GLOBAL WELL-POSEDNESS OF THE TWO-DIMENSIONAL STOCHASTIC NONLINEAR WAVE EQUATION ON AN UNBOUNDED DOMAIN

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We study the two-dimensional wave equation with cubic nonlinearity posed on \( \mathbb{R}^2 \), with space-time white noise forcing. After a suitable renormalisation of the nonlinearity, we prove global well-posedness for this equation for initial data in \( H^s \), \( s > \frac{4}{5} \).

1. Introduction. We consider the following stochastic nonlinear wave equation (SNLW) with additive space-time white noise forcing

\[
\begin{cases}
u_{tt} = \Delta u - u^3 + 3\infty \cdot u + dW,
\end{cases}
\]

where \( dW \) denotes a space-time white noise on \( \mathbb{R}^2 \). The term “\( -3\infty \cdot u \)” in the equation denotes a time-dependent renormalisation, which has been first introduced by Gubinelli, Koch and Oh in [11] for a family of stochastic wave equations with power nonlinearities. In this work, the authors prove local well-posedness for renormalised stochastic wave equations posed on \( T^2 \) with certain polynomial nonlinearities.

The necessity to apply such a renormalisation in order to get nontrivial solutions was highlighted in a series of works by Albeverio, Haba, Oberguggenberger, and Russo [1, 19, 20, 26], and more recently by Oh, Okamoto and Robert [21], who showed that without the renormalisation term, solutions to (1.1) must satisfy a linear wave equation.

The renormalisation that we apply in (1.1) is better described as follows. Recall that \( dW \) is defined to be a distribution-valued random variable such that for every test function \( \phi \),

\[
\mathbb{E} [ | \langle \phi, dW \rangle |^2 ] = \| \phi \|^2_{L^2} .
\]

Ignoring the term with \( \infty \) in (1.1), we consider a perturbative expansion \( u = v + \psi \), where

\[
\psi := \int_0^t \sin((t - t')|\nabla|) dW(t')\]

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\(^1\)We use the expression \( F(\nabla)u \) to denote the distribution whose Fourier transform is given by

\[
\hat{F}(\nabla)u(\xi) := F(\hat{\xi})\hat{u}(\xi).
\]
solves the linear wave equation

$$\psi_{tt} = \Delta \psi + dW.$$ 

Formally, the term $v$ would then solve the equation

$$v_{tt} - \Delta v = - (\psi + v)^3 = -\psi^3 - 3\psi^2 v - 3\psi v^2 - v^3.$$ 

However, because of the roughness of $dW$, it can be shown that the terms $\psi^3$, $\psi^2$ do not make sense as space-time distributions. Therefore, we introduce the Wick renormalisation

$$\begin{align*}
\psi^2 &:= \psi^2 - \mathbb{E}[|\psi|^2], \\
\psi^3 &:= \psi^3 - 3\mathbb{E}[|\psi|^2] \psi,
\end{align*}$$

and define $v = u - \psi$ to solve the equation

$$\begin{cases}
v_{tt} - \Delta v = - \psi^3: - 3 \psi^2: v - 3\psi v^2 - v^3, \\
v(0, \cdot) = u_0 \in H^s_{\text{loc}}(\mathbb{R}^2), \\
v_t(0, \cdot) = u_1 \in H^{s-1}_{\text{loc}}(\mathbb{R}^2).
\end{cases}$$

While both terms on the right hand side of (1.4) diverge (for both definitions), it is actually possible to give a meaning to the renormalised terms $\psi^2$, $\psi^3$: by first taking a smooth approximation of the noise $dW$ and then taking a limit in the space $W^{-\varepsilon, \infty}_{\text{loc}}$. This will be carried out explicitly in Section 2. Denoting (formally) $u^3 := u^3 - 3\mathbb{E}[|\psi|^2]u$, solving the equation (1.5) for $v$ corresponds to solving the equation

$$\begin{cases}
u_{tt} = \Delta u - u^3 + dW, \\
u(0, \cdot) = u_0 \in H^s_{\text{loc}}, \\
u_t(0, \cdot) = u_1 \in H^{s-1}_{\text{loc}},
\end{cases}$$

in the variable $u$. Since $\mathbb{E}[|\psi|^2] = +\infty$ for every $t > 0$, by inserting this into (SNLW) we obtain the formula (1.1). This kind of renormalisation is exactly the same that appears in [11] for the cubic wave equation on the torus.

Before stating our main result, we need to define what we mean by solutions of (1.1). As we already discussed, we write $u = \psi + v$, and we require $v$ to solve the mild formulation of (1.5),

$$v = \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|} u_1 + \int_0^t \sin((t - t')|\nabla|) \left(- \psi^3: (t') - 3 \psi^2: (t') v(t') - 3\psi v^2(t') - v^3(t') \right) dt'.$$

**Theorem 1.1.** Consider the equation (SNLW) on $\mathbb{R}^2$, and let $1 > s > \frac{4}{5}$. Then (SNLW) is almost surely globally well-posed. More precisely, for every $(u_0, u_1) \in H^s_{\text{loc}}$, there exists a unique $\psi + C(\mathbb{R}; H^s_{\text{loc}}(\mathbb{R}^2))$-valued random variable $u$ such that almost surely

- for every stopping time $T > 0$, $u|_{[-T,T]}$ is the unique solution to (SNLW) in the space $\psi + C([-T,T]; H^s_{\text{loc}}),$
• for every time $0 < T < \infty$, $u$ is continuous in the initial data $(u_0, u_1)$.

In the recent years, there have been many developments in the study of global solutions for parabolic stochastic SPDEs, both on a compact domain (see for instance [13, 22, 23] for a study of $\Phi^4_d$ models), and more recently, on the euclidian space ([8, 10]). However, the dispersive nature of the wave equation does not allow to extend the techniques developed for the study of stochastic quantisation equation to the study of equation 1.1.

The probabilistic study of dispersive equation, on the other hand, has been mostly focused on the analysis of deterministic equations with random initial data, often because of its links to the invariance of appropriate Gibbs-type measures. This direction of investigation was started by McKean and Vaninsky, [14] and Bourgain [2], and have seen a tremendous development in recent years. For a study of wave equations with random initial data, see for instance [4, 5, 6, 16, 17, 18] and references therein. In this setting, it is possible to find global well posedness results that apply to non-compact domains, as for the 1-dimensional wave equation on the real line [14], or for Schrödinger equations posed on $\mathbb{R}$ [3, 7].

More recently, there has been increasing interest in the analysis of stochastic dispersive equations. However, there are still not many global well-posedness results available, and this result is the first one the author is aware of that deals with a non compact domain. In [15], the author and his collaborators showed global well-posedness for the nonlinear stochastic beam equation posed on $T^3$. In [12], the author together with the authors of [11], showed a similar result to Theorem 1.1, by extending the solutions of the stochastic wave equation with cubic nonlinearity posed on $T^2$ built in [11] for infinite time. In [9], Forlano recently adapted the globalisation argument in this paper for the study the BBM equation with random initial data outside $L^2(T)$.

The main difference between the result in [12] and Theorem 1.1 is that we do not have a local well-posedness result available for (SNLW). Indeed, due to the unboundedness of the domain, one can show that for every $t > 0$, the stochastic convolution $\psi(t)$ satisfies $\|\psi(t)\|_X = +\infty$ for every translation invariant norm $\|\cdot\|_X$ defined on a subspace $X$ of distributions. This in turn implies that given $\sigma \in \mathbb{R}$, $t, R > 0$, a typical solution of (1.6) could satisfy a priori $\|u(t)\|_{H^\sigma(B)} \geq R$ on some ball $B$. Therefore, any perturbative argument for local well-posedness (such as a Banach fixed point argument) is bound to fail.

We therefore follow a different strategy in order to show global well-posedness for the equation (1.6) directly. Firstly, we build local solutions (in space and time) to the equation (1.6) by an approximation argument. We introduce a cutoff function $\rho$ with compact support $\rho \equiv 1$ on a big ball $B = B(0, R)$, and consider a localised version of the equation for $v$ given by

\begin{equation}
\tilde{v}_{tt} - \Delta \tilde{v} + \tilde{v}^3 + 3\tilde{v}^2 \rho \psi + 3\tilde{v} \rho : \psi^2 : + \rho : \psi^3 : = 0.
\end{equation}

Notice that in the ball $B$, this equation is exactly the same as (1.5). Sections 3 and 4 are dedicated to showing local and global well-posedness for the equation (1.7).

Let $\tilde{v}_{R_1}$ and $\tilde{v}_{R_2}$ denote the solution to (1.7) corresponding to two different values of $R$. Because of finite speed of propagation, we have that $\tilde{v}_{R_1}(t) = \tilde{v}_{R_2}(t)$ in the ball $B(0, \min(R_1, R_2) - t)$. See Proposition 5.1 and Corollary 5.2 for a precise statement. Therefore, we can take a limit as $R \to \infty$, and build a global solution $\tilde{v}$ to the original equation (1.6). Moreover, we can use the finite speed of propagation again to show that any solution
$v$ of (1.6) satisfies $v = \tilde{v}$, and obtain the existence and uniqueness results of Theorem 1.1. Lastly, since the unique solution $v$ coincides with the solution of (1.7) on bounded space-time domains, the continuity part of the statement of Theorem 1.1 will follow from the analogous result for the equation (1.7). This analysis is performed rigorously in Section 5.

The main technical difficulties arise in Section 4, i.e. in showing global well-posedness for the localised equation (1.7). In order to do so, we show an energy estimate. We would like to estimate the functional

$$E(\tilde{v}(t), \partial_t \tilde{v}(t)) := \frac{1}{2} \int |\partial_t \tilde{v}|^2 + \frac{1}{2} \int |\nabla \tilde{v}|^2 + \frac{1}{4} \int |\tilde{v}|^4,$$

since it is conserved by the flow of cubic wave equation with no forcing. However, as we will see in Section 2, $\psi \notin L^2_{\text{loc}}(\mathbb{R}^2)$, so we expect $\tilde{v} \notin H^1$, hence $E(\tilde{v}, \partial_t \tilde{v}) = +\infty$.

In order to deal with the lack of regularity of $\tilde{v}$, we make use of the I-method developed by Colliander, Keel, Staffilani, Takaoka and Tao in 2002 to show global well-posedness for dispersive equations with initial data of regularity below the energy space. We consider an operator $I = I_N$ given by the Fourier multiplier $m_N$ with

$$m_N(\xi) = \begin{cases} 1 & \text{for } |\xi| \leq N, \\ \left(\frac{N}{|\xi|}\right)^{1-s} & \text{for } |\xi| \geq 3N. \end{cases}$$

For every $N$, one has that $I\tilde{v} \in H^1$ if and only if $\tilde{v} \in H^s$, and $I$ is the identity at low frequencies. Therefore, we can bound $\|\tilde{v}\|_{H^s}$ by making use of the functional

$$F = E(I\tilde{v}(t), I\partial_t \tilde{v}(t)).$$

Of course, the functional $F$ will not be time independent, but we can bound it by making use of a Gronwall argument. We have that

$$\frac{d}{dt}F = \text{commutator terms} + \text{forcing terms}.$$ 

The commutator terms are typical of the I-method, and we can estimate them by making use of simple harmonic analytic techniques.

The terms coming from the forcing are more subtle to estimate, and in particular the term

$$-3 \int Iv_t(Iv)^2I(\rho\psi)$$

requires a very sharp large deviation estimate for the $L^p$ norm of the term $I(\rho\psi)$.

Putting the estimates together, for fixed $N$, allows us to show existence of $\tilde{v}$ for a random time $O(1)$. Iterating this procedure on a sequence $N_0 \ll N_1 \ll N_2 \ll \cdots$, we get global existence for $\tilde{v}$.

