AN OPTIMAL REGULARITY RESULT FOR KOLMOGOROV EQUATIONS AND WEAK UNIQUENESS FOR SOME CRITICAL SPDES

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We show uniqueness in law for the critical SPDE

\[ dX_t = AX_t dt + (-A)^{1/2}F(X_t) dt + dW_t, \quad X_0 = x \in H, \]

where \( A : \text{dom}(A) \subset H \to H \) is a negative definite self-adjoint operator on a separable Hilbert space \( H \) having \( A^{-1} \) of trace class and \( W \) is a cylindrical Wiener process on \( H \). Here \( F : H \to H \) can be continuous with at most linear growth (some functions \( F \) which grow more than linearly can also be considered). This leads to new uniqueness results for generalized stochastic Burgers equations and for three-dimensional stochastic Cahn-Hilliard type equations which have interesting applications. To get weak uniqueness we use an infinite dimensional localization principle and also establish a new optimal regularity result for the Kolmogorov equation

\[ \lambda u - Lu = f \]

associated to the SPDE when \( F = 0 \) (\( \lambda > 0 \), \( f : H \to \mathbb{R} \) Borel and bounded). In particular, we prove that the first derivative \( Du(x) \) belongs to \( \text{dom}((-A)^{1/2}) \), for any \( x \in H \), and

\[ \sup_{x \in H} |(-A)^{1/2} Du(x)|_H = \|(Au)^{1/2} Du\|_0 \leq C \|f\|_0. \]

1. Introduction. We establish weak uniqueness (or uniqueness in law) for critical stochastic evolution equations like

\[ dX_t = AX_t dt + (-A)^{1/2}F(X_t) dt + dW_t, \quad X_0 = x \in H. \]

Here \( H \) is a separable Hilbert space, \( A : D(A) \subset H \to H \) is a self-adjoint operator of negative type such that the inverse \( A^{-1} \) is of trace class (cf. Section 1.1 and see also Remark 3), \( W = (W_t) \) is a cylindrical Wiener process on \( H \), cf. [15], [16], [24] and the references therein. We may assume that

\[ F : H \to H \text{ is continuous and verifies } |F(x)|_H \leq C_F(1 + |x|_H), \quad x \in H, \]

for some constant \( C_F > 0 \). This allows to prove both weak existence and weak uniqueness for (1.1). Assumption (1.2) can be relaxed if we assume that weak existence holds for (1.1); see Section 7 where we consider \( F \) which is continuous and bounded on bounded sets of \( H \).

Using the analytic semigroup \( (e^{tA}) \) generated by \( A \) we consider mild solutions to (1.1), i.e.,

\[ X_t = e^{tA}x + \int_0^t (-A)^{1/2}e^{(t-s)A}F(X_s)ds + \int_0^t e^{(t-s)A}dW_s, \quad t \geq 0, \]

(cf. Section 1.1) and prove the following result.

**Theorem 1.** Under Hypothesis 1 and assuming (1.2), for any \( x \in H \), there exists a weak mild solution defined on some filtered probability space. Moreover uniqueness in law (or weak uniqueness) holds for (1.1), for any \( x \in H \).
Examples of SPDEs of the form (1.1) are considered in Section 2. In particular, we can deal with stochastic Burgers-type equations like
\[ du(t, \xi) = \frac{\partial^2}{\partial \xi^2} u(t, \xi) dt + \frac{\partial}{\partial \xi} h(u(t, \xi)) dt + dw_t(\xi), \quad u(0, \xi) = u_0(\xi), \quad \xi \in (0, \pi), \]
with suitable boundary conditions (cf. [23], [7] and [33]) and stochastic Cahn-Hilliard equations (cf. [17], [9], [31], [18]) like
\[ \frac{\partial}{\partial t} u(t, \xi) = -\Delta u(t, \xi) dt + \Delta h(u(t, \xi)) dt + dw_t(\xi), \quad t > 0, \quad u(0, \xi) = u_0(\xi) \text{ on } G, \]
with suitable boundary conditions (\( G \subset \mathbb{R}^3 \) is a regular bounded open set). We prove weak well-posedness for both SPDEs when \( h \) is continuous and has at most a linear growth (see Section 2). Such assumption does not cover classical stochastic Burgers equations (i.e., \( h(u) = \frac{1}{2} u^2 \)) and stochastic Cahn-Hilliard equations (i.e., \( h(u) = u^3 - u \)) for which strong existence and uniqueness can be proved by different methods (cf. [3] and [9]). On the other hand, in Section 7 we consider some perturbations of classical Burgers equations (cf. Proposition 25).

We mention that in [35], [1] and [2] weak uniqueness has been investigated for stochastic evolutions equations with H"older continuous coefficients and non-degenerate multiplicative noise when \((-A)^{1/2} F\) is replaced by \( F \). On the other hand, weak uniqueness for (1.1) follows by Section 4 of [7] assuming that \( F \) is \( \theta \)-H"older continuous and bounded, \( \theta \in (0, 1) \), with \( \| F \|_{C^\theta_0(H, H)} \) small enough. To prove weak uniqueness for (1.1) we first establish a new optimal regularity result for the infinite-dimensional Kolmogorov equation
\[ \lambda u(x) - Lu(x) = f(x), \quad x \in H. \]
where \( \lambda > 0, f : H \to \mathbb{R} \) is a given Borel and bounded function and \( L \) is an infinite-dimensional Ornstein-Uhlenbeck operator which is formally given by
\[ Lg(x) = \frac{1}{2} \text{Tr}(D^2 g(x)) + \langle Ax, Dg(x) \rangle, \quad x \in D(A), \]
where \( Dg(x) \) and \( D^2 g(x) \) denote respectively the first and second Fréchet derivatives of a regular function \( g \) at \( x \in H \) and \( \langle \cdot, \cdot \rangle \) is the inner product in \( H \) (for regularity results concerning \( L \) when \( H = \mathbb{R}^n \) see [27] and the references therein). According to Chapter 6 in [15] (see also [7] and [10]) we investigate properties of the bounded solution \( u : H \to \mathbb{R}, \)
\[ u(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt, \quad x \in H; \]
here \( (P_t) \) is the Ornstein-Uhlenbeck semigroup associated to \( L \). One has \( P_t f(x) = \mathbb{E}[f(Z_t^x)] = \mathbb{E} \left[ f(e^{t A} x + \int_0^t e^{(t-s) A} dw_s) \right]; \) \( Z_t^x \) denotes the Ornstein-Uhlenbeck process which solves (1.1) when \( F = 0 \) (cf. Section 1.2). It easy to prove that \( u \in C^1_0(H) \), i.e., \( u \) is continuous and bounded with the first Fréchet derivative \( Du : H \to H \) which is continuous and bounded.

The new regularity result we prove is that \( Du(x) \in D((-A)^{1/2}) \), for any \( x \in H \), and
\[ \sup_{x \in H} \|(-A)^{1/2} Du(x)\|_H \leq \frac{\pi}{\sqrt{2}} \sup_{x \in H} |f(x)|_H, \]
see Theorem 7 with \( z = 0 \) and compare with [1], [5], [7] and [30]. Note that (1.4) is a limit case of known estimates. Indeed if \( \theta \in (0, 1) \), and \( f : H \to \mathbb{R} \) is \( \theta \)-H"older continuous and bounded then
\[ \|(-A)^{1/2} Du\|_{C^\theta_0(H, H)} \leq c_\theta \| f \|_{C^\theta_0(H)} \]
is the main result in [7]. Similar regularity results have been already proved in \( L^p(H,\mu) \)-spaces with respect to the Gaussian invariant measure \( \mu \) for \((P_t)\) (cf. Section 3 of [6]):

\[
\|(-A)^{1/2}Du\|_{L^p(\mu)} \leq C_p\|f\|_{L^p(\mu)}, \quad 1 < p < \infty.
\]

Hence estimate (1.4) corresponds to the remaining case \( p = \infty \). We stress that when \( f \in L^2(\mu) \) the fact that the estimate \( \|(-A)^{1/2}Du\|_{L^2(\mu)} \leq C_2\|f\|_{L^2(\mu)} \) is sharp follows by Proposition 10.2.5 in [15]. Moreover, the bound

\[
\sup_{x \in H} \|(-A)^{1/2}DP_tf(x)\| = \|(-A)^{1/2}DP_tf\|_0 \sim \frac{C_1}{t}\|f\|_0, \quad t \to 0^+
\]

(\(C_1\) is given in (1.21)) containing the singular term \( \frac{1}{2} \) suggests that (1.4) cannot be improved replacing \((-A)^{1/2}\) by \((-A)\gamma\), \( \gamma \in (\frac{1}{2},1) \); see also Chapter 6 of [15] and Remark 2.

Theorem 7 is deduced by the crucial Lemma 6; the proof of such lemma uses the diagonal structure of \( A \). By Lemma 6 we also derive another new regularity result (cf. Theorem 9 with \( z = 0 \)):

\[
\|P_tDu - Du\|_0 \leq cs^{1/2}\|f\|_0, \quad s \in [0,1].
\]

In Appendix we show that (1.8) implies the \( C^1 \)-Zygmund regularity of \( Du \); such Zygmund regularity of \( Du \) has been obtained in [5] and [30] by different methods.

Concerning the SPDE (1.1) we first prove the weak existence in Section 4 (see also Remark (12)). To this purpose we adapt a compactness argument already used in [21] (see also Chapter 8 in [16]). The proof of the uniqueness part of Theorem 1 is more involved and it is done in various steps (see Sections 5 and 6). In the case when \( F \in C_b(H,H) \) we first consider equivalence between mild solutions and solutions to the martingale problem of Stroock and Varadhan [34] (cf. Section 5.1). This allows to use some uniqueness results available for the martingale problem (cf. Theorems 15, 16 and 17). On this respect we point out that an infinite-dimensional generalization of the martingale problem is given in Chapter 4 of [19].

In Section 5.3 we prove weak uniqueness assuming that there exists \( z \in H \) such that

\[
\sup_{x \in H} |F(x) - z|_H < 1/4.
\]

To this purpose we need a careful analysis of the Kolmogorov equation

\[
\lambda u - Lu - \langle z, (-A)^{1/2}Du \rangle = f + \langle F - z, (-A)^{1/2}Du \rangle
\]

under the condition (1.9) (see Section 5.2). This is based on the fact that the same estimate (1.4) holds more generally if \( u \) is replaced by the solution \( u^{(z)} \) of the following equation

\[
\lambda u - Lu - \langle z, (-A)^{1/2}Du \rangle = f,
\]

for any \( z \in H \) (cf. Theorem 7). In Section 5.4 we prove uniqueness in law when \( F \in C_b(H,H) \) (removing condition (1.9)). To this purpose we also adapt the localization principle which has been introduced in [34] (cf. Theorem 16). In Section 6 we complete the proof of Theorem 1, showing weak uniqueness under (1.2). To this purpose we truncate \( F \) and prove uniqueness for the martingale problem up to a stopping time (cf. Theorem 17). Section 7 considers the case of \( F \) which is continuous and locally bounded.

We finally mention that recent papers investigate pathwise uniqueness for SPDEs like (1.1) when \((-A)^{1/2}F\) is replaced by a measurable drift term \( F \) (cf. [10], [11] and also [4] for the case of semilinear stochastic heat equations and see the references therein). For such equations in infinite dimensions even if \( F \in C_b(H,H) \) pathwise uniqueness, for any initial \( x \in H \), is still not clear (however pathwise uniqueness holds for \( \mu \)-a.e. \( x \in H \)).
Remark 2. It is not clear if our uniqueness result holds for (1.1) when $(-A)^{1/2}$ is replaced by $(-A)^{\gamma}$, $\gamma \in (1/2, 1)$. We believe that for $\gamma \in (1/2, 1)$ there should exist a continuous and bounded drift $F_\gamma : H \to H$ and $x_\gamma \in H$ such that weak uniqueness fails for $dX_t = AX_t dt + (-A)^\gamma F_\gamma(X_t) dt + dW_t$, $X_0 = x$ (on the other hand, weak existence holds, cf. Remark 12). In this sense (1.1) can be considered as a critical SPDE.

1.1. Notations and preliminaries. Let $H$ be a real separable Hilbert space. Denote its norm and inner product by $\| \cdot \|_H$ and $\langle \cdot , \cdot \rangle$ respectively. Moreover $\mathcal{B}(H)$ indicates its Borel $\sigma$-algebra. Concerning (1.1) as in [7], [10] and [11] we assume

**Hypothesis 1.** $A : D(A) \subset H \to H$ is a negative definite self-adjoint operator with domain $D(A)$ (i.e., there exists $\omega > 0$ such that $\langle Ax, x \rangle \leq -\omega \| x \|_H^2$, $x \in D(A)$). Moreover $A^{-1}$ is a trace class operator.

In the sequel we will concentrate on an infinite dimensional Hilbert space $H$. Since $A^{-1}$ is compact, there exists an orthonormal basis $(e_k)$ in $H$ and an infinite sequence of positive numbers $(\lambda_k)$ such that

$$
(1.11) \quad Ae_k = -\lambda_k e_k, \quad k \geq 1, \text{ and } \sum_{k \geq 1} \lambda_k^{-1} < \infty.
$$

We denote by $\mathcal{L}(H)$ the Banach space of bounded and linear operators $T : H \to H$ endowed with the operator norm $\| \| : \| \mathcal{L} \|$. The operator $A$ generates an analytic semigroup $(e^{tA})$ on $H$ such that $e^{tA} e_k = e^{-\lambda_k t} e_k$, $t \geq 0$. Remark that

$$
(1.12) \quad \| (-A)^{1/2} e^{tA} \|_\mathcal{L} = \sup_{k \geq 1} \{ (\lambda_k)^{1/2} e^{-\lambda_k t} \} \leq \frac{c}{\sqrt{t}}, \quad t > 0,
$$

with $c = \sup_{a \geq 0} ae^{-u^2}$. We will also use orthogonal projections with respect to $(e_k)$:

$$
(1.13) \quad \pi_m = \sum_{j=1}^{m} e_j \otimes e_j, \quad \pi_m x = \sum_{k=1}^{m} x^{(k)} e_k, \quad \text{where } x^{(k)} = \langle x, e_k \rangle, x \in H, m \geq 1.
$$

Let $(E, \| \cdot \|_E)$ be a real separable Banach space. We denote by $B_b(H, E)$ the Banach space of all real, bounded and Borel functions on $H$ with values in $E$, endowed with the supremum norm $\| f \|_0 = \sup_{x \in H} |f(x)|_E$, $f \in B_b(H, E)$. Moreover $C_b(H, E) \subset B_b(H, E)$ indicates the subspace of all bounded and continuous functions. We denote by $C_b^k(H, E) \subset B_b(H, E)$, $k \geq 1$, the space of all functions $f : H \to E$ which are bounded and Fréchet differentiable on $H$ up to the order $k \geq 1$ with all the derivatives $D^j f$ bounded and continuous on $H$, $1 \leq j \leq k$. We also set $B_b(H) = B_b(H, \mathbb{R})$, $C_b(H) = C_b(H, \mathbb{R})$ and $C_b^k(H, \mathbb{R}) = C_b^k(H, \mathbb{R})$.

We will deal with the SPDE (1.1) where $W = (W_t) = (W(t))$ is a cylindrical Wiener process on $H$. Thus $W$ is formally given by $W_t = \sum_{k \geq 1} W^{(k)}_t e_k$ where $(W^{(k)})_{k \geq 1}$ are independent real Wiener processes and $(e_k)$ is the basis of eigenvectors of $A$ (cf. [15], [24] and [16]). The next definition is meaningful for $F : H \to H$ which is only continuous because of (1.12).

A weak mild solution to (1.1) is a sequence $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, X)$, where $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space on which it is defined a cylindrical Wiener process $W$ and an $\mathcal{F}_t$-adapted, $H$-valued continuous process $X = (X_t) = (X_t)_{t \geq 0}$ such that, $\mathbb{P}$-a.s.,

$$
(1.14) \quad X_t = e^{tA} x + \int_0^t (-A)^{1/2} e^{(t-s)A} F(X_s) ds + \int_0^t e^{(t-s)A} dW_s, \quad t \geq 0.
$$

(hence $X_0 = x$, $\mathbb{P}$-a.s.). We say that uniqueness in law holds for (1.1) for any $x \in H$ if given two weak mild solutions $X$ and $Y$ (possibly defined on different filtered probability spaces...
and starting at $x \in H$, we have that $X$ and $Y$ have the same law on $B(C([0, \infty); H))$ which is the Borel $\sigma$-algebra of $C([0, \infty); H)$ (this is the Polish space of all continuous functions from $[0, \infty)$ into $H$ endowed with the metric of the uniform convergence on bounded intervals; cf. [25] and [16]). Note that the stochastic convolution
\[
W_A(t) = \int_0^t e^{(t-s)A}dW_s = \sum_{k \geq 1} \int_0^t e^{(t-s)\lambda_k}ekdW^{(k)}(s)
\]
is well defined since the series converges in $L^2(\Omega; H)$, for any $t \geq 0$. Moreover $W_A(t)$ is a Gaussian random variable with values in $H$ with distribution $N(0, Q_t)$ where
\[
Q_t = \int_0^t e^{2sA}ds = (2A)^{-1}(I - e^{2tA}), \quad t \geq 0,
\]
is the covariance operator (see also [7]). Note that $(-A)^{1/2}W_A(t)$ has a continuous version with values in $H$ for $\gamma \in [0, 1/2)$. However, in general $(-A)^{1/2}W_A(t)$ has not a continuous version (see [12]).

Equivalence between different notions of solutions for (1.1) are clarified in [15] and [24] (see also [26] for a more general setting). If we write $X^{(k)}(t) = X_t^{(k)} = \langle X(t), e_k \rangle$, $k \geq 1$, (1.1) is equivalent to the system
\[
X_t^{(k)} = x^{(k)} - \lambda_k \int_0^t X_s^{(k)}ds + \lambda_{k/2} \int_0^t F(X_s)ds + W_t^{(k)}, \quad k \geq 1,
\]
or to $X_t^{(k)} = e^{-\lambda_k t}(x^{(k)}) + \int_0^t e^{-\lambda_k (t-s)}(\lambda_k)^{1/2}F(X_s)ds + \int_0^t e^{-\lambda_k (t-s)}dW_s^{(k)}$, for $k \geq 1$, $t \geq 0$, with $F(x) = \sum_{k \geq 1} F^{(k)}(x)e_k$, $x \in H$.

We will also use the natural filtration of $X$ which is denoted by $(F_t^X)$; $F_t^X = \sigma(X_s : 0 \leq s \leq t)$ is the $\sigma$-algebra generated by the r.v. $X_s$, $0 \leq s \leq t$ (cf. Chapter 2 in [19]).

**Remark 3.** We point out that Theorem 1 holds under the following more general hypothesis: $A : D(A) \subset H \rightarrow H$ is self-adjoint, $\langle Ax, x \rangle \leq 0$, $x \in D(A)$, and $(I - A)^{-1}$ is of trace class, with $I = I_H$. Indeed in this case one can rewrite equation (1.1) in the form
\[
dX_t = (A - I)X_tdt + (I - A)^{1/2}[(I - A)^{-1/2}X_t + (-A)^{1/2}(I - A)^{-1/2}F(X_t)]dt + dW_t,
\]
$X_0 = x$. Now the linear operator $\hat{A} = I - A$ and the nonlinear term $\tilde{F}(x) = ((I - A)^{-1/2}x + (-A)^{1/2}(I - A)^{-1/2}F(x))$, $x \in H$, verify Hypothesis 1 and condition (1.2) respectively.

