{-i} \left(1 - \frac{2}{d} \right) t^{-2} \beta, \quad 2^{-i} \frac{d-p}{d(p-1)} t^{-p} \beta \frac{p-1}{p-1} \right). \quad (67)
\]

We conclude the proof by considering three cases:

**Case 1.** \( d < p \). Assuming, as we may, that \( l \) is sufficiently large, let \( 0 \leq i_0 \leq l - 1 \) be such that \( t^d \leq 2^{i_0} < 2^{i_0} \) (recalling that \( t > 2 \)). We prove that \( p_{l-i_0} \gg 1 \) whence \( D^*(t) \gg t^d \) by (66). Indeed, by (67) and our assumption that \( d < p \),
\[
1 = p_1 + \cdots + p_l \ll \sum_{i=i_0}^{l-1} 2^{-i} \left(1 - \frac{2}{d} \right) t^2 \beta \frac{1}{t^2} + \sum_{i=0}^{i_0-1} 2^i \frac{p-d}{d(p-1)} \beta \frac{1}{t^p} \approx 2^{-i_0} \left(1 - \frac{2}{d} \right) t^2 \beta \frac{1}{t^2} + 2^{i_0} \frac{p-d}{d(p-1)} \beta \frac{1}{t^p} \approx \beta t^d + \frac{\beta}{t^d} \frac{1}{p-1}.
\]

Thus \( \beta \gg t^d \) so that \( p_{l-i_0} \gg 1 \) by (67).
Case 2. \(d = p\). Assuming, as we may, that \(l\) is sufficiently large, let \(0 \leq i_0 \leq l - 1\) be such that
\[
\frac{l^d}{(\ln t)^{(d-1)/(d-2)}} \leq 2^{i_0} < 2 \frac{l^d}{(\ln t)^{(d-1)/(d-2)}}
\]
(recalling that \(t > 2\)). We prove that \(p_{l-i} \gg \frac{1}{\ln t}\) for \(0 \leq i \leq i_0\) whence \(D^*(t) \gg \frac{l^d}{(\ln t)^{d-1}}\) by (66). Indeed, by (67) and our assumption that \(d = p\),
\[
1 = p_1 + \ldots + p_l \leq \sum_{i=0}^{i_0} 2^{-\frac{(1/2)}{t^2}} + \sum_{i=0}^{i_0} \left(\frac{\beta}{t^2}\right)^{\frac{1}{d-1}} \ll 2^{-i_0} \frac{(\beta)}{t^2} + i_0 \left(\frac{\beta}{t^2}\right)^{\frac{1}{d-1}}.
\]
Thus \(\beta \gg \frac{l^d}{(\ln t)^{d-1}}\) so that \(p_{l-i} \gg \frac{1}{\ln t}\) for \(0 \leq i \leq i_0\) by (67).

Case 3. \(d > p\). We prove that \(p_l \gg 1\) whence \(D^*(t) \gg t^p\) by (66). Indeed, by (67) and our assumption that \(d > p\),
\[
1 = p_1 + \ldots + p_l \ll \sum_{i=0}^{l-1} 2^{-\frac{d-p}{d(p-1)}} \left(\frac{\beta}{t^p}\right)^{\frac{1}{p-1}} \ll \left(\frac{\beta}{t^p}\right)^{\frac{1}{p-1}}.
\]
Thus \(\beta \gg t^p\) so that \(p_l \gg 1\) by (67) (as \(p > 2\)).

Theorem 1.5 is now proved in Setting 2 (the box \(\Lambda^d_L\)). Setting 1 is handled in the same way.

Remark. Theorem 1.5 discusses potentials of the form \(U(x) = |x|^p + x^2\). It is also natural to consider the family of potentials \(U_p(x) = |x|^p\) where we further assume \(p \geq 1\) to impose convexity. In the case \(p > 2\) we may define \(W_p\) via the recipe (58) (with respect to \(U_p\)) and observe that \(W_p(r) \approx r^p\).

With this observation we may follow the calculation in this section and obtain in dimensions \(d > p\) the tail probability decay \(P(|\varphi(v)| > t) \leq C_p \exp(-c_p t^p)\) (in the setup of Theorem 1.5). A more involved approach is required to handle dimensions \(d \leq p\) or the cases \(1 < p < 2\) and we do not pursue this direction in this paper.