We would like to point out that, as far as the author is aware, this is the first instance of an application of the I - method that requires to change the cutoff parameter $N$ in a time-dependent way.

**Remark 1.2.** The argument in this paper is completely pathwise, and after the analysis performed in Section 2, it relies only on deterministic estimates. In particular, we can replace $:\psi^j :$ by stochastic processes $X_j$, and consider the equation

(1.8) $$v_{tt} - \Delta v = -X_3 - 3X_2 v - 3X_1 v^2 - v^3.$$
As long as the processes $X_1, X_2, X_3$ satisfy the same estimates as in Proposition 2.1, (iv)-(v), by the same argument as in this paper, we can show an analogous of Theorem 1.1 for the equation (1.8).

**Remark 1.3.** One could wonder if it is possible to improve the result of Theorem 1.1 by extending the range of $s$ for which the theorem holds. Analysing the proof of Theorem 1.1, one can see that the restriction $s > \frac{4}{5}$ derives from the application of the I-method, and in particular from the estimate (4.13) for $k = 3$. While it does not seem to be possible to improve this particular estimate (see Remark 4.6), there are examples in which a more sophisticated application of the I-method can yield better results. See for instance [25] in the context of wave equation on the 2-dimensional torus.

However, pursuing this kind of improvement would substantially complicate the argument, which is already quite involved. Since our main goal in this paper is to present the first examples of a dispersive SPDE with space-time white noise forcing which is globally well posedness on an unbounded domain, we decided to keep the most technical part of the proof as simple as possible.

**Remark 1.4.** As one can observe by analysing the proof of Lemma 4.11, the energy estimate that leads to global well-posedness for the equation (1.7) relies heavily on the logarithmic divergence of $\psi$, and more specifically on the estimate

$$
\|I_N(\rho \psi)\|_{L^{log,n}} \lesssim \log N.
$$

At the moment, we do not know if it is possible to extend the argument in this paper to equations that present a stronger singularities. For instance, if we consider the equation

$$
\begin{align*}
\begin{cases}
    u_{tt} &= \Delta u - u^3 : + (\nabla)^{\epsilon} \, dW, \\
    u(0, \cdot) &= u_0 \in H^s_{loc}, \\
    u_t(0, \cdot) &= u_1 \in H^{s-1}_{loc},
\end{cases}
\end{align*}
$$

our argument does not apply.

However, it is still possible find examples in the literature of equations that satisfy a similar logarithmic divergence. One example is given by the BBM equation with random initial data, as explored by Forlano in [9]. Another example is given by the wave equation on $T^3$ with random initial data considered in [6]:

$$
(1.9)
\begin{align*}
\begin{cases}
    u_{tt} &= \Delta u - u^3, \\
    u(0, \cdot) &= u_0, \\
    u_t(0, \cdot) &= u_1,
\end{cases}
\end{align*}
$$

with

$$
\begin{align*}
    u_0(x) &= \Re \left( \sum_{n \in \mathbb{Z}^3} a_j g_j e^{i n \cdot x} \right), \\
    u_1(x) &= \Re \left( \sum_{n \in \mathbb{Z}^3} b_j h_j e^{i n \cdot x} \right),
\end{align*}
$$

where $(g_j)_j, (h_j)_j$ are i.i.d. standard complex-valued gaussian random variables, and

$$
\sum_n |a_j|^2 \langle n \rangle^{2s} + |b_j|^2 \langle n \rangle^{2s-2} < +\infty,
$$
so that \((u_0, u_1) \in \mathcal{H}^s\). In [6], the authors show that (1.9) is locally well-posed for \(s \geq 0\), and it is globally well-posed for \(s > 0\). The techniques developed in this paper should be able to extend the global well-posedness result to the full range \(s \geq 0\).

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2. Stochastic Convolution. In this section, we establish relevant estimates on the stochastic convolution
\[
\psi(t, \cdot) := \int_0^t \frac{\sin((t - t')|\nabla|)}{|\nabla|} dW_t.
\]
We recall that, in this section and throughout the paper, the notation \(F(\nabla)u\) denotes the distribution with Fourier transform given by
\[
\hat{F}(\nabla)u(\xi) := \hat{F}(\xi)\hat{u}(\xi).
\]
Let \(\psi_N(t, \cdot) := \int_0^t \frac{\sin((t - t')|\nabla|)}{|\nabla|} \chi_N(\nabla)dW_t\), where \(\chi_N\) is the indicator function of the ball of radius \(N\) in \(\mathbb{R}^2\). Following the construction in [11], we define \(\psi^l_N\): for \(0 \leq l \leq 3\) in the following way
\[
\begin{align*}
\psi^0_N: (x, t) &:= 1, \\
\psi^1_N: (x, t) &:= \psi_N(x, t), \\
\psi^2_N: (x, t) &:= \psi^2_N(x, t) - \mathbb{E}[\psi^2_N(x, t)], \\
\psi^3_N: (x, t) &:= \psi^3_N(x, t) - 3\mathbb{E}[\psi^2_N(x, t)]\psi_N(x, t).
\end{align*}
\]
This corresponds with the Wick renormalisation obtained through the Hermite polynomial described in [11]. We will prove the following:

**Proposition 2.1.** Let \(0 \leq l \leq 3\), let \(\rho: \mathbb{R}^2 \to \mathbb{R}\) a \(C^\infty\) function with compact support, let \(\varepsilon > 0\), and let \(T > 0\). Let \(1 \leq p, q, r < \infty\).

(i) For every \(N\), \(\psi^l_N \in C^{\infty}_{t,x}(\mathbb{R} \times \mathbb{R}^2)\) a.s.

(ii) The sequence \(\rho: \psi^l_N: \) is almost surely (uniformly) in \(L^q_T W^{-\varepsilon,r}_x\). More precisely,
\[
\left\| \rho: \psi^l_N: \right\|_{L^q_T W^{-\varepsilon,r}_x} \leq C_{T, p, q, \varepsilon, r}.\]

(iii) The sequence \(\rho: \psi^l_N: \) is Cauchy in \(L^p_\Omega L^q_T W^{-\varepsilon,r}_x\).

Because of the arbitrariness of \(\rho\), (iii) implies that \(\psi^l_N: \) has a limit in \(L^p(\Omega, L^q_T W^{-\varepsilon,r}_x)\).
We call the limit of this sequence \(\psi^l: \). We have that
(iv) \( \|\rho : \psi^t\|_{L^p W_x^{-\varepsilon, r}} < \infty \) a.s., and \( \rho : \psi^t : (t) \) is a.s. continuous in \( t \) with values in \( W_x^{-\varepsilon, r} \).

(v) Let \( m : \mathbb{R} \to [0, 1] \) be an even smooth function such that \( m(r) = 1 \) for every \( |r| \leq 1 \), and \( \partial^p_m(r) \leq r^{-\varepsilon-n} \) for \( r \geq 1, n \geq 0 \). Let \( I_N \) be the operator associated to the Fourier multiplier \( m_N(|\xi|) := m(|\xi|/N) \), i.e. \( \hat{I}_N \hat{\phi}(\xi) := m(|\xi|/N) \hat{\phi}(\xi) \).

Then \( \mathbb{P} \left( \|I_N(\rho\psi^t)\|_{L^p L^q([0, T]; \mathbb{R}^d)} > \lambda p^\frac{3}{2} \log^\frac{1}{2} N \right) \leq C_T, q, m \lambda^{-\rho} \).

In order to prove this proposition, we need some tools. Recall the Hermite polynomials generating function

\[
F(t, x; \sigma) := e^{tx - \frac{1}{2} \sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x; \sigma).
\]

For simplicity, let \( F(t, x) := F(t, x; 1) \), and let \( H_k(x) := H_k(x; 1) \). Fix \( d \in \mathbb{N} \). Consider the Hilbert space \( L^2(\mathbb{R}^d, \mu) \), where \( d\mu := (2\pi)^{-\frac{d}{2}} \exp(-|x|^2/2) \). Then Hermite polynomials satisfy

\[
\int_{\mathbb{R}} H_k(x) H_m(x) d\mu_1(x) = k! \delta_{km}
\]

for all \( k, m \in \mathbb{N} \). Define the homogeneous Wiener chaos of order \( k \) to be an element of the form \( \prod_{j=1}^{d} H_{k_j}(x_j) \), where \( k = k_1 + \cdots + k_d \). Denote by \( \mathcal{H}_k \) the closure of homogeneous Wiener chaoses of order \( k \) in \( L^2(\mathbb{R}^d, \mu_1) \). Then, from the property \( L^2(\mathbb{R}^d, \mu_1) = \bigotimes_{j=1}^{d} L^2(\mathbb{R}, \mu_1) \), we have the Ito-Wiener decomposition

\[
L^2(\mathbb{R}^d, \mu_1) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k.
\]

Consider the operator \( L := - (\Delta - x \cdot \nabla) \) (the Ornstein-Uhlenbeck operator). Then any element in \( \mathcal{H}_k \) is an eigenvector of \( L \) with eigenvalue \( k \), so \( \bigoplus_{k=0}^{\infty} \mathcal{H}_k \) is the spectral decomposition of \( L^2 \) associated to \( L \).

Moreover, we have the following hypercontractivity of the Ornstein-Uhlenbeck semigroup \( U(t) := e^{-tL} \) due to Nelson [24].

**Lemma 2.2.** Let \( q > 1 \) and \( p \geq q \). Then, for every \( u \in L^q(\mathbb{R}^d, \mu_d) \) and \( t \geq \frac{1}{2} \log(\frac{p-1}{q-1}) \), we have

\[
\|U(t)u\|_{L^p(\mathbb{R}^d, \mu_d)} \leq \|u\|_{L^q(\mathbb{R}^d, \mu_d)}.
\]

Notice that the constant of the inequality in (2.2) (i.e. 1) and the range of \( p, q, t \) do not depend on the dimension \( d \). As a consequence, the following holds.

**Lemma 2.3.** Let \( F \in \mathcal{H}_k \). Then, for \( p \geq 2 \), we have

\[
\|F\|_{L^p(\mathbb{R}^d, \mu_d)} \leq (p - 1)^{\frac{k}{2}} \|F\|_{L^2(\mathbb{R}^d, \mu_d)}.
\]

This estimate follows simply by applying (2.2) to \( F \), setting \( q = 2, t = \frac{1}{2} \log(p - 1) \), and recalling that \( F \) is an eigenvector of \( U(t) \) with eigenvalue \( e^{-kt} \). As a further consequence, we obtain the following lemma.
Lemma 2.4. Fix $k \in \mathbb{N}$ and $c(n_1, \ldots, n_k) \in \mathbb{C}$. Given $d \in \mathbb{N}$, let $\{g_n\}_{n=1}^d$ be a sequence of independent standard complex-valued Gaussian random variables. Define $S_k(\omega)$ by

$$S_k(\omega) = \sum_{\Gamma(k,d)} c(n_1, \ldots, n_k)g_{n_1}(\omega) \cdots g_{n_k}(\omega),$$

where $\Gamma(k,d)$ is defined by

$$\Gamma(k,d) = \{(n_1, \ldots, n_k) \in \{1, \ldots, d\}^k\}.$$

Then, for $p \geq 2$, we have

$$\|S_k\|_{L^p(\Omega)} \leq \sqrt{k + 1}(p - 1)^{\frac{1}{p}} \|S_k\|_{L^2(\Omega)}.$$  

Proof of Proposition 2.1.