1.2. A generalised Ornstein-Uhlenbeck semigroup. Let us fix $z \in H$. We will consider generalised Ornstein-Uhlenbeck operators like
\[
L^{(z)}g(x) = \frac{1}{2} \text{Tr}(D^2 g(x)) + \langle x, ADg(x) \rangle + \langle z, (-A)^{1/2}Dg(x) \rangle, \quad x \in H, \quad g \in C^2_{cil}(H).
\]
Here $C^2_{cil}(H)$ denotes the space of regular cylindrical functions. We say that $g : H \rightarrow \mathbb{R}$ belongs to $C^2_{cil}(H)$ if there exist elements $e_{i_1}, \ldots, e_{i_n}$ of the basis $(e_k)$ of eigenvectors of $A$ and a $C^2$-function $\tilde{g} : \mathbb{R}^n \rightarrow \mathbb{R}$ with compact support such that
\[
g(x) = \tilde{g}(\langle x, e_{i_1} \rangle, \ldots, \langle x, e_{i_n} \rangle), \quad x \in H.
\]
By writing the stochastic equation $dX_t = AX_tdt + (-A)^{1/2}zdt + dW_t$, $X_0 = x$ in mild form as $X_t = e^{tA}x + \int_0^t e^{(t-s)A}dW_s + \int_0^t (-A)^{1/2}e^{(t-s)A}zds$, one can easily check that the Markov semigroup associated to $L^{(z)}$ is a generalised Ornstein-Uhlenbeck semigroup $(P_t^{(z)})$:
\[
P_t^{(z)}f(x) = \int_H f(e^{tA}x + y + \Gamma_tz) N(0, Q_t)(dy), \quad f \in B_b(H), \quad x \in H,
\]
setting $\Gamma_t = (-A)^{1/2}\int_0^t e^{sA}ds$, $\Gamma_tz = (-A)^{-1/2}[z - e^{tA}z] = \sum_{k \geq 1} \frac{(1 - e^{-t\lambda_k})z^{(k)}}{(\lambda_k)^{1/2}}e_k$.
The case \( z = 0 \), i.e., \((P_t^{(0)}) = (P_t)\) corresponds to the well-known Ornstein-Uhlenbeck semigroup (see, for instance, [15], [16], [7], [10] and [11]) which has a unique invariant measure \( \mu = N(0, S) \) where \( S = -\frac{1}{2} A^{-1} \). It is also well-known (see, for instance, [15] and [16]) that under Hypothesis 1, \((P_t)\) is strong Feller, i.e., \( P_t(B_b(H)) \subset C_b(H), \ t > 0 \). Indeed we have \( e^{tA}(H) \subset Q_t^{1/2}(H), \ t > 0 \), or, equivalently,

\[
\Lambda_t = Q_t^{-1/2} e^{tA} = \sqrt{2} (-A)^{1/2} e^{tA} (I - e^{tA})^{-1/2} \in \mathcal{L}(H), \ t > 0.
\]

Moreover \( P_t(B_b(H)) \subset C_b^k(H), \ t > 0 \), for any \( k \geq 1 \). Following the same proof of Theorem 6.2.2 in [15] one can show that under Hypothesis 1, for any \( z \in H \), we have \( P_t^{(z)}(B_b(H)) \subset C_b^k(H), \ t > 0 \), for any \( k \geq 1 \). Moreover, for any \( f \in B_b(H), \ t > 0 \), the following formula for the directional derivative along a direction \( h \) holds:

\[
D_h P_t^{(z)} f(x) = \langle D_h P_t^{(z)} f(x), h \rangle = \int_H \langle \Lambda_t h, Q_t^{-\frac{1}{2}} y \rangle f(e^{tA} x + y + \Gamma_t z) \mu_t(dy), \ x, h \in H,
\]

where \( \mu_t = N(0, Q_t) \) (cf. (1.15)) and the mapping: \( y \mapsto \langle \Lambda_t h, Q_t^{-\frac{1}{2}} y \rangle \) is a centered Gaussian random variable on \((H, B(H), \mu_t)\) with variance \( |\Lambda_t h|^2 \) (cf. Theorem 6.2.2 in [15]). We have

\[
\Lambda_t e_k = \sqrt{2} (\lambda_k)^{1/2} e^{-t\lambda_k} (1 - e^{-2t\lambda_k})^{-1/2} e_k, \quad \|\Lambda_t\| \leq C_1 t^{-\frac{1}{2}}, \quad t > 0, \quad C_1 = \sqrt{2} \sup_{u \geq 0} [u e^{-u^2} (1 - e^{-2u^2})^{-1/2}],
\]

We deduce that, for \( t > 0, g \in B_b(H), h, k \in H, \)

\[
\|D_h P_t^{(z)} g\|_0 \leq \frac{C_0}{\sqrt{t}} \|h\| g\|_0, \quad \|D_{hk} P_t^{(z)} g\|_0 \leq \frac{\sqrt{7} C_0^2}{t} \|g\|_0 |h| |k|,
\]

where \( D_h P_t^{(z)} g = \langle D_t P_t^{(z)} g(\cdot), h \rangle, D_{hk} P_t^{(z)} g = \langle D^2 P_t^{(z)} g(\cdot), h, k \rangle \).

To study equation (1.10) we will investigate regularity properties of the function

\[
\nu^{(z)}(x) = \int_0^\infty e^{-\lambda t} P_t^{(z)} f(x) dt, \quad x \in H, \ f \in B_b(H)
\]

(we drop the dependence of \( u^{(z)} \) on \( \lambda \)); see also the remark below.

**Remark 4.** Let us fix \( z \in H \). We point out that under Hypothesis 1 when \( f \in B_b(H) \) and \( x \in H \), the mapping: \( t \mapsto P_t^{(z)} f(x) \) is right-continuous and bounded on \((0, \infty)\) by the semigroup property and the strong Feller property. Hence, for any \( \lambda > 0, \ u^{(z)} : H \to \mathbb{R} \) given in (1.23) belongs to \( C_b(H) \).

Moreover, also the mapping: \( t \mapsto D_h P_t^{(z)} f(x) \) is right-continuous on \((0, \infty), \) for \( x, h \in H \). To check this fact let us fix \( t > 0 \). Writing \( D_h P_t^{(z)} f(x) = D_h P_{t+s}^{(z)} [P_{t/2}^{(z)} f](x), \ s \geq 0, \) and using the strong Feller property we get easily the assertion.

Since \( \sup_{x \in H} |D_t P_t^{(z)} f(x)|_H \leq \frac{C \|f\|_0}{\sqrt{t}}, \ t > 0, \) differentiating under the integral sign, one shows that there exists the directional derivative \( D_h u^{(z)}(x) \) at any point \( x \in H \) along any direction \( h \in H \). Moreover, it is not difficult to prove that there exists the first Fréchet derivative \( Du^{(z)}(x) \) at any \( x \in H \) and \( Du^{(z)} : H \to H \) is continuous and bounded (cf. the proof of Lemma 9 in [10]). Finally we have the formula

\[
D_h u^{(z)}(x) = \int_0^\infty e^{-\lambda t} D_h P_t^{(z)} f(x) dt, \quad x, h \in H
\]

and the straightforward estimate \( \|Du^{(z)}\|_0 \leq c(\lambda) \|f\|_0 \) with \( c(\lambda) \) independent of \( z \in H \). We will prove a better regularity result for \( Du^{(z)} \) in Section 3.
Remark 5. In the final part of the proof of Lemma 6 we will need to use that \( \Gamma_t(H) \subset Q_t^{1/2}(H), t > 0 \) (cf. (1.2)). Note that this is equivalent to saying that \( Q_t^{-1/2} \Gamma_t \in \mathcal{L}(H), t > 0 \), and we have \( Q_t^{-1/2} \Gamma_t = \sqrt{2}(I - e^{2tA})^{-1/2}(I - e^{tA}) = \sqrt{2}(I + e^{tA})^{-1/2}(I - e^{tA})^{1/2} \in \mathcal{L}(H) \).

2. Examples.

2.0.1. One-dimensional stochastic Burgers-type equations. We consider

\[
(2.1) \quad du(t, \xi) = \frac{\partial^2}{\partial \xi^2} u(t, \xi) dt + \frac{\partial}{\partial \xi} h(\xi, u(t, \xi)) dt + dW_t(\xi), \quad u(0, \xi) = u_0(\xi), \quad \xi \in (0, \pi),
\]

with Dirichlet boundary condition \( u(t, 0) = u(t, \pi) = 0, t > 0 \) (cf. [23] and [7] and see the references therein). Here \( u_0 \in H = L^2(0, \pi) \) and \( A = \frac{d^2}{dx^2} \) with Dirichlet boundary conditions, i.e. \( D(A) = H^2(0, \pi) \cap H_0^1(0, \pi) \). It is well-known that \( A \) verifies Hypothesis 1. The eigenfunctions are \( e_k(\xi) = \sqrt{2/\pi} \sin(k \xi), \xi \in \mathbb{R}, \ k \geq 1 \).

The corresponding eigenvalues are \(-\lambda_k, \lambda_k = k^2\). The cylindrical noise is \( W_t(\xi) = \sum_{k \geq 1} W_t^{(k)} e_k(\xi) \) (cf. [16]). Classical stochastic Burgers equations with \( h(\xi, u) = \frac{u^2}{2} \) are examples of locally monotone SPDEs and strong uniqueness holds (cf. [3]). In [23] strong uniqueness is proved assuming that \( h(\xi, \cdot) \) is locally Lipschitz with a linearly growing Lipschitz constant.

Here we assume that \( h : (0, \pi) \times \mathbb{R} \to \mathbb{R} \) is continuous and there exists \( C > 0 \) such that

\[ |h(\xi, s)| \leq C(1 + |s|), \]

\( s \in \mathbb{R}, \xi \in (0, \pi) \) (more generally, one could impose Carathéodory type conditions on \( f \)).

It is well-known that the Nemiskii operator: \( x \in H \to h(\cdot, x(\cdot)) \in H \) is continuous from \( H \) into \( H \). To write (2.1) in the form (1.1) we define \( F : H \to H \) as follows

\[ F(x)(\xi) = (-A)^{-1/2} \partial_\xi [h(\cdot, x(\cdot))] (\xi), \quad x \in L^2(0, \pi) = H. \]

To check that \( F \) verifies (1.2) it is enough to prove that \( T = (-A)^{-1/2} \partial_\xi \) can be extended to a bounded linear operator from \( L^2(0, \pi) \) into \( L^2(0, \pi) \). We briefly verify this fact. Recall that the domain \( D((-A)^{1/2}) \) coincides with the Sobolev space \( H^1(0, \pi) \). Take \( y \in H^1_0(0, \pi) \) and \( x \in L^2(0, \pi) \). Define \( x_N = \pi_N x \) (cf. (1.13)). Using that \((-A)^{1/2}\) is self-adjoint and integrating by parts we find (we use inner product in \( L^2(0, \pi) \) and the fact that \( y(0) = y(\pi) = 0 \))

\[ \langle (-A)^{-1/2} \partial_\xi y, x_N \rangle = \langle \partial_\xi y, (-A)^{-1/2} x_N \rangle = - \langle y, \partial_\xi (-A)^{-1/2} x_N \rangle. \]

Now \( \partial_\xi (-A)^{-1/2} x_N(\xi) = \sqrt{2/\pi} \sum_{k=1}^N x^{(k)} \cos(k \xi) \) and so \( |\partial_\xi (-A)^{-1/2} x_N|^2_{L^2(0, \pi)} = |x_N|^2_{L^2(0, \pi)} \).

It follows that, for any \( N \geq 1 \), \( |\langle (-A)^{-1/2} \partial_\xi y, x_N \rangle| \leq |y|_{L^2(0, \pi)} |x_N|_{L^2(0, \pi)} \) and we easily get the assertion. Hence \( F \) verifies (1.2) and SPDE (2.1) is well-posed in weak sense, for any initial condition \( u_0 \in L^2(0, \pi) \).

Note that instead of \( h(\xi, u) \) one could consider different non local non-linearities like, for instance, \( u g(|u|) \) assuming that \( g : \mathbb{R} \to \mathbb{R} \) is continuous and bounded.

2.0.2. Three-dimensional stochastic Cahn-Hilliard equations. The Cahn-Hilliard equation is a model to describe phase separation in a binary alloy and some other media, in the presence of thermal fluctuations; we refer to [31] for a survey on this model. The stochastic Cahn-Hilliard equation has been recently much investigated under monotonicity conditions on \( h \) which allow to prove pathwise uniqueness; in one dimension a typical example is \( h(s) = s^3 - s \) (see [17], [9], [31], [18] and the references therein).
We can treat such SPDE in one, two or three dimensions. Let us consider Neumann boundary conditions in a regular bounded open set $G \subset \mathbb{R}^3$. For the sake of simplicity we concentrate on the cube $G = (0, \pi)^3$. The equation has the form

\begin{equation}
\begin{cases}
\frac{d}{dt}u(t, \xi) = -\triangle^2 u(t, \xi) dt + \triangle \xi h(u(t, \xi)) dt + dW_t(\xi), & t > 0, \ u(0, \xi) = u_0(\xi) \text{ on } G, \\
\frac{\partial}{\partial n} u = \frac{\partial}{\partial n} (\triangle u) = 0 \text{ on } \partial G,
\end{cases}
\end{equation}

where $\triangle^2$ is the bilaplacian and $n$ is the outward unit normal vector on the boundary $\partial G$. Let us introduce the Sobolev spaces $H^3(G) = W^{3,2}(G)$ and the Hilbert space $H$,

$$H = \{ f \in L^2(G) : \int_G f(\xi) d\xi = 0 \}.$$

We assume $u_0 \in H$ and define $D(A) = \{ f \in H^4(G) \cap H : \frac{\partial}{\partial n} f = \frac{\partial}{\partial n} (\triangle f) = 0 \text{ on } \partial G \},$ $Af = -\triangle^2 f$, $f \in D(A)$. Using also the divergence theorem, we have $A : D(A) \to H$.

The square root has domain $D((-A)^{1/2}) = \{ f \in H^2(G) \cap H : \frac{\partial}{\partial n} f = 0 \text{ on } \partial G \}; (-A)^{1/2} f = \triangle \xi f$, $f \in D((-A)^{1/2})$. Note that $A$ is self-adjoint with compact resolvent and it is negative definite with $\omega = 1$ (cf. Hypothesis 1). The eigenfunctions are

$$e_k(\xi_1, \xi_2, \xi_3) = (\sqrt{2/\pi})^3 \cos(k_1 \xi_1) \cos(k_2 \xi_2) \cos(k_3 \xi_3), \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3;$$

$k = (k_1, k_2, k_3) \in \mathbb{N}^3$, $k \neq (0, 0, 0) = 0^*$. The corresponding eigenvalues are $-\lambda_k$, where $\lambda_k = (k_1^2 + k_2^2 + k_3^2)^2$. Since $\sum_{k \in \mathbb{N}^3, k \neq 0} \lambda_k^{-1} < +\infty$ we see that $A$ verifies Hypothesis 1.

The cylindrical Wiener process is $W_t(\xi) = \sum_{k \in \mathbb{N}^3, k \neq 0} W_t^{(k)} e_k(\xi)$. Note that $\triangle \xi h(u(t, \xi)) = \triangle \xi [h(u(t, \xi)) - \int_G h(u(t, \xi)) d\xi]^2$.

Assuming that $h : \mathbb{R} \to \mathbb{R}$ in (2.2) is continuous and verifies $|h(s)| \leq c(1 + |s|)$, $s \in \mathbb{R}$, we can define $F : H \to H$ as follows:

$$F(x)(\xi) = h(x(\xi)) - \int_G h(u(x, \xi)) d\xi, \quad x \in H, \ \xi \in G.$$

It is not difficult to prove that $F$ verifies (1.2). Thus SPDE (2.2) is well-posed in weak sense, for any initial condition $u_0 \in H$.

3. A new optimal regularity result. Let $f \in B_b(H)$ and fix $z \in H$. Here we are interested in the regularity property of the function $u^{(z)} : H \to \mathbb{R}$ given in (1.23). By Remark 4 we know that $u^{(z)} \in C^1_b(H)$ and we have a formula for the directional derivative:

\begin{equation}
D_h u^{(z)}(x) = \langle Du^{(z)}(x), h \rangle = \int_0^\infty e^{-\lambda t} D_h P^{(z)}_t f(x) dt, \quad x, h \in H, \ \lambda > 0.
\end{equation}

In the sequel, for any $s \geq 0$, we will consider the bounded linear operator $(-A)^{-1/2} e^{sA} : H \to H$ defined as

$$(-A)^{-1/2} e^{sA} h = \sum_{k \geq 1} \lambda_k^{-1/2} e^{-s \lambda_k} h^{(k)} e_k,$$

$h \in H$. The following lemma will be important.

**Lemma 6.** For $s \geq 0$, $f \in B_b(H)$, $\lambda > 0$, $x, h, z \in H$, we have

$$\left| \int_0^\infty e^{-\lambda t} D_h P^{(z)}_{t+s} f(x) dt \right| \leq \frac{\pi}{\sqrt{2}} \| f \|_0 \| (-A)^{-1/2} e^{sA} h \|_H.$$
Proof. Let us fix \( f \in B_0(H), \) \( s \geq 0, \lambda > 0, \) and \( x \in H. \) We proceed in two steps.  

**I Step. We consider \( f \in C_b(H). \)** We know that the functional \( R_{x,f,s,\lambda}^{(z)} : H \to \mathbb{R}, \)

\[
R_{x,f,s,\lambda}^{(z)}(h) = \int_0^\infty e^{-\lambda t} D_h P_{t+s}^{(z)}(f(x)dt = \int_0^\infty e^{-\lambda t} \langle DP_{t+s}^{(z)}(f(x), h) \rangle dt, \quad h \in H,
\]

is linear and bounded, thanks to the estimate \( \sup_{x \in H} |D_P^{(z)}(f(x))| \leq \frac{\|f\|}{\sqrt{r}}, \) \( r > 0. \) Recall the projections \( \pi_N : H \to \text{Span}\{e_1, \ldots, e_N\} \) (see (1.13)); \( \pi_N h = \sum_{k=1}^N h(k) e_k, \) \( N \geq 1. \) The assertion for \( f \in C_b(H) \) follows if we prove that, for any \( h \in H, \) \( N \geq 1, \)

\[
|P_{x,f,s,\lambda}^{(z)}(\pi_N h)| \leq C\|(-A)^{-1/2}e^{\lambda A}h\|_{H} \|f\|_0,
\]

where \( C \) is independent of \( N \) and \( h. \) We fix \( h \in H \) and define

\[
\pi_N h = h_N.
\]

We first assume that \( f \in C_b(H) \) depends only on a finite numbers of coordinates. Identifying \( H \) with \( l^2(\mathbb{N}) \) through the basis \((e_k), \) we have

\[
f(x) = \tilde{f}(x^{(1)}, \ldots, x^{(m)}), \quad x \in H,
\]

for some \( m \geq 1, \) and \( \tilde{f} : \mathbb{R}^m \to \mathbb{R} \) continuous and bounded.

