MCKEAN–VLASOV OPTIMAL CONTROL: THE DYNAMIC PROGRAMMING PRINCIPLE*

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We study the McKean–Vlasov optimal control problem with common noise where allow the law of the control process to appear in the state dynamics, under various formulations: strong and weak ones, and Markovian or non-Markovian. By interpreting the controls as probability measures on an appropriate canonical space with two filtrations, we then develop the classical measurable selection, conditioning and concatenation arguments in this new context, and establish the dynamic programming principle under general conditions.

1. Introduction. We propose in this paper to study the problem of optimal control of mean-field stochastic differential equations, also called McKean–Vlasov stochastic differential equations in the literature. This problem is a stochastic control problem where the state process is governed by a stochastic differential equation (SDE for short), which has coefficients depending on the current time, the paths of the state process, but also its distribution (or conditional distribution in the case with common noise). Similarly, the reward functionals are allowed to be impacted by the distribution of the state process.

The pioneering works on McKean–Vlasov equations are due to McKean Jr. [36] and Kac [29], who were interested in studying uncontrolled SDEs, and in establishing general propagation of chaos results. Let us also mention the illuminating notes of Snitzman [45], which give a precise and pedagogical insight into this specific field. Though many authors have worked on this equation following the aforementioned papers, there has been a drastic surge of interest in the topic in the past decade, due to the connection that it shares with the so-called mean-field game (MFG for short) theory, introduced independently and simultaneously on the one hand by Lasry and Lions in [33] and on the other hand by Huang, Caines and Malhamé [28]. Indeed, the McKean–Vlasov equation naturally appears when one tries to describe the behaviour of many symmetric agents, which interact through the empirical distribution of their states, and who seek a Nash equilibrium (this is the competitive equilibrium case, leading to the MFG theory), or a Pareto equilibrium (this is the cooperative equilibrium case, associated to the optimal

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control of McKean–Vlasov stochastic equations). Though related, these two problems have some subtle differences which were investigated thoroughly by Carmona, Delarue and Lachapelle [14]. We also refer to Bensoussan, Frehse and Yam [3], Carmona and Delarue [17] for a general presentation of the recent development of the two problems.

Our goal in this paper, as already mentioned, is in the optimal control of McKean–Vlasov equations, and more precisely in the rigorous establishment of the dynamic programming principle (DPP for short), under conditions as general as possible. In a nutshell, the idea behind this principle is that the global optimisation problem can be solved by a recursive resolution of local optimisation problems. This fact is an intuitive result, which is often used as some sort of meta–theorem. Nevertheless, it may not be so easy to prove rigorously, especially in the context of the McKean–Vlasov optimal control problem. One of the main reasons is actually a very bleak one for us: due to the non-linear dependency with respect to the law of the process, the problem is actually a time inconsistent one (like the classical mean–variance optimisation problem in finance, see the recent papers by Björk and Murgoci [8], Björk, Khapko and Murgoci [7], and Hernández and Possamaï [27] for a more thorough discussion of this topic), and Bellman’s optimality principle does not hold. However, though the problem itself is time-inconsistent, one can recover some form of the DPP by extending the state space. This was first achieved by Lauriére and Pironneau [34], and later by Bensoussan, Frehse and Yam [3, 4, 5], who assumed the existence at all times of a density for the marginal distribution of the state process, and reformulated the problem as a density control problem, with a family of deterministic controls. Under this reformulation, they managed to prove a DPP and deduce a dynamic programming equation in the space of density functions. In a more general (non common noise) context without the density assumption, the DPP has been investigated by Pham and Wei [42] for closed-loop (or feedback) controls. For open-loop controls, DPP results can be in fact be deduced from the general results in Cosso and Pham [19] for McKean–Vlasov differential games, and from the so-called randomised DPP in Bayraktar, Cosso and Pham [2]. In the case with common noise, Pham and Wei [41] proved a DPP for the particular case where the control process is adapted to the common noise filtration. Let us also mention Bouchard, Djehiche and Kharroubi [9], who studied a stochastic McKean–Vlasov target problem, where one aims at optimally controlling a McKean–Vlasov equation so that the controlled process satisfies some target marginal constraints. They then established a general geometric dynamic programming, extending the seminal results of Soner and Touzi [46].

In this paper, our strategy is somewhat different, as we wish to establish the DPP using measurable selection techniques. For classical stochastic optimal control problems, this approach allows to obtain the desired result under very general conditions, avoiding for instance the Markovian property or regularity assumptions considered in the above literature. As a rule of thumb, it consists in two essential ingredients: first ensuring the stability of the controls with respect to conditioning and concatenation, and second the measurability of the associated value function. The use of measurable selection arguments makes it possible to provide an adequate framework for verifying these properties. This strategy was followed by Dellacherie [22], Bertsekas and Shreve [6] for discrete-time stochastic control problems. Later, El Karoui, Huu Nguyen and Jeanblanc-Picqué in [24] presented a framework for stochastic control problem in continuous-time (accommodating general Markovian processes). Thanks to the notion of relaxed control, that is to say the interpretation of a control as a probability measure on some canonical space, and using the notion of martingale problems, they proved
a DPP by simple and clear arguments. El Karoui and Tan [25, 26] extended this approach to the non-Markovian case. Similar results were obtained by several authors, among which we mention Nutz and Soner [39], Neufeld and Nutz [37, 38], Nutz and van Handel [40], Possamai, Royer and Touzi [43], Žitković [49], and Possamai, Tan and Zhou [44]. Following the framework in [24, 25], we develop in this paper a general analysis for the non-Markovian optimal control of McKean–Vlasov equations with common noise. In particular, we investigate the case where the drift and diffusion coefficients, as well as the reward functions, are allowed to depend on the joint conditional distribution of the path of the state process and of the control.

We shall consider both strong and weak formulations of the McKean–Vlasov stochastic control problem. Most of the literature falls in fact into the strong one. In this formulation, the (open-loop) control $\alpha$ is adapted to the filtration generated by both the Brownian motions $(W, B)$ (with $B$ being the common noise) and a random variable $\xi$ serving as initial condition. Notice that such a formulation is law invariant in the sense that it depends on the initial condition $\xi$ only through its distribution $\mathcal{L}(\xi)$. Furthermore, the ‘common noise’ filtration is generated by $B$ and the conditional distributions considered are associated to this common noise filtration, that is, $\mathcal{L}(X_{t\wedge\cdot}, \alpha_t|B)$, where $X$ is the state. Such a strong formulation does not enjoy good stability properties in general. To see this, it is enough to realise that the conditional distribution is not continuous with respect to the joint distribution: for instance the function $\mathcal{L}(X_t, B) \mapsto \mathbb{E}[(\mathbb{E}[X_t|B])^2]$ is not continuous. Consequently, the limit (in the sense of weak convergence) of strong solutions may not be itself a strong solution to the controlled McKean–Vlasov equation. For this reason, and also inspired by the notion of weak solution to classical SDEs as well as the work of Carmona, Delarue and Lacker [16], we introduce a notion of weak solution to the controlled McKean–Vlasov SDEs. The idea is to consider a more general filtration $\mathcal{F}$ describing the adaptability of the controls, and an extended common noise filtration $\mathcal{G}$. Additional conditions on $\mathcal{F}$ and $\mathcal{G}$ are needed to ensure that the weak solution remains first compatible with the strong solutions, second enjoys the good stability properties we already mentioned, and third that weak controls can be approximated ‘sufficiently well’ by strong controls.

By interpreting controls as probability measures on an appropriate canonical space, and using measurable selection arguments as in [25, 26], we then move on to prove the universal measurability of the associated value function, and derive the stability of controls with respect to conditioning and concatenation, and finally deduce the DPP for the weak formulation under only Borel-measurable conditions on the coefficient and reward functions. Our next result addresses the DPP for the classical strong formulation, where standard Lipschitz conditions are assumed on the drift and diffusion coefficients, so that a unique solution to the controlled McKean–Vlasov equation exists given an admissible control process. When the control processes are restricted to be adapted to the ‘common noise’ filtration (as in [41]), we obtain the DPP under very general conditions (in particular in a non-Markovian setting and without any regularity assumptions on the reward functions). In addition, for the general strong formulation, where the control processes are adapted to both $\xi, W$ and $B$, we obtain the DPP under some additional regularity conditions. These regularity conditions may seem unexpected at first sight, but they seem unavoidable due to the non-linear dependency of the drift and volatility coefficients with respect to the conditional distribution of $X$ (see Remark 10 for a more thorough discussion). Finally, the DPP results in the general non–Markovian context induce similar ones in the Markovian context.

We finally stress that our results and techniques are not direct extensions of those in the existing literature, although the very basic ideas may seem so. As we already
explained, in order to interpret a control as a probability measure on the canonical space, one needs to formulate an adequate notion of weak solutions which should enjoy more stability properties than the strong solutions, and at the same time be able to approximate these solutions by strong ones. Because of the coexistence of two filtrations (the general and the common noise filtrations), and the implicit compatibility conditions that should link them, it becomes much more delicate to develop the classical measurable selection, conditioning and concatenation arguments.

The rest of the paper is organised as follows. After recalling briefly some notations and introducing the probabilistic structure to give an adequate and precise definition of the tools that are used throughout the paper, we introduce in Section 2 several notions of weak and strong formulations for the McKean–Vlasov stochastic control problem with common noise in a non-Markovian framework, and prove some equivalence results. Next, in Section 3, we present the main result of this paper, the DPP for three formulations: weak formulation, strong formulation, and a \(\mathbb{B}\)-strong formulation where the control is adapted with respect to the ‘common noise’ filtration. We first provide all our results in the non-Markovian setting, and then specialise to the Markovian framework. Finally, Section 4 is devoted to the proof of our main results.

**Notations.** (i) Given a metric space \((E, \Delta)\) and \(p \geq 0\), we denote by \(\mathcal{P}(E)\) the collection of all Borel probability measures on \(E\), and by \(\mathcal{P}_p(E)\) the subset of Borel probability measures \(\mu\) such that \(\int_E \Delta(e, c_0)^p \mu(de) < \infty\) for some \(c_0 \in E\). When \(p \geq 1\), the space \(\mathcal{P}_p(E)\) is equipped with the Wasserstein distance \(W_p\) defined by

\[
W_p(\mu, \mu') := \left(\inf_{\lambda \in \Lambda(\mu, \mu')} \int_{E \times E} \Delta(e, e')^p \lambda(de, de')\right)^{1/p},
\]

where \(\Lambda(\mu, \mu')\) is the collection of all Borel probability measures \(\lambda\) on \(E \times E\) such that \(\lambda(de, E) = \mu(de)\) and \(\lambda(E, de') = \mu'(de')\). When \(E\) is a Polish space, \((\mathcal{P}_p(E), W_p)\) is a Polish space (see Villani [48, Theorem 6.16]). Given another metric space \((E', \Delta')\), we denote by \(\mu \otimes \mu' \in \mathcal{P}(E \times E')\) the product probability of any \((\mu, \mu') \in \mathcal{P}(E) \times \mathcal{P}(E')\).

(ii) Given a measurable space \((\Omega, \mathcal{F})\), we denote by \(\mathcal{P}(\Omega)\) the collection of all probability measures on \((\Omega, \mathcal{F})\). For any probability measure \(\mathbb{P} \in \mathcal{P}(\Omega)\), we denote by \(\mathcal{F}^\mathbb{P}\) the \(\mathbb{P}\)-completion of the \(\sigma\)-field \(\mathcal{F}\), and by \(\mathcal{F}^{\text{V}} := \bigcap_{\mathbb{P} \in \mathcal{P}(\Omega)} \mathcal{F}^\mathbb{P}\) the universal completion of \(\mathcal{F}\). Let \(\xi : \Omega \to \mathbb{R} \cup \{-\infty, +\infty\}\) be a random variable and \(\mathbb{P} \in \mathcal{P}(\Omega)\), we define, with the convention \(-\infty = -\infty\)

\[
\mathbb{E}^\mathbb{P}[\xi] := \mathbb{E}^\mathbb{P}[\xi_+] - \mathbb{E}^\mathbb{P}[\xi_-],\]

where \(\xi_+ := \xi \vee 0, \xi_- := (-\xi) \vee 0\).

We also use the following notation to denote the expectation of \(\xi\) under \(\mathbb{P}\) by \(\mathbb{E}^\mathbb{P}[\xi] = \langle \mathbb{P}, \xi \rangle = \langle \xi, \mathbb{P} \rangle\). When \(\Omega\) is a Polish space, a subset \(A \subseteq \Omega\) is called an analytic set if there is another Polish space \(E\), and a Borel subset \(B \subseteq \Omega \times E\) such that \(A = \{\omega \in \Omega : \exists e \in E, (\omega, e) \in B\}\). A function \(f : \Omega \to \mathbb{R} \cup \{-\infty, +\infty\}\) is called upper semi-analytic (u.s.a. for short) if \(\{\omega \in \Omega : f(\omega) > c\}\) is analytic for every \(c \in \mathbb{R}\). Any upper semi-analytic function is universally measurable (see e.g. [6, Chapter 7]).

(iii) Let \(\Omega\) be a metric space, \(\mathcal{F}\) its Borel \(\sigma\)-field and \(\mathcal{G} \subset \mathcal{F}\) be a countably generated sub-\(\sigma\)-field. Following [47], we say that \((\mathbb{P}^\mathcal{G}_\omega)_{\omega \in \Omega}\) is a family of r.c.p.d. (regular conditional probability distributions) of \(\mathbb{P}\) knowing \(\mathcal{G}\) if it satisfies

- the map \(\omega \mapsto \mathbb{P}^\mathcal{G}_\omega\) is \(\mathcal{G}\)-measurable, and for all \(A \in \mathcal{F}\) and \(B \in \mathcal{G}\), one has \(\mathbb{P}[A \cap B] = \int_B \mathbb{P}^\mathcal{G}_\omega[A] d\mathbb{P}(d\omega)\);
- \(\mathbb{P}^\mathcal{G}_\omega([\omega]_\mathcal{G}) = 1\) for all \(\omega \in \Omega\), where \([\omega]_\mathcal{G} := \{A \in \mathcal{F} : (A, \omega) \in \mathcal{G} \times A\}\).
Let $T > 0$ be given, and let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G} = (\mathcal{G}_t)_{t \in [0, T]})$ be a filtered probability space, $\mathbb{G} \subset \mathcal{F}$ be a sub-$\sigma$-field of $\mathcal{F}$, and $E$ a metric space. Then given a random element $\xi : \Omega \to E$, we use both the notations $\mathcal{L}_E^\xi(\omega)$ and $\mathbb{P}_\omega^\xi$ to denote the conditional distribution of $\xi$ knowing $\mathcal{G}$ under $\mathbb{P}$. Moreover, given a measurable process $X : [0, T] \times \Omega \to E$, we can always define for any $t \in [0, T]$, $\mu_t := \mathcal{L}_E^X(\mathcal{G}_t)$ to be a $\mathcal{P}(E)$-valued $\mathbb{G}$-optional process (see Lemma A.1).

(iv) Let $(E, \Delta)$ and $(E', \Delta')$ be two Polish spaces. We shall refer to $C_b(E, E')$ to designate the set of continuous functions $f : E \to E'$ such that $\sup_{e \in E} \Delta'(f(e), e'_0) < \infty$ for some $e'_0 \in E'$. Let $\mathbb{N}$ denote the set of non-negative integers. For $(k, q) \in \mathbb{N}^2$, we denote by $C^n_{b,k}(\mathbb{R}^k; \mathbb{R})$ the set of bounded continuous maps $f : \mathbb{R}^k \to \mathbb{R}$ possessing bounded continuous derivatives up to order $n$, and by $\partial_i f$ (resp $\partial^2_{ij}$) the partial derivative (resp. cross second partial derivative) with respect to $x^i$ (resp $(x^i, x^j)$) for $(i, j) \in \{1, \ldots, k\} \times \{1, \ldots, k\}$, and $x := (x^1, \ldots, x^k) \in \mathbb{R}^k$.

Given $(k, q) \in \mathbb{N} \times \mathbb{N}$, we denote by $S^{k \times q}$ the collection of all $k \times q$-dimensional matrices with real entries, equipped with the standard Euclidean norm that we will denote $| \cdot |$ when there is no ambiguity. Let us denote by $0_k$, $0_{k \times q}$ and $I_k$ the null matrix of dimension $k \times k$, the null matrix of dimension $k \times q$, and the identity matrix of dimension $k \times k$. For $T > 0$, and $(E, \Delta)$ a Polish space as above, we denote by $C([0, T], E)$ the space of all continuous functions on $[0, T]$ taking values in $E$, which is also a Polish space under the uniform convergence topology. When $E = \mathbb{R}^n$ for some $n \in \mathbb{N}$, and $\Delta$ is the usual Euclidean norm on $\mathbb{R}^n$, we simply write $C^n := C([0, T], \mathbb{R}^n)$ and denote by $\| \cdot \|$ the uniform norm. We also denote by $C^n_{b,k} := C([s, t]; \mathbb{R}^n)$ the space of all $\mathbb{R}^n$-valued continuous functions on $[s, t]$, for $0 \leq s \leq t \leq T$. When $n = 0$, the spaces $C^n$ and $C^n_{b,k}$ both degenerate to a singleton.

(v) Throughout the paper, we fix a constant $p \geq 0$, a nonempty Polish space $(U, \rho)$ and a point $u_0 \in U$. Notice that a Polish space is always isomorphic to a Borel subset of $[0, 1]$. Let us thus denote by $\pi$ such one (isomorphic) bijection between $U$ and $\pi(U) \subseteq [0, 1]$. We further extend the definition of $\pi^{-1}$ to $\mathbb{R} \cup \{-\infty, +\infty\}$ by setting $\pi^{-1}(x) := \bar{\partial}$ for all $x \notin \pi(U)$ and let $\bar{U} := U \cup \{\partial\}$, where $\partial$ is the usual cemetery point. Let $\nu \in \mathcal{P}(C^k)$ (resp $\bar{\nu} \in \mathcal{P}(C^k \times \bar{U})$) be a Borel probability measure on the canonical space $C^k$ (resp. $C^k \times \bar{U}$) equipped with the canonical process $X$ (resp. $(X, \alpha)$). We denote for each $t \in [0, T]$

$$\nu(t) := \nu \circ X_{t}^{-1} \quad \text{(resp. } \bar{\nu}(t) := \bar{\nu} \circ (X_{t}, \alpha)^{-1}).$$

2. Weak and strong formulations of the McKean–Vlasov control problem.

The main objective of this paper is to study the following (non-Markovian) McKean–Vlasov control problem, in both strong and weak formulations, of the form

$$\sup_{\alpha} \mathbb{E} \left[ \int_0^T L(t, X^\alpha_{t}, \mathcal{L}(X^\alpha_{t}, \alpha_t|\mathcal{G}_t), \alpha_t) dt + g(X^\alpha_{T}, \mathcal{L}(X^\alpha_{T}, \mathcal{G}_T)) \right];$$

where $\mathcal{G} := (\mathcal{G}_t)_{0 \leq t \leq T}$ is a filtration modelling the common noise $B$, $\mathcal{L}(X^\alpha_{t}, \alpha_t|\mathcal{G}_t)$ denotes the joint conditional distribution of $(X^\alpha_{t}, \alpha_t)$ knowing $\mathcal{G}_t$, and $(X^\alpha_{t})_{0 \leq t \leq T}$ is a McKean–Vlasov type process, controlled by $\alpha = (\alpha_t)_{0 \leq t \leq T}$ and driven by $W$ together with $B$

$$dX^\alpha_{t} = b(t, X^\alpha_{t}, \mathcal{L}((X^\alpha_{t}, \alpha_t)|\mathcal{G}_t), \alpha_t) dt + \sigma(t, X^\alpha_{t}, \mathcal{L}((X^\alpha_{t}, \alpha_t)|\mathcal{G}_t), \alpha_t) dW_t$$

(1)
We will provide in the following a precise definition to the above controlled SDE, depending on the strong/weak formulation considered. Let us first specify the dimensions and some basic conditions on the coefficient functions. Let \((n, \ell, d) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\). The coefficient functions
\[
(b, \sigma, \sigma_0) : [0, T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times U) \times U \rightarrow \mathbb{R}^n \times \mathbb{S}^{n \times d} \times \mathbb{S}^{n \times \ell},
\]
\[
L : [0, T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times U) \times U \rightarrow \mathbb{R},
\]
g : \(\mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n) \rightarrow \mathbb{R}\),
are all assumed to be Borel-measurable, and non-anticipative in the sense that: for all \((t, x, \nu, u) \in [0, T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times U) \times U\)
\[
(b, \sigma, \sigma_0, L)(t, x, \nu, u) = (b, \sigma, \sigma_0, L)(t, x_{t \wedge}, \nu(t), u).
\]
Moreover, for all the above functions \(\varphi\) defined on \(E \times U\) for some space \(E\), we extend its definition on \(E \times U\) by setting \(\varphi(\cdot, \partial) \equiv 0\).

### 2.1. Weak formulation

A weak formulation of the control problem is obtained by considering all weak solutions of the controlled McKean–Vlasov SDE (1). Here the word 'weak' refers to the fact that the probability space, as well as the equipped Brownian motion, is not assumed to be fixed, but is a part of the solution itself. This is of course consistent with the notion of weak solution in the classical SDE theory.

**Definition 2.1.** Let \((t, \nu) \in [0, T] \times \mathcal{P}(\mathcal{C}^n)\). We say that a term
\[
\gamma := (\Omega^\gamma, \mathcal{F}^\gamma, \mathbb{P}^\gamma, \mathbb{G}^\gamma = (\mathcal{G}^\gamma_s)_{0 \leq s \leq T}, X^\gamma, W^\gamma, \mathcal{P}^\gamma, \mu^\gamma, \alpha^\gamma),
\]
is a weak control associated with the initial condition \((t, \nu)\) if the following conditions are satisfied

(i) \((\Omega^\gamma, \mathcal{F}^\gamma, \mathbb{P}^\gamma)\) is a probability space, equipped with two filtrations \(\mathbb{F}^\gamma\) and \(\mathbb{G}^\gamma\) such that, for all \(s \in [0, T]\)

(2) \(\mathcal{G}^\gamma_s \subseteq \mathcal{F}^\gamma_s\), and \(\mathbb{E}^{\mathbb{P}^\gamma}[1_D | \mathcal{G}^\gamma_s] = \mathbb{E}^{\mathbb{F}^\gamma}[1_D | \mathcal{G}^\gamma_s]\), \(\mathbb{P}^\gamma\)-a.s., for all \(D \in \mathcal{F}^\gamma_s \lor \sigma(W^\gamma)\);

(ii) \(X^\gamma = (X^\gamma_s)_{s \in [0, T]}\) is an \(\mathbb{R}^n\)-valued, \(\mathbb{F}^\gamma\)-adapted continuous process, \(\alpha^\gamma := (\alpha^\gamma_s)_{0 \leq s \leq T}\) is an \(\mathbb{F}^\gamma\)-valued, \(\mathbb{F}^\gamma\)-predictable process, and for the fixed constant \(p \geq 0\), one has

\[
\mathbb{E}^{\mathbb{P}^\gamma}[\|X^\gamma\|^p] + \mathbb{E}^{\mathbb{P}^\gamma}\left[ \int_t^T (\rho(\alpha^\gamma_s, u_0))^p \, ds \right] < \infty;
\]

(iii) \((W^\gamma, B^\gamma)\) is an \(\mathbb{R}^d \times \mathbb{R}^\ell\)-valued, \(\mathbb{F}^\gamma\)-adapted continuous process; \((W^\gamma_t, B^\gamma_t) := ((W^\gamma_{s,t})_{0 \leq s \leq T}, (B^\gamma_{s,t})_{0 \leq s \leq T})\), defined by \(W^\gamma_{s,t} := W^\gamma_{s,t} - W^\gamma_s\), \(B^\gamma_{s,t} := B^\gamma_{s,t} - B^\gamma_s\), \(s \in [t, T]\), is a standard \((\mathbb{F}^\gamma, \mathbb{P}^\gamma)\)-Brownian motion on \([t, T]\); \(B^\gamma_t\) is \(\mathbb{G}^\gamma\)-adapted; \(\mathcal{F}^\gamma_t \lor \sigma(W^\gamma)\) is \(\mathbb{P}^\gamma\)-independent of \(\mathcal{G}^\gamma_t\), and \(\mu^\gamma = (\mu^\gamma_s)_{t \leq s \leq T}\) (resp. \(\mathcal{P}^\gamma = (\mathcal{P}^\gamma_s)_{t \leq s \leq T}\) is a \(\mathbb{G}^\gamma\)-predictable, \(\mathcal{P}(\mathcal{C}^n)\)-valued (resp. \(\mathcal{P}(\mathcal{C}^n \times U)\)-valued) process satisfying: for \(d\mathbb{P}^\gamma \otimes ds\)-a.e. \((s, \omega) \in [t, T] \times \Omega^\gamma\),

\[
\mu^\gamma_s = \mathcal{L}^{\mathbb{P}^\gamma}((X^\gamma_{s,t}, | \mathcal{G}^\gamma_s) (\text{resp. } \mathcal{P}^\gamma_s = \mathcal{L}^{\mathbb{P}^\gamma}((X^\gamma_{s,t}, | \mathcal{G}^\gamma_s) ));
\]

(iv) \(X^\gamma\) satisfies \(\mathbb{P}^\gamma \circ (X^\gamma_{t \wedge}, \cdot)^{-1} = \nu(t)\) and \(\mathbb{P}^\gamma\)-a.s., for all \(s \in [t, T]\),

\[
X^\gamma_s = X^\gamma_t + \int_t^s b(r, X^\gamma_r, \mu^\gamma_r, \alpha^\gamma_r) \, dr + \int_t^s \sigma(r, X^\gamma_r, \mu^\gamma_r, \alpha^\gamma_r) \, dW^\gamma_r + \int_t^s \sigma_0(r, X^\gamma_r, \mu^\gamma_r, \alpha^\gamma_r) \, dB^\gamma_r,
\]
where the integrals are implicitly assumed to be well-defined.
For all \((t, \nu) \in [0, T] \times \mathcal{P}(\mathcal{C}^n)\), let us denote
\[
\Gamma_W(t, \nu) := \{ \text{All weak controls with initial condition } (t, \nu) \}.
\]
Then, using the reward functions \(L : [0, T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times U) \times U \to \mathbb{R}\) and \(g : \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n) \to \mathbb{R}\), we introduce the value function of our McKean–Vlasov optimal control problem by
\[
V_W(t, \nu) := \sup_{\gamma \in \Gamma_W(t, \nu)} J(t, \gamma), \text{ with } J(t, \gamma) := \mathbb{E}^{\mathbb{P}^\gamma}\left[ \int_t^T L(s, X^\gamma, \mu_s^\gamma, \alpha_s^\gamma)ds + g(X^{\gamma}, \mu_T^\gamma) \right].
\]

**Remark 1.** In a weak control \(\gamma\), the filtration \(\mathbb{G}^\gamma\) is used to model the common noise. In particular, \(B^{\gamma, t}\) is adapted to \(\mathbb{G}^\gamma\), and \(W^{\gamma, t}\) is independent of \(\mathbb{G}^\gamma\). In the classical strong formulation, \(\mathbb{G}^\gamma\) is usually taken to be the filtration \(\mathbb{F}^{B^{\gamma, t}}\) generated by \(B^{\gamma, t}\), but for a general weak control, \(\mathbb{G}^\gamma\) may be larger than \(\mathbb{F}^{B^{\gamma, t}}\). This will be the main difference between the strong and weak formulations in our approach.