(i) Consider the operator $A_N(t) := \frac{\sin(t|\nabla|)}{|\nabla|} \chi_N(\nabla)$. We have that $A_N(t)\psi = \phi * a_N(t, \cdot)$, where $a_N(t, x) = \int_{B_N} \frac{\sin(t|\xi|)}{|\xi|} e^{2\pi i x \cdot \xi} d\xi$. Moreover,

$$a_N \in H^{2m}([-T, T] \times \mathbb{R}^2) \text{ for every } m \in \mathbb{N},$$

since

$$\|a_N\|_{H^{2m}}^2 \sim \|\partial^2_{t} a_N\|_{L^2}^2 + \|\Delta^m a_N\|_{L^2}^2 = 2 \int_{-T}^{T} \int_{B_N} \sin(t|\xi|)^2 |\xi|^{4m-2} d\xi < +\infty.$$ 

Therefore, for any ball $B \subseteq \mathbb{R}^2$,

$$\mathbb{E}\|\psi_N\|_{H^{2m}([-T, T] \times B)}^2 \sim \mathbb{E}\|\partial^2_{t} \psi_N\|_{L^2([-T, T] \times B)}^2 + \mathbb{E}\|\Delta^m \psi_N\|_{L^2([-T, T] \times B)}^2,$$

and

$$\mathbb{E}\|\partial^2_{t} \psi_N\|_{L^2([-T, T] \times B)}^2$$

$$= \int_{[-T,T] \times B} \mathbb{E} \left| \int_{0}^{t'} \int_{\mathbb{R}^2} \partial^2_{t} a_N(t - t', x - x') dW(t', x') \right|^2 dt dx$$

$$\leq \int_{[-T,T] \times B} \|a_N\|_{H^{2m}([-T, T] \times \mathbb{R}^2)}^2 dt dx$$

$$= 2T|B| \|a_N\|_{H^{2m}([-T, T] \times \mathbb{R}^2)}^2 < +\infty,$$
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and similarly,

$$
\mathbb{E} \left\| \Delta^m \psi_N \right\|_{L^2([-T,T] \times B)}^2
= \int_{[-T,T] \times B} \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^2} \Delta^m a_N(t-t', x-x') dW(t', x') \right] dt dx
\leq \int_{[-T,T] \times B} \left\| a_N \right\|_{H^{2m}([-T,T] \times \mathbb{R}^2)}^2 dt dx
= 2T |B| \left\| a_N \right\|_{H^{2m}([-T,T] \times \mathbb{R}^2)}^2
< +\infty,
$$

so $\mathbb{E} \left\| \psi_N \right\|_{H^{2m}([-T,T] \times B)}^2 < +\infty$ for every $m$ a.s., therefore $\psi_N \in C_{t,x}^\infty$ a.s.

(ii) We have that, for $s < t$, by Plancherel,

$$
\mathbb{E} \left[ \left\| \psi_N(s,x) \psi_N(t,y) \right\| \right]
= \mathbb{E} \left[ \int_s^t \int_{\mathbb{R}^2} a_N(s-t', x-x') dW(t', x') \int_s^t \int_{\mathbb{R}^2} a_N(t-t', y-x') dW(t', x') \right]
= \int_s^t \int_{\mathbb{R}^2} a_N(s-t', x-x') a_N(t-t', y-x') dt' dx'
= \int_s^t \int_{B_N} \sin((s-t')|\xi|) \sin((t-t')|\xi|) e^{-2\pi i \xi \cdot (x-y)} d\xi dt'.
$$

Define $\gamma(t, \xi)$ by

$$
\gamma(t, \xi) := \int_0^t \frac{\sin((t-t')|\xi|)^2}{|\xi|^2} dt'.
$$

By applying the Bessel potentials $\langle \nabla_x \rangle^{-\varepsilon}$ and $\langle \nabla_x \rangle^{-\varepsilon}$ of order $\varepsilon$ and setting $t = s$, $x = y$, we get

$$
\mathbb{E} \left[ \left\| \langle \nabla \rangle^{-\varepsilon} \psi_N(t, x) \right\| \right] = \int_{|\xi|<N} \langle \xi \rangle^{-2\varepsilon} \gamma(t, \xi) d\xi
\lesssim t^3 + t \int_{|\xi|>1} \frac{1}{(\xi)^{2+2\varepsilon}}
\lesssim t^3 + t.
$$

for any $\varepsilon > 0$, $x \in \mathbb{R}^2$, and uniformly in $N$. In particular, by hypercontractivity (Lemma 2.4), we have that

$$
\mathbb{E} \left[ \left\| \langle \nabla \rangle^{-\varepsilon} \psi_N(t, x) \right\|^p \right] \lesssim_{p,\varepsilon,t} 1,
$$

and thus, since $\rho \in C_c^\infty$,

$$
\mathbb{E} \left[ \left\| \rho(\cdot) \psi_N(t, \cdot) \right\|^p_{W^{-\varepsilon,p}} \right] = \mathbb{E} \left[ \left\| \langle \nabla \rangle^{-\varepsilon} \rho(\cdot) \psi_N(t, \cdot) \right\|^p_{L^p} \right] < +\infty
$$
for any $\varepsilon > 0$, $t > 0$, and $p \geq 1$, uniformly in $N \in \mathbb{N}$. 
By the properties of Wick products [27, Theorem I.22], we have that

$$
E \left[ \psi_N^l(t, x) :: \psi_N^l(t, y) : \right] \sim \{ E \left[ \psi_N(t, x) \psi_N(t, y) \right] \}^l
$$

$$
= \int_{|\xi_1|, \ldots, |\xi_l| \leq N} \prod_{j=1}^l \gamma(t, \xi_j) e^{-2\pi i \xi_j \cdot (x-y)} d\xi_j
$$

$$
= \int_{|\xi_1|, \ldots, |\xi_l| \leq N} e^{-2\pi i \sum_{j} \xi_j \cdot (x-y)} \prod_{j=1}^l \gamma(t, \xi_j) d\xi_j.
$$

Therefore, proceeding as before,

$$
E \left[ \left\| \langle \nabla \rangle^{-\varepsilon} : \psi_N(t, \cdot) : (x) \right\|^2 \right] = \int_{|\xi_1|, \ldots, |\xi_l| \leq N} \langle \xi_1 + \cdots + \xi_l \rangle^{-2\varepsilon} \prod_{j=1}^l \gamma(t, \xi_j) d\xi_j
$$

$$
\lesssim t \int_{|\xi_1|, \ldots, |\xi_l| \leq N} \langle \xi_1 + \cdots + \xi_l \rangle^{-2\varepsilon} \prod_{j=1}^l \langle \xi_j \rangle^{-2} d\xi_j < \infty.
$$

for any $\varepsilon > 0$, $t > 0$, uniformly in $x \in \mathbb{R}^2$ and $N$. Hence we have

$$
\left\| \rho(\cdot) : \psi_N(t, \cdot)^l : \right\|_{H^{-\varepsilon}}^2 \lesssim_{\rho, t} 1,
$$

and by hypercontractivity,

$$
E \left[ \left\| \rho(\cdot) : \psi_N(t, \cdot)^l : \right\|_{W^{-\varepsilon, p}}^p \right] \lesssim_{p, \rho, t} 1.
$$

Integrating this in time, we obtain

(2.7) $$
E \left[ \left\| \rho(x) : \psi_N(t, x)^l : \right\|_{L^p_x W^{-\varepsilon, p}([-T, T] \times \mathbb{R}^2)}^p \right] \lesssim_{p, \rho, T} 1.
$$

Moreover, since $\rho(\cdot) : \psi_N^l$ has compact support, $T < +\infty$, if $q, r, s \leq p$, it follows that

$$
\left\| \rho(x) : \psi_N(t, x)^l : \right\|_{L^q_x W^{-s, p}([-T, T] \times \mathbb{R}^2)} \lesssim_{p, \rho, T} 1.
$$

(iii) Following (ii), we have that for $N \leq M$:

$$
E[\psi_N(s, x) \psi_M(t, y)]
$$

$$
= \int_0^s \int_{B_N} \frac{\sin((s - t')|\xi|) \sin((t - t')|\xi|)}{|\xi|^2} e^{-2\pi i \xi \cdot (x-y)} d\xi dt'
$$

$$
= E[\psi_N(s, x) \psi_N(t, y)].
$$
Therefore, using the properties of Wick products [27, Theorem I.22] again,

\[
\mathbb{E} \left| \psi_M^j(t, x) : - : \psi_N^j(t, x) : \right|^2 \\
= \mathbb{E} \left| \psi_M^j(t, x) : \right|^2 - 2 \mathbb{E} \left[ \psi_M^j(t, x) : \psi_N^j(t, x) : \right] + \mathbb{E} \left| \psi_N^j(t, x) : \right|^2 \\
\sim \left\{ \mathbb{E} \left[ \psi_M(t, x)^2 \right] \right\}^l - 2 \left\{ \mathbb{E} \left[ \psi_M(t, x)\psi_N(t, x) \right] \right\}^l + \left\{ \mathbb{E} \left[ \psi_N(t, x)^2 \right] \right\}^l \\
= \left\{ \mathbb{E} \left[ \psi_M(t, x)^2 \right] \right\}^l - \left\{ \mathbb{E} \left[ \psi_N(t, x)^2 \right] \right\}^l \\
= \int_{N \leq |\xi_1|, \ldots, |\xi_l| \leq M} e^{-2\pi i \sum_j \langle \xi_j \rangle (x-y)} \prod_{j=1}^l \gamma(t, \xi_j) d\xi_j.
\]

Similarly, we have

\[
\mathbb{E} \left| \langle \nabla \rangle^{-\varepsilon} \left( \psi_M^j : - : \psi_N^j : \right) (t, x) \right|^2 \\
= \int_{N \leq |\xi_1|, \ldots, |\xi_l| \leq M} \langle \xi_1 + \cdots + \xi_l \rangle^{-2\varepsilon} \prod_{j=1}^l \gamma(t, \xi_j) d\xi_j \\
\lesssim_t \int_{|\xi_1|, \ldots, |\xi_l| \geq N} \langle \xi_1 + \cdots + \xi_l \rangle^{-2\varepsilon} \prod_{j=1}^l \langle \xi_j \rangle^{-2} d\xi_j \lesssim N^{-2\theta}.
\]

for every $0 < \theta < \varepsilon$. Integrating in space and time, we obtain

\[
\mathbb{E} \left[ \left\| \rho(x) : \psi_M(t, x)^l : - \rho(x) : \psi_N(t, x)^l : \right\|_{L_2^l H_x^{-\varepsilon}}^2 \right] \lesssim_{p, \rho, t} N^{-2\theta},
\]

from which, by hypercontractivity

\[
\mathbb{E} \left[ \left\| \rho(x) : \psi_M(t, x)^l : - \rho(x) : \psi_N(t, x)^l : \right\|_{W_{-\varepsilon, p}}^p \right] \lesssim_{p, \rho, t} N^{-\theta},
\]

and, arguing as in (ii),

\[
\left\| \rho(x) : \psi_M(t, x)^l : - \rho(x) : \psi_N(t, x)^l : \right\|_{L_p^l W_{-\varepsilon, q}([-T,T] \times \mathbb{R}^2)} \leq_{\text{max}(p, q), \rho, T} N^{-\theta}.
\]

(iv) Using the formula (2.5), we have that

\[
\mathbb{E} \left[ \psi_N^j(s, x)^l : \psi_N^j(t, y)^l : \right] \\
\sim \mathbb{E} \left[ \psi_N^j(s, x) \psi_N^j(t, y)^l : \right] \\
= \left( \int_0^{s\wedge T} \int_{B_N} \frac{\sin((s-t')|\xi|) \sin((t-t')|\xi|)}{|\xi|^2} e^{-2\pi i \langle \xi \rangle (x-y)} d\xi d\xi' \right)^l.
\]