Setting \( Q_t e_k = Q_t^k e_k, \) where \( Q_t^k = \int_0^t e^{-2\lambda k r} dr = \frac{1-e^{-2\lambda k t}}{2\lambda k} \) (cf. (1.15)) we consider

\[
p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad c_k(t) = \left(\frac{1-e^{-2\lambda k t}}{2\lambda k}\right)^{1/2} = \sqrt{Q_t^k}.
\]

Note that, for \( t > 0, \) the one dimensional Gaussian measure \( N(0, \int_0^t e^{-2\lambda k s} ds) \) has density \( p\left(\frac{x}{c_k(t)}\right)\frac{1}{c_k(t)} \) with respect to the Lebesgue measure on \( \mathbb{R}. \) Recall also that

\[
\Lambda_{t+s}^k = \sqrt{2} (\lambda k)^{1/2} e^{-t+s+1/2} (1 - e^{-2[t+s]\lambda k})^{-1/2},
\]

t > 0, \( k \geq 1 \) (see (1.19)). In the sequel we concentrate on the more difficult case \( N > m \) if \( N \leq m \) we can obtain (3.3) arguing similarly). By (1.20) we find, integrating over \( \mathbb{R}^N, \)

\[
R_{x,f,s,\lambda}^{(z)}(h_N) = \int_0^\infty e^{-\lambda t} \left( \int_H f(e(t+s)A_x + y + \Gamma_{t+s}) \left[ \sum_{k=1}^N \Lambda_{t+s}^k \frac{h(k) y_k}{c_k(t+s)} \right] N(0, Q_{t+s})dy \right) dt
\]

\[
= \int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^N} \tilde{f}(e^{-(t+s)\lambda_1} x^{(1)} + y_1 + \frac{1-e^{-(t+s)\lambda_1}}{\sqrt{\lambda_1}} z^{(1)}),
\]

\[
\ldots, e^{-(t+s)\lambda_m x^{(m)}} + y_m + \frac{1-e^{-(t+s)\lambda_m}}{\sqrt{\lambda_m}} z^{(m)}).
\]

Set \( \Gamma_k(r) = \left[\frac{1-e^{-r\lambda k}}{(\lambda k)^{1/2}}\right]^{1/2}, \) \( k \geq 1, \) \( r \geq 0. \) Since by the Fubini theorem

\[
0 = \int_{\mathbb{R}^{N-m}} \left[ \sum_{k=m+1}^N \Lambda_{t+s}^k \frac{h(k) y_k}{c_k(t+s)} \right] \prod_{k=m+1}^N p\left(\frac{y_k}{c_k(t+s)}\right) \frac{1}{c_k(t+s)} dy_{m+1} \ldots dy_N.
\]

\[
\cdot \int_{\mathbb{R}^m} \tilde{f}(e^{-(t+s)\lambda_1} x^{(1)} + y_1 + \Gamma_1 (t+s) z^{(1)}), \ldots, e^{-(t+s)\lambda_m x^{(m)}} + y_m + \Gamma_m (t+s) z^{(m)}).
\]

\[
\cdot \prod_{k=1}^m p\left(\frac{y_k}{c_k(t+s)}\right) \frac{1}{c_k(t+s)} dy_1 \ldots dy_m.
\]
we find
\[ R_{x,f,s,\lambda}^{(z)}(h_N) = \int_0^\infty e^{-t\lambda}D_{h_N}P_{t+s}f(x)dt \]
\[ = \int_0^\infty e^{-t\lambda}dt \int_{\mathbb{R}^m} \hat{f}(v_1, \ldots, v_m) \cdot \left[ \sum_{k=1}^m \Lambda_{t+s}^k \left( \frac{v_k - e^{-(t+s)\lambda_k x^{(k)}} - \Gamma_k(t + s)z^{(k)}}{c_k(t + s)} \right) \right] \cdot \prod_{k=1}^m p\left( \frac{v_k - e^{-(t+s)\lambda_k x^{(k)}} - \Gamma_k(t + s)z^{(k)}}{c_k(t + s)} \right) \frac{1}{c_k(t + s)} dv_1 \ldots dv_m. \]

In the sequel to simplify the notation we write
\[ \tilde{h} = (h^{(1)}, \ldots, h^{(m)}), \quad \tilde{x} = (x^{(1)}, \ldots, x^{(m)}), \]
\[ \tilde{v} = (v_1, \ldots, v_m), \quad \Lambda_{t+s} \tilde{h} = (\Lambda_{t+s}^1 h^{(1)}, \ldots, \Lambda_{t+s}^m h^{(m)}) \in \mathbb{R}^m. \]

By the Fubini theorem, we deduce
\[ R_{x,f,s,\lambda}^{(z)}(h_N) = \int_{\mathbb{R}^m} \hat{f}(\tilde{v})d\tilde{v} \int_0^\infty e^{-t\lambda} \left[ \sum_{k=1}^m \Lambda_{t+s}^k \left( \frac{v_k - e^{-(t+s)\lambda_k x^{(k)}} - \Gamma_k(t + s)z^{(k)}}{c_k(t + s)} \right) \right] \cdot \prod_{k=1}^m p\left( \frac{v_k - e^{-(t+s)\lambda_k x^{(k)}} - \Gamma_k(t + s)z^{(k)}}{c_k(t + s)} \right) \frac{1}{c_k(t + s)} dt. \]

Now, for any fixed \( \tilde{v} \in \mathbb{R}^m \), we have, recalling (3.4) and changing variable: \( u = \lambda_k t, \)
\[ \int_0^\infty e^{-t\lambda} \left[ \sum_{k=1}^m \Lambda_{t+s}^k \left( \frac{v_k - e^{-(t+s)\lambda_k x^{(k)}} - \Gamma_k(t + s)z^{(k)}}{c_k(t + s)} \right) \right] \cdot \prod_{k=1}^m p\left( \frac{v_k - e^{-(t+s)\lambda_k x^{(k)}} - \Gamma_k(t + s)z^{(k)}}{c_k(t + s)} \right) \frac{1}{c_k(t + s)} dt \]
\[ = \sqrt{2} \int_0^\infty \sum_{k=1}^m e^{-\lambda u/\lambda_k} \frac{e^{-u}}{(1 - e^{-2u}e^{-2s\lambda_k})^{1/2}} \cdot (\lambda_k)^{-1/2} e^{-s\lambda_k} \tilde{h}(k) \frac{v_k - e^{-(u/\lambda_k + s)\lambda_k x^{(k)}} - \Gamma_k(u/\lambda_k + s)z^{(k)}}{c_k(u/\lambda_k + s)} \cdot \prod_{k=1}^m p\left( \frac{v_k - e^{-(u/\lambda_k + s)\lambda_k x^{(k)}} - \Gamma_k(u/\lambda_k + s)z^{(k)}}{c_k(u/\lambda_k + s)} \right) \frac{1}{c_k(u/\lambda_k + s)} du. \]

By the Fubini theorem and using \( 1 - e^{-2u}e^{-2s\lambda_k} \geq 1 - e^{-2u}, u \geq 0, k \geq 1, \) we get
\[ |R_{x,f,s,\lambda}^{(z)}(h_N)| = \sqrt{2} \int_0^\infty du \int_{\mathbb{R}^m} \sum_{k=1}^m f(\tilde{v}) e^{-\lambda u/\lambda_k} \frac{e^{-u}}{(1 - e^{-2u}e^{-2s\lambda_k})^{1/2}} \cdot (\lambda_k)^{-1/2} e^{-s\lambda_k} \tilde{h}(k) \frac{v_k - e^{-(u/\lambda_k + s)\lambda_k x^{(k)}} - \Gamma_k(u/\lambda_k + s)z^{(k)}}{c_k(u/\lambda_k + s)} \cdot \prod_{k=1}^m p\left( \frac{v_k - e^{-(u/\lambda_k + s)\lambda_k x^{(k)}} - \Gamma_k(u/\lambda_k + s)z^{(k)}}{c_k(u/\lambda_k + s)} \right) \frac{1}{c_k(u/\lambda_k + s)} d\tilde{v}. \]

(3.5)
Let us fix $u \geq 0$; we have, changing variable in the integral over $\mathbb{R}^m$,

$$
\int_{\mathbb{R}^m} |f(\tilde{v})| \left| \sum_{k=1}^{m} (\lambda_k)^{-1/2} e^{-s\lambda_k} h^{(k)} \right| \left[ v_k - e^{-(u/\lambda_k + s)\lambda_k} x^{(k)} - \frac{\Gamma_k(u/\lambda_k + s) z^{(k)}}{c_k(u/\lambda_k + s)} \right].
$$

$$
\prod_{k=1}^{m} p\left( \frac{v_k - e^{-(u/\lambda_k + s)\lambda_k} x^{(k)} - \frac{\Gamma_k(u/\lambda_k + s) z^{(k)}}{c_k(u/\lambda_k + s)}}{c_k(u/\lambda_k + s)} \right) \frac{1}{c_k(u/\lambda_k + s)} d\tilde{v}
$$

$$
\leq \|f\|_0 \int_{\mathbb{R}^m} \left| \sum_{k=1}^{m} (\lambda_k)^{-1/2} e^{-s\lambda_k} h^{(k)} y_k \right| N(0, I_m)(d\tilde{y}),
$$

using the standard Gaussian measure $N(0, I_m)$ with density $\prod_{k=1}^{m} p(y_k)$. We find

$$
\int_{\mathbb{R}^m} \left| \sum_{k=1}^{m} (\lambda_k)^{-1/2} e^{-s\lambda_k} h^{(k)} y_k \right| N(0, I_m)(d\tilde{y})
$$

$$
\leq \left( \int_{\mathbb{R}^m} \sum_{k=1}^{m} (\lambda_k)^{-1} e^{-2s\lambda_k} [h^{(k)}]^2 y_k^2 N(0, I_m)(d\tilde{y}) \right)^{1/2} = \left( \sum_{k=1}^{m} (\lambda_k)^{-1} e^{-2s\lambda_k} [h^{(k)}]^2 \right)^{1/2}.
$$

By (3.5) it follows that

$$
|P_{x,f,s,\lambda}^{(z)}(h\lambda)| \leq \sqrt{2}\|f\|_0 \left( \sum_{k=1}^{m} (\lambda_k)^{-1} e^{-2s\lambda_k} [h^{(k)}]^2 \right)^{1/2} \int_{0}^{\infty} e^{-u(1 - e^{-2u})^{-1/2}} du
$$

$$
\leq \frac{\pi}{\sqrt{2}} \|f\|_0 \|(-A)^{-1/2} e^{sA} h\|_H
$$

and so (3.3) holds. Now we treat an arbitrary $f \in C_b(H)$. We introduce the cylindrical functions $f_n, h_n(x) = f(\pi_n x) = f\left( \sum_{k=1}^{n} x^{(k)} e_k \right)$, $n \in \mathbb{N}$, $x \in H$. It is clear that $\|f_n\|_0 \leq \|f\|_0, n \in \mathbb{N}$, and moreover $f_n(x) \to f(x)$, for any $x \in H$.

By the previous estimate with $f$ replaced by $f_n$, we get

(3.6) $$
\left| \int_{0}^{\infty} e^{-\lambda t} D_h P_{t+s}^{(z)} f_n(x) dt \right| \leq \frac{\pi}{\sqrt{2}} \|f\|_0 \|(-A)^{-1/2} e^{sA} h\|_H, \quad h \in H.
$$

Let $t > 0$ (recall that $s \geq 0$ is fixed). According to (1.20) we have

(3.7) $$
D_h P_{t+s}^{(z)} f_n(x) = \int_{H} \langle \Lambda_{t+s} h, Q_{t+s}^{-1/2} f_n(e^{(t+s)A} x + y + \Gamma_{t+s} z) \rangle N(0, Q_{t+s})(dy);
$$

we can easily pass to the limit as $n \to \infty$ in (3.7) by the Lebesgue convergence theorem and get $D_h P_{t+s}^{(z)} f_n(x) \to D_h P_{t+s}^{(z)} f(x)$. Similarly, using also the estimate $|D P_{t+s}^{(z)} f(x)|_H \leq \frac{\|f\|_0}{\sqrt{t}}$, $t > 0$, we can pass to the limit, as $n \to \infty$, in (3.6) and obtain (3.3) when $f \in C_b(H)$.

II Step. Let us consider $f \in B_0(H)$.

Here we use the invariant measure $\mu = N(0, -\frac{1}{2} A^{-1})$ for $(P_t)$ (i.e., $(P_t^{(z)})$ when $z = 0$). There exists a uniformly bounded sequence $(f_n) \subset C_b(H)$ such that $f_n(x) \to f(x)$, as $n \to \infty$, for any $x \in H$, $\mu$-a.s., and $\|f_n\|_0 \leq \|f\|_0$ (to this purpose it is enough to note that $P_{1/k} f \to f$ in $L^2(H, \mu)$ as $k \to \infty$).

It is well-known that for any $r > 0, x \in H, N(e^{rA} x, Q_r)$ is equivalent to $\mu$ (see [16] and [10]). This follows from the fact that $(P_t)$ is strong Feller and irreducible and so we can apply the Doob theorem (cf. Proposition 11.13 in [16]). We note that, for $r > 0, x \in H$,

$$
|P_r^{(z)} f_n(x) - P_r^{(z)} f(x)| \leq \int_{H} |f_n(y) - f(y)| N(e^{rA} x + \Gamma r z, Q_r)(dy).
$$
By Remark 5 we know $\Gamma_r z \in Q_r^{1/2}(H)$, where $Q_r^{1/2}(H)$ is the Cameron-Martin space of $N(e^{rA}x, Q_r)$. Applying the Feldman-Hajek theorem we find that $N(e^{rA}x + \Gamma_r z, Q_r)$ and $N(e^{rA}x, Q_r)$ are equivalent.

By the Lebesgue theorem we find, for $x \in H$, $r > 0$, $\lim_{n \to \infty} |P_r^{(z)} f_n(x) - P_r^{(z)} f(x)| = 0$. Now let us fix $t > 0$. Writing $D_h P_{t+s}^{(z)} f_n = D_h P_{t+s}^{(z)} f_{t/2}^{(z)} f_n$, we have

$$D_h P_{t+s}^{(z)} f_n(x) = \int_H (\Lambda_{t+s} h, Q^{-1/2}_{t+s} y) P_z^{(z)} f_n(e^{(t+s)/2} x + y + \Gamma_{t+s} z) N(0, Q_{t+s}^{1/2})(dy).$$

Since $P_{t/2} f_n(x) \to P_{t/2} f(x)$, for any $x \in H$, we find easily $D_h P_{t+s}^{(z)} f_n(x) \to D_h P_{t+s}^{(z)} f_{t/2}^{(z)} f(x) = D_h P_{t+s}^{(z)} f(x)$, as $n \to \infty$. Since, for any $n \geq 1$, $h \in H$,

$$(3.8) \quad \left| \int_0^\infty e^{-\lambda t} D_h P_{t+s}^{(z)} f_n(x) dt \right| \leq \frac{\pi}{\sqrt{2}} \|f\|_0 \|(-A)^{-1/2} e^{sA} h\|_H,$$

passing to the limit as $n \to \infty$ (using also (1.22)) we obtain the assertion.

By considering $s = 0$ in the previous lemma, we obtain

**Theorem 7.** Let $f \in B_0(H)$, $\lambda > 0$, $z \in H$ and consider $u^{(z)} \in C^1_b(H)$ given in (1.23). The following assertions hold.

(i) For any $x \in H$ we have $D u^{(z)}(x) \in D((-A)^{1/2})$, $(-A)^{1/2} D u^{(z)} : H \to H$ is Borel and bounded and

$$(3.9) \quad \sup_{x \in H} \left[ \sum_{k \geq 1} \lambda_k (D_{ek} u^{(z)}(x))^2 \right]^{1/2} \leq \|(-A)^{1/2} D u\|_0 \leq \frac{\pi}{\sqrt{2}} \|f\|_0.$$

(ii) Let $(f_n) \subset B_0(H)$ be such that $\sup_{n \geq 1} \|f_n\|_0 < C < \infty$ and $f_n(x) \to f(x)$, $x \in H$. Define

$$u_n^{(z)}(x) = \int_0^\infty e^{-\lambda t} P_t^{(z)} f_n(x) dt.$$

Then, for any $h \in H$,

$$(3.10) \quad \langle (-A)^{1/2} D u_n^{(z)}(x), h \rangle \to \langle (-A)^{1/2} D u^{(z)}(x), h \rangle, \quad \text{as } n \to \infty, \ x, z \in H.$$

**Proof.** (i) Set $B = (-A)^{1/2}$ with domain $D(B)$. Let us fix $x, z \in H$ and recall (3.1).

We know by Lemma 6 with $s = 0$ that $k_0 = D u^{(z)}(x)$ verifies $|\langle k_0, B h' \rangle| \leq \frac{\pi}{\sqrt{2}} \|f\|_0 \|h\|_H$, $h' \in D(B)$. This implies that $k_0 \in D(B^*)$ and $|B^* k_0|_H \leq \frac{\pi}{\sqrt{2}} \|f\|_0$. The assertion follows since $B$ is self-adjoint. Moreover, since $\sum_{k=1}^N \sqrt{\lambda_k} D_{ek} u^{(z)} e_k$ converges pointwise to $(-A)^{1/2} D u^{(z)}$ as $N \to \infty$ we get the desired Borel measurability.

(ii) To prove (3.10) we first note that, for $t > 0$, $D_h P_t^{(z)} f_n(x) \to D_h P_t^{(z)} f(x)$, as $n \to \infty$, $x, h \in H$ (see the argument after formula (3.7)). Moreover, $|D P_t^{(z)} f_n(x)|_H \leq \frac{\|f_n\|_0}{\sqrt{t}} \leq C \frac{\|f\|_0}{\sqrt{t}}$, $t > 0$. Hence we have, for any $k \in D((-A)^{1/2}), \ x \in H$,

$$\lim_{n \to \infty} \langle D u_n^{(z)}(x), (-A)^{1/2} k \rangle = \langle D u^{(z)}(x), (-A)^{1/2} k \rangle.$$

By the first assertion we deduce that $\lim_{n \to \infty} \langle (-A)^{1/2} D u_n^{(z)}(x), k \rangle = \langle (-A)^{1/2} D u^{(z)}(x), k \rangle$. It follows easily that (3.10) holds, for any $h \in H$. \qed
Remark 8. Note that if \( G \in B_b(H, H) \), then (3.10) implies that
\[
\lim_{n \to \infty} \langle (-A)^{1/2} D u_n^{(z)}(x), G(x) \rangle = \langle (-A)^{1/2} D u^{(z)}(x), G(x) \rangle, \quad x, z \in H.
\]
This fact will be useful in the sequel.

Recall that
\[
u^{(z)}(x) = \int_0^\infty e^{-\lambda t} P_t^{(z)} f(x) dt = R^{(z)}(\lambda) f(x)
\]
where the resolvent \( R^{(z)}(\lambda) : B_b(H) \to B_b(H) \) verifies the identity
\[
R^{(z)}(\mu) - R^{(z)}(\lambda) = (\lambda - \mu) R^{(z)}(\mu) R^{(z)}(\lambda), \quad \lambda, \mu > 0.
\]

By Lemma 6 we can also obtain the following new regularity result. It implies \( C^1 \)-Zygmund regularity for \( Du \); see Appendix (in finite dimension for Ornstein-Uhlenbeck semigroups such implication is proved in Lemma 3.6 and Proposition 3.7 with \( \theta = 1/2 \); see also Remark 29).