In addition, \((\mathbb{G}^\gamma, \mathbb{F}^\gamma)\) satisfies a sort of \((H)\)-hypothesis type condition given by (2), which is consistent with the classical strong formulation (see Section 2.2 below). This property will be crucial in our proof of the DPP result for the strong formulation of the control problem, as well as in the limit theory of the McKean–Vlasov control problem in our accompanying paper [23].

As in [24], one can also reformulate equivalently the weak control problem on an appropriate canonical space by considering the corresponding martingale problem, see Section 4.1.

**Remark 2.** (i) At this stage, the integrability condition (3) could be construed as artificial. Given more concrete properties of the coefficient functions \((b, \sigma, \sigma_0)\), it would serve as an admissibility criterion for the control processes, and ensure that the stochastic integrals in (4) are well-defined. As an example, consider the case where \(U = \mathbb{R}\) and \(u_0 = 0\). When \(b\), \(\sigma\), and \(\sigma_0\) are all uniformly bounded, one can choose \(p = 0\) so that all \(\mathbb{R}\)-valued predictable processes would then be admissible. When \(\sigma(t, x, u, \nu) = u\), one may choose \(p = 2\) to ensure that the stochastic integral \(\int_0^T \gamma s^2 dW_s^\gamma\) is well-defined and is a square-integrable martingale.

It is also possible to consider more general types of integrability conditions, such as
\[
\mathbb{E}^{\mathbb{P}^\gamma}\left[ \Phi \left( \int_t^T \Psi(\rho(u_0, \alpha_s^\gamma))ds \right) \right] < \infty,
\]
for given maps \(\Phi : [0, \infty) \to [0, \infty)\) and \(\Psi : [0, \infty) \to [0, \infty)\). This would for instance allow to consider exponential integrability requirements. For the sake of simplicity, we have chosen the condition in (3), but insist that it plays no essential role in the proof of the dynamic programming principle.

(ii) It can perfectly well happen that the set \(\Gamma_W(t, \nu)\) is empty, in which case \(V_W(t, \nu) = -\infty\) by convention. For example, when \(\int_{\mathcal{C}^n} ||x||^p \nu(dx) = \infty\), then \(\Gamma_W(t, \nu) = \emptyset\), since (3) cannot be satisfied. Nevertheless, \(\Gamma_W(t, \nu)\) is non-empty under either one of the following conditions (see for instance [23, Theorem A.2] for a brief proof)
- \((b, \sigma, \sigma_0)\) are bounded and continuous in \((x, \tilde{\nu}, u)\) and \(\nu \in \mathcal{P}(\mathcal{C}^n)\);
We further define a
\[ b(t, \mathbf{x}, \mathbf{\nu}, u) \leq C \left( 1 + \| \mathbf{x} \| + \left( \int_{C \times U} (\| \mathbf{x}' \|^p + \rho(u_0, u')^p) \mathbf{\nu}(d\mathbf{x}', du') \right)^{\frac{1}{p}} + \rho(u_0, u) \right) \]
and
\[ |(\sigma, \sigma_0)(t, \mathbf{x}, \mathbf{\nu}, u)|^2 \leq C \left( 1 + \| \mathbf{x} \|^p + \left( \int_{C \times U} (\| \mathbf{x}' \|^p + \rho(u_0, u')^p) \mathbf{\nu}(d\mathbf{x}', du') \right)^{\frac{2}{p}} + \rho(u_0, u)^2 \right). \]

**Remark 3.** It is perfectly possible for us to consider a slightly more general class of control problems allowing for exponential discounting. More precisely, we could have an additional Borel map \( k : [0, T] \times C \times \mathcal{P}(C^n \times U) \times U \rightarrow \mathbb{R} \) and consider, for fixed \((t, \nu) \in [0, T] \times \mathcal{P}(C^n)\), the problem of maximising over \( \gamma \in \Gamma_W(t, \nu) \) the functional
\[
\mathbb{E}_{P_\gamma} \left[ \int_t^T e^{-\int_t^s k(u, X_{\gamma_u}^\nu, \mathbf{\nu}_u, \alpha_u^\gamma)} ds + e^{-\int_t^T k(u, X_{\gamma_u}^\nu, \mathbf{\nu}_u, \alpha_u^\gamma)} g(X_{t\wedge \tau}^\gamma, \mu_T^\gamma) \right].
\]
We refrained from working at that level of generality for notational simplicity, but our results extend directly to this context.

We finish this section with some useful additional notations. We will represent the \( U \)-valued control process \( \alpha^\gamma \) using an \( \mathbb{R} \)-valued continuous process which we denote \( A^\gamma \). Towards this purpose, let us use the isomorphic bijection \( \pi : U \rightarrow \pi(U) \subset [0,1] \) to define, for \( dP^\gamma \otimes ds \)-a.e. on \( \Omega \times [t, T] \),
\[
(6) \quad A_t^\gamma := \int_t^{s \wedge t} \pi(\alpha^\gamma_s) \, ds, \quad s \in [0, T], \quad \text{so that} \quad \alpha^\gamma_s = \pi^{-1} \left( \lim_{n \to \infty} n (A^\gamma_s - A^\gamma_{s-(1/n)\wedge 0}) \right).
\]
We further define a \( \mathcal{P}(C^n \times C \times C^d \times C^\ell) \)-valued process \( \tilde{\mu}^\gamma = (\tilde{\mu}_t^\gamma)_{0 \leq t \leq T} \) by
\[
(7) \quad \tilde{\mu}_t^\gamma := \mathcal{L}^P \left( (X_{s \wedge t}^\gamma, A_{s \wedge t}^\gamma, W_t^\gamma, B_{s \wedge t}^\gamma) \mathbf{1}_{\{s \in [0, t] \}} \right) + \mathcal{L}^P \left( (X_{s \wedge t}^\gamma, A_{s \wedge t}^\gamma, W_t^\gamma, B_{s \wedge t}^\gamma) | \mathcal{G}_s^\gamma \right) \mathbf{1}_{\{s \in (t, T] \}}.
\]
The process \( \tilde{\mu}^\gamma \) can be defined to be a \( \mathcal{G}^\gamma \)-adapted and \( P^\gamma \)-a.s. continuous process (equipping \( \mathcal{P}(C^n \times C \times C^d \times C^\ell) \) with the weak convergence topology). Indeed, by (2), we have
\[
\tilde{\mu}_s^\gamma = \mathcal{L}^P \left( (X_{s \wedge t}^\gamma, A_{s \wedge t}^\gamma, W_t^\gamma, B_{s \wedge t}^\gamma) | \mathcal{G}_s^\gamma \right), \quad \text{\( P^\gamma \)-a.e.}, \quad \text{for all} \ s \in (t, T].
\]
Then by Lemma A.1, \( \tilde{\mu}^\gamma \) can be defined to be \( \mathcal{P}^\gamma \)-a.s. continuous on both \([0, t]\) and \((t, T]\). Moreover, using the independence property between \( \mathcal{F}_t^\gamma \wedge \sigma(W_t^\gamma) \) and \( \mathcal{G}_t^\gamma \), we have, \( \mathcal{P}^\gamma \)-a.s.
\[
\lim_{\mathcal{F}_r \wedge t \uparrow \mathcal{F}_t} \tilde{\mu}_r^\gamma = \lim_{\mathcal{F}_r \wedge t \uparrow \mathcal{F}_t} \mathcal{L}^P \left( (X_{r \wedge t}^\gamma, A_{r \wedge t}^\gamma, W_t^\gamma, B_{r \wedge t}^\gamma) | \mathcal{G}_t^\gamma \right) = \mathcal{L}^P \left( (X_r^\gamma, A_r^\gamma, W_t^\gamma, B_r^\gamma) | \mathcal{G}_t^\gamma \right) = \mathcal{L}^P \left( X_r^\gamma, A_r^\gamma, W_t^\gamma, B_r^\gamma \right) = \tilde{\mu}_t^\gamma.
\]
To apply the measurable selection arguments in the control problem, an important step is to introduce a canonical space (with a good topological structure) to represent the law of \((X^\gamma, \alpha^\gamma)\). For the diffusion process \( X^\gamma \), it is classical and natural to use the canonical space \( C^n \), which is a Polish space under the uniform convergence topology. For the \( U \)-valued control process \( \alpha^\gamma \), it is suggested in the seminal paper [24] to consider it as a measure-valued process and then use a sub-space of Borel measures on \( U \times [0, T] \) as canonical space, which is also a Polish space under the weak convergence topology. In our paper, we suggest the representation of \( \alpha^\gamma \) by the continuous process \( A^\gamma \), and then
use $\mathcal{C}$ as associated canonical space. This avoids the introduction of the measure-valued processes as well as the canonical space of measures on $U \times [0, T]$, which we feel would make the notations heavier.

At the same time, to recover the $U$-valued control process $\alpha^\gamma$ from the corresponding continuous process $A^\gamma$, one should realise that it is perfectly possible that $\lim_{n \to \infty} n (A^\gamma_s - A^\gamma_{(s-1/n)\wedge 0}) \notin \pi(U)$ on some null set. This is the main motivation for introducing the cemetery point $\partial$ at the end of the introduction, so as to extend the definition of $\pi^{-1}$ from $\pi(U)$ to $\mathbb{R} \cup \{-\infty, +\infty\}$. As such, $\pi^{-1}(\lim_{n \to \infty} n (A^\gamma_s - A^\gamma_{(s-1/n)\wedge 0}))$ is always well defined, even on these null sets. Moreover, for a function $\varphi$ defined on $U$, one can extend its definition on $\bar{U}$ by setting $\varphi(\partial) \equiv 0$ without changing the problem as $\pi^{-1}(\lim_{n \to \infty} n (A^\gamma_s - A^\gamma_{(s-1/n)\wedge 0})) = \alpha^\gamma_s \in U$, $d\pi^\gamma \otimes ds$–a.e.

2.2. Strong formulation.

2.2.1. Classical strong formulation. To introduce the strong formulation of the control problem, we shall consider a fixed probability space, equipped with fixed Brownian motions, on which the control processes as well as the controlled McKean–Vlasov processes are defined. Such a formulation is in the same spirit as the ones considered by Cosso and Pham [19], Pham and Wei [41], or Bouchard, Djehiche and Kharroubi [9].

Let us consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a $d$-dimensional standard Brownian motion $W$ and an $\ell$-dimensional standard Brownian motion $B$. The two Brownian motions $W$ and $B$ are $\mathbb{P}$-independent, and moreover, there exists a sub-sigma-field $\mathcal{F}_0 \subset \mathcal{F}$, $\mathbb{P}$-independent of $(W, B)$, such that the probability space $(\Omega, \mathcal{F}_0, \mathbb{P})$ is ‘rich enough’ in the sense that for any probability $\nu \in \mathcal{P}(\mathbb{C}^n)$, there exists an $\mathcal{F}_0$-measurable random variable $\xi : \Omega \to \mathbb{C}^n$ such that $\mathbb{P} \circ \xi^{-1} = \nu$.

Let us then define $\mathbb{F} := (\mathcal{F}_t)_{t \in [0,T]}$ the $\mathbb{P}$-augmentation of $\mathcal{F}^o = (\mathcal{F}_t^o)_{t \in [0,T]}$ where

\[ \mathcal{F}_t^o := \mathcal{F}_0 \vee \sigma((W_s, B_s) : s \in [0, t]), \quad t \in [0, T]. \]

Next, for each $(t, s) \in [0, T]^2$, we let $W^t_s := W_{s\wedge t} - W_t$, $B^t_s := B_{s\wedge t} - B_t$ be the increment of the Brownian motions $W$ and $B$ after time $t$, and then define the common noise filtration $\mathcal{G}^t := (\mathcal{G}_s^t)_{s \in [0,T]}$ the $\mathbb{P}$-augmentation of $\mathcal{G}^{t, o}$. $\mathcal{G}_s^{t, o} = (\mathcal{G}_r^{t, o})_{0 \leq r \leq s \leq T}$ where

\[ \mathcal{G}_s^{t, o} := \begin{cases} \{0, \Omega\}, & \text{if } 0 \leq s \leq t, \\ \sigma(B_r^t : r \in [t, s]), & \text{if } 0 \leq t \leq s \leq T. \end{cases} \]

Before we introduce the control and controlled processes on the fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we formulate the following standard Lipschitz and linear growth condition on the coefficient functions $(b, \sigma)$ so that the corresponding McKean–Vlasov equation has a unique solution (see e.g. Theorem A.3).

Assumption 2.2. Let the constant in (3) be $p = 2$. There exists a constant $C > 0$ such that, for all $(t, x, x', \tilde{v}, \tilde{v}', u) \in [0, T] \times \mathbb{C}^n \times \mathbb{C}^n \times \mathcal{P}_2(\mathbb{C}^n \times U) \times \mathcal{P}_2(\mathbb{C}^n \times U) \times U$, one has

\[ \| (b, \sigma, \sigma_0)(t, x, \tilde{v}, u) - (b, \sigma, \sigma_0)(t, x', \tilde{v}', u) \| \leq C (\| x - x' \| + \mathcal{W}_2(\tilde{v}, \tilde{v}')), \]

\[ \| (b, \sigma, \sigma_0)(t, x, \tilde{v}, u) \| ^2 \leq C \left( 1 + \| x \| ^2 + \int_{\mathbb{C}^n \times U} (\| y \| ^2 + \rho(\hat{u}, u_0)^2) \tilde{v}(dy, d\hat{u}) + \rho(\hat{u}, u_0)^2 \right). \]
For each \( t \in [0, T] \), let us denote by \( \mathcal{A}_t \) (resp. \( \mathcal{A}_t^\mathbb{F} \)) the collection of admissible (resp. \( \mathbb{F} \)-admissible) control processes, that is, \( U \)-valued processes \( \alpha = (\alpha_s)_{t \leq s \leq T} \) which are \( \mathbb{F} \)-predictable (resp. \( \mathcal{G}^2 \)-predictable) and such that

\[
\mathbb{E}^\mathbb{P} \left[ \int_t^T (\rho(u_0, \alpha_s))^2 \, ds \right] < \infty.
\]

Next, given \( \nu \in \mathcal{P}_2(C^\alpha) \), we denote by \( \mathcal{I}(\nu) \) the collection of all \( \mathcal{F}_0 \)-measurable random variables \( \zeta : \Omega \to C^\alpha \) such that \( \mathbb{P} \circ \zeta^{-1} = \nu \). Then given \((t, \zeta) \in [0, T] \times \mathcal{I}(\nu) \) and \( \alpha \in \mathcal{A}_t \), we denote by \( X_{t,\zeta}^{t,\zeta,\alpha} \) the unique strong solution (in the sense of Definition A.2) of the SDE with initial condition \( X_{t,\zeta}^{t,\zeta,\alpha}(0) = \zeta_t \), and for all \( s \in [t, T] \),

\[
X_{s}^{t,\zeta,\alpha} = \zeta_t + \int_t^s b(r, X_{r,\zeta}^{t,\zeta,\alpha}, \mu_r^{t,\zeta,\alpha}, \alpha_r) \, dr + \int_t^s \sigma(r, X_{r,\zeta}^{t,\zeta,\alpha}, \mu_r^{t,\zeta,\alpha}, \alpha_r) \, dW_r^t
\]

\[(9)\]

where \( \mu_r^{t,\zeta,\alpha} := \mathcal{L}^\mathbb{P} (X_{r,\zeta}^{t,\zeta,\alpha}, \alpha_r) \big| \mathcal{G}_t^\nu \), \( \mathbb{P} \times \text{d}r \)-a.e. on \( \Omega \times [t, T] \).

Notice that the existence and uniqueness of a solution to SDE (9) is ensured by Assumption 2.2 (see Theorem A.3). Finally, we denote \( \mu_s^{t,\zeta,\alpha} := \mathcal{L}^\mathbb{P} (X_{s,\zeta}^{t,\zeta,\alpha}, \mathcal{G}_t^\nu) \), \( s \in [t, T] \).

We can then introduce the following two strong formulations of the McKean–Vlasov optimal control problem for \((t, \nu) \in [0, T] \times \mathcal{P}_2(C^\alpha)\),

\[
V_S(t, \nu) := \sup_{\zeta \in \mathcal{I}(\nu)} \sup_{\alpha \in \mathcal{A}_t} J(t, \zeta, \alpha), \text{ and } V_{S}^\mathbb{P}(t, \nu) := \sup_{\zeta \in \mathcal{I}(\nu)} \sup_{\alpha \in \mathcal{A}_t^\mathbb{F}} J(t, \zeta, \alpha),
\]

where

\[
J(t, \zeta, \alpha) := \mathbb{E}^\mathbb{P} \left[ \int_t^T L(s, X_{s,\zeta}^{t,\zeta,\alpha}, \mu_s^{t,\zeta,\alpha}, \alpha_s) \, ds + g(X_{T,\zeta}^{t,\zeta,\alpha}, \mu_T^{t,\zeta,\alpha}) \right].
\]

**Remark 4.** In the literature for the McKean–Vlasov control problem, such as in Cosso and Pham [19], Cosso et al. [20], an alternative take is to define the value function \( V_S \) as a function of \((t, \zeta)\) for some random element \( \zeta \). It is therefore natural to ask whether the two formulations are equivalent or not, which amounts to proving a ‘law invariance property’, stating that the value function does not depend on \( \zeta \), rather than on its law. This was pioneered in [19], though the proof there has a gap which was pointed out and corrected in [20]. It is also shown in this last reference that one cannot expect this invariance property to hold without any regularity conditions on the reward functionals \( L \) and \( g \), see [20, Remark 3.7], and our Remark 6 below.

Notwithstanding, to prove the dynamic programming principle for \( V_S \), we will need to assume, see Assumption 2.3 below, some regularity conditions on \((L, g)\) (which are weaker than the ones in [20, Assumption \((A_{L, g})_{local}\)]) Under Assumption 2.3, we will be able to obtain the aforementioned law invariance property, so that \( \sup_{\alpha \in \mathcal{A}_t} J(t, \zeta, \alpha) \) depends only on the law of \( \zeta \) rather than on the random variable \( \zeta \) itself, as desired. In other words, our strong formulation for \( V_S \) is then equivalent to the strong formulation in [20], under their regularity assumptions. Here, we choose to define the value function \( V_S \) by taking supremum on \( \zeta \) as in (10) so that \( V_S \) is immediately a function of \((t, \nu)\).

We also point out that in [20] which appeared during the revision of our paper, the authors can prove, in a case without common noise, a dynamic programming principle for their alternative strong formulation for \( V_S \) without additional regularity conditions
Moreover, the map $C_F$ for some law invariance property the choice of the random variable singleton and hence a special case in our setting. Indeed, when $V$-Vlasov control problem without common noise has also largely been studied. This is Assumption 2.3 does not hold, since in this case the law invariance property is wrong in general.

**Remark 5 (The case without common noise: $\ell = 0$).** In the literature, the McKean–Vlasov control problem without common noise has also largely been studied. This is a special case in our setting. Indeed, when $\ell = 0$, the process $B_t$ degenerates to be a singleton and hence $G_s^\nu = \{\emptyset, \Omega\}$ for all $s \in [0,T]$. It follows that $\mathcal{F}^{\xi,\alpha}$ appearing in (9) turns out to satisfy

$$\mathcal{F}_s^{\xi,\alpha} = \mathcal{L}^\mathbb{P}(X_s^{\xi,\alpha}, \alpha_s), \quad d\mathbb{P} \otimes dt - \text{a.e. on } \Omega \times [t,T],$$

and the value function $V_S(t,\nu)$ in (10) turns out to be the standard formulation of the control problem without common noise (see e.g. [17]).

We next show that, in the two strong formulations in (10), it is equivalent to fix an arbitrary initial condition $\zeta \in \mathcal{I}(\nu)$ to define the value function. In particular, this shows that the value function depends essentially on the law of $\zeta$, rather than on the choice of the random variable $\zeta$ itself. This fact is sometimes referred to as the law invariance property (see Cosso and Pham [19]). First, let $t \in [0,T]$ and $\zeta \in \mathcal{I}(\nu)$ for some $\nu \in \mathcal{P}_2(C^n)$. We consider $\mathbb{F}^{\zeta}$ the $\mathbb{P}$-augmentation of $\mathbb{F}^{\xi,\zeta}$ where the filtration $\mathbb{F}^{\xi,\nu} := (\mathcal{F}_s^{\xi,\nu})_{s \in [0,T]}$ is generated by $(\zeta, W_t, B_t')$ i.e.

$$\mathcal{F}_s^{\xi,\nu} := \begin{cases} \sigma(\zeta_{s,n}), & \text{if } s \in [0,t], \\ \sigma((\zeta_{r,n}, W_r, B_r) : r \in [t,s]), & \text{if } s \in (t,T]. \end{cases}$$

We then introduce a subset of admissible control processes by

$$A^{\zeta}_t := \{\alpha \in A_t : \alpha \text{ is } \mathbb{F}^{\zeta}\text{-predictable}\}.$$

Under additional regularity conditions on the coefficient functions, one can further restrict the control processes $\alpha$ in the class $A^{\zeta}_t$ for the strong formulation problem $V_S$.

**Assumption 2.3.** For all $t \in [0,T]$, the functions $(b, \sigma, \sigma_0) : (x, \tilde{v}, u) \in C^n \times \mathcal{P}(C^n \times U) \times U \mapsto (b, \sigma, \sigma_0)(t, x, \tilde{v}, u) \in \mathbb{R}^n \times \mathbb{S}^{n \times d} \times \mathbb{S}^{n \times \ell}$, are continuous, and there exists a constant $C > 0$ such that, for all $(t, x, u, \tilde{v}) \in [0,T] \times C^n \times U \times \mathcal{P}(C^n \times U)$

$$| (L, g)(t, x, \tilde{v}, u) |^2 \leq C \left( 1 + ||x||^2 + \int_{C^n \times U} (||y||^2 + \rho(u', u_0)^2)\tilde{v}(dy, du') + \rho(u, u_0)^2 \right).$$

Moreover, the map $(x, \tilde{v}, u) \in C^n \times \mathcal{P}_2(C^n \times U) \times U \mapsto (L, g)(t, x, \tilde{v}, u) \in \mathbb{R} \times \mathbb{R}$, is lower semi-continuous for all $t \in [0,T]$.

**Proposition 2.4.** Let $(t, \nu) \in [0,T] \times \mathcal{P}_2(C^n)$, $\xi \in \mathcal{I}(\nu)$ be fixed, and let Assumption 2.2 hold true. Then

$$V^\mathbb{P}_S(t, \nu) = \sup_{\alpha \in A^\nu_t} J(t, \xi, \alpha).$$

Assume in addition that Assumption 2.3 holds, then

$$V_W(t, \nu) = V(t, \nu) = \sup_{\alpha \in A^\xi_t} J(t, \xi, \alpha).$$
such that \( L \) for fixed \( 12 \) then it follows by the strong uniqueness of SDE (9) that, for fixed \( t \in [0,T] \) and \( \alpha \in \mathcal{A}_t^B \), there exists a unique functional \( \Psi : C^* \times C^d \times C^l \rightarrow C^* \) such that \( X_{s,t}^{L,g,\alpha} = \Psi_s(\xi, W^i, B^j) \). Consequently, for all \( (\xi_1, \xi_2) \in \mathcal{I}(\nu)^2 \), one has \( \mathcal{L}^2(\Xi_{s,t}^{L,g,\alpha}, \nu) = \mathcal{L}^2(\Phi_s(\xi_1, W^i, B^j), \alpha(B^l)) = \mathcal{L}^2(\Phi_s(\xi_2, W^i, B^j), \alpha(B^l)) = \mathcal{L}^2(\Xi_{s,t}^{L,g,\alpha}, \alpha) \). It follows that, for all \( (\xi_1, \xi_2) \in \mathcal{I}(\nu)^2 \), and \( \alpha \in \mathcal{A}_t^B \), \( J(t, \xi_1, \alpha) = J(t, \xi_2, \alpha) \), which proves indeed (12).

(ii) For fixed \( \xi \in \mathcal{I}(\nu) \), and \( \alpha \in \mathcal{A}_t \), we denote \( \gamma := (\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, X_{s,t}^{L,g,\alpha}, W^i, B^j, \mathcal{P}^{L,g,\alpha}) \). Then it is straightforward to check that \( \gamma \) is a weak control rule (i.e. \( \gamma \in \Gamma_W(t, \nu) \)) such that \( J(t, \gamma) = J(t, \xi, \alpha) \). Therefore, one has

\[
V_W(t, \nu) \geq V_S(t, \nu) \geq \sup_{\alpha \in \mathcal{A}_t^*} J(t, \xi, \alpha).
\]

On the other hand, under Assumption 2.2 and Assumption 2.3, we can apply [23, Theorem 3.1] (letting \( \tilde{p} = p = 2 \) in their Assumption 2.1) to deduce that

\[
V_W(t, \nu) = \sup_{\alpha \in \mathcal{A}_t^*} J(t, \xi, \alpha).
\]

Therefore, (13) follows. \( \square \)

Remark 6. In Proposition 2.4, (12) and (13) imply that, for any \( (\xi_1, \xi_2) \in \mathcal{I}(\nu) \times \mathcal{I}(\nu) \), one has

\[
\sup_{\alpha \in \mathcal{A}_t^B} J(t, \xi_1, \alpha) = \sup_{\alpha \in \mathcal{A}_t^B} J(t, \xi_1, \alpha), \text{ and } \sup_{\alpha \in \mathcal{A}_t^*} J(t, \xi_1, \alpha) = \sup_{\alpha \in \mathcal{A}_t^*} J(t, \xi_2, \alpha).
\]

This is the so-called law invariance property we mentioned in Remark 4. Therefore, in our context, we obtain this invariance property for \( V_W^B \) with only Borel-measurability conditions on \( \langle L, g \rangle \), and for \( V_S \) with the additional growth and continuity conditions on \( \langle L, g \rangle \) in Assumption 2.3. This is consistent with [20, Remark 3.7], which shows that this property for \( V_S \) cannot be true in general without additional conditions on \( \langle L, g \rangle \).