Remark. The above proof of Theorem 1.5 may be extended to settings with non-zero boundary conditions. We demonstrate this on the \(d\)-dimensional box \(\Lambda^d_L\). Fix integers \(d \geq 3\), \(L \geq 2\) and real \(p > 2\). Consider the random surface measure \(\mu_{\Lambda^d_L, V_0, \varphi_0, U}\) where \(U(x) = |x|^p + x^2\), \(V_0\) are the vertices of \(V(\Lambda^d_L)\) which are adjacent in \(\mathbb{Z}^d\) to a vertex outside \(V(\Lambda^d_L)\) (as in Setting 2) and the boundary condition \(\varphi_0 : V_0 \to \mathbb{R}\) is general. Let \(\varphi\) be sampled from \(\mu_{\Lambda^d_L, V_0, \varphi_0, U}\) and let \(v \in V(\Lambda^d_L) \setminus V_0\). We claim that there exists \(C > 0\), depending on \(p\) and \(d\) but not on \(L\) or \(v\), such that for all real \(t > 0\),
\[
P(|\varphi(v) - \mathbb{E}(\varphi(v))| > 2(t + C)) \leq \exp(-2D(t))
\]
where \(D(t)\) is defined in (64). From the lower bounds above on \(D(t)\) we conclude that (68) extends Theorem 1.5 to non-zero boundary conditions, bounding the probability of deviation from \(\mathbb{E}(\varphi(v))\) with a bound having the same form as (18).

Let us prove (68). Let \(\alpha_v\) be the marginal density of \(\varphi(v)\). Lemma 4.1, applied to \(\alpha_v\), yields that for each \(s \in \mathbb{R}\) and \(t > 0\),
\[
\sqrt{\alpha_v(s-t)\alpha_v(s+t)} \leq \exp(-D(t)) \cdot \alpha_v(s).
\]
Denote \(M := \sup_{s \in \mathbb{R}} \alpha_v(s)\) and let \(s_0\) be such that \(\alpha_v(s_0) = M\). Using (69) with \(s = s_0 - t\) gives
\[
\alpha_v(s_0 - 2t)M = \sqrt{\alpha_v(s_0 - 2t)\alpha_v(s_0)} \leq \exp(-D(t)) \cdot \alpha_v(s_0 - t) \leq \exp(-D(t)) \cdot M.
\]
Together with the analogous calculation for \(s = s_0 + t\) we conclude that
\[
\max\{\alpha_v(s_0 - 2t), \alpha_v(s_0 + 2t)\} \leq M \exp(-2D(t))
\]
Thus, by item 3 of Proposition 2.1,
\[
P(|\varphi(v) - s_0| > 2t) \leq \mathbb{P}(\alpha_v(\varphi(v)) \leq M \exp(-2D(t))) \leq \exp(-2D(t)).
\]

(70)
To deduce (68), it remains to note that (70), together with the fast growth of $D(t)$, imply that $|s_0 - \mathbb{E}(\varphi(v))| \ll 1$.

4.2. Proof of Theorem 1.4. In this section we prove Theorem 1.4. We present two proofs: the first based on the quantile Brascamp–Lieb type inequality of Theorem 1.1 and the second based on the quantitative log-concavity result of Theorem 1.2. We will use the definitions of the random surface measure $\mu_{G,V_0,\varphi_0,U}$ and the set $\Omega_{G,V_0,\varphi_0}$, which the reader may recall from (13) and (14).

The next lemma is an adaptation of the quantile Brascamp–Lieb type inequality, Theorem 1.1, to the random surface setting.