Theorem 9. Let \( C_2 = \sqrt{2} \pi \frac{(1 + \omega)^{1/2}}{\omega^{1/2}} + 4C_1 \) where \( \omega > 0 \) and \( C_1 \) are respectively defined in Hypothesis 1 and formula (1.21). For any \( s \in [0, 1] \), \( x, z \in H \), \( f \in B_b(H) \), \( \lambda > 0 \), we have:
\[
|P_s^{(z)} Du^{(z)}(x) - Du^{(z)}(x)|_H \leq C_2 s^{1/2} \|f\|_0.
\]

Proof. It is enough to prove the assertion when \( \lambda \in (0, 1] \). Indeed once we have proved
\[
|P_s^{(z)} Du^{(z)}(x) - Du^{(z)}(x)|_H \leq C s^{1/2} \|f\|_0, \quad \lambda \in (0, 1],
\]
we write, for \( \lambda > 1 \), using the previous resolvent identity
\[
u^{(z)}(x) = \int_0^\infty e^{-\lambda t} P_t^{(z)} f(x) dt + (1 - \lambda) \int_0^\infty e^{-t} P_t^{(z)} u^{(z)}(x) dt.
\]
It follows that
\[
|P_s^{(z)} Du^{(z)}(x) - Du^{(z)}(x)|_H \leq C s^{1/2} \|f\|_0 + \frac{\lambda - 1}{\lambda} \|f\|_0 \leq 2 C s^{1/2} \|f\|_0,
\]
now we fix \( x, z \in H \), \( s \in (0, 1] \), \( \lambda \in (0, 1] \). We have
\[
\int_0^\infty e^{-\lambda t} P_t^{(z)} f(x) dt - \int_0^s e^{-\lambda t} P_t^{(z)} f(x) dt = \int_s^\infty e^{-\lambda t} P_t^{(z)} f(x) dt.
\]
Note that
\[
\int_0^\infty e^{-\lambda t} D_h P_t^{(z)} f(x) dt = \int_0^s e^{-\lambda t} D_h P_t^{(z)} f(x) dt + e^{-\lambda s} \int_s^\infty e^{-\lambda t} D_h P_{t+s}^{(z)} f(x) dt.
\]
Hence
\[
P_s^{(z)} D_h u^{(z)}(x) - D_h u^{(z)}(x) = \int_0^s e^{-\lambda t} P_t^{(z)} (D_h P_t^{(z)} f)(x) dt - e^{-\lambda s} \int_s^\infty e^{-\lambda t} D_h P_{t+s}^{(z)} f(x) dt
\]
\[
- \int_0^s e^{-\lambda t} D_h P_t^{(z)} f(x) dt.
\]
Since
\[
\int_0^s e^{-\lambda u} D_h P_u^{(z)} f(x) dt \leq 2C_1 \sqrt{s} \|h\|_H \|f\|_0,
\]
we concentrate on \( T_{x,s,f,\lambda}^{(z)} : H \to \mathbb{R} \):
\[
T_{x,s,f,\lambda}^{(z)}(h) = \int_0^\infty e^{-\lambda t} \frac{P_s^{(z)} (D_h P_t^{(z)} f)(x)}{\lambda} dt, \quad h \in H.
\]
This linear functional is well-defined because
\[ |P_s^{(z)}(D_hP_t^{(z)}f)(x) - e^{-\lambda_s D_hP_t^{(z)}f}(x)| \leq \|D_hP_t^{(z)}f\|_0 + c(\lambda)\|D_hP_t^{(z)}f\|_0 \leq \frac{c}{\sqrt{t}}|h|\|f\|_0, \]
t > 0, h \in H (using (1.21)). Moreover, it is easy to check that it is linear and bounded (to this purpose, note that, for any t > 0, the mapping h \mapsto P_s^{(z)}(D_hP_t^{(z)}f)(x) - e^{-\lambda_s D_hP_t^{(z)}f}(x) is linear). We will prove that

\[ |T_{x,s,f,\lambda}^{(z)}(h)| \leq \frac{\pi}{\sqrt{2}} \frac{(1 + \omega)^{1/2}}{(\omega)^{1/2}} s^{1/2} |h|\|f\|_0. \]

To this purpose let us consider h = e_k. Since in particular, for t > 0, P_t^{(z)}f \in C^1_b(H) we can differentiate under the integral sign and obtain
\[ D_{e_k}P_t^{(z)}f(x) = D_{e_k}P_t^{(z)}(P_t^{(z)}f)(x) = \int_H (DP_t^{(z)}f)(e^{sA}x + y + \Gamma_s z)e^{sA}e_k\mu_s(dy) = e^{-\lambda_k s}P_s^{(z)}(D_{e_k}P_t^{(z)}f)(x), \quad t > 0. \]

Hence, for any k \geq 1, t > 0,
\[ P_s^{(z)}(D_{e_k}P_t^{(z)}f)(x) - e^{-\lambda_s D_{e_k}P_t^{(z)}f}(x) = [e^{\lambda_k s} - e^{-\lambda_s}]D_{e_k}P_t^{(z)}f(x). \]

For h \in H we define \( \pi_N h = h_N = \sum_{k=1}^N h^{(k)}e_k \) (cf. (1.13)). We have
\[ \int_0^\infty e^{-\lambda t}[P_s^{(z)}(D_hN_{1,t}^{(z)}f)(x) - e^{-\lambda s}D_hN_{1,t}^{(z)}f(x)]dt = \sum_{k=1}^N h_k[e^{\lambda_k s} - e^{-\lambda_k}] \int_0^\infty e^{-\lambda t}D_{e_k}P_t^{(z)}f(x)dt. \]

Let \( S_N(s, \lambda) : H \to H, S_N(s, \lambda)h = \sum_{k=1}^N h_k(e^{s\lambda_k} - e^{-\lambda_k})e_k, h \in H. \) We have
\[ T_{x,s,f,\lambda}^{(z)}(h_N) = \sum_{k=1}^N h_k[e^{\lambda_k s} - e^{-\lambda_k}] \int_0^\infty e^{-\lambda t}D_{e_k}P_t^{(z)}f(x)dt = \int_0^\infty e^{-\lambda t}D_{S_N(s, \lambda)h}P_t^{(z)}f(x)dt. \]

Using Lemma 6 we find |\( T_{x,s,f,\lambda}^{(z)}(h_N)|^2 \leq \frac{\pi^2}{2} \|f\|_0^2 \|(-A)^{-1/2}e^{\lambda A}S_N(s, \lambda)\|_2^2 |h_N|^2. \]

Since \( \|(-A)^{-1/2}e^{\lambda A}S_N(s, \lambda)\|_2 \leq \sup_{k \geq 1} \{(1 - e^{-[\lambda + \lambda_k]s})^{1/2}\lambda_k^{-1/2}\} \leq s^{1/2}\sup_{k \geq 1} \{(1 + \lambda_k)^{1/2}/(\lambda_k)^{1/2}\}, \)
we get |\( T_{x,s,f}^{(z)}(h_N)|^2 \leq \frac{\pi^2}{2} \frac{1+\omega}{\omega} \|f\|_0^2 |h_N|^2 s. \) As \( N \to \infty \) we get (3.14). \qed

4. Proof of weak existence of Theorem 1. \ We will prove weak existence by adapting a compactness approach of [21]. This approach uses the factorization method introduced in [12] (this approach is also explained in Chapter 8 of [16]).

Let us fix \( x \in H. \) To construct the solution we start with some approximating mild solutions. We introduce, for each \( m \geq 1, \)
\[ A_m = A \circ \pi_m, \quad A_m e_k = -\lambda_k e_k, \quad k = 1, \ldots, m, \]

\( A_m e_k = 0, k > m; \) here \( \pi_m = \sum_{j=1}^m e_j \otimes e_j \) is the basis of eigenvectors of \( A; \) see (1.13)).

For each \( m \) there exists a weak mild solution \( X_m = (X_m(t))_{t \geq 0} \) on some filtered probability space, possibly depending on \( m \) (such solution can also be constructed by the Girsanov theorem, see [20], [16] and [10]).
Usually the mild solutions $X^m$ are constructed on a time interval $[0, T]$. However there is a standard procedure based on the Kolmogorov extension theorem to define the solutions on $[0, \infty)$. On this respect, we refer to Remark 3.7, page 303, in [25].

We know that

\begin{equation}
 X_m(t) = e^{tA}x + \int_0^t e^{(t-s)A}(-A_m)^{1/2}F(X_m(s))ds + \int_0^t e^{(t-s)A}dW_s, \quad t \geq 0.
 \end{equation}

Recall that, for any $t \geq 0$, the stochastic convolution $W_A(t) = \int_0^t e^{(t-s)A}dW_s$ is a Gaussian random variable with law $N(0, Q_t)$. Let $p > 2$ and $q = \frac{p}{p-1} < 2$. We find (using also (1.12) and the Hölder inequality)

\[
|X_m(t)|_H^p \leq c_p(|e^{tA}x|_H^p + \int_0^t e^{(t-s)A}(-A_m)^{1/2}F(X_m(s))ds|_H^p + |W_A(t)|_H^p)
\leq c_T|x|_H^p + c_T(\int_0^t (t-s)^{-q/2}ds)^{p/q} \cdot \int_0^t (1 + |X_m(s)|_H^p)ds + c_p|W_A(t)|_H^p
\leq C_T|x|_H^p + C_T + C_T \int_0^t |X_m(s)|_H^p ds + C_T|W_A(t)|_H^p, \quad t \in [0, T].
\]

By the Gronwall lemma we find the bound

\begin{equation}
\sup_{m \geq 1} \sup_{t \in [0, T]} E|X_m(t)|_H^p = C_T < \infty.
\end{equation}

The mild solution $X$ will be a weak limit of solutions $(X_m)$. To this purpose we need some compactness results. The next result is proved in [21] (the proof uses that $(e^{tA})$ is a compact semigroup).

**Proposition 10.** If $0 < \frac{1}{p} < \alpha \leq 1$ then the operator $G_\alpha : L^p(0, T; H) \to C([0, T]; H)$

\[
G_\alpha f(t) = \frac{\sin \pi \alpha}{\pi} \int_0^t (t-s)^{\alpha-1}e^{(t-s)A}f(s)ds, \quad f \in L^p(0, T; H), \quad t \in [0, T], \quad \text{is compact}.
\]

Below we consider a variant of the previous result. In the proof we use estimate (1.12).

**Proposition 11.** Let $p > 2$. Then the operator $Q : L^p(0, T; H) \to C([0, T]; H)$,

\[
Qf(t) = \int_0^t (-A)^{1/2}e^{(t-s)A}f(s)ds, \quad f \in L^p(0, T; H), \quad t \in [0, T], \quad \text{is compact}.
\]

**Proof.** Since the proof is similar to the one of Proposition 10 we only give a sketch of the proof. Denote by $\| \cdot \|_p$ the norm in $L^p(0, T; H)$. According to the infinite dimensional version of the Ascoli-Arzelá theorem one has to show that

(i) For arbitrary $t \in [0, T]$ the sets $\{Qf(t) : |f|_p \leq 1\}$ are relatively compact in $H$.

(ii) For arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that

\begin{equation}
|Qf(t) - Qf(s)|_H \leq \varepsilon, \quad \text{if } |f|_p \leq 1, \quad |t-s| \leq \delta, \quad s, t \in [0, T].
\end{equation}

To check (i) let us fix $t \in (0, T]$ and define operators $Q^t$ and $Q^{\varepsilon, t}$ from $L^p(0, T; H)$ into $H$, for $\varepsilon \in (0, t)$,

\[
Q^t f = Qf(t), \quad Q^{\varepsilon, t} f = \int_0^{t-\varepsilon} (-A)^{1/2}e^{(t-s)A}f(s)ds, \quad f \in L^p(0, T; H).
\]
Since $Q^{\varepsilon,t}f = e^{\varepsilon A} \int_0^{t-\varepsilon} (-A)^{1/2} e^{(t-s)-\varepsilon A} f(s) ds$ and $e^{\varepsilon A}, \varepsilon > 0$, is compact, the operators $Q^{\varepsilon,t}$ are compact. Moreover, by using (1.12) and the Hölder inequality (setting $q = \frac{p}{p-1} < 2$)

$$|Q^t f - Q^{\varepsilon,t} f|_H = \left| \int_{t-\varepsilon}^{t} (-A)^{1/2} e^{(t-s)-\varepsilon A} f(s) ds \right|_H$$

$$\leq M \left( \int_{t-\varepsilon}^{t} (t-s)^{-q/2} ds \right)^{1/q} \left( \int_{t-\varepsilon}^{t} |f(s)|_H^p ds \right)^{1/p} \leq Mq \varepsilon^{1/2+1/q} |f|_p$$

with $-\frac{1}{2} + \frac{1}{q} > 0$. Hence $Q^{\varepsilon,t} \to Q^t$, as $\varepsilon \to 0^+$, in the operator norm so that $Q^t$ is compact and (i) follows. Let us consider (ii). For $0 \leq t \leq t + u \leq T$ and $|f|_p \leq 1$, we have

$$|Q f(t + u) - Q f(t)|_H \leq \int_0^{t+u} \|(-A)^{1/2} e^{(t+s-u)A} - (-A)^{1/2} e^{(t-s)A}\|_L |f(s)|_H ds$$

$$+ \int_t^{t+u} \left\|(-A)^{1/2} e^{(t+u-s)A} f(s)\right\|_H ds$$

$$\leq M_p \left( \int_0^u s^{-q/2} ds \right)^{1/q} + \left( \int_0^T \|(-A)^{1/2} e^{(u+s)A} - (-A)^{1/2} e^{sA}\|_L^q ds \right)^{1/q} = I_1 + I_2.$$ 

It is clear that $I_1 = M_p u^{1/2-1/p} \to 0$ as $u \to 0$. Moreover, for $s > 0$, $(-A)^{1/2} e^{sA}$ is compact (indeed $(-A)^{1/2} e^{sA} e_k = (\lambda_k)^{1/2} e^{-\alpha s \lambda_k} e_k$ and $(\lambda_k)^{1/2} e^{-\alpha s \lambda_k} \to 0$ as $k \to \infty$). It follows that $\|e^{uA}(-A)^{1/2} e^{sA} - (-A)^{1/2} e^{sA}\|_L \to 0$ as $u \to 0$ for arbitrary $s > 0$. Since $\|(-A)^{1/2} e^{(u+s)A} - (-A)^{1/2} e^{sA}\|_L \leq \frac{(2M)^q}{s^{q/2}}, s > 0, u \geq 0, and q < 2$, by the Lebesgue’s dominated convergence theorem $I_2 \to 0$ as $u \to 0$. Thus the proof of (ii) is complete. 

Proof of the existence part of Theorem 1. Let $x \in H$. We proceed in two steps.

I Step. Let $(X_m)$ be solutions of (4.1). We prove that their laws $\{\mathcal{L}(X_m)\}$ form a tight family of probability measures on $B(\mathcal{C}([0,\infty); H))$.

To this purpose it is enough to show that for each $T > 0$ the laws $\{\mathcal{L}(X_m)\}$ form a tight family of probability measures on $B(\mathcal{C}([0,T]; H))$.

Let us fix $p > 2, T > 0$ and choose $\alpha$ such that $1/p < \alpha < 1/2$. We know by (4.2) that there exists a constant $c_p > 0$ such that $E|X_m(t)|_H^p \leq c_p, m \geq 1, t \in [0,T]$. It follows that

$$\sup_{m \geq 1} E \int_0^T |F_m(X_m(t))|_H^p < \infty.$$ 

with $\pi_m \circ F = F_m$, since $|F_m(x)|_H \leq C_F (1 + |x|_H)$, $m \geq 1$. By the stochastic Fubini theorem we have the following factorization formula (cf. Theorem 5.10 in [16])

$$W_A(t) = \int_0^t e^{(t-s)A} dW_s = G \alpha Y_t = G \alpha (Y)(t), \ t \in [0,T],$$

where $Y_t = \int_0^t (t-r)^{-\alpha} e^{(t-r)A} dW_r, t \in [0,T]$. Therefore

$$X_m(t) = e^{tA} x + Q(F_m(X_m))(t) + G \alpha (Y)(t), \ t \in [0,T].$$

Note that $Y_t$ is a Gaussian random variable with values in $H$, having mean 0 and covariance operator $R_t = \int_0^t s^{2\alpha} e^{2sA} ds$. Therefore it is easy to prove that

$$\int_0^T |Y_s|_H^p ds \leq T \sup_{t \in [0,T]} E[|Y_t|_H^p] < \infty.$$
Now we show tightness of \( \mathcal{L}(X_m) \) on \( \mathcal{B}(C([0,T];H)) \). It follows from \((4.4), (4.6)\) and Chebyshev's inequality that for \( \varepsilon > 0 \) one can find \( r > 0 \) such that for all \( m \geq 1 \)

\[
(4.7) \quad \mathbb{P}\left( \left( \int_0^T |Y_s|^p ds \right)^{1/p} \leq r \text{ and } \left( \int_0^T |F_m(X_m(s))|^p_H ds \right)^{1/p} \leq r \right) > 1 - \varepsilon.
\]

By Propositions 10 and 11 (recall that \(| \cdot |_p\) denotes the norm in \( L^p(0,T;H) \)) the set

\[
K = \{ e^{(\cdot)}x + G \alpha f(\cdot) + Qg(\cdot) : |f|_p \leq r, |g|_p \leq r \} \subset C([0,T];H)
\]
is relatively compact. It follows from \((4.5)\) that \( \mathbb{P}(X_m \in K) = \mathcal{L}(X_m)(K) > 1 - \varepsilon, m \geq 1. \) and the tightness follows by the Prokhorov theorem.

**II Step.** By the Skorohod representation theorem, possibly passing to a subsequence of \((X_m)\) still denoted by \((X_m)\), there exists a probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\) and random variables \( \hat{X} \) and \( \hat{X}_m, m \geq 1 \), defined on \( \hat{\Omega} \) with values in \( C(0, \infty; H) \) such that the law of \( X_m \) coincide with the law of \( \hat{X}_m, m \geq 1 \), and moreover

\[
\hat{X}_m \to \hat{X}, \quad \hat{\mathbb{P}} - a.s.
\]

Let us fix \( k_0 \geq 1 \). Let \( \hat{X}_{m(k_0)} = \langle \hat{X}_m, e_{k_0} \rangle \). Recall that \( \pi_m \circ F = F_m \). It is not difficult to prove that the processes \( (M_{m(k_0)})_{m \geq 1} \)

\[
M_{m(k_0)}(t) = \begin{cases} 
\hat{X}_{m(k_0)}(t) - x(k_0) + \lambda_{k_0} \int_0^t \hat{X}_{m(k_0)}(s) ds - \lambda_{k_0}^{1/2} \int_0^t F(k_0)(\hat{X}_m(s)) ds, & k_0 \leq m \\
\hat{X}_{m(k_0)}(t) - x(k_0) + \lambda_{k_0} \int_0^t \hat{X}_{m(k_0)}(s) ds, & k_0 > m, \ t \geq 0,
\end{cases}
\]

are square-integrable continuous \( \mathcal{F}_t \hat{\mathcal{X}}_m \)-martingales on \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\) with \( M_{m(k_0)}(0) = 0 \). Moreover the quadratic variation process \( \langle M_{m(k_0)} \rangle_t = t, m \geq 1 \) (cf. Section 8.4 in [16]).

Passing to the limit as \( m \to \infty \) we find that

\[
(4.8) \quad M(k_0)(t) = \hat{X}(k_0)(t) - x(k_0) + \lambda_{k_0} \int_0^t \hat{X}(k_0)(s) ds - \lambda_{k_0}^{1/2} \int_0^t F(k_0)(\hat{X}(s)) ds, \ t \geq 0,
\]
is a square-integrable continuous \( \mathcal{F}_t \hat{\mathcal{X}} \)-martingale with \( M(k_0)(0) = 0 \). To check the martingale property, let us fix \( 0 < s < t \). We know that \( \hat{\mathbb{E}}[M_{m(k_0)}(t) - M_{m(k_0)}(s)/\mathcal{F}_s \hat{\mathcal{X}}_m] = 0, m \geq 1. \)

Consider \( 0 \leq s_1 < \ldots < s_n \leq s, \ n \geq 1. \) For any \( h_j \in C_b(H) \), we have, for \( m \geq k_0, \)

\[
(4.9) \quad \hat{\mathbb{E}} \left[ \left( \hat{X}_{m(k_0)}(t) - \hat{X}_{m(k_0)}(s) + \lambda_{k_0} \int_s^t \hat{X}_{m(k_0)}(r) dr - \lambda_{k_0}^{1/2} \int_s^t F(k_0)(\hat{X}_m(r)) dr \right) \prod_{j=1}^n h_j(\hat{X}_{m}(s_j)) \right] = 0.
\]

Using that \( |F(k_0)(x)| \leq CF(1 + |x|_H) \) and that, for any \( T > 0 \), \( \sup_{m \geq 1, 1 \leq t \leq T} \sup \hat{\mathbb{E}}[|\hat{X}_m(t)|^p] \leq C < \infty \) (cf. \((4.2)\)) by the Vitali convergence theorem we get easily that \((4.9)\) holds when \( \hat{X}_m \) is replaced by \( \hat{X} \) (note that this assertion could be proved by using only the dominated convergence theorem). Then we obtain that \( M(k_0) \) is a square-integrable continuous \( \mathcal{F}_t \hat{\mathcal{X}} \)-martingale.

Moreover, by a limiting procedure, arguing as before, we find that \( ((M(k_0)(t))^2 - t) \) is a martingale. It follows that \( M(k_0) \) is a real Wiener process on \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\).

Hence, for any \( k \geq 1 \), we find that there exists a real Wiener process \( M(k) \) such that

\[
\hat{X}(k)(t) = x(k) - \lambda_k \int_0^t \hat{X}(k)(s) ds + \lambda_k^{1/2} \int_0^t F(k)(\hat{X}(s)) ds + M(k)(t).
\]
We prove now that \((M^{(k)})_{k \geq 1}\) are independent Wiener processes.
We fix \(N \geq 2\) and introduce the processes \((S^N_m(t))_{m \geq 1}, S^N_m(t) = (M^{(1)}_m(t), \ldots, M^{(N)}_m(t))_{t \geq 0}\),
with values in \(\mathbb{R}^N\). The components of \(S^N_m\) are square-integrable continuous \(\mathcal{F}_t^m\)-martingales.
Moreover the quadratic covariation \((M^{(i)}_m, M^{(j)}_m)_t = \delta_{ij}t\).