Therefore, calling

\[
\gamma_N(s, t; \xi) := \int_0^{s\wedge T} \frac{\sin((s-t')|\xi|) \sin((t-t')|\xi|)}{|\xi|^2} d\xi' \lesssim_{s, t} \langle \xi \rangle^{-2},
\]
and proceeding as in (iii), we have that

\[
E \left[ \left( \langle \nabla \rangle^{-\varepsilon} : \psi_N(s, \cdot)^l : (x) \right) \left( \langle \nabla \rangle^{-\varepsilon} : \psi_N(t, \cdot)^l : (x) \right) \right]
\sim \int_{|\xi_1|, \ldots, |\xi_l| \leq N} \langle \xi_1 + \cdots + \xi_l \rangle^{-2\varepsilon} \prod_{j=1}^l \gamma(s, t ; \xi_j) d\xi_j.
\]

Therefore,

\[
E \left[ \left| \left( \langle \nabla \rangle^{-\varepsilon} : \psi_N(t + h, \cdot)^l : (x) \right) - \left( \langle \nabla \rangle^{-\varepsilon} : \psi_N(t, \cdot)^l : (x) \right) \right|^2 \right]
\sim \int_{|\xi_1|, \ldots, |\xi_l| \leq N} \langle \xi_1 + \cdots + \xi_l \rangle^{-2\varepsilon} \prod_{j=1}^l \gamma(t + h, t + h ; \xi_j) - 2 \prod_{j=1}^l \gamma(t, t + h ; \xi_j) + \prod_{j=1}^l \gamma(t, t ; \xi_j) d\xi_j.
\]

Interpolating between

\[
\gamma(t + h, t + h ; \xi) - \gamma(t, t + h ; \xi), \gamma(t, t + h ; \xi) - \gamma(t, t ; \xi) \lesssim_t h |\xi|^{-1}
\]

and

\[
\gamma(t + h, t + h ; \xi) - \gamma(t, t + h ; \xi), \gamma(t, t + h ; \xi) - \gamma(t, t ; \xi) \lesssim_t \langle \xi \rangle^{-2},
\]

we get that, for every \(0 \leq \theta \leq 1\),

\[
\gamma(t + h, t + h ; \xi) - \gamma(t, t + h ; \xi), \gamma(t, t + h ; \xi) - \gamma(t, t ; \xi) \lesssim_t \min\{h |\xi|^{-1}, \langle \xi \rangle^{-2}\} \lesssim_t h^\theta |\xi|^{-2+\theta}.
\]

Choosing \(\theta < 2\varepsilon\), by the discrete Liebnitz formula we obtain

\[
E \left[ \left| \left( \langle \nabla \rangle^{-\varepsilon} : \psi_N(t + h, \cdot)^l : (x) \right) - \left( \langle \nabla \rangle^{-\varepsilon} : \psi_N(t, \cdot)^l : (x) \right) \right|^2 \right]
\lesssim_{t,l} \int_{|\xi_1|, \ldots, |\xi_l| \leq N} \langle \xi_1 + \cdots + \xi_l \rangle^{-2\varepsilon} |h|^\theta \langle \xi_1 \rangle^{-2+\theta} \prod_{j=2}^l \langle \xi_j \rangle^{-2} d\xi_j \lesssim_{t,l} |h|^\theta.
\]

Since this inequality passes to the limit, we have that

\[
E \left[ \left| \left( \langle \nabla \rangle^{-\varepsilon} : \psi(t + h, \cdot)^l : (x) \right) - \left( \langle \nabla \rangle^{-\varepsilon} : \psi(t, \cdot)^l : (x) \right) \right|^2 \right] \lesssim_t |h|^\theta
\]

for a.e. \(t\). Integrating and using hypercontractivity, we get

\[
E \left[ \left\| \rho(\cdot) \left( : \psi(t + h, \cdot)^l : - : \psi(t, \cdot)^l : \right) \right\|_{W^{-\varepsilon,p}}^p \right] \lesssim_{t,p} |h|^{\frac{\theta}{2(p+1)}}.
\]

Moreover, if \(|t| \leq T\), the implicit constant \(C\) can be chosen as \(C = C(T, p)\). Therefore, by Kolmogorov Continuity Theorem, for \(|t| < T\), we have that \(\rho : \psi^l : \in C^\alpha W^{-\varepsilon,p}\), for every \(\alpha < \frac{\theta}{2(p+1)}\), so \(\rho : \psi^l : \in C\tau W^{-\varepsilon,p}\) a.s..
(v) In order to prove this, we just need to prove
\[ \mathbb{E} \left[ \| I_N(\rho \psi) \|_{L^p_T}^p \right] \leq C(T, \rho, m)^p p^{\frac{p}{2}} \log^{\frac{p}{2}} N, \]
then (v) will follow from a straightforward application of Chebishev inequality.

Proceeding as in (2.5), we have that
\[
\mathbb{E} \left[ |I_N(\rho \psi)|^2(t, x) \right] = \mathbb{E} \left[ \left| \iint m_N(\xi + \eta) \hat{\psi}(\xi) \hat{\rho}(\eta) e^{i2\pi(\xi+\eta) \cdot x} d\xi d\eta \right|^2 \right]
\]
which is the inverse Fourier transform of the function \( F : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{C} \),
\[
F(\eta_1, \eta_2) = \hat{\rho}(\eta_1) \hat{\rho}(\eta_2) \int m_N(\xi + \eta_1) \overline{m_N(\xi + \eta_2)} \gamma(t, \xi) \hat{\rho}(\eta_1) \hat{\rho}(\eta_2) e^{i2\pi(\eta_1 - \eta_2) \cdot x} d\xi d\eta_1 d\eta_2,
\]
restricted to the plane \( \{(x, -x)\} \). We have that \( \gamma(t, \xi) \lesssim t \langle \xi \rangle^{-2} \), and that for \( n \geq 1 \), \( \partial^n_N m_N(\xi + \eta) \lesssim \langle \xi + \eta \rangle^{-n} \), therefore calling
\[
\phi(\eta_1, \eta_2) := \int m_N(\xi + \eta_1) \overline{m_N(\xi + \eta_2)} \gamma(t, \xi) d\xi,
\]
we have that
\[
\| \nabla^n_{\eta_1, \eta_2} \phi \|_{L^\infty} \lesssim n, m \begin{cases} \log N & \text{if } n = 0, \\ 1 & \text{if } n \geq 1, \end{cases}
\]
hence \( \| \phi \|_{C^k_{t, \rho, m}} \lesssim t, \rho, m \log N \). Since \( \hat{\rho}(\eta_1) \hat{\rho}(\eta_2) \) is a Schwartz function of \( \eta_1, \eta_2 \), this implies that
\[
(2.8) \quad \mathbb{E} \left[ \| I_N(\rho \psi) \|_{L^2_t}^2(t, x) \right] \lesssim t, \rho, m \frac{\log N}{\langle x \rangle^4}.
\]
By hypercontractivity, since \( I_N(\rho \psi) \) is gaussian, one gets that
\[
(2.9) \quad \mathbb{E} \left[ \| I_N(\rho \psi) \|_{L^p_t}^p(t, x) \right] \leq C(T, \rho, m)^p p^{\frac{p}{2}} \log^{\frac{p}{2}} N \langle x \rangle^{4p},
\]
hence, integrating,
\[
\mathbb{E} \left[ \| I_N(\rho \psi) \|_{L^p_{t,x}}^p \right] \leq C(T, \rho, m)^p p^{\frac{p}{2}} \log^{\frac{p}{2}} N.
\]

3. Local well-posedness for the localized equation. Take a compactly supported \( \rho \in C^\infty_c(\mathbb{R}^2) \), and consider the equation
\[
v_{tt} - \Delta v + v^3 + 3v^2 \rho \psi + 3v \rho : \psi^2 : + \rho : \psi^3 := 0, \quad v(t_0, x) = u_0(x), \quad \partial_t v(t_0, x) = u_1(x).
\]
Notice that (at least formally), whenever \( \rho = 1 \), we have that

\[
v^3 + 3v^2 \rho \psi + 3v \rho : \psi^2 + \rho : \psi^3 : = (v + \psi)^3 : \n
\]

Consider as well the mild formulation of (3.1),

\[
(\text{LSNLW}) \quad v(t) = \cos((t - t_0)|\nabla|)u_0 + \frac{\sin((t - t_0)|\nabla|)}{|\nabla|} u_1 + \int_{t_0}^{t} \frac{\sin((t - t')|\nabla|)}{|\nabla|} \left( v^3 + \sum_{j=0}^{2} \binom{3}{j} v^j \rho : \psi^{3-j} : \right) dt.
\]

The following local well-posedness result holds:

**Proposition 3.1 (Local Well-Posedness).** Let \( 1 > s \geq \frac{2}{3} \), and let \((u_0, u_1) \in \mathcal{H}^s\). Then there exists a time \( \tau = \tau(t_0, \omega, \|(u_0, u_1)\|_{\mathcal{H}^s}) > 0 \) a.s., nonincreasing in \( \|(u_0, u_1)\|_{\mathcal{H}^s} \), such that the equation (LSNLW) has a unique solution in the space \( C([t_0 - \tau, t_0 + \tau]; \mathcal{H}^s) \). Moreover, a.s. in \( \omega \), we have that \( \inf_{|t_0| < T} \tau(t_0, \omega, \|(u_0, u_1)\|_{\mathcal{H}^s}) > 0 \) for every \( T < \infty \).

**Remark 3.2.** For the sake of simplicity, in this work we will use just Sobolev embeddings to get the required estimates, obtaining the local well-posedness result in the space \( H^s \) for \( s \geq \frac{2}{3} \). Since the constraint coming from Section 5 is stronger than this one, it is enough for our purposes. However, improving the proof by making use of the Strichartz estimates for the wave equation, it is possible to relax the condition to \( s > \frac{1}{3} \). See [11] for this analysis, which can be applied with very little modifications to this problem.

**Proof.** This proposition will follow by a standard fixed point argument. Consider the map

\[
\Gamma(v) := \cos((t - t_0)|\nabla|)u_0 + \frac{\sin((t - t_0)|\nabla|)}{|\nabla|} u_1
\]

\[
+ \int_{t_0}^{t} \frac{\sin((t - t')|\nabla|)}{|\nabla|} \left( v^3 + \sum_{j=0}^{2} \binom{3}{j} v^j \rho : \psi^{3-j} : \right) dt.
\]

defined on functions \( v \in C([t_0 - \tau, t_0 + \tau]; \mathcal{H}^s) \). By the Sobolev embedding \( H^s \to H^{\frac{2}{3}} \to W^{0+, 6-} \to L^6 \), and using that \( s < 1 \) and the fact that \( \rho : \psi^k : \) is compactly supported, we have that

\[
\| \Gamma(v) \|_{H^s} \leq \|(u_0, u_1)\|_{\mathcal{H}^s} + 2\tau \| v^3 \|_{L^\infty_t L^{\frac{2}{3}}_{x} H^{-1}} + \sum_{j=0}^{2} \binom{3}{j} \| v^j \rho : \psi^{3-j} : \|_{L^\infty_t L^{\frac{2}{3}}_{x} H^{-1}}
\]

\[
\lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + 2\tau \| v \|_{L^\infty_t L^6}^3 + \sum_{j=0}^{2} \binom{3}{j} \| v \|_{L^\infty_t W^{0+, 6-}}^j \| \rho : \psi^{3-j} : \|_{L^1_t W^{0-, 6+t}}
\]

\[
\lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + 2\tau \| v \|_{L^\infty_t H^s}^3 + \sum_{j=0}^{2} \binom{3}{j} \| v \|_{L^\infty_t H^s}^j \| \rho : \psi^{3-j} : \|_{L^1_t W^{0-, 6+t}}
\]

\[
\lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + \tau \| v \|_{C_t H^s}^3 + (1 + \|v\|_{C_t H^s}^3) \max_{1 \leq j \leq 3} \| \rho : \psi^j : \|_{L^{3}_{t} W^{0-, 6+t}}.
\]
By Proposition 2.1, \( \| \psi^j \|_{L^2 W^{0,-\delta +}_x} \) is finite a.s. (locally in time), so it is possible to find \( \tau \ll 1 \) such that \( \max_{1 \leq j \leq 3} \| \rho \psi^j \|_{L^1 W^{0,-\delta +}_x} < \delta \). Therefore, for \( \tau \) small enough, we have that if \( \| v \|_{C_t H^s} \leq 2 \| (u_0, u_1) \|_{H^s} \), then \( \| \Gamma_{u_0, u_1} v \|_{C_t H^s} \leq 2 \| (u_0, u_1) \|_{H^s} \).