**Lemma 4.2.** There exists a universal constant $C > 0$ so that the following holds. Let $G$ be a finite connected graph, $V_0$ a proper subset of $V(G)$, $\varphi_0 : V_0 \to \mathbb{R}$ and $U : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be an even convex function which is not everywhere constant. Let $\eta \in \mathbb{R}^{V(G)}$ satisfy $\eta|_{V(G) \setminus V_0} \neq 0$. Let $\varphi$ be sampled from the random surface measure $\mu_{G,V_0,\varphi_0,U}$. Define, for each $\psi \in \Omega_{G,V_0,\varphi_0}$,

$$D_{\eta,\psi} := \inf_{\chi : V(G) \to \mathbb{R}} \sum_{\chi \equiv 0 \text{ on } V_0} U''(\nabla_e \psi) (\nabla_e \chi)^2.$$  

(71) Then for each $t > 0$,

$$\text{Var}(\langle \eta, \varphi \rangle) \leq \frac{Ct}{\mathbb{P}(D_{\eta,\varphi} \geq \frac{1}{t})^3}.$$  

(72)

In particular,

$$\text{Var}(\langle \eta, \varphi \rangle) \leq 8C \text{Med} \left( \frac{1}{D_{\eta,\varphi}} \right).$$  

(73)

where $\text{Med}(Y)$ is any median of the random variable $Y$, i.e., a number $t$ satisfying $\mathbb{P}(Y \geq t) \geq \frac{1}{2}$ and $\mathbb{P}(Y \leq t) \geq \frac{1}{2}$.

We note that the term $U''(s)$ appearing in (71) is defined for almost every $s \in \mathbb{R}$ by the convexity of $U$ (in the sense of second-order Taylor expansion; see (2)). The terms corresponding to $e \in E(G|_{V_0})$ in the sum (71) can be omitted due to having $\nabla_e \chi = 0$. The other terms are almost surely well defined when $\varphi$ is substituted for $\psi$, so that $D_{\eta,\varphi}$ is well defined.

**Proof.** As in the previous proof, identify $\Omega_{G,V_0,\varphi_0}$ with $\mathbb{R}^{V(G) \setminus V_0}$ in the canonical fashion (restricting the functions to $V(G) \setminus V_0$), and note that the distribution of $\varphi$ has the log-concave density $\exp(-f)$ given by (61). Noting that the statement of the lemma is unaffected by changes to the coordinates of $\eta$ on $V_0$, we also identify $\eta$ with its restriction to $V \setminus V_0$ when needed. With these in mind, the lemma follows from Theorem 1.1, when substituting $\varphi$ for $X$, after noting that

$$D_{\eta,\psi} = \frac{1}{\langle \eta, (\text{Hess } f)(\psi)^{-1}\eta \rangle},$$  

(74)

for the set of $\psi \in \Omega_{G,V_0,\varphi_0}$ for which $D_{\eta,\psi}$ is defined by (71). To see the equality, recall first the variational principle (see (26)),

$$\frac{1}{\langle \eta, (\text{Hess } f)(\psi)^{-1}\eta \rangle} = \inf_{\chi : V(G) \to \mathbb{R}} \langle \chi, \text{Hess}(f)(\psi)\chi \rangle$$  

\begin{equation}
\chi \equiv 0 \text{ on } V_0 \quad (\eta,\chi) = 1
\end{equation}

where, again, $\chi$ is identified with its restriction to $V(G) \setminus V_0$, and the left-hand side is interpreted as 0 when $\eta$ lies in the kernel of $(\text{Hess } f)(\psi)$. Equality (74) then follows by consideration of the formula (61) for $f$. \qed

We also adapt the quantitative log-concavity results, Theorem 1.2 and Lemma 1.3, to the random surface setting.
Lemma 4.3. Let $G$ be a finite connected graph, $V_0$ a proper subset of $V(G)$, $\varphi_0 : V_0 \to \mathbb{R}$ and $U : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be an even convex function which is not everywhere constant. Let $\eta \in \mathbb{R}^{V(G)}$ satisfy $\eta|_{V(G) \setminus V_0} \neq 0$. Let $\varphi$ be sampled from the random surface measure $\mu_{G,V_0,\varphi_0,U}$ and set $\alpha_\eta : \mathbb{R} \to [0,\infty)$ to be the (log-concave) density function of $\langle \eta, \varphi \rangle$. Define, for $\psi \in \Omega_{G,V_0,\varphi_0}$ and $t > 0$,

\[
D_{\eta,\psi}(t) := \inf_{\psi^+,\psi^- \in \Omega_{G,V_0,\varphi_0}} \sum_{e \in E(G)} \frac{1}{2} \left[ U(\nabla_e \psi^+) + U(\nabla_e \psi^-) \right] - U\left( \nabla_e \psi \right). \tag{75}
\]

