AN ELLIPTIC HARNACK INEQUALITY FOR DIFFERENCE EQUATIONS WITH RANDOM BALANCED COEFFICIENTS

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Abstract. We prove an elliptic Harnack inequality at large scale on the lattice \(\mathbb{Z}^d\), for non-negative solutions of a difference equation with balanced i.i.d. coefficients which are not necessarily elliptic. We also identify the optimal constant in the Harnack inequality. Our proof relies on a quantitative homogenization result of the corresponding invariance principle to Brownian motion, and on percolation estimates. As a corollary of our main theorem we derive an almost optimal Hölder estimate.

1. Introduction

This paper deals with difference equations on \(\mathbb{Z}^d\) which are defined as follows: Let \(B^{(d)} = \{\pm e_i : i = 1 \ldots d\}\) be the set of all of the nearest neighbors of the origin in \(\mathbb{Z}^d\). An environment is a function \(\omega : \mathbb{Z}^d \times B^{(d)} \to [0, \infty)\). We view an environment \(\omega\) as the coefficients of a difference equation

\[
(L_\omega f)(z) := \sum_{e \in B^{(d)}} \omega(z, e)(f(z + e) - f(z)) = 0. \tag{1.1}
\]

As we think of the environment \(\omega\) as random, we need to set an appropriate probability space. Let \(\mathcal{M}^d\) be the space of functions from \(B^{(d)}\) to \([0, \infty)\), equipped with the Euclidean topology and its Borel \(\sigma\)-algebra. Let \(\Omega = (\mathcal{M}^d)^{\mathbb{Z}^d}\), and let \(\mathcal{F}\) be the corresponding product \(\sigma\)-algebra. Let \(P\) be a probability measure on \((\Omega, \mathcal{F})\). Then \((\Omega, \mathcal{F}, P)\) is a probability space from which one can sample a random environment \(\omega\).

We write \(x \sim y\) if \(x, y \in \mathbb{Z}^d\) are nearest neighbors, i.e., \(|x - y|_1 = 1\). For \(A \subset \mathbb{Z}^d\), let

\[
\partial A = \{y \in \mathbb{Z}^d \setminus A : y \sim x\ \text{for some}\ x \in A\}
\]

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denote the discrete boundary of $A$, and write $\bar{A} = A \cup \partial A$. Then a function $f : \bar{A} \to \mathbb{R}$ is called $\omega$-harmonic in $A$ if \([1.1]\) holds for every $z \in A$.

The purpose of this paper is to study properties of $\omega$-harmonic functions under some assumptions that will be specified later.

We say that $\omega$ is non-null if for every $z \in \mathbb{Z}^d$ there exists $e \in B^d$ such that $\omega(z, e) > 0$.

To study $\omega$-harmonic functions, an important resource is the relation to random walk in random environment (RWRE). For a non-null environment $\omega$, the Random Walk on $\omega$ is a Markov chain $(X_n)$ jumping to the nearest neighbors with transition kernel

$$P_\omega (X_{n+1} = z + e | X_n = z) = \frac{\omega(z, e)}{\sum_{v \in B^d} \omega(z, v)} := \hat{\omega}(z, e).$$ \hspace{1cm} (1.2)

The law $P_\omega (\cdot | X_0 = x)$ is denoted by $P_x \omega$, and $P_0 \omega$ is simply written as $P_\omega$. Expectations with respect to $P_x \omega$ and $P_\omega$ are written as $E_x \omega$ and $E_\omega$, respectively. We note here that since we work with a discrete time random walk, we have that $P_x \omega = P_\omega \hat{\omega}$ and $E_x \omega = E_\omega \hat{\omega}$ for all $\omega$ and $x$.

Observe that for a fixed $\omega$, if $f$ is $\omega$-harmonic, then \(\{f(X_n) : n \geq 0\}\) is a martingale.

In this paper we will focus on a special class of environments: the balanced environment.

**Definition 1.1** An environment $\omega$ is said to be balanced if for every $z \in \mathbb{Z}^d$ and neighbor $e$ of the origin, $\omega(z, e) = \omega(z, -e)$.

We want to make sure that the environment does not disconnect hyperplanes in $\mathbb{Z}^d$:

**Definition 1.2** An environment $\omega$ is said to be genuinely $d$-dimensional if for every neighbor $e$ of the origin, there exists $z \in \mathbb{Z}^d$ such that $\omega(z, e) > 0$.

Unless otherwise specified, throughout this paper we make the following assumption.

**Assumption 1** $P$ is an i.i.d. measure. Further, $P$-almost surely, $\omega$ is non-null, balanced and genuinely $d$-dimensional.

For an i.i.d. measure $P$, Assumption [1] is equivalent with

$$P[\omega(0, e) = \omega(0, -e)] = 1, \hspace{1cm} P[\omega(0, e) > 0] > 0 \hspace{1cm} \text{and} \hspace{1cm} P[\sum_{u \in B^d} \omega(0, u) > 0] = 1.$$

for every $e \in B^d$. Note that we do not assume ellipticity, i.e. that $P[\omega(z, e) > 0] = 1$. Moreover, our results in this paper are robust with respect to the marginal distribution, in particular they do not depend on the tail $P(\omega(0, e) > L)$ as $L \searrow 0$.

**Example 1** Take $P = \nu \mathbb{Z}^d$, $d \geq 2$, where the measure $\nu$ is defined as

$$\nu \left[ \omega(z, e_i) = \omega(z, -e_i) = \frac{1}{2}, \omega(z, e_j) = \omega(z, -e_j) = 0, \forall j \neq i \right] = \frac{1}{2}, \hspace{1cm} i = 1, \ldots, d.$$

In this example, the environment chooses uniformly at random one of the $d$ directions to have non-zero transition probabilities. See Figure [1].
Note that in the balanced case $L_\omega$ can be rewritten as

$$L_\omega u(x) := \sum_{i=1}^{d} \omega(x, e_i)[u(x + e_i) + u(x - e_i) - 2u(x)].$$

This is the discrete counterpart of a purely second order generator of a diffusion process on $\mathbb{R}^d$ in non-divergence form (cf. Papanicolaou-Varadhan [30])

$$\mathcal{L}_\omega u(x) = \text{tr}(a(\omega, x) \cdot D^2 u(x)) = \sum_{i,j=1}^{d} a_{i,j}(\omega, x) \partial_{ij} u(x),$$

with a positive semi-definite symmetric matrix $a_{i,j}(\omega, x) = a_{i,j}(\tau_y \omega, 0)$ where $\tau_y : \Omega \rightarrow \Omega$ is the shift operator which is defined by

$$\tau_y \omega(z, e) = \omega(z - y, e)$$

for $e \in B^{(d)}$ and $z, y \in \mathbb{Z}^d$ (recall that $B^{(d)} \subseteq \mathbb{Z}^d$).

Another model that deals with diffusions in divergence form is generated by

$$\mathcal{\tilde{L}}_\omega u(x) = \nabla \cdot (a(\omega, x) \nabla u(x)) = \sum_{i,j=1}^{d} \partial_i \left( a_{i,j}(\omega, x) \partial_j u(x) \right).$$

This corresponds in the discrete setting to random walks among random conductances with symmetric jump rates

$$\tilde{\omega}(x, e) = \tilde{\omega}(x + e, -e).$$
In this paper we do not work on equations in divergence form, resp. the random conductance models. There is an abundance of results similar to ours in that setting, cf. e.g. surveys of [13] and [24]. Although the results for these two classes of models are quite similar, the methods of the proof are different. For the random conductance model, the measure $P$ is the unique probability measure on $\Omega$ which is invariant for the process of the environment viewed from the point of view of the particle, whereas in balanced environments, the existence and uniqueness of such a measure is one of the main challenges. On the other hand balanced random walks are martingales, while in the random conductance model one needs to decompose the walk into harmonic coordinates (i.e. martingale) and sub-linear corrector.

Our main objective in this paper is to show an elliptic Harnack inequality with optimal Hanack constant in the non-elliptic i.i.d. balanced environment. In the derivation of our result, we will be dealing with the following challenges:

- Estimating the homogenization in the degenerate environment.
- Understanding the geometry of a directed percolation in the non-reversible environment.
- Controlling the oscillation in a highly heterogeneous environment.

Classical proofs of the Harnack inequality for non-divergence form operators rely on the fact that values of a harmonic function $u > 0$ are comparable within a small ball, e.g., $u(x) \leq Cu(y)$, for $|x - y| = 1$, which is equivalent to $u(x) \leq e^{c|x-y|}u(y)$. For non-elliptic random conductance models, such estimate follows from the fact that the graph distance in the classical bond percolation is comparable to the Euclidean distance. However, in our non-reversible balanced environment, the difficulty comes not only from the presence of “holes” within the environment but also from the fact that distances have directions: a path from $x$ to $y$ may not lead us from $y$ back to $x$. Thus deeper understanding of the connectivity structure of directed percolation within the balanced environment is crucial.

In contrast to classical bond percolation, balanced directed percolation has not been studied before. We thus present in this paper new methods and new results which are of independent interest in addition to their crucial role in investigating the difference equations and the random walk.

1.1 Main results.

Our main result is a Harnack inequality for $\omega$-harmonic functions. The classical Harnack inequality was proved by Harnack in 1887, see [21]. We also refer to [22] for further discussion on the classical theory.

Let $\phi : \Omega \to \mathbb{R}$ be the normalized adjoint solution of the equation, namely $\phi$ is a measurable function satisfying the following conditions:

1. $\phi$ is non-negative, and $\int_{\Omega} \phi dP = 1$, and
2. for $P$-almost every $\omega$,

$$\sum e \left( \tau_e \omega(0, e) \phi(\tau_e \omega) - \omega(0, e) \phi(\omega) \right) = 0.$$
Note that if such $\phi$ exists and is unique, then the measure $Q$ defined by
\[
dQ = \phi dP
\]
is ergodic with respect to the point of view of the particle. That is, the sequence $(\tau_{-X_n,\omega})_{n \geq 0}$ under measure $Q \otimes P_\omega$ is stationary and ergodic with respect to the left shift. In this light, \cite[Proposition 5.7]{12} guarantees that $\phi$ exists and is unique up to measure zero. Recall $\tilde{\omega}$ in (1.2). Let $\Sigma = (\Sigma_{ij})$ be the diagonal positive-definite matrix
\[
\Sigma_{ij} = 2\mathbb{E}_Q[\tilde{\omega}(0,e_i)]\delta_{ij} = 2\delta_{ij}\int_{\Omega} \tilde{\omega}(0,e_i)\phi(\omega)dP(\omega), \quad i,j = 1,\ldots,d.
\]
(1.5)

Let
\[
B_R = \{z \in \mathbb{R}^d : \|z\|_2 < R\}
\]
be the ball of radius $R$ in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, let
\[
B_R(x) = x + B_R \quad \text{and} \quad B^\mathrm{dis}_R(x) = B_R(x) \cap \mathbb{Z}^d,
\]
and write $B^\mathrm{dis}_R = B_R \cap \mathbb{Z}^d$. Let $\mathcal{H}(\Sigma)$ denote the set of non-negative solutions $u \in C^2(\mathbb{R}^d)$ to the differential equation $\text{tr}(\Sigma \cdot D^2 u) = 0$ in $B_1$. For $\theta > 1$, the Harnack constant (see also \cite{21,22}) $M_\theta = M_\theta(\Sigma) > 1$ is defined as
\[
M_\theta = \sup_{u \in \mathcal{H}(\Sigma)} \frac{\sup_{B^\mathrm{dis}_{\theta R}} u}{\inf_{B^\mathrm{dis}_{\theta R}} u}.
\]
(1.6)

We prove the following which is our main result

**Theorem 1.3** Let $\Theta > 1$. Recall the Harnack constant $M_\Theta$ in (1.6). Under Assumption 1, for any $H > M_\Theta$, there exists a finite random variable $R_0 = R_0(\omega,H,d,P)$ such that for every $R > R_0$ and any non-negative $\omega$-harmonic function $f$ in $B^\mathrm{dis}_R$, we have
\[
\max_{B^\mathrm{dis}_R} f \leq H \min_{B^\mathrm{dis}_R} f.
\]
(1.7)

Furthermore, there exist $C = C(H,d,P)$ and $\gamma = \gamma(d,P) > 0$ such that for every $M > 0$,
\[
P(R_0 > M) < C \exp(-M^\gamma).
\]
(1.8)

**Remark 1.** $M_\Theta$ is optimal in the following sense: Let $H < M_\Theta$. Then, for a.e. $\omega$, at every large enough scale, there are $\omega$-harmonic functions which violate (1.7). We supply a proof to this statement in Appendix E.

Notice that in our non-elliptic case, one can only expect the inequality at large scale: consider in Example 1 the event $A_R = \{\omega(x,e_1) = \omega(x,-e_1) = 1/2, x \in B^\mathrm{dis}_R\}$, then $P(A_R) > 0$ and the Harnack inequality fails for the function $f(x) = 1_{\{x \neq 0\}}$ if $\omega \in A_R$.

**Remark 2.** Harnack inequalities for balanced environments in the uniform elliptic setting have been proven before, see \cite{25,27,31}. However, we believe that our result is the first Harnack inequality for random environment which is both non-reversible and non-elliptic. It is also to the best of our knowledge the first Harnack inequality in the context of RWRE where the optimal Harnack constant is established. Recently, David Criens and the first author also derived the corresponding parabolic Harnack inequality at large scale, cf. \cite{10}.

For random walks on the supercritical percolation cluster, both elliptic and parabolic Harnack inequalities have been shown in \cite{8}, with a stretched exponential control similar to ours, see also \cite{1,2} for results for non-elliptic ergodic random conductance model.
Remark 3. The stretched exponential control of the size is usually expected for strongly mixing environments, however in our situation, the i.i.d. assumption is crucial – when \( d \geq 3 \) one can find counter-examples to the Harnack inequality in balanced, genuinely \( d \)-dimensional, translation invariant environments that possess finite range dependence. In that example the Harnack inequality fails due to the presence of two disconnected sinks (see Definition 1.10 below). Such an example is sketched in Appendix F to this paper.

From Theorem 1.3 we get the following Hölder estimate for \( \omega \)-harmonic functions at large scale, which is analogous to the regularity theory of Krylov-Safonov for non-divergence form, \[23\], respectively of De Giorgi-Nash for divergence form, cf. \[16\]. Notice that, in contrast to the classical regularity theory in the deterministic uniform elliptic setting for operators in non-divergence form, cf. \[25\], we can state our result (at large scale with exponential decay) with the Hölder power \( 1 - \epsilon \) arbitrarily close to 1. We note that in the uniformly elliptic non-divergence form PDE case, Armstrong and Lin \[5, Theorem 3.1\] showed that at large scale harmonic functions can be uniformly approximated by affine functions, which thus yields an optimal Hölder exponent 1. Moreover, in \[4\], Armstrong and Dario obtained a uniform bound for the discrete gradient of harmonic functions at large scale with a stretched exponential control in the setting of conductance models on the percolation cluster, which thus implies an optimal \( C^{0,1} \) regularity result. In fact, the result in \[4\] yields a large-scale regularity of every order.

We believe that our methods also enable achieving Hölder estimates with power \( 1 - \epsilon \) at a variety of other settings, in particular in the case of super-critical Bernoulli percolation.

For any finite subset \( E \subset \mathbb{Z}^d \) and any function \( f : E \to \mathbb{R} \), we define the oscillation of \( f \) over the set \( E \) by

\[
\text{osc}_E f := \max_{x,y \in E} [f(x) - f(y)].
\] (1.9)

Corollary 1.4 For every \( \epsilon \in (0, 1) \), there exists a constant \( C = C(d, \epsilon, P) \), and a random variable \( R_0 = R_0(\omega, \epsilon, P) \) such that

1. \( R_0 \) has a stretched-exponential tail as in (1.8).
2. For \( P \)-almost all \( \omega \), for all \( R > r > R_0(\omega, \epsilon, P) \), and for every \( \omega \)-harmonic function \( f : \mathbb{Z}^d \to \mathbb{R} \),

\[
\text{osc}_{B_r^{\text{dis}}} f \leq C \left( \frac{r}{R} \right)^{1-\epsilon} \text{osc}_R f.
\] (1.10)

Notice that Corollary 1.4 immediately implies a weak Liouville property, namely that for \( P \)-almost every \( \omega \), if an \( \omega \)-harmonic function \( f \) satisfies \( \lim_{|x| \to \infty} |f(x)|/|x|^\delta = 0 \) for some \( \delta \in (0, 1) \), then \( f \) is a constant function. However, the Liouville property (with \( \delta = 1 \)) has been derived in \[9\]. For random conductance models, Liouville results have been proven in the percolation setting \[9\], uniformly elliptic \[6\] and degenerate setting \[29\] and \[4\] where a Liouville principle is derived for any harmonic function of polynomial growth on the percolation cluster.
A starting point in our proof of the Harnack inequality is a weak bound (that holds with high probability), cf. Theorem 1.5 below, for the difference between the discrete $\omega$-harmonic function and the corresponding homogenized function, where the two functions have the same (up to discretization) boundary conditions.

Let $\Sigma$ be the matrix as defined in (1.5). Let $F \in C^2(\overline{B}_1)$ be a solution of $\text{tr}(\Sigma \cdot D^2 F) = 0$ in $B_1$. Note that $F \in C^\infty(B_1)$. For given $R > 0$, we denote by $B^{\omega, \text{dis}}_R := \{y \in B_R^\text{dis} : y \sim x \text{ for all } y \sim x\}$ the biggest subset of $B^\text{dis}_R$ such that $B^{\omega, \text{dis}}_R = B^\text{dis}_R$. Define the function $F_R : B^\text{dis}_R \to \mathbb{R}$ by $F_R(x) = F(x/R)$ for every $x \in B^\text{dis}_R$.

Let $G_{R, \omega} : \overline{B}_1^\text{dis} \to \mathbb{R}$ be the solution of the Dirichlet problem

$$
\begin{cases}
L_\omega G = 0 & \text{on } B^\omega, \text{dis}_R \\
G = F_R & \text{on } \partial B^\omega, \text{dis}_R.
\end{cases}
$$

(1.11)

In other words, $G$ is the $\omega$-harmonic function on $B^\text{dis}_R$ whose boundary data on $\partial B^\omega, \text{dis}_R$ agrees with that of $F_R$.

For $i \in \mathbb{N}$, let $M^i_F$ denote the supremum (of the absolute values) of all $i$-th order partial derivatives of $F$ on $B_1$. We can thus state the following quantitative estimate.

**Theorem 1.5** For any $\epsilon > 0$, there exists $R_0 = R_0(\epsilon, P) > \epsilon^{-2}$, $C_i(\epsilon) > 0$, $i = 1, 2$, and $\delta = \delta(P) > 0$ such that for any $R > R_0$,

$$
P\left( \max_{x \in B^\omega, \text{dis}_R} |F_R(x) - G_{R, \omega}(x)| \geq \epsilon (M^3_F + M^3_F) \right) \leq C_1 \exp(-C_2 R^\delta).
$$

This weak quantitative result, without explicit dependence of $C_i(\epsilon)$ in $\epsilon$, is sufficient for our proof of the Harnack inequality. A stronger result for the same model is shown in a work in progress [11]. Quantitative homogenization results under mixing conditions have been obtained in the reversible setting for the random conductance models in several papers, see for example [17] in the uniformly elliptic case and [4] on the percolation cluster. Armstrong and Smart [7] and Armstrong and Lin [5] derived the best quantitative estimate for diffusions in non-divergence form, see also [19] for the corresponding result for random walks in i.i.d. elliptic balanced random environment.

