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REGULARIZATION BY NOISE AND FLOWS OF SOLUTIONS FOR A STOCHASTIC HEAT EQUATION

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Motivated by the regularization by noise phenomenon for SDEs we prove existence and uniqueness of the flow of solutions for the non-Lipschitz stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial z^2} + b(u(t,z)) + \dot{W}(t,z),$$

where $\dot{W}$ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$ and b is a bounded measurable function on $\mathbb{R}$. As a byproduct of our proof we also establish the so-called path–by–path uniqueness for any initial condition in a certain class on the same set of probability one. To obtain this results we develop a new approach that extends Davie's method (2007) to the context of stochastic partial differential equations.

1. Introduction. This work deals with the uniqueness theory for stochastic heat equations of the following form

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\frac{\partial^2 u}{\partial z^2} + b(u(t,z)) + \dot{W}(t,z), \quad t \geqslant 0,\ z \in \mathbb{R}, \\ u(0,z) &= q(z), \end{aligned} \tag{1.1}$$

where $\dot{W}$ is a Gaussian space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$, b is a bounded Borel measurable function on $\mathbb{R}$, and q is a Borel measurable function on $\mathbb{R}$ satisfying certain growth conditions. To be more precise we are going to construct the flow of solutions to (1.1) which is indexed by initial conditions q; we will establish uniqueness of the flow and show that in fact the flow can be constructed in a PDE sense on a set of full probability measure.

The equation (1.1) has been extensively studied in the SPDE literature. The strong existence and uniqueness (in a probability sense) to that equation has been shown by Gyöngy and Pardoux in [12] for bounded b and in [13]

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for some locally unbounded b. Later, in [2], the results were extended to the equations with the multiplicative noise. Note that in the above references the equations are defined for the spatial variable $z \in [0,1]$, but the results could be easily extended to our setting of $z \in \mathbb{R}$.

The strong uniqueness for (1.1) represents a phenomenon that is called "regularization by noise". This is the property that roughly speaking can be formulated as follows: deterministic equation without noise might not have uniqueness or existence property; however whenever the equation is perturbed by noise it has a unique solution, see the related discussion in a recent book of Flandoli [8]. This is the situation with (1.1): clearly one cannot make a general claim that equation (1.1) *without noise* at the right hand side has a unique solution whenever b is not Lipschitz, whereas, as we mentioned above, with the noise, uniqueness holds for a large class of drifts b. Note that whenever we say that there exists a unique strong solution to (1.1) we mean by this that on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathsf{P})$ there exists a unique *adapted* strong solution to that equation. That, in fact, implies that regularization by noise phenomenon happens in probability sense, as a regularization for Itô-Walsh stochastic equation.

On the other hand, one can ask the question whether the regularization effect takes place in a purely PDE setting. That is, one is interested whether it is possible to find a set $\Omega' \subset \Omega$ of full probability such that for almost every $\omega \in \Omega'$, given the path

$$(t,z) \mapsto V(t,z,\omega) := \int_0^t \int_{\mathbb{R}} p_{t-t'}(z-z') W(dt', dz', \omega)$$

(see the discussion in the beginning of Section 2 for the precise definition of V) equation (1.1) in the integral, or so-called, mild form (see equation (2.1) below), has a unique solution. Due to Flandoli's definition we will call the uniqueness of such kind the *path-by-path uniqueness*, see [8, Definition 1.5] and the discussion at [8, Section 1.3.3].

The problem of path-by-path uniqueness is interesting in itself. However it is closely related to another interesting question: existence and uniqueness of the flow of solutions indexed by initial conditions q of the equation. To the best of our knowledge, not much is known about existence and uniqueness of flows for SPDEs. Even if the drift and diffusion are very smooth functions, only the local flow property was established in [14, Corollary 1.10]. If the drift is Lipschitz and the diffusion coefficient is linear, the flow property was proved in [11]; see also [4] for related results. Linear systems were considered earlier by Flandoli [7]. We are not aware of any results in the literature concerning the case of non-Lipschitz coefficients; in the current paper we study an SPDE with a non–Lipschitz drift and an additive noise.

The question of regularization by noise for SDEs has been studied much more extensively. In particular, the following SDE has been thoroughly investigated:

$$dX_t = b(X_t)dt + dB_t, \tag{1.2}$$

where b is a measurable function and B is a d-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathsf{P})$. First, it was derived by Zvonkin in [27] for $d = 1$, that the above equation has a unique strong solution for a bounded measurable b. Then this result was generalized by Veretennikov in [23] for the multidimensional case, and later it was extended by Krylov and Röckner in [17] for the case of locally unbounded b under some integrability condition. The flow property of solutions to (1.2) was also established under essentially the same integrability condition, see [6], [9] and [26] for the case of non-constant diffusion coefficients. Note that the definition of stochastic flow in the above references requires that the solution $\{X_t, t \geqslant 0\}$ is adapted with respect to the filtration $\mathcal{F}_t$. In particular, the strong uniqueness, is by definition, the uniqueness among the adapted solutions. All the proofs use a Zvonkin-type transformation [27] that allows either to eliminate the "non-regular" drift or to make it more regular. For the related recent interesting works on flows of SDEs see also [19], [21].

If one asks the path-by-path uniqueness for (1.2) then the first result in this direction has been achieved by Davie in [5], who showed it for a fixed initial condition x. Later the result has been generalized by Shaposhnikov in [22], who established path-by-path uniqueness of solutions simultaneously for all initial conditions. Shaposhnikov also developed a new method that is based on the flow construction of Fedrizzi and Flandoli [6]. Recently the regularization by noise has been constructed also for equations driven by other types of noises, e.g. Lévy noises: see Priola [20], where Shaposhnikov's method is used. We would also like to mention a paper by Catellier and Gubinelli [3] where a number of very interesting results concerning regularization by noise and path–by–path uniqueness for ODEs were achieved. Recently, some results related to path–by–path uniqueness for Hilbert space–valued SDEs were obtained in [25] for fixed initial conditions. However the flow property is not obtained in that paper.

Now if we get back to our SPDE setting, we can say outright that we do not have a luxury of having a convenient Zvonkin-type transformation. That is why, we in a sense use the reverse argument: we first show path-by-path uniqueness together with some continuity with respect to initial conditions and based on this we show existence and uniqueness of the flow. To push the argument through we develop a new method that extends Davie's approach to the infinite-dimensional case. We believe that our method of proving existence of the flow is of independent interest.

In the next section we will present the main results of the paper.

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2. Main results. We study a one-dimensional stochastic heat equation on $\mathbb{R}$ with a drift (1.1). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathsf{P})$ be a probability space. Let $\dot{W}$ be a space-time white noise on this space adapted to the filtration. Let p be a standard heat kernel

$$p_t(z) = \frac{1}{\sqrt{2\pi t}} \exp(-z^2/2t), \quad t > 0,\ z \in \mathbb{R},$$

and V be a convolution of the heat kernel p with the white noise $\dot{W}(\cdot,\cdot,\omega)$, that is

$$V(s,t,z,\omega) := \int_s^t \int_{\mathbb{R}} p_{t-t'}(z-z')\, W(dt',dz'), \quad t \geqslant 0,\ s\in[0,t],\ z\in\mathbb{R}.$$

In case $s = 0$ for brevity we drop the first index and write $V(t,z,\omega) := V(0,t,z,\omega)$. Further we will frequently omit ω from the notation. Later on, in Lemma 4.7 we will show existence of a modification of V that is almost surely jointly continuous in (s,t,z); with some abuse of notation this modification will be denoted by the same symbol V. As usual, here and in the sequel we use the convention that $\int p_0(x-y)f(y)dy := f(x)$ for any measurable function f.

We say that a random function u solves (1.1) in the path–by–path sense, if $u(0,z) = q(z)$ and for P-almost surely ω the following holds for any $t > 0$, $z \in \mathbb{R}$

$$u(t,z,\omega) = \int_{\mathbb{R}} p_t(z-z')q(z')\, dz' + \int_0^t \int_{\mathbb{R}} p_{t-t'}(z-z') b(u(t',z',\omega))\, dz'dt' + V(0,t,z,\omega) \tag{2.1}$$

We will also consider a stochastic heat equation that starts with the initial condition q at time $s \geqslant 0$.

$$
\begin{aligned}
(2.2)\quad u(t,z,\omega) &= \int_{\mathbb{R}} p_{t-s}(z-z')q(z',\omega)dz' + \int_s^t \int_{\mathbb{R}} p_{t-t'}(z-z')b(u(t',z',\omega))dz'dt' \\
&\quad + V(s,t,z,\omega), \quad t > s,\ z \in \mathbb{R}, \\
u(s,z,\omega) &= q(z,\omega), \quad z \in \mathbb{R}.
\end{aligned}
$$

Sometimes, when there is an ambiguity, we denote a solution to (2.2) by $u_{s,q}(t,z,\omega)$, thus emphasizing the initial conditions. We see that for $s = 0$ (2.2) is just (2.1). We have to analyze $u_{s,q}$ for $s \geqslant 0$ (rather than just at $s = 0$) in order to prove the existence of the flow; see the proof of Theorem 2.2(a) below.

Note that the difference between the definition given above and the standard one (see, e.g., [16, Definition 6.3]) is that we do not require adaptiveness of the solution u to the filtration generated by $\dot{W}$. Instead of it, for each fixed $\omega \in \Omega$ we treat equation (2.2) separately as a deterministic PDE with a forcing term $V(s,\cdot,\cdot,\omega)$.

Let us now present the main results of the paper. First we define a class of functions that we take as initial conditions to (1.1).

Definition 2.1. 1) Let $\mu \geqslant 0$. We say that a measurable function $f\colon \mathbb{R} \to \mathbb{R}$ belongs to the class $\mathbf{B}(\mu)$, if there exists a constant $C > 0$ such that $|f(z)| \leqslant C(|z|^\mu \vee 1)$ for $z \in \mathbb{R}$.
2) We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ belongs to the class $\mathbf{B}(0+)$, if f belongs to the class $\mathbf{B}(\varepsilon)$ for all $\varepsilon > 0$.

For brevity, the class $\mathbf{B}(0)$ of measurable bounded functions on $\mathbb{R}$ will be denoted just by $\mathbf{B}$. If $f \in \mathbf{B}$ we define $\|f\|_\infty := \sup_{z\in\mathbb{R}} |f(z)|$.

Our first result proves existence and path–by–path uniqueness (see brief discussion on this concept in the introduction) of solutions to (1.1) on some "good" set of full probability measure simultaneously for **all** initial conditions in $\mathbf{B}(0+)$ and **all** starting times $s \geqslant 0$.

Theorem 2.1. *Let $b \in \mathbf{B}$. There exists a set $\Omega' = \Omega'(b) \subset \Omega$ with the following properties:*

1) $\mathsf{P}(\Omega') = 1$.
2) Let $\omega \in \Omega'$. Then for any initial condition $q \in \mathbf{B}(0+)$ and any $s \geqslant 0$ equation (2.2) *has a unique solution. This solution $u_{s,q}(t,\cdot) \in \mathbf{B}(0+)$ for any $t \geqslant s$.*

3) Let $\omega \in \Omega'$ and $q_1, q_2 \in \mathbf{B}(0+)$ be two initial conditions. If we have $q_1(z) = q_2(z)$ Lebesgue–almost everywhere in z, then $u_{s,q_1}(t, z, \omega) = u_{s,q_2}(t, z, \omega)$ for any $s \geqslant 0$, $t > s$, $z \in \mathbb{R}$.

Note that the class $\mathbf{B}(0+)$ of initial conditions (rather than, for example, $\mathbf{B}$) of initial conditions is chosen, since $u_{s,q}(t, \cdot) \in \mathbf{B}(0+)$. Thus, if one starts equation (2.2) from an initial condition in $\mathbf{B}(0+)$, then at any $t \geqslant 0$ the solution to this equation remains in the same class.

SKETCH OF THE PROOF OF THEOREM 2.1. The proof consists of three independent parts. First, in Section 4 we establish a number of useful regularity properties of V (on a certain "good" set) and prove that a certain auxiliary operator is continuous.

Then in Section 5.1 we prove existence of a solution to (2.2). Let $\mathbf{C}_0(\mathbb{R})$ be the space of all continuous functions $\mathbb{R} \to \mathbb{R}$ vanishing at infinity equipped with the standard sup-norm. Recall that it follows from Gyöngy and Pardoux [12] that for any $s \geqslant 0$, $q \in \mathbf{C}_0(\mathbb{R})$ there exists a set $\Omega_{s,q}$ of probability measure 1 such that on $\Omega_{s,q}$ equation (2.2) has a solution that starts with the initial condition q at time s. More precisely, although in [12] the equation is considered on $(t, z) \in \mathbb{R}_+ \times [0, 1]$, the methods that are used in that paper work exactly in the same way for $(t, z) \in \mathbb{R}_+ \times \mathbb{R}$. Our goal is to show that this "good" set $\Omega_{s,q}$ can be chosen to be the same for all $s \geqslant 0$, $q \in \mathbf{B}(0+)$.

To carry out this plan we fix a countable dense subset Ξ of $\mathbf{C}_0(\mathbb{R})$ and a countable dense subset Θ of $\mathbb{R}_+$. Since both Ξ and Θ are countable, we see that [12] implies that there exists a set Ω_E of probability measure 1 such that for any $\omega \in \Omega_E$, $s \in \Theta$, $q \in \Xi$ equation (2.2) has a solution that starts with the initial condition q at time s. Using continuity of a certain integral operator (Lemma 4.10), we will show that there exists a set of full measure $\Omega' \subset \Omega_E$ such that for any $\omega \in \Omega'$, $s \geqslant 0$, $q \in \mathbf{B}(0+)$ equation (2.2) has a solution that starts with the initial condition q at time s.

Finally, in Section 5.2 we prove uniqueness of a solution to (2.2). The proof extensively used smoothing properties of an integral operator involving white noise (Theorem 2.3 and Lemma 5.6). We develop a new approach motivated by the ideas of Davie [5]. □

The next theorem shows that there exists a unique flow of solutions to equation (1.1) and that this flow is continuous. We will see that this is a direct corollary of existence and path–by–path uniqueness of solutions to (1.1).

THEOREM 2.2. *Let $b \in \mathbf{B}$. Let $\Omega' \subset \Omega$ be from Theorem 2.1.*

a) [***Existence of the flow***] *There exists a mapping*

$$(s, t, q, \omega) \mapsto \varphi(s, t, q, \omega)$$

with values in $\mathbf{B}(0+)$ *defined for* $0 \leqslant s \leqslant t$, $q \in \mathbf{B}(0+)$, $\omega \in \Omega'$ *such that*

1. *For any* $s \geqslant 0$, $q \in \mathbf{B}(0+)$, $\omega \in \Omega'$ *the function* $u_{s,q}(t, \cdot) := \varphi(s, t, q, \omega)$ *is a unique solution to* (2.2) *that starts from the initial condition* q *at time* s*;*
2. *On* Ω' *we have for* $0 \leqslant r < s < t$

$$\varphi(r, t, q, \omega) = \varphi(s, t, \varphi(r, s, q, \omega), \omega);$$

b) [***Continuity of the flow***] *Let* φ *be the mapping defined in Part a) of the theorem. Let* $(q_n)_{n \in \mathbb{Z}_+}$ *be a sequence of functions from* $\mathbf{B}(0+)$*, that converges Lebesgue–almost everywhere to* $q \in \mathbf{B}(0+)$*. Assume that there exist constants* $C > 0$, $\mu > 0$ *such that for any* $n \in \mathbb{Z}_+$ *one has*

$$|q_n(z)| \leqslant C(|z|^\mu \vee 1), \quad z \in \mathbb{R}.$$

Then on Ω' *we have for* $0 \leqslant s < t$, $z \in \mathbb{R}$

$$\lim_{n \to \infty} \varphi(s, t, q_n, \omega)(z) = \varphi(s, t, q, \omega)(z).$$

Proof of Theorem 2.2(a). By Theorem 2.1, for any $\omega \in \Omega'$, $q \in \mathbf{B}(0+)$, $s \geqslant 0$ equation (2.2) has a unique solution $u_{s,q}$ that starts with initial condition q at time s.

Now for $0 \leqslant s \leqslant t$, $q \in \mathbf{B}(0+)$, $\omega \in \Omega'$ define

$$\varphi(s, t, q, \omega) := u_{s,q}(t, \cdot, \omega).$$

Let us check that φ satisfies all the properties of the flow formulated in Theorem 2.2(a). The first property is obvious. To check the second property we fix any $\omega \in \Omega'$, $0 \leqslant r < s$, $q \in \mathbf{B}(0+)$. For $t \geqslant s$ put $u_1(t, \cdot) := \varphi(r, t, q, \omega)$ and $u_2(t, \cdot) := \varphi(s, t, \varphi(r, s, q, \omega), \omega)$. Note that both u_1 and u_2 are solutions to equation (2.2) that starts with initial condition $\varphi(r, s, q, \omega)$ at time s. The initial condition $\varphi(r, s, q, \omega) = u_{r,q}(s, \cdot, \omega)$ is in $\mathbf{B}(0+)$ by Theorem 2.1. Therefore, by Theorem 2.1 the solutions u_1 and u_2 coincide. Thus,

$$\varphi(r, t, q, \omega) = \varphi(s, t, \varphi(r, s, q, \omega), \omega).$$

and φ is a flow of solutions to (2.2). □

The proof of Theorem 2.2(b) is given in Section 5.3.

The next theorem describes smoothing properties of the noise V that are crucial for the proof of Theorems 2.1 and 2.2. We are interested in the regularity properties of the mapping

$$(x,t,z)\mapsto \int_0^t b(V(r,z)+f(r,z)+x)\,dr, \tag{2.3}$$

where f belongs to a certain class of weighted Hölder functions with singularities defined below.

DEFINITION 2.2. 1) Let $h,\gamma\in[0,1]$, $T,M,\mu\geqslant 0$. We say that a measurable function $f\colon (0,T]\times\mathbb{R}\to\mathbb{R}$ is in the space $\mathcal{C}^h_{(0,T]}(\gamma,\mu,M)$ if

$$|f(t,z)-f(s,z)|\leqslant M|t-s|^h s^{-\gamma}(|z|^\mu\vee 1),\quad 0<s<t\leqslant T,\ z\in\mathbb{R}$$

and $|f(t,z)|\leqslant M(|z|^\mu\vee 1)$ for $z\in\mathbb{R}$, $t\in(0,T]$.

2) We say that a function $f\colon(0,T]\times\mathbb{R}\to\mathbb{R}$ is in the space $\mathcal{C}^{h-}_{(0,T]}(\gamma,0+)$ if for any $\varepsilon>0$ there exists $M>0$ such that $f\in\mathcal{C}^{h-\varepsilon}_{(0,T]}(\gamma,\varepsilon,M)$.

If there is no ambiguity in time interval, we will frequently drop the subscript $(0,T]$ and write $\mathcal{C}^h$ instead of $\mathcal{C}^h_{(0,T]}$.