Passing to the limit as before we obtain that also the \(\mathbb{R}^N\)-valued process \((S^N(t))\), \(S^N(t) = (M^{(1)}(t), \ldots, M^{(N)}(t))\), \(t \geq 0\), has components which are square-integrable continuous \(\mathcal{F}_t^m\)-martingales with quadratic covariation \((M^{(i)}, M^{(j)})_t = \delta_{ij}t\). Note that \(S^N(0) = 0\), \(\mathbb{P}\)-a.s.

By the Lévy characterization of the Brownian motion (see Theorem 3.16 in [25]) we have that \((M^{(1)}(t), \ldots, M^{(N)}(t))\) is a standard Wiener process with values in \(\mathbb{R}^N\). Since \(N\) is arbitrary, \((M^{(k)})_{k \geq 1}\) are independent real Wiener processes and the proof is complete.

**Remark 12.** Following the previous method one can prove existence of weak mild solution even for
\[
dX_t = AX_t dt + (-A)\gamma F(X_t) dt + dW_t, \quad X_0 = x \in H,
\]
with \(\gamma \in (0, 1)\) and \(F : H \to H\) continuous and having at most a linear growth.

5. **Proof of weak uniqueness when \(F \in C_b(H, H)\).** To get the weak uniqueness of Theorem 1 when \(F \in C_b(H, H)\) we first show the equivalence between martingale solutions and mild solutions. Indeed for martingale problems some useful uniqueness results are available even in infinite dimensions (see, in particular, Theorems 15, 16 and 17).

5.1. **Mild solutions and martingale problem.** We formulate the martingale problem of Stroock and Varadhan [34] for the operator \(\mathcal{L}\) given below in (5.1) and associated to (1.1). We stress that an infinite-dimensional generalization of the martingale problem is proposed in Chapter 4 of [19]. Here we follow Appendix of [32]. In such appendix some extensions and modifications of theorems given in Sections 4.5 and 4.6 of [19] are proved.

We use the space \(C^2(H)\) of regular cylindrical functions (cf. (1.18)). We deal with the following linear operator \(\mathcal{L} : D(\mathcal{L}) \subset C_b(H) \to C_b(H)\), with \(D(\mathcal{L}) = C^2(H)\) (recall that here \(F \in C_b(H, H)\)):

\[
\mathcal{L}f(x) = \frac{1}{2} Tr(D^2 f(x)) + \langle x, ADf(x) \rangle + \langle F(x), (-A)^{1/2} Df(x) \rangle = Lf(x) + \langle F(x), (-A)^{1/2} Df(x) \rangle, \quad f \in D(\mathcal{L}), \quad x \in H.
\]

**Remark 13.** We stress that the linear operator \((\mathcal{L}, D(\mathcal{L}))\) in (5.1) is countably pointwise determined, i.e., it verifies Hypothesis 17 in [32]. Indeed, arguing as in Remark 8 of [32], one shows that there exists a countable set \(\mathcal{H}_0 \subset D(\mathcal{L})\) such that for any \(f \in D(\mathcal{L})\), there exists a sequence \((f_n) \subset \mathcal{H}_0\) satisfying
\[
\lim_{n \to \infty} (\|f - f_n\|_0 + \|\mathcal{L}f_n - \mathcal{L}f\|_0) = 0. \quad \square
\]

Let \(x \in H\). An \(H\)-valued stochastic process \(X = (X_t) = (X_t)_{t \geq 0}\) defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with continuous trajectories is a **solution of the martingale problem** for \((\mathcal{L}, \delta_x)\) if, for any \(f \in D(\mathcal{L})\),
\[
M_t(f) = f(X_t) - \int_0^t \mathcal{L}f(X_s) ds, \quad t \geq 0, \quad \text{is a martingale}
\]
(with respect to the natural filtration \((\mathcal{F}_t^X)\)) and, moreover, \(X_0 = x, \mathbb{P}\)-a.s. If we do not assume that \(F\) is bounded then in general \(M_t(f)\) is only a local martingale because in general \(\mathcal{L}f\) is not a bounded function.
We say that the martingale problem for $\mathcal{L}$ is well-posed if, for any $x \in H$, there exists a martingale solution for $(\mathcal{L}, \delta_x)$ and, moreover, uniqueness in law holds for the martingale problem for $(\mathcal{L}, \delta_x)$.

Equivalence between mild solutions and martingale solutions has been proved in a general setting in [26] even for SPDEs in Banach spaces. We only give a sketch of the proof of the next result for the sake of completeness (see also Chapter 8 in [16]).

**Proposition 14.** Let $F \in C_0(H, H)$ and $x \in H$.

(i) If $X$ is a weak mild solution to (1.1) with $X_0 = x$, $\mathbb{P}$-a.s., then $X$ is also a solution of the martingale problem for $(\mathcal{L}, \delta_x)$.

(ii) Viceversa, if $X = (X_t)$ is a solution of the martingale problem for $(\mathcal{L}, \delta_x)$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ then there exists a cylindrical Wiener process on $(\Omega, \mathcal{F}, (\mathcal{F}_t^X), \mathbb{P})$ such that $X$ is a weak mild solution to (1.1) on $(\Omega, \mathcal{F}, (\mathcal{F}_t^X), \mathbb{P})$ with initial condition $x$.

**Proof.** (i) Let $X$ be a weak mild solution to (1.1) with $X_0 = x$, $\mathbb{P}$-a.s. defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Let $f \in D(\mathcal{L})$. Since $f$ depends only on a finite number of variables by the Itô formula we obtain that $f(X_t) - \int_0^t \mathcal{L}f(X_s)ds$ is an $\mathcal{F}_t$-martingale, for any $f \in D(\mathcal{L})$. We get easily the assertion since $\mathcal{F}_t^X \subset \mathcal{F}_t$, $t \geq 0$.

(ii) Let $X$ be a solution to the martingale problem for $(\mathcal{L}, \delta_x)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

I Step. Let $X_t^{(k)} = (X_t, e_k)$ and $F(x) = \sum_{k \geq 1} F^{(k)}(x)e_k$. We show that, for any $k \geq 1$,

$$X_t^{(k)} - x^{(k)} + \lambda_k \int_0^t X_s^{(k)}ds - \int_0^t (\lambda_k)^{1/2} F^{(k)}(X_s)ds$$

is a one-dimensional Wiener process $W^{(k)} = (W_t^{(k)})$.

Let $k \geq 1$. We will modify a well known argument (see, for instance, the proof of Proposition 5.3.1 in [19]). By the definition of martingale solution, it follows easily that if $f(x) = x^{(k)} = \langle x, e_k \rangle$, $x \in H$, the process

$$\mathbb{E}[f(X_t^{(k)})] = \mathbb{E}[f(X_t)]$$

which is $\mathcal{F}_t^X$-adapted, with $b_k(s) = (\lambda_k)^{1/2} F^{(k)}(X_s)$ (to this purpose one has to approximate the unbounded function $l_k(x) = \langle x, e_k \rangle$ by functions $l_k(x)\eta(\langle x, e_k \rangle)$, $n \geq 1$, where $\eta \in C_0^\infty(\mathbb{R})$ is such that $\eta(s) = 1$ for $|s| \leq 1$). Then using $f(x) = (\langle x, e_k \rangle)^2$, $x \in H$, we find that

$$N_t^k = \langle X_t^{(k)} \rangle^2 - \langle x^{(k)} \rangle^2 + 2\lambda_k \int_0^t (X_s^{(k)})^2 ds - 2 \int_0^t b_k(s) X_s^{(k)} ds - t,$$

is also a continuous local martingale. On the other hand, starting from (5.3) and applying the Itô formula (cf. Theorem 5.2.9 in [19]), we get

$$(X_t^{(k)})^2 = \langle x^{(k)} \rangle^2 - 2\lambda_k \int_0^t (X_s^{(k)})^2 ds + 2 \int_0^t b_k(s) X_s^{(k)} ds + 2 \int_0^t b_k(s) dM_s^{(k)} + \langle M^{(k)} \rangle_t,$$

where $(\langle M^{(k)} \rangle_t)$ is the variation process of $M^{(k)}$. Comparing this identity with (5.4) we deduce: $N_t^k - 2 \int_0^t b_k(s) dM_s^{(k)} = \langle M^{(k)} \rangle_t - t$ and so $\langle M^{(k)} \rangle_t = t$ (a continuous local martingale of bounded variation is constant). By the Lévy martingale characterization of the Wiener process (see Theorem 5.2.12 in [19]) we get that $M^{(k)}$ is a real Wiener process.

II Step. We prove that the previous Wiener processes $W^{(k)} = M^{(k)}$ are independent.

We fix any $N \geq 2$ and prove that $W^{(k)}$, $k = 1, \ldots, N$ are independent. We will argue similarly to the first step. By using functions $f(x) = x^j x^k$, $j, k \in \{1, \ldots N\}$, we get that $\langle W^{(j)} \rangle_t, W^{(k)} = \delta_{jk} t$. Again by the Lévy martingale characterization of the Wiener process (cf. Theorem 3.16 in [25]) we get that $(W^{(1)}, \ldots, W^{(N)})$ is an $N$-dimensional standard Wiener process. It follows that $\{W^{(k)}\}_{k=1,\ldots,N}$ are independent real Wiener processes.
For the martingale problem for $\mathcal{L}$ in (5.1) we have the following uniqueness result (we refer to Corollary 21 in [32]; see also Theorem 4.4.6 in [19] and Theorem 2.2 in [26]).

**Theorem 15.** Suppose the following two conditions:

(i) for any $x \in H$, there exists a martingale solution for $(\mathcal{L}, \delta_x)$;

(ii) for any $x \in H$, any two martingale solutions $X$ and $Y$ for $(\mathcal{L}, \delta_x)$ have the same one dimensional marginal laws (i.e., for $t \geq 0$, the law of $X_t$ is the same as $Y_t$ on $\mathcal{B}(H)$).

Then the martingale problem for $\mathcal{L}$ is well-posed.

Throughout Section 5 we will apply the previous result and also the next localization principle for $\mathcal{L}$ (cf. Theorem 26 in [32]).

**Theorem 16.** Suppose that for any $x \in H$ there exists a martingale solution for $(\mathcal{L}, \delta_x)$. Suppose that there exists a family $\{U_j\}_{j \in J}$ of open sets $U_j \subset H$ with $\bigcup_{j \in J} U_j = H$ and linear operators $L_j$ with the same domain of $\mathcal{L}$, i.e., $L_j : D(\mathcal{L}) \subset C_b(H) \rightarrow C_b(H)$, $j \in J$ such that

(i) for any $j \in J$, the martingale problem for $L_j$ is well-posed.

(ii) for any $j \in J$, $f \in D(\mathcal{L})$, we have $L_j f(x) = L f(x)$, $x \in U_j$.

Then the martingale problem for $\mathcal{L}$ is well-posed.

In Sections 6 and 7 we treat possibly unbounded $F$; we will prove uniqueness by truncating $F$ and using uniqueness for the martingale problem up to a stopping time. According to Section 4.6 of [19] this leads to the concept of stopped martingale problem for $\mathcal{L}$ which we define now.

Let us fix an open set $U \subset H$ and consider the Kolmogorov operator $\mathcal{L}$ in (5.1) with $F \in C_b(H, H)$.

Let $x \in H$. A stochastic process $Y = (Y_t)_{t \geq 0}$ with values in $H$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with continuous paths is a solution of the stopped martingale problem for $(\mathcal{L}, \delta_x)$ if $Y_0 = x$, $\mathbb{P}$-a.s. and the following conditions hold:

(i) $Y_t = Y_{t \wedge \tau}$, $t \geq 0$, $\mathbb{P}$-a.s, where

(5.5) \[
\tau = \tau^U_Y = \inf\{t \geq 0 : Y_t \notin U\}
\]

($\tau = +\infty$ if the set is empty; this exit time $\tau$ is an $\mathcal{F}_t^Y$-stopping time);

(ii) for any $f \in D(\mathcal{L}) = C^2_{cil}(H)$, $f(Y_t) - \int_0^{t \wedge \tau} \mathcal{L} f(Y_s) ds$, $t \geq 0$, is a $\mathcal{F}_t^Y$-martingale.

A key result says, roughly speaking, that if the (global) martingale problem for an operator is well-posed then also the stopped martingale problem for such operator is well-posed for any choice of the open set $U$ and for any initial condition $x$ (we refer to Theorem 22 in [32]; see also the beginning of Section A.3 for a comparison between this result and Theorem 4.6.1 in [19]). We state this result for the operator $\mathcal{L}$ in (5.1).

**Theorem 17.** Suppose that the martingale problem for $\mathcal{L}$ is well-posed.

Then also the stopped martingale problem for $(\mathcal{L}, \delta_x, U)$ is well-posed for any $x \in H$ and for any open set $U$ of $H$. In particular uniqueness in law holds for the stopped martingale problem for $(\mathcal{L}, \delta_x, U)$, for any $x \in H$ and $U$ open set in $H$.

In order to apply Theorems 15 and 16 we need existence of regular solutions for related Kolmogorov equations and some convergence results. This will be done in the next section.
5.2. On the Kolmogorov equation for $\mathcal{L}$ when $\|F - z\|_0 < 1/4$. Here we study the Kolmogorov equation

\begin{equation}
\lambda u - Lu - \langle (-A)^{1/2} Du, F \rangle = f, \tag{5.6}
\end{equation}

where $\lambda > 0$, $f \in C^2_b(H)$ and $F \in C_b(H, H)$ (cf. (5.1); $L$ is the Ornstein-Uhlenbeck operator).

We assume that there exists $z \in H$ such that

\begin{equation}
\sup_{x \in H} |F(x) - z|_H < 1/4. \tag{5.7}
\end{equation}

We will prove regularity and convergence results for solutions. Note that to study (5.6) we cannot proceed as in Proposition 5 of [10] because in general $\|(-A)^{1/2} Du\| \not\to 0$ as $\lambda \to \infty$.

We will rewrite the equation as

\begin{equation}
\lambda u(x) - Lu(x) - \langle (-A)^{1/2} Du(x), z \rangle = f(x) + \langle (-A)^{1/2} Du(x), F(x) - z \rangle. \tag{5.8}
\end{equation}

Let us introduce the Banach space $E = \{ v \in C^0_b(H), \ Dv(x) \in D((-A)^{1/2}), \ x \in H, \ (-A)^{1/2} Dv \in B_0(H, H) \}$ endowed with the norm

\begin{equation*}
\|v\|_E = \|v\|_0 + \|(-A)^{1/2} Dv\|_0, \ v \in E.
\end{equation*}

We first prove

**Lemma 18.** For any $z \in H$, $F \in C_b(H, H)$ which verify $\|F - z\|_0 < 1/4$, for any $\lambda \geq 1$, $g \in B_0(H)$, there exists a unique solution $u = u(z)$ in $E$ to the integral equation

\begin{equation*}
u^{(z)}(x) = \int_0^\infty e^{-\lambda t} P_t^{(z)}[g + \langle [F - z], (-A)^{1/2} Du^{(z)} \rangle](x) \, dt, \ x \in H
\end{equation*}

(we drop the dependence of $u^{(z)}$ from $\lambda$). Moreover

\begin{equation*}
\|(-A)^{1/2} Du^{(z)}\|_0 \leq 12\|g\|_0, \ \|u^{(z)}\|_0 \leq 4\|g\|_0.
\end{equation*}

**Proof.** We define $T : E \to E$, $Tu(x) = \int_0^\infty e^{-\lambda t} P_t^{(z)}[g + \langle [F - z], (-A)^{1/2} Du \rangle](x) \, dt$, $u \in E$, $x \in H$. Note that if $u \in E$ then $g + \langle [F - z], (-A)^{1/2} Du \rangle \in B_0(H)$ and so by Theorem 7, $Tu \in E$. We prove that $T$ is a strict contraction. Since

\begin{equation*}
Tu(x) - Tv(x) = \int_0^\infty e^{-\lambda t} P_t^{(z)}(\langle [F - z], (-A)^{1/2} (Du - Dv) \rangle)(x) \, dt,
\end{equation*}

we find $\|Tu - Tv\|_0 \leq \frac{1}{4\lambda}\|u - v\|_E \leq \frac{1}{4}\|u - v\|_E$ and by Theorem 7, since $\frac{\pi}{\sqrt{2}} < 3$,

\begin{equation*}
\|(-A)^{1/2} D[Tu] - (-A)^{1/2} D[Tv]\|_0 \leq \frac{3}{4}\|u - v\|_E.
\end{equation*}

Hence we have a unique fixed point $u^{(z)} \in E$ which solves the integral equation. Moreover

\begin{equation*}
\|(-A)^{1/2} Du^{(z)}\|_0 \leq 3(\|g\|_0 + \frac{1}{4}\|(-A)^{1/2} Du\|_0)
\end{equation*}

(see Theorem 7) and the assertion follows. \qed
Let $F \in C_b(H,H)$ which verifies (5.7). Set, for any $n \geq 1$, 

$$F_n(x) = \int_H F(e^{tA}x + y)N(0,Q_{1/n})(dy), \quad x \in H.$$  

Then $F_n$ is of $C^\infty$ class and all its derivatives are bounded. Moreover $\|F_n\|_0 \leq \|F\|_0$, $n \geq 1$. It is not difficult to prove that 

$$F_n(x) \to F(x), \quad x \in H,$$

as $n \to \infty$, and $\|F_n - z\|_0 < 1/4$, for any $n \geq 1$.

Recalling that in (5.6) $f \in C^2_b(H)$ we consider classical bounded solutions to the following finite-dimensional equations 

$$\lambda u_{nm} - Lu_{nm} - \langle (-A)^{1/2}\pi_mF_n \circ \pi_m, Du_{nm} \rangle = f \circ \pi_m, \quad n,m \geq 1, \quad \lambda \geq 1,$$

where $\pi_m = \sum_{j=1}^m e_j \otimes e_j$. We write $f_m = f \circ \pi_m$, $z_m = \pi_m z$ and $F_{nm} = \pi_m F_n \circ \pi_m$, $A_m = A_\pi m$. Note that 

$$\|F_{nm} - z_m\|_0 < 1/4, \quad n,m \geq 1.$$ 

We have $u_{nm} = u_{nm} \circ \pi_m$, $L u_{nm} = L_m u_{nm}$ with $L_m u_{nm} = \frac{1}{2} \text{Tr} [D^2 u_{nm}(x)] + \langle A_m, x, Du_{nm}(x) \rangle$, $x \in H$. Indeed, for $\lambda \geq 1$, $n,m \geq 1$, equation (5.11) can be solved by considering the associated equation in $\mathbb{R}^m$ which is like 

$$\lambda v(y) - \frac{1}{2} \Delta v(y) - \langle By, Dv(y) \rangle - \langle G(y), Dv(y) \rangle = g(y), \quad y \in \mathbb{R}^m,$$

where $B$ is a given $m \times m$ real matrix and $g,G$ are regular and bounded functions (to this purpose one can use, for instance, the Schauder estimates proved in [13]).