### 1.2 Structure of the paper.

In Section 1.3 we collect some results established in [12] that are useful later in the paper. In Section 2 we prove Theorem 1.5. In Section 3 we discuss the connectivity structure of the balanced directed percolation. While proving some results of independent interest, the main goal of this section is to provide percolation estimates necessary for the proof of Theorem 1.5. In Section 4 we prove an oscillation inequality, and then, finally, in Section 5 we combined the results of previous sections and use the ball reduction technique of Fabes and Stroock [16] to obtain Theorem 1.3.
homogenization estimates rather than heat kernel bounds, the ball reduction technique is able to give us an optimal Harnack constant.

Finally, some of the proofs in the paper, such as the proof of the percolation structure in 2 dimensions, were deferred to the appendix in order to make the main body of the paper focus on the main ideas.

1.3 Input from [12]: diffusive behavior and the sink.

In this section we review some useful definitions and results from [12].

Recall the following quenched invariance principle in [12]. Let \( \{X_n\}_{n=0}^{\infty} \) be the RWRE as defined in (1.2).

**Theorem 1.6** ([12]) Under Assumption 1, for \( P_0 \)-almost every \( \omega \), the process \( \{X_N(t) = N^{-1/2}X_{[Nt]} + N^{-1/2}(tN - [Nt])(X_{[Nt]+1} - X_{[Nt]}), t \geq 0\} \) converges weakly under \( P_0^\omega \) to a Brownian motion with deterministic non-degenerate covariance matrix \( \Sigma \) defined in (1.5).

The content of Theorem 1.6 is that on the large scale, the balanced RWRE behaves like a Brownian motion with covariance \( \Sigma \). Similar results have been obtained for balanced RW in elliptic RE by Lawler [26], Guo and Zeitouni [20] and for diffusions in non-divergence form in Papanicolaou-Varadhan [30] and Armstrong-Smart [7].

Next, we define the rescaled walk, which is a useful notion in the study of non-elliptic balanced RWRE, and recall some basic facts about it.

Let \( \alpha(n), n \geq 1 \), be the coordinate that changes between \( X_{n-1} \) and \( X_n \), i.e. \( \alpha(n) = i \) whenever \( X_n - X_{n-1} = \pm e_i \). The following is [12, Definition 3].

**Definition 1.7** The stopping times \( T_k, k \geq 0 \) are defined as follows: \( T_0 = 0 \). Then

\[
T_{k+1} = \min \{ n > T_k : \{\alpha(T_k + 1), \ldots, \alpha(n)\} = \{1, \ldots, d\} \} \leq \infty.
\]

We then define the rescaled random walk to be the sequence (no longer a nearest neighbor walk) \( Y_n = X_{T_n} \). Note that \( (Y_n) \) is defined as long as \( T_n \) is finite.

The following annealed and quenched estimates have been derived in [12], cf. Lemmas 2.1, 2.2, 2.3 and Corollary 2.4 in [12].

**Lemma 1.8** Let \( \mathbb{P}^x = P \otimes P_\omega^x \) denote the annealed measure. \( \mathbb{P}^0 \)-almost surely, \( T_k < \infty \) for every \( k \). There exists a constant \( C \) such that for every \( n \geq 1 \),

\[
\mathbb{P}^0(T_1 > n) < e^{-Cn^{3/2}}, \quad \text{and} \quad P(\{\omega : E_\omega(T_1) > n\}) \leq e^{-Cn^{3/2}}.
\]

Moreover for every \( 0 < p < \infty \),

\[
E[E_\omega(T_1^p)] < \infty.
\]

The maximum principle, originally due to Alexandrov-Bakelman-Pucci in the continuum and adapted to the discrete setting by Lawler [26], is one key analytical tool in the proof of the existence of the stationary measure \( Q \) for the walk. In our non-elliptic setting it can be restated in terms of the rescaled process as follows, cf. [12 Theorem 3.1].
For $N \in \mathbb{N}$ and $k = k(N) \in (0, N) \cap \mathbb{Z}$, let $T_1^{(N)} = T_1^{(N,k)} = \min(T_1, k)$. Let $h : \mathbb{Z}^d \to \mathbb{R}$ be a real valued function, and for every $z \in \mathbb{Z}^d$, let
$$L^{(N)}_\omega h(z) := h(z) - E^{z}_\omega [h(X_{T_1^{(N)},z})].$$

(1.12)

Let $\Delta \subseteq \mathbb{Z}^d$ be finite and connected, and let $\partial^{(k)} \Delta = \{ z \in \mathbb{Z}^d - \Delta : \exists x \in \Delta \| z - x \|_1 \leq k \}$. We say that a point $z \in \Delta$ is exposed if there exists $\beta = \beta(z,h) \in \mathbb{R}^d$ such that $h(z) - \langle \beta, z \rangle \geq h(x) - \langle \beta, x \rangle$ for every $x \in \Delta \cup \partial^{(k)} \Delta$. We let $D_h$ be the set of exposed points. Further, we define the upper contact set $I_h(z)$ as follows:
$$I_h(z) = \left\{ \beta \in \mathbb{R}^d : \forall x \in \Delta \cup \partial^{(k)} \Delta h(x) \leq h(z) + \langle \beta, x - z \rangle \right\}.$$  

(1.13)

This is the set of hyperplanes that touch the graph of $h$ at $(z, h(z))$ and are above the graph of $h$ all over $\Delta \cup \partial^{(k)} \Delta$. A point $z$ is exposed if and only if $I_h(z)$ is not empty.

**Theorem 1.9** (Maximum Principle, [12] Theorem 3.1) There exists $N_0$ such that for every $N > N_0$ and every $0 < k < N$, every balanced environment $\omega$, and every $\Delta$ of diameter $N$, if for every $z \in \Delta$
$$P^z_\omega (T_1 > k) < e^{-\log(N)^3},$$
then
$$\max \Delta - \max_{\partial^{(k)} \Delta} h \leq 6N \left( \sum_{z \in \Delta} 1_{z \in D_h} |L^{(N)}_\omega h(z)|^d \right)^{\frac{1}{d}}.$$ 

(1.15)

We now turn to some definitions and results pertaining to percolation.

**Definition 1.10** For $\omega \in \Omega$ and $x, y \in \mathbb{Z}^d$, we write $x \preceq y$ whenever $x \sim y$ and $\omega(x,y-x) > 0$. A sequence $(x_i)_{i=0}^n$ is called an “$\omega$-path” if $x_0 \preceq x_1 \cdots \preceq x_n$. We write $x \preceq y$ if there exists an $\omega$-path from $x$ to $y$. We say that a set $A \subseteq \mathbb{Z}^d$ is strongly connected if for every $x$ and $y$ in $A$, $x \preceq y$. A set $A \subseteq \mathbb{Z}^d$ is called a (\omega-) sink if it is strongly connected and $x \not\preceq y$ for every $x \in A$ and $y \not\in A$.

See Figure 1 for an illustration of the sink in Example 1.

Recall the density $\phi$ and the measure $Q$ in (1.4). We let $\text{supp} Q = \{ \omega : \phi(\omega) > 0 \}$ which is well-defined up to a set of $P$-measure zero. Define
$$\text{supp}^\omega Q = \{ z \in \mathbb{Z}^d : \tau_{-z}(\omega) \in \text{supp} Q \}.$$ 

In view of Corollary 4.12 and Lemma 5.6 of [12] we have the following lemma which says that, $P$-a.s. $\text{supp}^\omega Q$ is a finite union of sinks, each of which has lower density at least $\Omega$. For any discrete set $\Delta$, we denote the cardinality of $\Delta$ by $|\Delta|$.

**Lemma 1.11** (1) There exists $\Phi > 0$ such that for $P$-almost every $\omega$, every sink in $\mathbb{Z}^d$ has lower density at least $\Phi$, i.e. for every sink $C$,
$$\liminf_{n \to \infty} \frac{|C \cap [-n,n]^d|}{|[-n,n]^d|} \geq \Phi.$$
(2) For every ergodic $Q$ which is invariant with respect to the point of view of the particle and is absolutely continuous with respect to $P$, $P$-a.s., there are only finitely many sinks contained in $\text{supp}_Q$.

(3) $P$-a.s., every point in $\text{supp}_Q$ is contained in a sink.

As announced in \cite[Remark 3]{12}, let us now state the following proposition whose proof is given in Appendix A.

**Proposition 1.12** There exists a unique sink.

The last result that we need is the following lemma, which follows immediately from \cite[Proposition 5.9]{12} and Proposition 1.12.

**Lemma 1.13** Let $C$ be the (a.s. unique) sink. Then for $P$-a.e. $\omega$ and every $z \in \mathbb{Z}^d$, $P^x_\omega (\exists N \text{ s.t. } \forall n > N X_n \in C) = 1$.

### 2. Quantitative estimates for the invariance principle

Recall the notation from Section 1.1, and in particular the notation $\Sigma$, $B^\text{dis}_R$ and $B^\circ \text{dis}_R$.

Recall also that the function $F$ is harmonic on $B_1$, that $F_R$ is the stretching of $F$ to $B^\text{dis}_R$ and that $G_{R,\omega}$ is the $\omega$-harmonic function that agrees with $F_R$ on $\partial B^\circ \text{dis}_R$.

Our goal in this section is to obtain Theorem 1.5, namely that for any $\epsilon > 0$, there exist $R_0(\epsilon, P) > \epsilon^{-2}$ and $\delta(P)$ such that for $R > R_0$, with probability at least $1 - C \exp(-R^5)$,

$$\max_{x \in B_R^\text{dis}} |F_R(x) - G_{R,\omega}(x)| \leq \epsilon (M_2^F + M_3^F).$$

(2.1)

First, note that the “stretched” version $F_R$ of the function $F$ is very “flat” when $R$ is large. Indeed, for $x, y \in B^\text{dis}_R$, by Taylor expansion,

$$F_R(y) - F_R(x) = \frac{1}{R} (\nabla F_R(\hat{x}_R), y - x) + \frac{1}{2R^2} (y - x)^t D^2 F_R(\hat{x}_R)(y - x) + \rho_y \|y - x\|^3,$$

(2.2)

where the error term $\rho_y$ is bounded by $R^{-3}M_3^F$. Hence, we conclude that

$$|L_\omega F_R(x)| \leq \frac{M_2^F}{R^2} + \frac{2dM_3^F}{R^3} \leq \frac{C(M_2^F + 2dM_3^F)}{R^2} =: C_F$$

for all $x \in B^\circ \text{dis}_R$, (2.3)

where $\hat{\omega}$ is as defined in (1.2). Note that the constant $C$ may differ from line to line and so may $C_F$.

Our next observation is that since (by Theorem 1.6) the diffusion matrix of $X_n/\sqrt{n}$ converges to $\Sigma$, the function $F_R$ should be “approximately $\omega$-harmonic” in a sense that will be made precise in (2.6) below. Indeed, for $x \in \mathbb{Z}^d, n \in \mathbb{N}$, let $M_\omega^{(n)}(x)$ be the $d \times d$ covariance matrix with entries

$$(M_\omega^{(n)}(x))_{ij} := E_\omega[(X_n(i) - x)(X_n(j) - x)]/n, \quad 1 \leq i, j \leq n,$$
where $X_n(i)$ denotes the $i$-th coordinate of $X_n$. For any fixed $\epsilon > 0$ and $\alpha > 0$, by Theorem 1.6 there exists $n_0 = n_0(\epsilon, \alpha, P)$ such that for any $n \geq n_0$,  
\[ P \left[ \| M_n^{(n)}(x) - \Sigma \|_1 < \epsilon \right] > 1 - \alpha, \tag{2.4} \]
where by $\| \cdot \|_1$ we mean the $L^1$ norm in the space of matrices, thought of as $\mathbb{R}^{d}$. Moreover, for any $x \in B_{R}^{\epsilon, \text{dis}}$, by (2.2) we have
\[ E_{\omega}^{x}[F_{R}(X_{n_0}) - F_{R}(x)] = \frac{1}{2R^2} E_{\omega}[(X_{n_0} - x)^{T} D^{2} F(\frac{x}{R})(X_{n_0} - x)] + E_{\omega}^{x}[\rho X_{n_0} \| X_{n_0} - x \|^3] \]
\[ = \frac{n_0}{2R^2} \sum_{i,j=1}^{d} \partial_{ij} F(\frac{x}{R}) M_{\omega}^{(n_0)}(x)_{ij} + E_{\omega}^{x}[\rho X_{n_0} \| X_{n_0} - x \|^3], \]
where the linear term in (2.2) vanishes because the environment in balanced, which implies that $E_{\omega}^{x}(X_{n_0} - x) = 0$. Under the event
\[ A_{n_0}(x) = \{ \omega : \| M_{\omega}^{(n_0)}(x) - \Sigma \|_1 < \epsilon \}, \tag{2.5} \]
recalling that $\sum_{i,j=1}^{d} \partial_{ij} F(\frac{x}{R}) \Sigma_{ij} = 0$, we see that
\[ \left| \sum_{i,j=1}^{d} \partial_{ij} F(\frac{x}{R}) M_{\omega}^{(n_0)}(x)_{ij} \right| \leq \epsilon d^2 M_{F}^2. \]
Hence for $x \in B_{R}^{\epsilon, \text{dis}}$, when $A_{n_0}(x)$ occurs, we obtain, for $R > \frac{\sqrt{n_0}}{\epsilon} \vee n_0$,  
\[ |E_{\omega}^{x}[F_{R}(X_{n_0})] - F_{R}(x)| \leq \epsilon n_0 d^2 R^{-2} M_{F}^2 + C M_{\omega}^3 n_0^{3} R^{-3} \leq \epsilon n_0 R^{-2} C_{F}. \tag{2.6} \]

Recall the definitions of $n_0$ and $A_{n_0}$ in (2.4) and (2.5). We claim that, taking $k = \sqrt{R}$, (1.14) also happens with high probability. For fixed $\epsilon, \alpha > 0$, we define the events
\[ A_{R}^{(1)} = \left\{ \omega : \frac{\sum_{x \in B_{R}^{\epsilon, \text{dis}} 1_{\omega \in A_{n_0}(x)}}}{|B_{R}^{\text{dis}}|^{1/2}} > 1 - 2\alpha \right\}, \]
\[ A_{R}^{(2)} = \left\{ \omega : P_{\omega}(T_{1} > \sigma_{1/2}^{R}) < e^{-(\log R)^{3}} \text{ for all } x \in B_{R}^{\epsilon, \text{dis}} \right\}, \]
where the stopping time $T_{1}$ in the definition of $A_{R}^{(2)}$ is as in Definition 1.7.

**Lemma 2.1** Let $\epsilon \in (0, 1)$, $\alpha > 0$, $n_0 = n_0(\epsilon, \alpha, P)$ be the same as in (2.4). There exist $C = C(n_0, P)$ and $c = c(n_0, P)$ such that $P(A_{R}^{(1)} \cap A_{R}^{(2)}) > 1 - Ce^{-\epsilon R^{1/3}}$.

We postpone the proof of Lemma 2.1 until after the proof of Theorem 1.5. Throughout this section we always take $k = k(R) := \sqrt{R}$. To prove Theorem 1.5, we consider the error
\[ H(x) = H_{R,\omega}(x) := F_{R}(x) - G_{R,\omega}(x), \quad x \in B_{R}^{\text{dis}}. \]
Then $H$ is the solution of the discrete Dirichlet problem

$$
\begin{cases}
L_\omega H = L_\omega F_R & \text{on } B^\circ_{R,\text{dis}} \\
H = 0 & \text{on } \partial B^\circ_{R,\text{dis}}.
\end{cases}
$$

However, $H$ is not defined on $\partial (k) B^\circ_R$. To apply Theorem 1.9, we define an auxiliary function $H' : B^\circ_{R+k} \to \mathbb{R}$ to be the solution of the Dirichlet problem $H'|_{\partial B^\circ_{R+k}} = 0$ and

$$
L_\omega H' = \begin{cases}
L_\omega F_R & \text{on } B^\circ_{R,\text{dis}} \\
0 & \text{on } B^\circ_{R+k} \setminus B^\circ_R
\end{cases}
$$

Since $\hat{\omega}$ is defined in (1.2) as a normalization of $\omega$, we have

$$
L_\hat{\omega} H = L_\hat{\omega} H' = L_\omega F_R \text{ on } B^\circ_R.
$$

Hence,

$$
H'(x) = E^x_\omega \left[ \tau_{R+k} - \sum_{i=0}^{\tau_{R+k} - 1} -L_\omega F_R(X_i) 1_{X_i \in B^\circ_R} \right] \text{ for all } x \in \overline{B^\circ_{R+k}},
$$

and

$$
H(x) = E^x_\omega \left[ \sum_{i=0}^{\tau \omega - 1} -L_\omega F_R(X_i) \right] \text{ for all } x \in B^\circ_R,
$$

where $\tau_R := \tau(B^\circ_R)$ and

$$
\tau(\Delta) := \inf \{ n \geq 0 : X_n \notin \Delta \}
$$

denotes the exit time from $\Delta \subset \mathbb{R}^d$. Furthermore, by (2.3) and (2.7), for $k = \sqrt{R}$,

$$
\max_{\partial B^\circ_{R,\text{dis}}} |H'| \leq \max_{x \in \partial B^\circ_{R,\text{dis}}} C_F E^x_\omega [\tau_{R+k}] / R^2 \leq C_F k / R = C_F / \sqrt{R}
$$

and so

$$
\max_{B^\circ_{R,\text{dis}}} |H - H'| \leq \max_{\partial B^\circ_{R,\text{dis}}} |H - H'| = \max_{\partial B^\circ_{R,\text{dis}}} |H'| \leq C_F / \sqrt{R}.
$$

Now for any fixed small constant $\epsilon > 0$, we set

$$
\gamma := 3 \epsilon M^2 F R^{-2}
$$

and define $h : B^\circ_{R+k} \to \mathbb{R}$ as

$$
h(x) = H'(x) + \gamma \|x\|_2^2 \quad \forall x \in \overline{B^\circ_{R+k}}.
$$

Our first goal is to use Theorem 1.9 to estimate $\max_{B^\circ_{R,\text{dis}}} h$. We do so by showing that most of the points in $B^\circ_{R+k}$ are outside of $D_h$ (Lemma 2.2), and then controlling the $\omega$-Laplacian in the remaining points (Lemma 2.3 below).

**Lemma 2.2** Let $\epsilon, \alpha > 0$ be any fixed constants and let $n_0 = n_0(\epsilon, \alpha, P)$ be as in (2.4). Let $R > \sqrt{n_0} / \epsilon$. For any $x \in B^\circ_R$, if $\omega \in A_{n_0}(x)$, then $x \notin D_h$.