THEOREM 2.3. *Let $b\in\mathbf{B}$. There exists a set $\Omega''=\Omega''(b)\subset\Omega$ with the following properties:*

1) $\mathsf{P}(\Omega'')=1$;
2) *Let $\omega\in\Omega''$. Then for any $0<\varepsilon<3/4$, $T>0$, $h\in(1/2,1]$, $M>0$ there exists a constant $K_b=K_b(b,\omega,\varepsilon,T,M,h)<\infty$ such that for any $\gamma\in[0,1]$, $\mu>0$, $f\in\mathcal{C}^h_{(0,T]}(\gamma,\mu,M)$, $x,y,z\in\mathbb{R}$, $0\leqslant t_1\leqslant t_2\leqslant T$, $s\in[0,T]$ we have*

$$\Big|\int_{t_1}^{t_2}(b(V(t+s,z)+f(t,z)+x)-b(V(t+s,z)+f(t,z)+y))\,dt\Big|$$
$$\leqslant K_b(\omega)|x-y|(t_2-t_1)^{1-(\frac{\gamma}{4h-1}\vee\frac14)-\varepsilon}(|x|\vee|y|\vee 1)^{1+\varepsilon}(|z|^{3\mu+\varepsilon}\vee 1). \tag{2.4}$$

Furthermore, $\mathsf{E}K_b\leqslant\|b\|_\infty C(\varepsilon,T,M,h)$ for some function C that does not depend on b.

If the function b were a Lipschitz function, then the left–hand side of (2.4) would be bounded by $|t_1-t_2||x-y|$. In our case, when b is just a bounded function, the left–hand side of (2.4) is obviously bounded by $|t_1 - t_2|$. Theorem 2.3 implies that one can trade the regularity in t to gain the regularity in x. In particular, we see from the above theorem that P–almost surely the function

$$x \mapsto \int_{t_1}^{t_2} b(V(t,z)+x)\,dt, \quad x \in \mathbb{R},$$

is Lipschitz in x. Moreover, we have very good local control on coefficients.

Sketch of the proof of Theorem 2.3. The proof is based on an application of a suitable version of Kolmogorov continuity theorem to a corresponding moment bound. This is done in Section 3. The calculation of the moment bound turned out to be rather complicated and it involves a number of technical estimates. We do it thoroughly in Section 6 utilizing some ideas from [3]. □

Remark 2.4. We would like to note that whilst the good set Ω' in Theorems 2.1 and 2.2 can be chosen independently of the initial condition q, Ω' as well as Ω'' from Theorem 2.3 might still depend on the drift function b.

It is interesting to compare smoothing properties of operator (2.3) to the smoothing properties of a similar operator with a Brownian motion B in place of V, see [8, Corollary 2.2] and also [5, Lemmas 3.1 and 3.2]. We see that since V in the time variable is less regular than the Brownian motion, Theorem 2.3 guarantees a better smoothing.

The function f appears in (2.3) due to the presence of the drift in our main equation (2.2). Note that in the original Davie's paper [5] the smoothing is considered without the drift term (this corresponds to the case $f \equiv 0$). That was possible due to the use of the Girsanov transformation for eliminating the drift. In other words, the "good" set Ω'' in [5] depends on the drift f and the initial condition. Since we are aimed at establishing the flow property for (2.2) we have to prove path–by–path uniqueness simultaneously for all initial conditions, see the proof of Theorem 2.2(a). Thus, we have to prove that smoothing in Theorem 2.3 occurs simultaneously for all drifts f and this cannot be achieved with Girsanov's transformation.

The rest of the paper is devoted to the proofs of the main results and is organized as follows. In Section 3 we prove Theorem 2.3. The proof of Theorem 2.1 is rather large and is split into two parts for the convenience of the reader. Namely, in Section 4 we establish a number of auxiliary lemmas and present the main part of the proof in Section 5. Theorem 2.2 is also proved in Section 5. Section 1 contains a proof of a global version of the Kolmogorov continuity theorem that is used in the paper. An important moment bound that is exploited for the proof of Theorem 2.3 is derived in Section 6. A technical lemma that is applied to prove smoothing properties of the noise is established in Section 7. Finally, a number of technical estimates concerning the Gaussian kernel and related functions are obtained in the Appendix.

Convention on constants. Throughout the paper, C denotes a positive constant whose value may change from line to line. K denotes a random constant whose value might depend on $\omega \in \Omega$.

3. Proof of Theorem 2.3.

We start proving the main results by presenting a proof of Theorem 2.3. First we give here a version of the Kolmogorov theorem on a noncompact set (global version) that will be extensively used in this and other proofs in the paper.

Define for $w=(w_1,w_2)\in\mathbb{R}^2$, $a=(a_1,a_2)\in(0,1]^2$ a weighted norm d_a

$$d_a(w) := |w_1|^{a_1} + |w_2|^{a_2}. \tag{3.1}$$

LEMMA 3.1 (Kolmogorov Continuity Theorem). *Let $X(x,y)$, $x\in\mathbb{R}$, $y\in\mathbb{R}^2$, be a continuous random field with values in $\mathbb{R}$. Assume that there exist nonnegative constants $a=(a_1,a_2)\in(0,1]^2$, α, β_1, β_2, C such that the inequalities*

$$\begin{aligned}\mathsf{E}|X(x_1,y_1)-X(x_1,y_2)-X(x_2,y_1)+X(x_2,y_2)|^\alpha \\ \leqslant C|x_1-x_2|^{\beta_1} d_a(y_1-y_2)^{\beta_2},\end{aligned}$$

$$\mathsf{E}|X(x_1,y_1)-X(x_2,y_1)|^\alpha \leqslant C|x_1-x_2|^{\beta_1} \tag{3.2}$$

hold for any $x_1,x_2\in\mathbb{R}$, $y_1,y_2\in\mathbb{R}^2$, $|x_1-x_2|\leqslant 1$, $|y_1-y_2|\leqslant 1$.

Then for any $\gamma_1\in(0,(\beta_1-1)/\alpha)$ and $\gamma_2\in(0,(\beta_2-1/a_1-1/a_2)/\alpha)$ there exist a set $\Omega^*\subset\Omega$ with $\mathsf{P}(\Omega^*)=1$ and a random variable K with $\mathsf{E}K(\omega)^\alpha\leqslant C_1$ such that for any $\omega\in\Omega'$, $x_1,x_2\in\mathbb{R}$, $y_1,y_2\in\mathbb{R}^2$, $|x_1-x_2|\leqslant 1$, $|y_1-y_2|\leqslant 1$ we have

$$\begin{aligned}|X(x_1,y_1)-X(x_1,y_2)-X(x_2,y_1)+X(x_2,y_2)| \\ \leqslant K(\omega)(|x_1|\vee|y_1|\vee 1)^{3/\alpha}|x_1-x_2|^{\gamma_1} d_a(y_1-y_2)^{\gamma_2},\end{aligned} \tag{3.3}$$

and

$$|X(x_1,y_1)-X(x_2,y_1)| \leqslant K(\omega)(|x_1|\vee|y_1|\vee 1)^{3/\alpha}|x_1-x_2|^{\gamma_1}, \tag{3.4}$$

where the constant $C_1>0$ depends on the field X only through a, α, β_i, γ_i, C.

The proof of the lemma is based on a local version of the Kolmogorov continuity theorem and is given in the supplemental material [SM, Section 1].

The proof of Theorem 2.3 is based on the above mentioned version of the Kolmogorov theorem and the following moment bound.

PROPOSITION 3.2. *Let $b\colon \mathbb{R}\to\mathbb{R}$ be a bounded differentiable function with bounded derivative. Then for any $0\leqslant t_1\leqslant t_2\leqslant T$, $z, z_1, z_2, x, y\in\mathbb{R}$, $|z_1-z_2|\leqslant 1$, $\delta\in(0,1)$, $\delta'\in(0,\delta)$, $p>1$ we have*

$$\mathsf{E}\Big|\int_{t_1}^{t_2}\big(b'(V(t,z_1)+x)-b'(V(t,z_2)+y)\big)\,dt\Big|^p \leqslant C(t_2-t_1)^{p(3/4-\delta/4)}(|z_1-z_2|^{p\delta'/2}+|x-y|^{p\delta}); \tag{3.5}$$

$$\mathsf{E}\Big|\int_{t_1}^{t_2} b'(V(t,z))\,dt\Big|^p \leqslant C(t_2-t_1)^{p(3/4-\delta)}. \tag{3.6}$$

for some constant $C=C(p,T,\delta,\delta',\|b\|_\infty)>0$.

It is important to stress that the constant C from Proposition 3.2 depends only on $\|b\|_\infty$ but not on the function b itself and not on its derivative. The proof of Proposition 3.2 is postponed to Section 6.

Finally we need a technical estimate.

LEMMA 3.3. *Let $U\subset\mathbb{R}$ and assume that U has a Lebesgue measure 0. Then there exists a set $\Omega(U)\subset\Omega$ such that $\mathsf{P}(\Omega(U))=1$ and for any $\omega\in\Omega(U)$, $h>1/2$, $T>0$, $M>0$, $\mu>0$, $f\in\mathcal{C}^h_{(0,T]}(1,\mu,M)$, $z\in\mathbb{R}$, $s\in[0,T]$ we have*

$$\int_0^T \mathbb{1}_U(V(t+s,z,\omega)+f(t,z))\,dt=0.$$

This lemma is proved in Section 7.

PROOF OF THE THEOREM 2.3. First we consider the case when b is a bounded differentiable function with a continuous bounded derivative and $\|b\|_\infty=1$. In this case, we apply a version of the Kolmogorov continuity theorem (Lemma 3.1) to the random field

$$X(t,(z,x)):=\int_0^t b'(V(r,z)+x)\,dr.$$

Fix arbitrary $T > 0$. It follows from Proposition 3.2 and Lemma 3.1 that for any $\delta \in (0,1)$, $\varepsilon > 0$ there exist a set $\Omega_{T,\delta,\varepsilon} \subset \Omega$ with $\mathsf{P}(\Omega_{T,\delta,\varepsilon}) = 1$ and a random variable $K(\omega)$ such that for all $z_1, z_2, x, y \in \mathbb{R}$ with $|z_1 - z_2| + |x - y| \leqslant 1$ and $0 \leqslant t_1 \leqslant t_2 \leqslant 2T$, $\omega \in \Omega_{T,\delta,\varepsilon}$ one has

$$\Big|\int_{t_1}^{t_2} \big(b'(V(t,z_1)+x) - b'(V(t,z_2)+y)\big)\,dt\Big| \leqslant K(\omega)(t_2-t_1)^{(3/4-\delta/4-\varepsilon)}(|z_1-z_2|^{\delta/2} + |x-y|^{\delta})^{1-\varepsilon} \times (|x|^{\varepsilon} \vee |y|^{\varepsilon} \vee 1)(|z_1|^{\varepsilon} \vee |z_2|^{\varepsilon} \vee 1)$$

and $\mathsf{E}K \leqslant C$, where the constant $C = C(T,\delta,\varepsilon)$ does not depend on the function b.

We apply now the above inequality to $z_1 = z_2 = z$ and arbitrary $x, y \in \mathbb{R}$. That is, if $|x-y| \leqslant N$ we apply the above inequality N times. We get that on $\Omega_{T,\delta,\varepsilon}$ for all $x, y, z \in \mathbb{R}$, $0 \leqslant t_1 \leqslant t_2 \leqslant T$, $s \in [0,T]$

$$\Big|\int_{t_1}^{t_2} \big(b'(V(t+s,z)+x) - b'(V(t+s,z)+y)\big)\,dt\Big| \leqslant K_1(\omega)(|x|^{1+\varepsilon} \vee |y|^{1+\varepsilon} \vee 1)(|z|^{\varepsilon} \vee 1)(t_2-t_1)^{3/4-\delta/4-\varepsilon}|x-y|^{\delta(1-\varepsilon)}, \tag{3.7}$$

where we have also applied change of variables $t \to t+s$ in the integral. Here $\mathsf{E}K_1 \leqslant C = C(T,\delta,\varepsilon)$.

In a similar way, inequality (3.4) and Proposition 3.2 yield that for any $\varepsilon \in (0, 3/4)$ there exists a set $\widetilde{\Omega}_{T,\varepsilon}$ and a random variable $K_2(\omega)$ such that for any $\omega \in \widetilde{\Omega}_{T,\varepsilon}$, $z \in \mathbb{R}$, $0 \leqslant t_1 \leqslant t_2 \leqslant T$, $s \in [0,T]$ we have

$$\Big|\int_{t_1}^{t_2} b'(V(t+s,z))\,dt\Big| \leqslant K_2(\omega)(|z|^{\varepsilon} \vee 1)(t_2-t_1)^{3/4-\varepsilon}. \tag{3.8}$$

Again, $\mathsf{E}K_2 \leqslant C = C(T,\varepsilon)$.

This allows us to proceed to the next step. Fix $M > 0$ and take any function $f \in \mathcal{C}^h_{(0,T]}(\gamma,\mu,M)$. Fix $0 \leqslant t_1 \leqslant t_2 \leqslant T$. Consider the following binary partition of the interval $[t_1,t_2]$:

$$t_n^i := t_1 + (t_2-t_1)i2^{-n}, \quad n \in \mathbb{Z}_+,\ i = 0,1,\dots 2^n.$$

Let f_n be the following piecewise-constant approximation of f:

$$f_n(t,z) := \sum_{i=0}^{2^n-1} \mathbb{1}(t \in (t_n^i, t_n^{i+1}])f(t_n^{i+1},z), \quad t \in [t_1,t_2],\ z \in \mathbb{R}.$$

Clearly, the sequence of functions f_n converges pointwise to f on $[t_1, t_2]$. Thus, for arbitrary $s \in [0, T]$, $\omega \in \Omega$, $x, y, z \in \mathbb{R}$ we derive

$$\begin{aligned}(3.9)\quad \Big|\int_{t_1}^{t_2} \big(b(V(t+s,z)+f(t,z)+x) - b(V(t+s,z)+f(t,z)+y)\big)\,dt\Big| \\ = \Big|\int_x^y\int_{t_1}^{t_2} b'(V(t+s,z)+f(t,z)+r)\,dtdr\Big| \\ \leqslant \Big|\int_x^y\int_{t_1}^{t_2} b'(V(t+s,z)+f_0(t,z)+r)\,dtdr\Big| + \sum_{k=0}^{\infty} J(k,t_1,t_2) \\ =: I + \sum_{k=0}^{\infty} J(k,t_1,t_2),\end{aligned}$$

where in the first identity we have used Fubini's theorem (recall that we have assumed boundedness of the function b') and denoted for $k \in \mathbb{Z}_+$ and $l_1, l_2 \in [t_1, t_2]$

$$\begin{aligned}&J(k, l_1, l_2)\\ &\quad := \Big|\int_x^y\int_{l_1}^{l_2}\big(b'(V(t+s,z)+f_{k+1}(t,z)+r) - b'(V(t+s,z)+f_k(t,z)+r)\big)\,dtdr\Big|.\end{aligned}$$

It turns out that we need to apply (3.7) with different δ to estimate I and $J(k, t_1, t_2)$. Since, by definition, the function f_0 is constant on $[t_1, t_2]$, we see that (3.7) with $\delta = \varepsilon$ together with (3.8) yield for $\omega \in \widetilde{\Omega}_{T,\varepsilon} \cap \Omega_{T,\varepsilon,\varepsilon}$

$$\begin{aligned}(3.10)\quad I \leqslant &\Big|\int_x^y\int_{t_1}^{t_2} b'(V(t+s,z)+f_0(t,z)+r) - b'(V(t+s,z))\,dtdr\Big| \\ &+ \Big|\int_x^y\int_{t_1}^{t_2} b'(V(t+s,z))\,dtdr\Big| \\ \leqslant &K_3(\omega)(|x|\vee|y|\vee 1)^{1+2\varepsilon}(|z|^{2\mu+\varepsilon}\vee 1)(t_2-t_1)^{3/4-2\varepsilon}|x-y|\end{aligned}$$

and $\mathsf{E}K_3 \leqslant C(T, \varepsilon, M)$.

Recall that each function f_k is a piecewise constant function in t. Therefore to estimate $J(k, t_1, t_2)$ we split the integral over $[t_1, t_2]$ into integrals over intervals $(t_k^i, t_{k+1}^{2i+1}]$, $i = 0, 1, \ldots, 2^k{-}1$, where f_k and f_{k+1} are constant in t, and apply estimate (3.7) to each of these integrals. Note that $f_k = f_{k+1}$ on the complement of the union of these intervals. Thus, for any $k \in \mathbb{Z}_+$,

$i = 0, 1, \dots, 2^k–1$ we obtain on $\Omega_{T,\delta,\varepsilon}$

$$
\begin{aligned}
&J(k, t_k^i, t_k^{i+1})\\
&\quad\leqslant J(k, t_k^i, t_{k+1}^{2i+1}) + J(k, t_{k+1}^{2i+1}, t_k^{i+1})\\
&\quad=\Big|\int_x^y\int_{t_k^i}^{t_{k+1}^{2i+1}}\big(b'(V(t+s, z)+f(t_{k+1}^{2i+1},z)+r)-b'(V(t+s, z)+f(t_k^{i+1},z)+r)\big)\,dtdr\Big|\\
&\qquad+\Big|\int_x^y\int_{t_{k+1}^{2i+1}}^{t_k^{i+1}}\big(b'(V(t+s, z)+f(t_k^{i+1},z)+r)-b'(V(t+s, z)+f(t_k^{i+1},z)+r)\big)\,dtdr\Big|\\
&\quad\leqslant CK_1(\omega)(|x|\vee|y|\vee 1)^{1+\varepsilon}(|z|^{2\mu+\varepsilon}\vee 1)(t_2- t_1)^{3/4-\delta/4-\varepsilon}|x-y|\\
&\qquad\times 2^{-(3/4-\delta/4-\varepsilon)k}|f(t_{k+1}^{2i+1}, z) - f_k^{i+1}, z)|^{\delta-\varepsilon}\\
&\quad\leqslant CK_1(\omega)(|x|\vee|y|\vee 1)^{1+\varepsilon}(|z|^{3\mu+\varepsilon}\vee 1)(t_2- t_1)^{3/4+\delta(h-1/4-\gamma)-2\varepsilon}|x-y|\\
&\qquad\times 2^{-k(3/4-\delta(1/4-h+\gamma)-3\varepsilon)}(i + 1/2)^{-\gamma\delta},
\end{aligned}
$$

where in the last line we used the fact that $f \in \mathcal{C}^h_{(0,T]}(\gamma, \mu, M)$ and the constant $C = C(T, \varepsilon, M)$ does not depend on i and k. By summing the obtained inequality over all $k \in \mathbb{Z}_+$ and $i \in [0, 2^k - 1]$ we deduce on $\Omega_{T,\delta,\varepsilon}$

$$
\begin{aligned}
(3.11)\quad \sum_{k=0}^{\infty}J(k, t_1, t_2) &= \sum_{k=0}^{\infty}\sum_{i=0}^{2^k-1} J(k, t_k^i, t_k^{i+1})\\
&\leqslant CK_1(\omega)(|x|\vee|y|\vee 1)^{1+\varepsilon}(|z|^{3\mu+\varepsilon}\vee 1)(t_2- t_1)^{3/4+\delta(h-1/4-\gamma)-2\varepsilon}|x-y|\\
&\quad\times\sum_{k=0}^{\infty}2^{k(1/4-\delta(h-1/4)+3\varepsilon)}.
\end{aligned}
$$

Again, $C = C(T, \delta, \varepsilon, M)$. We see that in order for the sum in the right–hand side of (3.11) to be convergent we must necessarily have

$$\delta(h - 1/4) > 1/4 + 3\varepsilon.$$

We must also have $\delta < 1$. Recall that by assumption of the theorem $h > 1/2$. Thus one can take $\delta := 1/(4h - 1) + 24\varepsilon$. If, additionally, $\gamma \leqslant h - 1/4$, then combining (3.9), (3.10), (3.11) we finally obtain for $\varepsilon > 0$ on $\Omega^*_{T,\varepsilon,h} := \widetilde{\Omega}_{T,\varepsilon} \cap \Omega_{T,\varepsilon,\varepsilon} \cap \Omega_{T,1/(4h-1)+24\varepsilon,\varepsilon}$

$$
\begin{aligned}
&\Big|\int_{t_1}^{t_2}\big(b(V(t + s, z) + f(t, z) + x) - b(V(t + s, z) + f(t, z) + y)\big)\,dt\Big|\\
&\qquad\qquad\leqslant K_4(\omega)|x - y|(|x| \vee |y| \vee 1)^{1+2\varepsilon}(|z|^{3\mu+\varepsilon} \vee 1)(t_2 - t_1)^{3/4-2\varepsilon}.
\end{aligned}
$$

In case $\gamma > h - 1/4$, we obtain on $\Omega^*_{T,\varepsilon,h}$

$$\Big|\int_{t_1}^{t_2} \big(b(V(t+s,z)+f(t,z)+x) - b(V(t+s,z)+f(t,z)+y)\big)\,dt\Big| \\ \leqslant K_4(\omega)|x-y|(|x|\vee|y|\vee 1)^{1+2\varepsilon}(|z|^{3\mu+\varepsilon}\vee 1)(t_2-t_1)^{1-\gamma/(4h-1)-24\varepsilon}.$$

Note that in both cases we have $\mathsf{E}K_4(\omega) \leqslant C(T,\varepsilon,M,h)$. Now we set $\Omega^* := \cap\, \Omega^*_{T,\varepsilon,h}$ where the intersection is taken over all rational $T>0$, $\varepsilon>0$, $h \in (1/2,1]$. We see that on Ω^* the statement of the theorem holds. This concludes the proof of the theorem for the case where b is a bounded differentiable function with a continuous bounded derivative and $\|b\|_\infty = 1$.