We can also define the strong formulation equivalently by adding some measurability conditions on the weak controls. Given \( (t, \nu) \in [0,T] \times \mathcal{P}_2(C^*) \), and a weak control \( \gamma = (\Omega^\gamma, \mathcal{F}^\gamma, \mathbb{P}^\gamma, \mathcal{G}^\gamma, X^\gamma, W^\gamma, B^\gamma, \mathcal{P}^\gamma, \mu^\gamma, \alpha^\gamma) \in \Gamma_W(t, \nu) \), we define \( \mathcal{G}^\gamma \) and \( \mathcal{F}^\gamma \) as the \( \mathbb{P}^\gamma \)-augmentations of \( \mathcal{F}^\gamma := (\mathcal{F}^\gamma_s)_{s \in [0,T]} \) and \( \mathcal{G}^\gamma := (\mathcal{G}^\gamma_s)_{s \in [0,T]} \), which we define by

\[
\mathcal{F}^\gamma_s := \left\{ \sigma(X^{\gamma}_{s,t}), s < t, \sigma((X^{\gamma}_{s,t}, B^{\gamma,t}_{s,t}, W^{\gamma}_{s,t})), t \leq s \right\}, \text{ and } \mathcal{G}^\gamma_s := \left\{ \sigma(B^{\gamma}_{s,t}), t \leq s \right\}.
\]

Let \( \Gamma_S(t, \nu) := \{ \gamma \in \Gamma_W(t, \nu) : \mathcal{F}^\gamma = \mathcal{F}^\gamma = \mathcal{G}^\gamma = \mathcal{G}^\gamma \} \) and \( \Gamma_S^B(t, \nu) := \{ \gamma \in \Gamma_S(t, \nu) : \alpha^\gamma \text{ is } \mathcal{G}^\gamma \text{-predictable} \} \).

Proposition 2.5. Let \( (t, \nu) \in [0,T] \times \mathcal{P}_2(C^*) \), \( \xi \in \mathcal{I}(\nu) \) be fixed, and Assumption 2.2 hold true. Then we have

\[
\{ \mathcal{L}^\gamma(X^\gamma, W^\gamma, B^\gamma, \mathcal{P}^\gamma, \mu^\gamma, \alpha^\gamma) : \gamma \in \Gamma_S(t, \nu) \} = \{ \mathcal{L}^B(X^{L,g,\alpha}, W^i, B^j, \mathcal{P}^{L,g,\alpha}, \mu^{L,g,\alpha}, \alpha) : \alpha \in \mathcal{A}_t^L \}
\]

and

$$\{ \mathcal{L}^\gamma (X^\gamma, W^{\gamma,t}, B^{\gamma,t}, \overline{\nu}^\gamma, \mu^\gamma, \alpha^\gamma) : \gamma \in \Gamma_3^S(t, \nu) \}$$

$$= \{ \mathcal{L}^\mathbb{P} (X^{t,\xi,\alpha}, W^t, B^t, \overline{\mu}^{t,\xi,\alpha}, \mu^{t,\xi,\alpha}, \alpha) : \alpha \in \mathcal{A}_t^\mathbb{P} \}.$$ 

Consequently, under Assumption 2.3

(14) $$V_S(t, \nu) = \sup_{\gamma \in \Gamma_S(t, \nu)} J(t, \gamma),$$ and $$V_S^{\mathbb{P}}(t, \nu) = \sup_{\gamma \in \Gamma_3^S(t, \nu)} J(t, \gamma).$$

**Proof.** We will consider the case for $$\alpha \in \mathcal{A}_t^{f,\xi}$$ and $$V_S$$, since the one for $$\alpha \in \mathcal{A}_t^\mathbb{P}$$ and $$\Gamma_3^S$$ follows by almost the same arguments. First, for all $$\alpha \in \mathcal{A}_t^{f,\xi}$$, one has that $$\alpha$$ is $$\mathbb{F}^{t,\xi}$$-predictable. Then it is straightforward to check that

$$\gamma := (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}^{t,\xi}, \mathcal{G}^t, X^{t,\xi,\alpha}, W^t, B^t, \overline{\mu}^{t,\xi,\alpha}, \mu^{t,\xi,\alpha}, \alpha) \in \Gamma_S(t, \nu),$$

so that

$$\{ \mathcal{L}^\gamma (X^\gamma, W^{\gamma,t}, B^{\gamma,t}, \overline{\nu}^\gamma, \mu^\gamma, \alpha^\gamma) : \gamma \in \Gamma_S(t, \nu) \}$$

$$\supset \{ \mathcal{L}^\mathbb{P} (X^{t,\xi,\alpha}, W^t, B^t, \overline{\mu}^{t,\xi,\alpha}, \mu^{t,\xi,\alpha}, \alpha) : \alpha \in \mathcal{A}_t^{f,\xi} \}.$$ 

Next, for $$\gamma \in \Gamma_S(t, \nu),$$ since $$\alpha^\gamma$$ is $$\mathbb{F}^{t,\gamma}$$-predictable, we can always write $$\alpha^\gamma_s = \phi_s (X^\gamma_{t∧s}, W^{\gamma,t}_{t∧s}, B^{\gamma,t}_{s∧t})$$ for some Borel-measurable function $$\phi$$. Then there exists a functional $$\Psi : \mathcal{C}^n \times \mathcal{C}^d \times \mathcal{C}^t \to \mathcal{C}^n$$ such that $$X^\gamma = \Psi (X^\gamma_{t∧s}, W^{\gamma,t}_{t∧s}, B^{\gamma,t}_{s∧t})$$, $$\mathbb{P}$$-a.s. Further, on the space $$(\Omega, \mathcal{F}, \mathbb{P})$$, we define $$\alpha_s = \phi_s (\xi_{t∧s}, W^t_{t∧s}, B^t_{s∧t})$$. By uniqueness for SDE (4), one has $$X^{t,\xi,\alpha} = \Psi (\xi_{t∧s}, W^t_{t∧s}, B^t_{s∧t})$$, $$\mathbb{P}$$-a.s. Besides, recall that $$\xi_{t∧s}$$ is independent of $$(W^t, B^t)$$ under $$\mathbb{P}$$ and $$\mathbb{P} \circ \xi^{-1} = \nu$$, so that

$$\mathcal{L}^\mathbb{P} (\xi_{t∧s}, W^t, B^t) = \mathcal{L}^{\mathbb{P}^\gamma} (X^\gamma_{t∧s}, W^{\gamma,t}, B^{\gamma,t}).$$

Since $$(X^{t,\xi,\alpha}, \alpha) = (\Psi, \phi) (\xi_{t∧s}, W^t, B^t)$$ (resp. $$(X^\gamma, \alpha^\gamma) = (\Psi, \phi) (X^\gamma_{t∧s}, W^{\gamma,t}, B^{\gamma,t})$$) with the same functional $$\Psi$$ and $$\phi$$, it follows that

$$\mathcal{L}^\mathbb{P} (X^{t,\xi,\alpha}, W^t, B^t, \overline{\mu}^{t,\xi,\alpha}, \mu^{t,\xi,\alpha}, \alpha) = \mathcal{L}^{\mathbb{P}^\gamma} (X^\gamma, W^{\gamma,t}, B^{\gamma,t}, \overline{\nu}^\gamma, \mu^\gamma, \alpha^\gamma).$$

This implies that

$$\{ \mathcal{L}^\gamma (X^\gamma, W^{\gamma,t}, B^{\gamma,t}, \overline{\nu}^\gamma, \mu^\gamma, \alpha^\gamma) : \gamma \in \Gamma_S(t, \nu) \}$$

$$\supset \{ \mathcal{L}^\mathbb{P} (X^{t,\xi,\alpha}, W^t, B^t, \overline{\mu}^{t,\xi,\alpha}, \mu^{t,\xi,\alpha}, \alpha) : \alpha \in \mathcal{A}_t^{f,\xi} \}.$$ 

Finally, together with (13), the above equality of sets of laws implies immediately that

$$V_S(t, \nu) = \sup_{\gamma \in \Gamma_S(t, \nu)} J(t, \gamma).$$

**3. The dynamic programming principle.** The main results of our paper are related to the dynamic programming principle for the previously introduced formulations of the McKean–Vlasov control problem. We will first prove the DPP for the general strong and weak control problems introduced in Section 2, and then show how they naturally induce the associated results in the Markovian case. Finally, we also discuss heuristically the Hamilton–Jacobi–Bellman (HJB for short) equations which can be deduced for each formulation.
3.1. The dynamic programming principle in the general case. To provide the dynamic programming principle of the McKean–Vlasov control problem (5), let us introduce the canonical space

\[ \Omega^* := \mathcal{C}^f \times C([0, T], \mathcal{P}(\mathcal{C}^n \times \mathcal{C} \times \mathcal{C}^d \times \mathcal{C}^f)), \]

with canonical process \((B^*, \hat{\mu}^*)\), and canonical filtration \(G^* := (\mathcal{G}_s^r)_{0 \leq r \leq T} \) defined by \(\mathcal{G}_s^r := \sigma(\hat{\mu}_r, B_r^*) : r \in [0, s], s \in [0, T].\) Then, for every \(G^*\)-stopping time \(\tau^*\) (which can then be written as a function of \(B^*\) and \(\hat{\mu}^*\)), for all \((t, \nu) \in [0, T] \times \mathcal{P}(\mathcal{C}^n)\) and \(\gamma \in \Gamma_W(t, \nu)\), we define (recall that \(\hat{\mu}^\gamma\) is defined by (7))

\[ \tau^\gamma := \tau^*(B^\gamma, \hat{\mu}^\gamma). \]  

**Theorem 3.1.** The value function \(V_W : [0, T] \times \mathcal{P}(\mathcal{C}^n) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}\) of the weak McKean–Vlasov control problem (5) is upper semi-analytic. Moreover, let \((t, \nu) \in [0, T] \times \mathcal{P}(\mathcal{C}^n)\), \(\tau^*\) be a \(G^*\)-stopping time taking values in \([t, T]\), and \(\tau^\gamma\) be defined in (15). One has

\[ V_W(t, \nu) = \sup_{\gamma \in \Gamma_W(t, \nu)} \mathbb{E}^P \left[ \int_t^{\tau^\gamma} L(s, X^\gamma_{s, \Lambda}, \hat{\mu}^\gamma_s, \alpha_s) \, ds + V_W(\tau^{\gamma}, \mu^{\gamma}_\tau) \right]. \]  

**Remark 7.** Notice that Theorem 3.1 requires almost no conditions, except the Borel-measurability of the coefficient functions. In particular, the regularity conditions in Assumption 2.2 and Assumption 2.3 are not assumed. Of course, without additional conditions, it is possible that the set of weak controls \(\Gamma_W(t, \nu)\) is empty, in which case both sides of (16) are equal to \(-\infty\).

**Remark 8.** For every \(\gamma \in \Gamma_W(t, \nu)\), \(\tau^\gamma\) is a stopping time w.r.t. the common noise filtration \(G^\gamma\) in \((\Omega^*, \mathcal{F}^\gamma, \mathbb{P}^\gamma)\), and \(\mu^\gamma\) is a \(\mathcal{P}(\mathcal{C}^n)\)-valued, \(G^\gamma\)-adapted continuous process, which describes the conditional law of \(X^\gamma\) knowing the common noise. Conditionally on the common noise \(G^\gamma\), \(\mu^\gamma\) should be considered as a deterministic process, and \(\tau^\gamma\) as a deterministic time, and hence \(\mu^{\gamma}_\tau\) can be seen as a \(\mathcal{P}(\mathcal{C}^n)\)-valued random variable, representing the law of \(X^\gamma_{\tau, \Lambda}\), knowing the common noise.

In contrast to the strong formulation with a fixed probability space, the stopping times \(\tau^\gamma\) in (16) are defined with respect to different filtrations, in different probability spaces. Here, we define \(\tau^\gamma\) by (15) so that even when the weak control \(\gamma\) changes, the stopping times are given by the same function of the essential elements \((B^\gamma, \hat{\mu}^\gamma)\) in the common noise filtration \(G^\gamma\). Technically, we shall reformulate the weak control problem on a canonical space, on which the family \((\tau^\gamma)_{\gamma \in \Gamma_W(t, \nu)}\) can be represented by a unique stopping time w.r.t. a canonical common noise filtration. The DPP result in (16) can thus also be stated equivalently on the canonical space with a unique stopping time, see (39).

We next consider the two strong formulations \(V_S\) and \(V_S^B\) introduced in (10), or equivalently (14).

**Theorem 3.2.** Let Assumption 2.2 hold. Then the map \(V_S^B : [0, T] \times \mathcal{P}_2(\mathcal{C}^n) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}\) is upper semi-analytic. Moreover, let \((t, \nu) \in [0, T] \times \mathcal{P}_2(\mathcal{C}^n), \xi \in \mathcal{I}(\nu),\) and \(\tau\) be a \([t, T]\)-valued, \(G^{t, \nu}\)-stopping time on \((\Omega, \mathcal{F}, \mathbb{P})\). Then

\[ V_S^B(t, \nu) = \sup_{\xi \in \mathcal{I}(\nu)} \sup_{\alpha \in A_\nu} \mathbb{E}^P \left[ \int_t^{\tau} L(s, X^{t, \xi, \alpha}_{s, \Lambda}, \hat{\mu}^{t, \xi, \alpha}_s, \alpha_s) \, ds + V_S^B(\tau, \mu^{t, \xi, \alpha}_\tau) \right] \]

\[ = \sup_{\alpha \in A_\nu} \mathbb{E}^P \left[ \int_t^{\tau} L(s, X^{t, \xi, \alpha}, \hat{\mu}^{t, \xi, \alpha}_s, \alpha_s) \, ds + V_S^B(\tau, \mu^{t, \xi, \alpha}_\tau) \right]. \]
Theorem 3.3. Let Assumption 2.2 and Assumption 2.3 hold true. Let \((t,\nu) \in [0,T] \times \mathcal{P}_2(\mathcal{C}^n), \xi \in \mathcal{I}(\nu),\) and \(\tau\) be a \(\mathcal{G}^t\omega\)-stopping time on \((\Omega,\mathcal{F},\mathbb{P})\) taking values in \([t,T]\). Then \(V_S(t,\nu) = V_W(t,\nu),\) so that the value function \(V_S : [0,T] \times \mathcal{P}_2(\mathcal{C}^n) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}\) is upper semi-analytic, and

\[
V_S(t,\nu) = \sup_{\xi \in \mathcal{I}(\nu)} \sup_{\alpha \in \mathcal{A}^t} \mathbb{E}^{\mathbb{P}} \left[ \int_t^{\tau} L(s, X^{t,\xi,\alpha}_s, \mu_s^{t,\xi,\alpha}, \alpha_s) ds + V_S(\tau, \mu^{t,\xi,\alpha}_\tau) \right]
\]

(18)

Remark 9. (i) Our results for \(V_W\) and \(V_S\) in Theorem 3.1 and Theorem 3.3 are new in this general framework. For the result in Theorem 3.2, where the control is adapted to the common noise \(B\), the same DPP result has been obtained in Pham and Wei [41, Proposition 3.1]. However, our result is more general, since we do not require any regularity conditions on the reward functions \(L\) and \(g\), thanks to our use of measurable selection arguments.

(ii) From our point of view, the formulations \(V_W\) and \(V_S\) in (5) and (10) seem to be more natural than \(V^B_S\) defined in (10), because they should be the ones arising naturally as limit of finite population control problems (see Lacker [32] for the case without common noise, and Djete, Possamaï and Tan [23] for the context with common noise). Indeed, for the problem with a finite population \(N\), when the controller observes the evolution of the empirical distribution of \((X^1, \ldots, X^N)\), it is more reasonable to assume that he/she uses the information generated by both \((X^{t,\pi}_{s\downarrow}, W \circ s, B\) (as in the definition of \(V_S\)), rather than just the information from \(B\) (as in the definition \(V^B_S\)), to control the system.

Remark 10. The DPP result for \(V_S\) in Theorem 3.3 has been proved under the additional regularity conditions from Assumption 2.3. This should appear as a surprise to readers familiar with the measurable selection approach to the DPP for classical stochastic control problems. We will try here to give some intuition on why, at least if one uses our method of proof, there does not seem to be any way to make do without these additional assumptions.

As mentioned above, the two key steps to prove the DPP consist in proving the stability of the problem under conditioning and concatenation. For the conditioning argument, let us start with a control process \(\alpha := (\alpha_s)_{s \in [t,T]} \in \mathcal{A}^t_{t,\xi}\), which is adapted to the filtration generated by \((\xi_{t\downarrow s}, W^s_{t\downarrow s}, B^s_{t\downarrow s})_{s \in [t,T]}\), and fix some time \(t_0 \in (t,T]\), and the common noise filtration \(\mathcal{G}^t := (\mathcal{G}^t_s)_{s \in [t,T]}\), generated by \(B^t\). Then, under the conditional probability of \(\mathbb{P}\) given \(\mathcal{G}^t_{t_0}\), the process \((\alpha_s)_{s \in [t_0,T]}\) will be adapted to the filtration generated by \((\xi_{t_0 \downarrow s}, W^s_{t_0 \downarrow s}, B^s_{t_0 \downarrow s})_{s \in [t_0,T]}\) together with the additional random element \((W^s_{t_0 \downarrow s})_{s \in [t_0,T]}\). As such, after conditioning, \(\alpha\) should be considered as an element in \(\mathcal{A}_{t_0,\xi}\) rather than in \(\mathcal{A}^t_{t,\xi}\). In other words, the definition of \(V_S\) in (10) with admissible control set \(\mathcal{A}_{t}\) is appropriate to obtain the stability by conditioning.

However, to concatenate the controls on \([t_0,T]\) on \(\mathcal{F}^t_{t_0}\), it is necessary to fix the paths of the Brownian motions \((W, B)\) before \(t_0\) as deterministic paths (see e.g. in Stroock and Varadhan [47, Lemma 6.1.1 and Theorem 6.1.2]). This means that the control after \(t_0\) should be taken independent of the Brownian motions before time \(t_0\). In view of this, it is appropriate to define \(V_S\) with admissible control set \(\mathcal{A}^t_{t,\xi}\) in (11) to obtain the stability by concatenation.
To bypass this difficulty, we will need to use the equivalence result (13) in Proposition 2.4 to ensure the equivalence of the definition of $V_S$ with $A_t$ and $A_t^{\uparrow \downarrow}$. As an important technical step, we need to use the equivalence result $V_S = V_W$ proved in our accompanying paper [23] under the integrability and regularity conditions in Assumption 2.2 and Assumption 2.3.

**Remark 11.** In view of Proposition 2.5, the second equality in the DPP (18) is equivalent to

$$V_S(t, \nu) = \sup_{\gamma \in \Gamma_S(t, \nu)} \mathbb{E}^\nu \left[ \int_t^{\tau^\gamma} L(s, X_{\gamma, s}, \alpha_s^\gamma, \mu_s^\gamma) \, ds + V_W(\tau^\gamma, \mu_{\tau^\gamma}) \right].$$

An interesting consequence is that one can then formulate a control problem for an alternative set of controls $\Gamma(t, \nu)$ satisfying $\Gamma_S(t, \nu) \subset \Gamma(t, \nu) \subset \Gamma_W(t, \nu)$, so that the associated value function is by definition between $V_S(t, \nu)$ and $V_W(t, \nu)$. Then, as soon as $V_S = V_W$, and the DPP results in (16) hold, the same DPP is true for the control problem formulated with $\Gamma(t, \nu)$. For instance, this could be the case when one considers a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtrations larger than the one generated by $\mathcal{F}_0$ and $(W, B)$.

3.2. **Dynamic programming principle in the Markovian case.** With the DPP results in the general non-Markovian context of Theorem 3.1, Theorem 3.2 and Theorem 3.3, we can easily establish the DPP results for the control problems in the Markovian setting. In fact, we will consider a framework which is slightly more general than the classical Markovian formulation, by considering the so-called updating functions, as in Brunick and Shreve [11]. Let $E$ be a non-empty Polish space. A Borel-measurable function $\Phi : C^n \rightarrow C([0, T], E)$ is called an updating function if it satisfies

$$\Phi_t(x) = \Phi_t(x(t \wedge \cdot)), \text{ for all } (t, x) \in [0, T] \times C^n,$$

and for all $0 \leq s \leq t \leq T$, whenever $\Phi_s(x) = \Phi_s(x')$, and $(x(r) - x(s))_{r \in [s, t]} = (x'(r) - x'(s))_{r \in [s, t]}$, one has

$$(\Phi_t(x))_{r \in [s, t]} = (\Phi_t(x'))_{r \in [s, t]}.$$

The intuition of the updating function $\Phi$ is the following: the value of $\Phi_t(x)$ depends only on the path of $x$ up to time $t$, and for $0 \leq s < t$, $\Phi_t(x)$ depends only on $\Phi_s(x)$ and the increments of $x$ between $s$ and $t$. On the canonical space $C^n$, let $X := (X_t)_{t \in [0, T]}$ be the canonical process. We also define a new process $Z_t := \Phi_t(X)$, $t \in [0, T]$. Let us borrow some examples of updating functions from [11].

**Example.** (i) The most simple updating function is the running process itself, that is $\Phi_t(x) := x(t)$, with $E = \mathbb{R}^n$.

(ii) Let $M^i_t(x) := \max_{0 \leq s \leq t} x^i(s)$ for $i \in \{1, \ldots, n\}$, $t \in [0, T]$, and $A_t(x) := \int_0^t x(s) \, ds$, $t \in [0, T]$. Then the running process, together with the running maximum and running average process, are also examples of updating functions

$$\Phi_t(x) := (x(t), M_t(x), A_t(x)),$$

with $E = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$.

Throughout this subsection, we fix an updating function $\Phi$. In this context, one can in fact define the value function on $[0, T] \times \mathcal{P}(E)$ under some additional conditions.
Given $\nu \in \mathcal{P}(\mathcal{C}^n)$ (resp. $\tilde{\nu} \in \mathcal{P}(\mathcal{C}^n \times U)$), let us consider $X$ (resp. $(X, \alpha)$) as canonical element on the canonical space $\mathcal{C}^n$ (resp. $\mathcal{C}^n \times U$), and then define

$$[\tilde{\nu}]_t := \tilde{\nu} \circ (\Phi_t(X, \alpha))^{-1} \in \mathcal{P}(E \times U), \quad (\text{resp. } [\nu]_t := \nu \circ (\Phi_t(X))^{-1} \in \mathcal{P}(E)),$$

$t \in [0, T]$.

Denote further

$$\Phi_t(\mathcal{P}_2(\mathcal{C}^n)) := \{\nu \circ (\Phi_t(X))^{-1} : \nu \in \mathcal{P}(\mathcal{C}^n)\},$$

which is an analytic set as the image of $\mathcal{P}_2(\mathcal{C}^n)$ under the Borel map $\nu \mapsto \nu \circ (\Phi_t(X))^{-1}$.

**Assumption 3.4.** For a fixed updating function $\Phi : \mathcal{C}^n \longrightarrow C([0, T], E)$, there exist Borel-measurable functions $(b^o, \sigma^o, \mu^o, L^o, g^o) : [0, T] \times E \times U \times \mathcal{P}(E \times U) \longrightarrow \mathbb{R}^n \times \mathbb{S}^{n \times d} \times \mathbb{S}^{n \times \ell} \times \mathbb{R} \times \mathbb{R}$, such that for all $(t, x, u, \tilde{\nu}) \in [0, T] \times \mathcal{C}^n \times U \times \mathcal{P}(\mathcal{C}^n \times U),$

$$(b^o, \sigma^o, \mu^o, L^o, g^o)(t, x, u, \tilde{\nu}) = (b^o, \sigma^o, \mu^o, L^o, g^o)(t, \Phi_t(x), [\tilde{\nu}]_t, u).$$

Let $t \in [0, T]$ and $\nu^o \in \mathcal{P}(E)$, we define first the following sets

$$\nu(t, t^o) := \{\nu \in \mathcal{P}(\mathcal{C}^n) : [\nu]_t = \nu^o\}, \quad \mathcal{V}_2(t, t^o) := \{\nu \in \mathcal{P}_2(\mathcal{C}^n) : [\nu]_t = \nu^o\},$$

$$\Gamma^x_W(t, t^o) := \bigcup_{\nu \in \mathcal{V}_2(t, t^o)} \Gamma^W_W(t, \nu), \quad \Gamma^x_S(t, t^o) := \bigcup_{\nu \in \mathcal{V}_2(t, t^o)} \Gamma^S_S(t, \nu),$$

and $\Gamma^B_S(t, t^o) := \bigcup_{\nu \in \mathcal{V}_2(t, t^o)} \Gamma^B_S(t, \nu)$, as well as the value functions, with $J(t, \gamma)$ defined in (5)

$$V^x_W(t, t^o) := \sup_{\gamma \in \Gamma^x_W(t, t^o)} J(t, \gamma), \quad V^x_S(t, t^o) := \sup_{\gamma \in \Gamma^x_S(t, t^o)} J(t, \gamma),$$

and $V^B_S(t, t^o) := \sup_{\gamma \in \Gamma^B_S(t, t^o)} J(t, \gamma)$.

**Remark 12.** (i) Above, when $\nu^o \notin \Phi_t(\mathcal{P}_2(\mathcal{C}^n))$, one has $\mathcal{V}_2(t, \nu^o) = \emptyset$, so that $\Gamma^x_S(t, \nu^o) = \emptyset$ and hence $V^x_S(t, \nu^o) = V^B_S(t, \nu^o) = -\infty$ by convention.

(ii) When the updating process is the running process given by $\Phi_t(x) := x(t)$, the problems $V^x_W$, $V^x_S$ and $V^B_S$ are of course exactly the classical Markovian formulation of the control problems. In this case, one has simply $\Phi_t(\mathcal{P}_2(\mathcal{C}^n)) = \mathcal{P}_2(\mathbb{R}^n)$.

**Lemma 3.5.** Let Assumption 3.4 hold true, and fix some $t \in [0, T]$. Then, for any $(\nu_1, \nu_2) \in \mathcal{P}(\mathcal{C}^n) \times \mathcal{P}(\mathcal{C}^n)$ (resp. $(\nu_1, \nu_2) \in \mathcal{P}_2(\mathcal{C}^n) \times \mathcal{P}_2(\mathcal{C}^n)$) such that $[\nu_1]_t = [\nu_2]_t$, one has

$$V_W(t, \nu_1) = V_W(t, \nu_2) \quad (\text{resp. } V_S(t, \nu_1) = V_S(t, \nu_2), \quad \text{and } V^B_S(t, \nu_1) = V^B_S(t, \nu_2)).$$

Consequently, for all $\nu \in \mathcal{P}(\mathcal{C}^n)$ (resp $\nu \in \mathcal{P}_2(\mathcal{C}^n)$), one has

$$V_W(t, \nu) = V^x_W(t, [\nu]_t) \quad (\text{resp. } V_S(t, \nu) = V^x_S(t, [\nu]_t), \quad \text{and } V^B_S(t, \nu) = V^B_S(t, [\nu]_t)).$$

**Proof.** We will only consider the equality for $V_W$, since the arguments for $V_S$ and $V^B_S$ will be the same in view of their equivalent definition using $\Gamma^x_S(t, \nu)$ and $\Gamma^B_S(t, \nu)$ in Proposition 2.5. First, we can consider $\nu_2$ as a probability measure defined on the canonical space $\mathcal{C}^n$ with canonical process $X$, and containing the random variable $Z_t := \Phi_t(X)$. Then, on (a possibly enlarged) probability space $(\mathcal{C}^n, \mathcal{B}(\mathcal{C}^n), \nu_2)$, there exists a Borel-measurable function $\psi : E \times [0, 1] \longrightarrow \mathcal{C}^n$, together with a random variable $\eta$ with uniform distribution on $[0, 1]$, which is independent of $Z_t$,
such that \( \nu_2 \circ (Z_t, X)_t^{-1} = \nu_2 \circ (Z_t, \psi(Z_t, \eta))_t^{-1} \). Next, consider an arbitrary \( \gamma_1 := (\Omega^1, \mathcal{F}^1, \mathbb{P}^1, \mathcal{G}^1, X^1, W^1, B^1, \overline{\mathcal{P}}^1, \mu^1, \alpha^1) \in \Gamma_W(t, \nu_1) \). Without loss of generality (that is up to enlargement of the space), we can assume that there exists a random variable \( \eta \) with uniform distribution on \([0,1]\) in the probability space \((\Omega^1, \mathcal{F}^1_\circ, \mathbb{P}^1)\), and which is independent of the random variables \((X^1, W^1, B^1, \overline{\mathcal{P}}^1, \mu^1, \alpha^1)\).