Proceeding similarly, we have that
\[
\| \Gamma_{u_0, u_1} v - \Gamma_{u_0, u_1} w \| \lesssim \tau \| v - w \|_{C_t H^s} \left( \| v \|^2_{C_t H^s} + \| w \|^2_{C_t H^s} \right) + \| v - w \|_{C_t H^s} \left( 1 + \| v \|_{C_t H^s} + \| w \|_{C_t H^s} \right) \max_{j=1,2} \| \rho \psi^j \|_{L^1_1 W^{0,-\delta +}_x}.
\]

for the same reason, choosing \( \tau \ll 1 \) we have that \( \Gamma_{(u_0, u_1)} \) is a contraction on \( B_{2\| (u_0, u_1) \|_{H^s}} \subseteq C_t H^s \).

In order to finish the proof of this proposition, we just need to show that for every \( T > 0 \),
\[
\inf_{|t_0|<T} \tau(t_0, \omega, \| (u_0, u_1) \|_{H^s}) > 0.
\]

Notice that, by the previous argument, in order to have that \( \Gamma_{u_0, u_1} \) is a contraction, we just need that \( \tau, \| \rho \psi^j \|_{L^1_t W^{0,-\delta +}_x} \) for a certain fixed \( \delta \). By Proposition 2.1, we have that
\[
\| \rho \psi^j \|_{L^1_2 W^{0,-\delta +}_x([-T-1,T+1] \times \mathbb{R}^2)} < +\infty \text{ a.s.}
\]
As a consequence, we have that (when \( \tau < 1 \)),
\[
\| \rho \psi^j \|_{L^1_2 W^{0,-\delta +}_x([-t_0-\tau, t_0+\tau] \times \mathbb{R}^2)} \lesssim \tau^\frac{1}{2} \| \rho \psi^j \|_{L^1_2 W^{0,-\delta +}_x([-T-1,T+1] \times \mathbb{R}^2)}.
\]
Therefore, for fixed \( T, \tau \) can be chosen independently from \( t_0 \), and we have
\[
\inf_{|t_0|<T} \tau(t_0, \omega, \| (u_0, u_1) \|_{H^s}) > 0. \quad \square
\]

By this local well-posedness result, we obtain the following blow-up condition:

**Corollary 3.3 (Blow up condition).** Let \( T^* \) be the maximal time for which the solution to (LSNLW) exists in the interval \([0, T^*)\), in the sense that \((v, v_t) \in C([0, T^*); H^s)\) and for every \( \delta > 0 \), there is no function \((\tilde{v}, \tilde{v}_t)\) such that \((v, v_t) \in C([0, T^* + \delta); H^s)\) which solves (LSNLW). Suppose that \( T^* < +\infty \). Then we have
\[
\lim_{s \to T^*} \| (v(s), v_t(s)) \|_{H^s} = +\infty.
\]

**4. Global well-posedness for the localized equation.** In this section, we establish global well-posedness for the equation (LSNLW). In particular, we will prove the following

**Proposition 4.1.** Let \( s > \frac{4}{5} \). Then the solution to (SNLW) with \((u_0, u_1) \in H^s\) can be extended a.s. to a global solution \( u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R} \).

More precisely, for every \( T > 0 \), \( \delta > 0 \), there exists a set \( \Omega_{T, \delta} \) such that

- \( \mathbb{P}(\Omega_{T, \delta}) \leq \delta \),
- For every \( \omega \in \Omega_{T, \delta} \), there exists a unique solution \( u : [-T, T] \times \mathbb{R}^2 \to \mathbb{R} \) to (SNLW) such that \( v(0, x) = u_0(x), v_t(0, x) = u_1(x) \). Moreover, this solution satisfies the estimate \( \| v \|_{L^\infty([-T, T]; H^s_x)} \leq C(T, s, \delta) \).
Let \( m : \mathbb{R}^2 \to \mathbb{R} \) be a smooth radial function, \( 0 < m \leq 1 \), such that

\[
m(\xi) = \begin{cases} 
1 & \text{for } |\xi| \leq 1, \\
\frac{1}{|\xi|^\varepsilon} & \text{for } |\xi| \geq 3.
\end{cases}
\]

Let \( m_N(\xi) := m(\frac{\xi}{N}) \). This way, \( m_N \) will satisfy

\[
m_N(\xi) = \begin{cases} 
1 & \text{for } |\xi| \leq N, \\
\left(\frac{N}{|\xi|}\right)^\varepsilon & \text{for } |\xi| \geq 3N.
\end{cases}
\]

Let \( I_N \) the operator corresponding to the Fourier multiplier \( m_N \), i.e. \( \hat{I_N}f(\xi) = m_N(\xi)\hat{f}(\xi) \). By the definition of the multiplier, for every \( \sigma \in \mathbb{R}, 1 < p < +\infty, 0 \leq \delta \leq \varepsilon \), \( I_N \) will satisfy

\[
\|I_Nf\|_{W^{\sigma+\delta,p}} \lesssim N^\delta \|f\|_{W^{\sigma,p}},
\]

\[
\|f\|_{W^{\sigma,p}} \lesssim \|I_Nf\|_{W^{\sigma+\delta,p}}.
\]

One can establish these estimates by showing the analogous ones for \( I_1 \), and then the \( N \)-dependence will follow from a simple scaling argument.

From (4.3),(4.4), one has that, for fixed \( N \),

\[
\sup_{|s|<T} \|(I_Nv, I_Nv_t)\|_{H^{s+\varepsilon}} < +\infty.
\]

By taking \( \varepsilon = 1 - s \),

\[
E(v, v_t) := \frac{1}{2} \int v_t^2 + \frac{1}{2} \int v^2 + \frac{1}{2} \int |\nabla v|^2 + \frac{1}{4} \int v^4,
\]

we clearly have that

\[
E(I_Nv, I_Nv_t) \gtrsim \|(I_Nv, I_Nv_t)\|_{H^{s+\varepsilon}}^2 \gtrsim \|v, v_t\|_{H^s}^2.
\]

The goal of this section will henceforth be to show finiteness of \( E(I_Nv, I_Nv_t) \). In the following, we will abuse of notation and omit the subscript \( N \) whenever it is not important in the analysis, writing \( I \) instead of \( I_N \). Similarly, we will write \( E \) instead of \( E(I_Nv, I_Nv_t) \), and \( E(s) \) instead of \( E(I_Nv(s), I_Nv_t(s)) \).

\[\text{Lemma 4.2.}\]

\[
\frac{d}{dt} E(t) = -3 \int I_{v_t}(Iv)^2I(\rho\psi)
\]

\[
-3 \int I_{v_t}IvI(\rho : \psi^2 :) - \int I_{v_t}I(\rho : \psi^2 :)
\]

\[
+ \int I_{v_t}[(Iv)^3 - I(v^3)] + 3((Iv)^2I(\rho\psi) - I(v^2\rho\psi))
\]

\[
+ 3((Iv)I(\rho : \psi^2 :) - I(v\rho : \psi^2 :))
\]

\[
+ \int I_{v_t}Iv.
\]
Proof. We will show this proposition by a formal computation using (3.1). This computation can be made rigorous a posteriori by using the estimates of this section. We omit this part of the argument.

By (3.1),
\[
\frac{d}{dt} E(t) = \int \dot{I}v_t (Iv_t - I \Delta v + (Iv)^3 + Iv)
\]
\[
= \int Iv_t \left( - I(v^3 + 3v^2 \rho \psi + 3v \rho : \psi^2 : + \rho : \psi^3 :) + (Iv)^3 + Iv \right).
\]

The lemma follows by adding and subtracting the terms $3(Iv)^2 I(\rho \psi)$ and $3(Iv) I(\rho : \psi^2 : )$.

We will now proceed to estimate the various terms of the time derivative of $E(Iv, Iv_t)$, with the goal of applying a Gronwall argument. The terms (4.10),(4.8) are relatively harmless. Estimating the commutator terms in (4.9) is the core of the I-method, and will take most of this section. However, from a technical point of view, the hardest term to estimate will be (4.7), which will also give the main contribution to the estimate on the growth of $E$. This term is also what makes the iteration of the I-method with varying $N$ necessary.

Lemma 4.3.

(4.11) (4.10) = \int Iv_t Iv \leq E(Iv, Iv_t).

Proof. By Hölder,
\[
\int Iv_t Iv \leq \| Iv_t \|_{L^2} \| Iv \|_{L^2} \leq E(Iv, Iv_t).
\]

Lemma 4.4. For every $\gamma > 0$,

(4.12) $N^\gamma \left( E(Iv, Iv_t) \frac{3}{2} \| \rho : \psi^2 : \|_{W^{-\gamma,4}} + E(Iv, Iv_t) \frac{3}{2} \| \rho : \psi^3 : \|_{H^{-\gamma}} \right)$.

Proof. By Hölder and (4.3),
\[
-3 \int Iv_t Iv I(\rho : \psi^2 :) - \int Iv_t I(\rho : \psi^3 :)
\]
\[
\leq \| Iv_t \|_{L^2} \| Iv \|_{L^4} \| I(\rho : \psi^2 :) \|_{L^4} + \| Iv \|_{L^2} \| I(\rho : \psi^3 :) \|_{L^2}
\]
\[
\leq N^\gamma \left( E(Iv, Iv_t) \frac{1}{2} + \frac{3}{2} \| \rho : \psi^2 : \|_{W^{-\gamma,4}} + E(Iv, Iv_t) \frac{1}{2} \| \rho : \psi^3 : \|_{H^{-\gamma}} \right).
\]

Lemma 4.5. Let $k \leq 3$. Then

(4.13) $\| (Iv)^k - I(v^k) \|_{L^2} \leq s \ N^{-(k(1-s))} \| Iv \|^k_{H^1}$.
PROOF. Let \( v_{\leq N} = \int_{|\xi|<N/3} \hat{v}(\xi)e^{i\xi \cdot x} \), and let \( v_{\geq N} := v - v_{\leq N} \). Since \( \hat{v}_{\leq N}(\xi) \neq 0 \) only if \(|\xi| < N/3\), by definition of \( I \) we have that \( I v_{\leq N} = v_{\leq N} \). Similarly, \( v_{\leq N}^k(\xi) \neq 0 \) only if \(|\xi| < kN/3\), so \( I(v_{\leq N}^k) = v_{\leq N}^k \). Therefore, we have the following cancellation:

\[
(I v)^k - I(v^k) = (I(v_{\leq N} + v_{\geq N}))^k - I((v_{\leq N} + v_{\geq N})^k) = (v_{\leq N} + I(v_{\geq N}))^k - I((v_{\leq N} + v_{\geq N})^k) = v_{\leq N}^k - I(v_{\leq N}^k) + \sum_{l=0}^{k-1} \binom{k}{l} (I v_{\leq N}^l)(I v_{\geq N})^{k-l-1} - I(v_{\leq N}^l v_{\leq N}^k v_{\leq N}^{k-l-1})
\]

(4.14)