Define further, for each $D \geq 0$, $t > 0$ and $s \in \mathbb{R}$ satisfying that $\alpha_\eta(s) > 0$

\[
\gamma_\eta(D, s, t) := \mathbb{P}(D_{\eta,\varphi}(t) \geq D \mid \langle \eta, \varphi \rangle = s). \tag{76}
\]

Then the inequality

\[
\sqrt{\alpha_\eta(s-t)\alpha_\eta(s+t)} \leq (1 - \gamma_\eta(D, s, t)(1 - e^{-D})) \cdot \alpha_\eta(s) \tag{77}
\]

holds for every $D, t$ and $s$ as above. In particular, if $\mathbb{P}(D_{\eta,\varphi}(t) \geq D) \geq \frac{3}{4}$ for some $D, t > 0$ then

\[
\mathbb{P}\left( \sqrt{\alpha(\langle \eta, \varphi \rangle + t)\alpha(\langle \eta, \varphi \rangle - t)} \leq \left( 1 - \frac{1}{2} (1 - e^{-D}) \right) \alpha(\langle \eta, \varphi \rangle) \right) \geq \frac{1}{2} \tag{78}
\]

and consequently

\[
\text{Var}(\langle \eta, \varphi \rangle) \leq \left( \frac{Ct}{1 - e^{-D}} \right)^2 \tag{79}
\]

for an absolute constant $C > 0$.

Proof. The proof follows the same lines as the proof of Lemma 4.1. Identify $\Omega_{G,V_0,\varphi_0}$ with $\mathbb{R}^{V(G) \setminus V_0}$ in the canonical fashion (restricting the functions to $V(G) \setminus V_0$). Noting that the distribution of $\varphi$ is absolutely continuous with respect to the Lebesgue measure on $\Omega_{G,V_0,\varphi_0}$, with the log-concave density $\exp(-f)$ given by (61). With these definitions, $D_{\eta,\psi}(t)$ coincides with the formula (5) when substituting $\psi$ for $x$ and replacing the $\eta$ of the lemma by its restriction to $V(G) \setminus V_0$. Similarly, $\gamma_\eta(D, s, t)$ defined in (76) coincides with (6) when substituting $\varphi$ for $X$. The inequalities (77) and (78) thus follow from Theorem 1.2 and the conclusion (79) follows from Lemma 1.3. \hfill \square

4.2.1. The key lemma. We now specialize to the setting of Theorem 1.4 and state the key lemma for its proof.

Lemma 4.4. Suppose that $U : \mathbb{R} \to (-\infty, \infty]$ is such that $U(x) = U(-x)$ for all $x$, and, in addition, the following assumption is satisfied:

$U$ is convex and $U''(x) > 0$ Lebesgue almost-everywhere (a.e.) on $\{x : U(x) < \infty\}$. \tag{80}

Let $d \geq 2$ and $L \geq 2$ be integers and let $v \in V(\mathbb{T}_d^{2L}) \setminus \{0\}$. Define

(1) (Fluctuation growth): For $R > 0$,

\[
\tau_d(R) := \begin{cases} 
\sqrt{\ln(R+1)} & d = 2, \\
1 & d \geq 3. 
\end{cases} \tag{81}
\]

(2) (Effective conductance of a subgraph): For a set of edges $E \subseteq E(\mathbb{T}_d^{2L})$,

\[
D_{E,v} := \inf_{\chi : V(\mathbb{T}_d^{2L}) \to \mathbb{R}} \sum_{e \in E} (\nabla_e \chi)^2. \tag{82}
\]
(3) (Second-order ratio of $U$ at $s$): For $s \in \mathbb{R}$,
\[
\delta_U(s) := \inf_{t \in (0, \infty)} \frac{U(s + t) + U(s - t) - 2U(s)}{\min\{t^2, 1\}}.
\] (83)

(4) (Subgraph of edges with large second-order ratio): For $\psi : V(T_{2L}^d) \to \mathbb{R}$ and $\delta > 0$,
\[
E(\psi, \delta) = \{e \in E(G) : \delta_U(\nabla_e \psi) \geq \delta\}.
\] (84)