We then define \( \gamma_2 \) as follows. Let \( Z_1 := \Phi_s(X^1) \), for all \( s \in [0,T] \), so that, by definition, \( \mathbb{P}^1 \circ (Z_1^1)_t^{-1} = [\nu_1 t] = [\nu_2 t] \). Next, let

\[
X^2_s := \psi_s(Z^1_s, \eta)1_{\{s \in [0,t]\}} + (X^2_t + X^1_s - X^1_t)1_{\{s \in (t,T]\}}.
\]

It follows by the properties of \( \psi \) and those of the updating function \( \Phi \) that

\[
\mathbb{P}^1 \circ (X^2_t)_t^{-1} = \nu_2(t), \quad \text{and} \quad \Phi_s(X^2) = \Phi_s(X^1), \quad s \in [t,T].
\]

Let \( \overline{\mu}^2 := \mathcal{L}^{\mathbb{P}^1}(X^2_s, \alpha^1)^{\mathcal{G}^1}_s, \mu^2 := \mathcal{L}^{\mathbb{P}^1}(X^2_s, \alpha^1)^{\mathcal{G}^1}_s, \) for \( s \in [t,T] \), and

\[
\gamma_2 := (\Omega^1, \mathcal{F}^1, \mathbb{P}^1, \mathcal{G}^1, X^2, W^1, B^1, \overline{\mu}^2, \mu^2, \alpha^1).
\]

Using Assumption 3.4 and (19), we have \( \gamma_2 \in \Gamma_W(t, \nu_2) \) and \( J(t, \gamma_2) = J(t, \gamma_1) \), implying \( V_W(t, \nu_1) = V_W(t, \nu_2) \).

Now we provide the dynamic programming principle for the Markovian control problem under Assumption 3.4.

**Corollary 3.6.** Let Assumption 3.4 hold true, and fix \( t \in [0,T] \) and \( \nu^o \in \mathcal{P}(E) \). Let \( \tau^* \) be a \( \mathcal{G}^t \)-stopping time taking values in \([t,T]\) on \( \Omega^o \), \( (\gamma^\tau)_\tau \in \Gamma_W(t, \nu^o) \) be defined from \( \tau^* \) as in (15). Further, \( \tau \) be a \( \mathcal{G}^{t,\alpha} \)-stopping time taking values in \([t,T]\) on \( \Omega \), and \( \xi : \Omega \rightarrow \mathcal{C}^n \) satisfies \( \mathbb{P} \circ \xi^{-1} = \nu \) for some \( \nu \in \mathcal{P}_2(\mathcal{C}^n) \) with \( [\nu^o ]^{[\nu]} = \nu^o \). Then one has the following dynamic programming results.

(i) The function \( V^\circ_W : [0,T] \times \mathcal{P}(E) \rightarrow \mathbb{R} \cup \{-\infty, \infty\} \) is upper semi-analytic and, with \( Z^\circ_s := \Phi_s(X^\gamma_s) \)

\[
V^\circ_W(t, \nu^o) = \sup_{\gamma \in \Gamma_W(t, \nu^o)} \mathbb{E}^{\mathbb{P}^o}\left[ \int_t^{\tau^*} L^o(s, Z^\gamma_s, [\overline{\mu}^o]^{[\nu]}_s, \alpha^\gamma_s)ds + V^\circ_W(\tau^*, [\mu^o]^{[\nu]}_{\tau^*}) \right].
\]

(ii) Let Assumption 2.2 hold true, then \( V^\circ_S(t, \nu) := \sup_{\nu \in \mathcal{P}_2(\mathcal{C}^n) \cap \mathcal{P}(E)} \mathbb{E}^\nu \left[ \int_t^T L^o(s, Z^\xi_s, [\overline{\mu}^o]^{[\nu]}_s, \alpha_s)ds + V_S^\circ(\tau, [\mu^o]^{[\nu]}_{\tau}) \right] \)

\[
= \sup_{\alpha \in A_\xi^\circ} \mathbb{E}^\nu \left[ \int_t^T L^o(s, Z^\xi_s, [\overline{\mu}^o]^{[\nu]}_s, \alpha_s)ds + V_S^\circ(\tau, [\mu^o]^{[\nu]}_{\tau}) \right].
\]

(iii) Let Assumption 2.2 and Assumption 2.3 hold, then \( V_S^\circ(t, \nu^o) = V^\circ_W(t, \nu^o) \), and with \( Z^\circ_{t,\xi} := \Phi_s(X^\xi_{t,\xi}) \), one has

\[
V^\circ_S(t, \nu) = \sup_{\nu \in \mathcal{P}_2(\mathcal{C}^n) \cap \mathcal{P}(E)} \mathbb{E} \left[ \int_t^T L^o(s, Z^\xi_s, [\overline{\mu}^o]^{[\nu]}_s, \alpha_s)ds + V_S^\circ(\tau, [\mu^o]^{[\nu]}_{\tau}) \right]
\]

\[
= \sup_{\alpha \in A_\xi^\circ} \mathbb{E} \left[ \int_t^T L^o(s, Z^\xi_s, [\overline{\mu}^o]^{[\nu]}_s, \alpha_s)ds + V_S^\circ(\tau, [\mu^o]^{[\nu]}_{\tau}) \right].
\]
such that, for all \( (t, \nu, \nu^o) \in [0, T] \times \mathcal{P}(C^n) \times \mathcal{P}(E) : [\nu]^o = \nu^o \). Notice that \( \Phi : C^n \rightarrow C([0, T], E) \) is Borel, then \( (t, \nu) \rightarrow [\nu]^o \) is also Borel, and hence \( [\mathcal{V}] \) is a Borel subset of \([0, T] \times \mathcal{P}(C^n) \times \mathcal{P}(E)\). Further, one has \( V_W^o(t, \nu^o) = \sup_{(t, \nu) \in [\mathcal{V}]} V_W(t, \nu) \) from Lemma 3.5, and \( V_W^o \) is upper semi-analytic by Theorem 3.1. By [25, Proposition 2.17], \( V_W^o : (t, \nu^o) \in [0, T] \times \mathcal{P}(E) \rightarrow V_W^o(t, \nu^o) \in \mathbb{R} \cup \{-\infty, \infty\} \) is thus also upper semi-analytic. Finally, using the DPP results in Theorem 3.1, we have

\[
V_W^o(t, \nu^o) = \sup_{\nu \in V(t, \nu^o)} V_W(t, \nu)
= \sup_{\nu \in V(t, \nu^o)} \sup_{\gamma \in \Gamma_W(t, \nu)} \mathbb{E}^{\nu} \left[ \int_t^\tau L(s, X^\nu_s, \alpha^\nu_s) \, ds + V_W(\tau, \mu^\nu_\tau) \right]
= \sup_{\nu \in V(t, \nu^o)} \sup_{\gamma \in \Gamma_W(t, \nu)} \mathbb{E}^{\nu} \left[ \int_t^\tau L^o(s, Z^\nu_s, [\alpha^\nu_s]_o, \alpha^\nu_s) \, ds + V_W^o(\tau, [\mu^\nu_\tau]_o) \right]
= \sup_{\gamma \in \Gamma_W(t, \nu)} \mathbb{E}^{\nu} \left[ \int_t^\tau L^o(s, Z^\nu_s, [\alpha^\nu_s]_o, \alpha^\nu_s) \, ds + V_W^o(\tau, [\mu^\nu_\tau]_o) \right].
\]

\( \square \)

3.3. Discussion: from dynamic programming to the HJB equation. One of the classical applications of the DPP consists in giving some local characterisation of the value function, such as in proving that it is the viscosity solution of an HJB equation. This was achieved in Pham and Wei [42] for the control problem \( V^o_S \) in the setting with \( \sigma_0 \equiv 0 \), and in Pham and Wei [41] for the control problem \( V^o_{S^g} \) (with \( \Phi_t(x) := x(t) \) and \( E = \mathbb{R}^n \)). It relies essentially on the notion of differentiability with respect to probability measures due to Lions (see e.g. [35] and Cardaliaguet’s notes [13, Section 6]), and Itô’s formula along a measure (see e.g. Carmona and Delarue [15, Proposition 6.5 and Proposition 6.3]). We will now provide some heuristic arguments to derive the HJB equation from our DPP results for both \( V^o_{S^g} \) and \( V^o_S \), with updating function \( \Phi_t(x) = x(t) \).

Let us first recall briefly the notion of the derivative, in sense of Fréchet, \( \partial_\nu V(\nu) \) for a function \( V : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R} \). Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) rich enough so that, for any \( \nu \in \mathcal{P}_2(\mathbb{R}^n) \), there exists a random variable \( Z : \Omega \rightarrow \mathbb{R}^n \) such that \( \mathcal{L}^2(Z) = \nu \). We denote by \( \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \) the space of square-integrable random variables on \((\Omega, \mathcal{F}, \mathbb{P})\). Let \( V : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R} \), we consider \( \tilde{V} : \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}^n \), the lifted version of \( V \), defined by \( \tilde{V}(X) := V(\mathcal{L}^2(X)) \). Recall that \( \tilde{V} \) is said to be continuously Fréchet differentiable, if there exists a unique continuous application \( D\tilde{V} : \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \), such that, for all \( Z \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \),

\[
\lim_{\|Y\|_2 \rightarrow 0} \frac{\tilde{V}(Z + Y) - \tilde{V}(Z) - \mathbb{E}[Y^\top D\tilde{V}(Z)]}{\|Y\|_2} = 0,
\]

where \( \|Y\|_2 := \mathbb{E}[\|Y\|^2]^{1/2} \) for any \( Y \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \). We say that \( V \) is of class \( C^1 \) if \( \tilde{V} \) is continuously Fréchet differentiable, and denote for any \( \nu \in \mathcal{P}_2(\mathbb{R}^n) \), \( \partial_\nu V(\nu)(Z) := D\tilde{V}(Z), \mathbb{P}\)-a.s., for any \( Z \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \) such that \( \mathcal{L}^2(Z) = \nu \). Notice that one has \( \partial_\nu V(\nu) : \mathbb{R}^n \ni y \rightarrow \partial_\nu V(\nu)(y) \in \mathbb{R}^n \) and this function belongs to \( \mathcal{L}^2(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \nu) \). Besides, the law of \( D\tilde{V}(Z) \) is independent of the choice of \( Z \). Similarly, we also define the derivatives \( \partial_\nu \partial_\nu V(\nu)(y, y') \) and \( \partial_\nu \partial_\nu V(\nu)(y, y') := \partial_\nu [\partial_\nu V(\nu)(y)](y') \in \mathbb{S}^n \).
3.3.1. HJB equation for the common noise strong formulation. Let us consider the control problem $V_S^{B,o}$ and repeat the arguments in [41] in a heuristic way. Given a ‘smooth function’ $V : [0, T] \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}$, $(t, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^n)$, $\xi : \Omega \to \mathbb{C}^n$ such that $\mathbb{P} \circ \xi_t = \nu$, and $\alpha \in A_\mathbb{B}$, it follows from Itô’s formula that, for $s \in [t, T]$.

\begin{equation}
V(s, \mu_r^{t,\xi,\alpha}) = V(t, \nu)
\end{equation}

\begin{align*}
&+ \int_t^s \int_{\mathbb{R}^n} \left( \partial_t V(r, \mu_r^{t,\xi,\alpha}) + \partial_r V(r, \mu_r^{t,\xi,\alpha})(y) \cdot b(r, y, \mu_r^{t,\xi,\alpha} \otimes \delta_{\alpha_r}, \alpha_r) \right) \mu_r^{t,\xi,\alpha}(dy) dr \\
&+ \frac{1}{2} \int_t^s \int_{\mathbb{R}^n} \text{Tr}[\partial_x \partial_r V(r, \mu_r^{t,\xi,\alpha})(y) (\sigma^\top \sigma + \sigma_0^\top \sigma_0)(r, y, \mu_r^{t,\xi,\alpha} \otimes \delta_{\alpha_r}, \alpha_r)] \mu_r^{t,\xi,\alpha}(dy) dr \\
&+ \frac{1}{2} \int_t^s \int_{\mathbb{R}^n} \text{Tr} \left[ \partial^2_{x} V(r, \mu_r^{t,\xi,\alpha})(y, y') \right] \\
&\quad \cdot \sigma_0(r, y, \mu_r^{t,\xi,\alpha} \otimes \delta_{\alpha_r}, \alpha_r) \sigma_0(r, y', \mu_r^{t,\xi,\alpha} \otimes \delta_{\alpha_r}, \alpha_r) \mu_r^{t,\xi,\alpha}(dy) \mu_r^{t,\xi,\alpha}(dy') dr \\
&+ \int_t^s \int_{\mathbb{R}^n \times U} \partial_y V(r, \mu_r^{t,\xi,\alpha})(x) \cdot (\sigma_0(r, y, \mu_r^{t,\xi,\alpha} \otimes \delta_{\alpha_r}, \alpha_r) \mu_r^{t,\xi,\alpha}(dy) dB_t
\end{align*}

As $\alpha \in A_\mathbb{B}$, for Lebesgue–almost every $r \in [t, T]$, $\alpha_r$ is a measurable function of $(B_u - B_t)_{u \in [t, r]}$. Considering piecewise constant control process, $\alpha$ would be a deterministic constant on a small time horizon $[t, t + \varepsilon]$. Recall the map $L^\circ$ used in the definition of $V_S^{B,o}$ and in Corollary 3.6. Replacing $V$ in (23) by $V_S^{B,o}$ and taking supremum as in Corollary 3.6 (over constant control processes), this leads to the Hamiltonian $H_B^\circ[V](t, \nu, u) := \sup_{u \in U} \tilde{H}_B^\circ[V](t, \nu, u)$, with

$$
\tilde{H}_B^\circ[V](t, \nu, u) := \left\{ \int_{\mathbb{R}^n} \left( \left( L^\circ + [V]^1 \right)(t, y, \nu \otimes \delta_{\alpha}, \alpha) \right) \nu(dy) \\
\quad + \int_{\mathbb{R}^n} [V]^2(t, y, y', \nu \otimes \delta_{\alpha}, \alpha \nu(dy') \nu(dy') \right\},
$$

where for any $(r, y, y', \nu) \in [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n)$

$$
[V]^1(r, y, \bar{\nu}, u) := \partial_y V(r, \nu)(y) \cdot b(r, y, \bar{\nu}, u) + \frac{1}{2} \text{Tr}[\partial_x \partial_r V(r, \nu)(y) (\sigma^\top \sigma + \sigma_0^\top \sigma_0)(r, y, \bar{\nu}, u)]
$$

and

$$
[V]^2(r, y, y', \bar{\nu}, u') := \frac{1}{2} \text{Tr}[\partial^2_{x} V(r, \nu)(y, y') \sigma_0^\top (r, y, \bar{\nu}, u) \sigma_0(r, y', \bar{\nu}, u')].
$$

Heuristically, $V_S^{B,o}$ should satisfy the HJB equation

$$
-\partial_t V_S^{B,o}(t, \nu) - H_B^\circ[V_S^{B,o}](t, \nu) = 0, (t, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^n), V_S^{B,o}(T, \cdot) = \hat{g}(\cdot),
$$

where $\hat{g}(\nu) := \langle g(\cdot, \nu), \nu \rangle$ for each $\nu \in \mathcal{P}(\mathbb{C}^n)$. We refer to [41] for a detailed rigorous proof of the fact that $V_S^{B,o}$ is a viscosity solution of the above HJB equation under some technical regularity conditions.
3.3.2. HJB equation for the general strong formulation. Similarly, for the control problem \( V_S^\circ \), we consider a strong control \( \alpha \in \mathcal{A}_t^{\xi} \) for some \( \xi \in \mathcal{I}(\nu) \), where for Lebesgue–almost every \( r \in [t, T] \), the control process \( \alpha_r \) is now a measurable function of \((W_u - W_t)_{u \in [t, r]}\), \((B_u - B_t)_{u \in [t, r]}\) and the initial condition random variable \( \xi_{\mathfrak{I} \lambda} \). As the control process \( \alpha \) is adapted to the filtration generated by \((\xi_{\mathfrak{I} \lambda}, W^t, B^t)\), by considering adapted piecewise constant control processes, the control process on the first small interval \([t, t + \varepsilon]\) should be a measurable function of \( \xi_{\mathfrak{I} \lambda} \). Similarly to Pham and Wei [42] in a non-common noise setting, and by considering Itô’s formula (23), this would formally lead to the Hamiltonian \( H[V](t, \nu) = \sup_{a \in \mathcal{L}_2} \tilde{H}[V](t, \nu, a) \), where

\[
\tilde{H}[V](t, \nu, a) := \left\{ \int_{\mathbb{R}^n} \left( L^0 + [V]_1 \right)(t, y, \nu \circ (\tilde{a})^{-1}, a(y))\nu(dy) + \int_{\mathcal{L}_2^n} [V]^2(t, y, a(y), y', a(y'), \nu \circ (\tilde{a})^{-1})\nu(dy)\nu(dy') \right\},
\]

where \( \tilde{a} : \mathbb{R}^n \ni x \mapsto (x, a(x)) \in \mathbb{R}^n \times U \), and \( \mathcal{L}_2^n \) consists of all \( \nu \)-square integrable functions \( a : (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \nu) \rightarrow U \). Heuristically, \( V_S^\circ \) should be a solution of the HJB equation

\[
(24) \quad - \partial_t V_S^\circ(t, \nu) - H[V_S^\circ](t, \nu) = 0, \quad (t, \nu) \in [0, T) \times \mathcal{P}_2(\mathbb{R}^n), \quad V_S^\circ(T, \cdot) = \tilde{g}(\cdot).
\]

As explained above, the difference between the HJB equations for \( V_S^{B, \circ} \) and \( V_S^\circ \) comes mainly from the fact that the control processes \( \alpha \in \mathcal{A}_t^{\xi} \) depend on the initial condition \( \xi \), which in turn modifies the Hamiltonian function which appears in the PDE (see Cosso and Pham [19] for some related observations). We also refer to Wu and Zhang [50] for a discussion of the McKean–Vlasov control problem in a non-Markovian framework without common noise, and to Burzoni et al. [12] for a notion of viscosity solution of HJB equation without using a lifting to an Hilbert space in the context of McKean–Vlasov jump processes.

As in the classical optimal control theory, one can obtain the optimal control from a smooth solution of the HJB equation by a so-called verification argument. We provide below such a verification result, given a smooth solution \( V_S^\circ \) of the HJB equation (24) (the case for \( V_S^{B, \circ} \) is similar). Here, \( v \) is a smooth solution to (24) means that all the Fréchet derivatives involved are well-defined, bounded and continuous.

**Proposition 3.7** (Verification). Assume that \((b, \sigma, \sigma_0)\) are bounded. Let \( v : [0, T] \times \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R} \) be a smooth solution of Equation (24), and suppose that there exists a Borel map \( a : [0, T] \times \mathcal{P}(\mathbb{R}^n) \rightarrow C(\mathbb{R}^n; U) \) such that

(i) for all \((t, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^n)\), one has \( \tilde{H}[v](t, \nu, a(t, \nu)) = H[v](t, \nu) \);

(ii) for all \((t, \nu) \in [0, T], \xi \in \mathcal{I}(\nu)\), the following SDE, with the initial condition \( X_{\mathfrak{I} \lambda}^\alpha = \xi_{\mathfrak{I} \lambda} \), has a unique strong solution \( X^\alpha \)

\[
X_s^\alpha = X_0^\alpha + \int_0^s b(r, X_r^\alpha, \mu_r^\alpha, \alpha_r^\star)dr + \int_0^s \sigma(r, X_r^\alpha, \mu_r^\alpha, \alpha_r^\star)dW^t_r
\]

\[
+ \int_0^s \sigma_0(r, X_r^\alpha, \mu_r^\alpha, \alpha_r^\star)dB^s_r, \quad s \in [t, T], \mathbb{P}-a.s.,
\]

where \( \mu_r^\alpha = \mathbb{P}^{\mathcal{G}^t} \circ (X_r^\alpha, \alpha_r^\star)^{-1}, \mu_r^\star = \mathbb{P}^{\mathcal{G}^t} \circ (X_r^\alpha)^{-1} \) and \( (\alpha_r^\star = a(r, \mu_r^\alpha))_{r \in [t, T]} \) belongs to \( \mathcal{A}_t^{\xi} \). Then \( v = V_S^\circ \) and \( \alpha^\star \) is an optimal control.
and \(3.99\)). We introduce a second canonical space \(\mathbb{P}^T\). Denote by \(\mathbb{P}^T\) the control problems on a canonical space. This is going to be achieved the usual way, that is to say by considering appropriately defined controlled martingale problems. Recall that \(\mathcal{A}_T^{\xi,\alpha}\) and \(\mathcal{A}_T^{\xi,\alpha}\), we write \((X^\alpha, \mu^\alpha, \bar{\mu}^\alpha)\) to denote \((X^{\xi,\alpha}, \mu^{\xi,\alpha}, \bar{\mu}^{\xi,\alpha})\). Then it follows by Itô’s formula that

\[
v(t, \nu) = v(T, \mu_T^\alpha) - \int_t^T \int_{\mathbb{R}^n \times \mathcal{U}} \left( \partial_x v(r, \mu_r^\alpha) + \partial_
u v(r, \mu_r^\alpha)(y) \cdot b(r, y, \bar{\mu}_r^\alpha, u) \right) \bar{\mu}_r^\alpha(dy, du)dr
- \frac{1}{2} \int_t^T \int_{\mathbb{R}^n \times \mathcal{U}} \text{Tr} \left[ \partial_x \sigma_r^\alpha(y, y') \right] \bar{\mu}_r^\alpha(dy, du)dr
- \frac{1}{2} \int_t^T \int_{\mathbb{R}^n \times \mathcal{U}^2} \partial_x v(r, \mu_r^\alpha)(x) \cdot \sigma_0(r, y, \bar{\mu}_r^\alpha, u) \bar{\mu}_r^\alpha(dy, du)dB_r^1.
\]

As \(v\) is solution of Equation (24), one obtains that

\[
v(t, \nu) \geq \langle g(\cdot, \mu_t^\alpha), \mu_t^\alpha \rangle
+ \int_t^T \int_{\mathbb{R}^n \times \mathcal{U}} \left( L^\alpha(r, y, \bar{\mu}_r^\alpha, u) \bar{\mu}_r^\alpha(dy, du)dr - \partial_x v(r, \mu_r^\alpha)(x) \cdot \sigma_0(r, y, \bar{\mu}_r^\alpha, u) \bar{\mu}_r^\alpha(dy, du)dB_r \right).
\]

By taking expectation, this implies that \(v(t, \nu) \geq V^\alpha_S(t, \nu)\). When we use the control \(\alpha^*\), the inequality above becomes an equality, so that \(v(t, \nu) = V^\alpha_S(t, \nu)\), and \(\alpha^*\) is an optimal control. \(\blacksquare\)

4. Proofs of the main results. We now provide the proofs of our main DPP results in Theorems 3.1, 3.2 and 3.3, where a key ingredient is the measurable selection argument. We will first reformulate the control problems on an appropriate canonical space in Section 4.1, and then provide some technical lemmata for the problems formulated on the canonical space in Section 4.2, and finally give the proofs of the main results themselves in Section 4.3.