From Lemma 2.2 we see that such $x$ is not counted in the sum in (1.15).
Proof of Lemma 2.3 When $R > \frac{\sqrt{n_0}}{\epsilon} \vee n_0$, $x \in B_R^{\text{dis}}$ and $\omega \in A_{n_0}(x)$, by (2.6) and our definition of $\gamma$,

$$E^x_\omega[h(X_{n_0}) - h(x)] = E^x_\omega[H'(X_{n_0}) - H'(x)] + \gamma E^x_\omega(||X_{n_0}||^2 - ||x||^2)$$

$$= E^x_\omega[F_R(X_{n_0}) - F_R(x)] + \gamma E^x_\omega(||X_{n_0}||^2 - ||x||^2)$$

$$\geq -2\epsilon n_0 R^{-2} M^2_F + \gamma n_0 > 0.$$ 

Thus $E^x_\omega[h(X_{n_0})] > h(x)$. In particular, since $E^x_\omega[X_{n_0}] = x$, for every $\beta \in \mathbb{R}^d$ we get $E^x_\omega[h(X_{n_0}) + \langle \beta, X_{n_0} - x \rangle] > h(x)$. So for every $\beta \in \mathbb{R}^d$ there exists $y$ in the support of $X_{n_0}$ with $h(y) + \langle \beta, y - x \rangle > h(x)$, which implies $x \notin D_h$. \qquad \Box

Our next step is to control the $\omega$-Laplacian of the function $h$. For $x \in B_{R+k}$,

$$|E^x_\omega[h(X_1) - h(x)]| = |\omega \cdot h| \leq |\gamma - 1_{B_R^{\text{dis}}} C_F/R^2| \leq C_F/R^2.$$ 

For $p > 0$ and $K > 0$, we define the event

$$A^{(3)}_R(p,K) := \left\{ \omega : \frac{1}{|B_R^{\text{dis}}|} \sum_{x \in B_R^{\text{dis}}} (E^x_\omega[T_1])^p \leq K \right\}.$$

Lemma 2.3 Let $p > d$. There exist constants $\delta, C > 0$ depending on $(p,P)$ such that

$$P\left(A^{(3)}_R(p,C)\right) > 1 - Ce^{-R^\delta}.$$ 

We postpone the proof of Lemma 2.3 until after the proof of Theorem 1.5.

We are now ready to bound the function $H$ on $B_R^{\text{dis}}$.

Proof of Theorem 1.5 Let $\alpha = \alpha(\epsilon) > 0$ be a small constant to be determined later. Let $n_0 = n_0(\epsilon, \alpha, P)$ be as in (2.4) and $R > \frac{\sqrt{n_0}}{\epsilon} \vee n_0$. We only need to prove (2.1) for $\omega \in A^{(1)}_R \cap A^{(2)}_R \cap A^{(3)}_R$.

First we estimate $\max_{x \in B_R^{\text{dis}}} h(x)$. By Lemma 2.2, $1_{x \in D_h} L^{(R)}_\omega h(x) \neq 0$ only if $\omega \notin A_{n_0}(x)$ or $x \in J_k := B_R^{\text{dis}} \setminus B_{R-k}^{\text{dis}}$. Note that $|J_k| \leq CK^d R^{d-1}$.
By Theorem 1.9, on the event \( A_R^{(2)} \),

\[
\max_{B^R_{\text{dis}}} h - \max_{\partial(k)B^R_{\text{dis}}} h \leq 6R \left( \sum_{x \in \partial B^R_{\text{dis}}} 1_{x \in D_R(-L_\omega h(x))^2} \right)^{\frac{1}{3}} \tag{2.11}
\]

\[
\leq CR^2 \left( \frac{1}{|B^R_{\text{dis}}|} \sum_{x \in \partial B^R_{\text{dis}}} 1_{x \in J_k \text{ or } \omega \notin A_{n_0}(x)} \left| L_\omega^{(R)} h(x) \right|^d \right)^{\frac{1}{2}} \leq CR^2 \left( \frac{1}{|B^R_{\text{dis}}|} \sum_{x \in \partial B^R_{\text{dis}}} \left| L_\omega^{(R)} h(x) \right|^d \right)^{\frac{1}{2}} \tag{2.12}
\]

On the event \( A_R^{(1)} \),

\[
\sum_{x \in \partial B^R_{\text{dis}}} 1_{x \in J_k \text{ or } \omega \notin A_{n_0}(x)} \leq |J_k| + \sum_{x \in \partial B^R_{\text{dis}}} 1_{\omega \notin A_{n_0}(x)} \leq (\frac{k}{R} + 2\alpha)|B^R_{\text{dis}}|.
\]

Recall that \( k = \sqrt{R} \). Hence, for \( R > C/\alpha^2 \),

\[
\frac{1}{|B^R_{\text{dis}}|} \sum_{x \in \partial B^R_{\text{dis}}} 1_{x \in J_k \text{ or } \omega \notin A_{n_0}(x)} \leq 3\alpha. \tag{2.13}
\]

Applying (2.10) and Lemma 2.3 on the event \( A_R^{(3)} \), by (2.12) and (2.11),

\[
\max_{x \in B^R_{\text{dis}}} h(x) - \max_{x \in \partial(k)B^R_{\text{dis}}} h(x) \leq CR^2(3\alpha)^{1/(d+1)}d(C_F R^{-2}) = C_F \alpha^{1/(d+1)}d. \tag{2.14}
\]

Moreover,

\[
\max_{\partial(k)B^R_{\text{dis}}} h \leq \gamma(R + k)^2 + \max_{\partial(k)B^R_{\text{dis}}} H' \leq \gamma(R + k)^2 + \max_{\partial B^R_{\text{dis}}} H' \leq C_F(\epsilon + \frac{k}{R}), \tag{2.15}
\]

where in the second inequality we used the fact that \( H' \) is \( \omega \)-harmonic in \( B^{\text{dis}}_{R + k} \setminus B^R_{\text{dis}} \) and that an \( \omega \)-harmonic function achieves its maximum on the boundary.

Therefore, from (2.11), (2.12) and (2.13), we have

\[
\max_{B^R_{\text{dis}}} H \leq \max_{B^R_{\text{dis}}} H' + \frac{C_F}{\sqrt{R}} \leq \max_{B^R_{\text{dis}}} h + C_F(\epsilon + \frac{k}{R}) + C_F \alpha^{1/qd}. \tag{2.16}
\]

Replacing \( F \) by \( -F \), similar arguments also give the same upper bound for \( \max_{B^R_{\text{dis}}} (-H) \).

Theorem 1.5 now follows by taking \( \alpha = \epsilon^{(d+1)/2} \) and \( R \geq \epsilon^{-2} + n_0 \). \( \square \)

We now prove Lemmas 2.1 and 2.3.

**Proof of Lemma 2.7.** We start with estimating the probability of \( A_R^{(2)} \). By Lemma 1.8 (Recall the notation \( \mathbb{P}^x \) therein), for every \( x \in B^R_{\text{dis}} \),

\[
\mathbb{P}^x(T_1 > R^{1/2}) \leq e^{-cR^{1/6}},
\]
and thus by Markov’s inequality
\[ P \left( \{ \omega : P^x(T_1 > R^{1/2}) \geq e^{-(\log R)^3} \} \right) < e^{(\log R)^3 - cR^{1/6}}, \]
and a union bound over all possible values of \( x \) yields
\[ P(A_R^{(2)}) \geq 1 - Ce^{-R^{1/7}}. \quad (2.14) \]

Next we estimate the probability of the event \( A_R^{(1)} \). Note that the event \( A_{n_0}(x) \) is determined by \( \omega|_{y||y-x|| \leq n_0} \), and therefore \( (A_{n_0}(x_i))_{i \in I} \) are independent events whenever \( \forall i,j \in I \|x_i - x_j\| > 2n_0 \). For every \( z \in [-n_0, n_0]^d \) we write \( I_z = (z + (2n_0 + 1)\mathbb{Z}^d) \cap B_R^{\text{dis}} \). Then for every \( z \) the events \( (A_{n_0}(x_i))_{i \in I_z} \) are independent, and each happens with probability at least \( 1 - \alpha \). Therefore by Chernoff’s inequality, for every \( z \in [-n_0, n_0]^d \),
\[ P\left( \sum_{x \in I_z} \frac{1}{\omega_{A_{n_0}(x)}} > 1 - 2\alpha \right) > 1 - e^{-C|I_z|}. \]
Remembering that we have only finitely many choices of \( z \), and that \( |I_z| = \Theta(R^d) \) for every \( z \), a union bound gives us
\[ P(A_R^{(1)}) > 1 - e^{-CR^d}. \quad (2.15) \]

The lemma now follows from (2.14) and (2.15).

Proof of Lemma 2.3: For \( W = R^{1/2} \) we write \( E^x_{\omega}[T_1] = E_1(x) + E_2(x) \) with \( E_1(x) = E^x_\omega[T_1 \cdot 1_{T_1 \leq W}] \) and \( E_2(x) = E^x_\omega[T_1 \cdot 1_{T_1 > W}] \). We separately control
\[ \frac{1}{|B_R^{\text{dis}}|} \sum_{x \in B_R^{\text{dis}}} (E_1(x))^p \quad \text{and} \quad \frac{1}{|B_R^{\text{dis}}|} \sum_{x \in B_R^{\text{dis}}} (E_2(x))^p. \]

To control the empirical \( L^p \) norm of \( E_1(x) \) we note that \( E_1(x) \leq E^x_\omega[T_1] \), and that, as in the proof of Lemma 2.1, \( E_1(x_1) \) and \( E_1(x_2) \) are independent whenever \( \|x_1 - x_2\| > 2W \). Therefore, following the same decomposition as in the proof of Lemma 2.1 for
\[ K_1 = 2E \left[ (E^x_\omega(T_1))^p \right] \]
we get
\[ P\left( \frac{1}{|B_R^{\text{dis}}|} \sum_{x \in B_R^{\text{dis}}} (E_1(x))^p > K_1 \right) < e^{-CR^{d'}} \quad (2.16) \]
with \( d' > 0 \) determined by \( p \), by the choice of \( W \) and by the dimension. We now turn to control \( E_2(x) \). We note that by Lemma 1.8, \( E \left[ E^x_\omega(T_1^2) \right] < \infty \), and that \( E \left[ E^x_\omega(1_{T_1 > W}) \right] < e^{-cW^{1/3}} = e^{-cR^{1/6}} \). Thus by Cauchy-Schwarz,
\[ E \left[ E_2(x) \right] \leq C e^{-cR^{1/6}}. \quad (2.17) \]

From (2.17) we learn that
\[ P \left( \exists x \in B_R^{\text{dis}} E_2(x) > 1 \right) \leq |B_R^{\text{dis}}| \cdot e^{-cR^{1/6}} \leq e^{-R^{1/7}}. \quad (2.18) \]
From (2.18) we get that
\[ P\left( \frac{1}{|B_R^{\text{dis}}|} \sum_{x \in B_R^{\text{dis}}} (E_2(x))^p > 1 \right) \leq e^{-R^{3/7}}. \] (2.19)

The lemma now follows from (2.16) and (2.19) once we remember that \( E_{\omega}^x(T_1)^p \leq p(E_1(x)^p + E_2(x)^p) \) for all \( x \).

One corollary of Theorem 1.5 is particularly useful for us. This is Corollary 2.4 below. For \( A \subset \partial B_1 \) and a ball \( B = B_R^{\text{dis}}(x) \), denote the part of \( \partial B \) that correspond to \( A \) by
\[ \partial_A B := \{ y \in \partial B : \text{dist}(y, \partial B_1 \setminus A) \geq \epsilon \} . \] (2.20)

For \( x \in \mathbb{R}^d \), we let \( P^x_{\text{BM}} \) denote the law of the Brownian motion with covariance matrix \( \Sigma \) and starting point \( x \). We may describe an event without mentioning the underlying Brownian motion. E.g, it should be clear what \( P^x_{\text{BM}}(\text{exits the ball } B_1 \text{ through } A) \) means.

**Corollary 2.4** Let \( A \subseteq \partial B_1 \) be open in the relative topology of \( \partial B_1 \). Assume also that the boundary of \( A \) with respect to the topology of \( \partial B_1 \) has measure zero under the \((d-1)\text{-dimensional}) \) Lebesgue measure on \( \partial B_1 \). For \( x \in B_1 \), let
\[ \chi_A(x) = P^x_{\text{BM}}(\text{exits } B_1 \text{ through } A) \]

Let \( \delta > 0 \) be as in Theorem 1.5. For \( \epsilon, r \in (0, 1) \) and \( R > 0 \), let
\[ G(A, R, r, \epsilon) = \left\{ \omega : \text{max}_{x \in B_R^{\text{dis}}} |\chi_A(x) - P^x_{\omega}(X, \text{exits } B_R^{\text{dis}} \text{ through } \partial_B B_R^{\text{dis}})| \leq \epsilon \right\} . \]

Then, there are constants \( c, C \) depending on \( (A, r, \epsilon) \) such that for every \( R \geq 1 \),
\[ P(G(A, R, r, \epsilon)) \geq 1 - C \exp(-cR^\delta). \] (2.21)

**Proof.** We fix \( \epsilon \in (0, 1) \). Recall the constant \( R_0 \) in Theorem 1.5. It suffices to prove the lemma for all \( R \geq R_0(\epsilon^4, P) \). Our proof consists of several steps.

Step 1. First, we will define two functions \( F^{(1)}, F^{(2)} \) with smooth boundary data such that \( F^{(1)} \leq \chi_A \leq F^{(2)} \). For \( A \subset \partial B_1 \), we define subsets \( A^{-}_\epsilon, A^{+}_\epsilon \) of \( \partial B_1 \) as
\[ A^{-}_\epsilon = \{ x \in A : \text{dist}(x, \partial B_1 \setminus A) \geq \epsilon \} , \]
\[ A^{+}_\epsilon = \{ x \in \partial B_1 : \text{dist}(x, A) \leq \epsilon \} . \]

Clearly, \( A^{-}_2 \subset A^{-}_\epsilon \subset A \subset A^{+}_\epsilon \subset A^{+}_2 \). We can construct two smooth functions \( f^{(\ell)} : \partial B_1 \to [0, 1], \ell = 1, 2 \) such that all of their \( i \)-th order partial derivatives have absolute values less than \( \epsilon^{-i} \) for \( i = 1, 2, 3 \), and
\[ \begin{cases} f^{(1)}|_{A^{-}_2} = 1 \\ f^{(1)}|_{\partial B_1 \setminus A^{-}_2} = 0 \end{cases} \quad \text{and} \quad \begin{cases} f^{(2)}|_{A^{+}_2} = 1 \\ f^{(2)}|_{\partial B_1 \setminus A^{+}_2} = 0 \end{cases} . \]
Then $f^{(1)}$ is supported on $A_{1}^-$ and $f^{(2)}$ is supported on $A_{2\epsilon}^\pm$. Now for $\ell = 1, 2$, let $F^{(\ell)} : B_1 \to [0, 1]$ be the solution of the Dirichlet problem

$$\begin{cases}
\sum_{i,j=1}^{d} \Sigma_{ij} \partial_{ij}F^{(\ell)} = 0 & \text{in } B_1 \\
F^{(\ell)} = f^{(\ell)} & \text{on } \partial B_1.
\end{cases}$$

Note that $f^{(1)} \leq 1_A \leq f^{(2)}$ on $\partial B_1$ and so

$$F^{(1)} \leq \chi_A \leq F^{(2)} \text{ in } \overline{B_1}.$$ 

Note also that by the definitions of $f^{(\ell)}$, we have for $i = 1, 2, 3$ and $\ell = 1, 2$,

$$M^i_{F^{(\ell)}} \leq C/\epsilon^3,$$

where $M^i_F$ is as defined above Theorem 1.5. Moreover, for $\ell = 1, 2$,

$$\sup_{B_r} |\chi_A - F^{(\ell)}| \leq \sup_{x \in B_r} P^\epsilon_{BM} (\text{exits } \partial B_1 \text{ from } A_{2\epsilon}^+ \setminus A_{2\epsilon}^-) \xrightarrow{\epsilon \to 0} 0. \tag{2.22}$$

Step 2. Next, we will define two $\omega$-harmonic functions on $B_{R+1}^{\text{dis}}$ whose boundary values agree with that of $F^{(\ell)}$, $\ell = 1, 2$. Let $B_{R+1}^{\text{dis}} = \{ x \in B_{R+1}^{\text{dis}} : y \in B_{R+1}^{\text{dis}}, \text{ for all } y \sim x \}$. Note that $B_{R+1}^{\text{dis}} = B_{R+1}^{\text{dis}}$. For $\ell = 1, 2$, let $G^{(\ell)}_{R+1} : B_{R+1}^{\text{dis}} \to [0, 1]$ be the solution of the Dirichlet problem

$$\begin{cases}
L_\omega G^{(\ell)}_{R+1} = 0 & \text{in } B_{R+1}^{\text{dis}} \\
G^{(\ell)}_{R+1} = F^{(\ell)} & \text{on } \partial B_{R+1}^{\text{dis}}.
\end{cases}$$

Recall that $F^{(\ell)}(x) := F^{(\ell)}(x/(R + 1))$ for $x \in B_{R+1}^{\text{dis}}$. Then, for any $R \geq R_0(\epsilon^4, P)$, by Theorem 1.5 with probability at least $1 - C \exp(-CR^\delta)$ we have

$$\max_{B_{R+1}^{\text{dis}}} |F^{(\ell)}_{R+1} - G^{(\ell)}_{R+1}| \leq \epsilon^3 (M^2_{F^{(\ell)}} + M^3_{F^{(\ell)}}) \leq C \epsilon, \quad \ell = 1, 2. \tag{2.23}$$

Step 3. We will show for all $R \geq R_0(\epsilon^4, P) \geq \epsilon^{-8}$,

$$F^{(1)}_{R+1} - c \epsilon \leq 1_{B^{\text{dis}}_R} \leq F^{(2)}_{R+1} + C \epsilon \text{ on } \partial B^{\text{dis}}_R. \tag{2.24}$$

First, for any $x \in \partial B^{\text{dis}}_R \setminus \partial B^{\text{dis}}_R$, since $R \geq \epsilon^{-8}$, we have dist$(\frac{x}{R + 1}, \partial B_1 \setminus A_{1}^-) \leq 1/R \leq \epsilon^8$ and dist$(\frac{x}{R + 1}, A_{1}^-) \geq C \epsilon$. Hence for $x \in \partial B^{\text{dis}}_R \setminus \partial B^{\text{dis}}_R$,

$$1 - F^{(1)}_{R+1}(x) \geq P^\epsilon_{BM} (\text{exits } \partial B_1 \text{ at } \partial B_1 \setminus A_{1}^-) \geq 1 - c \epsilon^7,$$

which implies $F^{(1)}_{R+1} - c \epsilon \epsilon \leq 1_{B^{\text{dis}}_R} \text{ on } \partial B^{\text{dis}}_R$. The lower bound of (2.24) is proved.

Similarly, to obtain the upper bound of (2.24), notice that for any $x \in \partial B^{\text{dis}}_R$, dist$(\frac{x}{R + 1}, A_{1}^+)$ $\leq 1/R \leq \epsilon^8$ and dist$(\frac{x}{R + 1}, \partial B_1 \setminus A_{1}^+) \geq C \epsilon$. Hence

$$F^{(2)}_{R+1}(x) \geq P^\epsilon_{BM} (\text{exits } \partial B_1 \text{ at } A_{1}^+) \geq 1 - C \epsilon^7 \quad \text{for } x \in \partial B^{\text{dis}}_R,$$

which implies that $F^{(2)}_{R+1} + C \epsilon \epsilon \geq 1_{B^{\text{dis}}_R} \text{ on } \partial B^{\text{dis}}_R$. Display (2.24) is proved.
Step 4. With (2.24), assuming (2.23) we have
\[ G_{R+1}^{(1)} - c\epsilon \leq 1_{\partial B^\text{dis}_R} \leq G^{(2)}_{R+1} + C\epsilon \quad \text{on} \quad \partial B^\text{dis}_R \]
and so on \( B^\text{dis}_R \),
\[ G_{R+1}^{(1)} - c\epsilon \leq P^x(\omega)(x \text{ exits } B^\text{dis}_R \text{ through } \partial A \cap B^\text{dis}_R) \leq G^{(2)}_{R+1} + C\epsilon. \]

This inequality, together with (2.22) and (2.23), yields
\[ \max_{x \in B^\text{dis}_R} |\chi_A(x) - P^x(\omega)(x \text{ exits } B^\text{dis}_R \text{ through } \partial A \cap B^\text{dis}_R)| \leq C\epsilon. \]

Since (2.23) occurs with probability at least \( 1 - C \exp(-cR^\delta) \), display (2.21) follows. □

3. Percolation estimates for the sink

In this section we study connectivity properties of the balanced directed percolation at \( \omega \). Recall the notation \( x \bowtie y \) and the definition of a sink in Definition 1.10, and that we use \( \mathcal{C} \) to denote the unique sink (see Lemma 1.13).