Then there exist $\delta_0, c > 0$ depending only on $d$ and $U$ (but not on $L$ and $v$) such that when $\varphi$ is randomly sampled from $\mu_{T_{2L}^d, U}$,
\[
\mathbb{P}\left(D_{E(\varphi, \delta_0), v} \geq \frac{c}{\tau_d(\|v\|_1)^2}\right) \geq \frac{3}{4}.
\] (85)

4.2.2. Deduction of Theorem 1.4 from Lemma 4.4. We explain here how Theorem 1.4 follows from the key lemma using either the quantile Brascamp–Lieb result of Lemma 4.2 or the quantitative log-concavity result of Lemma 4.3. Let $\Omega = \{\psi : V(T_{2L}^d) \to \mathbb{R} : \psi(0) = 0\}$.

Let $\eta_v : V(T_{2L}^d) \to \mathbb{R}$ be defined by $\eta_v(v) = 1$ and $\eta_v(w) = 0$ for $w \in V(T_{2L}^d) \setminus \{v\}$, so that $\langle \eta_v, \psi \rangle = \psi(v)$ for $\psi \in \Omega$. We apply the aforementioned results with $\eta = \eta_v$.

Proof of Theorem 1.4 using the quantile Brascamp–Lieb approach. Observe that if $U''(s)$ is defined at an $s \in \mathbb{R}$ then
\[
U''(s) \geq \delta_U(s).
\] (86)

Thus, for $\psi \in \Omega$, with the definition of $D_{\eta_v, \psi}$ from (71), the definition of $D_{E, v}$ from (82) and the definition of $E(\psi, \delta)$ from (84),
\[
D_{\eta_v, \psi} \geq \delta \cdot D_{E(\psi, \delta), v}.
\] (87)

As the conclusion (85) of the key lemma implies that when $\varphi$ is randomly sampled from $\mu_{T_{2L}^d, U}$ then
\[
\text{Med}\left(\frac{1}{D_{\eta_v, \varphi}}\right) \leq \frac{\tau_d(\|v\|_1)^2}{c}
\]
for at least one median, we conclude from Lemma 4.2 that
\[
\text{Var}(\varphi(v)) \leq C'\tau_d(\|v\|_1)^2
\]
for an absolute constant $C' > 0$, as required. \qed

Proof of Theorem 1.4 using the quantitative log-concavity approach. We aim to give a lower bound for the quantity $D_{\eta_v, \psi}(t)$ of (75) (with $\eta = \eta_v$) in terms of the quantity $D_{E, v}$ of (82) and the set of
The assumption (91) implies that the graph $\Lambda_d$ vertices of the embedded graph, denote holds for every

Now on, let also notation of that lemma, let

Proof of Lemma 4.4.

We rely on Lemma 3.3 to introduce an auxiliary constant

Proof of the key lemma. 4.2.3.

for an absolute constant $C$ result of Lemma 4.3 we conclude that

As this is the sufficient condition for the variance bound (79) stated in the quantitative log-concavity result of Lemma 4.3 we conclude that

for an absolute constant $C' > 0$, as required.

4.2.3. Proof of the key lemma.

Proof of Lemma 4.4. We rely on Lemma 3.3 to introduce an auxiliary constant $p_0 \in (0, 1)$. In the notation of that lemma, let $p_0$ be chosen so close to 1 as to satisfy the inequality $q(p_0) \geq \frac{11}{12}$. From now on, let also $R := \|v\|_\infty$.

Assume that, with some choice of $\delta_0 > 0$ the inequality

holds for every $E' \subseteq E(T^d_{2L})$. Considering the embedding $\Lambda^{d}_{R+1} \hookrightarrow T^d_{2L}$ such that both $0$ and $v$ are vertices of the embedded graph, denote