4.1. Reformulation of the control problems on the canonical space.

4.1.1. Canonical space. In order to prove the dynamic programming results in Section 3, we first reformulate the controlled McKean–Vlasov SDE problems on a canonical space. This is going to be achieved the usual way, that is to say by considering appropriately defined controlled martingale problems. Recall that \(n, d\) and \(\ell\) are the dimensions of the spaces in which \(X, W\) and \(B\) take values, \(\mathcal{U} = U \cup \{\partial\}\) and that \(\pi^{-1}\) maps \(\mathbb{R} \cup \{\infty, -\infty\}\) to \(\mathcal{U}\). Let us introduce a first canonical space

\[
\hat{\Omega} := C^n \times C \times C^d \times C^\ell, \text{ with canonical process } (\hat{X}, \hat{A}, \hat{W}, \hat{B})
\]

and

\[
\hat{\alpha}_t := \pi^{-1}\left( \lim_{n \to +\infty} n(\hat{A}_t - \hat{A}_{0\nu(t+1/n)}) \right), t \in [0, T].
\]

Denote by \(C([0, T], \mathcal{P}(\hat{\Omega}))\) the space of all continuous paths on \([0, T]\) taking values in \(\mathcal{P}(\hat{\Omega})\), which is a Polish space for the uniform topology (see e.g. [1, Lemmata 3.97, 3.98, and 3.99]). We introduce a second canonical space

\[
\bar{\Omega} := \hat{\Omega} \times C([0, T], \mathcal{P}(\hat{\Omega})), \text{ with canonical process } (X, A, W, B, \bar{\mu}),
\]
We next define a process $\tau_{[0,T]}$ and then, for all $t \in [0, T]$, let us first introduce the corresponding generator. For all $t \in [0, T]$, we introduce the processes $W^t := (W^t_s)_{s \in [0, T]}$ and $B^t := (B^t_s)_{s \in [0, T]}$ by

$$B^t_s := B_s + B_t, \quad W^t_s := W_s - W_t, \quad s \in [0, T],$$

and the filtration $\mathcal{G}^t_s := (\mathcal{G}^t_s)_{0 \leq s \leq T}$ by

$$\mathcal{G}^t_s := \left\{ \begin{array}{ll}
\{\emptyset, \Omega\}, & \text{if } s \in [0, t), \\
\sigma(\{B^t_s, \hat{\mu}_r : r \in [0, s]\}), & \text{if } s \in [t, T].
\end{array} \right.$$ 

### 4.1.2. Controlled martingale problems on the canonical space.

We now reformulate the strong/weak control problem as a controlled martingale problem on the canonical space $\bar{\Omega}$, where a control (term) can be considered as a probability measure on $\bar{\Omega}$. To this end, let us first introduce the corresponding generator. For all $(t, x, w, b, \nu, u) \in \{0, T\} \times \mathcal{C}^n \times \mathcal{C}^d \times \mathcal{C}^\ell \times \mathcal{P}(\mathcal{C}^n \times U) \times U$, let

$$\bar{b}(t, (x, w, b), \nu, u) := (b, 0_d, 0_\ell)(t, x, \nu, u),$$

and then introduce the generator $\mathcal{L}$, for all $\varphi \in C^2_b(\mathbb{R}^{n+d+\ell})$

$$\mathcal{L}_t \varphi(x, w, b, \nu, u) :=
\sum_{i=1}^{n+d+\ell} \bar{b}_i(t, (x, w, b), \nu, u) \partial_i \varphi(x_t, w_t, b_t) + \frac{1}{2} \sum_{i,j=1}^{n+d+\ell} \bar{a}_{i,j}(t, (x, w, b), \nu, u) \partial_{i,j} \varphi(x_t, w_t, b_t).$$

We next define a process $|\mathbb{S}| = (|\mathbb{S}|_t)_{0 \leq t \leq T}$ by

$$|\mathbb{S}|_t := \int_0^t (|b| + |b|)(s, X, W, B, \bar{\mu}, \bar{\pi}_s) ds,$$

and then, for all $\varphi \in C^2_b(\mathbb{R}^{n+d+\ell})$, let $\mathcal{S}^\varphi = (\mathcal{S}^\varphi_t)_{t \in [0, T]}$ be defined by

$$\mathcal{S}^\varphi_t := \varphi(X_t, W_t, B_t) - \int_0^t \mathcal{L}_s \varphi(X, W, B, \bar{\mu}, \bar{\pi}_s) ds, \quad t \in [0, T],$$

where for $\phi : [0, T] \to \mathbb{R}$, $\int_0^t \phi(s) ds := \int_0^t \phi^+(s) ds - \int_0^t \phi^-(s) ds$ with the convention $\infty - \infty = -\infty$. Notice that on $\{|\mathbb{S}|_T < \infty\}$, the process $\mathcal{S}^\varphi$ is $\mathbb{R}$-valued. To localise the process $\mathcal{S}^\varphi$, we also introduce, for each positive integer $m$

$$\tau_m := \inf \{t \in [0, T] : |\mathbb{S}|_t \geq m\}, \quad \text{and} \quad \mathcal{S}^{\varphi,m}_t := \mathcal{S}^\varphi_{t \wedge \tau_m} = \mathcal{S}^\varphi_t \mathbf{1}_{\{\tau_m \geq t\}} + \mathcal{S}^\varphi_{\tau_m} \mathbf{1}_{\{\tau_m < t\}}, \quad t \in [0, T].$$

Notice that the process $|\mathbb{S}|$ is left-continuous, $\tau_m$ is an $\mathbb{F}^+$-stopping time on $\bar{\Omega}$, and $\mathcal{S}^{\varphi,m}$ is an $\mathbb{F}$-adapted uniformly bounded process.
**Definition 4.1.** Let \((t, \tilde{\nu}) \in [0, T] \times \mathcal{P}(\Omega)\). A probability \(\widehat{\mathbb{P}}\) on \((\Omega, \mathcal{F})\) is called a weak control rule with initial condition \((t, \tilde{\nu})\) if

(i) the process \(\overline{\sigma} = (\overline{\sigma}_t)_{t \leq s \leq T}\) satisfies

\[
\mathbb{P}[\overline{\sigma}_s \in U] = 1, \text{ for Lebesgue–a.e. } s \in [t, T], \text{ and } \mathbb{E}^{\mathbb{P}} \left[ \int_t^T (\rho(u_0, \overline{\sigma}_s)) d\mathbb{P}_s \right] < \infty;
\]

(ii) the process \(\widehat{\mu} = (\overline{\mu}_s)_{0 \leq s \leq T}\) satisfies \(\mathbb{P}\)-a.s.,

\[
\widehat{\mu}_s = \mathbb{P} \circ (X_{s \wedge \cdot}, A_{s \wedge \cdot}, W, B_{s \wedge \cdot})^{-1} 1_{\{s \in [0, t]\}} + \mathbb{P} \circ (X_{t \wedge \cdot}, A_{t \wedge \cdot}, W, B_{t \wedge \cdot})^{-1} 1_{\{s \in (t, T]\}}
\]

with \(\mathbb{P} \circ (X_{t \wedge \cdot}, A_{t \wedge \cdot}, W, B_{t \wedge \cdot})^{-1} = \tilde{\nu}(t)\);

(iii) \(\mathbb{E}^{\mathbb{P}}[||X||^p] < \infty, \mathbb{P}[\mathcal{F}_T < \infty] = 1, \text{ and for all } \varphi \in C^2_b(\mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^d)\) the process \(\mathcal{S}^\varphi\) is an \((\mathbb{P}, \mathbb{P})\)-local martingale on \(t, T]\).

Given \(\nu \in \mathcal{P}(\mathcal{C}^d)\), we denote by \(\mathcal{V}(\nu)\) the collection of all probability measures \(\tilde{\nu} \in \mathcal{P}(\Omega)\) such that \(\tilde{\nu} \circ \bar{X}^{-1} = \nu\), and let \(\mathcal{P}_W(t, \nu) := \bigcup_{\nu \in \mathcal{V}(\nu)} \mathcal{P}_W(t, \nu)\) and

\[
\mathcal{P}_W(t, \tilde{\nu}) := \{\text{All weak control rules } \mathbb{P} \text{ with initial condition } (t, \tilde{\nu})\}.
\]

**Remark 13.** Let \(\mathbb{P} \in \mathcal{P}_W(t, \nu)\) for some \(t \in [0, T]\) and \(\nu \in \mathcal{P}(\mathcal{C}^d)\). Notice that for \(s \in (t, T]\), \(\mu_s\) is \(\mathcal{G}^\nu_s\)-measurable, then by (30), one has

\[
\widehat{\mu}_s = \mathbb{P}^{\mathcal{G}^\nu_s} \circ (X_{s \wedge \cdot}, A_{s \wedge \cdot}, W, B_{s \wedge \cdot})^{-1}, \mathbb{P}\text{-a.s.}
\]

Besides, since the canonical process \((\mu_s)_{s \in [0, T]}\) is continuous, it follows that

\[
\mathcal{L}^{\mathbb{P}}(X_{t \wedge \cdot}, A_{t \wedge \cdot}, W, B_{t \wedge \cdot}) = \mu_t = \lim_{s \uparrow t} \mu_s = \lim_{s \uparrow t} \mathcal{L}^{\mathbb{P}}((X_{s \wedge \cdot}, A_{s \wedge \cdot}, W, B_{s \wedge \cdot})|\mathcal{G}^\nu_s) = \mathcal{L}^{\mathbb{P}}((X_{t \wedge \cdot}, A_{t \wedge \cdot}, W, B_{t \wedge \cdot})|\mathcal{G}^\nu_t), \mathbb{P}\text{-a.s.}
\]

This implies that \(\mathcal{F}_t \vee \sigma(W) = \sigma(X_{t \wedge \cdot}, A_{t \wedge \cdot}, W, B_{t \wedge \cdot})\) is independent of \(\mathcal{G}^\nu_T\), which is consistent with the conditions in Definition 2.1.

Finally, under \(\mathbb{P}\), \((\mu_s)_{s \in [0, t]}\) is completely determined by \(\tilde{\nu}(t)\). More precisely, one has

\[
\widehat{\mu}_t(dx, da, dw, db) = \int_{\mathbb{R} \times \mathbb{R}^d} \delta_{(x', a', w_1 + w_2, b')} (dx, da, dw, db) \tilde{\nu}(t)(dx', da', dw', db') \mathcal{L}^{\mathbb{P}}(W^t)(dw^*), \mathbb{P}\text{-a.s.}
\]

where for all \(s \in [0, T]\), \((w' \oplus_t w^*)_s := w'_s 1_{s \in [0, t]} + (w'_s - w'_s + w'_t) 1_{s \in [t, T]}\), and \(W^t\) is an \((\mathbb{F}, \mathbb{P})\)-Brownian motion on \([t, T]\), by the martingale problem property in Definition 4.1.

**Definition 4.2.** Let \((t, \nu) \in [0, T] \times \mathcal{P}(\mathcal{C}^d)\). A probability \(\mathbb{P}\) on \((\Omega, \mathcal{F})\) is called a strong control rule (resp. \(\mathbb{P}\text{–}\)strong control rule) with initial condition \((t, \nu)\), if \(\mathbb{P} \in \mathcal{P}_W(t, \nu)\) and moreover there exists some Borel-measurable function \(\phi : [0, T] \times \mathcal{C} \rightarrow U\) (resp. \(\phi : [0, T] \times \mathcal{C}_T \rightarrow U\)) such that

\[
\overline{\sigma}_s = \phi(s, X_{s \wedge \cdot}, W^t_{s \wedge \cdot}, B^{t}_{s \wedge \cdot}) \text{ (resp. } \phi(s, B^t_{s \wedge \cdot}))\text{, } \mathbb{P}\text{-a.s.}, \text{ for all } s \in [t, T].
\]

Let us then denote by \(\mathcal{P}_S(t, \nu)\) (resp. \(\mathcal{P}^\mathbb{P}_S(t, \nu)\)) the collection of all strong (resp. \(\mathbb{P}\text{–}strong\) control rules with initial condition \((t, \nu)\).
4.1.3. Equivalence of the reformulation. We now show that every strong/weak control (term) induces a strong/weak control rule on the canonical space, and any strong/weak control rule on the canonical space can be induced by a strong/weak control (term).

**Lemma 4.3.** (i) Let $t \in [0, T]$ and $\nu \in \mathcal{P}(\mathcal{C}^n)$. Then for every $\gamma \in \Gamma_W(t, \nu)$, one has

$$\mathbb{P}^\gamma := \mathbb{P} \circ (X^\gamma, A^\gamma, W^\gamma, B^\gamma, \hat{\mu}^\gamma)^{-1} \in \mathcal{P}_W(t, \nu).$$

Conversely, given $\mathbb{P} \in \mathcal{P}_W(t, \nu)$, there exists some weak control $\gamma \in \Gamma_W(t, \nu)$ such that $\mathbb{P} \circ (X^\gamma, A^\gamma, W^\gamma, B^\gamma, \hat{\mu}^\gamma)^{-1} = \mathbb{P}$.

(ii) Let $t \in [0, T]$, $\nu \in \mathcal{P}_2(\mathcal{C}^n)$, and Assumption 2.2 hold true. Then for every $\gamma \in \Gamma_S(t, \nu)$ (resp. $\Gamma^B_S(t, \nu)$), one has

$$\mathbb{P}^\gamma := \mathbb{P} \circ (X^\gamma, A^\gamma, W^\gamma, B^\gamma, \hat{\mu}^\gamma)^{-1} \in \mathcal{P}_S(t, \nu) \text{ (resp. } \mathcal{P}^B_S(t, \nu)).$$

Conversely, given $\mathbb{P} \in \mathcal{P}_S(t, \nu)$ (resp. $\mathcal{P}^B_S(t, \nu)$), there exists $\gamma \in \Gamma_S(t, \nu)$ (resp. $\Gamma^B_S(t, \nu)$) such that $\mathbb{P} \circ (X^\gamma, A^\gamma, W^\gamma, B^\gamma, \hat{\mu}^\gamma)^{-1} = \mathbb{P}$.

**Proof.** (i) First, let $\gamma \in \Gamma_W(t, \nu)$ and $\mathbb{P}^\gamma := \mathbb{P} \circ (X^\gamma, A^\gamma, W^\gamma, B^\gamma, \hat{\mu}^\gamma)^{-1}$. It is straightforward to check that

$$\mathbb{P}^\gamma[\bar{\alpha}_s \in U] = 1, \text{ for } dt\text{-a.e. } s \in [t, T], \mathbb{E}^\gamma[||X||^p] < \infty \text{ and } \mathbb{E}^\gamma\left[\int_{t}^{T} (\rho(u_s, \bar{\alpha}_s))^p ds\right] < \infty.$$ 

Further, as the integrals in (4) are well defined, one has $|S|_T < \infty, \mathbb{P}^\gamma$-a.s. Moreover, by Itô’s formula, the process $(\bar{S}_s^\gamma)_{s \in [t, T]}$ defined in (28) is an $(\mathbb{F}, \mathbb{P}^\gamma)$-local martingale, for every $\varphi \in C^2_\delta(\mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^\ell)$.

Next, notice that $B^\gamma_t$ and $\hat{\mu}^\gamma_t$ are adapted to $\mathcal{G}^\gamma$, one has, for all $(s, \beta, \psi) \in (t,T) \times C_0(\bar{\Omega}) \times C_0(\mathbb{C}^\ell \times C([0,T]; \mathcal{P}(\bar{\Omega})))$

$$\mathbb{E}^\gamma[(\beta, \hat{\mu}_s)\psi(B^\gamma_{T \wedge \cdot}, \hat{\mu}_{T \wedge \cdot})] = \mathbb{E}^\gamma[(\beta, \hat{\mu}_s)\psi(B^\gamma_{T \wedge \cdot}, \hat{\mu}_{T \wedge \cdot})]$$

This implies that $\hat{\mu}_s = \mathcal{L}^\gamma((X_s, A_s, W, B_{s \wedge \cdot})[\mathcal{G}^\gamma_T]), \mathbb{P}^\gamma$-a.s. for all $s \in (t, T)$. By the same argument and using the fact that $\mathcal{F}^\gamma_t \vee \sigma(W^\gamma)$ is independent of $\mathcal{G}^\gamma_T$, one can easily check that $\hat{\mu}_s = \mathcal{L}^\gamma((X_s, A_s, W, B_{s \wedge \cdot})[\mathcal{G}^\gamma_T])$ for $s \in [0, t]$, and that $\mathbb{P}^\gamma \circ X_{t \wedge \cdot}^{-1} = \nu$. This implies that $\mathbb{P}^\gamma \in \mathcal{P}_W(t, \nu)$.

Assume in addition that $\gamma \in \Gamma_S(t, \nu)$ so that $\alpha^\gamma$ is $\mathbb{P}^\gamma$-predictable. Then there exists a Borel-measurable function $\phi : [t, T] \times \Omega^T \rightarrow U$ such that $\alpha^\gamma_s = \phi(s, X^\gamma_{t \wedge \cdot}, W^\gamma_{s \wedge \cdot}, B^\gamma_{s \wedge \cdot})$, for all $s \in [t, T], \mathbb{P}^\gamma$-a.s. (see e.g. Claisse, Talay and Tan [18, Proposition 10]). This implies that $\pi_s = \phi(s, X_{t \wedge \cdot}, W^\gamma_{s \wedge \cdot}, B^\gamma_{s \wedge \cdot}, \mathcal{P}^\gamma_t)$, $\mathbb{P}^\gamma$-a.s. for all $s \in [t, T]$, and it follows that $\mathbb{P}^\gamma \in \mathcal{P}_S(t, \nu)$. 


(ii) Let \( \overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(t, \nu) \) for some \( \nu \in \mathcal{P}(\mathcal{C}_0^\alpha) \). By Stroock and Varadhan [47, Theorem 4.5.2], one knows that \((W, B)\) are \((\overline{\mathbb{P}}, \overline{\mathbb{P}})\)-Brownian motions on \([t, T]\), and \(\overline{\mathbb{P}}\)-a.s.,

\[
X_s = X_t + \int_t^s b(r, X_r, \overline{\mu}_r, \overline{\alpha}_r)dr + \int_t^s \sigma(r, X_r, \overline{\mu}_r, \overline{\alpha}_r)dW_r + \int_t^s \sigma_0(r, X_r, \overline{\mu}_r, \overline{\alpha}_r)dB_r,
\]

Moreover, with the filtration \(\overline{\mathcal{G}}^t\) defined in (25), and in view of Remark 13, it is straightforward to check that

\[
\gamma := (\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}, \overline{\mathcal{G}}^t, X, W, B, \overline{\mu}, \overline{\alpha}) \in \Gamma_W(t, \nu).
\]

If, in addition, \(\mathbb{P} \in \mathcal{P}_S(t, \nu)\), so that \(A\) is a \((\sigma(X_t\wedge T, W_t\wedge T), B^\mathbb{P}_r)\)\(_{r \in [t, T]}\) adapted continuous process. Using Corollary A.4 and the fact that one has the equality \(\hat{\mu}_s = \mathcal{L}^\mathbb{P}((X_{s\wedge \cdot}, A_{s\wedge \cdot}, W, B_{s\wedge \cdot})|B^\mathbb{P}_r_{r \in [t, s]} , \hat{\mu}_{s\wedge \cdot} )\), \(\mathbb{P}\)-a.s., for all \(s \in [t, T]\), one can deduce that \(\hat{\mu}_s = \mathcal{L}^\mathbb{P}((X_{s\wedge \cdot}, A_{s\wedge \cdot}, W, B_{s\wedge \cdot})|B^\mathbb{P}_r_{r \in [t, s]} , \hat{\mu}_{s\wedge \cdot} )\), \(\mathbb{P}\)-a.s., for all \(s \in [t, T]\). Let \(\hat{\mathcal{G}}^t\) be the filtration generated by \(B^\mathbb{P}_t\), \(\hat{\mathcal{G}}^t\) be the filtration generated by \((X_{t\wedge \cdot}, W^\mathbb{P}_t, B^\mathbb{P}_t)_t\), and \(\hat{\mathcal{G}}^{t, \mathbb{P}}, \hat{\mathcal{G}}^{t, \mathbb{P}}\) be the corresponding \(\mathbb{P}\)-augmented filtrations. Then \(\hat{\mu}\) is \(\hat{\mathcal{G}}^{t, \mathbb{P}}\)-predictable, and \(X\) is \(\hat{\mathcal{G}}^{t, \mathbb{P}}\)-predictable. Then it follows that

\[
\gamma' := (\overline{\Omega}, \overline{\mathcal{F}}, \hat{\mathcal{G}}^{t, \mathbb{P}}, , \overline{\mathcal{G}}^{t, \mathbb{P}}, X, W^\mathbb{P}_t, B^\mathbb{P}_t, \overline{\mu}, \overline{\alpha}) \in \Gamma_S(t, \nu).
\]

(iii) Finally, the results related to \( \overline{\mathcal{P}}_S^B(t, \nu) \) and \( \Gamma_S^B(t, \nu) \) can be deduced by almost the same arguments as for \( \overline{\mathcal{P}}_S(t, \nu) \) and \( \Gamma_S(t, \nu) \).

\[\square\]

**Remark 14.** From Lemma 4.3, we can easily deduce that under Assumption 2.2, for \(\mathbb{P} \in \mathcal{P}_S(t, \nu)\) or \(\mathbb{P} \in \overline{\mathcal{P}}_S(t, \nu)\), the canonical process \(\hat{\mu}\) satisfies

\[
\hat{\mu}_s = \mathcal{L}^\mathbb{P}((X_{s\wedge \cdot}, A_{s\wedge \cdot}, W, B_{s\wedge \cdot})|B^\mathbb{P}_r_{r \in [t, s]} , \hat{\mu}_{s\wedge \cdot} ) \quad = \mathcal{L}^\mathbb{P}((X_{s\wedge \cdot}, A_{s\wedge \cdot}, W, B_{s\wedge \cdot})|B^\mathbb{P}_t), \quad \mathbb{P}\)-a.s., for all \(s \in [0, T]\).
\]

A direct consequence of Lemma 4.3 is that we can now reformulate equivalently the weak/strong formulation of the McKean–Vlasov control problem on the canonical space.

**Corollary 4.4.** Let \((t, \nu) \in [0, T] \times \mathcal{P}(\mathcal{C}_0^\alpha)\), one has

\[
V_W(t, \nu) = \sup_{\mathbb{P} \in \mathcal{P}_W(t, \nu)} J(t, \mathbb{P}), \text{ where } J(t, \mathbb{P}) := \mathbb{E}^\mathbb{P}\left[\int_t^T L(s, X_{s\wedge \cdot}, \overline{\mu}_s, \overline{\alpha}_s)ds + g(X_{T\wedge \cdot}, \mu_T)\right].
\]

Moreover, let \(\nu \in \mathcal{P}_2(\mathcal{C}_0^\alpha)\) and Assumption 2.2 (resp. and Assumption 2.3) hold true, then

\[
V_S^B(t, \nu) = \sup_{\mathbb{P} \in \mathcal{P}_S^B(t, \nu)} J(t, \mathbb{P}) \quad \text{(resp. } V_S(t, \nu) = \sup_{\mathbb{P} \in \mathcal{P}_S(t, \nu)} J(t, \mathbb{P})\text{)}.
\]

**4.2. Technical lemmata.** We provide in this section some technical results related to the sets \(\overline{\mathcal{P}}_W(t, \nu)\). In the proof of the next lemma and quite often in the following, we will use without further comments the fact that given a Polish space \(E\), we can find a countable set \(\mathcal{D} \subset C_b(E)\) such that for any \((m_1, m_2) \in \mathcal{P}(E)^2\), one has \(m_1 = m_2\) if and only if \(\langle f, m_1 \rangle = \langle f, m_2 \rangle\) for all \(f \in \mathcal{D}\) (see for instance [6, Propositions 7.18, 7.19]). We will refer to \(\mathcal{D}\) as a characterising set.
Lemma 4.5. Both graph sets

\[ [\tilde{P}_W] := \{(t, \tilde{\nu}, \mathbb{P}) : \mathbb{P} \in \tilde{P}_W(t, \tilde{\nu})\} \]

and \([\mathcal{P}_W] := \{(t, \nu, \mathbb{P}) : \mathbb{P} \in \mathcal{P}_W(t, \nu)\}\),

are analytic subsets of, respectively, \([0, T] \times \mathcal{P}(\tilde{\Omega}) \times \mathcal{P}(\tilde{\Omega})\) and \([0, T] \times \mathcal{P}(C^n) \times \mathcal{P}(\Omega)\). Moreover, the value function

\[ V_W : [0, T] \times \mathcal{P}(C^n) \rightarrow \mathbb{R} \cup \{-\infty, \infty\} \]

is upper semi-analytic.

Proof. For \(0 \leq r \leq s \leq T, m \geq 1, \chi \in C_b(\tilde{\Omega}), \varphi \in C^0_b(\mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^\ell), \phi \in C_b(\tilde{\Omega}), \psi \in C_b(C^\ell \times C([0, T]; \mathcal{P}(\tilde{\Omega})))\), we define

\[ \xi_{r\wedge} := \chi(X_{r\wedge}, A_{r\wedge}, W_{r\wedge}, B_{r\wedge}, \tilde{\mu}_{r\wedge}), \]

and the following Borel-measurable subsets of \([0, T] \times \mathcal{P}(\tilde{\Omega}) \times \mathcal{P}(\tilde{\Omega})\)

\(K^1 := \{(t, \tilde{\nu}, \mathbb{P}) : \int_t^T \mathbb{P}[\pi_\theta \in U]d\theta = T - t, \mathbb{E}^\mathbb{P}[||X||^t] < \infty, \mathbb{E}^\mathbb{P}[\int_t^T (\rho(u_0, \pi_\theta))^t d\theta] < \infty\}, \)

\(K^2_{r,s,m}[\chi, \varphi] := \{(t, \tilde{\nu}, \mathbb{P}) : \mathbb{E}^\mathbb{P}[\mathbb{S}_r^{\phi,m}\xi_{r\wedge}] = \mathbb{E}^\mathbb{P}[\mathbb{S}_r^{\phi,m}\xi_{r\wedge}]\}, \)

\(K^3_2[\phi] := \{(t, \tilde{\nu}, \mathbb{P}) : \mathbb{E}^\mathbb{P}[\langle \phi, \tilde{\mu}_{t\wedge} \rangle - \mathbb{E}^\mathbb{P}[\phi(X_{t\wedge}) A_{t\wedge}, W, B_{t\wedge}]]) = 0\}, \)

and

\(K^4_2[\phi, \psi] := \{(t, \tilde{\nu}, \mathbb{P}) : \mathbb{E}^\mathbb{P}[\langle \phi, \tilde{\mu}_{t\wedge} \rangle \psi(B^t) \tilde{\mu}] = \mathbb{E}^\mathbb{P}[\phi(X_{t\wedge} A_{t\wedge}, W, B_{t\wedge} \psi(B^t) \tilde{\mu})\}. \)

The above Borel-measurable sets allow to characterise the graph set \([\tilde{P}_W]\). Indeed, \(K^1\) contains the probabilities on \(\tilde{\Omega}\) such that the canonical element \(\pi\) takes its values in \(U\) and not in \(U \cup \{\partial\}\), \(K^2_{r,s,m}[\chi, \varphi]\) reduces the set \(\mathcal{P}(\tilde{\Omega})\) to the set of probabilities on \(\tilde{\Omega}\) that solves a (local) martingale problem, while the probabilities which satisfy the ‘fixed point property’, i.e. the canonical process \(\tilde{\mu}\) is equal to the conditional distribution of canonical process \((X, A, W, B)\), are contained in \(K^3_2[\phi]\) and \(K^4_2[\phi, \psi]\).

By using a countable subset of \(C_b(\tilde{\Omega}) \times C^0_b(\mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^\ell) \times C_b(\tilde{\Omega}) \times C_b(C^\ell \times C([0, T]; \mathcal{P}(\tilde{\Omega})))\) characterising probability measures as mentioned in the beginning of Section 4.2, combined with a countable dense subset of \([0, T]^2 \times \mathbb{N}\), we consider a countable subset \(X\) of \((r, s, m, \chi, \varphi, \phi, \psi)\) in

\([0, T]^2 \times \mathbb{N} \times C_b(\tilde{\Omega}) \times C^0_b(\mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^\ell) \times C_b(\tilde{\Omega}) \times C_b(C^\ell \times C([0, T]; \mathcal{P}(\tilde{\Omega}))), \)

where \(0 \leq r \leq s \leq T\). It follows that

\[ [\tilde{P}_W] = \bigcap_{X} \{K^1[h] \cap K^2_{r,s,m}[\chi, \varphi] \cap K^3_2[\phi] \cap K^4_2[\phi, \psi]\}, \]

and hence it is a Borel subset of \([0, T] \times \mathcal{P}(\tilde{\Omega}) \times \mathcal{P}(\tilde{\Omega})\). Furthermore, since \(\mathcal{P}(\tilde{\Omega}) \ni \tilde{\nu} \mapsto \tilde{\nu} \circ (X)^{-1} \in \mathcal{P}(C^n)\) is continuous, the set

\[ [\mathcal{P}_W] = \{(t, \nu, \mathbb{P}) : (t, \tilde{\nu}, \mathbb{P}) \in [\tilde{P}_W], \tilde{\nu} \circ (X)^{-1} = \nu\}, \]
is an analytic subset of $[0, T] \times \mathcal{P}(\mathcal{C}^n) \times \mathcal{P}(\Omega)$. Finally, using the (analytic) measurable selection theorem (see e.g. El Karoui and Tan [25, Proposition 2.17]), it follows that

$$V_W(t, \nu) = \sup_{(t, \nu, \tilde{\nu}) \in \tilde{P}_W} J(t, \tilde{\nu}),$$

is upper semi-analytic as desired. \qed

We next prove a stability result w.r.t. the ‘conditioning’ of $\tilde{P}_W(t, \tilde{\nu})$.