The main results of this section are the following two theorems.

**Theorem 3.1** There exists \( \alpha > 0 \) depending only on the dimension and a constant \( C < \infty \) depending on \( P \), such that for every \( k \)
\[ P \left[ \text{there exists } x \text{ such that } 0 \bowtie x, \ x \notin \mathcal{C} \text{ and } \|x\| = k \right] < Ce^{-k^\alpha}. \]

**Theorem 3.2** For \( x, y \in \mathbb{Z}^d \) we define the distance \( \text{dist}_\omega(x, y) \leq \infty \) as the length of a shortest \( \omega \)-path from \( x \) to \( y \). Note that in general \( \text{dist}_\omega(x, y) \) may be different from \( \text{dist}_\omega(y, x) \). Then for the same \( \alpha > 0 \) as in Theorem 3.1 and some constant \( C < \infty \) depending on \( P \), we have that for every \( x \) and \( y \),
\[ P \left[ \text{dist}_\omega(x, y) > C\|x - y\| ; \ x, y \in \mathcal{C} \right] < Ce^{-\|x - y\|^\alpha}. \]

Unfortunately we need to provide different proofs for Theorem 3.1 in two dimensions and in larger dimensions. The reason is that our high-dimensional proof uses the fact that Bernoulli percolation in \( d - 1 \) dimensions has a non-trivial critical value, whereas our 2-dimensional proof relies heavily on planarity. We give here the proof in dimensions 3 and higher, and defer the two-dimensional proof to Appendix B.

### 3.1 Proof of Theorems 3.1 and 3.2 in three or more dimensions.

For a set \( S \subset \mathbb{Z}^d \), we say “\( x \bowtie y \) in \( S \)” if there is an \( \omega \)-path from \( x \) to \( y \) which is completely contained in \( S \). For \( A \subset S \), we say that “\( A \) is a sink of \( S \)” if for every \( x, y \in A \), \( x \bowtie y \) in \( S \), and \( x \bowtie z \) in \( S \) for every \( x \in A \) and \( z \in S \setminus A \).

**Claim 3.3** Let \( A(n) \) be the number of sinks of \([-n, n]^d \). Then \( P \)-a.s. there exists some \( n_0 \) such that \( A(n+1) \leq A(n) \) for all \( n > n_0 \). In particular, the limit \( A = \lim_{n \to \infty} A(n) \) exists \( P \)-a.s.
Proof. Recall the constant \( \Phi \) in Lemma 1.11. Let \( n_0 \) be so large that

1. For all \( n > n_0 \), every sink in \([-n, n]^d\) has density at least \( \Phi \).
2. For all \( n > n_0 \),
   \[
   \frac{|[-n-1, n+1]^d \setminus [-n, n]^d|}{|[-n-1, n+1]^d|} < \frac{\Phi}{2}.
   \]

Lemma 1.11 guarantees that almost surely \( n_0 < \infty \).

Fix \( n > n_0 \). Then every sink in \([-n-1, n+1]^d\) intersects \([-n, n]^d\), and thus contains at least one sink in \([-n, n]^d\). Therefore \( A(n+1) \leq A(n) \). □

We can also identify the value of the limit \( A = \lim_{n \to \infty} A(n) \).

**Claim 3.4** \( \lim_{n \to \infty} A(n) = 1 \).

**Proof.** By Proposition 1.12, the infinite sink is unique. Assume for contradiction that \( A > 1 \). Then there exists some \( N > n_0 \) (where \( n_0 \) is from the proof of Claim 3.3) such that \( A(n) = A > 1 \) for all \( n > N \). By induction on the previous argument, for every \( n \geq N \), every sink in \([-n, n]^d\) intersects \([-N, N]^d\). Furthermore, every sink in \([-N, N]^d\) is contained in a sink in \([-n, n]^d\), because \( A(n) = A(N) \). Let \( x_1 \) and \( x_2 \) belong to two distinct sinks in \([-N, N]^d\). Then the set of points in \( \mathbb{Z}^d \) reachable from \( x_1 \) is disjoint of the set of points reachable from \( x_2 \). This stands in contradiction to Lemma 1.13 which implies that every point in the (infinite) sink is reachable from all points in \( \mathbb{Z}^d \), and is thus reachable from both \( x_1 \) and \( x_2 \). □

As an immediate consequence of Claim 3.4 we get the following Grimmett-Marstrand type lemma.

**Lemma 3.5**

\[
\lim_{n \to \infty} P(A(n) = 1) = 1.
\]

Typically, the sink is ubiquitous in the cube. We make precise a weak sense of this statement.

**Lemma 3.6** For any \( k \in \mathbb{N} \) and \( \epsilon \in (0, 1) \), there exists \( N = N(k, \epsilon, P) \) such that for all \( n \geq N \), with \( P \)-probability greater than \( 1 - \epsilon \) the following happens:

1. There is a unique sink in \([-n, n]^d\).
2. The sink intersects every sub-cube of side length \( n/k \).

**Proof.** By Lemma 3.5, Item (1) holds with high probability for large enough \( n \). To see that Item (2) holds too, we note that it is enough to intersect \((4k)^d\) cubes of side length \( n/2k \). If \( n \) is large enough then each of them has a unique sink, and by the same induction as in the proof of Claim 3.3 the sink in \([-n, n]^d\) intersects each of them (note that their number does not grow with \( n \)). □

We will prove Theorems 3.1 and 3.2 for dimensions \( d \geq 3 \) by considering a percolation process of large scale boxes. To this end, for \( z \in \mathbb{Z}^d \) and \( n > 0 \), we let \( \Box_n = [-n, n]^d \),

\[
\Box_n(z) := z + \Box_n, \quad \text{and} \quad Q_n(z) = \Box_n((2n+1)z).
\]  (3.1)
Now set $k = 10$ and let $N = N(10, \epsilon, P)$ be as in Lemma 3.6 with $\epsilon = \epsilon(d) > 0$ to be determined later. Set $n^* = 10N$. For any cube $\Box$, we write $C(\Box)$ for the sink in $\Box$. If there are more than one, we use some arbitrary scheme to choose one. (In fact, we will only use this definition for cubes that have a unique sink.)

We define a notion of a point or a cube being $\omega$-good, and then we will see that most cubes are indeed $\omega$-good.

**Definition 3.7** A point $z \in \mathbb{Z}^d$ and the cube $Q_{n^*}(z)$ are both called $\omega$-good if the following requirements are satisfied:

- $Q_{n^*}(z)$ contains a unique sink,
- for every nearest neighbor $y$ of $z$, $Q_{n^*}(y)$ contains a unique sink, and
- for every nearest neighbor $y$ of $z$, $C(Q_{n^*}(y))$ connects to $C(Q_{n^*}(z))$, namely there are two $\omega$-paths, contained in $Q_{n^*}(y) \cup Q_{n^*}(z)$, one going from $C(Q_{n^*}(y))$ to $C(Q_{n^*}(z))$, and the other going from $C(Q_{n^*}(z))$ to $C(Q_{n^*}(y))$.

Then by Lemma 3.6 for every $z$, with probability greater than $1 - 4d\epsilon$, we have that $Q_{n^*}(z)$ is $\omega$-good. This allows us to prove the following corollary.

**Corollary 3.8** If we choose $\epsilon = \epsilon(d) > 0$ small enough and let $n^* = n^*(\epsilon, P) = n^*(d, P)$ be as mentioned above Definition 3.4, then the percolation process $\{ z \in \mathbb{Z}^d : z$ is $\omega$-good $\}$ stochastically dominates a Bernoulli site percolation with parameter $p_0(\epsilon) \in (0, 1)$ which is supercritical for dimension $d - 1$ (remember that $d \geq 3$). Moreover, $p_0 > p_c(\mathbb{Z}^{d-1})$ can be as close to 1 as we want.

**Remark 4.** Corollary 3.8 will be used in the proofs of Claim 3.9 and Lemma 3.11 below. To prove Claim 3.9 we only need to dominate a super critical percolation in $\mathbb{Z}^d$. Note that $p_c(\mathbb{Z}^{d-1}) > p_c(\mathbb{Z}^d)$. However, we need to dominate a super critical percolation in dimension $d - 1$ in the proof of Lemma 3.11.

**Proof of Corollary 3.8.** Let $X_z = 1_{\{z \text{ is } \omega \text{-good}\}}$ for $z \in \mathbb{Z}^d$. Then for every $z$, $P(X_z = 1) \geq 1 - 4d\epsilon$, with $n^*$ goes to infinity as $\epsilon$ going to zero, and for every two sets $S_1, S_2 \subseteq \mathbb{Z}^d$, if the distance between $S_1$ and $S_2$ is greater than or equal to 2, then $\sigma(X_z, z \in S_1)$ is independent of $\sigma(X_z, z \in S_2)$. Thus the dependence degree $\Delta$ as in 25 is bounded by $2d + 1$, and thus by Theorem 1.3 of 28, the distribution of $(X_z : z \in \mathbb{Z}^d)$ stochastically dominates a Bernoulli product measure with parameter $p_0 = p_0(\epsilon)$, with $p_0 \to 1$ as $\epsilon \to 0$. As $d \geq 3$, we have $p_c(\mathbb{Z}^{d-1}) < 1$. Thus for $\epsilon > 0$ small enough, $(X_z : z \in \mathbb{Z}^d)$ stochastically dominates a super-critical Bernoulli site percolation in dimension $d - 1$. $\square$

Another important fact is the following. Recall the notation $x \preceq y$ in Definition 1.10.

**Claim 3.9** Let $n^*$ be as in Corollary 3.8. If we choose $\epsilon = \epsilon(d) > 0$ small enough, then there exists $\xi = \xi(\epsilon, P) > 0$ such that for any $n \geq n^*$, $z \in \mathbb{Z}^d$, and all $x \in \partial \Box_n(z)$,

$$P(y \preceq x \text{ for some } y \text{ in the unique sink of } \Box_n(z)) \geq \xi.$$
We first consider the case \( n = n^* \). For simplicity of notations, in the case \( n = n^* \), we assume that \( 0 = x \in \partial \Box_n(z) \) and \( -e_1 \in \Box_n(z) \). For \( y \in \mathbb{Z}^d \), let

\[
j = j(y) = \begin{cases} 
  d, & \text{if } y_1 + \cdots + y_d \equiv 0 \mod d, \\
  \mod d, & \text{otherwise}.
\end{cases}
\]

Let \( A_x(z) \) denote the event

\[
A_x(z) = \{ y \ni x \text{ for some } y \in Q_{n^*}(z), \text{ and } C(Q_{n^*}(z)) = Q_{n^*}(z) \}.
\] (3.2)

If \( \omega(y, e_j(y)) > 0 \) for all \( y \in \Box_{n^*}(z) \), then \( -e_1 z_0 \) and \( C(\Box_{n^*}(z)) = \Box_{n^*}(z) \). Hence

\[
P(A_x(z)) \geq P[\omega(y, e_j(y)) > 0 \text{ for all } y \in \Box_{n^*}(z)] \geq \kappa^{(2n^*+1)^d},
\] (3.3)

where \( \kappa = \min \{ P(\omega(0, e_i) > 0) : i = 1, \ldots d \} \). The claim is proved for the case \( n = n^* \).

Next, we consider the case \( n > n^* \). For simplicity of notations, we assume that \( z = 0 \), \( (2n^* + 1) \mid (2n + 1) \), and break \( Q_n(0) = \Box_n(z) \) into cubes

\[
\{ Q_{n^*}(a) : a \in \Box((2n+1)/(2n^*+1)) \}.
\]

Let \( x \in \partial Q_n(0) \) and let \( z_0 \) be such that \( x \in Q_{n^*}(z_0) \). Let \( A_1 \) be the event

\[A_1 = \{ \omega : \text{ there is a path of good points from } 0 \text{ to } z_0 \},\]

and let \( A_2 \) be the event \( A_2 = \{ C(Q_{n^*}(z_0)) \cap C(\Box_n) \neq \emptyset \} \).

By the proof of Claim 3.4, \( P(A_2) \) can be made as close as we want to 1 by taking \( \epsilon > 0 \) small. For \( p < 1 \) large enough, the probability of \( A_1 \) under Bernoulli \( p \)-percolation is greater than \( 1 - \epsilon \) independently of \( n \). Thus, for any sufficiently small \( \epsilon > 0 \), we have (using Corollary 3.8 to dominate Bernoulli percolation) \( P(A_1 \cap A_2) > 1 - 2\epsilon \). Recall that \( x \in \partial Q_{n^*}(z_0) \). Then, the probability in Claim 3.9 can be bounded from below by

\[
P(A_1 \cap A_2 \cap A_x(z_0)) \geq P(A_1 \cap A_2)P(A_x(z_0)) \geq (1 - 2\epsilon)\kappa^{(2n^*+1)^d},
\]

where the second inequality used (3.3) and the fact that \( A_1 \cap A_2 \) and \( A_x(z_0) \) are positively correlated. Our proof is complete. \( \square \)

The following is a fact for classical site percolation. See Appendix C for its proof.

**Claim 3.10** Let \( d \geq 2 \). For \( n \in \mathbb{N} \) and a Bernoulli site percolation on \( \mathbb{Z}^d \) with parameter \( p \), we let \( C_{x,n} \) denote the connected component within \( \Box_n \) that contains \( x \). There exists a constant \( \rho \in (0,1) \) such that if \( p \geq \rho \), then for any \( x \in \Box_n \) and \( n \geq 1 \),

\[
P(\| C_{x,n} \| \geq n) > 0.5.
\]

We now advance towards proving Theorem 3.1. Let \( \mathcal{D} \) denote the infinite cluster of the percolation of \( \omega \)-good points mentioned in Corollary 3.8. We write

\[
\hat{\mathcal{C}} = \hat{\mathcal{C}}(\omega) = \bigcup_{z \in \mathcal{D}} C(Q_{n^*}(z)),
\] (3.4)

and note that \( \hat{\mathcal{C}} \subseteq \mathcal{C} \). We start by bounding the directed distance from \( \hat{\mathcal{C}} \) to a point.
Lemma 3.11  Recall the directed distance $\text{dist}_\omega(\cdot, \cdot)$ defined in Theorem 3.2. For $k \in \mathbb{N}, z \in \mathbb{Z}^d$, write $\Psi_\omega(k; z) := \{x : \text{dist}_\omega(x, z) \geq k\}$. Let $\text{dist}_\omega(A, B) = \inf_{x \in A, y \in B} \text{dist}_\omega(x, y)$ for $A, B \subset \mathbb{Z}^d$. There exist constants $\alpha, C > 0$ depending on $(d, P)$ such that

$$P(\Psi_\omega(k; z) \neq \emptyset, \text{dist}_\omega(\hat{C}, z) \geq k) < Ce^{-k^\alpha}$$

(3.5)

for any $k \in \mathbb{N}, z \in \mathbb{Z}^d$. In particular,

$$P(\text{dist}_\omega(\hat{C}, z) \geq k; z \in \mathcal{C}) < Ce^{-k^\alpha},$$

(3.6)

$$P(\Psi_\omega(k; z) \neq \emptyset; z \notin \hat{C}) < Ce^{-k^\alpha}.$$  

(3.7)

Proof. Without loss of generality, let $z = 0$, and simply write $\Psi_\omega(k) := \Psi_\omega(k; 0)$. We consider points that can access 0 within large scale boxes $\square_n := \bigcup_{a \in \square_n} Q_{n^*}(a)$, $n \geq 0$. Note that $\bigcap (n) = \square_{R_n}$, with $R_n := n^* + n(2n^* + 1)$.

First we arbitrarily enumerate $\mathbb{Z}^d$, i.e. take a bijection $\pi : \mathbb{N} \rightarrow \mathbb{Z}^d$ with $\pi(0) = 0$. For $A \subset \mathbb{Z}^d$, we let $\partial I A$ be the inner boundary of $A$, namely the vertices in $A$ that have a neighbor in $A^c$. We say “$x \nsim 0$ in $A$” if there is an $\omega$-path from $x$ to 0 which is completely contained in $A$. With the convention $\inf \emptyset = \infty$, we define, for $n \geq 0$,

$$k_n = \inf\{k : \pi(k) \in \partial I \square (n), \pi(k) \nsim 0 \text{ in } \square (n)\}.$$  

If $k_n < \infty$, we let $x_n = \pi(k_n) \in \partial I \square (n)$ and let $a_n \in \partial I \square_n$ be such that $x_n \in Q_{n^*}(a_n)$. In this situation, with $e_{-i} := -e_i$, we let

$$\iota(n) = \min\{j \in \{\pm 1, \ldots, \pm d\} : \langle a_n, e_j \rangle = n\}.$$  

If $k_n = \infty$, we simply put $x_n = a_n = \iota_n = \infty$. We write

$$W = \sup\{n \geq 0 : k_n < \infty\}.$$

Note that if $\Psi_\omega(R_{k_n}^d) \neq \emptyset$, then $W \geq n$.