We proceed to estimating the elements of this sum individually. In particular, (4.13) follows if we prove that for every \( l \leq k-1 \),

\[
\left\| (I v_{\leq N}^l)(I v_{\geq N})^{k-l-1} \right\|_{L^2} \lesssim N^{-(1-k(1-s))} \| Iv \|_{H^1}^k
\]

(4.15)

and

\[
\left\| I(v_{\leq N}^l v_{\leq N}^k v_{\leq N}^{k-l-1}) \right\|_{L^2} \lesssim N^{-(1-k(1-s))} \| Iv \|_{H^1}^k.
\]

(4.16)

Let \( \varepsilon := 1 - s \). By Sobolev embeddings, we have that \( \| f \|_{L^\frac{2}{k-\varepsilon}} \lesssim \| f \|_{H^{1-\varepsilon}} \). By Hölder, we have

\[
\| f \|_{L^\frac{2}{k-\varepsilon}} \lesssim \| f \|_{L^2}^{\frac{k-\varepsilon}{k}} \| f \|_{L^\frac{2}{k}}^{\frac{(k-1)\varepsilon}{k}} \lesssim \| f \|_{L^2}^{\frac{k-\varepsilon}{k}} \| f \|_{H^{1-\varepsilon}}^{\frac{(k-1)\varepsilon}{k}}.
\]

Therefore, again by Hölder, we have:

\[
\left\| (I v_{\leq N}^l)(I v_{\geq N})^{k-l-1} \right\|_{L^2} \lesssim \| I v_{\leq N} \|_{L^\left(\frac{1}{k-\varepsilon}\right)} \| I v_{\leq N} \|_{L^\frac{2}{k}} \| I v_{\geq N} \|_{L^\frac{2}{k}}^{k-l-1} \lesssim \| I v_{\leq N} \|^{\frac{1-k\varepsilon}{k-\varepsilon}}_{L^2} \| I v_{\geq N} \|^{\frac{k-1\varepsilon}{1-\varepsilon}}_{H^{1-\varepsilon}} \| I v \|^{k-1}_{H^{1-\varepsilon}} \lesssim N^{-(1-\varepsilon)} \| I v \|^{k}_{H^{1-\varepsilon}}
\]

Proceeding similarly and using (4.4), we have

\[
\left\| I(v_{\leq N}^l v_{\leq N}^k v_{\leq N}^{k-l-1}) \right\|_{L^2} \lesssim \left\| v_{\leq N}^l v_{\leq N}^k v_{\leq N}^{k-l-1} \right\|_{L^2} \lesssim N^{-(1-k\varepsilon)} \| v \|^{k}_{H^{1-\varepsilon}} \lesssim N^{-(1-k\varepsilon)} \| Iv \|_{H^1}
\]

\( \square \)
Remark 4.6. We do not expect any extra cancellation in the summands of (4.14). To see this, let us writing one summand in Fourier transform, corresponding to the case \( k = 3, l = 1 \),

\[
\left( (Iv_{\geq N})^2 v_{\geq N}^2 - I(v_{\geq N}^2 v_{\leq N}^2) \right)^{\wedge} (\xi) = \int_{|\xi_1|,|\xi_2| > N/3, |\xi_1| \leq N/3, \xi_1 + \xi_2 + \xi_3 = \xi} \left( m(\xi_1) m(\xi_2) m(\xi_3) - m(\xi_1 + \xi_2 + \xi_3) \right) \hat{v}(\xi_1) \hat{v}(\xi_2) \hat{v}(\xi_3) d\xi_1 d\xi_2.
\]

In the regime \(|\xi_1|, |\xi_2|, |\xi_1 + \xi_2| \gg N\), we have that

\[
m(\xi_1) m(\xi_2) m(\xi_3) - m(\xi_1 + \xi_2 + \xi_3) = \left( \frac{N}{|\xi_1|} \right)^{1-s} \left( \frac{N}{|\xi_2|} \right)^{1-s} - \left( \frac{N}{|\xi_1 + \xi_2 + \xi_3|} \right)^{1-s} \sim - \left( \frac{N}{|\xi_1 + \xi_2|} \right)^{1-s} \sim - m(\xi_1 + \xi_2 + \xi_3),
\]

which corresponds to the “worst” term of the difference that appears in (4.14).

Lemma 4.7. For every \( \gamma > 0, 0 < \tilde{s} < 1 \), there exist \( p(\gamma) > 1, \eta(\gamma) > 0 \) such that

\[
\| (I f)(I g) - I(f g) \|_{L^2} \lesssim_{\gamma, \tilde{s}} N^{\gamma - \frac{1-\tilde{s}}{2}} \| f \|_{H^{1-\tilde{s}}} \| g \|_{W^{-\eta(\gamma), 1} W^{-\eta(\gamma), p(\gamma)}}
\]

Proof. As in the proof of Lemma 4.5, let us define

\[
u_{\leq M} := \int_{|\xi| < M/3} \hat{u}(\xi) e^{i\xi \cdot x}
\]

and \( u_{\geq M} := u - u_{\leq M} \). Writing \( f = f_{\leq N^{\frac{1}{2}}} + f_{\geq N^{\frac{1}{2}}} \) and \( g = g_{\leq N} + g_{\geq N} \), we have that

\[
(I f)(I g) - I(f g) = (I f_{\leq N^{\frac{1}{2}}})(I g_{\leq N}) - I(f_{\leq N^{\frac{1}{2}}} g_{\leq N}) + (I f_{\leq N^{\frac{1}{2}}})(I g_{\geq N}) - I(f_{\leq N^{\frac{1}{2}}} g_{\geq N}) + (I f_{\geq N^{\frac{1}{2}}})(I g) - I(f_{\geq N^{\frac{1}{2}}} g)
\]

We have that

- \( I = 0 \), since \( I f_{\leq N^{\frac{1}{2}}} = f_{\leq N^{\frac{1}{2}}}, I g_{\leq N} = g_{\leq N} \) by definition of \( I \), and \( (f_{\leq N^{\frac{1}{2}}} g_{\leq N})^\wedge (\xi) \neq 0 \) only for \(|\xi| \leq (N + N^{\frac{1}{2}})/3 < N\), so \( I(f_{\leq N^{\frac{1}{2}}} g_{\leq N}) = f_{\leq N^{\frac{1}{2}}} g_{\leq N} \) as well.
- \( \| II \|_{L^2} \) can be written as \( \sup_{p: \| h \|_{L^2} = 1} \int_{\mathbb{R}^2} h \cdot (II) \). Calling \( f_{\leq N^{\frac{1}{2}}} = a, g_{\geq N} = b \), expanding II in Fourier series and using Plancherel, we have to estimate

\[
\left\| \int_{|\xi_1| < N^{\frac{1}{2}}/3, |\xi_2| > N/3} a(\xi_1) b(\xi_2) \left( m_N(\xi_1) m_N(\xi_2) - m_N(\xi_1 + \xi_2) \right) h(\xi_1 + \xi_2) d\xi_1 d\xi_2 \right\|
\]
Using the fact that on the considered interval \(m(\xi_1) \equiv 1\) and that, by the mean value theorem, \(|m(\xi_1 + \xi_2) - m(\xi_1)| \lesssim N^{1-s}|\xi_1||\xi_2|^{-2+s}\), we have that

\[
\int_{\mathbb{R}^2} h \cdot (\text{II}) \leq \int_{\xi_1 < N^{1/2}, \xi_2 > N/3} a(\xi_1)b(\xi_2) \left( m_N(\xi_1)m_N(\xi_2) - m_N(\xi_1 + \xi_2) \right) \hat{h}(\xi_1 + \xi_2) d\xi_1d\xi_2
\]

\[
 \lesssim N^{(1-s)} \int_{\xi_1 < N^{1/2}, \xi_2 \geq N/2} \left| a(\xi_1) \frac{\hat{b}(\xi_2)}{|\xi_2|} \right| \delta \frac{1+\frac{\delta}{2}}{|\xi_1|^{2+\delta}} |\hat{\hat{h}}(\xi_1 + \xi_2)| d\xi_1d\xi_2
\]

\[
 \lesssim N^{-\left(\frac{1-2s}{2} - \frac{3}{2}\delta\right)} \int_{\xi_1 < N^{1/2}, \xi_2 \geq N/2} \frac{a(\xi_1)}{|\xi_1|^{2+\delta}} \frac{b(\xi_2)}{|\xi_2|^\delta} |\hat{\hat{h}}(\xi_1 + \xi_2)| d\xi_1d\xi_2
\]

\[
 \lesssim N^{-\left(\frac{1-2s}{2} - \frac{3}{2}\delta\right)} \|f\|_{H^1_{-\delta}} \|g\|_{H^{1-s}} \|h\|_{L^2},
\]

therefore \(\|\text{II}\|_{L^2} \lesssim N^{-\left(\frac{1-2s}{2} - \frac{3}{2}\delta\right)} \|f\|_{H^1_{-\delta}} \|g\|_{H^{1-s}}\).

- By Hölder, Sobolev embeddings and (4.3),

\[
\|\text{III}\|_{L^2} \lesssim \|I f_{\leq N^{1/2}}\|_{L^2} \|Ig\|_{L^\infty}
\]

\[
 \lesssim N^{-\left(\frac{1-2s}{2} + \delta\right)} \|f\|_{H^{1-s}} \|Ig\|_{W^{3\delta,\delta-1}}
\]

\[
 \lesssim N^{-\left(\frac{1-2s}{2} + 4\delta\right)} \|f\|_{H^{1-s}} \|g\|_{W^{3\delta,\delta-1}}.
\]

- By duality (like for II), self-adjointness of \(I\), fractional Liebnitz inequality, Sobolev embeddings and (4.3), we have

\[
\int h \cdot (\text{IV}) \leq -\int hI(f_{\geq N^{1/2}}g) = -\int I(h)f_{\geq N^{1/2}}g
\]

\[
 \lesssim \|I(h)f_{\geq N^{1/2}}\|_{W^{2\delta,(1-\delta)-1}} \|g\|_{W^{-2\delta,\delta-1}}
\]

\[
 \lesssim \|g\|_{W^{-2\delta,\delta-1}} \left( \|I(h)\|_{H^{2\delta}} \|f_{\geq N^{1/2}}\|_{L^{(\frac{1}{2}-\delta)-1}} + \|I(h)\|_{L^{(\frac{1}{2}-\delta)-1}} \|f_{\geq N^{1/2}}\|_{H^{2\delta}} \right)
\]

\[
 \lesssim \|g\|_{W^{-2\delta,\delta-1}} \|I(h)\|_{H^{2\delta}} \|f_{\geq N^{1/2}}\|_{H^{2\delta}}
\]

\[
 \lesssim N^{2\delta} N^{-\left(\frac{1-2s}{2} - 2\delta\right)} \|g\|_{W^{-2\delta,\delta-1}} \|h\|_{L^2} \|f_{\geq N^{1/2}}\|_{H^{1-s}}
\]

\[
 \lesssim N^{-\left(\frac{1-2s}{2} - 3\delta\right)} \|g\|_{W^{-2\delta,\delta-1}} \|h\|_{L^2} \|f_{\geq N^{1/2}}\|_{H^{1-s}},
\]

so \(\|\text{IV}\|_{L^2} \lesssim N^{-\left(\frac{1-2s}{2} - 3\delta\right)} \|g\|_{W^{-2\delta,\delta-1}} \|f\|_{H^{1-s}}\).
Therefore, by choosing \( \delta \) such that \( \gamma \geq 4\delta \), \( \eta = \delta \) and \( p = \delta^{-1} \), we obtain (4.17).