**Lemma 4.6.** Let $(t, \tilde{\nu}) \in [0, T] \times \mathcal{P}(\tilde{\Omega})$, $\tilde{P} \in \tilde{P}_W(t, \tilde{\nu})$, $\tilde{\tau}$ be a $\mathbb{T}$-stopping time taking values in $[t, T]$, and $(\tilde{P}_1^{\tilde{\omega}})_{\tilde{\omega} \in \tilde{\Omega}}$ be a family of r.c.p.d. of $\tilde{P}$ knowing $\tilde{G}_\tau$. Then

$$\tilde{P}_1^{\tilde{\omega}} \in \tilde{P}_W(\tilde{\tau}(\tilde{\omega}), \tilde{\mu}_{\tilde{\tau}(\tilde{\omega})}(\tilde{\omega})), \text{ for } \tilde{P} \text{-a.e. } \tilde{\omega} \in \tilde{\Omega}.$$

**Proof.** Let $\tilde{P} \in \tilde{P}_W(t, \tilde{\nu})$. First, it is easy to check that for $\tilde{P}$-a.e. $\tilde{\omega} \in \tilde{\Omega}$, one has the equality $\tilde{P}_1^{\tilde{\omega}}[\tau_s \in U] = 1$, for Lebesgue–almost every $s \in [\tilde{\tau}(\tilde{\omega}), T]$, and $\mathbb{E}^{\tilde{P}_1^{\tilde{\omega}}}[\int_T^{\tilde{\tau}(\tilde{\omega})} (\rho(u_0, \tilde{\nu})) \tilde{s} \, ds] < \infty$.

Next, notice that for all $s \in [0, T]$, $Z \in \tilde{G}_\tau$, $\beta \in C_b(\mathcal{C}^n \times \mathcal{C} \times \mathcal{C}^d \times \mathcal{C})$ and $\psi \in C_b(\mathcal{C} \times \mathcal{C}([0, T], \mathcal{P}(\tilde{\Omega})))$, we have

$$\mathbb{E}^{\tilde{P}}[\langle \beta, \tilde{\mu}_s \rangle \psi(B^\tau_{\tilde{\nu}}) 1_{\tilde{\tau}(\tilde{\omega}) \leq s}] = \mathbb{E}^{\tilde{P}}[\beta(X_{s, \tilde{\omega}}, A_{s, \tilde{\omega}}, W, B_{s, \tilde{\omega}}) \psi(B^\tau_{\tilde{\nu}}) 1_{\tilde{\tau}(\tilde{\omega}) \leq s}],$$

so that, for $\tilde{P}$-a.e. $\tilde{\omega} \in \tilde{\Omega}$ and any $\tilde{\tau}(\tilde{\omega}) \leq s \leq T$

$$\mathbb{E}^{\tilde{P}_1^{\tilde{\omega}}}[\langle \beta, \tilde{\mu}_s \rangle \psi(B^\tau_{\tilde{\nu}})] = \mathbb{E}^{\tilde{P}_1^{\tilde{\omega}}}[\beta(X_{s, \tilde{\omega}}, A_{s, \tilde{\omega}}, W, B_{s, \tilde{\omega}}) \psi(B^\tau_{\tilde{\nu}})].$$

By considering a characterising set (see the beginning of Section 4.2) of $(\beta, \psi) \in C_b(\mathcal{C}^n \times \mathcal{C} \times \mathcal{C}^d \times \mathcal{C} \times \mathcal{C}([0, T], \mathcal{P}(\tilde{\Omega})))$ it follows that

$$\tilde{\mu}_s = \mathcal{L}^{\tilde{P}_1^{\tilde{\omega}}}(X_{s, \tilde{\omega}}, A_{s, \tilde{\omega}}, W, B_{s, \tilde{\omega}})|_{\tilde{G}^{\tilde{\tau}(\tilde{\omega})}}, \text{ for all } s \geq \tilde{\tau}(\tilde{\omega}), \text{ for } \tilde{P} \text{-a.e. } \tilde{\omega} \in \tilde{\Omega}.$$

Similarly, one can prove that, for $\tilde{P}$-a.e. $\tilde{\omega}$, and $s \leq \tilde{\tau}(\tilde{\omega})$, $\tilde{P}_1^{\tilde{\omega}}$-a.s.,

$$\tilde{\mu}_s = \mathcal{L}^{\tilde{P}_1^{\tilde{\omega}}}(X_{s, \tilde{\omega}}, A_{s, \tilde{\omega}}, W, B_{s, \tilde{\omega}}),$$

and hence, for $\tilde{P}$-a.e. $\tilde{\omega} \in \tilde{\Omega}$, $\tilde{P}_1^{\tilde{\omega}}$-a.s.

$$\tilde{\mu}_s = \mathcal{L}^{\tilde{P}_1^{\tilde{\omega}}}(X_{s, \tilde{\omega}}, A_{s, \tilde{\omega}}, W, B_{s, \tilde{\omega}}) 1_{\{s \in [0, \tilde{\tau}(\tilde{\omega})]\}} + \mathcal{L}^{\tilde{P}_1^{\tilde{\omega}}}(X_{s, \tilde{\omega}}, A_{s, \tilde{\omega}}, W, B_{s, \tilde{\omega}})|_{\tilde{G}^{\tilde{\tau}(\tilde{\omega})}} 1_{\{s \in (\tilde{\tau}(\tilde{\omega}), T]\}}.$$

Finally, it is clear that for $\tilde{P}$-a.e. $\tilde{\omega}$, one has $\mathbb{E}^{\tilde{P}_1^{\tilde{\omega}}}$ $\|X_{\tilde{\tau}(\tilde{\omega})}\| < \infty$ and $\mathbb{P}^{\tilde{P}_1^{\tilde{\omega}}}$ $\|S_{\tilde{\tau}(\tilde{\omega})}\| < \infty = 1$. Moreover, let $\varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^\ell)$, so that the localised process $S^\varphi_{\tilde{\tau}(\tilde{\omega})} = S^\varphi_{\tilde{\tau}(\tilde{\omega})}$ is an $(\tilde{P}, \tilde{P})$-martingale on $[t, T]$. Fix $T \geq r > s \geq t$, $J \in \tilde{F}_s$ and $K \in \tilde{G}_\tau$, we have

$$\mathbb{E}^{\tilde{P}}[\mathbb{E}^{\tilde{P}_1^{\tilde{\omega}}}[S^\varphi_{\tilde{\tau}(\tilde{\omega})} 1_{J} 1_{K \cap (s \leq \tilde{\tau}(\tilde{\omega}))}] = \mathbb{E}^{\tilde{P}}[S^\varphi_{\tilde{\tau}(\tilde{\omega})} 1_{J} 1_{K \cap (s \leq \tilde{\tau}(\tilde{\omega}))}]$$

$$= \mathbb{E}^{\tilde{P}}[S^\varphi_{\tilde{\tau}(\tilde{\omega})} 1_{J} 1_{K \cap (s \leq \tilde{\tau}(\tilde{\omega}))}] = \mathbb{E}^{\tilde{P}_1^{\tilde{\omega}}}[S^\varphi_{\tilde{\tau}(\tilde{\omega})} 1_{J} 1_{K \cap (s \leq \tilde{\tau}(\tilde{\omega}))}].$$
Moreover, as in Lemma 4.5, the graph set
\[ \mathcal{P}_{t}^{M} := \left\{ \mathbb{P} \in \mathcal{P}(\bar{\Omega}) : \mathbb{E}^{\mathbb{P}} \left[ \int_{t}^{T} (\rho(u_{s}, x_{s}))^{p} ds \right] \leq M \right\}, \]
and
\[ V_{M}^{W}(t, \nu) := \mathcal{P}_{W}(t, \nu) \cap \mathcal{P}_{t}^{M}, \]
and
\[ V_{M}^{M}(t, \nu) := \sup_{\mathbb{P} \in \mathcal{P}_{W}(t, \nu)} J(t, \mathbb{P}). \]
Notice that
\[ (t, \nu, M) \rightarrow V_{M}^{W}(t, \nu) \in \mathbb{R} \cup \{-\infty, \infty\} \]
is upper semi-analytic.

**Lemma 4.7.** Let \( t \in [0, T], (\tilde{\nu}_{1}, \tilde{\nu}_{2}) \in \mathcal{P}(\bar{\Omega}) \times \mathcal{P}(\bar{\Omega}), \) and \( \nu \in \mathcal{P}(\mathbb{C}^{n}) \) be such that \( \tilde{\nu}_{1} \circ X_{t_{\nu_{1}}}^{-1} = \tilde{\nu}_{2} \circ X_{t_{\nu_{2}}}^{-1} = \nu(t) \). Then for all \( \mathbb{P}_{1} \in \mathcal{P}_{W}(t, \tilde{\nu}_{1}), \) there exists \( \mathbb{P}_{2} \in \mathcal{P}_{W}(t, \tilde{\nu}_{2}) \)
satisfying
\[ \mathbb{P}_{1} \circ (X, A^{t}, W^{t}, B^{t})^{-1} = \mathbb{P}_{2} \circ (X, A^{t}, W^{t}, B^{t})^{-1}, \]
where \( A^{t} := A_{\nu_{1}} - A_{t}, \) so that \( J(t, \mathbb{P}_{1}) = J(t, \mathbb{P}_{2}). \)
Consequently, one has
\[ V_{W}(t, \nu) = \sup_{\mathbb{P} \in \mathcal{P}_{W}(t, \tilde{\nu}_{1})} J(t, \mathbb{P}), \]
and
\[ V_{M}^{W}(t, \nu) = \sup_{\mathbb{P} \in \mathcal{P}_{W}(t, \tilde{\nu}_{1})} J(t, \mathbb{P}). \]

The proof is almost the same as that of Lemma 3.5, and hence it is omitted.

**Lemma 4.8.** Let \( (t, \nu) \in [0, T] \times \mathcal{P}(\mathbb{C}^{n}), \mathbb{P} \in \mathcal{P}_{W}(t, \nu), \tau \) be a \( \mathcal{G}_{t}^{\nu} \)-stopping time taking values in \([t, T], \) and \( \varepsilon > 0. \) Then there exists a family of probability measures \( \left( \mathbb{Q}_{t, \nu, M} \right)_{(t, \nu, M) \in [0, T] \times \mathcal{P}(\bar{\Omega}) \times \mathbb{R}_{+}} \) such that \( (t, \nu, M) \mapsto \mathbb{Q}_{t, \nu, M} \) is universally measurable, and for every \( (t, \nu, M) \) such that \( \mathcal{P}_{W}^{M}(t, \tilde{\nu}) \neq \emptyset, \) one has, with \( \nu := \tilde{\nu} \circ \hat{X}^{-1}, \)
\[ (36) \quad \mathbb{Q}_{t, \nu, M} \in \mathcal{P}_{W}^{M}(t, \tilde{\nu}), \]
and
\[ J(t, \mathbb{Q}_{t, \nu, M}) \geq \begin{cases} V_{W}^{M}(t, \nu) - \varepsilon, & \text{when } V_{W}^{M}(t, \nu) < \infty, \\ 1/\varepsilon, & \text{when } V_{W}^{M}(t, \nu) = \infty. \end{cases} \]

Furthermore, there exists a \( \mathbb{P} \)-integrable, \( \mathcal{G}_{t}^{\nu} \)-measurable random variable \( \hat{M} : \bar{\Omega} \rightarrow \mathbb{R}_{+} \) such that for all constant \( M \geq 0, \) one can find a probability measure \( \mathbb{P}_{M, \varepsilon}^{\nu} \in \mathcal{P}_{W}(t, \nu) \)
satisfying \( \mathbb{P}_{M, \varepsilon}^{\nu} \big|_{\mathcal{F}_{t}} = \mathbb{P} \big|_{\mathcal{F}_{t}} \) and
\[ (37) \quad \mathbb{Q}_{t, \nu, M} \in \mathcal{P}_{W}^{M, \varepsilon}(t, \tilde{\nu}), \]
where \( \mathbb{Q}_{t, \nu, M} \) is a version of the r.c.p.d. of \( \mathbb{P}_{M, \varepsilon}^{\nu} \) knowing \( \mathcal{G}_{t}^{\nu}. \)
Proof. The existence of the family of probability measures
\[ ((\mathbb{Q}_t^{p,M})_{(t,p,M) \in [0,T] \times \mathbb{R}_+})_{\omega \in \Omega}, \]
satisfying (36) follows by (35) and Lemma 4.7, together with the measurable selection theorem (see e.g. [25, Proposition 2.21]).

With \( \mathbb{P} \in \mathbb{P}_W(t, \nu) \), we consider a family of r.c.p.d. \( (\mathbb{P}_\omega)_{\omega \in \Omega} \) of \( \mathbb{P} \) knowing \( \mathbb{G}_t^\omega \), and define
\[ \hat{M}(\omega) := \mathbb{E}^{\mathbb{P}_\omega} \left[ \|X\|^p + \int_0^T (\rho(\bar{\pi}_s, u_0))^p ds \right], \]
so that \( \mathbb{P}_\omega \in \mathbb{P}_W^M(\tau(\omega), \bar{\mu}_\omega(\omega)) \) for \( \mathbb{P} \)-a.e. \( \omega \), by Lemma 4.6. In particular, \( \hat{M}(\omega) = \mathbb{P}_\omega \circ (X_{\tau(\omega)\lambda}, A_{\tau(\omega)\lambda}, W, B_{\tau(\omega)\lambda})^{-1} = \mathbb{P}_\omega \circ (X_{\tau(\omega)\lambda}, A_{\tau(\omega)\lambda}, W, B_{\tau(\omega)\lambda})^{-1} \)
so that
\[ \mathcal{L}^\mathbb{P}_\omega (X_{\tau(\omega)\lambda}, A_{\tau(\omega)\lambda}, W_{\tau(\omega)\lambda}, B_{\tau(\omega)\lambda}) = \mathcal{L}^\mathbb{P}_\omega (X_{\tau(\omega)\lambda}, A_{\tau(\omega)\lambda}, W_{\tau(\omega)\lambda}, B_{\tau(\omega)\lambda}) \otimes \mathcal{L}^\mathbb{P}_\omega (\bar{\mu}_{\tau(\omega)\lambda}) \]
\[ = \mathcal{L}^{\mathbb{P}_\omega} (X_{\tau(\omega)\lambda}, A_{\tau(\omega)\lambda}, W_{\tau(\omega)\lambda}, B_{\tau(\omega)\lambda}, \bar{\mu}_{\tau(\omega)\lambda}). \]
In particular, one has
\[ \mathcal{Q}_\omega [B^t_{\tau(\omega)\lambda} = (\bar{\omega}^t_{\tau(\omega)\lambda}, \bar{\mu}_{\tau(\omega)\lambda} = \bar{\omega}^t_{\tau(\omega)\lambda} \epsilon 1,] \text{ for } \mathbb{P} \text{-a.e. } \omega = (\bar{\omega}^x, \bar{\omega}^a, \bar{\omega}^w, \bar{\omega}^h, \bar{\omega}^p) \in \Omega. \]

Let us then define a probability measure \( \mathbb{P}^{M, \epsilon} \) on \( \Omega \) by
\[ \mathbb{P}^{M, \epsilon}[K] := \int_\Omega \mathbb{Q}_\omega (K) \mathbb{P}(d\omega), \text{ for all } K \in \mathcal{F}. \]
By (37), one has \( \mathbb{P}^{M, \epsilon} = \mathbb{P} \) on \( \mathcal{F}_t \), and moreover, \( (\mathbb{Q}_\omega)_{\omega \in \Omega} \) is a family of r.c.p.d. of \( \mathbb{P}^{M, \epsilon} \) knowing \( \mathbb{G}_t^\omega \). To conclude the proof, it is enough to check that \( \mathbb{P}^{M, \epsilon} \in \mathbb{P}_W(t, \nu) \). First, it is clear that \( \mathbb{P}^{M, \epsilon}[\bar{\pi}_s \in U] = 1 \), for Lebesgue-almost every \( s \in [t, T] \), and
\[ \mathbb{E}^{\mathbb{P}^{M, \epsilon}} \left[ \int_t^T (\rho(u_0, \bar{\pi}_s))^p ds \right] \leq \mathbb{E}^{\mathbb{P}^{M, \epsilon}} \left[ \int_t^T (\rho(u_0, \bar{\pi}_s))^p ds \right] + \mathbb{E}^{\mathbb{P}}[\hat{M}] < \infty. \]

Next, for each \( \beta \in C_b(\bar{\Omega}), \psi \in C_b(C^\nu \times \mathcal{P}(\bar{\Omega})), h \in C_b(\mathbb{C}^\lambda) \) and \( s \in [t, T] \), one has
\[ \mathbb{E}^{\mathbb{P}^{M, \epsilon}} \left[ \beta(X_{s \lambda}, A_{s \lambda}, W, B_{s \lambda} \lambda) \psi(B_{s \lambda}^\mu, \bar{\mu}) h(B_{s \lambda}^\mu, \bar{\mu}) \right] \]
\[ = \int_\omega \mathbb{E}^{\mathbb{P}_\omega} \left[ \beta(X_{s \lambda}, A_{s \lambda}, W, B_{s \lambda} \lambda) \psi(B_{s \lambda}^\mu, \bar{\mu}) h(B_{s \lambda}^\mu, \bar{\mu}) \right] \mathbb{P}(d\omega) \]
\[ = \int_\omega \mathbb{E}^{\mathbb{P}_\omega} \left[ \beta(X_{s \lambda}, A_{s \lambda}, W, B_{s \lambda} \lambda) \psi(B_{s \lambda}^\mu, \bar{\mu}) h(B_{s \lambda}^\mu, \bar{\mu})(\omega) \right] \mathbb{P}(d\omega) \]
\[ = \int_\omega \mathbb{E}^{\mathbb{P}_\omega} \left[ \mathbb{E}^{\mathbb{P}_\omega} \left[ \beta(X_{s \lambda}, A_{s \lambda}, W, B_{s \lambda} \lambda) (\mathbb{G}_t^\omega)^T \right] \psi(B_{s \lambda}^\mu, \bar{\mu}) h(B_{s \lambda}^\mu, \bar{\mu})(\omega) \right] \mathbb{P}(d\omega) \]
\[ = \int_\omega \mathbb{E}^{\mathbb{P}_\omega} \left[ \left( \beta, \bar{\mu}_s \psi(B_{s \lambda}^\mu, \bar{\mu}) \right) h(B_{s \lambda}^\mu, \bar{\mu})(\omega) \right] \mathbb{P}(d\omega) \]
\[
E^\mathbb{P}_{\omega} \left[ \langle \beta, \check{\mu}_s \rangle \psi(B^s, \check{\mu}) h(B^s_{t \land \bar{\tau}}) \right] \mathbb{P}(d\omega)
\]

where the second and fifth equalities are due to Equation (38), and the fourth follows by the fact that \( \mathcal{Q}^{\varepsilon}_{\omega} \in \mathcal{P}_W(\bar{\tau}(\omega), \check{\mu}_{\bar{\tau}(\omega)}(\omega)) \). Notice that \( B^u_t = B^u_{t \land \bar{\tau}} + B^\varepsilon_u \) for any \( u \in [t, T] \), so that the above equality implies that

\[
\hat{\mu}_s = \mathcal{L}^{M,\varepsilon} \left( X_{s \land \bar{\tau}}(s, \check{\mu}_s, W, B_{s \land \bar{\tau}}(\mathcal{G}_T) \right), \mathbb{P}^{M,\varepsilon} - \text{a.s.}
\]

Finally, we easily check that \( \mathbb{P}^{M,\varepsilon}[|S|_T < \infty] = 1 \) and \( \mathbb{E}^{\mathbb{P}^{M,\varepsilon}}[||X||^p] \leq \mathbb{E}^{\mathbb{P}}[||X||^p] + M < \infty \).

For a fixed test function \( \varphi \in C^2_{\text{c}}(\mathbb{R}^{n+d+\ell}) \), we consider the localised stopping times \( \tau_m \) defined in (29) and \( \tau^\omega_k(\bar{\tau}) := \bar{\tau}(\omega) \vee \tau_k(\omega') \) for each \( \omega \in \Omega \). We know that \( \tau^\omega_k \sim \tau^\omega_{k+1} \), for any \( k \in \mathbb{N} \), that \( \tau^\omega_k \xrightarrow{k \to \infty} \infty \), and that \( (S^\varphi_{s \land \tau^\varphi_k})_{s \in [\tau(\omega), T]} \) is an \( (\mathbb{F}, \mathcal{Q}^{\varepsilon}_{\omega}) \)-martingale for all \( k \in \mathbb{N} \). Notice that for all \( s \in [t, T] \) and \( A \) in \( \mathcal{F}_s \), the map

\[
\omega \mapsto \mathbb{E}^{\mathbb{Q}^{\varepsilon}_{\omega}} \left[ S^\varphi_{s \land \tau^\varphi_k} A \right] \text{ is } \mathcal{G}^{\varphi}_{s \land \tau^\varphi_k}-\text{measurable.}
\]

Then for \( s \leq r \leq T \)

\[
E^{M,\varepsilon}_{S_{s \land \tau^\varphi_k}} A = E^{M,\varepsilon}_{S_{s \land \tau^\varphi_k}} 1_{A_{s \leq r}} + E^{M,\varepsilon}_{S_{s \land \tau^\varphi_k}} 1_{A_{s > r}}
\]

\[
E^{M,\varepsilon}_{S_{s \land \tau^\varphi_k}} A_{s \leq r} = E^{M,\varepsilon}_{S_{s \land \tau^\varphi_k}} 1_{A_{s \leq r}} + E^{M,\varepsilon}_{S_{s \land \tau^\varphi_k}} 1_{A_{s > r}}
\]

\[
E^{M,\varepsilon}_{S_{s \land \tau^\varphi_k}} A_{s > r} = E^{M,\varepsilon}_{S_{s \land \tau^\varphi_k}} 1_{A_{s \leq r}} + E^{M,\varepsilon}_{S_{s \land \tau^\varphi_k}} 1_{A_{s > r}}
\]

which means that \( (S^\varphi_u)_{u \in [t, T]} \) is an \( (\mathbb{F}, \mathbb{P}^{M,\varepsilon}) \)-local martingale, and hence \( \mathbb{P}^{M,\varepsilon} \in \mathcal{P}_W(t, \nu) \). \( \square \)

4.3. Proof of the main results.

4.3.1. Proof of Theorem 3.1. First, \( V_W \) is upper semi-analytic by Lemma 4.5.

Further, let \( \bar{\tau} \) be a \( \mathcal{G}^{\varphi}_{s \land \tau^\varphi_k} \)-stopping time taking value in \([t, T]\), which can be considered as a \( \mathcal{G}^\varphi \)-stopping time \( \tau^\varphi \) on \( \Omega^\varphi \). By the way how \( \tau^\varphi \) is defined from \( \tau^\varphi \) in (15) and Lemma 4.3, to prove the DPP result in (16), it is equivalent to prove that

\[
V_W(t, \nu) = \sup_{\mathbb{P} \in \mathcal{P}_W(t, \nu)} \mathbb{E}^{\mathbb{P}} \left[ \int^\bar{\tau}_t L(s, X_{s \land \tau^\varphi}, \check{\mu}_s, \check{\alpha}_s) ds + V_W(\bar{\tau}, \varphi) \right].
\]

Using Lemma 4.6, it follows that, for every \( \mathbb{P} \in \mathcal{P}_W(t, \nu) \)

\[
J(t, \mathbb{P}) = \mathbb{E}^{\mathbb{P}} \left[ \int^\bar{\tau}_t L(s, X_{s \land \tau^\varphi}, \check{\mu}_s, \check{\alpha}_s) ds + \mathbb{E}^{\mathbb{P}} \left[ \int^T_{\bar{\tau}} L(s, X_{s \land \tau^\varphi}, \check{\mu}_s, \check{\alpha}_s) ds + g(X_{T \land \tau^\varphi}) \right] \right]
\]
\[
\leq \mathbb{E}^F \left[ \int_t^\tau L(s, X_{s\wedge \cdot}, \bar{\mu}_s, \bar{\pi}_s) ds + V_W(\bar{\tau}, \mu_\tau) \right]
\]
\[
\leq \sup_{\mathbb{F} \in \mathcal{F}_W(t, \nu)} \mathbb{E}^{\mathbb{F}} \left[ \int_t^\tau L(s, X_{s\wedge \cdot}, \bar{\mu}_s, \bar{\pi}_s) ds + V_W(\bar{\tau}, \mu_\tau) \right].
\]

Then
\[
(40) \quad V_W(t, \nu) \leq \sup_{\mathbb{F} \in \mathcal{F}_W(t, \nu)} \mathbb{E}^{\mathbb{F}} \left[ \int_t^\tau L(s, X_{s\wedge \cdot}, \bar{\mu}_s, \bar{\pi}_s) ds + V_W(\bar{\tau}, \mu_\tau) \right].
\]

We now consider the reverse inequality, for which one can assume w.l.o.g. that
\[
(41) \quad V_W(t, \nu) < \infty, \quad \text{and} \quad \sup_{\mathbb{F} \in \mathcal{F}_W(t, \nu)} \mathbb{E}^{\mathbb{F}} \left[ \int_t^\tau L(s, X_{s\wedge \cdot}, \bar{\mu}_s, \bar{\pi}_s) ds + V_W(\bar{\tau}, \mu_\tau) \right] > -\infty.
\]

Let \( \mathbb{F} \in \mathcal{F}_W(t, \nu) \) be a weak control rule, then by Lemma 4.8, for some \( \mathcal{F}_\mu \)-measurable \( \mathbb{F} \)-integrable r.v. \( \bar{M} : \bar{\Omega} \rightarrow \mathbb{R}_+ \), one has a family of probability measures \( \{\mathbb{P}^{\bar{M}, \varepsilon}\}_{\bar{M} \geq 0} \) in \( \mathcal{F}_W(t, \nu) \) such that
\[
\mathbb{E}^{\bar{\mathbb{P}}} \left[ \int_t^\tau L(s, X_{s\wedge \cdot}, \bar{\mu}_s, \bar{\pi}_s) ds + \left( V^M_{W+\bar{M}}(\bar{\tau}, \mu_\tau) - \varepsilon \right) 1_{\{V^M_{W+\bar{M}}(\bar{\tau}, \mu_\tau) < \infty\}} \right]
\]
\[
\leq \mathbb{E}^{\bar{\mathbb{P}}} \left[ \int_t^\tau L(s, X_{s\wedge \cdot}, \bar{\mu}_s, \bar{\pi}_s) ds + \mathbb{E}^{\bar{\mathbb{P}}, \bar{\mu}, \bar{M}} \left[ \int_t^\tau L(s, X_{s\wedge \cdot}, \bar{\mu}_s, \bar{\pi}_s) ds + g(X_{T\wedge \cdot}, \mu_T) \right] \right]
\]
\[
= \mathbb{E}^{M, \varepsilon} \left[ \int_t^\tau L(s, X_{s\wedge \cdot}, \bar{\mu}_s, \bar{\pi}_s) ds + g(X_{T\wedge \cdot}, \mu_T) \right] \leq V_W(t, \nu).
\]

If \( \mathbb{P}^{V^M_{W+\bar{M}}(\bar{\tau}, \mu_\tau) = \infty}(\infty) > 0 \) for some \( M \geq 0 \), then by taking \( \varepsilon \rightarrow 0 \), one finds \( V_W(t, \nu) = \infty \) which is in contradiction to (41). When \( \mathbb{P}^{V^M_{W+\bar{M}}(\bar{\tau}, \mu_\tau) = \infty} = 0 \) for all \( M \geq 0 \), we let \( M \rightarrow \infty \), so that \( V^M_{W+\bar{M}} \) increases to \( V_W \) pointwise by (34). Then by monotone convergence theorem, one obtains that
\[
\mathbb{E}^\mathbb{F} \left[ \int_t^\tau L(s, X_{s\wedge \cdot}, \bar{\mu}_s, \bar{\pi}_s) ds + \lim_{M \rightarrow \infty} \left( V^M_{W+\bar{M}}(\bar{\tau}, \mu_\tau) - \varepsilon \right) 1_{\{V^M_{W+\bar{M}}(\bar{\tau}, \mu_\tau) < \infty\}} \right] \leq V_W(t, \nu).
\]

Take the supremum over all \( \mathbb{F} \in \mathcal{F}_W(t, \nu) \), it follows that
\[
\sup_{\mathbb{F} \in \mathcal{F}_W(t, \nu)} \mathbb{E}^\mathbb{F} \left[ \int_t^\tau L(s, X_{s\wedge \cdot}, \bar{\mu}_s, \bar{\pi}_s) ds + V_W(\bar{\tau}, \mu_\tau) \right] - \varepsilon \leq V_W(t, \nu).
\]

Notice that \( \varepsilon > 0 \) is arbitrary, and again by the way how \( \tau^\gamma \) is defined from \( \tau^* \) (equivalent to \( \bar{\tau} \) on \( \bar{\Omega} \)) and Lemma 4.3, we can conclude the proof with (40).