The assumption (91) implies that the graph $\Lambda^{d}_{R+1,p_0}$ satisfies the condition of Lemma 3.3 with $p = p_0$. Consequently, writing

\begin{align*}
\mathcal{E}_0 &= \{ \text{event (a) of Lemma 3.3 for $\Lambda^{d}_{R+1,p_0}$ with $a = 0$ and $b = v$ occurs} \}, \\
\mathcal{E}_1 &= \{ \text{event (b) of Lemma 3.3 for $\Lambda^{d}_{R+1,p_0}$ with $a = 0$ and $b = v$ occurs} \}, \\
\mathcal{E}_2 &= \{ \text{event (b) of Lemma 3.3 for $\Lambda^{d}_{R+1,p_0}$ with $a = v$ and $b = 0$ occurs} \},
\end{align*}

edges $E(\psi, \delta)$ of (84). To this end, we write for $\psi \in \Omega$ and $t, \delta > 0$,

$$
D_{\eta,\psi}(t) \equiv \inf_{\psi^+, \psi^- \in \Omega \atop \psi^+ - \psi^- = 2\psi} \sum_{e \in E(G)} \frac{1}{2} \left[ U(\nabla_e \psi^+) + U(\nabla_e \psi^-) \right] - U(\nabla_e \psi)
$$

\begin{align}
&\geq \inf_{\psi^+, \psi^- \in \Omega \atop \psi^+ - \psi^- = 2\psi} \sum_{e \in E(\psi, \delta)} \frac{1}{2} \left[ U(\nabla_e \psi^+) + U(\nabla_e \psi^-) \right] - U(\nabla_e \psi) \\
&\geq \inf_{\chi \in \Omega \atop \chi(v) = 1} \sum_{e \in E(\psi, \delta)} \frac{1}{2} \left[ U(\nabla_e \psi + t\nabla_e \chi) + U(\nabla_e \psi - t\nabla_e \chi) \right] - U(\nabla_e \psi) \\
&\geq \inf_{\chi \in \Omega \atop \chi(v) = 1} \frac{\delta}{2} \min \left\{ t^2 \sum_{e \in E(\psi, \delta)} (\nabla_e \chi)^2, 1 \right\} \\
&\geq \inf_{\chi \in \Omega \atop \chi(v) = 1} \frac{\delta}{2} \min \left\{ t^2 \sum_{e \in E(\psi, \delta)} (\nabla_e \chi)^2, 1 \right\} \\
&\geq \inf_{\chi \in \Omega \atop \chi(v) = 1} \frac{\delta}{2} \min \left\{ t^2 D_{E(\psi, \delta), v}, 1 \right\}
\end{align}

where the inequality (b) uses the fact that $\frac{1}{2}(U(s + t) + U(s - t)) - U(s) \geq 0$ for all $s, t \in \mathbb{R}$ by the convexity of $U$, the equality (c) follows by defining $\chi = \frac{\psi^+ - \psi^-}{2t}$ and the inequality (d) holds by the definition of $E(\psi, \delta)$ (see (84)). Let $\delta_0, c > 0$ be as in the key lemma, Lemma 4.4. Thus, the conclusion (85) of the key lemma implies that when $\varphi$ is randomly sampled from $\mu_{T^d_{2L}, U}$ then

$$
\mathbb{P} \left( D_{\eta,\varphi}(\tau_0(\|v\|_1)) \geq \frac{\delta}{2} \min\{c, 1\} \right) \geq \frac{3}{4}. \tag{89}
$$

As this is the sufficient condition for the variance bound (79) stated in the quantitative log-concavity result of Lemma 4.3 we conclude that

$$
\text{Var}(\varphi(v)) \leq C' \tau_0(\|v\|_1)^2 \tag{90}
$$

for an absolute constant $C' > 0$, as required. □

4.2.3. Proof of the key lemma.

Proof of Lemma 4.4. We rely on Lemma 3.3 to introduce an auxiliary constant $p_0 \in (0, 1)$. In the notation of that lemma, let $p_0$ be chosen so close to 1 as to satisfy the inequality $q(p_0) \geq \frac{11}{12}$. From now on, let also $R := \|v\|_\infty$.