If $\|b\|_\infty = 0$, then there is nothing to prove. If b is a bounded differentiable function with a continuous bounded derivative but $\|b\|_\infty \neq 1$, $\|b\|_\infty > 0$, then the statement of the theorem also holds. Indeed, we can renormalize b and consider $b_1(x) := b(x)/\|b\|_\infty$.

Finally, to prove the theorem in the general case (for bounded but not necessarily differentiable b with arbitrarily $\|b\|_\infty$) we use approximations. It follows from Lusin's theorem that there exists a sequence $(b_n)_{n\in\mathbb{Z}_+}$ of bounded differentiable function with continuous bounded derivatives such that

$$\lim_{n\to\infty} b_n(x) = b(x) \ \text{ Lebesgue-almost everywhere in } x;\ x \in \mathbb{R}$$

and $\sup_n \|b_n\|_\infty \leqslant 2\|b\|_\infty$. Put $U := \{x \in \mathbb{R}\colon \lim_{n\to\infty} b_n(x) \neq b(x)\}$, $\widetilde{b}(x) := \lim_{n\to\infty} b_n(x)$. We see that the set U is of Lebesgue measure 0.

Let Ω_n be the "good" set for the function b_n (i.e., the set such that the statement of the theorem is satisfied for the function b_n). By above, $\mathsf{P}(\Omega_n) = 1$. Take

$$\Omega_\infty := \bigcap_{n=1}^{\infty} \Omega_n \cap \Omega(U),$$

where the set $\Omega(U)$ is defined in Lemma 3.3. Clearly, $\mathsf{P}(\Omega_\infty) = 1$. Take arbitrary $T > 0$, $M > 0$, $h > 1/2$, $0 < \varepsilon < 3/4$. By the dominated convergence theorem and Lemma 3.3 we have on Ω_∞ for any $\gamma \in [0,1]$, $\mu > 0$,

$f \in \mathcal{C}^h_{(0,T]}(\gamma,\mu,M)$, $x,y,z \in \mathbb{R}$, $0 \leqslant t_1 \leqslant t_2 \leqslant T$, $s \in [0,T]$

$$\begin{aligned}
&\Big|\int_{t_1}^{t_2} \big(b(V(t+s,z)+f(t,z)+x) - b(V(t+s,z)+f(t,z)+y)\big)\,dt\Big|\\
&\quad\leqslant \Big|\int_{t_1}^{t_2} \big(\widetilde{b}(V(t+s,z)+f(t,z)+x) - \widetilde{b}(V(t+s,z)+f(t,z)+y)\big)\,dt\Big|\\
&\qquad + 3\int_{t_1}^{t_2} \mathbb{1}_U(V(t+s,z)+f(t,z)+x)\,dt\\
&\qquad + 3\int_{t_1}^{t_2} \mathbb{1}_U(V(t+s,z)+f(t,z)+y)\,dt\\
&\quad= \liminf_{n\to\infty}\Big|\int_{t_1}^{t_2} \big(b_n(V(t+s,z)+f(t,z)+x) - b_n(V(t+s,z)+f(t,z)+y)\big)\,dt\Big|\\
&\quad\leqslant |x-y|(|x|\vee|y|\vee 1)^{1+\varepsilon}(|z|^{3\mu+\varepsilon}\vee 1)(t_2-t_1)^{1-1/4(\frac{\gamma}{h-1/4}\vee 1)-\varepsilon}\liminf_{n\to\infty} K_{b_n}(\omega),
\end{aligned}$$

where K_{b_n} is the corresponding constant from Theorem 2.3. For $\omega \in \Omega_\infty$ put $K_b(\omega) := \liminf_{n\to\infty} K_{b_n}(\omega)$. By Fatou's lemma,

$$\mathsf{E}K_b(\omega) \leqslant \liminf_{n\to\infty} \mathsf{E}K_{b_n}(\omega) \leqslant C(\varepsilon,T,M,h)\sup_n \|b_n\|_\infty.$$

Thus the random variable $K_b(\omega)$ has a finite expectation. Hence there exists a set $\Omega'' \subset \Omega_\infty$ such that $\mathsf{P}(\Omega'') = 1$ and on Ω'' we have $K_b(\omega) < \infty$. This together with the above estimate concludes the proof of the theorem. □

4. Preparation steps for proving Theorem 2.1. In this section we prepare for the proof of our main result, that is, Theorem 2.1. In particular, we will select a specific "good" set Ω' of full probability measure and in the next section we will prove that equation (2.2) indeed has a unique solution on Ω'.

First we need to introduce approximation operator in the following way. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. We define a piecewise-constant approximation of f as follows. For $n \in \mathbb{Z}_+$ put

$$\lambda_n(f)(t) := \sum_{i=0}^{2^n-1} \mathbb{1}_{(i2^{-n},(i+1)2^{-n}]}(t) f((i+1)2^{-n}). \tag{4.1}$$

In other words, $\lambda_n(f)$ is a piecewise-constant function that takes constant values on intervals of length 2^{-n}.

Many times in the proofs of the theorems it will be convenient to work with a shifted solution to (2.2). Thus, we define

$$u^*_{s,q}(t,\cdot) := u_{s,q}(t+s,\cdot), \quad t \geqslant 0, \tag{4.2}$$

where we recall that $u_{s,q}$ stands for the solution to (2.2) that starts from the initial condition q at times s. It is easy to see that $u^*_{s,q}$ satisfies the following equation for any $t > 0$, $z \in \mathbb{R}$

$$\begin{aligned} (4.3)\quad u^*_{s,q}(t,z,\omega) &= \int_{\mathbb{R}} p_t(z-z')q(z',\omega)\,dz' \\ &\quad + \int_0^t\int_{\mathbb{R}} p_{t-t'}(z-z')b(u^*_{s,q}(t',z',\omega))\,dz'dt' + V(s,t+s,z,\omega), \\ u^*_{s,q}(0,z,\omega) &= q(z,\omega). \end{aligned}$$

REMARK 4.1. Clearly, equation (2.2) has a unique solution if and only if equation (4.3) has a unique solution.

We introduce also the notation for the difference between two Gaussian kernels by setting

$$(4.4)\qquad \Delta p_t(z_1,z_2) := p_t(z_1) - p_t(z_2), \quad t>0,\ z_1,z_2\in\mathbb{R}.$$

Further, we will need to consider weighted norms. So for $\delta \geqslant 0$ we define weight function

$$(4.5)\qquad \Lambda_\delta(x) := e^x x^\delta, \quad x \geqslant 0.$$

Consider also a class of globally Lipschitz functions.

DEFINITION 4.1. We say that a function $f \in \mathbf{B}$ belongs to the class $\mathcal{C}^b_{Lip}$, if there exists a constant $C > 0$ such that $|f(z_1) - f(z_2)| \leqslant C|z_1 - z_2|$ for any $z_1, z_2 \in \mathbb{R}$.

Finally, we will also need the following process:

$$(4.6)\quad \mathcal{V}(r,s,t,z) := \int_r^s \int_{\mathbb{R}} p_{t-t'}(z-z')\,W(dt',dz'), \quad 0\leqslant r\leqslant s\leqslant t,\ z\in\mathbb{R}.$$

We see that, by definition, $V(s,t,z) = \mathcal{V}(s,t,t,z)$.

4.1 Estimates involving Gaussian density. First, let us give a number of very simple lemmas involving the Gaussian kernel. Their proofs are standard; for the sake of completeness we provide their proofs in the supplemental material [SM, Section 2].

LEMMA 4.2. *For any $\delta_1, \delta_2 \in [0,1]$ there exists $C = C(\delta_1, \delta_2)$ such that for any $a_1, a_2 \in \mathbb{R}$, $t > 0$ we have the following bounds:*

$$\int_{\mathbb{R}} |p_t(x+a_1) - p_t(x)|\, dx \leqslant C|a_1|^{\delta_1} t^{-\delta_1/2}; \tag{4.7}$$

$$\int_{\mathbb{R}} |\frac{\partial p_t}{\partial x}(x+a_1) - \frac{\partial p_t}{\partial x}(x)|\, dx \leqslant C|a_1|^{\delta_1} t^{-(1+\delta_1)/2}; \tag{4.8}$$

$$\begin{aligned}\int_{\mathbb{R}} |\frac{\partial p_t}{\partial x}(x+a_1+a_2) - \frac{\partial p_t}{\partial x}(x+a_1) - \frac{\partial p_t}{\partial x}(x+a_2) + \frac{\partial p_t}{\partial x}(x)|\, dx \\ \leqslant C|a_1|^{\delta_1}|a_2|^{\delta_2} t^{-(1+\delta_1+\delta_2)/2}.\end{aligned} \tag{4.9}$$

LEMMA 4.3. *For any $T > 0$, $\delta \geqslant 0$ there exists $C = C(T, \delta) > 0$ such that for any $s, t \in [0, T]$, we have*

$$\int_{\mathbb{R}} |p_t(z) - p_s(z)|\ (|z|^\delta \vee 1)\, dz \leqslant C|\log t - \log s|.$$

LEMMA 4.4. *For any $\delta \in (0, 1/6)$, $T > 0$ there exists $C = C(T, \delta) > 0$ such that for any $0 < t_1 < t_2 < t$, $z_1, z_2, z \in \mathbb{R}$ we have*

$$\int_{\mathbb{R}} p_t(z-z')\Lambda_\delta(|z'| \vee 1) dz' \leqslant C\Lambda_\delta(|z| \vee 1); \tag{4.10}$$

$$\begin{aligned}\int_{\mathbb{R}} \int_{t_1}^{t_2} \Big|\frac{\partial}{\partial t'} p_{t-t'}(z-z')\Big|(t_2-t')^{2/3-\delta}\Lambda_\delta(|z'| \vee 1)\, dt' dz' \\ \leqslant C|t_2-t_1|^{2/3-\delta}\Lambda_\delta(|z| \vee 1);\end{aligned} \tag{4.11}$$

$$\begin{aligned}\int_{\mathbb{R}} |p_t(z_1-z') - p_t(z_2-z')|\Lambda_\delta(|z'| \vee 1) dz' \\ \leqslant Ct^{-1/2}|z_1-z_2|\Lambda_\delta(|z_1| \vee |z_2| \vee 1);\end{aligned} \tag{4.12}$$

$$\begin{aligned}\int_{\mathbb{R}}\int_{t_1}^{t_2} \Big|\frac{\partial}{\partial t'}(p_{t-t'}(z_1-z') - p_{t-t'}(z_2-z'))\Big|(t_2-t')^{2/3-\delta}\Lambda_\delta(|z'| \vee 1)\, dt' dz' \\ \leqslant C(t_2-t_1)^{2/3-\delta}(t-t_1)^{-1/2}|z_1-z_2|\Lambda_\delta(|z_1| \vee |z_2| \vee 1).\end{aligned} \tag{4.13}$$

LEMMA 4.5. *Let $f\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Define*

$$h(t,z) := \int_0^t \int_{\mathbb{R}} p_{t-t'}(z-z') f(t',z')\, dz'\, dt', \quad z \in \mathbb{R},\ t \geqslant 0$$

Then for any $T > 0$, $\delta > 0$, there exists a constant $C = C(T, \delta)$ such that for any $t_1, t_2 \in [0, T]$, $z_1, z_2 \in \mathbb{R}$ we have

$$|h(t_1, z_1) - h(t_2, z_2)| \leqslant C\|f\|_\infty(|z_1-z_2| + |t_1-t_2|^{1-\delta}). \tag{4.14}$$

LEMMA 4.6. *Let $\mu \geqslant 0$ and let $q \in \mathbf{B}(\mu)$, that is, for some $M > 0$ we have $|q(z)| \leqslant M(|z|^\mu \vee 1)$, $z \in \mathbb{R}$. Then for any $T > 0$ there exists a constant $C = C(T,\mu)$ such that function*

$$h(t,z) := \int_{\mathbb{R}} p_t(z-z')q(z')\,dz' \quad z \in \mathbb{R},\, t \in [0,T]$$

belongs to the class $\mathcal{C}^1_{(0,T]}(1,\mu,CM)$. The constant C does not depend on the function q.

Further, if $q \in \mathcal{C}^b_{Lip}$, then there exists a constant $C_1 = C_1(T,q)$ such that for any $t_1,t_2 \in [0,T]$, $z_1,z_2 \in \mathbb{R}$ we have

$$|h(t_1,z_1) - h(t_2,z_2)| \leqslant C_1(|z_1 - z_2| + |t_1 - t_2|^{1/2}).$$

4.2 Existence of a regular version of V and its properties. The next lemma establishes the global regularity properties of the noise process $\mathcal{V}$ (recall the definition of $\mathcal{V}$ given in (4.6)). The proof of the lemma is technical and follows the usual line of argument. We provide it in the supplemental material [SM, Section 3].

LEMMA 4.7. *There exists a set $\Omega_V \subset \Omega$ with the following properties:*

1) $\mathsf{P}(\Omega_V) = 1$;

2) Let $\omega \in \Omega_V$. Then the functions $(s,t,z) \mapsto V(s,t,z,\omega)$ and $(r,s,t,z) \mapsto \mathcal{V}(r,s,t,z,\omega)$ are continuous. Furthermore, for any $T > 0$, $\varepsilon \in (0,1/2)$, $p > 0$ there exists $K(\omega) = K(\omega,p,T,\varepsilon)$ such that for any $0 \leqslant s \leqslant t \leqslant T$, $0 \leqslant s < t_1 < t_2 \leqslant T$, $z,z_1,z_2 \in \mathbb{R}$ we have

$$|\mathcal{V}(0,s,t,z_1) - \mathcal{V}(0,s,t,z_2)| \leqslant K(\omega)|z_1 - z_2|^{1/2-\varepsilon}(|z_1|^{2\varepsilon} \vee |z_2|^{2\varepsilon} \vee 1); \tag{4.15}$$

$$|\mathcal{V}(0,s,t_1,z) - \mathcal{V}(0,s,t_2,z)| \leqslant K(\omega)|t_1 - t_2|(t_1 - s)^{-1}(|z|^\varepsilon \vee 1); \tag{4.16}$$

$$|V(s,t,z)| \leqslant K(\omega)(|z|^\varepsilon \vee 1). \tag{4.17}$$

Moreover, $\omega \mapsto K(\omega)$ is a random variable with $\mathsf{E}K(\omega)^p < \infty$.

Let us emphasize that the result is of course not surprising: it is well-known that the convolution of the white noise with the heat kernel is *locally* Hölder $(1/2 - \varepsilon)$ continuous in space and Hölder $(1/4 - \varepsilon)$ continuous in time (see, e.g., [16, Exercise 6.9]). The lemma gives uniform *global* control on Hölder coefficients. As one can expect, the price to pay is that Hölder coefficients are no longer bounded but grow as $|z|^\varepsilon$ if $z \to \infty$ (in other words, the convolution of the white noise with the heat kernel is not a *globally* Hölder function).

The next lemma provides some useful simple bounds for the variance of $\mathcal{V}$. Its proof is very standard and straightforward; thus we also put it in the supplemental material [SM, Section 3].

LEMMA 4.8. *For any $\delta \in [0,1]$ there exists a constant $C = C(\delta) > 0$ such that for any $0 \leqslant r < s \leqslant t$, $z, z_1, z_2 \in \mathbb{R}$ we have*

$$\operatorname{Var} \mathcal{V}(r,s,t,z) \leqslant C(s-r)(t-r)^{-1/2}; \tag{4.18}$$

$$\operatorname{Var}\big(\mathcal{V}(r,s,t,z_1) - \mathcal{V}(r,s,t,z_2)\big) \leqslant C|z_1 - z_2|^{\delta}(s-r)(t-s)^{-1/2-\delta/2}; \tag{4.19}$$

$$\operatorname{Var}\big(\mathcal{V}(r,s,t,z_1) - \mathcal{V}(r,s,t,z_2)\big) \leqslant C|z_1 - z_2|. \tag{4.20}$$

4.3 Continuity lemmas. As we mentioned before, in order to prove Theorem 2.1 we approximate the drift in equation (2.2) by a sequence of piecewise continuous functions and pass to the limit (see the proof of Lemma 5.9 below). If the function b were continuous, this would not require any additional clarifications. However in our case when we assume that b is just a measurable bounded function we need to explain why the passage to the limit is justified here.

Recall the definition of the approximation operator λ_n given in (4.1).