4.3.2. Proof of Theorem 3.3. Let \((t, \nu) \in [0,T] \times \mathcal{P}_2(C^n)\) and Assumption 2.3 hold, by [23, Theorem 3.1] (letting \( \hat{p} = p = 2 \) in their Assumption 2.1), one has
\[
V_S(t, \nu) = V_W(t, \nu).
\]

Therefore \( V_S : [0,T] \times \mathcal{P}_2(C^n) \rightarrow \mathbb{R} \cup \{-\infty, \infty\} \) has the same measurability as \( V_W : [0,T] \times \mathcal{P}_2(C^n) \rightarrow \mathbb{R} \cup \{-\infty, \infty\} \).
Next, let \( \tau \) be a \( \mathcal{G}^{t,\alpha} \)-stopping time on \((\Omega,\mathcal{F},\mathbb{P})\) taking values in \([t,T]\), we denote \( \tau^\gamma := \tau(B^t) \) for each \( \gamma \in \Gamma_W(t,\nu) \). Then using Proposition 2.5, Lemma 4.6 and the fact that \( V_S = V_W \), it follows that

\[
V_S(t,\nu) \leq \sup_{\tau \in \Gamma_W(t,\nu)} \mathbb{E}^\mathbb{P} \left[ \int_t^\tau L(s, X_s^{t,\xi,\alpha}, \overline{p}^{t,\xi,\alpha}, \alpha_s) \, ds + V_W(\tau^\gamma, \mu^\gamma) \right]
\]

Further, we denote

\[
\Gamma_S(t,\nu) := \{ \gamma = (\Omega, \mathcal{F}, \mathbb{P}, \mathcal{G}^t, X^{t,\xi,\alpha}, W^t, B^t, \overline{p}^{t,\xi,\alpha}, \mu^{t,\xi,\alpha}, \alpha) : \zeta \in \mathcal{I}(\nu), \alpha \in \mathcal{A}_t \},
\]

which clearly satisfies that \( \Gamma_S(t,\nu) \subset \Gamma_W(t,\nu) \). Then by Theorem 3.1, we have

\[
V_S(t,\nu) = V_W(t,\nu) = \sup_{\gamma \in \Gamma_W(t,\nu)} \mathbb{E}^\mathbb{P} \left[ \int_t^\tau L(s, X_s^{t,\xi,\alpha}, \overline{p}^{t,\xi,\alpha}, \alpha_s) \, ds + V_W(\tau^\gamma, \mu^\gamma) \right]
\]

4.3.3. Proof of Theorem 3.2. In this part, we use the results and techniques of Theorem 3.1 to show the DPP for \( V_S^B \). We start by proving the universal measurability of \( V_S^B \). For this, we consider an equivalent formulation of \( V_S^B \), which is more appropriate for our purpose.

An equivalent reformulation for \( V_S^B \). Let \( \tilde{\Omega^*} := \mathcal{C}^t \) be the canonical space with canonical process \( \tilde{B}^t \), and \( \tilde{\mathbb{P}^*} \) be the Wiener measure, under which \( \tilde{B}^t \) is an \( \ell \)-dimensional standard Brownian motion. Let \( \tilde{\mathbb{F}^*} = (\tilde{\mathcal{F}}^t)_{t \in [0,T]} \) be the canonical filtration. Recall that we consider a fixed Borel map \( \pi : U \cup \{ \emptyset \} \to \mathbb{R} \cup \{-\infty, \infty\} \). We denote by \( \mathcal{U} \) the set of \( \mathbb{P}^* \)-predictable processes \( \theta \) taking values in \( \mathbb{R} \), such that \( \mathbb{E}^{\mathbb{P}^*} \left[ \int_0^T |\theta_t|^2 \, dt \right] < \infty \). Define a metric \( d^* \) on \( \mathcal{U} \) by

\[
d^*(\eta, \theta)^2 := \mathbb{E}^{\tilde{\mathbb{P}^*}} \left[ \int_0^T |\eta_t - \theta_t|^2 \, dt \right], \text{ for all } (\eta, \theta) \in \mathcal{U} \times \mathcal{U},
\]

so that \((\mathcal{U}, d^*)\) is a Polish space (see e.g. Brezis [10, Theorems 4.8 and 4.13]). Next, let \( \theta \in \mathcal{U} \), and define \( A^t_\theta := \int_0^t \theta_s \, ds, t \in [0,T] \). We consider then the map \( \Upsilon : \mathcal{U} \to \mathcal{P}(\mathcal{C}^t \times \mathcal{C}) \)
defined by
\[ \Upsilon(\theta) := \bar{\mathbb{P}}^* \circ \left( \bar{B}^*, A^{\theta(\bar{B}^*)} \right)^{-1}, \theta \in \mathcal{U}. \]

Let us introduce, for all \( t \in [0,T] \), \( \nu \in \mathcal{P}_2(C^n) \) and \( \bar{\nu} \in \mathcal{P}_2(\bar{\Omega}) \) such that \( \nu = \bar{\nu} \circ \hat{X}^{-1} \)
\[ \mathcal{P}^*_S(t,\nu) := \{ \mathbb{P} \in \mathcal{P}_W(t,\nu) : \mathbb{P} \circ (B_t, A)\} \subseteq \Upsilon(\mathcal{U}), \text{ and } B_{t_A}, \mathbb{P}-\text{independent of } (B^t, A), \]
and
\[ \mathcal{P}^*_S(t,\bar{\nu}) := \mathcal{P}_W(t,\bar{\nu}) \cap \mathcal{P}^*_S(t,\nu). \]

**Lemma 4.9.** Let \((t,\nu) \in [0,T] \times \mathcal{P}_2(C^n) \) and \( \bar{\nu} \in \mathcal{P}(\bar{\Omega}) \) be such that \( \bar{\nu} \circ \hat{X}^{-1} = \nu \).
Then under Assumption 2.2, one has \( \mathcal{P}^*_S(t,\nu) \subseteq \mathcal{P}^{\bar{\nu}}_S(t,\nu) \) and
\[ V^\gamma_S(t,\nu) = \sup_{\mathbb{P} \in \mathcal{P}^*_S(t,\nu)} J(t,\mathbb{P}). \]

**Proof.** First, take \( \gamma \in \Gamma^{\bar{\nu}}_S(t,\nu) \). W.l.o.g., we can assume that there exists an independent Brownian motion \( \bar{B} \) in the space \((\mathcal{V}^\gamma, \mathcal{F}^\gamma, \mathbb{P}^\gamma)\), and let \( B^{\gamma,\gamma} := B^\gamma_{t_A} - B^\gamma_t + \bar{B}, \) then
\[ \gamma' := (\Omega^\gamma, \mathcal{F}^\gamma, \mathbb{P}^\gamma, \mathbb{G}^\gamma, X^\gamma, \mathcal{W}^\gamma, B^{\gamma,\gamma}, \mu^\gamma, \alpha^\gamma) \in \mathcal{P}^{\bar{\nu}}_S(t,\nu). \]

Recall that \( \alpha^\gamma \) is \( \mathcal{G}^\gamma \)-predictable and \( \mathcal{G}^\gamma \) is the augmented filtration generated by \( B^{\gamma,\gamma} \), then for some Borel function \( \phi : [t,T] \times \mathcal{C}^d \rightarrow \mathcal{U} \), one has \( \alpha^\gamma_s = \phi(s, B^{\gamma,\gamma}_s), s \in [t,T], \) \( B^{\gamma,\gamma} \)-a.s. Let
\[ A^\gamma := \int_0^T \pi(\phi(s, B^{\gamma,\gamma}_s))ds \text{ and } \bar{\mu}^\gamma \text{ be defined as in (7), it follows that } \mathbb{P} := \mathbb{P}^\gamma \circ \mathcal{P}^\gamma, A^\gamma, W^\gamma, B^{\gamma,\gamma}, \bar{\mu}^\gamma, \alpha^\gamma \rightarrow \mathcal{P}^*_S(t,\nu), \]
\( B^\gamma_{t_A}, \mathbb{P}-\text{independent of } (B^t, A), \) then \( J(t,\gamma) = J(t,\mathbb{P}) \leq \sup_{\mathbb{P} \in \mathcal{P}^*_S(t,\nu)} J(t,\mathbb{P}) \) and hence \( V^\gamma_S(t,\nu) \leq \sup_{\mathbb{P} \in \mathcal{P}^*_S(t,\nu)} J(t,\mathbb{P}). \)

Next, given \( \mathbb{P} \in \mathcal{P}^*_S(t,\nu) \), since \( \mathbb{P} \circ (B, A)^{-1} \in \Upsilon(\mathcal{U}) \), there exists \( \theta^* \in \mathcal{U} \) such that
\[ \mathbb{P} \circ (B, A)^{-1} = \theta^* \circ \left( B^*, A^{\theta^*(\bar{B}^*)} \right)^{-1}. \]
Thus \( \pi(\mathbb{P}_s(\omega)) = \theta^*(B^*_s, \omega) \), for \( \mathbb{P}\otimes dt \)-a.e. \((s, \omega) \in [t,T] \times \bar{\Omega} \). As \( \mathbb{P} \in \mathcal{P}_W(t,\nu) \), we know \( \mathbb{P}[\mathbb{P}_s \in U] = 1 \) for \( dt \)-a.e. \( s \in [0,T] \), therefore \( \pi(\mathbb{P}_s(\omega)) = \theta^*_s(B^*_s, \omega) \in \pi(U) \) and \( \mathbb{P}[\mathbb{P}_s \in U] = \pi^{-1}(\mathbb{P}[\mathbb{P}_s \in U]) \) \( \subseteq \mathcal{P}_W(t,\nu) \), for \( \mathbb{P} \otimes dt \)-a.e. \((s, \omega) \in [0,T] \times \bar{\Omega} \). Further, since \( (B^t, A) \) is \( \mathbb{P}-\text{independent of } B_{t_A}, \) it follows that there is a Borel-measurable function \( \phi : [0,T] \times \mathcal{C}^d \rightarrow \mathbb{R} \) such that \( A_s = \phi(s, B^t_s, \omega) \), \( s \in [0,T], \) \( \mathbb{P}-\text{a.s.} \), and therefore \( \mathbb{P} \in \mathcal{P}^{\bar{\nu}}_S(t,\nu) \). This implies that \( \mathcal{P}^*_S(t,\nu) \subseteq \mathcal{P}^{\bar{\nu}}_S(t,\nu) \), and the equality (43).

We are now ready to prove the measurability of \( V^\gamma_S \).

**Lemma 4.10.** The graph sets
\[ \mathcal{P}^*_S := \{ (t,\nu, \mathbb{P}) \in [0,T] \times \mathcal{P}_2(C^n) \times \mathcal{P}(\bar{\Omega}) : \mathbb{P} \in \mathcal{P}^*_S(t,\nu) \}, \]
and \( \mathcal{P}^{\bar{\nu}}_S := \{ (t,\bar{\nu}, \mathbb{P}) \in \mathcal{P}^{\bar{\nu}}_S(t,\nu) \} \) are analytic sets in respectively \([0,T] \times \mathcal{P}_2(C^n) \times \mathcal{P}(\bar{\Omega}) \) and \([0,T] \times \mathcal{P}_2(\bar{\Omega}) \times \mathcal{P}(\bar{\Omega}) \). Consequently, \( V^\gamma_S : [0,T] \times \mathcal{P}_2(C^n) \rightarrow \mathbb{R} \cup \{-\infty, \infty\} \) is upper semi-analytic.

**Proof.** We will only consider the case of \( \mathcal{P}^*_S \), while the proof is almost the same for \( \mathcal{P}^{\bar{\nu}}_S \). First, notice that
\[ \Upsilon : \mathcal{U} \rightarrow \mathcal{P}(\mathcal{C}^d \times \mathcal{C}), \]
is continuous and injective, so that \( \Upsilon(\mathcal{U}) \) is a Borel subset of \( \mathcal{P}(\mathcal{C}^\ell \times \mathcal{C}) \) (see e.g. Kechris [30, Theorem 15.1]). It follows that

\[
\mathcal{D}^1 := \{ (t, \nu, \mathbb{P}) \in [0, T] \times \mathcal{P}_2(\mathcal{C}^\nu) \times \mathcal{P}(\bar{\Omega}) : \mathbb{P} \circ (B, A)^{-1} \in \Upsilon(\mathcal{U}) \},
\]

is a Borel subset of \([0, T] \times \mathcal{P}(\mathcal{C}^\nu) \times \mathcal{P}(\bar{\Omega})\), as the map

\[
\Gamma_1 : [0, T] \times \mathcal{P}_2(\mathcal{C}^\nu) \times \mathcal{P}(\bar{\Omega}) \ni (t, \nu, \mathbb{P}) \longmapsto \mathbb{P} \circ (B, A)^{-1} \in \mathcal{P}(\mathcal{C}^\ell \times \mathcal{C}),
\]

is Borel-measurable. Similarly

\[
\mathcal{D}^2 := \{ (t, \nu, \mathbb{P}) \in [0, T] \times \mathcal{P}_2(\mathcal{C}^\nu) \times \mathcal{P}(\bar{\Omega}) : B_{t, \lambda} \text{ is } \mathbb{P}-\text{independent of } (B^t, \mu) \},
\]

is also a Borel subset of \([0, T] \times \mathcal{P}(\mathcal{C}^\nu) \times \mathcal{P}(\bar{\Omega})\). Indeed, for all \((h, \psi) \in \mathcal{C}_b(\mathcal{C}^\ell) \times \mathcal{C}_b(\mathcal{C}^\ell \times \mathcal{C})\), the function

\[
\Gamma_{h, \psi} : (t, \nu, \mathbb{P}) \longmapsto (\mathbb{E}^{\mathbb{P}}[h(B_{t, \lambda})\psi(B^t, A)] - \mathbb{E}^{\mathbb{P}}[h(B_{t, \lambda})\mathbb{E}^{\mathbb{P}}[\psi(B^t, A)])
\]

is continuous. By considering a characterising subset \( \mathcal{R} \subset \mathcal{C}_b(\mathcal{C}^\ell) \times \mathcal{C}_b(\mathcal{C}^\ell \times \mathcal{C}) \) (see the beginning of Section 4.2), it follows that

\[
\mathcal{D}^2 = \bigcap_{(h, \psi) \in \mathcal{R}} \Gamma_{h, \psi}^{-1}\{0\},
\]

is a Borel set. Finally, notice that

\[
|\mathcal{P}_S^\nu| = |\mathcal{P}_W^\nu| \cap \mathcal{D}^1 \cap \mathcal{D}^2,
\]

and we then conclude the proof by Lemma 4.5 and Lemma 4.9.

Recall that for each \( M > 0 \), \( \mathcal{P}^M_t \) is defined in Section 4.2, we similarly introduce, for \((t, \nu) \in [0, T] \times \mathcal{P}_2(\mathcal{C}^\nu) \) and \( \nu \in \mathcal{P}_2(\bar{\Omega})\) such that \( \nu = \bar{\nu} \circ \bar{X}^{-1}\)

\[
\mathcal{P}^M_S(t, \nu) := \mathcal{P}_S^\nu(t, \nu) \cap \mathcal{P}^M_t, \mathcal{P}^\nu_S(t, \nu) := \mathcal{P}_S^\nu \cap \mathcal{P}^M_S(t, \nu),
\]

\[
\mathcal{P}^M_S(t, \nu) := \mathcal{P}_S^\nu(t, \nu) \cap \mathcal{P}^M_t, \mathcal{P}^\nu_S(t, \nu) := \mathcal{P}_S^\nu \cap \mathcal{P}^M_S(t, \nu).
\]

By Lemma 4.7, it is clear that

\[
V^M_S(t, \nu) := \sup_{\mathbb{P} \in \mathcal{P}^M_S(t, \nu)} J(t, \mathbb{P}) = \sup_{\mathbb{P} \in \mathcal{P}^M_S(t, \nu)} J(t, \mathbb{P}) = \sup_{\mathbb{P} \in \mathcal{P}^M_S(t, \nu)} J(t, \mathbb{P}) = \sup_{\mathbb{P} \in \mathcal{P}^M_S(t, \nu)} J(t, \mathbb{P})
\]

and \( V^M_S(t, \nu) \not\to V^M_S(t, \nu) \), as \( M \to \infty \).

**Lemma 4.11.** (i) Let \((t, \nu) \in [0, T] \times \mathcal{P}_2(\mathcal{C}^\nu)\), \( \mathbb{P} \in \mathcal{P}^\nu_S(t, \nu)\), \( \bar{\tau} \) a \( \mathcal{G}^\nu \)-stopping time taking values in \([t, T]\), and \((\mathcal{P}^\nu_S)_{\omega \in \bar{\Omega}}\) be a family of r.c.p.d. of \( \mathbb{P} \) knowing \( \mathcal{G}_\omega^\nu\). Then

\[
\mathcal{P}^\nu_S(\bar{\tau}(\omega), \mu_{\bar{\tau}(\omega)}(\omega)), \text{ for } \mathbb{P}-\text{a.e. } \omega \in \bar{\Omega}.
\]

(ii) The graph set \( \{(t, \nu, \mathcal{M}, \mathbb{P}) : \mathbb{P} \in \mathcal{P}^\nu_S(t, \nu)\} \) is analytic. Further, let \((t, \nu) \in [0, T] \times \mathcal{P}_2(\mathcal{C}^\nu)\), \( \mathbb{P} \in \mathcal{P}^\nu_S(t, \nu)\), \( \bar{\tau} \) be a \( \mathcal{G}^\nu \)-stopping time taking values in \([t, T]\), and \( \varepsilon > 0 \). Then there exists a family of probability measures and a family of probability measures \((\mathcal{Q}^\nu_{t, \nu, \mathcal{M}})_{(t, \nu, \mathcal{M}) \in [0, T] \times \mathcal{P}(\bar{\Omega}) \times \mathcal{R}}\) such that \((t, \nu, \mathcal{M}) \longmapsto \mathcal{Q}^\nu_{t, \nu, \mathcal{M}}\) is universally measurable, and for every \((t, \nu, \mathcal{M})\) s.t. \( \mathcal{P}^\nu_S(t, \nu) \neq \emptyset\), one has, for \( \nu = \nu \circ \bar{X}^{-1}\),

\[
\mathcal{Q}^\nu_{t, \nu, \mathcal{M}} \in \mathcal{P}^\nu_S(t, \nu), \text{ and } J(t, \mathcal{Q}^\nu_{t, \nu, \mathcal{M}}) = \begin{cases} V^M_S(t, \nu) - \varepsilon, & \text{when } V^M_S(t, \nu) < \infty, \\ \frac{1}{2}, & \text{when } V^M_S(t, \nu) = \infty. \end{cases}
\]
Moreover, there is a $\mathcal{G}_s^\ell$-measurable and $\mathbb{P}$-integrable r.v. $\tilde{M} : \Omega \to \mathbb{R}_+$ such that for all constant $M > 0$, there exists $\mathbb{P}_M^\varepsilon \in \mathcal{P}_S^\mathbb{B}(t, \nu)$ such that $\mathbb{P}_M^\varepsilon | \mathcal{F}_s = \mathbb{P}| \mathcal{F}_s$ and
\[
\left( \mathcal{U}_{\tau(\omega), \tilde{M}(\omega)} \circ \mathcal{M}(\omega) \right)_{\omega \in \Omega} \text{ is a version of the r.c.p.d. of } \mathbb{P}_M^\varepsilon \text{ knowing } \mathcal{G}_s^\ell. \]

**Proof.** (i) Let $\mathbb{P} \in \mathcal{P}_S^\mathbb{B}(t, \nu)$, then there exists a Borel-measurable function $\phi : [t, T] \times \mathcal{C}_s \to U$ such that
\[
\tilde{\pi}_s = \phi(s, B_{s_a}), \text{ for all } s \in [t, T], \mathbb{P}\text{-a.s.}
\]
Let us consider the concatenated path $(\tilde{\omega} \circ \tilde{\tau})_s := \tilde{\omega}_{ts} + \tilde{w}_{s \wedge t} - \tilde{w}_t$ and define a Borel-measurable function $\phi^\tau$ by
\[
\phi^\tau(s, \tilde{w}^b) := \phi(s, \tilde{\omega} \circ \tilde{\tau}(\tilde{\omega}) \tilde{w}^b), \text{ for } s \in [\tilde{\tau}(\tilde{\omega}), T]
\]
where $\tilde{\omega} = (\tilde{\omega}^x, \tilde{\omega}^a, \tilde{\omega}^w, \tilde{\omega}^{\hat{b}}, \tilde{\omega}^\mu)$, $w = (w^x, w^a, w^w, w^{\hat{b}}, w^\mu) \in \Omega$. Then by a classical conditioning argument, it is easy to check that for $\mathbb{P}$-a.e. $\tilde{\omega} \in \Omega$,
\[
\tilde{\pi}_s = \phi^\tau(s, B_{s_a}^\tau(\tilde{\omega})), \text{ for all } s \in [\tilde{\tau}(\tilde{\omega}), T], \mathbb{P}^{\tilde{\tau}^{-1}, \omega}-a.s.
\]
Using Lemma 4.6 and Definition 4.2, it follows that $\mathbb{P}^{\tilde{\tau}}_{\tilde{\omega}} \in \mathcal{P}_S^\mathbb{B}(\tilde{\tau}(\tilde{\omega}), \mu_{\tilde{\tau}}(\tilde{\omega})(\tilde{\omega}))$, for $\mathbb{P}$-a.e. $\tilde{\omega} \in \Omega$.

(ii) Using Lemma 4.10, it is easy to see that the graph set $\{(t, \nu, M, \mathbb{P}) : \mathbb{P} \in \mathcal{P}_S^M(t, \nu)\}$ is analytic. Then one can apply the same arguments as in Lemma 4.8 to obtain a measurable family $(\mathcal{U}_{t, v, M}(t, \nu, M)_{(t, \nu, M) \in [0, T] \times \mathcal{P}(C_s) \times \mathbb{R}_+})$ such that
\[
\mathcal{U}_{t, v, M} \in \mathcal{P}_S^M(t, \nu), \text{ and } J(t, \mathcal{U}_{t, v, M}) \geq (V_{S}^B_{M}(t, \nu) - \varepsilon)1_{\{V_{S}^B_{M} < \infty\}} + \frac{1}{\varepsilon}1_{\{V_{S}^B_{M} = \infty\}}.
\]
To proceed, we will define a family $(\mathcal{U}_{t, v, M}^{\tilde{\tau}}(t, \tilde{\nu}, M)_{(t, \tilde{\nu}, M) \in [0, T] \times \mathcal{P}(\tilde{\Omega}) \times \mathbb{R}_+})$ from the family $(\mathcal{U}_{t, v, M}^{\tilde{\tau}}(t, \tilde{\nu}, M)_{(t, \tilde{\nu}, M) \in [0, T] \times \mathcal{P}(\tilde{\Omega}) \times \mathbb{R}_+})$ as follows. For all $(t, \tilde{\nu}) \in [0, T] \times \mathcal{P}(\tilde{\Omega})$, let $\nu := \tilde{\nu} \circ \tilde{\tau}^{-1}$. Then on the probability space $(\tilde{\Omega}, \mathcal{F}_T, \mathcal{U}_{t, v, M}^{\tilde{\tau}})$, we consider a $\mathcal{F}_T$-measurable random element $(A'_s, W'_s, B'_s)_{s \in [0, t]}$ such that
\[
\mathcal{U}_{\tilde{\tau}^{-1}, \nu, M} \circ (X_{tA}, A'_{tA}, W'_{tA}, B'_{tA})^{-1} = \tilde{v}(t).
\]
Define
\[
A'_s := A'_t + A_s - A_t, \quad W'_s := W'_t + W'_s, \quad B'_s := B'_t + B'_s, \quad \text{for } s \in [t, T],
\]

\[
\tilde{\nu}_s := \mathcal{L}^{\mathcal{U}_{t, v, M}}(X_{sA'}, A'_{sA'}, W', B'_{sA'})1_{\{s \in [0, t]\}} + \mathcal{L}^{\mathcal{U}_{t, v, M}}(X_{sA'}, A'_{sA'}, W', B'_{sA'}, \mathcal{U}_s^{\tilde{\tau}})1_{\{s \in (t, T]\}}.
\]
Let
\[
\mathcal{U}_{\tilde{\tau}, \tilde{\nu}, M} := \mathcal{U}_{t, v, M} \circ (X, A', W', B', \tilde{\nu}), \text{ so that } J(t, \mathcal{U}_{\tilde{\tau}, \tilde{\nu}, M}) = J(t, \mathcal{U}_{t, v, M})
\]
and hence satisfies (44). Let $\mathbb{P} \in \mathcal{P}_S^B(t, \nu)$, as in Lemma 4.8, for $M > 0$, we let
\[
\tilde{M}(\tilde{\omega}) := \mathbb{E}\left[\|X\|^p + \int_{\tilde{\tau}}^{T} (\rho(\tilde{\pi}_s, u_0))^p ds \right]|_\mathcal{F}_s (\tilde{\omega}), \text{ and } \tilde{\mathcal{U}}_{\tilde{\tau}} := \mathcal{U}_{\tilde{\tau}, \tilde{\nu}(\tilde{\tau}), \tilde{M}(\tilde{\omega}) + M}.
\]
Again, as in the proof of Lemma 4.8, one has $\mathcal{U}_{\tilde{\tau}}$ satisfies (37) and (38), which allows defining $\mathbb{P}_M^\varepsilon$ by
\[
\mathbb{P}_M^\varepsilon [K] := \int_K \mathcal{U}_{\tilde{\tau}}(d\omega), \text{ for all } K \in \mathcal{F},
\]
so that $\mathbb{F}^{M,\varepsilon} \in \mathcal{P}_W(t,\nu)$, $\mathbb{F}^{M,\varepsilon} = \mathbb{F}$ on $\mathcal{F}_\omega$ and $(\overline{\mathcal{G}}_\omega)_{\omega \in \Omega}$ is a family of r.c.p.d. of $\mathbb{F}^{M,\varepsilon}$ knowing $\mathcal{G}_\omega^t$.