**Lemma 4.8.** Let \( 0 < k \leq 2 \). For every \( \gamma > 0 \), there exist \( p(\gamma) > 1 \), \( \eta(\gamma) > 0 \) such that

\[
(4.18) \quad \| I(v^k \rho : \psi^{3-k} : ) - (Iv)^k I(\rho : \psi^{3-k} : ) \|_{L^2} \lesssim_{s,\gamma} N^{-\frac{1-k(1-s)}{2}} + \gamma \| Iv \|_{H^1}^k \| \rho : \psi^{3-k} : \|_{W^{-\eta(\gamma),1} \cap W^{-\eta(\gamma),p(\gamma)}}.
\]

**Proof.** We have that

\[
\| I(v^k \rho : \psi^{3-k} : ) - (Iv)^k I(\rho : \psi^{3-k} : ) \|_{L^2} \lesssim \| I(v^k \rho : \psi^{3-k} : ) - I(v^k) I(\rho : \psi^{3-k} : ) \|_{L^2} + \| (I(v^k) - (Iv)^k) I(\rho : \psi^{3-k} : ) \|_{L^2}.
\]

- By Sobolev embeddings and fractional Liebnitz, we have that

\[
\| v^k \|_{H^{1-k(1-s)}} \lesssim \| v^k \|_{W^{1 + k(1-s)(1-s), (1-s)}} \lesssim \| v \|_{H^s} \| v^k \|_{H^{k(1-s)}} \lesssim \| v \|_{H^s}^k.
\]

Therefore, by (4.17),

\[
\| I \|_{L^2} \lesssim N^{-\frac{1-k(1-s)}{2}} + \gamma \| Iv \|_{H^s}^k \| \rho : \psi^{3-k} : \|_{W^{-\eta(\gamma),p(\gamma)}}.
\]

From (4.4), we have that \( \| v \|_{H^s} \lesssim \| Iv \|_{H^1} \) , so

\[
\| I \|_{L^2} \lesssim_{(1-s),\gamma} N^{-\frac{1-k(1-s)}{2}} + \gamma \| Iv \|_{H^s}^k \| \rho : \psi^{3-k} : \|_{W^{-\eta(\gamma),p(\gamma)}}.
\]

- From Hölder, (4.13) and Sobolev embeddings, we have

\[
\| I \|_{L^2} \lesssim \| I(v^k) - (Iv)^k \|_{L^2} \| I(\rho : \psi^{3-k} : ) \|_{L^\infty} \lesssim_{s,\delta} N^{-(1-k(1-s))} \| Iv \|_{H^1}^k \| I(\rho : \psi^{3-k} : ) \|_{W^{\delta,\delta^{-1}}} \lesssim_{s,\delta} N^{-(1-k(1-s))} N^{4\delta} \| Iv \|_{H^1}^k \| \rho : \psi^{3-k} : \|_{W^{\delta,\delta^{-1}}}.
\]

Choosing \( \delta \) small enough, we have that \( 1 - k(1-s) - 4\delta > \frac{1-k(1-s)}{2} - \gamma \), so the main contribution comes from \( I \). We get (4.18) by taking \( \gamma' = \geq 4\delta, p(\gamma') = \max(p(\gamma), \delta^{-1}) \), \( \eta(\gamma') = \max(\eta(\gamma), \delta) \), and then renaming \( \gamma = \gamma' \).

**Lemma 4.9.** There exists \( c > 0 \) such that for every \( 0 < \eta < \frac{1}{8} \),

\[
(4.19) \quad \int_{t_0}^T (4.7)(s)ds = \int_{t_0}^T \left( -3 \int Iv(t)(Iv(s))^2(I(\rho \psi(s))) \right)ds \lesssim \left( 1 + \int_{t_0}^T E^{1+c\eta} \right) \| \psi \|_{L_{t,x}^{-1}}.
\]
Proof. Let $0 < \theta < 1$. Since we have
\[
\|Iv\|_{H^1} \lesssim E^\frac{1}{2},
\]
\[
\|Iv\|_{L^4} \lesssim E^\frac{1}{4},
\]
by Gagliardo-Niremberg we have $\|Iv\|_{W^{\frac{1}{\theta},\frac{1}{1+\theta}}} \lesssim E^{\frac{1+\theta}{4}}$. Therefore, by Sobolev inequality, we have that $\|Iv\|_{L^4(t)} \lesssim E^{\frac{1}{4} + \eta}$, and the implicit constant is uniform in $\theta$ as long as $0 \leq \theta \leq \theta_{\text{max}} < 1$. Take $\theta = 4\eta$. Therefore, by Hölder,
\[
\left| \int_{t_0}^T \int_{T^2} I_{v_t}(Iv)^2 I_\psi \right| \lesssim \int_{t_0}^T \|Iv_t\|_{L^2} \|Iv\|_{L^4} \|I_\psi\|_{L^{4^{-1}}} \lesssim \int_{t_0}^T E^{\frac{1}{2}} E^{\frac{1}{4}} (E^{\frac{1+\theta}{4}}) \|I_\psi\|_{L^{4^{-1}}} \lesssim \int_{t_0}^T (E^{1+\eta}) \|I_\psi\|_{L^{4^{-1}}} \lesssim \left( \int_{t_0}^T (E^{1+\eta}) \|I_\psi\|_{L^{4^{-1}}} \right)^{1-\eta} \|I_\psi\|_{L^{4^{-1},t}} \lesssim \left( 1 + \int_{t_0}^T E^{1+\eta} \right) \|I_\psi\|_{L^{4^{-1},t}}.
\]
Therefore, choosing $c = \max_{\eta \in [0, \frac{1}{8}]} \eta^{-1} \left( \frac{1+\eta}{1-\eta} - 1 \right)$, we have
\[
\left| \int_{t_0}^T \int_{T^2} I_{v_t}(Iv)^2 I_\psi \right| \lesssim \left( 1 + \int_{t_0}^T E^{1+\eta} \right) \|I_\psi\|_{L^{4^{-1},t}},
\]
which gives (4.19). □

Lemma 4.10. Let $T > 0$. For every $|t - t_0| \leq T$, for every $0 < \eta \leq \frac{1}{8}$, we have that for every $\gamma > 0$,
\[
E(t) - E(t_0) \lesssim_{s, \gamma, T} \left( 1 + \int_{t_0}^t E^{1+\gamma} \right) \|I_\psi\|_{L^{4^{-1},t}}
\]
\[
+ \int_{t_0}^t N^{-(1-3(1-s))} E^2
\]
\[
+ \sum_{k=0}^2 \int_{t_0}^t N^{-\frac{1-k(1-s)}{2}} E^{\frac{k+1}{2}} \|\rho : \psi^2 : \|_{L^{\infty}W^{-\eta(\gamma)},1} \|_{W^{-\eta_k(\gamma),\eta_k(\gamma),p(\gamma)}}
\]
\[
+ \int_{t_0}^t N^{\gamma} \left( E^{\frac{3}{2}} \|\rho : \psi^2 : \|_{W^{-\gamma,4}} + E^{\frac{1}{2}} \|\rho : \psi^3 : \|_{H^{-\gamma}} \right)
\]
\[
+ \int_{t_0}^t E(s) ds,
\]
where $c$ is the one given by Lemma 4.9 and $\eta(\gamma), p(\gamma)$ are the ones given by Lemma 4.8.
Proof. By Lemma 4.2,

\[ E(t) - E(t_0) = \int_{t_0}^t (4.7)(s) + (4.8)(s) + (4.9)(s) + (4.10)(s) \, ds. \]

We have that

- From (4.19),
  \[ \int_{t_0}^t (4.7)(s) \, ds = \int_0^T \left( -3 \int Iv_t(s) (Iv(s))^2 I(\rho \psi(s)) \right) \, ds \lesssim \left( 1 + \int_0^T E^{1+c_\eta} \right) \|I \psi\|_{L^{\eta^{-1}}_{t,x}}. \]

- From (4.12), \( \int_{t_0}^t (4.8)(s) \, ds \lesssim (4.23) \),
- From (4.13) for the first term and (4.18) for the second and third terms respectively, and Hölder inequality, \( \int_{t_0}^t (4.9)(s) \, ds \lesssim (4.21) + (4.22) \)
- From (4.11),
  \[ \int_{t_0}^t (4.10)(s) \, ds = \int_{t_0}^t \left( \int Iv_t(s) Iv(s) \right) \, ds \leq \int_{t_0}^t E(Iv(s), Iv_t(s)) \, ds = (4.24). \]

\[ \square \]

Lemma 4.11. Let \( T > 0 \), and let

\[ (4.25) \quad A(N) = \frac{\|I_N \rho \psi\|_{L^{\log N}([-T,T] \times \mathbb{R}^2)}}{\log N}. \]

For \( \gamma > 0 \), \( M \geq 1, \Lambda \geq 1 \), define

\[ (4.26) \quad \Omega^\gamma_M := \left\{ \max_k \left\| \rho \psi^{3-k} \right\|_{L^{\infty}([-T,T] : W^{-\left(\max(p_{k_{\gamma}(\gamma)}), \max(p_{k_{\gamma}(\gamma)})\right), \max(p_{k_{\gamma}(\gamma)})}} \leq M \right\}, \]

\[ (4.27) \quad \Omega_{\Lambda}(N) := \{ A(N) \leq \Lambda \}. \]

Then for \( \gamma = \gamma(s) \) small enough, \( \alpha < 1 - 3(1-s) \), \( \delta < \beta < \alpha \), \( \omega \in \Omega^\gamma_M \), there exists \( \tau = \tau(s,M,\Lambda,\alpha - \beta) \) such that if \( \omega \in \Omega^\gamma_M \cap \Omega_{\Lambda}(N) \), \( E(t_0) \leq N^{\beta}/2 \), \( |t_0| \leq T \), \( N \geq N_0 = N_0(s,T,M,\Lambda) \), then \( E(t) \leq N^{\alpha} \) for every \( t \) such that \( |t| \leq T \) and \( |t - t_0| \leq \tau \).

Proof. By Lemma 4.10, as long as \( E \leq N^\alpha \), since \( \alpha < 1 - 3(1-s) \), for \( N \) big enough we have that \( (4.21) + (4.22) \leq 1 + \int_0^T E \). Similarly, from Young’s inequality, for some universal constant \( C \), we have

\[ (4.23) \quad \leq \int_0^T E(s) \, ds + CN^{4\alpha} M^4. \]

Choosing \( \eta = (\log N)^{-1} \) in (4.20), as long as \( E \leq N^\alpha \), we get

\[ (4.20) \quad \leq A(N) \log N \left( 1 + \int_0^T E^{1+c(\log N)^{-1}}(s) \, ds \right) \leq \Lambda \log N \left( 1 + e^{c\alpha} \int_0^T E(s) \, ds \right). \]
Therefore, as long as \( E(t) \leq N^\alpha \), for \( N \) big enough (depending on \( s, T, M, \Lambda \)),

\[
E(t) \leq E(t_0) + C(s, \gamma, T) \Lambda \log N \left(1 + \int_{t_0}^{T} E(s) \, ds\right) + \left(2 + C(s, \gamma, T) \Lambda \log N \right) \int_{t_0}^{T} E(s) \, ds
\]

Let \( \bar{t} = \max\{s : t_0 \leq s \leq T, E(s) \leq N^\alpha\} \), \( \tilde{t} = \min\{s : t_0 \geq s \geq -T, E(s) \leq N^\alpha\} \). Then the lemma is proven if we show that if \( \bar{t} \neq T \), then \( |\tilde{t} - t_0| \geq \tau(s, \gamma, T, \Lambda) \) and similarly if \( \bar{t} \neq -T \), then \( |\tilde{t} - t_0| \geq \tau(s, \gamma, T, \Lambda) \). For \( \tilde{t} \leq t \leq \bar{t} \), by definition, (4.28) holds, so by Gronwall

\[
E(t) \leq N^\beta \exp\left(|t - t_0| C'(s, \gamma, T, \Lambda) \log N\right).
\]

Suppose that \( \bar{t} \neq T \). Then one must have \( E(\bar{t}) = N^\alpha \). Therefore, by (4.29), \( |\tilde{t} - t_0| \geq \tau := \frac{(\alpha - \beta)}{C'(s, \gamma, T, \Lambda)} \). The same holds for \( \bar{t} \), and the lemma is proven.