LEMMA 4.9. *For any $\varepsilon > 0$, $M > 0$, $h > 1/2$, $N \in \mathbb{N}$, $T > 0$, $\mu > 0$ there exists $\delta > 0$ such that for each open set $U \subset \mathbb{R}$ with $|U| \leqslant \delta$ we have with probability greater or equal than $1 - \varepsilon$*

$$\int_0^T \mathbb{1}_U(V(t+s,z,\omega) + f_1(t,z) + \lambda_r(f_2)(t,z))\, dt \leqslant \varepsilon. \tag{4.21}$$

simultaneously for all $z \in [-N,N]$, $r \in \mathbb{N}$, $s \in [0,T]$ and all $f_1, f_2 \in \mathcal{C}^h_{(0,T]}(1,\mu,M)$.

The proof of the lemma is given in Section 7.

LEMMA 4.10. *Let $b \in \mathbf{B}$. Then there exists a set $\Omega_C \subset \Omega$ with the following properties:*

1) $\mathsf{P}(\Omega_C) = 1$;
2) *Let $\omega \in \Omega_C$. Then for any $T > 0$, $h > 1/2$, $M > 0$, $\mu > 0$, $0 \leqslant t_1 \leqslant t_2 \leqslant T$, $z \in \mathbb{R}$, $\theta \in \mathbf{B}$, function $\psi \in \mathcal{C}^h_{(0,T]}(1,\mu,M)$, any sequence $(s_n)_{n\in\mathbb{Z}_+}$, $s_n \in [0,T]$ converging to s and any sequence of functions*

$f_n \in \mathcal{C}_{(0,T]}(h,1,\mu,M)$ converging pointwise on $(0,T]\times\mathbb{R}$ to a limit f we have

$$\lim_{n\to\infty}\int_{t_1}^{t_2}\theta(t)b(V(t+s_n,z,\omega)+f_n(t,z)+\lambda_n(\psi)(t,z))\,dt$$
$$=\int_{t_1}^{t_2}\theta(t)b(V(t+s,z,\omega)+f(t,z)+\psi(t,z))\,dt.$$

Proof. The proof is based on the ideas from the proofs of [5, Lemmas 3.3, 3.4]. Fix $h>1/2$, $\mu>0$ and integers $N,M,T>0$. Take arbitrary $\varepsilon>0$. Then there exists $\delta>0$ such that statement of Lemma 4.9 is satisfied.

By Lusin's theorem there exists a continuous bounded function $\widetilde{b}\colon\mathbb{R}\to\mathbb{R}$ and an open set U such that $|U|<\delta$, $\|\widetilde{b}\|_\infty\leqslant 2\|b\|_\infty$ and $b(x)=\widetilde{b}(x)$ for all $x\notin U$. Thus, we have the bound

$$|b(x)-\widetilde{b}(x)|=\mathbb{1}(x\in U)|b(x)-\widetilde{b}(x)|\leqslant 3\|b\|_\infty\,\mathbb{1}(x\in U). \tag{4.22}$$

Further, by Lemma 4.9, there exists a set Ω_ε with $\mathsf{P}(\Omega_\varepsilon)\geqslant 1-\varepsilon$ such that bound (4.21) holds on Ω_ε. Take now any $\omega\in\Omega_\varepsilon$, $0\leqslant t_1\leqslant t_2\leqslant T$, $s,s_n\in[0,T]$, $s_n\to s$, $z\in[-N,N]$, $\theta\in\mathbf{B}$, a function $\psi\in\mathcal{C}^h_{(0,T]}(1,\mu,M)$ and any sequence of functions $f_n\in\mathcal{C}^h_{(0,T]}(1,\mu,M)$ converging pointwise to a limit $f\in\mathcal{C}^h_{(0,T]}(1,\mu,M)$. Taking into account (4.22), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{t_1}^{t_2}\theta(t)b(V(t+s_n,z,\omega)+f_n(t,z)+\lambda_n(\psi)(t,z))\,dt\\
&\quad\leqslant\int_{t_1}^{t_2}\theta(t)\widetilde{b}(V(t+s,z,\omega)+f(t,z)+\psi(t,z))\,dt\\
&\qquad+3\|b\|_\infty\|\theta\|_\infty\limsup_{n\to\infty}\int_{t_1}^{t_2}\mathbb{1}_U(V(t+s_n,z,\omega)+f_n(t,z)+\lambda_n(\psi)(t,z))\,dt\\
&\quad\leqslant\int_{t_1}^{t_2}\theta(t)\widetilde{b}(V(t+s,z,\omega)+f(t,z)+\psi(t,z))\,dt+3\|b\|_\infty\|\theta\|_\infty\varepsilon\\
&\quad\leqslant\int_{t_1}^{t_2}\theta(t)b(V(t+s,z,\omega)+f(t,z)+\psi(t,z))\,dt+3\|b\|_\infty\|\theta\|_\infty\varepsilon\\
&\qquad+3\|b\|_\infty\|\theta\|_\infty\int_{t_1}^{t_2}\mathbb{1}_U(V(t+s,z,\omega)+f(t,z)+\psi(t,z))\,dt\\
&\quad\leqslant\int_{t_1}^{t_2}\theta(t)b(V(t+s,z,\omega)+f(t,z)+\psi(t,z))\,dt+6\|b\|_\infty\|\theta\|_\infty\varepsilon,
\end{aligned}$$

where the second and the last inequalities follow from Lemma 4.9.

By a similar argument, we have on Ω_ε

$$\liminf_{n\to\infty}\int_{t_1}^{t_2}\theta(t)b(V(t+s_n,z,\omega)+f_n(t,z)+\lambda_n(\psi)(t,z))\,dt \geqslant \int_{t_1}^{t_2}\theta(t)b(V(t+s,z,\omega)+f(t,z)+\psi(t,z))\,dt-6\|b\|_\infty\|\theta\|_\infty\varepsilon.$$

Since ε was arbitrary, and since $\mathsf{P}(\Omega_\varepsilon)\geqslant 1-\varepsilon$ we see that there exists a set $\Omega(N,M,T,h,\mu)$ such that $\mathsf{P}(\Omega(N,M,T,h,\mu))=1$ and

$$\lim_{n\to\infty}\int_{t_1}^{t_2}\theta(t)b(V(t+s_n,z,\omega)+f_n(t,z)+\lambda_n(\psi)(t,z))\,dt = \int_{t_1}^{t_2}\theta(t)b(V(t+s,z,\omega)+f(t,z)+\psi(t,z))\,dt.$$

for any $\omega\in\Omega(N,M,T,h,\mu)$, $0\leqslant t_1\leqslant t_2\leqslant T$, $s,s_n\in[0,T]$, $s_n\to s$, $\theta\in\mathbf{B}$, function $\psi\in\mathcal{C}^h_{(0,T]}(1,\mu,M)$, $z\in[-N,N]$ and any sequence of functions $f_n\in\mathcal{C}^h_{(0,T]}(1,\mu,M)$ converging pointwise to a limit $f\in\mathcal{C}^h_{(0,T]}(1,\mu,M)$.

To complete the proof of the lemma it remains to take

$$\Omega_C:=\cap\Omega(N,M,T,h,\mu),$$

where the intersection is over all positive integers N,M,T and rational $h>1/2$, $\mu>0$. □

5. Proofs of Theorem 2.1 and Theorem 2.2(b). Most of the section is devoted to the proof of Theorem 2.1. Fix a bounded measurable function b. Without loss of generality and to ease the notation we assume in this section that $\|b\|_\infty\leqslant 1$. Now with such b at hand we take for the rest of the section

$$\Omega':=\Omega_E\cap\Omega''\cap\Omega_V\cap\Omega_C\subset\Omega, \tag{5.1}$$

where the set Ω_E is defined in the sketch of the proof of Theorem 2.1 in Section 2, Ω'' is from Theorem 2.3, Ω_V is from regularity Lemma 4.7 and Ω_C is from Lemma 4.10. Thus, on Ω' the statements of the aforementioned theorems and lemmas are satisfied and $\mathsf{P}(\Omega')=1$.

We begin this section with an easy observation.

PROPOSITION 5.1. *Let $s\geqslant 0$, $q\in\mathbf{B}(0+)$. Let $u_{s,q}$ be any solution to (2.2) that starts with initial condition q at time s. Then $u_{s,q}(t,\cdot,\omega)\in\mathbf{B}(0+)$ for any $\omega\in\Omega'$, $t\geqslant s$.*

PROOF. This statement immediately follows from equation (2.2) and estimate (4.17). □

5.1 Existence part of Theorem 2.1. In this subsection we present the proof of the existence part of Theorem 2.1. Our main tool is the following lemma that establishes continuity of solution to (2.2) with respect to the initial condition. Recall that the set Ω' is defined in (5.1).

LEMMA 5.2. *Let $\omega \in \Omega'$. Let $(s_n)_{n\in\mathbb{Z}_+}$, $s_n \geqslant 0$ be a sequence that converges to s, as $n \to \infty$. Let $(q_n)_{n\in\mathbb{Z}_+}$ be a sequence of measurable functions $\mathbb{R} \to \mathbb{R}$ such that $q_n(z) \to q(z)$ as $n \to \infty$ Lebesgue–almost everywhere in z. Assume that there exist $C > 0$, $\mu > 0$ such that for any $n \in \mathbb{Z}_+$ one has*

$$|q_n(z)| \leqslant C(|z|^\mu \vee 1), \quad z \in \mathbb{R}.$$

For each $n \in \mathbb{Z}_+$ let $u_{s_n,q_n}(\cdot,\cdot,\omega)$ be a solution to (2.2) that starts with the initial condition q_n at time s_n.

Then there exists a solution $u_{s,q}(\cdot,\cdot,\omega)$ to (2.2) that starts with the initial condition q at time s. Moreover, there exists a subsequence $(n_k)_{k\in\mathbb{Z}_+}$ such that for any $t > 0$, $z \in \mathbb{R}$ we have

$$u_{s_{n_k},q_{n_k}}(s_{n_k} + t, z, \omega) \to u_{s,q}(s + t, z, \omega) \quad \textit{as } k \to \infty.$$

This lemma implies that for any $\omega \in \Omega'$ a sequence of solutions to equation (2.2) that start at time s_n from the initial condition q_n has a subsequence that converges pointwise to a solution of equation (2.2) that starts at time s from the initial condition q.

PROOF. Fix $\omega \in \Omega'$, the sequences $(s_n)_{n\in\mathbb{Z}_+}$, $(q_n)_{n\in\mathbb{Z}_+}$ as in the lemma and also any $T > 0$. By the definition of $u^*_{s_n,q_n}$ (recall equation (4.2)), we have for any $t \in (0, T]$, $z \in \mathbb{R}$

$$u^*_{s_n,q_n}(t,z) = \int_{\mathbb{R}} p_t(z-z')q_n(z')\,dz' + \int_0^t \int_{\mathbb{R}} p_{t-t'}(z-z')b(u^*_{s_n,q_n}(t',z'))\,dz'dt' + V(s_n, t+s_n, z)$$

and $u^*_{s_n,q_n}(0,z) = q_n(z)$. For $n \in \mathbb{Z}_+$ set

$$h_n(t,z) := \int_0^t \int_{\mathbb{R}} p_{t-t'}(z-z')b(u^*_{s_n,q_n}(t',z'))\,dz'\,dt', \quad t \in [0,T],\ z \in \mathbb{R}.$$

Clearly, the sequence $(h_n)_{n\in\mathbb{Z}_+}$ is uniformly bounded. Indeed, for any $n \in \mathbb{Z}_+$ we have $\|h_n\|_\infty \leqslant T\|b\|_\infty$. Further, it follows from Lemma 4.5 that for any $t_1, t_2 \in [0,T]$, $z_1, z_2 \in \mathbb{R}$

$$|h_n(t_1,z_1) - h_n(t_2,z_2)| \leqslant C\|b\|_\infty(|z_2 - z_1| + |t_2 - t_1|^{3/4}) \tag{5.2}$$

for some $C = C(T) > 0$ that is independent of n. Hence the Arzelà–Ascoli theorem for locally compact metric spaces (see, e.g., [10, Theorem 4.44]) implies that there exists a subsequence $(n_k)_{k\in\mathbb{Z}_+}$, such that h_{n_k} converges pointwise to some function h. We simplify the notation by assuming that we have already started with such a subsequence and that $n_k = k$. Hence, (5.2) yields

$$|h(t_1,z_1)-h(t_2,z_2)| \leqslant C\|b\|_\infty(|z_2-z_1|+|t_2-t_1|^{3/4}),\ \ t_1,t_2 \in [0,T],\ \ z_1,z_2 \in \mathbb{R}.$$

For $t \in (0,T]$, $z \in \mathbb{R}$ put

$$\begin{aligned} u^*_{s,q}(t,z) &:= \int_{\mathbb{R}} p_t(z-z')q(z')\,dz' + h(t,z) + V(s,t+s,z), \\ u^*_{s,q}(0,z) &:= q(z), \quad z \in \mathbb{R}. \end{aligned} \tag{5.3}$$

We claim now that $u^*_{s,q}$ is a solution to (4.3) on $[0,T]$. Indeed, we observe that $u^*_{s_n,q_n}$ can be written as follows:

$$u^*_{s_n,q_n}(t,z) = V(t+s_n,z) + g_n(t,z), \quad t \in [0,T],\ z \in \mathbb{R},$$

where for $t \in [0,T]$, $z \in \mathbb{R}$ we defined

$$g_n(t,z) := \int_{\mathbb{R}} p_t(z-z')q_n(z')\,dz' + h_n(t,z) - \mathcal{V}(0,s_n,t+s_n,z). \tag{5.4}$$

It follows from Lemma 4.6, Lemma 4.7 and inequality (5.2) that there exists $M > 0$ such that for any $n \in \mathbb{Z}_+$ the function $g_n \in \mathcal{C}^{3/4}_{(0,T]}(1,\mu,M)$. By our assumptions and the dominated convergence theorem, the first term at the right-hand side of (5.4) converges pointwise to $\int_{\mathbb{R}} p_t(z-z')q(z')\,dz'$ for $(t,z) \in (0,T]\times\mathbb{R}$. By Lemma 4.7, $\mathcal{V}(0,s_n,t+s_n,z) \to \mathcal{V}(0,s,t+s,z)$ as $n \to \infty$ for $(t,z) \in [0,T]\times\mathbb{R}$. This together with h_n converging pointwise to h implies that

$$\begin{aligned} \lim_{n\to\infty} g_n(t,z) &= \int_{\mathbb{R}} p_t(z-z')q(z')\,dz' + h(t,z) - \mathcal{V}(0,s,t+s,z) \\ &=: g(t,z), \quad t \in (0,T],\ z \in \mathbb{R}. \end{aligned}$$

On the other hand,

$$\begin{aligned} h(t,z) &= \lim_{n\to\infty} h_n(t,z) \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} \int_0^t p_{t-t'}(z-z')b(u^*_{s_n,q_n}(t',z'))\,dt'\,dz' \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} \int_0^t p_{t-t'}(z-z')b(V(t+s_n,z)+g_n(t,z))\,dt'\,dz'. \end{aligned}$$

Note that the function b is not necessarily continuous so we cannot pass to the limit directly. Therefore to pass to the limit we employ Lemma 4.10 with the following set of parameters: $f_n \leftarrow g_n$, $f \leftarrow g$, $\psi \leftarrow 0$, $\theta \leftarrow p_{t-\cdot}(z-z')$, $t_2 \leftarrow t$, $t_1 \leftarrow 0$. Since $g_n \in \mathbf{H}_T(3/4, 1, \mu, M)$ and since for fixed t, $z \neq z'$ the function $t' \mapsto p_{t-t'}(z-z')$ is bounded, we see that all conditions of Lemma 4.10 are satisfied. We apply the dominated convergence theorem (this is possible due to the fact that b is bounded) and continue the identity above as follows:

$$
\begin{aligned}
h(t,z) &= \int_{\mathbb{R}} \int_0^t p_{t-t'}(z-z') b(V(t+s,z) + g(t,z))\, dt'\, dz' \\
&= \int_{\mathbb{R}} \int_0^t p_{t-t'}(z-z') b(u^*_{s,q}(t',z'))\, dt'\, dz',
\end{aligned}
\tag{5.5}
$$

where we also used that by (5.3) $u^*_{s,q}(t,z) = V(t+s,z) + g(t,z)$. Obtained identity (5.5), combined with (5.3), implies that $u^*_{s,q}$ is indeed a solution to (4.3). Hence the function $u_{s,q}(t,\cdot) := u^*_{s,q}(t+s,\cdot)$ solves equation (2.2) that starts with the initial condition q at time s.

We note that the convergence of g_n to g and continuity of V imply that

$$
\lim_{n\to\infty} u_{s_n,q_n}(s_n+t,z) = \lim_{n\to\infty} u^*_{s_n,q_n}(t,z) = u^*_{s,q}(t,z) = u_{s,q}(s+t,z). \tag{5.6}
$$

for any $t \in (0,T]$, $z \in \mathbb{R}$. Finally, by the standard diagonalization argument, we see that there exists a subsequence (n_k) such that identity (5.6) is valid for any $t \in (0,\infty)$. □

Proof of existence part of Theorem 2.1. We recall that we have already fixed Ξ, a countable dense subset of $\mathbf{C}_0(\mathbb{R})$, and Θ, a countable dense subset of $\mathbb{R}_+$ (see the sketch of the proof of Theorem 2.1 in Section 2). Since $\Omega' \subset \Omega_E$, we see that for any $\omega \in \Omega'$, $s \in \Theta$, $q \in \Xi$ equation (2.2) has a solution that starts with the initial condition q at time s.

Fix any $\omega \in \Omega'$. Let q now be an arbitrary element of $\mathbf{B}(0+)$, let $s \in \mathbb{R}$. Let $(q_n)_{n\in\mathbb{Z}_+}$ be a sequence of elements in Ξ that converge Lebesgue–almost everywhere to q and such that for some $C > 0$, $\mu > 0$ one has $q_n(z) \leqslant C(|z| \vee 1)^\mu$ uniformly over all n. The existence of such a sequence is clear and can be shown by the standard argument. Let $(s_n)_{n\in\mathbb{Z}_+}$ be a sequence of elements in Θ that converges to s. By above, equation (2.2) has a solution that starts with the initial condition q_n at time s_n. Hence, by Lemma 5.2, equation (2.2) has a solution that starts with the initial condition q at time s.

Since q and s were arbitrary elements of $\mathbf{B}(0+)$ and $\mathbb{R}_+$, respectively, this concludes the proof of the existence part of Theorem 2.1. □

5.2 Uniqueness part of Theorem 2.1. Recall that by Remark 4.1 it is sufficient to show that on Ω' equation (4.3) has a unique solution. This will straightforwardly imply that the original equation (2.2) has also a unique solution on Ω'.