Thus, for any $m,n \geq 1$, there exist classical cylindrical functions $u_{nm} \in C^2_b(H)$ which solve (5.11). Such functions are the unique bounded classical solutions; however in order to prove uniqueness for SPDE (1.1) it is important to show existence of classical solutions. We can rewrite (5.11) as 

$$\lambda u_{nm}(x) - Lu_{nm}(x) - \langle (-A)^{1/2}z_m, Du_{nm}(x) \rangle$$

$$= f \circ \pi_m(x) + \langle (-A)^{1/2}[\pi_m F_n \circ \pi_m - z_m], Du_{nm}(x) \rangle, \quad x \in H,$$

and so we obtain the following finite-dimensional representation formula:

$$u_{nm}(x) = \int_0^\infty e^{-\lambda t} P_t^{(z_m)}[f_m + \langle [\pi_m F_n \circ \pi_m - z_m], (-A)^{1/2}Du_{nm} \rangle](x)dt, \quad x \in H.$$ 

Note that, since $f_m = f \circ \pi_m$, 

$$P_t^{(z_m)} f_m(x) = P_t^{(z_m)} f_m(x) = \int_H f(e^{tA}x + \pi_m y + (-A_m)^{-1/2}[z_m - e^{tA_m}z_m]) N(0,Q_t)(dy).$$ 

By Lemma 18 we have the bounds 

$$\|(-A)^{1/2}Du_{nm}\|_0 \leq 12 \|f\|_0, \quad \|u_{nm}\|_0 \leq 4 \|f\|_0.$$ 

Now let us introduce, for $x \in H$, $m \geq 1$, $\lambda \geq 1$, the solution $u_m = u_m^{(z)} \in E$ to 

$$u_m(x) = \int_0^\infty e^{-\lambda t} P_t^{(z_m)}[f_m + \langle [\pi_m F \circ \pi_m - z_m], (-A)^{1/2}Du_m \rangle](x)dt, \quad x \in H.$$ 

By Lemma 18, we know that 

$$\|(-A)^{1/2}Du_m\|_0 \leq 12 \|f\|_0.$$
Lemma 19. Let \( \lambda \geq 1, z \in H, f \in C^2_b(H) \) and consider classical bounded solutions \( u_{nm} \) of equation (5.11) when \( F \in C_b(H, H) \) verifies (5.7). We have, for any \( x \in H, m \geq 1 \),

\[
\lim_{n \to \infty} u_{nm}(x) = u_m(x) \quad \text{and} \quad \sup_{n,m \geq 1} \| u_{nm} \|_0 = C < \infty,
\]

Proof. We only need to prove the first assertion. Let us fix \( m \geq 1 \).

\[
u_{nm}(x) - u_m(x) = \int_0^\infty e^{-\lambda t} P_t^{(z)} \left( \left[ \pi_m F \circ \pi_m - \pi_m F \circ \pi_m \right], (-A)^{1/2} Du_{nm} \right) + \left( [\pi_m F \circ \pi_m - z_m], (-A)^{1/2} Du_{nm} - (-A)^{1/2} Du_m \right) \)(x)dt.
\]

Using the uniform bound on \( \|(-A)^{1/2} Du_{nm}\|_0 \) and the fact that \( (\pi_m F \circ \pi_m - \pi_m F \circ \pi_m) \) is uniformly bounded and converges pointwise to zero as \( n \to \infty \) we obtain

\[
\left| \int_0^\infty e^{-\lambda t} P_t^{(z)} \left( \left[ \pi_m F \circ \pi_m - \pi_m F \circ \pi_m \right], (-A)^{1/2} Du_{nm} - (-A)^{1/2} Du_m \right) \)(x)dt \right| \to 0, \quad \text{as} \quad n \to \infty.
\]

It remains to prove that

\[
\lim_{n \to \infty} \left| \int_0^\infty e^{-\lambda t} P_t^{(z)} \left( \left[ \pi_m F \circ \pi_m - z_m \right], (-A)^{1/2} Du_{nm} - (-A)^{1/2} Du_m \right) \)(x)dt \right| = 0.
\]

Using the bound (5.17) and Theorem 7 (see also (3.11)) we can apply the dominated convergence theorem and obtain the assertion.

Lemma 20. Let \( \lambda \geq 1, z \in H, f \in C^2_b(H) \) and consider \( u_m \) given in (5.16) with \( F \) verifying (5.7). We have, for any \( x \in H \),

\[
\lim_{m \to \infty} u_m(x) = u(x) \quad \text{and} \quad \sup_{m \geq 1} \| u_m \|_0 < \infty,
\]

Proof. The second bound is clear by Lemma 18. We prove the first assertion.

\[
u_m(x) - u(x) = \int_0^\infty e^{-\lambda t} P_t^{(z)} \left( f_m - f + \left( \pi_m F \circ \pi_m - z_m \right) - (F - z), (-A)^{1/2} Du_m \right) \)(x)dt + \int_0^\infty e^{-\lambda t} P_t^{(z)} \left( (F - z), (-A)^{1/2} Du_m - (-A)^{1/2} Du \right) \)(x)dt.
\]

Using the uniform bound on \( \|(-A)^{1/2} Du_m\|_0 \), the fact that \( ([\pi_m F \circ \pi_m - z_m] - [F - z]) \) and \( (f_m - f) \) are uniformly bounded and both converge pointwise to zero as \( m \to \infty \) we obtain

\[
\int_0^\infty e^{-\lambda t} P_t^{(z)} \left( f_m - f + \left( \pi_m F \circ \pi_m - z_m \right) - (F - z), (-A)^{1/2} Du_m \right) \)(x)dt \to 0,
\]

\[x \in H, \text{as} \ m \to \infty. \]

It remains to prove that

\[
\lim_{m \to \infty} \int_0^\infty e^{-\lambda t} P_t^{(z)} \left( (F - z), (-A)^{1/2} Du_m - (-A)^{1/2} Du \right) \)(x)dt = 0, \quad x \in H.
\]

Using the bounds (5.17) and Theorem 7 (see also (3.11)) we can apply the dominated convergence theorem and obtain the assertion.
5.3. Weak uniqueness when \( \|F - z\|_0 < 1/4 \). Here we will apply the regularity results of the previous section to obtain

**Lemma 21.** Let \( x \in H \) and consider the SPDE (1.1). If there exists \( z \in H \) such that (5.7) holds then we have uniqueness in law for (1.1).

**Proof.** By Section 4, for any \( x \in H \), there exists a weak mild solution starting at \( x \in H \). Equivalently, by Proposition 14, for any \( x \in H \), there exists a solution to the martingale problem for \( \mathcal{L}_x \).

We will prove that given two weak mild solutions \( X \) and \( Y \) which both solve (1.1) and start at \( x \) we have that the law of \( X_t \) coincides with the law of \( Y_t \) on \( \mathcal{B}(H) \), for any \( t \geq 0 \). By Theorem 15 we will deduce that \( X \) a \( Y \) have the same law on \( \mathcal{B}(C([0, \infty); H)) \).

Let us fix \( x \in H \) and let \( X = (X_t) \) be a weak mild solution starting at \( x \in H \). We proceed in two steps. First we prove useful formulas for finite-dimensional approximations of \( X_t \) and then we pass to the limit obtaining a basic identity for \( X \).

**Step 1. Some useful formulas for finite-dimensional approximations of \( X_t \).**

For any \( m \in \mathbb{N} \) we set \( X_{t,m} := \pi_m X_t \), where \( \pi_m = \sum_{j=1}^m e_j \otimes e_j \) (cf. formula (1.13)). We have

\[
X_{t,m} = e^{t\mathcal{A}_{\pi_m}} x + \int_0^t e^{(t-s)\mathcal{A}} (-A_m)^{1/2} F(X(s))ds + \int_0^t \pi_m e^{(t-s)\mathcal{A}} dW_s, \quad t \geq 0,
\]

where \( A_m = A\pi_m \). Writing \( \pi_m W_t = \sum_{k=1}^m W_t^{(k)} e_k \) it follows that

\[
X_{t,m} = \pi_m x + \int_0^t A_m X_s ds + \int_0^t (-A_m)^{1/2} F(X_s)ds + \pi_m W_t.
\]

Let \( f \in C_0^2(H) \). As in (5.11) and (5.14) we denote by \( u_{nm} \) the classical solution of the equation

\[
\lambda u_{nm} - Lu_{nm} - \langle (-A)^{1/2} \pi_m F_n \circ \pi_m, Du_{nm} \rangle = f \circ \pi_m, \quad \lambda \geq 1.
\]

Applying a finite-dimensional Itô’s formula to \( u_{nm}(X_{t,m}) = u_{nm}(X_t) \) yields

\[
\begin{align*}
du_{nm}(X_{t,m}) &= \frac{1}{2} \text{Tr} \left[ D^2 u_{nm}(X_{t,m}) \right] dt \\
&+ \langle Du_{nm}(X_{t,m}), A_m X_t + (-A_m)^{1/2} F(X_t) \rangle dt + \langle Du_{nm}(X_{t,m}), \pi_m dW_t \rangle.
\end{align*}
\]

On the other hand, by (5.21) we have

\[
\begin{align*}
\lambda u_{nm}(X_{t,m}) - \frac{1}{2} \text{Tr} \left[ D^2 u_{nm}(X_{t,m}) \right] \\
&- \langle Du_{nm}(X_{t,m}), A_m X_{t,m} + (-A_m)^{1/2} F_n(X_{t,m}) \rangle = f(X_{t,m}).
\end{align*}
\]

Taking into account (5.22) and the fact that \( u_{nm}(\pi_m y) = u_{nm}(y), y \in H, \ n, m \geq 1 \), yields

\[
\begin{align*}
u_{nm}(X_t) - u_{nm}(x) &= \lambda \int_0^t u_{nm}(X_s)ds - \int_0^t f(X_{s,m})ds \\
&+ \int_0^t \langle (-A)^{1/2} Du_{nm}(X_s), (F(X_s) - F_n(X_{s,m})) \rangle ds + \int_0^t \langle Du_{nm}(X_s), \pi_m dW_s \rangle,
\end{align*}
\]

where \( \lambda \) is such that \( \lambda \geq 1 \).

Thus we have that for any \( \lambda \geq 1 \)

\[
\lambda u_{nm}(X_{t,m}) - \frac{1}{2} \text{Tr} \left[ D^2 u_{nm}(X_{t,m}) \right] \\
- \langle Du_{nm}(X_{t,m}), A_m X_{t,m} + (-A_m)^{1/2} F_n(X_{t,m}) \rangle = f(X_{t,m}),
\]

and for any \( \lambda \geq 1 \)

\[
\lambda u_{nm}(X_{t,m}) - \frac{1}{2} \text{Tr} \left[ D^2 u_{nm}(X_{t,m}) \right] \\
- \langle Du_{nm}(X_{t,m}), A_m X_{t,m} + (-A_m)^{1/2} F_n(X_{t,m}) \rangle = f(X_{t,m}).
\]

Therefore, the desired result holds.
t ≥ 0. By (5.15) we deduce that \( \langle f_t(Du_{nm}(X_s), \pi_m dW_s) \rangle_{t≥0} \) is a martingale. Hence

\[
E[u_{nm}(X_t) - u_{nm}(x)] = \lambda \int_0^t E[u_{nm}(X_s)]ds - \int_0^t E[f(X_{s,m})]ds + \int_0^t E[(-A)^{1/2}Du_{nm}(X_s), (F(X_s) - F_n(X_{s,m}))]ds.
\]

(5.23)

Step 2. Passing to the limit in (5.23) as \( n, m \to \infty \).

We apply the convergence results of Lemmas 19 and 20. To this purpose note the pointwise convergence

\[ \pi_m F_n \circ \pi_m \to F \]

first as \( n \to \infty \) and then as \( m \to \infty \) (according to the convergence used in the previous section). Moreover \( \sup_{n,m \geq 1} \| \pi_m F_n \circ \pi_m \|_0 \leq \| F \|_0 \) and \( u_m(\pi_m y) = u_m(y), y \in H, m \geq 1 \).

Let us fix \( m \geq 1 \). First we can pass to the limit as \( n \to \infty \) in (5.23) by the Lebesgue convergence theorem and get

\[
E[u_{nm}(X_t)] - u_m(x) = \lambda \int_0^t E[u_m(X_s)]ds - \int_0^t E[f(X_{s,m})]ds + \int_0^t E[(-A)^{1/2}Du_m(X_s), (F(X_s) - F \circ \pi_m(X_s))]ds.
\]

Then, using also Lemma 20, we pass to the limit as \( m \to \infty \) and arrive at

\[ E[u(X_t)] - u(x) = \lambda \int_0^t E[u(X_s)]ds - \int_0^t E[f(X_s)]ds. \]

Integrating both sides over \([0, \infty)\) with respect to \( e^\lambda t dt \) and using the Fubini theorem we arrive at the basic identity

\[ u(x) = \int_0^\infty e^{-\lambda s} E[f(X_s)]ds, \lambda \geq 1. \]

Now if \( Y \) is another weak mild solution starting at \( x \) and defined on \((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P})}\). We obtain, for any \( f \in C^2_b(H) \),

\[
\int_0^\infty e^{-\lambda s} E[f(X_s)]ds = \int_0^\infty e^{-\lambda s} E[f(Y_s)]ds, \lambda \geq 1.
\]

By the uniqueness of the Laplace transform and using an approximation argument we find that \( E[g(X_s)] = \tilde{E}[g(Y_s)] \), for any \( g \in C_b(H), s \geq 0 \). Applying Proposition 14 and Theorem 15 we find that \( X \) and \( Y \) have the same law on \( \mathcal{B}(C([0, \infty); H)) \).

5.4. Weak uniqueness when \( F \in C_b(H, H) \). Here we prove uniqueness using the localization principle (cf. Theorem 16 and Lemma 21).

LEMMA 22. Let \( x \in H \) and consider the SPDE (1.1). If \( F \in C_b(H, H) \) then we have uniqueness in law for (1.1).

PROOF. By Proposition 14 it is enough to show that the martingale problem for \( \mathcal{L} \) is well-posed (cf. (5.1)). By Section 4, for any \( x \in H \), there exists a solution to the martingale problem for \((\mathcal{L}, \delta_x)\).

In order to apply Theorem 16 we proceed into two steps. In the first step we construct a suitable covering of \( H \); in the second step we define suitable operators \( \mathcal{L}_j \) according to Lemma 21 such that the martingale problem associated to each \( \mathcal{L}_j \) is well-posed.
I Step. There exists a countable set of points \((x_j) \subset H, j \geq 1\), and numbers \(r_j > 0\) with the following properties:

(i) the open balls \(U_j = B(x_j, \frac{r_j}{2}) = \{x \in H : |x - x_j|_H < r_j/2\}\) form a covering for \(H\);

(ii) we have: \(|F(x) - F(x_j)| < 1/4, \ x \in B(x_j, r_j)\).

Using the continuity of \(F\); for any \(x\) we find \(r(x) > 0\) such that

\[ |F(y) - F(x)|_H < 1/4, \ y \in B(x, r(x)). \]

We have a covering \(\{U_x\}_{x \in H}\) with \(U_x = B(x, \frac{r(x)}{2})\). Since \(H\) is a separable Hilbert space we can choose a countable subcovering \((U_j)_{j \geq 1}\), with \(U_j = B(x_j, \frac{r(x_j)}{2}) = B(x_j, \frac{r_j}{2})\).

II Step. We construct \(\mathcal{L}_j\) in order to apply the localization principle.

Let us consider the previous covering \((B(x_j, r_j/2))\). We take \(\rho \in C_0^\infty(\mathbb{R}_+)\), \(0 \leq \rho \leq 1\), \(\rho(s) = 1, 0 \leq s \leq 1\), \(\rho(s) = 0\) for \(s \geq 2\). Define

\[ \rho_j(x) = \rho(4r_j^{-2}|x - x_j|^2), \ x \in H. \]

Now \(\rho_j = 1\) in \(B(x_j, \frac{r_j}{2})\) and \(\rho_j = 0\) outside \(B(x_j, r_j)\). Set \(F_j(x) := \rho_j(x)F(x) + (1 - \rho_j(x))F(x_j), \ x \in H\), so that

\[ \sup_{x \in H}|F_j(x) - F(x_j)|_H = \sup_{x \in B(x_j, r_j)}|F(x) - F(x_j)|_H < 1/4 \]

and \(F_j(x) = F(x), \ x \in B(x_j, \frac{r_j}{2}) = U_j\). Define \(D(\mathcal{L}_j) = C^2_{cil}(H), j \geq 1, \)

\[ \mathcal{L}f_j(x) = \frac{1}{2}Tr(D^2f(x)) + \langle x, ADf(x) \rangle + \langle (-A)^{1/2}F_j(x), Df(x) \rangle, \ f \in C^2_{cil}(H), \ x \in H. \]

We have \(\mathcal{L}_j f(x) = \mathcal{L}f(x), \ x \in U_j, \ f \in C^2_{cil}(H)\) and the martingale problem for each \(\mathcal{L}_j\) is well-posed by Lemma 21 (with \(F = F_j\) an \(z = F(x_j)\)). By Theorem 16 we find the assertion.

6. Proof of weak uniqueness of Theorem 1. Here we prove uniqueness in law for (1.1) assuming that \(F : H \rightarrow H\) is continuous and has at most linear growth, i.e., it verifies (1.2). To this purpose we will use Lemma 22 and Theorem 17.

Let \(X = (X_t)_{t \geq 0}\) be a mild solution of (1.1) starting at \(x \in H\) (under the assumption (1.2)) defined on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) on which it is defined a cylindrical \(\mathcal{F}_t\)-Wiener process \(W\); see Section 4. For a cylindrical function \(f \in C^2_{cil}(H)\) in general \(\mathcal{L}f\) (see (5.1)) is not a bounded function on \(H\) because \(F\) can be unbounded. However we know by a finite-dimensional Itô’s formula that

\[ M_t(f) = f(X_t) - \int_0^t \mathcal{L}f(X_s)ds = f(x) + \int_0^t Df(X_s)dW_s \]

is still a continuous square integrable \(\mathcal{F}_t\)-martingale. Note that we can apply Itô’s formula because there exists \(m \geq 1\) such that \(f(x) = f(\pi_m x), \ x \in H, \) and so \(f(X_t) = f(\pi_m X_t)\) (cf. formula (5.20)).

Now let us consider \(B(0, n) = \{x \in H : |x|_H < n\}\) and define continuous and bounded functions \(F_n : H \rightarrow H\) such that \(F_n(y) = F(y), y \in B(0, n), n \geq 1\).

To this purpose one can take \(\eta \in C_0^\infty(\mathbb{R})\) such that \(0 \leq \eta(s) \leq 1, s \in \mathbb{R}, \eta(s) = 1\) for \(|s| \leq 1\) and \(\eta(s) = 0\) for \(|s| \geq 2\), and set \(F_n(y) = F(y)\eta\left(\frac{|y|_H}{n}\right), y \in H\). Define

\[ \mathcal{L}_n f(y) = \frac{1}{2}Tr(D^2f(y)) + \langle y, ADf(y) \rangle + \langle F_n(y), (-A)^{1/2}Df(y) \rangle, \ f \in C^2_{cil}(H), y \in H. \]
Let us introduce the exit time \( \tau^X_n = \inf\{t \geq 0 : |X_t|_H \geq n\} \) \( \tau^X_n = +\infty \) if the set is empty; cf. (5.5)) for each \( n \geq 1 \). It is an \( \mathcal{F}_t \)-stopping time (cf. Proposition II.1.5 in [19]). By the optional sampling theorem (cf. Theorem II.2.13 in [19]) we know that

\[
M_{t \wedge \tau^X_n}(f) = f(X_{t \wedge \tau^X_n}) - \int_0^{t \wedge \tau^X_n} \mathcal{L}f(X_s)ds = f(X_{t \wedge \tau^X_n}) - \int_0^{t \wedge \tau^X_n} \mathcal{L}_n f(X_{s \wedge \tau^X_n})ds, \quad t \geq 0,
\]

is a martingale with respect to the filtration \( (\mathcal{F}_{t \wedge \tau^X_n})_{t \geq 0} \); note that the process \( (X_{t \wedge \tau^X_n})_{t \geq 0} \) is adapted with respect to \( (\mathcal{F}_{t \wedge \tau^X_n}) \) (see Proposition II.1.4 in [19]).

Thus \( (X_{t \wedge \tau^X_n})_{t \geq 0} \) is a solution to the stopped martingale problem for \( (\mathcal{L}_n, \delta_x, B(0,n)) \). By Lemma 22 the martingale problem for each \( \mathcal{L}_n \) is well-posed because \( F_n \in C_b(H,H) \). By Theorem 17 also the stopped martingale problem for \( (\mathcal{L}_n, \delta_x, B(0,n)) \) is well-posed, \( n \geq 1 \).

Let \( Y \) be another mild solution starting at \( x \in H \). Then \( (Y_{t \wedge \tau^X_n})_{t \geq 0} \) also solves the stopped martingale problem for \( (\mathcal{L}_n, \delta_x, B(0,n)) \). By weak uniqueness of the stopped martingale problem it follows that, for any \( n \geq 1 \), \( (X_{t \wedge \tau^X_n})_{t \geq 0} \) and \( (Y_{t \wedge \tau^X_n})_{t \geq 0} \) have the same law. Now it is not difficult to prove that \( X \) and \( Y \) have the same law on \( \mathcal{B}(C([0,\infty);H)) \) and this finishes the proof.