Finally, it is enough to prove that $\mathbb{F}^{M,\varepsilon} \in \mathcal{P}^B_S(t,\nu)$. Let $s \in [t,T]$, $h \in C_b(\mathbb{R})$ and $\psi \in C_b(\mathcal{C}_s)$, then

$$
\mathbb{E}[A_s h(A_s) \psi(B_{s \wedge \cdot}^t) 1_{s > \tau}] = \mathbb{E}[h(A_s) \psi(B_{s \wedge \cdot}^t) 1_{s > \tau}]
$$

Putting $\phi(s,B_{s \wedge \cdot}^t) = h(A_s) \psi(B_{s \wedge \cdot}^t)$, it follows that

$$
\mathbb{E}[\phi(s,B_{s \wedge \cdot}^t)] = \mathbb{E}^{\mathbb{F}^{M,\varepsilon}}[\phi(s,B_{s \wedge \cdot}^t)]
$$

where the second equality holds by the fact that $\mathbb{F}^{M,\varepsilon} \in \mathcal{P}^B_S(\tau(\omega),\nu')$ for some $\nu' \in \mathcal{P}(\mathcal{C}^n)$ and hence $h(A_s) = h(\phi(B_{s \wedge \cdot}^t))$, $\mathbb{Q}_\omega$-a.s., for some Borel-measurable function $\phi$, and the last equality follows by the fact that on $\{s > \tau\}$, $\mathbb{E}^{\mathbb{F}^{M,\varepsilon}}[\phi(s,B_{s \wedge \cdot}^t)] = \mathbb{E}^{\mathbb{F}^{M,\varepsilon}}[\phi(s,B_{s \wedge \cdot}^t)]$. Further, as $\mathbb{P}^{M,\varepsilon}_{\mathcal{F}_\omega} = \mathbb{P}_{\mathcal{F}_\omega}$ and $\mathbb{P} \in \mathcal{P}^B_S(t,\nu)$, one can use similar arguments to find that

$$
\mathbb{E}^{\mathbb{F}^{M,\varepsilon}}[A_s h(A_s) \psi(B_{s \wedge \cdot}^t) 1_{s > \tau}] = \mathbb{E}^{\mathbb{F}^{M,\varepsilon}}[A_s B_{s \wedge \cdot}^t h(A_s) \psi(B_{s \wedge \cdot}^t) 1_{s > \tau}].
$$

This implies that

$$
\mathbb{E}^{\mathbb{F}^{M,\varepsilon}}\left[(A_s - \mathbb{E}^{\mathbb{F}^{M,\varepsilon}}[A_s B_{s \wedge \cdot}^t]) h(A_s) \psi(B_{s \wedge \cdot}^t)\right] = 0,
$$

hence $A_s = \mathbb{E}^{\mathbb{F}^{M,\varepsilon}}[A_s B_{s \wedge \cdot}^t]$, $\mathbb{P}^{M,\varepsilon}$-a.s.

In other words, $A$ is a continuous process, adapted to the $\mathbb{F}^{M,\varepsilon}$-augmented filtration generated by $B^t$, then there exists a Borel-measurable function $\phi : [t,T] \times \mathcal{C}_t \rightarrow U$ such that $A_s = \phi(s,B_{s \wedge \cdot}^t)$, for all $s \in [t,T]$, $\mathbb{P}^{M,\varepsilon}$-a.s., and hence $\mathbb{P}^{M,\varepsilon} \in \mathcal{P}^B_S(t,\nu)$, which concludes the proof.

**Proof of Theorem 3.2.** The proof is almost the same as that of Theorem 3.1. First, one has the measurability of $V^B_S$ by Lemma 4.10. Next, notice that a $\mathcal{G}^{t,\omega}$-stopping time $\tau$ on $\Omega^t$ can be considered as a $\mathcal{F}^t$-stopping time $\overline{\tau}$ on $\overline{\Omega}$, then using the conditioning argument in Lemma 4.11, it follows that

$$
V^B_S(t,\nu) \leq \sup_{\mathbb{F} \in \mathcal{P}^B_S(t,\nu)} \mathbb{E}\left[\int_t^T L(s,X_{s \wedge \cdot},\overline{\sigma}_s,\overline{\sigma}_s)ds + V^B_S(\tau,\mu_\tau)\right].
$$

Finally, it is enough to use the concatenation argument in Lemma 4.11 and sending $M$ to $\infty$ to obtain the reverse inequality

$$
V^B_S(t,\nu) \geq \sup_{\mathbb{F} \in \mathcal{P}^B_S(t,\nu)} \mathbb{E}\left[\int_t^T L(s,X_{s \wedge \cdot},\overline{\sigma}_s,\overline{\sigma}_s)ds + V^B_S(\tau,\mu_\tau)\right].
$$

\[\square\]
APPENDIX A: SOME TECHNICAL RESULTS ON CONTROLLED MCKEAN–VLASOV SDES

Here, we fix an abstract complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) as in Section 2.2. Nevertheless, instead of the filtrations \(\mathbb{F}\) (resp. \(\mathbb{G}\)) generated by \(\mathcal{F}_0\) and the Brownian motions \((W, B)\) (resp. \(B\)), we consider more general filtrations \(\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \in [0,T]}\) and \(\tilde{\mathbb{G}} = (\tilde{\mathcal{G}}_t)_{t \in [0,T]}\) on \((\Omega, \mathcal{F}, \mathbb{P})\), whose properties will be precised depending on the context.

Let us first recall a technical optional projection result.

**Lemma A.1.** Let \(E\) be a Polish space, \(\tilde{\mathbb{G}} := (\tilde{\mathcal{G}}_t)_{t \geq 0}\) be a complete filtration.

(i) Given an \(E\)-valued measurable process \((X_t)_{t \in [0,T]}\), there exists a \(\mathcal{P}(E)\)-valued \(\tilde{\mathbb{G}}\)-optional process \(\beta\) such that

\[
\beta_t = \mathcal{L}^E(X_t | \mathcal{G}_t), \quad \mathbb{P}\text{-a.s., for all } \tilde{\mathbb{G}}\text{-stopping times } \tau.
\]

(ii) Assume in addition that \(X\) is a continuous process, and that the \(\tilde{\mathbb{G}}\)-optional \(\sigma\)-field is identical to the \(\tilde{\mathbb{G}}\)-predictable \(\sigma\)-field. Then one can choose \(\beta\) to be an \(\mathbb{P}\)–continuous process.

**Proof.** (i) The existence of such process \(\beta\) is ensured by, e.g. Kurtz [31, Theorem A.3] or Yor [51, Proposition 1].

(ii) When \(X\) is a continuous process, it follows again by [31, Theorem A.3] (or [51, Proposition 1]) that \(\beta\) is càdlàg \(\mathbb{P}\)-a.s. Further, let \(\varphi \in C_b(E)\) and \((\tau_n)_{n \geq 1}\) be an increasing sequence of uniformly bounded \(\tilde{\mathbb{G}}\)-stopping times, which is \(\tilde{\mathbb{G}}\)-predictable as soon as the \(\tilde{\mathbb{G}}\)-optional \(\sigma\)-field is identical to the \(\tilde{\mathbb{G}}\)-predictable \(\sigma\)-field. One has

\[
\mathbb{E}^\mathbb{P}[\varphi(X_{\tau_n}) | \tilde{\mathcal{G}}_{\tau_n}], \quad \mathbb{P}\text{-a.s., and hence } \lim_{n \to \infty} \mathbb{E}^\mathbb{P}[\varphi(\beta_{\tau_n})] = \mathbb{E}^\mathbb{P}[\varphi(\beta_{\lim_{n \to \infty} \tau_n})].
\]

Then it follows by Dellacherie [21, Theorem IV–T24] that \((\langle \varphi, \beta_t \rangle)_{t \in [0,T]}\) is \(\mathbb{P}\)-a.s. left-continuous. By considering a characterising set of functions \(\varphi\) in \(C_b(E)\) (see the the beginning of Section 4.2), one concludes that \(\beta\) is also left-continuous \(\mathbb{P}\)–a.s.

We now assume that \(\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_s)_{s \geq 0}\) is a complete filtration, \(\tilde{\mathbb{G}} := (\tilde{\mathcal{G}}_s)_{s \geq 0}\) is a complete sub-filtration of \(\tilde{\mathbb{F}}\), \(W\) and \(B\) are two independent \(\mathbb{R}^d\)- and \(\mathbb{R}^t\)-valued Brownian motions w.r.t. \(\tilde{\mathbb{F}}\). Let us fix an \(\mathbb{R}^n\)-valued, \(\tilde{\mathbb{F}}\)-adapted continuous process \(\xi = (\xi_s)_{s \geq 0}\), an \(\mathbb{R}^t\)-valued \(\tilde{\mathbb{F}}\)-predictable process \(\alpha = (\alpha_s)_{s \geq 0}\), and denote \(W^t_s := W_{s \wedge t} - W_t\) and \(B^t_s := B_{s \wedge t} - B_t\). We will study the following SDE with data \((t, \xi, \alpha, \tilde{\mathbb{G}})\), that is to say \(X_s = \xi_s\), for all \(s \in [0,t]\), and using the notation \(\mu_s := \mathcal{L}^F(X_{r \wedge }, \alpha_r | \tilde{\mathcal{G}}_r), \mathbb{P}\text{-a.s., } s \in [t, T]\),

\[
X_s = \xi_t + \int_t^s b(r, X_{r \wedge }, \mu_r, \alpha_r) \, dr + \int_t^s \sigma(r, X_{r \wedge }, \mu_r, \alpha_r) \, dW^t_r + \int_t^s \sigma_0(r, X_{r \wedge }, \mu_r, \alpha_r) \, dB^t_r
\]

**Definition A.2.** A strong solution of SDE (45), with data \((t, \xi, \alpha, \tilde{\mathbb{G}})\), on \([0, T]\), is an \(\mathbb{R}^n\)-valued \(\tilde{\mathbb{F}}\)-adapted continuous process \(X = (X_t)_{t \geq 0}\) s.t. \(\mathbb{E}^\mathbb{P}[\sup_{s \in [0,T]} |X_s|^2] < \infty\), \(X_s = \xi_s\), for \(s \in [0,t]\), and (45) holds true.

**Theorem A.3.** Let Assumption 2.2 hold true, fix some \(t \in [0,T]\), and assume that \(\mathbb{E}^\mathbb{P}[\sup_{s \in [0,T]} |\xi_s|^p] < \infty\) and \(\mathbb{E}^\mathbb{P}[\int_0^T \rho(u_0, \alpha_s) \, ds] < \infty\) for some \(p \geq 2\). Then

(i) there exists a unique strong solution \(X^{t,\xi,\alpha}\) of (45) on \([0,T]\) with data \((t, \xi, \alpha, \tilde{\mathbb{G}})\). Moreover, it holds that \(\mathbb{E}^\mathbb{P}[\sup_{s \in [0,T]} |X^{t,\xi,\alpha}_s|^p] < \infty\);
(ii) assume in addition that \((\xi_{t,\lambda}, W^t, B^t_{r,\lambda})\) is independent of \(\tilde{G}_t\), and \(B^t\) is \(\tilde{G}_t\)-adapted, and there exists a Borel-measurable function \(\phi : [0, T] \times C^n \times C^d \times C^d \rightarrow U\) such that
\[
\alpha_r = \phi(s, \xi_{t,\lambda}, W^t_{s,\lambda}, B^t_{r,\lambda}), \quad \mathbb{P}\text{-a.s.}, \quad \text{for all } s \in [0, T].
\]
Then, with \(A_s := \int_t^{s\vee t} \pi(\alpha_r) \text{d}r\), there exists a continuous process \((\tilde{\mu}_t)_{t \in [0, T]}\) such that for all \(s \in [0, T]\)
\[
\tilde{\mu}_s = \mathcal{L}^p((X^t_{s,\lambda}, A_{s,\lambda}, W^t, B^t_{s,\lambda})|\tilde{G}_T) = \mathcal{L}^p((X^t_{s,\lambda}, A_{s,\lambda}, W^t, B^t_{s,\lambda})|\tilde{G}_s) = \mathcal{L}^p((X^t_{s,\lambda}, A_{s,\lambda}, W^t, B^t_{s,\lambda})|\tilde{G}_s), \quad \mathbb{P}\text{-a.s.}
\]

**PROOF.** (i) We follow [47, Theorem 5.1.1] to prove well-posedness for (45). Let \(S^p\) be defined by
\[
S^p := \{ Y := (Y_s)_{s \in [0, T]} : \mathbb{R}^n\text{-valued } \tilde{G}_t\text{-adapted continuous process s.t. } \mathbb{E}^p[||Y||^p] < \infty \},
\]
where \(||x||_s := \sup_{r \in [0, s]} |x_r|\) for \(s \in [0, T]\) and \(x \in C^n\). For all \(Y \in S^p\), we define, with \(\tilde{\mu}_s := \mathcal{L}^p(\tilde{Y}_s, \alpha_s|\tilde{G}_T), \Psi(Y) := (\Psi(Y)_s)_{0 \leq s \leq T}\) by: \(\Psi(Y)_{[t, \lambda]} := \xi_{t,\lambda}\) and for \(s \in [t, T]\)
\[
\Psi(Y)_s := \xi_t + \int_t^s b(r, Y_r, \tilde{\mu}_r, \alpha_r) \text{d}r + \int_t^s \sigma(r, Y_r, \tilde{\mu}_r, \alpha_r) \text{d}W^t_r + \int_t^s \sigma_0(r, Y_r, \tilde{\mu}_r, \alpha_r) \text{d}B^t_r.
\]
Then, for \((Y^1, Y^2) \in S^p \times S^p\), with \(\tilde{\mu}^i_s := \mathcal{L}^p(\tilde{Y}^i_s, \alpha_s|\tilde{G}_T), \ i \in \{1, 2\}\), one has
\[
\mathbb{E}^p[||\Psi(Y^1) - \Psi(Y^2)||^p_s] \leq 3^p \mathbb{E}^p\left[ \sup_{r \in [t, s]} \left| \int_t^r \sigma(r, Y^1, \tilde{\mu}^1_r, \alpha_r) - \sigma(r, Y^2, \tilde{\mu}^2_r, \alpha_r) \right| \text{d}W^t_r \right]^p
\]
\[
+ 3^p \mathbb{E}^p\left[ \sup_{r \in [t, s]} \left| \int_t^r \sigma_0(r, Y^1, \tilde{\mu}^1_r, \alpha_r) - \sigma_0(r, Y^2, \tilde{\mu}^2_r, \alpha_r) \right| \text{d}B^t_r \right]^p
\]
\[
+ 3^p \mathbb{E}^p\left[ \int_t^s |b(r, Y^1, \tilde{\mu}^1_r, \alpha_r) - b(r, Y^2, \tilde{\mu}^2_r, \alpha_r)| \text{d}r \right]^p.
\]
Notice that, for all \(r \in [t, T],\)
\[
W^2_p(\tilde{\mu}^1_r, \tilde{\mu}^2_r)^p \leq W^p_p(\tilde{\mu}^1_r, \tilde{\mu}^2_r)^p = W_p^p(\mathcal{L}^p((W^1_{r, \lambda}, \alpha_r)|\tilde{G}_r), \mathcal{L}^p((W^2_{r, \lambda}, \alpha_r)|\tilde{G}_r))^p \leq \mathbb{E}^p[||Y^1_{r, \lambda} - Y^2_{r, \lambda}||^p|\tilde{G}_r].
\]
Then by Burkholder–Davis–Gundy inequality, Jensen’s inequality and Assumption 2.2, there is some constant \(C_T > 0\) such that
\[
\mathbb{E}^p[||\Psi(Y^1) - \Psi(Y^2)||^p_s] \leq C_T \int_t^s \mathbb{E}^p[||Y^1_{r, \lambda} - Y^2_{r, \lambda}||^p]|\tilde{G}_r|\text{d}r.
\]
Besides, by Assumption 2.2,
\[
\mathbb{E}[||\Psi(0)||^p] \leq C \left( 1 + \mathbb{E}^p\left[ \sup_{r \in [0, t]} |\xi_r|^p \right] + \mathbb{E}^p\left[ \int_t^T \rho(u, \alpha_r)^p \text{d}r \right] \right).
\]
Then by taking \(Y^2 = 0\), (46) implies that \(\Psi(Y) \in S^p\) whenever \(Y \in S^p\). Moreover, for any positive integer \(n\)
\[
\mathbb{E}^p[||\Psi^n(Y^1) - \Psi^n(Y^2)||^p_s] \leq C_T \int_t^s \mathbb{E}^p[||Y^{n-1}(Y^1) - Y^{n-1}(Y^2)||^p]|\tilde{G}_r|\text{d}r.
\]
\[ \leq (C_T)^2 \int_t^s \int_t^r \mathbb{E}^P \left[ \| \Psi^{n-2}(Y^1) - \Psi^{n-2}(Y^2) \|_p^p \right] \, dv \, dr \]
\[ \leq (C_T)^n \int_1 \left\{ s \geq n \geq \ldots \geq v_n \geq t \right\} \mathbb{E}^P \left[ \| Y^1 - Y^2 \|_{v_n}^p \right] \, dv_1 \ldots \, dv_n \]
\[ \leq (C_T)^n \mathbb{E}^P \left[ \| Y^1 - Y^2 \|_p^p \right] \frac{(s - t)^n}{n!} \]

Let \( Y \in \mathcal{S}^p, X^0 := Y \), and \( X^n := \Psi^n(Y) \), for \( n \geq 1 \), it follows that
\[ \mathbb{E}^P \left[ \| X^n - X^{n+1} \|_p^p \right] \leq (C_T)^n \mathbb{E}^P \left[ \| Y - \Psi(Y) \|_p^p \right] \frac{(s - t)^n}{n!} \]

and hence \( \mathbb{E}^P \left[ \sum_{n \geq 1} \| X^n - X^{n+1} \|_p^p \right] < \infty \), which implies that the sequence \( (X^n)_{n \geq 1} \) converges uniformly, \( \mathbb{P} \)-a.s., to some \( X \in \mathcal{S}^p \). Finally, it is straightforward to see that \( X \) is the unique strong solution of (45) with data \((t, \xi, \alpha, \hat{\mathbb{G}})\).

(ii) Let us consider the filtration \( \mathbb{G}^t = (\mathcal{G}^t_s)_{s \in [0,T]} \), which is the \( \mathbb{P} \)-augmented filtration generated by \((B^t_s)_{s \in [0,T]}\), see Section 2.2. Then there is a unique solution \( X^\alpha \) of Equation (45) associated with \((t, \xi_{t \wedge \cdot}, \alpha, \hat{\mathbb{G}}^t)\), and by the uniqueness of SDE Equation (45), there exists an \( \mathbb{P} \)-measurable function \( \Psi : [0,T] \times \mathcal{C}^\alpha \times \mathcal{C}^d \times \mathcal{C}^t \rightarrow \mathbb{P}^n \) such that
\[ X^\alpha_s = \Psi (\xi_{t \wedge \cdot}, W^t_{s \wedge \cdot}, B^t_{s \wedge \cdot}), \quad s \in [0,T], \ \mathbb{P} \text{-a.s.} \]

Further, as \((\xi_{t \wedge \cdot}, W^t_{s \wedge \cdot}, B^t_{s \wedge \cdot})\) is independent of \( \mathbb{G} \), and \( B^t \) is \( \mathbb{G} \)-adapted, one has, for all \( s \in [t,T] \),
\[ \mathbb{P} \left( (X^{\alpha}_{s \wedge \cdot}, \alpha_s) | \mathcal{G}^t_s \right) = \mathbb{L}^P \left( (X^{\alpha}_{s \wedge \cdot}, \alpha_s) | \mathcal{G}^t_s \right) = \mathbb{L}^P \left( (\Psi (\xi_{t \wedge \cdot}, W^t_{s \wedge \cdot}, \rho_s, \alpha_s) \right) \mathcal{L}^P \left( B^t_{s \wedge \cdot} | \mathcal{G}^t_s \right) \]
\[ = \mathbb{L}^P \left( (X^{\alpha}_{s \wedge \cdot}, \alpha_s) | \mathcal{G}^t_s \right), \ \mathbb{P} \text{-a.s.} \]

This implies that \( X^\alpha \) is also a solution of Equation (45) associated with \((t, \xi_{t \wedge \cdot}, \alpha, \hat{\mathbb{G}})\), and hence \( X^{t \wedge \cdot, \alpha} = X^\alpha \) by uniqueness of solution to Equation (45).

Moreover, as \( A_s := \int_t^s \pi(\alpha_\tau) \, d\tau \) is a \( \mathbb{P} \)-measurable function of \((\xi_{t \wedge \cdot}, W^t_{s \wedge \cdot}, B^t_{s \wedge \cdot})\), it follows by the same argument that, for all \( s \in [0,T] \),
\[ \mathbb{P} \left( (X^{\alpha}_{s \wedge \cdot}, A_{s \wedge \cdot}, W^t_{s \wedge \cdot}, B^t_{s \wedge \cdot}) = \mathbb{L}^P \left( (X^{\alpha}_{s \wedge \cdot}, A_{s \wedge \cdot}, W^t_{s \wedge \cdot}, B^t_{s \wedge \cdot}) | \mathcal{G}_s \right), \ \mathbb{P} \text{-a.s.} \]

Finally, using Lemma A.1, one can choose the process \( \hat{\mu} \) to be continuous. \( \square \)

We finally consider a variation of the system of the controlled McKean–Vlasov SDE. Given the initial condition \((t, \xi)\) and control process \( \alpha \in \mathcal{A} \), we say \((X, \hat{\mu}) = (X_s, \hat{\mu}_s)_{s \in [0,T]} \) is a solution to (47) if they are \( \mathbb{F} \)-adapted continuous such that
\[ \mathbb{E}^P \left[ \| X \|_2^2 + W_2(\hat{\mu}, \hat{\nu}_0) \right] < \infty, \text{ for some } \hat{\nu}_0 \in \mathcal{P}(\mathcal{C}^\alpha \times \mathcal{C} \times \mathcal{C}^t), \]

\((\xi_{t \wedge \cdot}, W^t)\) is independent of \((B^t, \hat{\mu})\), and \( \mathbb{P} \)-a.s., with \( \pi_r := \mathbb{L}^P (X_{t \wedge \cdot}, \alpha_r | B^t_{t \wedge \cdot}, \hat{\mu}_{t \wedge \cdot}) \) and \( A_s := \int_t^s \pi(\alpha_r) \, d\tau \)
(47)
\[
\begin{cases}
X_s = \xi_s, & s \in [0,t], \text{ for all } s \in [t,T],

X_s = \xi + \int_t^s b(r, X_r, \pi_r, \alpha_r) \, dr + \int_t^s \sigma(r, X_r, \pi_r, \alpha_r) \, dW^t_r + \int_t^s \sigma_0(r, X_r, \pi_r, \alpha_r) \, dB^t_r,

\hat{\mu}_s = \mathbb{L}^P (X_{s \wedge \cdot}, A_{s \wedge \cdot}, W^t_{s \wedge \cdot}, B^t_{s \wedge \cdot}, \hat{\mu}_{s \wedge \cdot}).
\end{cases}
\]
COROLLARY A.4. Let Assumption 2.2 hold true, and assume that there exists a Borel-measurable function \( \phi : [0,T] \times \mathcal{C}^n \times \mathcal{C}^d \times \mathcal{C}^\ell \rightarrow U \) such that
\[
\alpha_s = \phi(s, \xi_{t\wedge \cdot}, W^t_{s\wedge \cdot}, B^t_{s\wedge \cdot}), \text{ for } d\mathbb{P} \otimes dt \text{-a.e. } (s, \omega) \in [t,T] \times \Omega,
\]
and \( \mathbb{E} \left[ \int_t^T \rho(u_s, \alpha_s)^2 ds \right] < \infty \). Then, Equation (47) has a unique solution \( (X, \hat{\mu}) \), where \( X \) is the strong solution of Equation (45) with data \( (t, \xi, \alpha, \mathcal{G}^t) \), with \( \mathcal{G}^t \) being the \( \mathbb{P} \)-augmented filtration generated by \( B^t \), and
\[
\hat{\mu}_s = \mathcal{L}^P \left( X_{s\wedge \cdot}, A_{s\wedge \cdot}, W^t, B^t_{s\wedge \cdot}, |\mathcal{G}^t_s \right), s \in [t,T], \mathbb{P}-\text{a.s.}
\]

PROOF. Given a solution \( (X, \hat{\mu}) \) to Equation (47), we notice that \( X \) is a strong solution of Equation (45) associated with data \( (t, \alpha, \xi, \mathcal{G}^t) \), where \( \tilde{\mathcal{G}}^t := (\tilde{\mathcal{G}}^t_s)_{s \in [0,T]} \) is defined by \( \tilde{\mathcal{G}}^t_s := \sigma(B^t_{s\wedge \cdot}, \hat{\mu}_{s\wedge \cdot}) \). As \( (\xi_{t\wedge \cdot}, W^t) \) is independent of \( (B^t, \hat{\mu}) \), it is then enough to apply Theorem A.3 to conclude that \( X \) is also the unique strong solution of Equation (45) with data \( (t, \xi, \alpha, \mathcal{G}^t) \).

REFERENCES