Till the end of this section we fix arbitrary $\omega \in \Omega'$, $s \geqslant 0$, $q \in \mathbf{B}(0+)$. Without loss of generality, we assume $s \in [0,1]$. Let v and w be any two solutions to (4.3) with the initial condition q for our fixed ω, s. To prove the theorem it is sufficient to show that $v(t,z) = w(t,z)$ for $z \in \mathbb{R}$, $t \in [0,T]$ for any $T > 0$. We will verify this statement for $T = 1$; the proof for other values of T is exactly the same.

We observe that for $t \in [0,1]$, $z \in \mathbb{R}$ we have

$$v(t,z) - w(t,z) = \int_0^t \int_{\mathbb{R}} p_{t-t'}(z-z')(b(v(t',z')) - b(w(t',z')))\,dz'dt'.$$

We denote $\psi(t,z) := v(t,z) - w(t,z)$ and rewrite the above equation in the following form:

$$\psi(t,z) = \int_0^t \int_{\mathbb{R}} p_{t-t'}(z-z')(b(w(t',z') + \psi(t',z)) - b(w(t',z')))\,dz'dt',$$

where $t \in [0,1]$, $z \in \mathbb{R}$. It is easy to check, that for any $r \in [0,1]$ the function ψ satisfies also a more general equation

$$\begin{aligned}(5.7)\quad \psi(t,z) &= \int_{\mathbb{R}} p_{t-r}(z-z')\psi(r,z')\,dz' \\ &\quad + \int_{\mathbb{R}}\int_r^t p_{t-t'}(z-z')\big(b(w(t',z')+\psi(t',z'))-b(w(t',z'))\big)\,dt'dz', \\ \psi(0,z) &= \phi(z).\end{aligned}$$

where $t \in [r,1]$, $z \in \mathbb{R}$.

Our goal is to prove that the only solution to this equation with the initial condition $\phi(z) = 0$ is identically zero (this would immediately imply uniqueness of solution to (4.3)). To show this we have to analyze equation (5.7) with a more general class of initial conditions. Namely, we assume that the function $\phi \in \mathcal{C}^b_{Lip}$ (recall that the class $\mathcal{C}^b_{Lip}$ is introduced in Definition 4.1). Note also that the functions ψ, v, w depend also on fixed ω, s, q. In order not to overcrowd the notation, we write $\psi(t,z)$ for $\psi(t,z,\omega,s,q)$ and so on.

To show that equation (5.7) has only a trivial solution we will need to control the norm of $\psi(t,\cdot)$. We will work with a weighted Hölder norm. The use of a weighted norm is natural here since we work with functions defined on a noncompact space $\mathbb{R}$. Thus, for a function $f\colon \mathbb{R} \to \mathbb{R}$ we put

$$\|f\|_{0,w} = \sup_{z\in\mathbb{R}} |f(z)|e^{-|z|}.$$

For $\delta > 0$ consider a weighted Lipschitz coefficient of f

$$[f]_{1,\delta} := \sup_{z_1 \neq z_2} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|\Lambda_\delta(|z_1| \vee |z_2| \vee 1)}.$$

Recall that the function Λ_δ was defined in (4.5).

Finally, define a weighted Lipschitz norm of f by

$$\|f\|_{1,\delta} := \|f\|_{0,w} + [f]_{1,\delta}.$$

We have to use the additional factor z^δ in the weight of the Lipschitz coefficient because this factor appears in the right-hand side of our main bound (2.4) in Theorem 2.3 (see also Lemma 5.6 below).

The function w can be represented as $w(t,z) = V(t+s,z) + g(t,z)$, where

$$\begin{aligned} g(t,z) := &\int_{\mathbb{R}} p_t(z-z')q(z')\,dz' + \int_0^t \int_{\mathbb{R}} p_{t-t'}(z-z')b(w(t',z'))\,dz'dt' \\ &- \mathcal{V}(0,s,t+s,z), \quad t \in [0,1],\ z \in \mathbb{R}, \end{aligned} \tag{5.8}$$

and $\mathcal{V}$ was defined in (4.6). We used here the identity

$$V(s,t+s,z) = \mathcal{V}(s,t+s,t+s,z) = \mathcal{V}(0,t+s,t+s,z) - \mathcal{V}(0,s,t+s,z).$$

The next two lemmas establish useful properties of the functions g and ψ.

LEMMA 5.3. *The function g defined in* (5.8) *belongs to* $\mathcal{C}^{1-}(1,0+)$.

PROOF. The lemma follows immediately from Lemmas 4.5, 4.6, 4.7. □

REMARK 5.4. Note that at any time $t > s$ a solution to (2.2), $u_{s,q}(t,\cdot)$, is much more regular than its initial condition $q \in \mathbf{B}(0+)$. Namely, $u_{s,q}(t,\cdot)$ is a Hölder function with exponent $1/2-$. If one starts with such a "regular" initial condition q (Hölder with exponent $1/2-$), then it is possible to show that g is more regular than it is shown in Lemma 5.3. Namely, $g \in \mathcal{C}^{1-}(3/4,0+)$. However we will not use this improvement of regularity of g in our proof and will continue to consider solutions to (2.2) that start from the initial condition $q \in \mathbf{B}(0+)$.

We will need to obtain the bounds on the norm of ψ both in the weighted and in the standard Hölder spaces. As one might expect, there is a certain trade-off between having a singular weight and a better regularity.

LEMMA 5.5. *Assume that $\phi \in \mathcal{C}^b_{Lip}$. Then any solution ψ to (5.7) belongs to $\mathcal{C}^{1-}(1,0)$. Further, there exists a constant $C = C(\phi)$ such that*

$$\sup_{\substack{t\in[0,1]\\ z\in\mathbb{R}}} |\psi(t,z)| \leqslant C; \tag{5.9}$$

$$\sup_{\substack{t_1,t_2\in[0,1]\\ z_1,z_2\in\mathbb{R}}} \frac{|\psi(t_1,z_1)-\psi(t_2,z_2)|}{|t_1-t_2|^{1/2}+|z_1-z_2|} \leqslant C. \tag{5.10}$$

That is, ψ is a bounded function on $[0,1]\times\mathbb{R}$, which is Lipschitz in space and Hölder in time with exponent $1/2$. In particular, $\psi(t,\cdot)\in\mathcal{C}^b_{Lip}$ for any $t\in[0,1]$.

PROOF. Take in (5.7) $r=0$. Then, by Lemmas 4.5 and 4.6, $\psi\in\mathcal{C}^{1-}(1,0)$. Bound (5.9) obviously follows from boundedness of the functions b and ϕ. Estimate (5.10) is obtained by a straightforward application of Lemmas 4.5 and 4.6. □

The next lemma establishes "smoothing" properties of the operator

$$(x(\cdot),s,t,z)\mapsto \int_{\mathbb{R}}\int_0^s p_{t-t'}(z-z')b(V(t'+s,z')+f(t',z')+x(z'))\,dt'dz', \tag{5.11}$$

where $x\in\mathbf{B}$ and $0\leqslant s\leqslant t$, simultaneously for all $f\in\mathcal{C}^{1-}(1,0+)$. To simplify the notation, further we will denote the sum of V and the function $f\colon[0,1]\times\mathbb{R}\to\mathbb{R}$ (that will correspond to the drift term) by

$$V_{f,s}(t,z) := V(t+s,z)+f(t,z),\quad t\in[0,1],\ z\in\mathbb{R}. \tag{5.12}$$

Recall the definition of the difference of two Gaussian kernels Δp_t from (4.4).

LEMMA 5.6. *For any $\delta\in(0,1/6)$, $N>0$, and any function $f\in\mathcal{C}^{1-}(1,0+)$ there exists a constant $C=C(\omega,N,f,\delta)<\infty$ such that for any $s\in[0,1]$, $0\leqslant t_1\leqslant t_2\leqslant t\leqslant 1$, $z,z_1,z_2\in\mathbb{R}$, and any $x,y\in\mathbf{B}$ with $\|x\|_\infty,\|y\|_\infty\leqslant N$*

we have

$$\int_{\mathbb{R}}\left|\int_{t_1}^{t_2} p_{t-t'}(z-z')\big(b(V_{f,s}(t',z')+x(z'))-b(V_{f,s}(t',z')+y(z'))\big)\,dt'\right|dz' \tag{5.13}$$
$$\leqslant C\|x-y\|_{0,w}|t_2-t_1|^{2/3-\delta}\Lambda_\delta(|z|\vee 1);$$

$$\int_{\mathbb{R}}\left|\int_{t_1}^{t_2} \Delta p_{t-t'}(z_1-z',z_2-z') \tag{5.14}$$
$$\times\big(b(V_{f,s}(t',z')+x(z'))-b(V_{f,s}(t',z')+y(z'))\big)\,dt'\right|dz'$$
$$\leqslant C\|x-y\|_{0,w}(t-t_1)^{-1/2}|t_2-t_1|^{2/3-\delta}|z_1-z_2|\Lambda_\delta(|z_1|\vee|z_2|\vee 1).$$

REMARK 5.7. If $f\in\mathcal{C}^{1-}(3/4,0+)$ (see Remark 5.4 where it is explained why this is relevant), then the operator (5.11) is more regular in time. Namely the term $|t_2-t_1|$ in the right-hand side of (5.13) and (5.14) has the exponent $3/4-\delta$ instead of $2/3-\delta$.

PROOF OF LEMMA 5.6. Fix $\delta\in(0,1/6)$, $N>0$ and a function $f\in\mathcal{C}^{1-}(1,0+)$. Consider a function

$$B(r_1,r_2,s,\alpha,\beta,z):=\int_{r_1}^{r_2}\big(b(V_{f,s}(t',z)+\alpha)-b(V_{f,s}(t',z)+\beta)\big)\,dt',$$

defined for $\alpha,\beta,z\in\mathbb{R}$, $0\leqslant r_1\leqslant r_2\leqslant 1$, $s\in[0,1]$.

It follows from Theorem 2.3 that there exists a constant $C(\omega,N,f,\delta)$ such that for any $\alpha,\beta,z\in\mathbb{R}$, $|\alpha|,|\beta|\leqslant N$, $0\leqslant r_1\leqslant r_2\leqslant 1$, $s\in[0,1]$ we have

$$|B(r_1,r_2,s,\alpha,\beta,z)|\leqslant C(\omega,N,f,\delta)|r_2-r_1|^{2/3-\delta}|\alpha-\beta|(|z|^\delta\vee 1). \tag{5.15}$$

To simplify the notation, for the rest of the proof we drop the variables ω, N, f, δ and write C instead of $C(\omega,N,f,\delta)$.

Fix $0\leqslant t_1\leqslant t_2\leqslant 1$. Let $(t',z')\mapsto h(t',z')$, $t'\in[t_1,t_2]$, $z'\in\mathbb{R}$ be a continuously differentiable function in t' for $z'\in\mathbb{R}\setminus E$, where the Lebesgue measure of E is 0. Then for any $\alpha,\beta\in\mathbb{R}$, $z'\in\mathbb{R}\setminus E$ integration by parts gives

$$\int_{t_1}^{t_2} h(t',z')\big(b(V_{f,s}(t',z')+\alpha)-b(V_{f,s}(t',z')+\beta)\big)\,dt'$$
$$=-\int_{t_1}^{t_2} h(t',z')\,d_{t'}B(t',t_2,s,\alpha,\beta,z')$$
$$=h(t_1,z')B(t_1,t_2,s,\alpha,\beta,z')+\int_{t_1}^{t_2} B(t',t_2,s,\alpha,\beta,z')\frac{\partial h}{\partial t'}(t',z')\,dt'.$$

We integrate over z' and apply estimate (5.15) to derive for any $x, y \in \mathbf{B}$

$$\begin{aligned}(5.16)\quad &\int_{\mathbb{R}}\int_{t_1}^{t_2} h(t',z')\big(b(V_{f,s}(t',z')+x(z'))-b(V_{f,s}(t',z')+y(z'))\big)\,dt' \\ &\leqslant C\|x-y\|_{0,w}|t_2-t_1|^{2/3-\delta}\int_{\mathbb{R}}|h(t_1,z')|\Lambda_\delta(|z'|\vee 1)dz' \\ &+C\|x-y\|_{0,w}\int_{\mathbb{R}}\int_{t_1}^{t_2}\Big|\frac{\partial h}{\partial t'}(t',z')\Big||t_2-t'|^{2/3-\delta}\Lambda_\delta(|z'|\vee 1)\,dt'dz'.\end{aligned}$$

For any $t \geqslant t_2$, $z \in \mathbb{R}$ we can apply this formula to the function $h(t',z') := p_{t-t'}(z-z')$ (indeed, for $z' \in \mathbb{R}\setminus z$ this function is continuously differentiable in t'). Using estimates (4.10) and (4.11) from Lemma 4.4, we get (5.13). In a similar way, for $t \geqslant t_2$, $z_1, z_2 \in \mathbb{R}$ we apply formula (5.16) to the function $h(t',z') := p_{t-t'}(z_1-z') - p_{t-t'}(z_2-z')$. Using estimates (4.12) and (4.13) from Lemma 4.4, we obtain (5.14). □

Remark 5.8. Let us explain how Lemmas 5.3, 5.5, and 5.6 will be used in the proof. Fix initial condition $\phi \in \mathcal{C}^b_{Lip}$. It follows from Lemma 5.5 that there exists a constant $C_1 = C_1(\phi)$ such that inequalities (5.9) and (5.10) hold. Recall again that the solution w can be represented as

$$(5.17)\qquad w(t,z) = V(t+s,z) + g(t,z),$$

where g was defined in (5.8). By Lemma 5.3, $g \in \mathcal{C}^{1-}(1,0+)$. Thus, we can apply Lemma 5.6 with $f \leftarrow g$ and $N \leftarrow C_1$. We see that there exists a constant $C_2 = C_2(C_1,\phi)$ such that the estimates (5.13) and (5.14) are satisfied with C_2 instead of C. We will use further the constant $C_\phi := \max(1, C_1, C_2)$.

Now, we apply Lemma 5.6 to analyze the behavior of ψ on a small interval $[k2^{-m},(k+1)2^{-m}]$. More precisely, for any $t \in [k2^{-m},(k+1)2^{-m}]$ we will derive bounds on $\|\psi(t,\cdot)\|_{1,\delta}$ in terms of $\|\psi(k2^{-m},\cdot)\|_{1,\delta}$. This lemma will be crucial for the whole argument. Namely, we will just apply the bound from Lemma 5.9 consecutively 2^m times to prove later the uniqueness part of Theorem 2.1.

Lemma 5.9. *For any $\delta \in (0,1/6)$ and any initial condition $\phi \in \mathcal{C}^b_{Lip}$ there exist constants $C = C(\delta,\phi)$, $m_0 = m_0(\delta,\phi)$ such that for any integers $m > m_0$, $r \in [0, 2^m-1]$ we have the following estimate*

$$(5.18)\qquad \sup_{t\in[\frac{r}{2^m},\frac{r+1}{2^m}]}\|\psi(t,\cdot)\|_{1,\delta} \leqslant C\|\psi(\frac{r}{2^m},\cdot)\|_{1,\delta} + Ce^{-2^{m/(2\delta)}}.$$

In particular,

$$\|\psi(\frac{r+1}{2^m},\cdot)\|_{1,\delta} \leqslant C\|\psi(\frac{r}{2^m},\cdot)\|_{1,\delta} + Ce^{-2^{m/(2\delta)}}. \tag{5.19}$$

PROOF. Fix $\delta \in (0, 1/6)$, the initial condition ϕ and integer $m > 0$. All the constants that will appear in the proof will depend only on δ and ϕ but not on m or r.

To simplify the notation we will show (5.18) for $r = 0$. The proof for other values of $r = 1, 2, \dots, 2^m - 1$ is exactly the same.

Recall that we already know from Lemma 5.5 and Remark 5.8 that

$$\sup_{\substack{t_1,t_2\in[0,2^{-m}]\\ z\in\mathbb{R}}} \frac{|\psi(t_1,z)-\psi(t_2,z)|}{|t_1-t_2|^{1/2}} \leqslant C_\phi. \tag{5.20}$$

This bound is rather rough; our goal is to obtain a much finer bound that we can later iterate over r. We will show in the proof that if $\|\psi(0,\cdot)\|_{1,\delta}$ is small, then the left-hand side of (5.20) is also very small. This would imply (5.18).

Our proof strategy consists of three steps. First, following a standard technique (see, e.g., [5, proof of Lemma 3.1]), we show that it is sufficient to estimate the supremum in the left-hand side of (5.20) only for those $t_1, t_2 \in [0, 2^{-m}]$ that are dyadic neighbors. This would imply a corresponding bound for any $t_1, t_2 \in [0, 2^{-m}]$. As is common in the PDE literature, to get a "time" bound we need to obtain first a "space" bound. This is done in the second step using approximation technique and estimate (5.14). Finally, using again approximation technique and estimate (5.13) we get the required "time" bound (5.20) with much smaller constant.

In the proof of the theorem we will be working with binomial partitions of the interval $[0, 1]$. So, for integers $n \geqslant 0$, $k \in [0, 2^n]$ put

$$t_n^k := k2^{-n}; \quad Lip_n^k := [\psi(t_n^k,\cdot)]_{1,\delta}. \tag{5.21}$$

By Lemma 5.5, Lip_n^k are finite for any $n \geqslant 0$, $k \in [0, 2^n]$.

As explained above, we will consider the differences $|\psi(t_1, z) - \psi(t_2, z)|$, $t_1, t_2 \in [0, 2^{-m}]$, where t_1 and t_2 are dyadic neighbors. Thus, we define α as the smallest number such that for any integers $n \geqslant m$, $k \in [0, 2^{n-m} - 1]$ we have

$$\|\psi(\frac{k+1}{2^n},\cdot) - \psi(\frac{k}{2^n},\cdot)\|_{0,w} \leqslant \alpha 2^{-n/2}. \tag{5.22}$$

Note that such an α exists and is finite. Indeed thanks to Remark 5.8, the left-hand side of (5.22) is bounded by $C_\phi 2^{-n/2}$.

Consider a binary notation of $k2^{-n}$, where $k \in [0, 2^{n-m}-1]$. We have $k2^{-n} = \sum_{i=m+1}^{n} d_i 2^{-i}$, where each d_i equals 0 or 1. Define approximations of $k2^{-n}$ by

$$k_m := 0,\, k_j := \sum_{i=m+1}^{j} d_i 2^{-i}, \quad j \in [m+1, n].$$

It follows from the definition that either $k_j = k_{j-1}$ or $k_j = k_{j-1} + 2^{-j}$. Therefore, we can apply estimate (5.22) $n-m$ times to derive

$$\|\psi(\frac{k}{2^n}, \cdot) - \psi(0, \cdot)\|_{0,w} \leqslant \sum_{j=m+1}^{n} \|\psi(k_j, \cdot) - \psi(k_{j-1}, \cdot)\|_{0,w} \leqslant \alpha \sum_{j=m+1}^{n} 2^{-j/2}.$$

Hence, for any $n \geqslant m$ and $0 \leqslant k \leqslant 2^{n-m}-1$ we get

$$\|\psi(\frac{k}{2^n}, \cdot)\|_{0,w} \leqslant \|\psi(0, \cdot)\|_{0,w} + 3\alpha 2^{-m/2}.$$

Since the function ψ is continuous we get the following bound for any $t \in [0, 2^{-m}]$.

$$\|\psi(t, \cdot)\|_{0,w} \leqslant \|\psi(0, \cdot)\|_{0,w} + 3\alpha 2^{-m/2} \tag{5.23}$$

Thus, we can effectively bound $\|\psi(t, \cdot)\|_{0,w}$ for any $t \in [0, 2^{-m}]$.