### 7. An extension to locally bounded functions \( F : H \rightarrow H \)

Assuming weak existence for (1.1) one can obtain the following extension of Theorem 1.

**Theorem 23.** Let us consider (1.1) under Hypothesis 1 and fix \( x \in H \). Assume that

- \( H1 \): \( F : H \rightarrow H \) is continuous and bounded on bounded sets of \( H \) (or locally bounded);
- \( H2 \): there exists a weak mild solution \( (X_t)_{t \geq 0} \) starting at \( x \in H \).

Under the previous assumptions weak uniqueness holds, i.e., all weak mild solutions starting at \( x \in H \) have the same law on \( \mathcal{B}(C([0,\infty);H)) \).

**Proof.** The proof is similar to the one of Section 6. We give some details for the sake of completeness. Let \( X = (X_t)_{t \geq 0} \) be a mild solution of (1.1) starting at \( x \in H \) (defined on some filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}) \)). We have that \( M_t(f) \) in (6.1) is a continuous square integrable \( \mathcal{F}_t \)-martingale, for any \( f \in C^2_{\text{loc}}(H) \). Using the continuity and the local boundedness of \( F \), we obtain that the functions \( F_n(y) = F(y) \eta\left(\frac{\|y\|}{n}\right), y \in H \), are continuous and bounded from \( H \) into \( H \).

By the optional sampling theorem we find that \( (X_{t \wedge \tau^X_n})_{t \geq 0} \) is a solution to the stopped martingale problem for \( (\mathcal{L}_n, \delta_x, B(0,n)) \). Using Lemma 22 and Theorem 17 we know that the stopped martingale problem for \( (\mathcal{L}_n, \delta_x, B(0,n)) \) is well-posed, \( n \geq 1 \). Proceeding as in the final part of Section 6 we obtain the assertion. \( \square \)

#### 7.1. Singular perturbations of classical stochastic Burgers equations

Here we show that Theorem 23 can be applied to SPDEs (1.1) in cases where \( F \) grows more than linearly. As an example we consider

\[
du(t,\xi) = \frac{\partial^2}{\partial \xi^2} u(t,\xi)dt + h(u(t,\xi)) \cdot g\left(\|u(t,\cdot)\|_{H_0^1}\right)dt + \frac{1}{2} \frac{\partial}{\partial \xi} \left(u^2(t,\xi)\right)dt + \sum_{k \geq 1} \frac{1}{k} \frac{dW_t^{(k)}}{\xi} e_k(\xi),
\]

\[
u(t,\xi) = u(0,\xi), \quad \xi \in (0,\pi),
\]

\[
u(t,0) = u(t,\pi) = 0, \quad t > 0, \quad u_0 \in H_0^1(0,\pi); \quad g : \mathbb{R} \rightarrow \mathbb{R} \text{ is continuous and } h : \mathbb{R} \rightarrow \mathbb{R} \text{ is a } C^1 \text{-function. Moreover to get existence of solutions we require}
\]

\[
\sup_{s \in \mathbb{R}} |h'(s)| \cdot \sup_{s \in \mathbb{R}} |g(s)| \leq 1.
\]
For instance, we can consider \( h(u(t, \xi)) = g(\|u(t, \cdot)\|_{H^1_0}) = u(t, \xi) \cdot (\sqrt{|u(t, \cdot)|_{H^1_0}} \wedge 1) \).

Recall that \( e_k(\xi) = \sqrt{2/\pi} \sin(k \xi), \xi \in [0, \pi], \ k \geq 1 \) (cf. Section 2; note that in (7.1) the noise is “more regular” than the one in (2.1)).

We first establish existence of mild solutions with values in \( H^1_0(0, \pi) \) when \( g = 0 \) (see Proposition 24). This is needed in other to show that the classical Burgers equation can be considered in the form (1.1) with a suitable \( F = F_0 : H^1_0(0, \pi) \to H^1_0(0, \pi) \) continuous and locally bounded (see (7.20)). To this purpose we follow the approach in Chapter 14 of [14].

Then to get well-posedness of (7.1) (see Proposition 25) we will apply the Girsanov theorem using an exponential estimate proved in [8]. Such Girsanov theorem provides existence of weak solutions (cf. Remark 26). Uniqueness in law is obtained using Theorem 23.

We need to review basic facts about fractional powers of the operator \( A = \frac{d^2}{dx^2} \) with Dirichlet boundary conditions, i.e., \( D(A) = H^2(0, \pi) \cap H^1_0(0, \pi) \) (cf. Section 2). The eigenfunctions are \( e_k(\xi), k \geq 1 \), with eigenvalues \( -k^2 \) (we set \( \lambda_k = -k^2 \)). For \( v \in L^2(0, \pi) \) we write \( v_k = \langle v, e_k \rangle = \int_0^\pi v(x)e_k(x)dx, k \geq 1 \).

We introduce for \( s > 0 \) the Hilbert spaces

\[
(7.3) \quad \mathcal{H}_s = D((-A)^s) = \left\{ u \in L^2(0, \pi) : \sum_{k \geq 1} \lambda_k^{2s} u_k^2 \leq \sum_{k \geq 1} k^{4s} u_k^2 < \infty \right\}.
\]

Moreover, for any \( u \in \mathcal{H}_s \), \((-A)^s u = \sum_{k \geq 1} k^{2s} u_k e_k \). He also set \( \mathcal{H}_0 = L^2(0, \pi) \). We have \( \langle u, v \rangle_{\mathcal{H}_s} = \sum_{k \geq 1} k^{4s} u_k v_k \) (note that \( |u|_{L^2} \leq |(-A)^s u|_{L^2} = |u|_{\mathcal{H}_s}, u \in \mathcal{H}_s, s > 0 \)).

If \( u \in H^1_0(0, \pi) \), \( |u|_{H^1_0(0, \pi)} = |u'|_{L^2(0, \pi)} \), where \( u' \) is the weak derivative of \( u \). We have

\[
(7.4) \quad H^{1/2} = H^1_0(0, \pi) \quad \text{with equivalence of norms;}
\]

\[
(7.5) \quad \mathcal{H}_{1/2} \subset L^4(0, \pi)
\]

(with continuous inclusion, i.e., there exists \( C > 0 \) such that \( |u|_{L^4} \leq C|u|_{\mathcal{H}_{1/2}}, u \in \mathcal{H}_{1/2} \)). Assertion (7.5) follows by a classical Sobolev embedding theorem (cf. Theorem 6.16 and Remark 6.17 in [24]). We only note that if \( u \in \mathcal{H}_{1/2} \) one can consider the odd extension \( \bar{u} \) of \( u \) to \((-\pi, \pi)\); it is easy to check that \( \bar{u} \) belongs to the space \( H^{1/4}(\pi, \pi) \) considered in [24].

We also have with continuous inclusion (cf. Lemma 6.13 in [24])

\[
(7.6) \quad \mathcal{H}_s \subset \{ u \in C([0, \pi]), u(0) = u(\pi) = 0 \}, \quad s > 1/4.
\]

Now let us consider the linear bounded operator \( T : \mathcal{H}_{1/2} \to \mathcal{H}_{1/2}, Tu = (-A)^{-1/2} \partial_x u, u \in \mathcal{H}_{1/2} \); \( T \) can be extended to a linear and bounded operator \( T : \mathcal{H}_0 = L^2(0, \pi) \to \mathcal{H}_0 \), see Section 2.0.1. By interpolation it follows that

\[
(7.7) \quad T = (-A)^{-1/2} \partial_x \quad \text{is bounded linear operator from } \mathcal{H}_s \text{ into } \mathcal{H}_s, s \in [0, 1/2].
\]

Indeed by Theorem 4.36 in [29] we know that \( \mathcal{H}_{s/2} \) can be identified with the real interpolation space \( (\mathcal{H}_0, \mathcal{H}_{1/2})_{s/2}, s \in (0, 1) \). Applying Theorem 1.6 in [29] we deduce (7.7).

Let \( T > 0 \). For \( g \in C([0, T]; \mathcal{H}_0) \) we define \( (Sg)(t) = \int_0^t e^{(t-s)A}g(s)ds, t \in [0, T] \). One can prove that \( Sg \in C([0, T]; \mathcal{H}_s) \), for any \( s \in [0, 1) \). More precisely,

\[
(7.8) \quad S \text{ is a bounded linear operator from } C([0, T]; \mathcal{H}_0) \text{ into } C([0, T]; \mathcal{H}_s), s \in [0, 1].
\]

This result can be also deduced from Proposition 5.9 in [16] with \( \alpha = 1, E_1 = \mathcal{H}_s \) and \( E_2 = \mathcal{H}_0 \). We only remark that, for any \( p > 1 \), \( L^p(0, T; \mathcal{H}_0) \subset C([0, T]; \mathcal{H}_0) \) (with continuous inclusion) and \( |(-A)^s e^{A_x}x|_{\mathcal{H}_0} = |e^{A_x}x|_{\mathcal{H}_s} \leq \frac{C}{\pi} |x|_{\mathcal{H}_0} \) (see Proposition 4.37 in [24]).

In the next proposition, assertion (i) extends a result of [14] which actually shows the existence of a mild solution to the stochastic Burgers equations with continuous path in \( \mathcal{H}_s \), \( s \in (0, 1/4) \). Assertion (ii) is proved in [8].
Proposition 24. Let us consider (7.1) with \( g = 0 \). Then the following assertions hold:

i) for any \( u_0 \in \mathcal{H}_{1/2} \) there exists a pathwise unique mild solution \( Y = (Y_t) = (Y_t)_{t \geq 0} \) with continuous paths in \( \mathcal{H}_{1/2} \).

ii) The following estimate holds, for any \( T > 0 \),

\[
\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T |Y_s|^2_{\mathcal{H}_{1/2}} ds \right) \right] < \infty.
\]

Proof. According to [14] and [8], setting \( u(t, \cdot) = Y_t \) we write (7.1) with \( g = 0 \) as

\[
Y_t = e^{-A} u_0 + \frac{1}{2} \int_0^t e^{(t-s)A} \partial_\xi (Y^2_s) ds + \int_0^t e^{(t-s)A} \sqrt{C} dW_s, \quad t \geq 0,
\]

where \( W_t = \sum_{k \geq 1} W^{(k)}_t e_k \) is a cylindrical Wiener process on \( \mathcal{H}_0 = L^2(0, \pi) \) and \( C = (-A)^{-1} : \mathcal{H}_0 \to \mathcal{H}_0 \) is symmetric, non-negative and of trace class, \( Ce_k = \frac{1}{k^2} e_k, \ k \geq 1 \).

(i) In Theorem 14.2.4 of [14] (see also the references therein) it is proved that, for any \( T > 0 \), there exists a pathwise unique solution \( Y \) to (7.10) on \( [0, T] \) such that, \( \mathbb{P} \)-a.s., \( Y \in C([0, T]; \mathcal{H}_0) \cap L^2(0, T; \mathcal{H}_{1/2}) \) (i.e., \( \mathbb{P} \)-a.s. the paths of \( Y \) are continuous with values in \( \mathcal{H}_0 \) and square-integrable with values in \( \mathcal{H}_{1/2} \)) such result holds even if we replace \( C \) by identity \( I \). By a standard argument based on the pathwise uniqueness, we get a solution \( Y \) defined on \( [0, \infty) \) which verifies \( Y \in C([0, \infty); \mathcal{H}_0) \cap L^2_{\text{loc}}(0, \infty; \mathcal{H}_{1/2}), \mathbb{P}\text{-a.s.} \).

Let us fix \( T > 0 \). To prove our assertion, we will show that

\[
Y \in C([0, T]; \mathcal{H}_{1/2}), \quad \mathbb{P}\text{-a.s.}
\]

Note that the stochastic convolution \( W_A(t) = \int_0^t e^{(t-s)A} \sqrt{C} dW_s \) has a modification with continuous paths with values in \( \mathcal{H}_{1/2} \) (cf. Lemma 5.17 in [15]). Moreover in Lemma 14.2.1 of [15] it is proved that the operator \( R \),

\[
(Rv)(t) = \int_0^t e^{(t-s)A} \partial_\xi v(s) ds, \quad t \in [0, T], \ v \in C([0, T]; \mathcal{H}_{1/2}),
\]

can be extended to a linear and bounded operator from \( C([0, T]; L^1(0, \pi)) \) into \( C([0, T]; \mathcal{H}_s) \), \( s \in (0, 1/4) \). Since the mapping \( h \mapsto h^2 \) is continuous from \( C([0, T]; \mathcal{H}_0) \) into \( C([0, T]; L^1(0, \pi)) \), we obtain that

\[
u \mapsto R(u^2) \text{ is continuous from } C([0, T]; \mathcal{H}_0) \text{ into } C([0, T]; \mathcal{H}_s).
\]

We deduce from Lemma 14.2.1 of [15] that the solution \( Y \in C([0, T]; \mathcal{H}_s), \mathbb{P}\text{-a.s.}, s \in (0, 1/4) \).

To get more spatial regularity for \( Y \) we proceed in two steps.

I Step. We show that, \( \mathbb{P} \)-a.s, \( Y \subset C([0, T]; \mathcal{H}_s), \ s \in (0, 1/2) \).

Let us fix \( s = 1/8 \). By (7.5) we know that the mapping \( h \mapsto h^2 \) is continuous from \( C([0, T]; \mathcal{H}_{1/8}) \) into \( C([0, T]; \mathcal{H}_0) \). Moreover, using (7.7) we can write, for \( w \in C([0, T]; \mathcal{H}_0) \),

\[
(Rw)(t) = \int_0^t e^{(t-s)A} \partial_\xi w(s) ds = \int_0^t e^{(t-s)A} (-A)^{1/2} [(-(A)^{-1/2} \partial_\xi)] w(s) ds, \quad t \in [0, T].
\]

Note that \( [(-(A)^{-1/2} \partial_\xi)] w \in C([0, T]; \mathcal{H}_0) \). By (7.8) we know that, for any \( \epsilon \in (0, 1) \), \( t \mapsto (-(A)^{-1/2} \partial_\xi)] w(s) ds \) belongs to \( C([0, T]; \mathcal{H}_0) \). Hence

\[
(-A)^s Rw \in C([0, T]; \mathcal{H}_0), \quad s \in (0, 1/2), \ \text{i.e.,} \ Rw \in C([0, T]; \mathcal{H}_s), \ s \in (0, 1/2).
\]

Using this fact we easily obtain that, \( \mathbb{P} \)-a.s, \( Y \subset C([0, T]; \mathcal{H}_s), \ s \in (0, 1/2) \).

II Step. We show that \( Y \subset C([0, T]; \mathcal{H}_{1/2}), \mathbb{P}\text{-a.s.} \).
Let us fix \( s \in (1/4, 1/2) \) and recall (7.6). According to [22] the space \( \mathcal{H}_s \) can be identified with \( \{ u \in W^{2s,2}(0, \pi) : u(0) = u(\pi) = 0 \} \), where

\[
W^{2s,2}(0, \pi) = \{ u \in \mathcal{H}_0 : \| u \|_{W^{2s,2}(0, \pi)}^2 = \int_0^\pi \int_0^\pi |u(x) - u(y)|^2 |x - y|^{-1-4s} \, dx \, dy < \infty \}
\]

is a Sobolev-Slobodeckij space; the norm \( |u|_{W^{2s,2}(0, \pi)} = |u|_{\mathcal{H}_0} + |u|_{W^{2s,2}(0, \pi)} \) is equivalent to \( |u|_{\mathcal{H}_s} \) (see also Theorem 3.2.3 in [28], taking into account that \( \mathcal{H}_s \) can be identified with the real interpolation space \( (\mathcal{H}_0, D(A))_{s,2} \) by Theorem 4.36 in [29]).

Using the previous characterization and (7.6) it is easy to prove that if \( u \in \mathcal{H}_s \) and \( v \in \mathcal{H}_s \) then the pointwise product \( uv \in \mathcal{H}_s \). Indeed we have

\[
|u(x)v(x) - u(y)v(y)| \leq \| u \|_0 |v(x) - v(y)| + \| v \|_0 |u(x) - u(y)|, \quad x, y \in [0, \pi],
\]

and so \( \|uv\|_{W^{2s,2}(0, \pi)} \leq c \| u \|_{W^{2s,2}(0, \pi)} \| v \|_{W^{2s,2}(0, \pi)} \leq c' \| u \|_{\mathcal{H}_s} \| v \|_{\mathcal{H}_s} \). It follows that \( |uv|_{\mathcal{H}_s} \leq C |u|_{\mathcal{H}_s} \| v \|_{\mathcal{H}_s} \).

Let now \( u \in C([0, T]; \mathcal{H}_s) \). Using that \( |u^2(t) - u^2(r)|_{\mathcal{H}_s} \leq |u(t) - u(r)|_{\mathcal{H}_s} |u(t) + u(r)|_{\mathcal{H}_s} \leq 2|u|_{C([0, T]; \mathcal{H}_s)} |u(t) - u(r)|_{\mathcal{H}_s} \), \( t, r \in [0, T] \), we see that the mapping:

\[
u \mapsto u^2 \text{ is continuous from } C([0, T]; \mathcal{H}_s) \text{ into } C([0, T]; \mathcal{H}_s).
\]

Hence, taking into account I Step, to get the assertion it is enough to prove that

\[
R\eta \in C([0, T]; \mathcal{H}_{1/2}) \text{ if } \eta \in C([0, T]; \mathcal{H}_s), \quad s \in (1/4, 1/2).
\]

This would imply \( R(\eta^2) \in C([0, T]; \mathcal{H}_{1/2}) \) if \( \eta \in C([0, T]; \mathcal{H}_s) \) and so \( Y \in C([0, T]; \mathcal{H}_{1/2}) \), \( \mathbb{P}\text{-a.s.} \). Let us fix \( \eta \in C([0, T]; \mathcal{H}_s) \). Using (7.7) we can write

\[
(R\eta)(t) = \int_0^t e^{(t-s)A} \partial \eta(s) ds = \int_0^t e^{(t-s)A} (-A)^{1/2} (-A)^{-1/2} \partial \eta(s) ds, \quad t \in [0, T],
\]

where \((-A)^{-1/2} \partial \eta \in C([0, T]; \mathcal{H}_s)\). Hence \( \theta(r) = (-A)^s [(-A)^{-1/2} \eta](r) \in C([0, T]; \mathcal{H}_0) \).

Writing

\[
(R\eta)(t) = \int_0^t e^{(t-r)A} (-A)^{1/2} \eta(r) dr, \quad t \in [0, T],
\]

and using (7.8), we find that \((-A)^{1/2} R\eta \in C([0, T]; \mathcal{H}_0)\) and this shows (7.14).

\( \text{(ii)} \) A similar estimate is proved in Propositions 2.2 and 2.3 in [8]. However in [8] equation (7.10) is considered in \( L^2(0, 1) \) (instead of \( L^2(0, \pi) \)); the authors prove that \( \mathbb{E} [e^{\int_0^t |\eta|_{H^s_{1/2}}^2 \, ds} ] < \infty \) if \( \epsilon \leq \epsilon_0 = \pi^2/2 \| C \| \) (using the operator norm \( \| C \| \) of \( C \)).

The condition \( \epsilon \leq \epsilon_0 \) is used in the proof of Proposition 2.2 in order to get the inequality

\[-|x|^2_{H^s_{1/2}} + 2c|\sqrt{C} x|_{L^2}^2 \leq 0, \quad x \in H^s_{1/2}. \]

In our case \( \epsilon_0 = 1/2 \) since \( \| C \| = 1 \). \( \Box \)

In the remaining part we consider

\[
\mathcal{H} = H^1_0(0, \pi) = \mathcal{H}_{1/2}
\]

as the reference Hilbert space and study the SPDE (7.1) in \( \mathcal{H} \).