For $n \geq 0$, if $\{W \geq 2n\}$ happens, we can define

$$\ell_{2n} = \{a \in \mathbb{Z}^d : \langle a, e_{\iota(2n)} \rangle = 2(n + 1)\}, \quad \mathcal{H}_{2n} = \ell_{2n} \cap \partial I \square_{2(n+1)}.$$  

That is, $\mathcal{H}_{2n}$ is the surface of $\square_{2(n+1)}$ which is perpendicular to the $\iota(2n)$-th direction and is “two levels away” from $a_{2n}$. Recall that by Corollary 3.8, there exists $p_0 > p_c(\mathbb{Z}^d)$ such that $\omega$-good points dominate a Bernoulli site percolation with parameter $p_0$. For $A \subset \mathbb{Z}^d$ and $x \in A$, write $C_{x,A}$ for the connected component (of this Bernoulli percolation) within $A$ that contains $x$, with the understanding that $C_{x,A} = \emptyset$ when $x$ is not good. If $\{W \geq 2n\}$ happens, we let $F_n \in \sigma(\omega|\square(2n+2))$ be the event

$$F_n = \{|C_{a_{2n} + 2e_{\iota(2n)}, 2}\mathcal{H}_{2n}| > n, \exists x \in Q_{n^*}(a_{2n} + 2e_{\iota(2n)}) \text{ s.t. } x \nsim 0 \text{ in } \square (2n)\}.$$  

Hence, recalling $A_2(a)$ in (3.2), by Claim 3.10 and (3.3), for $n \geq 0$,

$$P(F_n|\sigma(\omega|\square(2n)))|W \geq 2n) \geq P(|C_{a_{2n} + 2e_{\iota(2n)}, \mathcal{H}_{2n}}| > n, A_{x_{2n}}(a_{2n} + e_{\iota(2n)})$$

$$\geq P(|C_{a_{2n} + 2e_{\iota(2n)}, \mathcal{H}_{2n}}| > n) P(A_{x_{2n}}(a_{2n} + e_{\iota(2n)}))$$

$$\geq \rho$$

for some $\rho = \rho(d, P) > 0$, where in the second inequality we used the fact that $\{|C_{a_{2n} + 2e_{\iota(2n)}, \mathcal{H}_{2n}}| > n\}$ and $A_{x_{2n}}(a_{2n} + e_{\iota(2n)})$ are positively correlated.
Now, writing $F_n$ for the complement of $F_n \in \sigma(\omega|_{\mathbb{E}(2n+2)})$, by the above inequality,

$$P\left(\Psi_\omega(P_{2n+2}^d) \neq \emptyset, \text{dist}_\omega(\hat{C}, 0) \geq R_{2n+2}^d\right)$$

$$\leq P(W \geq 2n, F_n) + P(F_n, |C_{0n+1}\omega(2n)_\ell \omega| < \infty, W \geq 2n)$$

$$\leq (1 - \rho)^n + P(n < |C_{0n+1}\omega(2n)_\ell \omega| < \infty),$$

and (3.5) follows since by [18, Theorem 8.65] the last probability $P(n < |C_{0n+1}\omega(2n)_\ell \omega| < \infty)$ decays stretched exponentially. (3.6) and (3.7) follow rather easily from (3.5). □

**Proof of Theorem 3.1.** Using (3.7), for any $k \geq 1$, $x \in \mathbb{Z}^d$,

$$P\left(\text{there exists } z \text{ such that } x \not\sim z, z \notin \hat{C} \text{ and } \|z - x\| = k\right)$$

$$\leq \sum_{z: \|z - x\| = k} P(\Psi_\omega(k; z) \neq \emptyset, z \notin \hat{C}) \leq Ck^6 e^{-k^\alpha}.$$

Note that this inequality is stronger than Theorem 3.1 with $C$ replaced by $\hat{C} \subset C$. □

**Proof of Theorem 3.2.** First, by the Antal-Pisztora theorem [3, Theorem 1.1], there exist $C$ and $\gamma$ such that

$$P\left[\text{dist}_\omega(z, w) > C\|w - z\|, z, w \in \hat{C}\right] < C e^{-\gamma \|z - w\|}. \quad (3.8)$$

In view of (3.6), we only need to show that for all $x$,

$$P(\text{dist}_\omega(x, \hat{C}) \geq k; x \in C) < C e^{-k^\alpha}. \quad (3.9)$$

To this end, we write $1$ for the vector $(1, 1, \ldots, 1) \in \mathbb{Z}^d$ and then define a sequence $(x_n)_{n \geq 0}$ as follows. $x_0 = 0$. Given $x_n$ we choose $x_{n+1}$ as a nearest neighbor of $x_n$ satisfying

1. $\langle x_{n+1}, 1 \rangle > \langle x_n, 1 \rangle$, and
2. $x_n \not\sim x_{n+1}$. That is, $\omega(x_n, x_{n+1} - x_n) > 0$.

If there are more than one such neighbors, we apply some arbitrary scheme to choose one. The existence of $(x_n)_{n \geq 0}$ is guaranteed by the balancedness of the environment. Let $D(j, 1) \subset D$ denote the infinite connected component of $D$ restricted to $\{a \in \mathbb{Z}^d : \|a, 1\| \leq 1\}$. By Corollary 3.8 for every $j \in \mathbb{N}$, $D(j, 1)$ exists and is unique a.-s.. Let

$$\mathbb{D}(j) = \bigcup_{z \in D(j, 1)} C(Q_n(z)).$$

Then, the events $E_j = \{x_{3jn} \in \mathbb{D}(3j)\}, j \geq 1$, are independent and of positive probability. Display (3.9) follows.

Theorem 3.2 now follows from (3.8), (3.6) and (3.9). □

### 3.2 Relation to harmonic functions.

From Theorems 3.1 and 3.2 we can now prove a corollary which states that with high probability, in a large enough ball, values of a positive harmonic function are comparable via a multiplicative factor that grows exponentially with the radius. Note that such a statement is trivial in the uniformly elliptic setting. In our non-elliptic setting, the i.i.d.
structure of the environment is crucial. The example in Appendix F is a finite-range dependent counterexample to the following corollary.

**Corollary 3.12** There exist constants \( \alpha > 0, C_0 > 2 \) depending on \( (d, P) \) such that for any \( R \geq 1 \),

\[
P \left( \max_{B_R^\text{dis}} h \leq e^{C_0 R} \min_{B_R^\text{dis}} h \right) \quad \forall \omega \text{-harmonic function } h \geq 0 \text{ on } B_{C_0 R}^\text{dis} \geq 1 - C_0 e^{-R^\alpha}.
\]  

(3.10)

**Proof.** First, we will show that \( B_R^\text{dis} \) is surrounded by a subset of \( B_{2R}^\text{dis} \) with good connectivity. Let \( \epsilon = \epsilon(d) > 0 \) be a constant to be determined. By Corollary 3.8 for \( n^* = n^*(\epsilon, P) \), we have \( P(Q_{n^*}(0) \text{ is } \omega\text{-good}) \geq 1 - 4d\epsilon \). Recall the notation \( \hat{\omega} \) in (1.2). Then \( P \left( \exists x \in Q_{n^*}(0), i \in (0, \xi) \right) \leq |Q_{n^*}(0)| \sum_{i=1}^d P(\omega(x, e_i) \in (0, \xi)) \), so

\[
\lim_{\xi \to 0} \sum_{i=1}^d P(\omega(x, e_i) \in (0, \xi)) = |Q_{n^*}(0)| \sum_{i=1}^d P(\omega(x, e_i) \in (0, \xi)) = 0
\]

and there exists \( \xi = \xi(n^*, d) = \xi(d, P) > 0 \) such that \( P(\forall x \in Q_{n^*}(0), i \in (0, \xi)) > 1 - 4d\epsilon \). Let us call a point \( z \in \mathbb{Z}^d \) and the cube \( Q_{n^*}(z) \) very good if \( Q_{n^*}(z) \) is \( \omega \)-good and \( \hat{\omega}(x, e_i) \notin (0, \xi) \) for all \( x \in Q_{n^*}(z) \) and all \( i \). Then, \( P(z \text{ is very good}) > 1 - 5d\epsilon \). Now we take \( \epsilon(d) \) to be small enough such that, as in Corollary 3.8, the set of very good points still dominates a Bernoulli site percolation with parameter \( p_0 > p_c(d-1) \). Note that the values of \( \xi, n^* \) are redefined accordingly. Now, with \( D_{\text{vg}} \) denoting the infinite cluster of this Bernoulli percolation, we let

\[
\hat{C}_{\text{vg}} = \bigcup_{z \in \partial D_{\text{vg}}} C(Q_{n^*}(z)).
\]

It is known that all connected components of the complement of the infinite cluster of the Bernoulli percolation are small. A precise way of saying this is that the probability that there exists connected component of the complement inside \( B_{2R}(0) \) with diameter larger than \( R/10 \) decays exponentially with \( R \).

Then for all \( R \),

\[
P(\text{there exists an } \omega\text{-path from } B_R(0) \text{ to } \partial B_{2R}(0) \text{ avoiding } \hat{C}_{\text{vg}}) \leq e^{-C R},
\]

Hence, with \( P \)-probability at least \( 1 - CR^d e^{-R^\alpha} \), \( \omega \) satisfies, for all \( x \in B_R^\text{dis} \),

\[
P_\omega^x (\text{the random walk visits } \hat{C}_{\text{vg}} \text{ before exiting } B_{2R}^\text{dis}) = 1
\]

and thus, for any \( h \) which is harmonic on \( B_{2R}^\text{dis} \),

\[
\min_{B_{2R}^\text{dis} \cap \hat{C}_{\text{vg}}} h \leq h(x) \leq \max_{B_{2R}^\text{dis} \cap \hat{C}_{\text{vg}}} h, \quad \forall x \in B_R^\text{dis}.
\]  

(3.11)

Next, we will show that \( h \)-values within \( B_{2R}^\text{dis} \cap \hat{C}_{\text{vg}} \) are comparable via a multiplicative factor that grows exponentially. Indeed, for any \( z, w \in \mathbb{Z}^d \), we let \( Q_{n^*}(q_z), Q_{n^*}(q_w) \) denote
the large-scale cubes that contain $z,w$, respectively. By the Antal-Pisztora theorem [3, Theorem 1.1], there exist $C$ and $\gamma$ such that the probability

$$P(\exists \text{a path within } D^\text{vg} \text{ with length } \leq C\|q_z - q_w\| \text{ that connects } q_z \text{ and } q_w \mid q_z, q_w \in D^\text{vg})$$

is at least $1 - Ce^{-\gamma\|q_z - q_w\|}$. Recall that, by definition, any two points within the sink of the same very good cube or within sinks of neighboring very good cubes can be connected by a $\omega$-path (with length at most $2(2n^*)^d$) whose jump probabilities (of all steps) are at least $\xi = \xi(d,P)$. Hence, with $P$-probability at least $1 - CR^2de^{-\gamma R}$, any pair of points $x,y \in B_{\text{dis}}2R \cap \hat{C}^\text{vg}$ can be connected by an $\omega$-path with length at most $C\|x - y\|$ whose jump probabilities are at least $\xi = \xi(d,P)$. Here $C = C(n^*,d,P)$ could differ from line to line. This implies, with $P$-probability at least $1 - CR^2de^{-\gamma R}$, we have

$$\max_{B_{\text{dis}}CR} h \leq \xi - CR \min_{B_{\text{dis}}CR} h$$

for any $\omega$-harmonic $h \geq 0$ on $B_{\text{dis}}CR$. Display (3.10) follows from (3.11) and (3.12). \qed

4. An oscillation inequality

The goal of this section is to obtain an oscillation estimate (Theorem 4.1 below) for $\omega$-harmonic functions. Our argument is robust enough so that we can drop the balancedness Assumption 1 and use instead the following weaker assumption.

**Assumption 2** $P$ is an i.i.d. measure, and the statements in Corollary 2.4 and Corollary 3.12 hold.

Of course, Assumption 1 implies Assumption 2. Recall the notation $\text{osc}_E$ in (1.9).

For $z \in \mathbb{Z}^d$, $0 < \alpha < 1 < \beta < \infty$ and $R > 0$, we let

$$H_{z,R}^{\alpha,\beta} = \left\{ \omega \in \Omega : \text{osc}_{B_{\text{dis}}(z)} f \leq \alpha \text{ osc}_{B_{\text{dis}}(z)} f \text{ for all } \omega\text{-harmonic functions } f \text{ on } B_{\text{dis}}^\beta(z) \right\}.$$

**Theorem 4.1** Under Assumption 2, there exist constants $\upsilon < 1$, $\Psi > 1$, $\gamma > 0$, and $C$ depending on $(d,P)$ such that $P(H_{0,R}^{\upsilon,\Psi}) > 1 - e^{-R^\gamma}$ for all $R \geq C$.

4.1 Main idea of the proof.

The proof of Theorem 4.1 is based on coupling. For any $R > 1$, $z \in \mathbb{Z}^d$, define the hitting time of the inner-boundary of the ball $B_{\text{dis}}^R(z)$ as

$$\tau_{R,z} = \tau_{R,z}(X) = \inf\{n \geq 0 : X_n \in \partial(\mathbb{Z}^d \setminus B_{\text{dis}}^R(z))\}.$$  \hspace{1cm} (4.1)

The underlying process of the stopping time $\tau_{R,z}$ should be understood from the context. For instance, the subscripts of $X_{\tau_{R,z}}$ and $Y_{\tau_{R,z}}$ represent two different stopping times $\tau_{R,z}(X)$ and $\tau_{R,z}(Y)$, respectively.

For $\omega \in \Omega$, we will have $\omega \in H_{z,R}^{\alpha,\gamma}$ if, for any $x,y \in B_{\text{dis}}^R(z)$, there is a coupling of two paths $(X_n), (Y_n)$ in $B_{\text{dis}}^R(z)$ such that
(a) The marginal distributions of \((X_n)\) and \((Y_n)\) are \(P^x_\omega\) and \(P^y_\omega\), respectively. With abuse of notation, we use \(P^{x,y}_\omega\) to denote the joint law of \((X_n, Y_n)_{n \geq 0}\).

(b) \(P^{x,y}_\omega(X_{\tau_{\gamma R,z}} = Y_{\tau_{\gamma R,z}}) > 1 - \alpha\).

Indeed, for any \(\omega\)-harmonic function \(f\), \(f(X_n)\) is a martingale under the quenched law \(P^x_\omega\). Hence, by the optional stopping theorem, for any \(x, y \in B_{\text{dis}} R(z)\),

\[
f(x) - f(y) = E^{x,y}_\omega[f(X_{\tau_{\gamma R,z}}) - f(Y_{\tau_{\gamma R,z}})] \\
\leq P^{x,y}_\omega(X_{\tau_{\gamma R,z}} \neq Y_{\tau_{\gamma R,z}}) \text{ osc } B_{\gamma R}^{\text{dis}}(z) f \\
\leq \alpha \text{ osc } B_{\gamma R}^{\text{dis}}(z) f. \tag{4.2}
\]

We now begin working towards constructing the coupling that proves Theorem 4.1.

4.2 The coupling in large balls.

In this subsection, we will construct the desired coupling mentioned in Subsection 4.1.

We let \(M = M(d, P) > 100\) be a large constant integer whose value will be determined at the end of Section 4. Our construction relies on the fact that, with high probability, within a large ball of radius \(M\), we can force two random walks to meet, and, within larger balls \(B_{\text{dis}} R\) with radius \(R > M\), we can force the two walks to become closer so that they are in a ball of radius \(R/M\) before exiting \(B_{\text{dis}} R\). Hence starting from any radius \(R > M\), as long as we are in a good enough environment, we can control the distance of the walks by shrinking (if the walks get closer) or inflating the balls by a factor of \(M\). We hope that this radius will eventually become less than \(M\), at which point the walks can be forced to meet.

We will define the goodness of balls. Let

\[
\Gamma = \{A_1, A_2, \ldots, A_k\}, \quad A_j \subseteq \partial B_1(0) \text{ for all } j = 1, \ldots, k \tag{4.3}
\]

be such that

1. \(\Gamma\) is a covering of \(\partial B_1(0)\), namely

\[
\partial B_1(0) = \bigcup_{j=1}^k A_j;
\]

2. for every \(j = 1, \ldots, k\), \(A_j\) is closed in the topology of \(\partial B_1(0)\);
3. the interiors (w.r.t. the topology of \(\partial B_1(0)\)) of \(A_j, j = 1, \ldots, k\), are pairwise disjoint;
4. for every \(j = 1, \ldots, k\), \(\partial A_j\) is a measure zero set, where the boundary is w.r.t. the topology of \(\partial B_1(0)\), and the measure that we consider is the \(d - 1\) dimensional Lebesgue measure on \(\partial B_1(0)\);
5. for every \(j = 1, \ldots, k\), the diameter of \(A_j\) is smaller than \(1/M^2\).

Recall notations of the boundary \(\partial_1 B\) in (2.20) and the exit time \(\tau(B)\) in (2.8). For a ball \(B = B_{\text{dis}} R(x)\) and a point \(y \in B\), we define a distribution \(D^y_\omega(B)\) on \(\Gamma\) by

\[
D^y_\omega(B)(A) = P^y_\omega \left[ X_{\tau(B)} \in \partial_1 B \right], \quad A \in \Gamma. \tag{4.4}
\]
For \( z \in B_1 \), we denote by \( D^x_\Sigma \) the distribution on the set \( \Gamma \) with

\[
D^x_\Sigma(A) = P^x_{BM}(\text{exits } B_1 \text{ through } A), \quad A \in \Gamma,
\]

where \( P^x_{BM} \) is the law of the Brownian motion as defined below (2.20).

We now define the notion of goodness of a ball \( B^{\text{dis}}_R(x) \), \( R \geq M \).

**Definition 4.2** For a ball \( B^{\text{dis}}_R(x) \), denote by \( P^y_\omega \) the quenched exit distribution of the RWRE from \( B^{\text{dis}}_M(x) \), namely \( P^y_\omega(z) := P^y_\omega(X_{\tau(B^{\text{dis}}_M(x))} = z) \) for all \( z \in \partial B^{\text{dis}}_M(x) \).

1. If \( R = M \), we say that the ball \( B = B^{\text{dis}}_R(x) \) is good if

\[
\max_{x,y \in B} \left\| P^x_\omega - P^y_\omega \right\|_{TV} \leq 1 - e^{-M^2}.
\]

2. If \( R > M \), we say that the ball \( B = B^{\text{dis}}_R(x) \) is good if

\[
\max_{x,y \in B} \left\| D^y_\omega(B^{\text{dis}}_M(x)) - D^x_\Sigma(x,\eta / MR) \right\|_{TV} < \frac{1}{\delta}.
\]

In particular, \( \max_{y,z \in B} \left\| D^y_\omega(B^{\text{dis}}_M(x)) - D^z_\omega(B^{\text{dis}}_M(x)) \right\|_{TV} < \frac{1}{5} \).

Notice that the goodness of \( B^{\text{dis}}_R(x) \), \( R > M \), depends on the environment in a larger ball \( B^{\text{dis}}_{(M+1)}(x) \). The following claim follows from Corollaries 3.12 and 2.4.

**Claim 4.3** Under Assumption \( \Box \) there exists \( \delta(d, P) > 0 \) such that for every \( R \geq M \),

\[
P\left( B^{\text{dis}}_R \text{ is good} \right) > 1 - \exp(-R^\delta).
\]

**Proof.** Take \( M \) to be greater than the constant \( C_0 = C_0(d, P) \) in (3.10). Since, for every \( z \in \partial B^{\text{dis}}_{C_0 M} \) fixed, \( x \mapsto P^x_\omega(z) \) is an \( \omega \)-harmonic function on \( B^{\text{dis}}_{C_0 M} \), by (3.10), we conclude that the event in (4.6) happens with \( P \)-probability at least \( 1 - e^{-M^2} \).

For \( R > M \), the claim follows from (2.21) and the fact that, with \( M(d, P) \) being large, we have \( P^w_{BM}(\text{exits } B_1 \text{ through } A_j) < C/M \) for all \( w \in B_{1/M} \). \( \square \)

We start with a notion of a basic coupling \( \hat{\mu} \), and will afterwards compose the desired coupling \( \mu \) by concatenating many basic couplings.

**Definition 4.4** Let \( R \geq M \) and \( y, z \in B^{\text{dis}}_R(x) \). The basic coupling \( \hat{\mu}^{(x,R;z,y)} \) is a joint distribution of two paths \((\hat{Y}_n)_{n=0}^{\hat{T}_y}, (\hat{Z}_n)_{n=0}^{\hat{T}_z})\) with lengths \( \hat{T}_y \) and \( \hat{T}_z \) defined as follows.