Assume that, with some choice of $\delta_0 > 0$ the inequality

$$
\mathbb{P} \left( E(\varphi, \delta_0) \cap E' = \emptyset \right) \leq (1 - p_0)^{|E'|} \tag{91}
$$

holds for every $E' \subseteq E(T^d_{2L})$. Considering the embedding $\Lambda^{d}_{R+1} \hookrightarrow T^d_{2L}$ such that both $0$ and $v$ are vertices of the embedded graph, denote

$$
\Lambda^{d}_{R+1,p_0} := \left( V(\Lambda^{d}_{R+1}), E(\Lambda^{d}_{R+1}) \cap E(\varphi, \delta_0) \right).
$$

The assumption (91) implies that the graph $\Lambda^{d}_{R+1,p_0}$ satisfies the condition of Lemma 3.3 with $p = p_0$. Consequently, writing

$$
\mathcal{E}_0 := \{ \text{event (a) of Lemma 3.3 for $\Lambda^{d}_{R+1,p_0}$ with $a = 0$ and $b = v$ occurs} \}, \\
\mathcal{E}_1 := \{ \text{event (b) of Lemma 3.3 for $\Lambda^{d}_{R+1,p_0}$ with $a = 0$ and $b = v$ occurs} \}, \\
\mathcal{E}_2 := \{ \text{event (b) of Lemma 3.3 for $\Lambda^{d}_{R+1,p_0}$ with $a = v$ and $b = 0$ occurs} \},
$$

edges $E(\psi, \delta)$ of (84). To this end, we write for $\psi \in \Omega$ and $t, \delta > 0$,
we get that $\mathbb{P}(\mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2) \geq \frac{3}{4}$.

As in Lemma 3.3, denote by $G_v$ the connected component of the vertex $u$ in $\Lambda^d_{R+1,p_0}$. By definition of the event $\mathcal{E}_0$ it follows that $G_0 = G_v$ whenever this event occurs. In addition, by definition of the events $\mathcal{E}_1$ and $\mathcal{E}_2$, it holds that, whenever the event $\mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2$ occurs, the graph $G_0 = G_v$ satisfies the condition of Corollary 3.8. As a result, whenever the event $\mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2$ occurs, we have

$$D_{E(\varphi,\delta_0),v} \overset{(a)}{=} \inf_{\chi:V(\mathcal{T}_d^2) \to \mathbb{R}} \sum_{e \in E(\varphi,\delta_0)} (\nabla e \chi)^2 \overset{(b)}{=} \inf_{\chi:V(\mathcal{T}_d^2) \to \mathbb{R}} \sum_{e \in E(G_0)} (\nabla e \chi)^2 \geq \frac{c}{\tau_d(\|\|v\|\|^2),$$

where the inequality (c) follows from Corollary 3.8. Hence

$$\mathbb{P}(D_{E(\varphi,\delta_0),v} \geq \frac{c}{\tau_d(\|\|v\|\|^2)}) \geq \mathbb{P}(\mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2) \geq \frac{3}{4},$$

and (85) indeed follows from (91).

Therefore, in order to finish the proof, it is sufficient to choose $\delta_0 > 0$ so that (91) holds. The remaining part of the argument pursues that goal.

Denote

$$S_U(\delta) := \{s \in \mathbb{R} : U(s) < \infty, \delta U(s) < \delta\}.$$ 

The set $S_U(\delta)$ is 0-symmetric; for convenience, we also denote $S_U^+(\delta) := S_U(\delta) \cap [0, \infty)$. It is also straightforward that $\delta U(s) > 0$ if $U''(s) > 0$, which implies that the identity

$$\lim_{\delta \downarrow 0} 1\{s \in S_U(\delta)\} = 0$$

holds Lebesgue-a.e. on the set $\{s : U(s) < \infty\}$. By the dominated convergence theorem, we get

$$\lim_{\delta \downarrow 0} \int_{S_U^+(\delta)} e^{-U(t)} dt = \lim_{\delta \downarrow 0} \int_0^\infty e^{-U(t)} 1\{t \in S_U^+(\delta)\} dt = 0.$$ 

Consequently, with $p(S,d,U)$ defined as in Lemma 3.9, we have

$$\lim_{\delta \downarrow 0} p(S_U^+(\delta),d,U) = \lim_{\delta \downarrow 0} \left( C(d,U) \cdot \int_{S_U^+(\delta)} e^{-U(x)} dx \right) c(d) = 0.$$ 

Thus there exists $\delta_0 > 0$ depending only on $d$ and $U$ such that $p(S_U^+(\delta_0),d,U) < 1 - p_0$. For this choice of $\delta_0$ the property (91) follows directly from Lemma 3.9, which concludes the proof.

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