To approximate the solutions to (5.7) we consider piecewise approximations defined above in (4.1). Namely, we introduce a sequence of piecewise–constant (in time) functions

$$\psi_n(\cdot, z) := \lambda_n(\psi(\cdot, z)), \quad z \in \mathbb{R},$$

where $n \geqslant m$. We see that $\psi_n(t, z)$ is equal to $\psi((k+1)2^{-n}, z)$ whenever $t \in (k2^{-n}, (k+1)2^{-n}]$, $k = 0, 1, \ldots, 2^{n-m}-1$. In particular, the function $\psi_m(t, z)$ is constant in t on the interval $(0, 2^{-m}]$.

We start with an estimation of the weighted Lipschitz coefficient (with respect to the space variable) of the function ψ. We want to do it in all binary points of our initial interval $[0, 2^{-m}]$, i.e., in all points of the form $t_n^k = k2^{-n}$. Here $n \geqslant m$, $0 \leqslant k \leqslant 2^{n-m}-1$. Recall the definition of the function g and the process $V_{g,s}$ given at (5.8) and (5.12). respectively. Taking into account (5.17), we derive from (5.7)

$$\psi(t_n^{k+1}, z_1) - \psi(t_n^{k+1}, z_2) = I_1 + I_2, \tag{5.24}$$

where

$$I_1 := \int_{\mathbb{R}} p_{2^{-n}}(z')(\psi(t_n^k, z_1 - z') - \psi(t_n^k, z_2 - z'))\, dz'$$

and

$$I_2 := \int_{\mathbb{R}} \int_{t_n^k}^{t_n^{k+1}} \Delta p_{t_n^{k+1}-t}(z_1 - z', z_2 - z')\big(b(V_{g,s}(t, z') + \psi(t, z')) - b(V_{g,s}(t, z'))\big) dt dz'.$$

First, let us bound I_1. By definition of Lip_n^k (see (5.21)), we have

$$\begin{aligned}(5.25)\quad |I_1| &\leqslant Lip_n^k |z_1 - z_2| \int_{\mathbb{R}} p_{2^{-n}}(z') \Lambda_\delta(|z_1 - z'| \vee |z_2 - z'| \vee 1)\, dz' \\ &\leqslant Lip_n^k |z_1 - z_2| \Lambda_\delta(|z_1| \vee |z_2| \vee 1) \int_{\mathbb{R}} p_{2^{-n}}(z') e^{|z'|}(|z'|^\delta + 1)\, dz' \\ &\leqslant Lip_n^k |z_1 - z_2| \Lambda_\delta(|z_1| \vee |z_2| \vee 1) e^{2^{-n/2}} (1 + C 2^{-n\delta/2}),\end{aligned}$$

where in the second inequality we also used the fact that the function Λ is increasing and $|z_1 - z'| \vee |z_2 - z'| \vee 1 \leqslant (|z_1| \vee |z_2| \vee 1) + |z'|$.

To handle I_2 we apply Lemma 4.10 with the following set of parameters: $f_n \leftarrow g$, $f \leftarrow g$, $\psi \leftarrow \psi$, $\theta \leftarrow \Delta p_{t_n^{k+1}-\cdot}(z_1 - z', z_2 - z')$, $s_n \leftarrow s$, $t_2 \leftarrow t_n^{k+1}$, $t_1 \leftarrow t_n^k$. Since $\psi, g \in \mathcal{C}^{1-}(1, 0+)$ (see Lemmas 5.3 and 5.5), and since for fixed $z' \neq z_1, z_2$ the function $\Delta p_{t_n^{k+1}-\cdot}(z_1 - z', z_2 - z')$ is bounded, we see that all the conditions of Lemma 4.10 are satisfied. Since the function b is also bounded, we can apply dominated convergence theorem to obtain

$$\begin{aligned}(5.26)\quad I_2 &= \lim_{l \to \infty} \int_{\mathbb{R}} \int_{t_n^k}^{t_n^{k+1}} \Delta p_{t_n^{k+1}-t}(z_1 - z', z_2 - z') \\ &\qquad \times \big(b(V_{g,s}(t, z') + \psi_l(t, z')) - b(V_{g,s}(t, z'))\big)\, dt dz' \\ &= \int_{\mathbb{R}} \int_{t_n^k}^{t_n^{k+1}} \Delta p_{t_n^{k+1}-t}(z_1 - z', z_2 - z') \\ &\qquad \times \big(b(V_{g,s}(t, z') + \psi_n(t, z')) - b(V_{g,s}(t, z'))\big)\, dt dz' \\ &\quad + \sum_{l=n}^{\infty} I_{22}(l) \\ &=: I_{21} + \sum_{l=n}^{\infty} I_{22}(l),\end{aligned}$$

where we denoted

$$\begin{aligned} I_{22}(l) := \int_{\mathbb{R}} \int_{t_n^k}^{t_n^{k+1}} &\Delta p_{t_n^{k+1}-t}(z_1 - z', z_2 - z') \\ &\times \big(b(V_{g,s}(t, z') + \psi_{l+1}(t, z')) - b(V_{g,s}(t, z') + \psi_l(t, z'))\big)\, dt dz'.\end{aligned}$$

Thus, Lemma 4.10 allowed us to pass from continuous function ψ to its piecewise-constant approximations ψ_l. This is crucial due to the fact that our main tool, Lemma 5.6, works only for constant in t functions x and y.

Estimation of I_{21} is straightforward. It follows from the definition of approximation ψ_n, that $\psi_n(t,z)=\psi(t_n^{k+1},z)$ for any $t\in(t_n^k,t_n^{k+1}]$, $z\in\mathbb{R}$. Taking into account Remark 5.8, we make use of the bounds in (5.14) and (5.23) to obtain

$$\begin{aligned}(5.27)\quad |I_{21}|&\leqslant C_\phi|z_1-z_2|2^{-(1/6-\delta)n}\|\psi(t_n^{k+1},\cdot)\|_{0,w}\Lambda_\delta(|z_1|\vee|z_2|\vee 1)\\ &\leqslant CC_\phi|z_1-z_2|2^{-(1/6-\delta)n}\Lambda_\delta(|z_1|\vee|z_2|\vee 1)(\|\psi(0,\cdot)\|_{0,w}+\alpha 2^{-m/2}).\end{aligned}$$

Now let's do the most tricky part and work with term $I_{22}(l)$ in (5.24). As mentioned above, the functions ψ_l and ψ_{l+1} are piecewise-constant. Thus, we split the integral over $(t_n^k,t_n^{k+1}]$ into 2^{l-n} integrals over smaller subintervals $(t_l^i,t_l^{i+1}]$, $k2^{l-n}\leqslant i<(k+1)2^{l-n}$. On each such subinterval function ψ_l is constant in t and is equal to $\psi(t_l^{i+1},z)$. The function ψ_{l+1} also equal to the same value $\psi(t_l^{i+1},z)$ for $t\in(t_l^{i+1/2},t_l^{i+1}]$ and to $\psi(t_l^{i+1/2},z)$ for $t\in(t_l^i,t_l^{i+1/2}]$. Thus, we get

$$\begin{aligned}I_{22}(l)&=\sum_{i=k2^{l-n}}^{(k+1)2^{l-n}-1}\int_{\mathbb{R}}\int_{t_l^i}^{t_l^{i+1}}\Delta p_{t_n^{k+1}-t}(z_1-z',z_2-z')\\ &\qquad\times\big(b(V_{g,s}(t,z')+\psi_{l+1}(t,z'))-b(V_{g,s}(t,z')+\psi_l(t,z'))\big)\,dtdz'\\ &=\sum_{i=k2^{l-n}}^{(k+1)2^{l-n}-1}\int_{\mathbb{R}}\int_{t_l^i}^{t_l^{i+1/2}}\Delta p_{t_n^{k+1}-t}(z_1-z',z_2-z')\\ &\qquad\times\big(b(V_{g,s}(t,z')+\psi(t_l^{i+1/2},z'))-b(V_{g,s}(t,z')+\psi(t_l^{i+1},z))\big)dtdz'.\end{aligned}$$

We apply our main estimate (5.14) and assumption (5.22) to this identity. We derive

$$\begin{aligned}|I_{22}(l)|\leqslant&C_\phi|z_1-z_2|\Lambda_\delta(|z_1|\vee|z_2|\vee 1)2^{-l(2/3-\delta)}\\ &\times\sum_{i=k2^{l-n}}^{(k+1)2^{l-n}-1}(t_n^{k+1}-t_l^i)^{-1/2}\|\psi(t_l^{i+1},\cdot)-\psi(t_l^{i+1/2},\cdot)\|_{0,w}\\ &\leqslant C_\phi|z_1-z_2|\Lambda_\delta(|z_1|\vee|z_2|\vee 1)\alpha 2^{-l(2/3-\delta)}\sum_{i=k2^{l-n}}^{(k+1)2^{l-n}-1}((k+1)2^{l-n}-i)^{-1/2}\\ &\leqslant CC_\phi|z_1-z_2|\Lambda_\delta(|z_1|\vee|z_2|\vee 1)\alpha 2^{-l(1/6-\delta)}2^{-n/2}.\end{aligned}$$

Therefore

$$\sum_{l=n}^{\infty} I_{22}(l) \leqslant CC_\phi \alpha |z_1 - z_2| \Lambda_\delta(|z_1| \vee |z_2| \vee 1) 2^{-n(2/3-\delta)}. \tag{5.28}$$

Combining (5.24), (5.25), (5.26), (5.27), and (5.28), we finally obtain

$$\begin{aligned} Lip_n^{k+1} &\leqslant e^{2^{-n/2}}(1 + C2^{-n\delta/2}) Lip_n^k + CC_\phi 2^{-n(1/6-\delta)}(\|\psi(0,\cdot)\|_{0,w} + \alpha 2^{-m/2}) \\ &\quad + CC_\phi \alpha 2^{-n(2/3-\delta)} \\ &=: a_n Lip_n^k + b_n. \end{aligned}$$

We consider again binary approximations to t_n^k and employ again the same argument as we used in the proof of (5.23). We get

$$Lip_n^{k+1} \leqslant \left(\prod_{i=m+1}^{n} a_i \right) \left(Lip_m^0 + \sum_{i=m+1}^{n} b_i \right)$$

Since

$$\prod_{i=m+1}^{n} a_i = \prod_{i=m+1}^{n} e^{2^{-i/2}}(1 + C2^{-i\delta/2}) \leqslant \exp(\sum_{i=m+1}^{n} (2^{-i/2} + C2^{-i\delta/2}))$$

is bounded uniformly in n, we derive

$$Lip_n^k \leqslant CC_\phi \|\psi(0,\cdot)\|_{1,\delta} + CC_\phi \alpha 2^{-m(2/3-\delta)}, \quad n \geqslant m,\ k \in [0, 2^{n-m} - 1].$$

Using again continuity of the function ψ and the fact that the constants C, C_ϕ in the bound above do not depend on k, n, we arrive to the bound

$$[\psi(t,\cdot)]_{1,\delta} \leqslant CC_\phi \|\psi(0,\cdot)\|_{1,\delta} + CC_\phi \alpha 2^{-m(2/3-\delta)}, \quad t \in [0, 2^{-m}]. \tag{5.29}$$

Now we return to the main line of the proof. Recall that our aim is to estimate left-hand side of (5.22) and bound α. We treat large $|z|$ and small $|z|$ differently. Therefore, we fix a large threshold $M > 1$. The precise value of M will depend on m and will be chosen later.

If $|z|$ is large enough, we use very rough estimates from Lemma 5.5:

$$\sup_{|z| \geqslant M} e^{-|z|} |(\psi(t_n^{k+1}, z) - \psi(t_n^k, z))| \leqslant C_\phi e^{-M} 2^{-n/2}. \tag{5.30}$$

For small z we estimate the same quantity more precisely. Arguing similarly to (5.26) and using (5.7) and Lemma 4.10, we obtain for any $z \in \mathbb{R}$, and integers $n \geqslant m$, $k \in [0, 2^{n-m} - 1]$

$$\begin{aligned}\psi(t_n^{k+1}, z) - \psi(t_n^k, z) &= \int_{\mathbb{R}} p_{2^{-n}}(z')(\psi(t_n^k, z - z') - \psi(t_n^k, z))\, dz' \\ &+ \int_{\mathbb{R}} \int_{t_n^k}^{t_n^{k+1}} p_{t_n^{k+1}-t}(z - z')\big(b(V_{g,s}(t, z') + \psi_n(t, z')) - b(V_{g,s}(t, z'))\big) dt dz' \\ &+ \sum_{l=n}^{\infty} \int_{\mathbb{R}} \int_{t_n^k}^{t_n^{k+1}} p_{t_n^{k+1}-t}(z - z') \\ &\qquad \times \big(b(V_{g,s}(t, z) + \psi_{l+1}(t, z)) - b(V_{g,s}(t, z) + \psi_l(t, z))\big)\, dt dz' \\ &=: J_1(k, z) + J_2(k, z) + J_3(k, z)\end{aligned} \tag{5.31}$$

By the definition of Lip_n^k and (5.29) we have

$$\begin{aligned}\sup_{|z| \leqslant M} e^{-|z|} |J_1(k, z)| &\leqslant C Lip_n^k M^\delta 2^{-n/2} \\ &\leqslant C C_\phi M^\delta 2^{-n/2} (\|\psi(0, \cdot)\|_{1,\delta} + \alpha 2^{-m(2/3-\delta)}).\end{aligned} \tag{5.32}$$

To bound $e^{-|z|} J_2(k, z)$ we apply estimate (5.13). We get

$$\begin{aligned}&\sup_{|z| \leqslant M} e^{-|z|} |J_2(k, z)| \\ &\quad = \sup_{|z| \leqslant M} e^{-|z|} \Big| \int_{\mathbb{R}} \int_{t_n^k}^{t_n^{k+1}} p_{t_n^{k+1}-t}(z - z') \\ &\qquad\qquad \times \big(b(V_{g,s}(t, z') + \psi(t_n^{k+1}, z')) - b(V_{g,s}(t, z'))\big)\, dt dz' \Big| \\ &\quad \leqslant C_\phi M^\delta \|\psi(t_n^{k+1}, \cdot)\|_{0,w} 2^{-n(2/3-\delta)}.\end{aligned}$$

Hence, by (5.23) we have

$$\sup_{|z| \leqslant M} e^{-|z|} |J_2(k, z)| \leqslant C C_\phi 2^{-n(2/3-\delta)} M^\delta (\|\psi(0, \cdot)\|_{0,w} + \alpha 2^{-m/2}). \tag{5.33}$$

Finally, let's work with J_3. We estimate it using (5.13) and (5.22). We derive (similarly to derivation of the bound on $I_{22}(l)$)

$$\begin{aligned}(5.34)\quad \sup_{|z|\leqslant M} & e^{-|z|}|J_3(k,z)| \\ &\leqslant C_\phi M^\delta \sum_{l=n}^{\infty} \sum_{i=k2^{l-n}}^{(k+1)2^{l-n}-1} 2^{-l(2/3-\delta)} \|\psi(t_l^{i+1},\cdot)-\psi(t_l^{i+1/2},\cdot)\|_{0,w} \\ &\leqslant C_\phi M^\delta \alpha \sum_{l=n}^{\infty} 2^{-n} 2^{-l(1/6-\delta)} \\ &\leqslant C C_\phi M^\delta \alpha 2^{-n(7/6-\delta)}.\end{aligned}$$

Thus, it remains just to combine (5.30) and the obtained estimates of terms in the right-hand side of (5.31) (we use (5.32), (5.33), and (5.34)) to obtain for any integers $n \geqslant m$, $k \in [0, 2^{n-m}-1]$

$$\begin{aligned}\|\psi(t_n^{k+1},\cdot)-\psi(t_n^k,\cdot)\|_{0,w} \leqslant & C C_\phi 2^{-n/2} M^\delta (\|\psi(0,\cdot)\|_{1,\delta} + \alpha 2^{-m(2/3-\delta)}) \\ & + C_\phi e^{-M} 2^{-n/2}.\end{aligned}$$

Comparing this with (5.22) and using the definition of α, we get

$$\alpha \leqslant C C_\phi M^\delta (\|\psi(0,\cdot)\|_{1,\delta} + \alpha 2^{-m(2/3-\delta)}) + C_\phi e^{-M}.$$

Now we pick $M = 2^{m/(2\delta)}$ and rewrite the obtained bound

$$\alpha \leqslant C C_\phi 2^{-m(1/6-\delta)} \alpha + C C_\phi 2^{m/2} \|\psi(0,\cdot)\|_{1,\delta} + C_\phi e^{-2^{m/(2\delta)}}.$$

Recall that the constants C, C_ϕ do not depend on m. Thus, if we choose now $m_0 = m_0(\delta,\phi)$ large enough such that $C C_\phi 2^{-m_0(1/6-\delta)} < 1/2$, then for any $m \geqslant m_0$ we finally deduce

$$\alpha \leqslant C C_\phi 2^{m/2} \|\psi(0,\cdot)\|_{1,\delta} + C C_\phi e^{-2^{m/(2\delta)}}. \tag{5.35}$$

Let us stress once again, that the constants that appear in the right-hand side of the above equation (C, C_ϕ, m_0) depend only on δ and ϕ, but not on r (r was assumed to be equal to 0 in the beginning of the proof). The proof for other values of r is exactly the same with exactly the same final constants C, C_ϕ, m_0.

To complete the proof it remains just to substitute the obtained estimate of α (5.35) into estimates (5.23) and (5.29). Thus, we get the following final bound on the Hölder norm of $\psi(t,\cdot)$

$$\|\psi(t,\cdot)\|_{1,\delta} = \|\psi(t,\cdot)\|_{0,w} + [\psi(t,\cdot)]_{1,\delta} \leqslant C C_\phi \|\psi(0,\cdot)\|_{1,\delta} + C C_\phi e^{-2^{m/(2\delta)}},$$

valid for all $t \in [0, 2^{-m}]$. This proves (5.18). Estimate (5.19) is an immediate corollary of (5.18). □

Now we can straightforwardly estimate the behavior of ψ on bigger intervals and give a proof of uniqueness part of Theorem 2.1.

PROOF OF UNIQUENESS PART OF THEOREM 2.1. We apply Lemma 5.9 with $\delta = 1/10$ and zero initial condition $\phi = 0$. For large enough $m \geqslant m_0$ we apply bound (5.19) consequently 2^m times. We get that there exists a constant $C > 0$ such that for any integer $k \in [0, 2^m]$

$$\|\psi(k2^{-m}, \cdot)\|_{1,\delta} \leqslant C^{2^m} \exp(-2^{m/(2\delta)}).$$

Note that the constant C does not depend on m or k. Therefore, by letting $m \to \infty$ we get $\|\psi(t, \cdot)\|_{1,\delta} = 0$ for any dyadic $t \in [0, 1]$. By the continuity of ψ, we see that $\|\psi(t, \cdot)\|_{1,\delta} = 0$ for any $t \in [0, 1]$, which implies that ψ is identically 0 on $[0, 1] \times \mathbb{R}$.