1. If \( R > M \), then \((\hat{Y}_n)_{n=0}^{\hat{T}_y}\) and \((\hat{Z}_n)_{n=0}^{\hat{T}_z}\) are sampled as random walks in the environment \( \omega \), starting respectively at \( y \) and \( z \), with \( \hat{T}_y \) and \( \hat{T}_z \) being the respective stopping times of reaching \( \partial B^{\text{dis}}_{RM}(x) \), where the two walks are coupled in a way that maximizes the probability that \( \hat{Y}_{\hat{T}_y} = \hat{Z}_{\hat{T}_z} \).
2. If \( R = M \) then \((\hat{Y}_n)_{n=0}^{\hat{T}_y}\) and \((\hat{Z}_n)_{n=0}^{\hat{T}_z}\) are sampled as random walks in the environment \( \omega \), starting respectively at \( y \) and \( z \), with \( \hat{T}_y \) and \( \hat{T}_z \) being the stopping times of reaching \( \partial B^{\text{dis}}_{RM}(x) \), where the two walks are coupled in a way that maximizes the probability that \( \hat{Y}_{\hat{T}_y} = \hat{Z}_{\hat{T}_z} \).
On good balls, the basic coupling has a relatively good success probability, as evident by the following lemma, which follows immediately from the definition of good balls.

**Lemma 4.5** Let $B_R^{dis}(x)$ be a good ball, and let $y, z \in B_R^{dis}(x)$.

1. If $R > M$ then
   \[
   \tilde{\mu}^{(x, R; z, y)}\left(\|Y_{T_y} - \tilde{Z}_{T_z}\| < R/M\right) > \frac{2}{3}.
   \]

2. If $R = M$ then
   \[
   \tilde{\mu}^{(x, R; z, y)}\left(Y_{T_y} = \tilde{Z}_{T_z}\right) > e^{-M^2}.
   \]

We now concatenate basic couplings, and get the following construction. Denote the set of powers of $M$ by

\[M^* = \{M^n : n \in \mathbb{N}\}.
\]

**Definition 4.6** Let $R \in M^*$, $x \in \mathbb{Z}^d$, and let $y, z \in B_R^{dis}(x)$. For $\omega \in \Omega$, the coupling $\mu^{(x, R; z, y)}_\omega = \mu^{(x, R; z, y)}_\omega$ is a joint distribution of two paths $(Y_n)_{n=0}^\infty$, $(Z_n)_{n=0}^\infty$ and a sequence of balls $(B_R^{(m)}(x_m))_{m=0}^\infty$ that is constructed as follows.

Set $x_0 = x, y_0 = y, z_0 = z, R^{(0)} = R$ and $T_y^{(0)} = T_z^{(0)} = 0$.

Inductively, for $m \geq 1$, when $\{(Y_n)_{n=0}^{T_y^{(m-1)}}, (Z_n)_{n=0}^{T_z^{(m-1)}}, B_R^{(m-1)}(x_{m-1})\}$ is constructed, we first sample $\{(Y_n)_{n=T_y^{(m-1)}}^{T_y^{(m)}}, (Z_n)_{n=T_z^{(m-1)}}^{T_z^{(m)}}\}$ according to the law $\tilde{\mu}^{(x_{m-1}, R^{(m-1)}, y_{m-1}, z_{m-1})}$, and then, with $y_m := Y_{T_y^{(m)}}$ and $z_m := Z_{T_z^{(m)}}$, we do the following:

- When $R^{(m-1)} > M$, the ball $B_R^{(m)}(x_m)$ is defined as follows. If $\|y_m - z_m\| < R^{(m-1)}/M$ then we take $R^{(m)} = R^{(m-1)}/M$, and $x_m \in B_{MR^{(m-1)}}(x_{m-1})$ such that $y_m, z_m \in B_{R^{(m)}}(x_m)$. Else we take $R^{(m)} = MR^{(m-1)}$ and $x_m = x_{m-1}$.
- When $R^{(m-1)} = M$, we set $R^{(n)} = R^{(m-1)}$ and $x_n = x_{m-1}$ for all $n \geq m$. If $y_m = z_m$, we let $(Y_i)_{i=T_y^{(m)}}^{\infty} = (Z_j)_{j=T_z^{(m)}}^{\infty}$ be the same path with law $P_{y_m}^\omega$.
  Otherwise, we let $(Y_i)_{i=T_y^{(m)}}^{\infty}$ and $(Z_j)_{j=T_z^{(m)}}^{\infty}$ be two independent paths with laws $P_{y_m}^\omega$ and $P_{z_m}^\omega$, respectively. Then, our definition is finished.

Starting with $R^{(0)} = R \in M^*$, we want the process of balls $B_R^{(m)}(x_m)$ to eventually shrink into a good ball with radius $M$, so that the two paths can stick together. To this end, we define, for $m \geq 0$,

\[L_m := \log_M (R^{(m)}/R^{(0)}) \in \mathbb{Z}.
\]

Note that $(L_m)$ is a random path on $\mathbb{Z}$ with $L_0 = 0$ and absorbing state $\log_M (M/R^{(0)}) \leq 0$. Moreover, if $B_{R^{(m)}}(x_m)$ is good and $R^{(m)} > M$, then, by Lemma 4.5, $L_{m+1} = L_m - 1$ with probability at least $2/3$. Based on this observation, we will define the third and last
coupling and dominate \((L_m)\) by a biased simple random walk (SRW). To be specific, let

\[
\mathbb{U} = \inf\{n \geq 0 : B_{R(\nu)}^{\text{dis}}(x_n) \text{ is bad}\}, \tag{4.7}
\]

\[
\tilde{m}(r) = \inf\{n \geq 0 : R^{(n)} = r\}, \text{ for } r \in M^*, \tag{4.8}
\]

and write \(\mathcal{F}_m\) for the \(\sigma\)-algebra generated by the environment \(\omega\) and by \((x_k, R^{(k)})\) as well as \((Y_n)_{n \leq T^{(m)}_y}\) and \((Z_n)_{n \leq T^{(m)}_z}\). For any \(y, z \in B_{R(\nu)}^{\text{dis}}(x)\), by Lemma 4.5,

\[
\mu^{(x, R; y, z)}(L_{m+1} = L_m - 1 \mid \mathcal{F}_m; B_{R^{(m)}(x_m)}^{\text{dis}} \text{ is good and } R^{(m)} > M) > \frac{2}{3}.
\]

Hence we can couple the process \((L_n)_{n \geq 0}\) with a biased SRW \((\xi_n)_{n \geq 0}\) such that

1. \((\xi_{n+1} - \xi_n)\) is an iid sequence satisfying \(P(\xi_{n+1} - \xi_n = 1) = 1/3\) and \(P(\xi_{n+1} - \xi_n = -1) = 2/3\), \(\xi_0 = 0\), and
2. for every \(0 \leq n < \mathbb{U} \cap \tilde{m}(M)\) we have \(L_{n+1} - L_n \leq \xi_{n+1} - \xi_n\).

Note that \((\xi_n)\) is transient to \(-\infty\), and its probability of ever visiting a positive integer decays exponentially. Using the stochastic domination, we get the following estimate.

**Lemma 4.7** Let \(R \in M^*, x \in \mathbb{Z}^d\), and let \(z, y \in B_{R(\nu)}^{\text{dis}}(x)\). For every \(j \in \mathbb{N}\), \(\omega \in \Omega\),

\[
\mu^{(x, R; z, y)}(\sup\{L_m : m \leq \mathbb{U} \cap \tilde{m}(M)\} > j) \leq 2^{-j}.
\]

Since the stochastic domination stops when a bad ball is hit, we would like all balls \(B_{R^{(n)}(x)}^{\text{dis}}(x_n)\) to be good. To see that with high \(P\)-probability, the coupling only sees good balls, we will consider a multi-scale structure in the subsection below.

### 4.3 Multi-scale structure.

For any \(r \in M^*\) and \(x \in \mathbb{Z}^d\), we define the stopping time

\[
T(x, r) = \inf\{n \geq 0 : B_{R^{(n)}(x)}(x_n) \not\subset B_r(x)\}. \tag{4.9}
\]

We say that the coupling \(\mu^{(x, R; z, y)}, z, y \in B_{R(\nu)}^{\text{dis}}(x)\), is successful if the following are true.

1. It only sees good balls. I.e., \(\mathbb{U} = \infty\).
2. All balls are within \(B_{2MR}(x)\). I.e., \(\tilde{m}(M) < T(x, 2MR)\).
3. The two paths meet when they exit the last ball. I.e., \(Y_{T}^{\tilde{m}(M) + 1} = Z_{T}^{\tilde{m}(M) + 1}\).

Intuitively, to have \(\mathbb{U} = \infty\), we need all balls within a relatively large scale to be good, and the “majority” of balls under certain smaller scale to have nice properties, so that bad balls are hard to be hit.

We can now define our multi-scale structure. Let us start with setting the scales.

Let \(K = K(M) \in 2\mathbb{N}\) be a large even number whose value is to be determined later. Define, for \(n \geq 0\), a sequence of radii that grows double-exponentially

\[
\mathcal{R}_n := M^{K^n} \in M^*
\]

so that \(\mathcal{R}_0 = M\) and \(\mathcal{R}_{n+1} = \mathcal{R}_n^K\). We will define the notion of admissibility for balls with radius \(\mathcal{R}_k^2\), and then estimate the probability that a ball is admissible.

**Definition 4.8** We choose a parameter \(\nu < \delta/K\), where \(\delta\) is as in Claim 4.3.
(1) A ball of radius $R_0^2 = M^2$ is called admissible if all of its sub-balls with radius $M$ are good.

(2) A ball of radius $R_k^2$, $k \geq 1$, is called admissible if
(a) every sub-ball with integer radius greater or equal to $R_{k-1}$ is good, and
(b) there are at most $|R_{k-1}^2|$ sub-balls of radius $R_{k-1}$ whose concentric ball with radius $R_{k-1}^2$ is non-admissible.

We denote by $A(x,k)$ the event that $B^{\text{dis}}_{R_k^2}(x)$ is admissible.

Note that the admissibility of a ball with radius $R_k^2$, $k \geq 1$, depends on the environment in a concentric ball with radius $R_k^2 + R_{k-1}^2 < 2R_k^2$.

**Lemma 4.9** Under Assumption 2 (in Page 25), for every $x \in \mathbb{Z}^d$ and $k \geq 0$,

$$P(B^{\text{dis}}_{R_k^2}(x) \text{ is not admissible}) < e^{-R_k^2/2}.$$  

**Proof.** For $k = 0$ this follows from Claim 4.3 and the fact that $\nu < \delta$. For $k \geq 1$ we prove the lemma by induction. Let $A$ be the event that there exists a sub-ball of $B^{\text{dis}}_{R_k^2}(x)$ with radius at least $R_{k-1}$ which is not good, and let $B$ be the event that there are more than $R_k^2$ non-admissible balls centered in $\square_{R_k^2}(x)$ and with radius $R_{k-1}^2$. We estimate the probabilities of $A$ and $B$.

We start with estimating the probability of $A$. There are less than $(2R_k^2)^{d+1}$ sub-balls of $B^{\text{dis}}_{R_k^2}(x)$ with integer radius at least $R_{k-1}$, and each of them is bad with probability less than $e^{-R_{k-1}^2} < e^{-R_k^2}$. Thus

$$P(A) \leq (2R_k^2)^{d+1}e^{-R_k^2}.$$  

We continue with estimating the probability of $B$. To this end we partition $\square_{R_k^2}(x)$ into $(2R_{k-1}^2)^d$ subsets $L_1, L_2, \ldots, L_{(2R_{k-1}^2)^d}$ such that for every $j$ and every $z, y \in L_j$, the events $A(z, k-1)$ and $A(y, k-1)$ are independent. For given $j \leq (2R_{k-1}^2)^d$, we write

$$U(j) = \text{the number of non-admissible balls with radius } R_{k-1}^2 \text{ centered at points in } L_j.$$  

Then $U(j)$ is a binomial ($|L_j|$, $P(A(0, k-1))$) random variable, and by the induction hypothesis is dominated by a binomial $((2R_{k-1}^2)^d, e^{-R_{k-1}^2/2})$ variable. Let $\nu - \nu/2K < \nu' < \nu$. Then, using the fact that for any binomial variable $Y$ and number $\ell \geq 0$, we have $P(Y \geq \ell) \leq E(Y)^\ell$, we get that for every $j$,

$$P(U(j) > R_{k-1}^2) \leq (2R_{k-1}^2)^{2d}e^{-R_{k-1}^2/2} e^{-0.9R_{k-1}^2} \leq e^{-0.9R_{k-1}^2},$$  

and so $P(B) \leq (2R_{k-1}^2)^d e^{-0.9R_{k-1}^2}$. Therefore,

$$P(A^c(x,k)) \leq P(A) + P(B) \leq e^{-R_k^2/2}. \quad \Box$$  

Recall the stopping times $U, \bar{m}, \mathbb{T}$ in (4.7), (4.8), (4.9).
Lemma 4.10 Let \( k \geq 0 \), and let \( x, y, z \in \mathbb{Z}^d \) be such that \( y, z \in B_{k+1}^{\text{dis}}(x) \). Assume that \( B_{k+1}^{\text{dis}}(x) \) is admissible. Then

(a) for any \( R \in (\mathcal{R}_k, \mathcal{R}_{k+1}] \cap M^* \), we have
\[
\mu^{(x, R, z, y)}(\bar{m}(\mathcal{R}_k) \geq U) \leq \mu^{(x, R, z, y)}(\bar{m}(\mathcal{R}_k) \cup \mathcal{T}(0, \mathcal{R}_k^2) < 2R_k^{-\sqrt{K}};
\]

(b) for every \( w \in B_{k+1}^{\text{dis}}(x) \),
\[
\mu^{(x, R, k+1; z, y)}(\bar{m}(\mathcal{R}_k) < U; w \in B_{k+1}^{\text{dis}}(x \bar{m}(\mathcal{R}_k))) < R_k^{-\sqrt{K}';}
\]

(c) \( \mu^{(x, R, k+1; z, y)}(A(x \bar{m}(\mathcal{R}_k), k); \bar{m}(\mathcal{R}_k) < U) \geq 1 - 3R_k^{-\sqrt{K}/2} \).

Proof. Without loss of generality, assume \( x = 0 \). We fix \( k, y, z \) and simply write the coupling \( \mu^{(x, R, z, y)} \) as \( \mu \). Recall the stopping time \( \mathcal{T} \) in (4.9).

To prove (a), note that, by the assumption \( \omega \in A(0, k+1) \), the event \( \{ \bar{m}(\mathcal{R}_k) \geq U \} \) implies \( \{ \bar{m}(\mathcal{R}_k) \wedge U \geq \mathcal{T}(0, \mathcal{R}_k^2) \} \). Further,
\[
\mu(\bar{m}(\mathcal{R}_k) \wedge U \geq \mathcal{T}(0, \mathcal{R}_k^2)) \leq \mu(\bar{m}(\mathcal{R}_k) \wedge \mathcal{T}(0, \mathcal{R}_k^2) \geq \bar{m}(\mathcal{R}_k^3/2)) + \mu(\bar{m}(\mathcal{R}_k^3/2) \wedge \bar{m}(\mathcal{R}_k) \wedge U \geq \mathcal{T}(0, \mathcal{R}_k^2)).
\]

Since we see only good balls before time \( \bar{m}(\mathcal{R}_k) \wedge \mathcal{T}(0, \mathcal{R}_k^2) \), by Lemma 4.7,
\[
\mu(\bar{m}(\mathcal{R}_k) \wedge \mathcal{T}(0, \mathcal{R}_k^2) \geq \bar{m}(\mathcal{R}_k^3/2)) \leq 2^{1-\log_M(\mathcal{R}_k^{3/2}/\mathcal{R}_k^2)} \leq R_k^{-\sqrt{K}}.
\]

To estimate the second term on the right side of (4.10), we write \( \bar{m} := \bar{m}(\mathcal{R}_k^3/2) \wedge \bar{m}(\mathcal{R}_k) \wedge U \). Since \( (L_n) \) is dominated by \( (\xi_n) \) before time \( U \wedge \bar{m}(M) \geq \bar{m} \), we have
\[
\bar{m} \leq \mathcal{S} := \min\{n \geq 0 : \xi_n \leq \log_M(\mathcal{R}_k/\mathcal{R}_k^3/2)\}.
\]

Hence, using a fact of the biased SRW, we get \( E_M[\bar{m}] \leq E[S] = 3\log_M(\mathcal{R}_k/\mathcal{R}_k^3/2) \). Since
\[
\max\{v : v \in B_{R_i}^{\text{dis}}(x_i), 0 \leq i \leq \bar{m} \} \leq \sum_{i=0}^{\bar{m}} R_i^{\epsilon_i} \leq M^{3/2} + 1 + \bar{m},
\]
applying this inequality and Markov’s inequality we get
\[
\mu(\bar{m} \geq \mathcal{T}(0, \mathcal{R}_k^3/2)) = \mu(\max\{|v| : v \in B_{R_i}^{\text{dis}}(x_i), 0 \leq i \leq \bar{m} \} \geq \mathcal{R}_k^3/2)
\leq \mathcal{R}_k^{-2}\mathcal{R}_k^{3/2}(1 + E_M[\bar{m}])
\leq M\mathcal{R}_k^{-1/2}[1 + 3\log_M(\mathcal{R}_k/\mathcal{R}_k^3/2)] \leq \mathcal{R}_k^{-K/3}.
\]

Recalling (4.10), statement (a) is proved.

To prove (b), we call \( n \geq 0 \) a regeneration if \( \xi_n > \xi_{n+1} > \xi_{n+2} \) and
\[
\sum_{j \geq n+2} M^{\xi_{j-\xi_{n+1}}} < 1.
\]
(Notice that our definition of regenerations is quite different from the standard definitions in the literature). Note that if \( n \) is a regeneration, then for any \( m \in [n + 1, U \wedge \tilde{m}(M)] \),

\[
\|x_m - x_{n+1}\| \leq \sum_{j=n+1}^{m-1} MR^{(j)} \leq 2R^{(n)}.
\]  

\[ (4.12) \]

We use \( K(n) \) to denote the event that \( n \geq 0 \) is a regeneration. We need to use Claim 4.11 below which says that there are plenty of regenerations. It is a statement regarding biased simple random walks on \( \mathbb{Z} \).

Claim 4.11 There exist \( \kappa > 0 \) and \( \nu > 0 \) such that for every \( N \in \mathbb{N} \),

\[
P\left( \sum_{m=1}^{N} 1_{K(m)} < \kappa N \right) < e^{-\nu N}.
\]

Note that by definition, \( \tilde{m}(\mathcal{R}_k) \geq \log_M(\mathcal{R}_{k+1}/\mathcal{R}_k) \geq \frac{K}{2\log M} \log \mathcal{R}_k \). Thus, on the event \( \{\tilde{m}(\mathcal{R}_k) < U\} \), by Claim 4.11 with \( \mu \)-probability at least \( 1 - \mathcal{R}_k^{-\nu K/(2\log M)} \), there are at least \( \frac{KN}{2\log M} \) log \( \mathcal{R}_k \) regenerations in \([0, \tilde{m}(\mathcal{R}_k)]\). Moreover, for every regeneration \( n \in [0, \tilde{m}(\mathcal{R}_k)] \), by (4.12),

\[
\|x_{\tilde{m}(\mathcal{R}_k)} - x_{n+1}\| \leq 2R^{(n)}.
\]

Hence \( \|w - x_{\tilde{m}(\mathcal{R}_k)}\| < \mathcal{R}_k \) only if \( \|w - x_{n+1}\| < 3R^{(n)} \) for every regeneration \( n \leq \tilde{m}(\mathcal{R}_k) \), and this happens with \( \mu \)-probability at most \( \frac{1}{2} + \frac{C}{M} < e^{-1} \) for every such \( n \). For such events to happen for all \( n \leq \tilde{m}(\mathcal{R}_k) \), the \( \mu \)-probability is at most \( (\frac{1}{2} + \frac{C}{M})^K \mathcal{R}_k/\log M \leq \mathcal{R}_k^{-\sqrt{K}} \).