**Proof of Proposition 4.1.** Let \( \varepsilon > 0, T > 0 \), let \( 2(1 - s) - \beta < \alpha < 1 - 3(1 - s) \), and let \( \gamma \) as in Lemma 4.11.

By Proposition 2.1,(iv), \( \|\rho_{\psi}^{3-k}\|_{L^\infty([-T,T];W^{-\left(\max(\eta_k(\gamma),\gamma)\right),\max(\rho_k(\gamma),\gamma)\right)} < +\infty \) a.s., so there exists \( M = M(T, \varepsilon) \) such that \( P(\Omega_M^T) \geq 1 - \frac{\varepsilon}{2} \), where \( \Omega_M^T \) is defined in (4.26). Moreover, by Proposition 2.1,(v),

\[
P(\Omega_N^T e) \leq \frac{N^\log N}{T,\rho, m} \Lambda^{-\log N} = N^{\log C_T,\rho, m - \log \Lambda}.
\]

If \( \Lambda = C_T,\rho, m e^2 \), this expression is summable in \( N \), so for \( N \) big enough, \( N \geq \bar{N} = \bar{N}(\varepsilon) \),

\[
P(\bigcap_{N \geq \bar{N}} \Omega_N^T) \geq 1 - \frac{\varepsilon}{2}.
\]

Let \( \Omega_{T,\varepsilon} := \Omega_M^T \cap \bigcap_{N \geq \bar{N}} \Omega_N^T \). By inclusion-exclusion, we have that \( P(\Omega_{T,\varepsilon}) \geq 1 - \varepsilon \).

For this choice of \( M, \Lambda, \gamma, \alpha, \beta \), let \( N_0 \) and \( \tau \) be the ones given by Proposition 4.11, and take \( (u_0, u_1) \in \mathcal{H}^s \). Define a sequence \( N_k \) of integers recursively. Take \( N_1 \) such that \( N_1 \geq \max(N_0, \bar{N}) \) and

\[
N_1^{2(1-s)} \|(u_0, u_1)\|_{\mathcal{H}^s}^2 + \|u_0\|_{\mathcal{H}^s}^4 \ll N_1^{\beta}.
\]

By Sobolev embeddings, (4.3) and (4.4),

\[
E(I_{N_1} v(0), I_{N_1} v_1(0)) = E(I_{N_1} u_0, I_{N_1} u_1)
\]

\[
\lesssim \|I_{N_1} u_0, I_{N_1} u_1\|_{\mathcal{H}^1}^2 + \|I_{N_1} u_0\|_{\mathcal{H}^4}^4
\]

\[
\lesssim N_1^{2(1-s)} \|(u_0, u_1)\|_{\mathcal{H}^s}^2 + \|I_{N_1} u_0\|_{\mathcal{H}^s}^4
\]

\[
\lesssim N_1^{2(1-s)} \|(u_0, u_1)\|_{\mathcal{H}^s}^2 + \|u_0\|_{\mathcal{H}^s}^4
\]
so we will have $E(I_{N_k}v(t_0), I_{N_k}v(t_0)) \leq \frac{1}{2}N_k^{2\alpha}$, therefore by Lemma 4.11 one has that
\[
E((v(t), v(t)))_{L^2} \lesssim E(I_{N_k}v(t), I_{N_k}v(t)) \leq N^\alpha \text{ for } t \leq T, t_0 \leq t \leq t_0 + \tau, \text{ and similarly backwards in time.}
\]
Then take $N_{k+1} \gg N_k$ such that
\[
(4.31) \quad N_{k+1}^{2(1-s)} N_k^\alpha + N_k^{2\alpha} \ll N_{k+1}^\beta.
\]
Take $k \in \mathbb{N}$ such that $t_0 + (k - 1)\tau \leq T$. If one has that, for every such $k$,
\[
(4.32) \quad E(I_{N_k}v(t_0 + (k - 1)\tau), I_{N_k}v(t_0 + (k - 1)\tau)) \leq \frac{1}{2}N_k^\beta,
\]
then by Lemma 4.11,
\[
(4.33) \quad \exists_{t_0 + (k - 1)\tau \leq s \leq \min(t_0 + k\tau, T)} E(I_{N_k}v(s), I_{N_k}v(s)) \leq N_k^\alpha,
\]
and similarly backwards in time. Therefore, because of the blowup condition (3.3), Proposition 4.1 is shown if we show (4.32). Proceeding inductively, we know (4.32) for $N_1$, and proceeding as in (4.30),
\[
E(I_{N_{k+1}}v(t_0 + k\tau), I_{N_{k+1}}v(t_0 + k\tau)) \lesssim N_{k+1}^{2(1-s)} \|(v(t_0 + k\tau), v(t_0 + k\tau))_{L^2} + \|v(t_0 + k\tau)\|_{H^s}^4
\]
\[
\lesssim N_{k+1}^{2(1-s)} E(I_{N_k}v(t_0 + k\tau), I_{N_k}v(t_0 + k\tau)) + E(I_{N_k}v(t_0 + k\tau), I_{N_k}v(t_0 + k\tau))^2
\]
\[
\lesssim N_{k+1}^{2(1-s)} N_k^\alpha + N_k^{2\alpha} \ll N_{k+1}^\beta,
\]
so $E(I_{N_{k+1}}v(t_0 + k\tau), I_{N_{k+1}}v(t_0 + k\tau)) \leq \frac{1}{2}N_{k+1}^\beta$ and we have (4.32). \qed

5. Independence from the cutoff and global well-posedeness for the global equation. In this section, we prove that on appropriate space-time regions, the solution to (LSNLW) does not depend on the particular choice of the cutoff $\rho$, and proceed to the proof of Theorem 1.1.

PROPOSITION 5.1 (Finite speed of propagation for SNLW). Let $R, T > 0$, $x_0 \in \mathbb{R}^2$, $t_0 \in \mathbb{R}$. Let $u_1, u_2$ be solutions to (SNLW) on $B(x_0, R)$ for a time $T$, in the sense that $u_j|_{B(x_0, R)} = \psi|_{B(x_0, R)} + v_j|_{B(x_0, R)}$, $v_j \in C([t_0 - T, t_0 + T]; H^s_{loc})$, $s > \frac{2}{3}$, and for every $|t - t_0| \leq T$,
\[
(5.1) \quad v_j(t)|_{B(x_0, R)} = \left(\cos((t - t_0)\nabla) v_j(t_0) + \frac{\sin((t - t_0)\nabla)}{\nabla} \partial_t v_j(t_0)
\right.
\]
\[
\left. + \int_{t_0}^t \frac{\sin((t - t')\nabla)}{\nabla} : (\psi + v_j)^3 : (t') dt' \right)|_{B(x_0, R)}.
\]
Suppose moreover that
\[
v_1(0) = v_2(0), \partial_t v_1(0) = \partial_t v_2(0) \text{ on } B(x_0, R), \quad v_j \in C([t_0 - T, t_0 + T]; H^s(\mathbb{R}^2)), s > \frac{2}{3}.
\]
Then $v_1(t)|_{B(x_0, R - |t - t_0|)} = v_2(t)|_{B(x_0, R - |t - t_0|)}$ for every $|t - t_0| \leq T$. 

Proof. Without loss of generality, assume that $t_0 = 0$, $x_0 = 0$. Let $D_t = B_{R-|t|}$. Recalling that the kernels of $\cos(s|\nabla|)$, $\frac{\sin(s|\nabla|)}{|\nabla|}$ are distributions supported in $B_s$, from (SNLW) we have that

\[ v_1 - v_2|_{D_t} = \int_0^t \sin((t-t')|\nabla|) \left[(v_1 - v_2)(3 : \psi^2 : + 3\psi(v_1 + v_2) + v_1^2 + v_2^2 + v_1v_2)\right]|_{D_t} dt'. \]

Proceeding as in Proposition 3.1, we obtain that

\[
\|v_1(t) - v_2(t)\|_{H^s(D_t)} \lesssim_R \int_0^t \|v_1 - v_2\|_{H^s(D_t')} (1 + \|v_1\|^2_{C([-R,R];H^s(B_R))} + \|v_2\|^2_{C([-R,R];H^s(B_R))}) \times \max_{1 \leq j \leq 3} \|\rho : \psi^j : \|_{L^1_t W^{0,-s+}}
\]

for any $\rho \in C^\infty_c$ with $\rho \equiv 1$ on $B_R$. Therefore by Gronwall,

\[
\|v_1(t) - v_2(t)\|_{H^s(D_t)} \leq C(R, v_1, v_2, \omega) \|v_1(0) - v_2(0)\|_{H^s(B_R)} = 0.
\]

From this proposition, we obtain immediately the following:

Corollary 5.2. Let $T > 0$, let $t_0 = 0$, and let $\rho_1$, $\rho_2$ be two cutoff functions such that $\rho_1(x) = \rho_2(x) = 1$ for every $x \in B_{2T}$. Let $s > \frac{4}{3}$. Let $(u_0, u_1) \in \mathcal{H}_loc^s$, and let $v_1$, $v_2$ be respectively the solutions to (LSNLW) with cutoff function $\rho_1$ and $\rho_2$ and initial data respectively $(\rho_1 u_0, \rho_1 u_1)$ and $(\rho_2 u_0, \rho_2 u_1)$. Then $v_1(t, x) = v_2(t, x)$ for every $|x|, |t| < T$.

Proof of Theorem 1.1. Let $\rho_n$ be a cutoff function such that $\rho_N(x) = 1$ for every $|x| \leq n$. Let $(u_0, u_1) \in \mathcal{H}_loc^s$, and let $v_n$ be the solution of (LSNLW) with cutoff function $\rho_n$ and initial data $(\rho_n u_0, \rho_n u_1)$. By Proposition 4.1, we will have $v_n \in C(\mathbb{R}; \mathcal{H}^s)$. By Corollary 5.2,

\[
v := \lim_{n \to +\infty} v_n
\]

is well defined, and we have that $v|_{[-T,T] \times B_R} = v|_{[\frac{T}{\sqrt{R}},\frac{T}{\sqrt{R}}] \times B_R}$. By Proposition 4.1 again, $v$ will also be a continuous function of $(u_0, u_1)$ with values in $C([-T,T]; \mathcal{H}_loc^s)$. Therefore the theorem is proven if we show that every solution $\tilde{u} = \psi + \tilde{v}$ of (SNLW) with $v \in C([-T,T]; \mathcal{H}_loc^s)$ satisfies $\tilde{v}(t) = v(t)$ for every $t \leq T$. Let $\phi \in C^\infty_c((-T,T) \times \mathbb{R}^2)$ be a test function. Let $n \in \mathbb{N}$ be such that $\text{supp}(\phi) \subseteq [-n,n] \times B_n$. By Proposition 5.1, we have that

\[
\tilde{v}|_{[-n,n] \times B_n} = v_n|_{[-n,n] \times B_n} = v|_{[-n,n] \times B_n}.
\]

Therefore, $\langle \tilde{v}, \phi \rangle = \langle v, \phi \rangle$, so $\tilde{v} = v$ as space-time distributions. Since they both belong to the space $C([-T,T]; \mathcal{H}_loc^s)$, the equality must hold in the space $C([-T,T]; \mathcal{H}_loc^s)$ as well, hence $\tilde{v}(t) = v(t)$ for every $t \leq T$. \qed
References.


[27] B. Simon, The $P(\varphi)_2$ Euclidean (quantum) field theory, Princeton Series in Physics. Princeton Universi-

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