We will consider the following restriction of \( A \) :

\[
A = \frac{\partial^2}{\partial \xi^2} \quad \text{with } D(A) = \{ u \in H^3(0, \pi) : u, \frac{\partial^2 u}{\partial \xi^2} \in H^1_0(0, \pi) \}; \quad A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}.
\]

Eigenfunctions of \( A \) are \( \tilde{\chi}_k(\xi) = \sqrt{2/\pi} k \sin(k \xi) = \frac{1}{k} \tilde{\chi}_k(\xi) \) with eigenvalues \(-k^2, \ k \geq 1\).
It is clear that $\mathcal{A}$ verifies Hypothesis 1 when $H = \mathcal{H}$. Moreover $(\phi_k) = (\hat{e}_k)$ forms an orthonormal basis in $\mathcal{H}$. The noise in (7.1) will be indicated by $W$; it is a cylindrical Wiener process on $\mathcal{H}$:

\begin{equation}
W_t(\xi) = \sum_{k \geq 1} \frac{1}{k} W_t^{(k)} e_k(\xi) = \sum_{k \geq 1} W_t^{(k)} \hat{e}_k(\xi), \quad t \geq 0, \xi \in [0, \pi].
\end{equation}

Let $D_0$ be the space of infinitely differentiable functions vanishing in a neighborhood of 0 and $\pi$. Such functions are dense in $\mathcal{H}$. The operator

\begin{equation}
(-\mathcal{A})^{1/2} \partial_\xi : D_0 \to \mathcal{H}
\end{equation}

can be extended to a bounded linear operator from $\mathcal{H}$ into $\mathcal{H}$. To check this fact we consider $y \in D_0$ and $x \in \mathcal{H}$. Define $x_N = \sum_{k=1}^N x_k \hat{e}_k$, with $x_k = \langle x, \hat{e}_k \rangle_{\mathcal{H}}$, $N \geq 1$. Using that $(-\mathcal{A})^{1/2}$ is self-adjoint on $\mathcal{H}$ and integrating by parts we find

\begin{align*}
\langle (-\mathcal{A})^{-1/2} \partial_\xi y, x_N \rangle_{\mathcal{H}} &= \langle \partial_\xi y, (-\mathcal{A})^{-1/2} x_N \rangle_{\mathcal{H}} = \langle \partial_\xi y, \partial_\xi x_N \sum_{k=1}^N \frac{x_k}{k} \hat{e}_k \rangle_{L^2(0,\pi)} \\
&= -\langle \partial_\xi y, \partial_\xi \sum_{k=1}^N x_k \sin(k \cdot) \hat{e}_k \rangle_{L^2(0,\pi)} = \langle \partial_\xi y, \sum_{k=1}^N x_k \sin(k \cdot) \rangle_{L^2(0,\pi)}.
\end{align*}

Hence $|\langle (-\mathcal{A})^{-1/2} \partial_\xi y, x_N \rangle_{\mathcal{H}}| \leq |y|_{\mathcal{H}} \ (\sum_{k=1}^N x_k^2)^{1/2} \leq |y|_{\mathcal{H}} |x|_{\mathcal{H}}$ and we get the assertion. Let us introduce, for any $x \in \mathcal{H}$,

\begin{equation}
F_0(x) = \frac{1}{2} \mathcal{A}^{-1/2} \partial_\xi [x^2].
\end{equation}

Since the mapping $x \mapsto x^2$ is continuous and locally bounded from $\mathcal{H}$ into $\mathcal{H}$ (recall that $|x^2|_{\mathcal{H}} = 2|x_0|_{L^2(0,\pi)}$) it is clear that

\begin{equation}
F_0 : \mathcal{H} \to \mathcal{H}
\end{equation}

is continuous and locally bounded.

The mild solution $Y$ of Proposition 24 with paths in $C([0,\infty);\mathcal{H})$ verifies, $\mathbb{P}$-a.s.,

\begin{equation}
Y_t = e^{t\mathcal{A}} x + \int_0^t (-\mathcal{A})^{1/2} e^{(t-s)\mathcal{A}} F_0(Y_s) ds + \int_0^t e^{(t-s)\mathcal{A}} dW_s; \quad t \geq 0,
\end{equation}

where $\mathcal{A}$ is defined in (7.15) and we have set $u_0 = x \in \mathcal{H}$.

We consider the following SPDE which includes (7.1) as a special case:

\begin{equation}
X_t = e^{t\mathcal{A}} x + \int_0^t (-\mathcal{A})^{1/2} e^{(t-s)\mathcal{A}} F_0(X_s) ds + \int_0^t e^{(t-s)\mathcal{A}} B(X_s) ds + \int_0^t e^{(t-s)\mathcal{A}} dW_s,
\end{equation}

$t \geq 0$. Here

\begin{equation}
B : \mathcal{H} \to \mathcal{H} \text{ is continuous and } |B(x)|_{\mathcal{H}} \leq c_0 + |x|_{\mathcal{H}}, \quad x \in \mathcal{H},
\end{equation}

for some $c_0 \geq 0$. In (7.1) we have $B(x) = h(x)g(|x|_{\mathcal{H}})$, $x \in \mathcal{H}$, and (7.22) holds with $c_0 = 0$ (we only note that $|B(x)|^2 \leq \|g\|^2_0 \int_0^1 |h'(x)| \cdot \frac{dx}{|x|_{\mathcal{H}}^2} d\xi \leq \|g\|^2_0 \|h'\|^2_0 |x|^2_{\mathcal{H}} \leq |x|^2_{\mathcal{H}}, \ x \in \mathcal{H}$).

Condition (7.2) is used to guarantee the bound on $B$ in (7.22). This bound is used to check the Novikov condition (7.23) and prove the existence part in the following result.

**Proposition 25.** Let us consider (7.21) on $H = H^1_0(0,\pi)$ with $A$ given in (7.15) and the cylindrical Wiener process $W$ on $\mathcal{H}$ given in (7.16) ($W^{(k)}_{k \geq 1}$ are independent real Wiener processes). Let $F_0$ as in (7.18) and suppose that $B : \mathcal{H} \to \mathcal{H}$ verifies (7.22). Then the following assertions hold.

i) For any $x \in \mathcal{H}$, there exists a weak mild solution $(X_t)_{t \geq 0}$.

ii) Weak uniqueness holds for (7.21) for any $x \in \mathcal{H}$. 


Proof. i) Let us fix \( x \in \mathcal{H} \). We will use the Girsanov theorem as in Appendix A.1 of [10], using the reference Hilbert space \( \mathcal{H} \).

Let \( Y = (Y_t) \) be the unique solution to the Burgers equation (7.20) with values in \( \mathcal{H} \) and such that \( Y_0 = x \). This is defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) on which it is defined the cylindrical Wiener process \( \mathcal{W} \) on \( \mathcal{H} \). Set

\[
    b(s) = B(Y_s), \quad s \geq 0, 
\]

and note that \( |b(s)|_\mathcal{H} \leq c_0 + |Y_s|_\mathcal{H}, \quad s \geq 0, \) by (7.22). The process \( (b(s)) \) is progressively measurable and verifies

\[
    \mathbb{E} \int_0^T |b(s)|^2_\mathcal{H} \, ds < \infty, \quad T > 0 \quad \text{(see (7.9) and recall that \( e^r \geq 1 + r \)).}
\]

Moreover, by (7.9) and (7.22) it follows that, for any \( T > 0, \)

\[
    \mathbb{E} \left[ e^{\frac{1}{2} \int_0^T |b(s)|^2_\mathcal{H} \, ds} \right] \leq C_T \mathbb{E} \left[ e^{\frac{1}{2} \int_0^T |Y_s|^2_\mathcal{H} \, ds} \right] < \infty. \tag{7.23}
\]

Let \( U_t = \sum_{k \geq 1} \int_0^t (b(s), \tilde{e}_k)_\mathcal{H} \, dW_s^{(k)} \), \( t \geq 0, \) and fix \( T > 0. \) By Proposition 17 in [10] we know that \( \tilde{W}_t^{(k)} = W_t^{(k)} - \int_0^t (\tilde{e}_k, b(s))_\mathcal{H} \, ds, \) \( t \in [0, T], \) \( k \geq 1, \) are independent real Wiener processes on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \tilde{\mathbb{P}})\), where the probability measure

\[
    \tilde{\mathbb{P}} = e^{UT - \frac{1}{2} \int_0^T |b(s)|^2_\mathcal{H} \, ds} \cdot \mathbb{P}
\]

is equivalent to \( \mathbb{P} \) (the quadratic variation process \( (U)_t = \int_0^t |b(s)|^2_\mathcal{H} \, ds, \) \( t \in [0, T] \)).

Hence \( \tilde{W}_t = \sum_{k \geq 1} \tilde{W}_t^{(k)} \tilde{e}_k, \) \( t \in [0, T], \) is a cylindrical Wiener process on \( \mathcal{H} \) defined on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \tilde{\mathbb{P}})\). Arguing as in Proposition 21 of [10] we obtain that

\[
    Y_t = e^{tA}x + \int_0^t (-A)^{1/2} e^{(t-s)A} F_0(Y_s) \, ds + \int_0^t e^{(t-s)A} B(Y_s) \, ds + \int_0^t e^{(t-s)A} d\tilde{W}_s, \quad t \in [0, T],
\]

\( \mathbb{P} \)-a.s. Thus \( Y \) a mild solution on \([0, T]\) to (7.21) defined on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \tilde{\mathbb{P}})\).

Since \( T > 0 \) is arbitrary, using a standard procedure based on the Kolmogorov extension theorem, one can prove the existence of a weak mild solution \( X \) to (7.21) on \([0, \infty)\). On this respect, we refer to Remark 3.7, page 303, in [25] (cf. the beginning of Section 4).

ii) We use Theorem 23 with \( H = \mathcal{H}, \) \( A = A \) and \( W = W \). Indeed, (7.21) can be rewritten as

\[
    X_t = e^{tA}x + \int_0^t (-A)^{1/2} e^{(t-s)A} F(X_s) \, ds + \int_0^t e^{(t-s)A} dW_s, \quad t \geq 0,
\]

where \( F(x) = F_0(x) + (-A)^{-1/2} B(x), \) \( x \in \mathcal{H} \). The function \( F : \mathcal{H} \to \mathcal{H} \) is continuous and locally bounded (cf. (7.19) and (7.22)). \( \square \)

Remark 26. Assertion (ii) in Proposition 25 cannot be deduced directly from the Girsanov theorem as in Appendix A.1 of [10]. To this purpose, one should prove that

\[
    \mathbb{E} \left[ e^{\frac{1}{2} \int_0^T |B(X_s)|^2_\mathcal{H} \, ds} \right] < \infty, \quad \text{for any weak mild solution} \ X \ \text{to (7.21) starting at} \ x \ \text{in} \ \mathcal{H}. \quad \text{A sufficient condition would be} \ \mathbb{E} \left[ e^{\frac{1}{2} \int_0^T |X_s|^2_\mathcal{H} \, ds} \right] < \infty. \quad \text{It seems that such estimate does not hold under our assumptions. Note that since the nonlinearity of the Burgers equation grows quadratically one cannot follow the proof of Proposition 22 of [10] to derive} \ \mathbb{E} \left[ e^{\frac{1}{2} \int_0^T |X_s|^2_\mathcal{H} \, ds} \right] < \infty. \]
APPENDIX A: A FURTHER REGULARITY RESULT ON THE KOLMOGOROV EQUATION

When \( z = 0 \) one can show that \( Du \) (see (3.1)) belongs to the so-called Zygmund \( C^1 \)-space (see [5] and [30]). We will provide an alternative proof which is inspired by Lemma 2.2 in [1]. The Zygmund regularity will follow by Theorem 9, taking into account the estimate

\[
\|D^2 P_t^{(z)} \varphi\|_0 \leq Ct^{-1}\|\varphi\|_0, \quad t > 0, \quad \varphi \in B_b(H).
\]

(see Section 1.2). Let \( E \) be a separable Hilbert space. The Zygmund space \( C^1(H, E) \) is the space of all continuous and bounded function \( f : H \to E \), i.e., \( f \in C_b(H, E) \), such that

\[
[f]_{C^1} = \sup_{x, h \in H, h \neq 0, |h| \leq 1} \frac{|f(x + h) - 2f(x) + f(x - h)|_E}{|h|_E} < \infty.
\]

This is a Banach space endowed with the norm \( \|f\|_{C^1} = [f]_{C^1} + \|f\|_0, \quad f \in C^1(H, E) \). As usual we set \( C^1(H) = C^1(H, \mathbb{R}) \).

**Lemma 27.** Let us consider a semigroup of linear contractions \( (R_t), R_t : C_b(H) \to C_b(H), \ t \geq 0, \) such that \( R_t(C_b(H)) \subset C_b^2(H) \), \( t > 0 \), and there exists \( C_0 > 0 \) such that

\[
\|D^2 R_t \varphi\|_0 \leq C_0 t^{-1}\|\varphi\|_0, \quad t \in (0, 1], \ \varphi \in C_b(H).
\]

Let \( f \in C_b(H) \). If there exists a constant \( N > 0 \) such that

\[
\sup_{x \in H} |R_t f(x) - f(x)| = \|R_t f - f\|_0 \leq N t^{1/2}, \quad t \in [0, 1].
\]

Then \( f \in C^1(H) \). Moreover, \( [f]_{C^1} \leq 16(C_0 + 1)(N + \|f\|_0) \).

**Proof.** I Step. We introduce the semigroup \( (\hat{R}_t), \ \hat{R}_t = e^{-t} R_t, \ t \geq 0 \), and prove that

\[
\|D^2 \hat{R}_t f\|_0 \leq \frac{4C_0 (N + \|f\|_0)}{\sqrt{t}}, \quad t \in (0, 1/2].
\]

First note that \( |\hat{R}_t f(x) - R_t f(x) + R_t f(x) - f(x)| \leq |1 - e^{-t}|\|R_t f\|_0 + N t^{1/2}, \ t \in [0, 1], \) and so (using also that \( \|R_t f - f\|_0 \leq 2\|f\|_0 \))

\[
\|\hat{R}_t f - f\|_0 \leq (N + 2\|f\|_0) t^{1/2}, \quad t \geq 0.
\]

Let now \( \varphi \in C_b(H), \ t > 1 \). We can write \( D^2 \hat{R}_t \varphi = D^2 \hat{R}_1 \hat{R}_{t-1} \varphi \) and so

\[
\|D^2 \hat{R}_t \varphi\|_0 \leq C_0 e^{-t}\|\varphi\|_0, \quad t > 1.
\]

It follows that, for any \( \varphi \in C_b(H) \),

\[
\|D^2 \hat{R}_t \varphi\|_0 \leq C_0 t^{-1}\|\varphi\|_0, \quad t > 0.
\]

Now by (A.5) we obtain (A.4) as follows.

Let \( x \in H \). Let \( k \geq 0 \) be an integer and fix \( t \in (0, 1/2] \). Using the semigroup law and (A.5), we write

\[
\|D^2 \hat{R}_{t2^k} f(x) - D^2 \hat{R}_{t2^k} f(x)\|_L = \|D^2 \hat{R}_{t2^k+t2^k} f(x) - D^2 \hat{R}_{t2^k} f(x)\|_L
\]

\[
= \|D^2 \hat{R}_{t2^k} [R_{t2^k} f - f](x)\|_L
\]

\[
\leq \frac{C_0}{t2^k} \|\hat{R}_{t2^k} f - f\|_0 \leq \frac{C_0(N + 2\|f\|_0)t^{1/2}}{2^k t} \leq \frac{C_0(N + 2\|f\|_0)}{t^{1/2} 2^{k/2}}.
\]
Now $D^2 \hat{R}_{t_2N+1} f(x) - D^2 \hat{R}_t f(x) = \sum_{k=0}^N [D^2 \hat{R}_{t_2k+1} f(x) - D^2 \hat{R}_{t} f(x)]$. Since we know by (A.5) that $\lim_{N \to \infty} D^2 \hat{R}_{t_2N+1} f(x) = 0$, for any $x \in H$, we obtain
\[
D^2 \hat{R}_t f(x) = \sum_{k \geq 0} [D^2 \hat{R}_{t_2k} f(x) - D^2 \hat{R}_{t} f(x)], \quad x \in H,
\]
and we deduce (A.4) since $\sup_{x \in H} \|D^2 \hat{R}_t f(x)\| \leq \frac{C_0(N+2\|f\|_0)}{t^{1/2}} \sum_{k \geq 0} \frac{1}{tk^{1/2}}$. Formula (A.4) implies, for any $t \in (0,1]$, (A.6)
\[
\|D^2 \hat{R}_t f\|_0 \leq 16C_0(N + \|f\|_0) t^{-1/2}, \quad t \in (0,1].
\]

**II Step.** Let us check that $f \in C^1(H)$ using (A.6).

Fix $h \in H$ with $|h|_H \leq 1$ and set $t = |h|_H^2$. We write $f = [f - R_t f] + R_t f = l_t + g_t$. Since $\|l_t\|_0 = \|f - R_t f\|_0 \leq N|h|_H$ we consider $g_t$. Setting
\[
\triangle_h f(x) = f(x + h) - 2f(x) + f(x - h),
\]
we get $\triangle_h f(x) = \triangle_h l_t(x) + \triangle_h g_t(x)$ and $\|\triangle_h l_t\|_0 \leq 4N|h|_H$. By the Taylor formula and (A.6) we find
\[
|\triangle_h g_t(x)|_E \leq \|D^2 \hat{R}_t f\|_0 |h|_H \leq \frac{16C_0(N + \|f\|_0)}{|h|_H^2} |h|_H^2 = 16C_0(N + \|f\|_0)|h|_H.
\]
Hence $|f|_C \leq 16(C_0 + 1)(N + \|f\|_0)$. \(\square\)

Combining the previous lemma and Theorem 9 we find that $Du(z) \in C^1(H, H)$ with a bound on $[Du^{(z)}]_{C^1}$ independent of $z$.

**Theorem 28.** Let $f \in B_{h}(H)$ and consider $Du^{(z)}$ given in (3.1). Then with $c_1 = 16[C^2_1 + 1][C^2_1 + 1]$ (C_1 and C_2 are the same of Theorem 9) we have
\[
|Du^{(z)}(x + k) - 2Du^{(z)}(x) + Du^{(z)}(x - k)|_H \leq c_1 \|f\|_0, \quad x \in H, \quad k \in H, \quad |k| \leq 1, \tag{A.7}
\]

**Proof:** Recall that by (1.22) $\sup_{x \in H} |D^2 P^{(z)}_t \varphi(x)| = \|D^2 P^{(z)}_t \varphi\|_0 \leq \sqrt{2}C_1^2 t^{-1} \|\varphi\|_0, \quad t \in (0,1], \quad \varphi \in C_b(H)$ (the estimates holds more generally for any $t > 0$). We know by Theorem 9 that
\[
|P^{(z)}_s(\langle Du^{(z)}(\cdot), h\rangle(x) - \langle Du^{(z)}(x), h\rangle| \leq c_2 |h| s^{1/2} \|f\|_0.
\]
We obtain by Lemma 27, for any $k \in H$ with $|k|_H \leq 1$,
\[
|\langle Du^{(z)}(x + k) - 2Du^{(z)}(x) + Du^{(z)}(x - k), h\rangle| \leq c_1 |h|_H \|f\|_0.
\]
After taking the supremum over $\{|h|_H \leq 1\}$ we obtain the assertion. \(\square\)

**Remark 29.** Let us consider the Ornstein-Uhlenbeck semigroup $(P_t)$ (i.e., $(P^{(z)}_t)$ when $z = 0$). One may ask if a kind of converse of Lemma 27 holds. In other word if $g$ belongs to the Zygmund space $C^1(H)$ we may ask if $g$ verifies
\[
\sup_{s \in (0,1)} s^{-1/2} \|P_s g - g\|_0 < \infty. \tag{A.8}
\]
Arguing as in the proof of Lemma 3.6 and Proposition 3.7 of [13] (considering $\theta = 1/2$ in such results) one can prove that if $g \in C^1(H)$ and in addition we have
\[
\sup_{s \in (0,1)} s^{-1/2} \|g(e^{sA}(\cdot)) - g\|_0 < \infty \tag{A.9}
\]
then (A.8) holds. We point out that in infinite dimensions under Hypotheses 1 it is not clear if (A.9) holds when $g$ is replaced by the derivative $Du$ ($Du$ is given in (3.1) with $z = 0$ and $f \in B_{h}(H)$).
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