Statement (b) of Lemma 4.10 is proved.

Statement (c) follows from (b) and a union bound. Indeed, by the definition of \( \mathcal{A}(x, k + 1) \), there are at most \( \mathcal{R}_k^{\mu} \) subballs \( B_{\mathcal{R}_k}^{\text{dis}}(w) \) of \( B_{\mathcal{R}_k^{\mu}}^{\text{dis}} \) such that \( B_{\mathcal{R}_k}^{\text{dis}}(w) \) is non-admissible.

Hence, recalling that \( \nu < \delta/K \), we get

\[
\mu \left[ \tilde{m}(\mathcal{R}_k) < U; \mathcal{A}(x, \tilde{m}(\mathcal{R}_k), k) \right] \leq \mathcal{R}_k^{-\sqrt{K}} \mathcal{R}_k^{\nu} \mathcal{R}_k^{\nu} < \mathcal{R}_k^{-\sqrt{K}/2}
\]

\[ (4.13) \]

for \( K \) large enough. Item (c) then follows from (4.13) and (a). \( \square \)

We can now proceed with the proof of Theorem 4.1. We estimate the success probability of the coupling in Lemma 4.12 below, and then we use this lemma to prove Theorem 4.1.

Lemma 4.12 Let \( R > \mathcal{R}_1 \) and \( k = k(R) = \max\{n : R > \mathcal{R}_n\} \). Let \( z, y \in B_{\mathcal{R}_R}^{\text{dis}} \). There exists \( \rho = \rho(d, P) > 0 \) such that if \( w \in \mathcal{A}(0, k + 1) \cap \{\mathcal{A}(x, k), \forall x \in B_{\mathcal{R}_k^{\mu}}^{\text{dis}}\} \), then there is a coupling \( \mu \) of two random walks \( (Y_n) \) and \( (Z_n) \) with laws \( P_y^\mu \) and \( P_z^\mu \) such that

\[
\mu \left[ Z_{\tau_{2MR,0}} = Y_{\tau_{2MR,0}} \right] \geq \rho
\]

where \( \tau_{2MR,0} \) is as defined in (4.1).

Proof. We can assume \( R \in M^* \). We consider the coupling \( \mu^{(0, R; y, z)} \) and simply write \( \mu = \mu^{(0, R; y, z)} \) for convenience. By our assumption and Lemma 4.10, with \( \mu \)-probability at least \( 1 - 3\mathcal{R}_1^{-\sqrt{K}} \), the event \( \mathcal{A}(x, \tilde{m}(\mathcal{R}_k), k) \cap \{\tilde{m}(\mathcal{R}_k) < U\} \) occurs. Moreover, by
Lemma 4.10, with \( \mu \)-probability at least \( 1 - 3 \mathcal{R}_n^{\sqrt{K}/2} \), \( 1 \leq n \leq k(R) \), the event \( \mathcal{A}(x \tilde{m}(\mathcal{A}_n), n) \cap \{ \tilde{m}(\mathcal{A}_n) < U \} \) implies the event \( \mathcal{A}(x \tilde{m}(\mathcal{A}_{n-1}), n-1) \cap \{ \tilde{m}(\mathcal{A}_{n-1}) < U \} \). Recursively applying this relation until time \( \tilde{m}(M) \), we have

\[
\mu(\mathcal{U} = \infty) = \mu(\mathcal{U} > \tilde{m}(M)) \geq 1 - 3 \mathcal{R}_1^{\sqrt{K} - \sum_{n=0}^{k-1} 3 \mathcal{R}_n^{\sqrt{K}/2}} \geq 1 - \frac{1}{4T}.
\]

Recall the definition of regeneration in (4.11). Note that there is a constant \( c > 0 \) such that \( \mu(0 \text{ is a regeneration of } \xi) > c \). So \( \mu(\mathcal{U} = \infty; 0 \text{ is a regeneration}) > c - \frac{1}{4T} > \frac{c}{2} \).

Further, by Lemma 4.5, we have

\[
\mu(\mathcal{U} = \infty; 0 \text{ is a regeneration}; Y_{\tau_{2MR},0} = Z_{\tau_{2MR},0}) > \frac{c}{2} e^{-M^2}.
\]

The lemma follows by noticing that, on \( \{ \mathcal{U} = \infty; 0 \text{ is a regeneration} \} \), the whole process of balls stay within the ball centered at 0 and with radius \( \sum_{n=0}^{\tilde{m}(M)} MR(n) < 2MR \).

With Lemma 4.12 we can prove Theorem 4.1.

**Proof of Theorem 4.1.** For \( R > \mathcal{R}_1 \), we assume \( R \in M^* \) for the simplicity of notations. Let \( k = k(R) \geq 1 \) be such that \( \mathcal{R}_k < R \leq \mathcal{R}_{k+1} \). By Lemma 4.9, the event

\[
\omega \in \mathcal{A}(0, k+1) \cap \{ \mathcal{A}(x, k) \text{ for all } x \in B_{2MR} \}
\]

occurs with \( P \)-probability at least \( 1 - e^{-R^3/3K} \). Under this event, by Lemma 4.12, for any \( y, z \in B_{2MR} \), we can use the coupling \( \mu = \mu^{(0; R, y, z)} \) of two paths \( (Y_n), (Z_n) \) with laws \( P^\omega, P^\omega_z \) to obtain \( \mu(\mathcal{Z}_{\tau_{2MR},0} = \mathcal{Z}_{\tau_{2MR},0}) \geq \varrho \). Then, repeating the calculation in (4.2), for every \( \omega \)-harmonic function \( f \) on \( B_{2MR} \) and any \( z, y \in B_{2MR} \),

\[
f(z) - f(y) \leq (1 - \varrho) \text{ osc } f_{B_{2MR}}.
\]

In other words, \( P(H_{1-\varrho; 2MR}) \geq 1 - e^{-R^3/3K} \).

### 5. Proof of the Harnack Inequality

In this section we prove Theorem 1.3 and Corollary 1.4. We use a modified Fabes-Stroock argument and apply the quantitative estimate (Theorem 1.5), Corollary 3.12, and the oscillation inequality (Theorem 4.1) to derive a contradiction. Notice that our argument is robust so that we only need Assumption 2 (in Page 25) instead of Assumption 1.

#### 5.1 Preparation and notations for the proof of Theorem 1.3

To prove Theorem 1.3 we only consider the case \( \Theta = 2 \). Recall the Harnack constant \( M_\varrho \) in (1.6). We fix the parameter \( H > M_2 \), and take \( \tilde{e} = \tilde{e}(H) \in (0, 1) \) such that
Let $H > M_2 + 3\varepsilon$. Let $\tilde{a} = \tilde{a}(H) \in (1, 2)$ be such that $M_2 \leq M_\tilde{a} \leq M_2 + \varepsilon$. Recall the event $H_{x,F}^{\Psi}$ in Theorem 4.1 and the constant $C_0 = C_0(d, P)$ in Corollary 3.12.

We choose two constants $N \in \mathbb{N}$ and $\nu > 0$ depending on $(H, d, P)$ such that
\begin{equation}
\nu^{-N}(M_2 + 2\varepsilon)^{-1} \varepsilon > 4 + 2(M_2 + 2\varepsilon - 1),
\end{equation}
\begin{equation}
\nu \chi = \frac{1}{4} \wedge \frac{2-\tilde{a}}{3\varepsilon}.
\end{equation}

The parameters $\nu, \Psi, C_0, H$ (and hence $\varepsilon, \tilde{a}, N, \nu$ as well) are fixed throughout the section.

Let $\Gamma = \{A_1, A_2, \ldots, A_k\}$ be a fixed covering of $\partial B_1$ as in (4.3), except that the diameters of the sets $A_1, A_2, \ldots, A_k$ are bounded by $\nu$. For $A \in \Gamma$, let $D_\nu^{\beta}(B)(A)$ be as in (4.4) and recall the notation $D^\beta(\omega)$ in (2.20). For any ball $B = B_{\nu}^{\text{dis}}(x)$, define the events
\begin{align*}
\mathcal{G}_1(B) &= H_{x,F}^{\Psi} \cap \left\{ \forall y, z \in B \forall A \in \Gamma, \frac{D_\nu^{\beta}(B_{\nu}^{\text{dis}}(x))(A)}{D^\beta(\omega)(B_{\nu}^{\text{dis}}(x))(A)} < M_2 + 2\varepsilon \right\}, \\
\mathcal{G}_2(B) &= \left\{ \max_B h \leq e^{C_0(\nu)} \min_B h \text{ for every function } h \geq 0 \text{ that is } \omega \text{-harmonic on } B_{\nu}^{\text{dis}}(x) \right\}.
\end{align*}

We fix a constant $\Xi \in (0, 1/4)$, and let $\mathcal{G}_\Xi(R), R \geq 1$, be the event that $\mathcal{G}_1(B) \cap \mathcal{G}_2(B)$ happens for every sub-ball $B$ of $B_{\nu}^{\text{dis}}$ with radius greater or equal to $\nu(R/2)^{\Xi}$.

**Claim 5.1** Under Assumption 3 (in Page 25), there exist constants $C = C(H, d, P), \gamma = \gamma(d, P) > 0$ such that for any $R \geq C$,
\begin{equation}
P\left[ \bigcap_{r \geq R} \mathcal{G}_\Xi(r) \right] \geq 1 - e^{-R^\gamma}.
\end{equation}

**Proof.** Recall the notation $D^\beta(\omega)$ in (4.5). Note that $\min_{x \in B_{1/\tilde{a}}, A \in \Gamma} D_\Sigma^\beta(A) \geq C_1(d, \Gamma, H, P) > 0$. By Corollary 2.4, with $P$-probability $\geq 1 - C e^{-R^4}$, we have $\min_{y \in B_{\nu}^{\text{dis}}, A \in \Gamma} D_\nu^{\beta}(B_{\nu}^{\text{dis}}(x))(A) \geq C_2 \nu$ and $\max_{y \in B_{\nu}^{\text{dis}}, A \in \Gamma} D_\nu^{\beta}(B_{\nu}^{\text{dis}}(x))(A) \leq \frac{\varepsilon}{4M_\alpha} \cdot \frac{C_2}{2} = \nu \cdot \frac{C_2}{2}$. Under this event, for any $y, z \in B_{\nu}^{\text{dis}}$ and $A \in \Gamma$,
\begin{equation}
D_\nu^{\beta}(B_{\nu}^{\text{dis}}(x))(A) \leq (1 + \epsilon)D_\Sigma^{\beta/\tilde{a}}(A) \leq (1 + \epsilon)M_\alpha D_\Sigma^{\beta/\tilde{a}}(A) \leq (1 + \epsilon)^2 M_\alpha D_\Sigma^{\beta}(B_{\nu}^{\text{dis}}(x))(A).
\end{equation}

The above inequality, together with Theorem 4.1 yields $P(\mathcal{G}_1(B_{\nu}^{\text{dis}})) \geq 1 - C e^{-R^4}$. With $C, \delta$ properly redefined, similar inequality holds for $\mathcal{G}_2(B_{\nu}^{\text{dis}})$ (by (3.10)), and hence for $\mathcal{G}_\Xi(R)$ as well. The claim then follows by a union bound. \hfill \Box

**Claim 5.2** If $\omega \in \mathcal{G}_\Xi(2R)$, then for any non-negative $\omega$-harmonic function $f$ on $B_{2R}^{\text{dis}}$ and any sub-ball $B_{\nu}^{\text{dis}}(x)$ of $B_{2R}^{\text{dis}}$ with $r > \nu R^\Xi$, there exists $A \in \Gamma$ such that
\begin{equation}
\max_{y, z \in \partial A B_{\nu}^{\text{dis}}(x)} \frac{f(y)}{f(z)} > \frac{1}{M_\alpha R^{2R}} \max_{y, z \in B_{\nu}^{\text{dis}}(x)} \frac{f(y)}{f(z)}.
\end{equation}
Proof. Write \( C_1 := \max \{ \max_{y,z \in \partial B^\text{dis}_{\alpha r}(x)} \frac{f(y)}{f(z)} : A \in \Gamma \} \) and \( B := B^\text{dis}_\alpha(x) \). Recall the exit time \( \tau(B) \) in (2.8). Then, by the definitions of \( G_{\Xi}(2R) \) and \( C_1 \), for any \( y, z \in B^\text{dis}_r(x) \),

\[
f(y) = E^y_\omega[f(X_{\tau(B)})] \leq \sum_{A \in \Gamma} D^\alpha_x(B)(A) \max_{\partial_t B} f < (M_2 + 2\tilde{\epsilon}) C_1 \sum_{A \in \Gamma} D^\alpha_x(B)(A) \min_{\partial_t B} f \leq (M_2 + 2\tilde{\epsilon}) C_1 f(z).
\]

Hence \( C_1 > (M_2 + 2\tilde{\epsilon})^{-1} f(y)/f(z) \) for all \( y, z \in B^\text{dis}_r(x) \). The claim is proved. \( \square \)

5.2 Proof of Theorem 1.3 and Corollary 1.4

We will prove that, under Assumption 2, the Harnack inequality (Theorem 1.3) holds for every large enough \( R \) satisfying \( \omega \in G_{\Xi}(2R) \).

Proof of Theorem 1.3: For the simplicity of notations we only consider the case \( \Theta = 2 \). The argument for general \( \Theta > 1 \) is the same. Let \( R_0 = R_0(\omega, H, d, P) = \inf\{n \geq 1 : \omega \in G_{\Xi}(2r) \} \) for all \( r \geq n \). By Claim 5.1, \( R_0 \) has a stretch exponential tail.

Assume for contradiction that \( \max_{x,y \in \partial B^\text{dis}_{R}} f(x)/f(y) > H \) for some \( R > R_0 \) and some non-negative function \( f \) which is \( \omega \)-harmonic on \( B^\text{dis}_{2R} \).

First, will define a sequence of sub-balls \( B^\text{dis}_{R}(x_j) \) of \( B^\text{dis}_{2R} \). Let \( \delta := \frac{2}{\omega} - 1 > 0 \). Define a sequence of radii \( r_0, r_1, \ldots, r_\ell \) as follows: \( r_0 = R, r_1 = \delta R/3, \) and then \( r_j = r_1/j^2 \). We take \( \ell \) to be the largest integer such that \( r_\ell > R^{\Xi} \). Note that (redefine \( R_0 \) if necessary)

\[
\ell = \ell(R) > CR^{(1-\Xi)/2} > R^{1/3}.
\]

We now define the centers \( x_j \) inductively. Let \( x_0 = 0 \). Provided that we know \( x_j, 0 \leq j < \ell \), by Claim 5.2 there exists \( A \in \Gamma \) such that (5.3) holds with \( (x, r) \) replaced by \( (x_j, r_j) \). Note that the diameter of \( \partial B^\text{dis}_{\alpha r_j}(x_j) \) is less than \( 2\nu r_j \). Thus there exists \( x_{j+1} \in B^\text{dis}_{r_j}(x_j) \) such that \( \partial B^\text{dis}_{\alpha r_j}(x_{j+1}) \subset B^\text{dis}_{r_j}(x_j+1) \). In particular, we get

\[
\max_{y,z \in \partial B^\text{dis}_{r_j}(x_{j+1})} \frac{f(y)}{f(z)} > M_2^{\alpha r_j} \max_{y,z \in \partial B^\text{dis}_{r_j}(x_j)} \frac{f(y)}{f(z)}.
\]

Notice that \( B^\text{dis}_{\alpha r_j}(x_j) \subset B^\text{dis}_{2R} \) for all \( 0 \leq j \leq \ell \), and \( B^\text{dis}_{C_0r_j}(x_j) \subset B^\text{dis}_{2R} \). Indeed, for any \( z \in B^\text{dis}_{C_0r_j}(x_j) \), redefining \( R_0 \) if necessary, we have

\[
\|z\| \leq \sum_{i=0}^{\ell} \alpha r_i + C_0 r_\ell \leq \alpha (1 + \frac{2}{3} \delta) R + 4 C_0 R^{\Xi} < 2R.
\]

Next, by our choice of \( \nu \) in (5.2), \( \nu \Phi^N \leq \frac{1}{4} \wedge \frac{3}{2} \leq \min_{0 \leq j < \ell} \frac{r_{j+1}}{r_j} \). Hence, we can apply the oscillation inequality (inside the definition of \( H^\nu_{x,j} \) in Page 25) repeatedly to obtain

\[
u^{-N} \frac{\text{osc}}{B^\text{dis}_{\alpha r_j}(x_{j+1})} f \leq \frac{\text{osc}}{B^\text{dis}_{\nu r_j}(x_{j+1})} f \leq \frac{\text{osc}}{B^\text{dis}_{r_{j+1}}(x_{j+1})} f.
\]
Finally, we will derive a contradiction. We write $B^j := B^\text{dis}_{r^j}(x_j)$, and $\beta_j = \frac{\text{osc}_{B^j} f}{\min_{B^j} f} = \frac{\max_{B^j} f}{\min_{B^j} f} - 1$. By our assumption, $\beta_0 > H - 1$. Moreover, by (5.5) and (5.4),

$$
\beta_{j+1} \geq \frac{v^{-N} \text{osc}_{B^j_{r^j}^{(x_{j+1})}} f}{\min_{B^j_{r^j}^{(x_{j+1})}} f} \\
\geq v^{-N} \left[ (M_2 + 2\varepsilon)^{-1} \frac{\max_{B^j} f}{\min_{B^j} f} - 1 \right] \\
= v^{-N}(M_2 + 2\varepsilon)^{-1} [\beta_j - (M_2 + 2\varepsilon - 1)]
$$

Since $\beta_0 - (M_2 + 2\varepsilon - 1) > \varepsilon$, we can take $N$ big enough so that $\beta_1 > 2(M_2 + 2\varepsilon - 1)$ and $\frac{1}{2} v^{-N} (M_2 + 2\varepsilon)^{-1} > 2$. Then it follows that $\beta_2 > 2\beta_1$, and inductively, $\beta_{j+1} > 2\beta_j$ for $1 \leq j < \ell$. Therefore, if $N$ is chosen so that (5.1) holds, then $\beta \ell > 2^\ell \varepsilon$, which implies

$$
\max_{B^\text{dis}_{r^\ell}(x_\ell)} f \geq (2^{\ell} \varepsilon + 1) \min_{B^\text{dis}_{r^\ell}(x_\ell)} f \geq \left(2^{(r/4)^{1/(3\Xi)}} + 1 \right) \min_{B^\text{dis}_{r^\ell}(x_\ell)} f
$$

where in the last inequality we used the fact that $\ell > R^{1/3} \geq (r/4)^{1/(3\Xi)}$.