Thus, equation (5.7) has only the trivial solution and therefore on Ω' equation (2.2) has a unique solution. This solution is in $\mathbf{B}(0+)$ by Proposition 5.1.

Finally, let us prove the last part of the theorem. Let $q_1, q_2 \in \mathbf{B}(0+)$ be two initial conditions and $q_1(z) = q_2(z)$ Lebesgue–almost everywhere. Let u_{s,q_1} and u_{s,q_2} be the solutions to (2.2) that start with initial conditions q_1 and q_2 correspondingly. Note that for $t \in (0, 1]$, $z \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} p_t(z - z')q_1(z')\, dz' = \int_{\mathbb{R}} p_t(z - z')q_2(z')\, dz'. \tag{5.36}$$

Consider now the function $v(t, z) := u_{s,q_2}(t, z, \omega)$ for $t \in (0, 1]$, $z \in \mathbb{R}$, and $v(0, z) := q_1(z)$, $z \in \mathbb{R}$. Identity (5.36) implies that v is another solution to (2.2) that start with initial condition q_1 at time s. By uniqueness, $v = u_{s,q_1}$. Thus, $u_{s,q_1}(t, z) = u_{s,q_2}(t, z)$ for $t \in (0, 1]$, $z \in \mathbb{R}$. □

5.3 Proof of Theorem 2.2(b): continuity of the flow.

PROOF OF THEOREM 2.2(B). Fix $\omega \in \Omega'$, $s \geqslant 0$, $t > s$, $z \in \mathbb{R}$ and the initial conditions $(q_n)_{n \in \mathbb{Z}_+}$ satisfying the assumptions of the theorem. It follows from the definition of φ in Part (a) of the theorem that $\varphi(s, \cdot, q_n, \omega)$ is a solution to (2.2) that starts at time s with the condition q_n. Since all the assumptions of Lemma 5.2 are met, we see that there exist a subsequence $(n_k)_{k \in \mathbb{Z}_+}$ such that

$$\lim_{k \to \infty} \varphi(s, t, q_{n_k}, \omega)(z) = \widetilde{u}(t, z, \omega)$$

for some $\widetilde{u}$ and $\widetilde{u}(\cdot, \cdot, \omega)$ solves (2.2) that starts at time s with the initial condition q. On the other hand, $\varphi(s, \cdot, q, \omega)$ is also a solution to (2.2) that starts at time s with the initial condition q. Therefore, by Theorem 2.1, these solutions coincide and $\widetilde{u}(t,z,\omega)=\varphi(s,t,q,\omega)(z)$.

Thus, $(\varphi(s,t,q_n,\omega)(z))_{n\in\mathbb{Z}_+}$ is a sequence of real numbers such that each subsequence of it has a converging sub-subsequence. Since all the limiting points coincide (and equal to $\varphi(s,t,q,\omega)(z)$), we see by the standard argument that

$$\lim_{n\to\infty}\varphi(s,t,q_n,\omega)(z)=\varphi(s,t,q,\omega)(z). \qquad \square$$

6. Proof of Proposition 3.2 (Moment bound). The proof of this proposition is long and tedious. Note that all the difficulties come from the fact that the function $t\mapsto V(t,z)$ is not a semimartingale; if this were the case, then an application of Itô's lemma would imply the required bound. In our proof we were inspired by the ideas from [3, Section 4].

We begin by observing that the process $(V(\cdot,z))_{z\in\mathbb{R}}$ is stationary. Hence for any $z_1,z_2\in\mathbb{R}$

$$\mathrm{Law}(V(\cdot,z_1),V(\cdot,z_2))=\mathrm{Law}(V(\cdot,z_1-z_2),V(\cdot,0)). \tag{6.1}$$

Therefore, it will be sufficient to establish inequality (3.5) in Proposition 3.2 only for $z_1=z$, $z_2=0$, $z\in\mathbb{R}$. Since we have assumed that $|z_1-z_2|\leqslant 1$, it is enough to prove (3.5) for $z_1=z$, $z_2=0$, $|z|\leqslant 1$.

To present the proof we need to introduce a number of new objects. We fix $T>0$, the function b that appears in the statement of Proposition 3.2 and consider the random function

$$H(t,z,\alpha,\beta):=b'(V(t,z)+\alpha)-b'(V(t,0)+\beta),\quad t\in[0,T],\ z,\alpha,\beta\in\mathbb{R}. \tag{6.2}$$

Recall that it is assumed that b is a bounded differentiable function with bounded derivative. Without loss of generality we suppose that

$$\|b\|_\infty\leqslant 1 \tag{6.3}$$

(otherwise we consider the function $\widetilde{b}:=b/\|b\|_\infty$ instead of b). All constants that appear in this section do not depend on the function b (satisfying condition (6.3)).

Fix $z\in[-1,1]$, $\alpha,\beta\in\mathbb{R}$, $t_1,t_2\in[0,T]$, $t_1<t_2$ and define a martingale $M^{t_1,t_2}=(M_t^{t_1,t_2})_{t_1\leqslant t\leqslant t_2}$, where

$$M_t^{t_1,t_2}:=\mathsf{E}\Big[\int_{t_1}^{t_2}H(r,z,\alpha,\beta)\,dr\Big|\mathcal{F}_t\Big],\quad t_1\leqslant t\leqslant t_2. \tag{6.4}$$

Recall that $(\mathcal{F}_t)_{t\geqslant 0}$ is the filtration associated with $\dot{W}$. By definition, for any $z\in[-1,1]$, $0\leqslant s\leqslant t$ the random variable $V(s,t,z,\omega)$ is $\mathcal{F}_t$-measurable.

Using the new notation and taking into account formula (6.1), we can rewrite the left-hand side of inequality (3.5) in Proposition 3.2 as $\mathsf{E}|M_{t_2}^{t_1,t_2}|^p$. We clearly have

$$\mathsf{E}|M_{t_2}^{t_1,t_2}|^p \leqslant C\mathsf{E}|M_{t_1}^{t_1,t_2}|^p + C\mathsf{E}|M_{t_2}^{t_1,t_2} - M_{t_1}^{t_1,t_2}|^p. \tag{6.5}$$

The first term in the right-hand side of (6.5) is easy to bound. Namely we first estimate $\mathsf{E}\big[H(r,z,\alpha,\beta)|\mathcal{F}_{t_1}\big]$ for $r \in [t_1,t_2]$ (this is done in Lemma 6.2) and then apply the integral Minkowski inequality.

To estimate the second term in the right-hand side of (6.5), we first calculate the quadratic variation of the martingale M^{t_1,t_2} (that is, $[M^{t_1,t_2}]_t$) and then apply the Burkholder–Davis–Gundy inequality.

Let us briefly explain how we estimate $[M^{t_1,t_2}]_t$. To simplify the notation, if there is no ambiguity, further we drop the superindex (t_1,t_2) and write M_t instead of $M_t^{t_1,t_2}$. Recall that for continuous martingales quadratic variation $[\cdot]_t$ equals predictable quadratic variation $\langle\cdot\rangle_t$. To calculate $\langle M\rangle_t$ we use the following identity valid for $t_1 \leqslant r \leqslant s \leqslant t_2$

$$\begin{aligned} M_s - M_r &= \int_{t_1}^{t_2} \mathsf{E}([H(t,z,\alpha,\beta)\big|\mathcal{F}_s] - \mathsf{E}[H(t,z,\alpha,\beta)\big|\mathcal{F}_r])\,dt \\ &= \int_{s}^{t_2} \mathsf{E}([H(t,z,\alpha,\beta)\big|\mathcal{F}_s] - \mathsf{E}[H(t,z,\alpha,\beta)\big|\mathcal{F}_r])\,dt \\ &\quad + \int_r^s H(t,z,\alpha,\beta)\,dt - \int_r^s \mathsf{E}[H(t,z,\alpha,\beta)\big|\mathcal{F}_r]\,dt \\ &=: I_1(r,s,t_2) + I_2(r,s) - I_3(r,s). \end{aligned} \tag{6.6}$$

Note that the terms I_1, I_2, I_3 depend also on α,β,z. To simplify the notation we omit this dependence.

Thus, we can bound the term $\mathsf{E}\big((M_s - M_r)^2\big|\mathcal{F}_r\big)$, which is the main ingredient of $\langle M\rangle_t$, as follows

$$\mathsf{E}\big((M_s - M_r)^2\big|\mathcal{F}_r\big) \leqslant C\mathsf{E}(I_1(r,s,t_2)^2|\mathcal{F}_r) + C\mathsf{E}(I_2(r,s)^2|\mathcal{F}_r) + CI_3(r,s)^2,$$

where we have also used $\mathcal{F}_r$-measurability of $I_3(r,s)$. The corresponding bounds for the terms in right–hand side of the above inequality are obtained in Lemmas 6.2–6.4. We combine all these bounds together in Lemma 6.5.

To prove that the martingale M is continuous we split it into two parts $M := L + N$, where

$$L_t := I_2(t_1,t) = \int_{t_1}^t H(r,z,\alpha,\beta)\,dr, \quad t_1 \leqslant t \leqslant t_2; \tag{6.7}$$

$$N_t := \mathsf{E}\Big[\int_t^{t_2} H(r,z,\alpha,\beta)\,dr\big|\mathcal{F}_t\Big], \quad t_1 \leqslant r \leqslant t_2. \tag{6.8}$$

Since H is bounded, the process L is obviously continuous. To prove that N is also continuous we employ the Kolmogorov continuity theorem. We use the identity $N_s - N_r = I_1(r,s,t_2) - I_3(r,s)$ valid for $t_1 \leqslant r \leqslant s \leqslant t_2$. This is done in Lemma 6.6.

Finally we calculate quadratic variation $[M]_t$, which, by the continuity of M, is equal to $\langle M\rangle_t$. This is done also in Lemma 6.6. At the end of the section we combine the obtained estimates and prove Proposition 3.2.

We begin with the following simple estimate.

LEMMA 6.1. *Let b be a bounded differentiable function with bounded derivative and $\|b\|_\infty \leqslant 1$. Then for any $\delta_1, \delta_2, \delta_3 \in [0,1]$ there exists a constant $C = C(\delta_1,\delta_2,\delta_3)$ such that for any $a_0, a_1, a_2, a_3 \in \mathbb{R}$ and any Gaussian random variable X with zero mean and variance $\operatorname{Var} X = \sigma^2$ we have the following bounds:*

$$\Bigl|\mathsf{E}\bigl(b'(X+a_0+a_1) - b'(X+a_0)\bigr)\Bigr| \leqslant C\sigma^{-1-\delta_1}|a_1|^{\delta_1}; \tag{6.9}$$

$$\begin{aligned}\Bigl|\mathsf{E}\bigl(b'(X+a_0+a_1+a_2) - b'(X+a_0+a_1) - b'(X+a_0+a_2) + b'(X+a_0)\bigr)\Bigr|\\ \leqslant C\sigma^{-1-\delta_1-\delta_2}|a_1|^{\delta_1}|a_2|^{\delta_2};\end{aligned} \tag{6.10}$$

$$\begin{aligned}\Bigl|\mathsf{E}\bigl(b'(X+a_0+a_1+a_2) - b'(X+a_0+a_3) - b'(X+a_0+a_2) + b'(X+a_0)\bigr)\Bigr|\\ \leqslant C\sigma^{-1-\delta_3}|a_1-a_3|^{\delta_3} + C\sigma^{-1-\delta_1-\delta_2}|a_1|^{\delta_1}|a_2|^{\delta_2}.\end{aligned} \tag{6.11}$$

PROOF. First, we establish bound (6.10). Fix arbitrary $\delta_1, \delta_2 \in [0,1]$. We use integration by parts and rewrite the left-hand side of (6.10) in the following form:

$$\begin{aligned}&\Bigl|\mathsf{E}\bigl(b'(X+a_0+a_1+a_2) - b'(X+a_0+a_1) - b'(X+a_0+a_2) + b'(X+a_0)\bigr)\Bigr|\\ &= \Bigl|\int_{\mathbb{R}}\bigl(b(x+a_0+a_1+a_2) - b(x+a_0+a_2) - b(x+a_0+a_1) + b(x+a_0)\bigr)p'_{\sigma^2}(x)\,dx\Bigr|\\ &= \Bigl|\int_{\mathbb{R}} b(x+a_0)\bigl(p'_{\sigma^2}(x-a_1-a_2) - p'_{\sigma^2}(x-a_2) - p'_{\sigma^2}(x-a_1) + p'_{\sigma^2}(x)\bigr)\,dx\Bigr|\\ &\leqslant C\sigma^{-1-\delta_1-\delta_2}|a_1|^{\delta_1}|a_2|^{\delta_2},\end{aligned}$$

where for the last inequality we used boundedness of the function b and Lemma 4.2. Estimate (6.9) is derived by the same argument. Estimate (6.11) is a direct corollary of (6.9) and (6.10). □

Now we are moving on to calculating the quadratic variation of the martingale M. In the next three lemmas we will obtain a moment bound on I_1

(recall its definition in (6.6)). Recall also the definition of $\mathcal{V}$ and H in (4.6) and (6.2), respectively.

LEMMA 6.2. *Let $\delta \in [0,1]$. There exists $C = C(T,\delta)$ such that for any $0 \leqslant r \leqslant s \leqslant t \leqslant T$, $z,\alpha,\beta \in \mathbb{R}$ we have*

$$\begin{aligned}(6.12)\quad &\Big|\mathsf{E}\big[H(t,z,\alpha,\beta)|\mathcal{F}_s\big] - \mathsf{E}\big[H(t,z,\alpha,\beta)|\mathcal{F}_r\big]\Big| \\ &\leqslant C(t-s)^{-1/2}|\mathcal{V}(r,s,t,z) - \mathcal{V}(r,s,t,0)| \\ &\quad + C(t-s)^{-1/2-\delta/4}(|\mathcal{V}(0,r,t,z)-\mathcal{V}(0,r,t,0)|^\delta + |\alpha-\beta|^\delta) \\ &\quad \times (\mathsf{E}|\mathcal{V}(r,s,t,0)| + |\mathcal{V}(r,s,t,0)|);\end{aligned}$$

$$\begin{aligned}(6.13)\quad &\Big|\mathsf{E}\big[H(t,z,\alpha,\beta)|\mathcal{F}_s\big]\Big| \\ &\leqslant C(t-s)^{-(1+\delta)/4}\big(|\mathcal{V}(0,s,t,z) - \mathcal{V}(0,s,t,0)|^\delta + |\alpha-\beta|^\delta\big).\end{aligned}$$

PROOF. We start with the proof of inequality (6.12). Fix $0 \leqslant r \leqslant s \leqslant t \leqslant T$, $z,\alpha,\beta \in \mathbb{R}$. For $i = 1,2$ introduce the following random variables:

$$X_i := \mathcal{V}(0,r,t,z_i);\ Y_i := \mathcal{V}(r,s,t,z_i);\ Z_i := \mathcal{V}(s,t,t,z_i),$$

where $z_1 := 0$, $z_2 := z$. We clearly have $V(t,z_i) = X_i + Y_i + Z_i$, $i = 1,2$. Note also that the random vectors (X_1,X_2), (Y_1,Y_2) and (Z_1,Z_2) are Gaussian and independent. Additionally, $\mathrm{Law}(X_1) = \mathrm{Law}(X_2)$, $\mathrm{Law}(Y_1) = \mathrm{Law}(Y_2)$, $\mathrm{Law}(Z_1) = \mathrm{Law}(Z_2)$. Define for $x,y \in \mathbb{R}$

$$J(x,y) := \mathsf{E}(b'(x+Z_1) - b'(y+Z_1)).$$

With this notation in hand we rewrite

$$\begin{aligned}(6.14)\quad \mathsf{E}\big[H(t,z,\alpha,\beta)|\mathcal{F}_s\big] &= \mathsf{E}\big[b'(X_1+Y_1+Z_1+\alpha) - b'(X_2+Y_2+Z_2+\beta)\big|X_1,X_2,Y_1,Y_2\big] \\ &= \mathsf{E}\big[b'(X_1+Y_1+Z_1+\alpha) - b'(X_2+Y_2+Z_1+\beta)\big|X_1,X_2,Y_1,Y_2\big] \\ &= J(X_1+Y_1+\alpha, X_2+Y_2+\beta)\end{aligned}$$

and

$$\begin{aligned}(6.15)\quad \mathsf{E}\big[H(t,z,\alpha,\beta)|\mathcal{F}_r\big] &= \mathsf{E}\big[b'(X_1+Y_1+Z_1+\alpha) - b'(X_2+Y_2+Z_2+\beta)\big|X_1,X_2\big] \\ &= \mathsf{E}\big[b'(X_1+Y_1+Z_1+\alpha) - b'(X_2+Y_1+Z_1+\beta)\big|X_1,X_2\big] \\ &= [\mathsf{E}J(x_1+Y_1+\alpha, x_2+Y_1+\beta)]\Big|_{\substack{x_1=X_1\\x_2=X_2}} \\ &= \int_{\mathbb{R}} J(X_1+y_1+\alpha, X_2+y_1+\beta)p_{\mathrm{Var}\,Y_1}(y_1)\,dy_1.\end{aligned}$$

Now take any $c_1, c_2, d_1, d_2 \in \mathbb{R}$. We apply inequality (6.11) with the following set of parameters: $\delta_1 \leftarrow 1$, $\delta_2 \leftarrow \delta$, $\delta_3 \leftarrow 1$, $a_0 \leftarrow d_2$, $a_1 \leftarrow c_1 - d_1$, $a_2 \leftarrow d_1 - d_2$, and $a_3 \leftarrow c_2 - d_2$. We obtain

$$\begin{aligned}(6.16)\quad &|J(c_1,c_2) - J(d_1,d_2)| \\ &\qquad \leqslant C(t-s)^{-1/2}|c_1-d_1-c_2+d_2| + C(t-s)^{-1/2-\delta/4}|c_1-d_1||d_1-d_2|^{\delta},\end{aligned}$$

where we also used the fact that

$$\operatorname{Var} Z_1 = \operatorname{Var} \mathcal{V}(s,t,t,0) = C(t-s)^{1/2}. \tag{6.17}$$

Combining (6.14), (6.15) and (6.16), we deduce

$$\begin{aligned}\Big|\mathsf{E}\big[H(t,z,\alpha,\beta)|\mathcal{F}_s\big] - \mathsf{E}\big[H(t,z,\alpha,\beta)|\mathcal{F}_r\big]\Big| \\ \leqslant C(t-s)^{-1/2}|Y_1 - Y_2| \\ + C(t-s)^{-1/2-\delta/4}(|Y_1| + \mathsf{E}|Y_1|)(|X_1 - X_2|^{\delta} + |\alpha - \beta|^{\delta}),\end{aligned}$$

which is (6.12).