Since $r^\ell > R_0^\Xi$ and $1/(3\Xi) > 4/3$, by redefining $R_0$ to be sufficiently large if necessary, inequality (5.6) contradicts with the event $\Theta_2(B^\text{dis}_{r^\ell}(x_\ell))$. \qed

**Proof of Corollary 1.5.** We choose a constant $D > 0$ such that for all $\Theta \geq 2$,

$$M_\Theta < 1 + D\Theta^{-1},$$

where $M_\Theta$ is as defined in (1.6). We will determine the value of $\Theta = \Theta(\epsilon, d)$ later. Let $M_\Theta < H < 1 + 2D\Theta^{-1}$, and let $R_0$ be as in Theorem 1.3. For $r > R_0$, we claim that for every $\omega$-harmonic $f : B^\text{dis}_{r^\Theta} \to \mathbb{R}$, we have $\text{osc}_{B^\text{dis}_{r^\Theta}} f \leq (H - 1) \text{osc}_{B^\text{dis}_{r^\Theta}} f$. Indeed, assume without loss of generality that $\text{inf}_{B^\text{dis}_{r^\Theta}} f = 0$. Then, by the Harnack inequality,

$$\text{osc}_{B^\text{dis}_{r^\Theta}} f = \sup_{B^\text{dis}_{r^\Theta}} f - \inf_{B^\text{dis}_{r^\Theta}} f \leq (H - 1) \inf_{B^\text{dis}_{r^\Theta}} f \leq 2D\Theta^{-1} \text{osc}_{B^\text{dis}_{r^\Theta}} f. \quad (5.7)$$

Fix $\Theta$ so large that $2D\Theta^{-\epsilon} < 1$. For every $r > R_0$, we iterate the bound (5.7) to get

$$\text{osc}_{B^\text{dis}_{r^\Theta}} f \leq (2D\Theta^{-1})^n \text{osc}_{B^\text{dis}_{r^{\Theta n}} f} \leq \left(\frac{r}{\Theta^n r} \right)^{1-\epsilon} \text{osc}_{B^\text{dis}_{r^{\Theta n}} f}$$

for all $n \in \mathbb{N}$. The Hölder estimate then follows by interpolation. \qed

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**References**


Assume w.l.o.g. that \( \langle e_i, z - w \rangle > 0 \), and let \( z' = z + e_i \). Then \( z' \) is in no sink in \( \omega' \) because it is too close to \( S_2 \). Note that \( z \notin z' \) for every \( x \in S_1 \). Thus no point in \( S_1 \) is in a sink in \( \omega' \). Further, no point outside of \( S = \bigcup_{j=1}^{k} S_j \) is in a sink in \( \omega' \). Indeed, if \( y \in \mathbb{Z}^d \setminus S \) is in a sink \( W \), then \( W \) cannot intersect \( S \), and thus \( W \) is a sink in \( \omega \) other than \( S_1, \ldots, S_k \). However, there is no such sink, thus \( W \) does not exist.
Thus there are only $k-1$ sinks in $\omega'$, in contradiction to $P_2$ being absolutely continuous with respect to $P$. \hfill $\Box$

B. Proof of Theorems 3.1 and 3.2 in two dimensions

A sequence $(x_i)_{i=0}^n$ is called a path if $x_0 \sim x_1 \ldots \sim x_n$. A subset $S \subset \mathbb{Z}^2$ is said to be connected if for any $x, y \in S$, there is a path $(x_i)_{i=0}^n \subset S$ with $x_0 = x$ and $x_n = y$.

Proof of Theorem 3.1 for $d = 2$: First, we will show that the “holes” outside of the sink are rectangles.

Let $C$ be a connected (in the sense of $\sim$) component of $\mathbb{Z}^2 \setminus C$. For the sake of clarity, we color the unit square centered at $x$ (with sides parallel to the lattice) with white color if $x \notin C$, and with blue color if $x \in C$. Now consider the interface between the white and blue areas. The border of the blue area may consist of straight lines and angles with degrees $90^\circ$ or $270^\circ$. See Figure 2. However, case (C) in Figure 2 is impossible, since every point in the sink has at least two neighbors in opposite sides that are in $C$.

Therefore, the border of $C$ consists of only straight lines and right angles. In other words, it is a rectangle. The rectangle is of finite size almost surely, because the probability of an infinite line with no bond orthogonal emanating from it is zero, and there are countably many such lines.

The probability for a given rectangle $L$ to be a connected component of $\mathbb{Z}^2 \setminus C$ is exponentially small in the length of the boundary of $L$. This proves Theorem 3.1 in two dimensions with $\alpha = 1/2$. \hfill $\Box$

Proof of Theorem 3.2 for $d = 2$: Our proof consists of several steps. In Steps 1–3, we define and estimate several geometric quantities in $\omega$. In Step 4 we will prove Theorem 3.2 using these geometric estimates.

Step 1. First, we define a few terms. For a fixed environment $\omega$, the ES-stair (ES stands for east-south) is defined to be the infinite path starting from $o$ which goes first upwards until it has the possibility to move right, and then it takes every opportunity to move right and moves downwards if a step to the right is not possible. The part of the ES-stair above the horizontal line is called the ES-path. See Figure 3. We estimate the length $L^{ES}(\omega)$ of the ES-path. To do this, we set $V_0 = V_0(\omega) = \inf\{n \geq 0 : \omega(ne_2,e_1) > 0\}$, $H_0 := 1$, and define recursively

$H_j := \inf\{n > 0 : \omega((n+H_0+\ldots+H_{j-1})e_1+(V_0-j)e_2,e_2) > 0\}, \forall j \geq 0.$
Figure 3. The ES-(EN-) path is marked with a solid red (blue) line. The shaded region is an E-bubble.

We have the length of the ES-path

\[ L^{ES}(\omega) = 2V_0 + \sum_{j=0}^{V_0-1} H_j. \]

Observe that \( V_0, H_1, H_2, \ldots \) are independent (under \( P \)) geometric random variables, and \( (H_j)_{j=1}^{\infty} \) are identically distributed. Hence we conclude that \( L^{ES} \) has exponential tail. That is, for \( x \geq 0 \),

\[ P(L^{ES} > x) \leq Ce^{-cx}. \]

Similarly, we can define the EN-stair, EN-path and \( L^{EN} \). The shorter one among the EN- and ES-path (If \( L^{ES} = L^{EN} \) then take the EN-path) is simply called the E-path. The E-path has length

\[ L^{E} = L^{E}(\omega) := L^{EN} \land L^{ES}, \]

which also has exponential tail

\[ P(L^{E} > x) \leq Ce^{-cx}, \quad \forall x \geq 0. \quad (B.1) \]

Step 2. The set of vertices that lie below (or on) the ES-stair and above (or on) the EN-stair is called the E-bubble, denoted by \( B^{E}(\omega) \). In other words, \( B^{E}(\omega) \) is the area enclosed by the EN- and ES-stairs. Clearly, \( \#B^{E}(\omega) \leq (L^{ES} + L^{EN})^2 \) and so it has stretched exponential tail. That is, for \( x \geq 0 \),

\[ P(\#B^{E} > x) \leq Ce^{-c\sqrt{x}}. \quad (B.2) \]

Here \( \#B^{E} \) denotes the cardinality of the set \( B^{E} \).

Step 3. Now we will define the east-tadpole. Recall \( \tau \) in (1.3). We denote the E-path by \( p^{E}(\omega) \) and its end-point by \( R^{E}(\omega) \in \mathbb{N}e_1 \). Set \( R_0^{E} = 0 \) and define for \( j \geq 0 \)

\[ R_{j+1}^{E} := R_j^{E} + E(\tau_{-R_j^{E}}\omega). \]

In other words, \( R_j^{E} \) is end-point of the concatenation \( p_j^{E}(\omega) := \bigcup_{i=0}^{j-1} p^{E}(\tau_{-R_i^{E}}\omega) \) of \( j \) consecutive E-paths. For any \( n \geq 0 \), let

\[ M(n) := \inf\{i \geq 1 : R_i^{E} \cdot e_1 \geq n\} \]
and define the east-tadpole $T^E(n) = T^E_{\omega}(n)$ to be the union of the first $M(n) - 1$ E-paths and the $M(n)$-th E-bubble. Namely,

$$T^E(n) := p^E_{M(n)-1}(\omega) \bigcup B^E(\tau^E_{-R^E_{M(n)-1}} \omega).$$

Since $M(n) \leq n$, we have

$$\#T^E(n) \leq \sum_{i=0}^{n-1} L^E(\tau^E_{-R^E_i} \omega) + \max_{j=0,...,n-1} \#B^E(\tau^E_{-R^E_j} \omega),$$

where the right side is a sum of 1-dependent random variables. It then follows from (B.1) and (B.2) that for $n \in \mathbb{N}$,

$$P(\#T^E(n) > Cn) \leq Ce^{-c\sqrt{n}}.$$

As the most important property of the tadpole, notice that for any $x \in T^E_{\omega}(n)$, if $o \preceq x$, then we can find a $\omega$-path in $T^E(n)$ from $o$ to $x$.

Step 4. Finally, we are ready to prove the theorem. Without loss of generality, assume that $x = o$ and $y = (y_1,y_2)$ with $y_1,y_2 \geq 0$. Let $z = (y_1,0)$. We define the north-tadpole $T^N(n)$ similarly as in Step 3. By the remark at the end of Step 3, if $o \preceq y$, then we can find a $\omega$-path in $T^E_{\omega}(y_1) \cup T^N_{\tau^E_{-\omega}}(y_2)$ from $o$ to $y$. Therefore,

$$P[\text{dist}_\omega(o,y) > C\|y\| ; o \preceq y] \leq P(\#T^E(y_1) + \#T^N_{\tau^E_{-\omega}}(y_2) \geq C(|y_1| + |y_2|)] \leq Ce^{-c\sqrt{n}}. \quad \Box$$

C. Proof of Claim 3.10

For $x, y \in \mathbb{Z}^d$, we write $x \preceq y$ if $\|x - y\|_\infty = 1$. We call a sequence $(x_i)_{i=0}^n$ a $*$-path if $x_i \preceq x_{i+1}$ for all $0 \leq i < n$. Recall that in the Bernoulli percolation, a site is “open” with probability $p$ and “closed” with probability $1 - p$.

**Proof of Claim 3.10**. It suffices to consider the case $d = 2$. We fix $x \in \square_n$.

First, observe that if $|C_{x,n}| = k$, $k < n$, then there exists a $*$-path of closed points within $\partial C_{x,n} \cap \square_n$ whose length is $\lfloor \sqrt{k} \rfloor$. Indeed, we denote the surfaces of $\square_n$ by

$$S^\pm_i = \{ z \in \square_n : z \cdot e_i = \pm n \}, \quad i = 1,2.$$

Note that $C_{x,n}$ is connected to at most one of the parallel surfaces $S^+_1$, $S^-_1$, because otherwise $|C_{x,n}| \geq n$. The same is true for $S^+_2$. Without loss of generality, we can assume that $C_{x,n}$ is not connected to $S^+_1$ and $S^+_2$. Note also that the projection of $C_{x,n}$ on one of the two surfaces $S^+_1$, $S^+_2$ should have length at least $\sqrt{k}$, say, its projection on $S^+_1$ has length at least $\lfloor \sqrt{k} \rfloor$. Then all points in $\partial C_{x,n} \cap \square_n$ that lie between $C_{x,n}$ and $S^+_1$ are closed points.

Next, notice that if $|C_{x,n}| = k$, $k < n$, then there are at most $(2k + 2)^28\sqrt{k}$ possibilities for a $*$-path of closed points within $\partial C_{x,n} \cap \square_n$ whose length is $\lfloor \sqrt{k} \rfloor$. Indeed, such a path should be inside $\square_{k+1}(x)$, since $C_{x,n} \subset \square_k(x)$. Further, for any $z \in \square_{k+1}(x)$, there are
at most $8\sqrt{k}$ *-paths with starting point $z$ and length $\lfloor \sqrt{k} \rfloor$. Hence the number of such paths is at most $(2k+2)^28\sqrt{k}$.

Using the observations as above, we have, for $1 \leq k < n$,

$$P(|E_{x,n}| = k) \leq (2k+2)^28\sqrt{k}(1-p)^{\lfloor \sqrt{k} \rfloor} \leq (1-p)^{\sqrt{k}/2}$$

if $p < 1$ is sufficiently close to 1. Therefore $P(|E_{x,n}| < n) \leq (1-p) + \sum_{k \geq 1} (1-p)^{\sqrt{k}/2}$ can be as close to 0 as we want, if $p < 1$ is sufficiently close to 1. □

D. Proof of Claim 4.11

We call $n \geq 0$ a renewal if $\xi_m > \xi_n$ for all $m < n$ and $\xi_m < \xi_{n+1} < \xi_n$ for all $m > n+1$. Denote by $\tau_k$ the $k$th renewal. Then $(\tau_{k+1} - \tau_k)_{k \geq 1}$ is an i.i.d. sequence and $\tau_1$, as well as $\tau_2 - \tau_1$ have exponential tails. Write $U_k = \xi_{\tau_k}$ and $V_k = \tau_{k+1} - \tau_k$. Then $(V_k)_{k \geq 1}$ is an i.i.d. sequence and there exists $\nu > 0$ such that $P(V_k > l) < e^{-\nu l}$ for every $l$ and $k \geq 0$. In addition, $U_{k+1} - U_k \leq -1$ for every $k$. Hence for $j \geq k \geq 0$ and all $m \in (\tau_j, \tau_{j+1}]$, we have $\xi_m - \xi_{\tau_k} = (\xi_m - \xi_{\tau_j}) + (\xi_{\tau_j} - \xi_{\tau_k}) \leq -1 + (k-j)$. Thus, for $k \geq 0$, we have

$$\sum_{m=2+\tau_k}^{\infty} M^{\xi_m - \xi_{1+\tau_k}} = M\sum_{m=2+\tau_k}^{\infty} M^{\xi_m - \tau_k} \leq M[V_k M^{-1} + \sum_{j=k+1}^{\infty} V_j M^{k-j-1}]$$

where we also used the fact that $\xi_m - \xi_{\tau_k} \leq -2$ for all $m \geq 2 + \tau_k$. So, in particular, $\tau_k$ is a regeneration if $V_j M^{-i} < 3^{-i}$ for every $i \geq 1$. For $j \geq k$, we say that $j$ influences $k$, and denote it by $\mathcal{I}[j \rightarrow k]$, if $V_j M^{k-j-1} \geq 3^{k-j-1}$. So $\tau_k$ is a regeneration if it is not influenced by any $j \geq k$. Let

$$I(j) = \sum_{k \leq j} 1_{\mathcal{I}[j \rightarrow k]}.$$

Then $(I(j))_{j \geq 1}$ is an i.i.d. sequence with exponential tails, and for $M$ large enough we have $E[I(j)] < 1$. Thus by a large deviation estimate, with probability exponentially close to 1, $\sum_{j=1}^{J} I(j) < CJ$ for some $C < 1$, and under this event there are at least $(1-C)J$ regenerations. □

E. Proof of the optimality of the Harnack constant in Theorem 1.3

We will prove the statement in the first paragraph of Remark 1. Without loss of generality, we consider $\Theta = 2$. Recall the notations $M_\Theta$ and $\mathcal{H}(\Sigma)$ in (1.6).

For any $H < M_2$, by the definition of $M_2$, there exist $F \in \mathcal{H}(\Sigma) \cap C^\infty(B_1)$ and two points $x_0, y_0 \in B_{1/2}$ such that $F(x_0) > HF(y_0)$. Let $G_{R,\omega}$ be the solution of (1.11). Then,

$$\lim_{R \to \infty} G_{R,\omega}([R x_0]) - H G_{R,\omega}([R y_0]) = F(x_0) - HF(y_0) > 0$$

by Theorem 1.5, where $[x]$ is a closest point in $\mathbb{Z}^d$ to $x$. Hence $\max_{B_{R/2}} G_{R,\omega} > H \min_{B_{R/2}} G_{R,\omega}$ for all sufficiently large $R$, and the optimality is proved. □
F. An example

In this appendix we provide an example of a translation invariant genuinely $d$-dimensional finite range dependent environment where the Harnack principle fails. We mention that in this example there is a non-degenerate quenched functional CLT, but we do not provide here a proof for this fact.

We work in three dimension.

Let $\tilde{\omega}$ be the following i.i.d. (but not genuinely 3-dimensional) balanced environment on $\mathbb{Z}^3$: for every $x$, with probability $1/2$ we have $\tilde{\omega}(x, \pm e_1) = 1/2$ and $\tilde{\omega}(x, \pm e_2) = \tilde{\omega}(x, \pm e_3) = 0$, and with probability $1/2$ we have $\tilde{\omega}(x, \pm e_2) = 1/2$ and $\tilde{\omega}(x, \pm e_1) = \tilde{\omega}(x, \pm e_3) = 0$. In other words, the arrows are either pointing in the east-west axis or in the south-north axis, but never in the up-down axis. We call a point isolated if no other point points to it. In other words, $x$ is isolated if $\tilde{\omega}(x - e_1, e_1) = \tilde{\omega}(x + e_1, -e_1) = \tilde{\omega}(x - e_2, e_2) = \tilde{\omega}(x + e_2, -e_2) = 0$ (see figure 4).

We now say that $x$ is a special point if $x + e_3$ and $x - e_3$ are isolated, while $x, x + 2e_3, x + 3e_3, x - 2e_3$ and $x - 3e_3$ are not isolated and $\tilde{\omega}(x, \pm e_1) = 1/2$.

We then define $\omega$ as follows (see figure 5). For any $x \in \mathbb{Z}^3$, $i \in \{1, 2, 3\}$, let

$$
\omega(x, \pm e_i) = \begin{cases} 
\frac{1_{\{i=3\}}}{2} & x + e_3 \text{ or } x - e_3 \text{ is special} \\
\frac{1_{\{i\neq 2\}}}{4} & x \text{ is special} \\
\tilde{\omega}(x, \pm e_i) & \text{otherwise.}
\end{cases}
$$

Let $P$ be the distribution of $\omega$. Clearly, $P$ is genuinely $d$-dimensional and has finite range dependence.

Claim F.1 $P$-almost surely there are two distinct sinks.

Given Claim F.1 it is easy to see that the Harnack inequality fails: write $S_1$ and $S_2$ for the two sinks, and let $T = \inf\{n \geq 0 : X_n \in S_1 \cup S_2\} \leq \infty$. Write

$$
f(x) = P_{\omega}^x(X_T \in S_1).
$$

Then $f$ is non-negative $\omega$-harmonic with $f|_{S_1} = 1$ and $f|_{S_2} = 0$, which contradicts the Harnack inequality.
Figure 5. An illustration of the change in the environment from $\tilde{\omega}$ to $\omega$ around a special point. Here a circle represents an isolated point. The special point is the point in the middle.

**Sketch of the proof of Claim F.1.** First note that (while constructing $\omega$ from $\tilde{\omega}$) arrows are only being removed if they emanate from isolated points. In particular, for every $z$, no arrow in the sink in $\mathbb{Z} \times \mathbb{Z} \times \{z\}$ is removed. Note that the modification as in Figure 5 adds “bridges” between sinks in even levels, and bridges between sinks in odd levels. More precisely, in $\omega$, for every $z$ there is a path from the sink in $\mathbb{Z} \times \mathbb{Z} \times \{z\}$ to the sink in $\mathbb{Z} \times \mathbb{Z} \times \{z+2\}$ and back, but no path from the sink in $\mathbb{Z} \times \mathbb{Z} \times \{z\}$ to the sink in $\mathbb{Z} \times \mathbb{Z} \times \{z+1\}$. Thus there are $P$-a.s. two sinks: $S_1$ denotes the union of sinks in even levels, together with the bridges between them, and $S_2$ denotes its counterpart for odd levels. □