To establish inequality (6.13) we just note that for any $c_1, c_2 \in \mathbb{R}$, $\delta \in [0,1]$ it follows from (6.9) and (6.17) that

$$|J(c_1,c_2)| \leqslant C(\operatorname{Var} Z_1)^{-1/2-\delta/2}|c_1 - c_2|^{\delta} \leqslant C(t-s)^{-1/4-\delta/4}|c_1 - c_2|^{\delta}.$$

Combining this with (6.14), we come to (6.13). □

The next statement can be called the conditional integral Minkowski inequality. This inequality is definitely not new; however we were not able to find its proof in the literature. So we provide it here for the completeness of the argument.

Lemma 6.3. *Let $(\xi(t))_{t\geqslant 0}$ be a random process. Then for any σ–field $\mathcal{G} \subset \mathcal{F}$ and $0 \leqslant a \leqslant b$ we have*

$$\mathsf{E}\Big[\Big(\int_a^b \xi(t)\,dt\Big)^2\Big|\mathcal{G}\Big] \leqslant \Big(\int_a^b (\mathsf{E}[\xi(t)^2|\mathcal{G}])^{1/2}\,dt\Big)^2.$$

Proof. By the Fubini theorem, we have

$$\begin{aligned}\mathsf{E}\Big[\Big(\int_a^b \xi(t)\,dt\Big)^2\Big|\mathcal{G}\Big] &= \int_a^b\int_a^b \mathsf{E}[\xi(t)\xi(s)|\mathcal{G}]\,dtds \\ &\leqslant \int_a^b\int_a^b (\mathsf{E}[\xi(t)^2|\mathcal{G}])^{1/2}(\mathsf{E}[\xi(s)^2|\mathcal{G}])^{1/2}\,dtds \\ &= \Big[\int_a^b (\mathsf{E}[\xi(t)^2|\mathcal{G}])^{1/2}\,dt\Big]^2.\end{aligned}$$

□

The next lemma provides estimates on the moments of I_1. Recall its definition in (6.6).

LEMMA 6.4. *Let $p>0$, $\delta\in(0,1)$, $\delta'\in(0,\delta)$. Then there exist a random variable $K(\omega)$ and a constant $C=C(p,T,\delta,\delta')>0$ such that $\mathsf{E}K(\omega)^p\leqslant C$ and for any $0\leqslant r\leqslant s\leqslant t\leqslant T$, $z,\alpha,\beta\in\mathbb{R}$, $|z|\leqslant 1$ we have almost surely*

$$\mathsf{E}[I_1(r,s,t)^2|\mathcal{F}_r]\leqslant K(\omega)(s-r)(t-s)^{1/2-\delta/2}\big(|z|^{\delta'}+|\alpha-\beta|^{2\delta}\big). \tag{6.18}$$

We also have for $\delta\in(0,1/2)$

$$\mathsf{E}(I_1(r,s,t))^4\leqslant C(s-r)^2(t-s)^{1-\delta}(|z|^{2\delta}+|\alpha-\beta|^{4\delta}). \tag{6.19}$$

PROOF. Fix $p>0$, $\delta\in(0,1)$, $\delta'\in(0,\delta)$. We employ estimate (6.12) from Lemma 6.2 and use the corresponding estimates from Lemma 4.7 and Lemma 4.8 to obtain that there exists a random variable $K(\omega)=K(\omega,p,T,\delta,\delta')>0$ and a constant $C=C(p,T,\delta,\delta')$ such that $\mathsf{E}K(\omega)^p\leqslant C$ and

$$\begin{aligned}\mathsf{E}\Big[\Big(\mathsf{E}\big(H(t',z,\alpha,\beta)|\mathcal{F}_s\big)-\mathsf{E}\big(H(t',z,\alpha,\beta)|\mathcal{F}_r\big)\Big)^2\Big|\mathcal{F}_r\Big]&\\ \leqslant K(\omega)(t'-s)^{-3/2-\delta/2}(s-r)\big(|z|^{\delta'}+|\alpha-\beta|^{2\delta}\big)&\end{aligned} \tag{6.20}$$

for any $0\leqslant r\leqslant s\leqslant t'$, $z,\alpha,\beta\in\mathbb{R}$, $|z|\leqslant 1$. An application of the conditional integral Minkowski inequality (Lemma 6.3) to (6.20) leads to (6.18).

Similarly, to establish bound (6.19) we also use estimate (6.12) from Lemma 6.2 and Lemma 4.8. We get

$$\begin{aligned}\mathsf{E}\Big(\mathsf{E}\big(H(t',z,\alpha,\beta)|\mathcal{F}_s\big)-\mathsf{E}\big(H(t',z,\alpha,\beta)|\mathcal{F}_r\big)\Big)^4&\\ \leqslant C(t'-s)^{-3-\delta}(s-r)^2\big(|z|^{2\delta}+|\alpha-\beta|^{4\delta}\big).&\end{aligned}$$

The proof is competed by an application of the integral Minkowski inequality. □

We are almost ready to carry out the main goal of this subsection, that is calculating quadratic variation of the martingale M^{t_1,t_2} (recall that this martingale is defined in (6.4)). As mentioned before, a remaining technical step is to prove that this martingale is continuous. Recall that to do it we have split $M=M^{t_1,t_2}$ into two parts: $M=N+L$, see their definitions in (6.7) and (6.8).

LEMMA 6.5. *Let $p > 0$, $\delta \in (0,1)$, $\delta' \in (0,\delta)$. Then there exist a random variable $K(\omega) > 0$ and a constant $C = C(p,T,\delta,\delta')$ such that $\mathsf{E}K(\omega)^p \leqslant C$ and for any $0 \leqslant t_1 \leqslant r \leqslant s \leqslant t_2 \leqslant T$, $z,\alpha,\beta \in \mathbb{R}$, $|z| \leqslant 1$ we have*

$$\begin{aligned}(6.21)\quad & \mathsf{E}[(M_s - M_r)^2|\mathcal{F}_r] \\ & \qquad \leqslant K(\omega)(s-r)(t_2-r)^{1/2-\delta/2}(|z|^{\delta'} + |\alpha-\beta|^{2\delta}) + 8\|b'\|_\infty^2(s-r)^2;\\ (6.22)\quad & \mathsf{E}(N_s - N_r)^4 \leqslant C(s-r)^2\big(1 + |\alpha-\beta|^{4\delta} + \|b'\|_\infty^4\big).\end{aligned}$$

PROOF. Recall that according to our definitions (see the beginning of this section) we have

$$\mathsf{E}\big((M_s - M_r)^2\big|\mathcal{F}_r\big) \leqslant C\mathsf{E}(I_1(r,s,t_2)^2|\mathcal{F}_r) + C\mathsf{E}(I_2(r,s)^2|\mathcal{F}_r) + CI_3(r,s)^2.$$

By Lemma 6.4,

$$\mathsf{E}[I_1(r,s,t_2)^2|\mathcal{F}_r] \leqslant K(\omega)(s-r)(t_2-s)^{1/2-\delta/2}\big(|z|^{\delta'} + |\alpha-\beta|^{2\delta}\big).$$

Note that the terms I_2 and I_3 are of order $s-r$. Therefore they will not impact the quadratic variation. Thus, we estimate them using a very rough estimate:

$$|I_2(r,s)| \leqslant 2\|b'\|_\infty(s-r); \quad |I_3(r,s)| \leqslant 2\|b'\|_\infty(s-r). \tag{6.23}$$

Hence

$$\mathsf{E}[I_2(r,s)^2|\mathcal{F}_r] + I_3(r,s)^2 \leqslant 8\|b'\|_\infty^2(s-r)^2.$$

Thus, we have

$$\mathsf{E}[(M_s-M_r)^2|\mathcal{F}_r] \leqslant K(\omega)(s-r)(t_2-s)^{1/2-\delta/2}(|z|^{\delta'}+|\alpha-\beta|^{2\delta})+8\|b'\|_\infty^2(s-r)^2,$$

from which (6.21) follows immediately.

In a similar manner,

$$\begin{aligned}\mathsf{E}[(N_s - N_r)^4] &= \mathsf{E}(I_1(r,s,t_2) - I_3(r,s))^4\\ &\leqslant C(\mathsf{E}I_1(r,s,t_2)^4 + \mathsf{E}I_3(r,s)^4)\\ &\leqslant C(s-r)^2(t_2-s)^{1-\delta}\big(|z|^{2\delta} + |\alpha-\beta|^{4\delta}\big) + C\|b'\|_\infty^4(s-r)^4,\end{aligned}$$

where we have used (6.23) and estimate (6.19) from Lemma 6.4. This implies (6.22). □

Now we have all the tools to bound the quadratic variation of M^{t_1,t_2}.

LEMMA 6.6. *For any* $0 \leqslant t_1 \leqslant t_2 \leqslant T$, $z, \alpha, \beta \in \mathbb{R}$, $|z| \leqslant 1$ *the martingale* M^{t_1,t_2} *defined in* (6.4) *is continuous. Moreover, for any* $\delta \in (0,1)$, $\delta' \in (0,\delta)$ *and any* $p > 0$ *there exist a random variable* $K(\omega) > 0$ *and a constant* $C = C(p,T,\delta,\delta')$ *such that* $\mathsf{E}K(\omega)^p \leqslant C$ *and*

$$[M^{t_1,t_2}, M^{t_1,t_2}]_{t_2} \leqslant K(\omega)(|z|^{\delta'} + |\alpha-\beta|^{2\delta})(t_2-t_1)^{3/2-\delta/2}. \tag{6.24}$$

PROOF. Fix $0 \leqslant t_1 \leqslant t_2 \leqslant T$, $z, \alpha, \beta \in \mathbb{R}$, $|z| \leqslant 1$. First we prove that the martingale M^{t_1,t_2} is continuous. We make use of Lemma 6.5 to obtain for $r, s \in [t_1, t_2]$

$$\mathsf{E}(N_s - N_r)^4 \leqslant C(T,\alpha,\beta,z)(\|b'\|_\infty^4 \vee 1)(s-r)^2.$$

Hence, by the Kolmogorov continuity theorem, the process $(N_s)_{t_1\leqslant s\leqslant t_2}$ is continuous. The process $(L_s)_{t_1\leqslant s\leqslant t_2}$ is also continuous since the function H is bounded. Thus, the martingale M^{t_1,t_2} is continuous as a sum of two continuous processes.

We move on and calculate predictable quadratic variation of the martingale M^{t_1,t_2}. We employ Lemma 6.5 to get

$$\begin{aligned}
&\langle M^{t_1,t_2}, M^{t_1,t_2}\rangle_{t_2}\\
&\quad= \lim_{n\to\infty}\sum_{k=0}^{n-1}\mathsf{E}\big[(M^{t_1,t_2}_{t_1+(k+1)(t_2-t_1)/n} - M^{t_1,t_2}_{t_1+k(t_2-t_1)/n})^2\big|\mathcal{F}_{t_1+k(t_2-t_1)/n}\big]\\
&\quad\leqslant K(\omega)(|z|^{\delta'} + |\alpha-\beta|^{2\delta})(t_2-t_1)^{3/2-\delta/2}\lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}(n-k)^{1/2-\delta/2}}{n^{3/2-\delta/2}}\\
&\qquad + 8\|b'\|_\infty^2(t_2-t_1)^2\lim_{n\to\infty} n^{-1}\\
&\quad\leqslant K(\omega)(|z|^{\delta'} + |\alpha-\beta|^{2\delta})(t_2-t_1)^{3/2-\delta/2}.
\end{aligned}$$

By above, the martingale M^{t_1,t_2} is continuous. Hence its quadratic variation is equal to its predictable quadratic variation, that is $[M^{t_1,t_2}, M^{t_1,t_2}]_t = \langle M^{t_1,t_2}, M^{t_1,t_2}\rangle_t$. This proves (6.24). □

Finally, we can prove Proposition 3.2.

PROOF OF PROPOSITION 3.2. Fix $p \geqslant 1$. As we already pointed out at the beginning of this section, it is sufficient to show (3.5) only for $z_1 = z$, $z_2 = 0$, where $|z| \leqslant 1$. Note also that

$$\mathsf{E}\Big(\int_{t_1}^{t_2}\big(b'(V(u,z)+\alpha) - b'(V(u,0)+\beta)\big)\,du\Big)^p = \mathsf{E}|M^{t_1,t_2}_{t_2}|^p.$$

It follows from the Burkholder–Davis–Gundy inequality and Lemma 6.6 that

$$\mathsf{E}|M^{t_1,t_2}_{t_2}-M^{t_1,t_2}_{t_1}|^p \leqslant C\mathsf{E}[M^{t_1,t_2},M^{t_1,t_2}]^{p/2}_{t_2} \leqslant C(t_2-t_1)^{p(3/4-\delta/4)}(|z|^{p\delta'/2}+|\alpha-\beta|^{p\delta}), \tag{6.25}$$

where we have also used the finiteness of the $p/2$-th moment of $K(\omega)$. By the integral Minkowski inequality,

$$\mathsf{E}|M^{t_1,t_2}_{t_1}|^p = \|M^{t_1,t_2}_{t_1}\|^p_p = \Big\|\int_{t_1}^{t_2} \mathsf{E}\big[H(t,z,\alpha,\beta)\big|\mathcal{F}_{t_1}\big]\,dt\Big\|^p_p \leqslant \Big(\int_{t_1}^{t_2} \|\mathsf{E}\big[H(t,z,\alpha,\beta)\big|\mathcal{F}_{t_1}\big]\|_p\,dt\Big)^p \tag{6.26}$$

We employ estimate (6.13) from Lemma 6.2 and Lemma 4.8 to get

$$\|\mathsf{E}\big[H(t,z,\alpha,\beta)\big|\mathcal{F}_{t_1}\big]\|_p \leqslant C(t-t_1)^{-(1+\delta)/4}(|z|^{\delta/2}+|\alpha-\beta|^{\delta}).$$

Combining this inequality with (6.26) we obtain

$$\mathsf{E}|M^{t_1,t_2}_{t_1}|^p \leqslant C(t_2-t_1)^{p(3/4-\delta/4)}(|z|^{p\delta/2}+|\alpha-\beta|^{p\delta}).$$

Inequality (3.5) follows now from this, (6.25) and the following simple observation:

$$\mathsf{E}|M^{t_1,t_2}_{t_2}|^p \leqslant C(\mathsf{E}|M^{t_1,t_2}_{t_1}|^p + \mathsf{E}|M^{t_1,t_2}_{t_2}-M^{t_1,t_2}_{t_1}|^p).$$

The second part of Proposition 3.2 (inequality (3.6)) is established along the same lines as inequality (3.5). □

7. Proofs of Lemma 4.9 and Lemma 3.3.

Proof of Lemma 4.9. The proof is based on Proposition 3.2 and the ideas from the proofs of [5, Lemma 3.3] and [22, Lemma 3.4]. Before we begin the proof let us just note that in the case $f_1 = const$, $f_2 = const$ inequality (4.21) is almost obvious; one should just calculate the corresponding expected value and apply the Chebyshev inequality, see inequality (7.1) below. If f_1 and f_2 are piecewise constant functions, establishing (4.21) is also not very difficult. Thus to prove (4.21) for the general case, we first establish it for a suitable piecewise-continuous approximations of f_1, f_2 and then pass to the limit. Let us carry out this plan.

Fix $\varepsilon > 0$, $M > 0$, $N \in \mathbb{N}$, $h > 1/2$ and take sufficiently small δ. Without loss of generality and to simplify the notation, we assume $T = 1$, $\mu = 1$; the proof for other values of T, μ is exactly the same. We will choose a specific δ later. Let U be any set such that $|U|\leqslant\delta$. Let us verify that inequality (4.21) holds on a large enough set for all $z\in[-N,N]$, $r\in\mathbb{N}$, $f_1,f_2\in\mathcal{C}^h(1,1,M)$. Note that by definition of the class $\mathcal{C}^h$, we have

$$\sup_{\substack{z\in[-N,N]\\ t\in[0,1]}}|f_i(t,z)|\leqslant NM,\quad i=1,2.$$

The proof strategy relies on two observations. First, we note that the random variable $V(t,z)$ has a Gaussian distribution with mean 0 and variance $\sqrt{t/\pi}$, see (4.18). Hence for any $x\in\mathbb{R}$, $z\in\mathbb{R}$, $0\leqslant t_1<t_2\leqslant 2$ we have

$$\begin{aligned}\mathsf{E}\int_{t_1}^{t_2}\mathbb{1}_U(V(t,z)+x)dt&=\int_{t_1}^{t_2}\int_{\mathbb{R}}\mathbb{1}_U(y+x)p_{\sqrt{t/\pi}}(y)\,dydt\\ &\leqslant C_1\Big(\int_{\mathbb{R}}\mathbb{1}_U(y)dy\Big)^{1/2}\int_{t_1}^{t_2}t^{-1/8}dt\\ &\leqslant C_1\sqrt{\delta}|t_2-t_1|^{7/8}.\end{aligned}\tag{7.1}$$

We fix a large integer m (the precise value of m will be chosen later) and split the interval $[0,2]$ into 2^{m+1} smaller subintervals. For $k\in[0,2^{m+1}-1]$ consider the event

$$A_m^k(\varepsilon,x,z):=\Big\{\int_{k2^{-m}}^{(k+1)2^{-m}}\mathbb{1}_U(V(t,z)+x)\,dt<\varepsilon 2^{-m}\Big\}.$$

By the Chebyshev inequality, (7.1) implies $\mathsf{P}(A_k(x,\varepsilon,z))\geqslant 1-C_1\sqrt{\delta}\varepsilon^{-1}2^{m/8}$. Thus, for the event

$$A_m(\varepsilon):=\bigcap_{k=0}^{2^{m+1}-1}\ \bigcap_{i=-MN2^{4m+1}}^{MN2^{4m+1}}\ \bigcap_{j=-N2^{8m}}^{N2^{8m}}A_m^k(\varepsilon,i2^{-4m},j2^{-8m})$$

we have

$$\mathsf{P}(A_m(\varepsilon))\geqslant 1-C_1MN^2 2^{14m}\sqrt{\delta}\varepsilon^{-1}.$$

Second, we fix $\rho\in(1/2,1)$ and $\theta\in(0,1)$ such that

$$\rho(h-1/4)-1/4-\theta>\theta.\tag{7.2}$$

Since we have assumed that $h>1/2$, we see that such ρ, θ exist. We derive from Proposition 3.2 and the Kolmogorov continuity theorem that for $x,y\in[-2MN,2MN]$, $z_1,z_2\in[-N,N]$, $0\leqslant t_1<t_2\leqslant 2$ we have

$$\begin{aligned}&\Big|\int_{t_1}^{t_2}(\mathbb{1}_U(V(t,z_1)+x)-\mathbb{1}_U(V(t,z_2)+y))\,dt\Big|\\ &\qquad\leqslant K(\omega)(t_2-t_1)^{3/4-\rho/4-\theta}(|x-y|^{\rho}+|z_1-z_2|^{\rho/2}),\end{aligned}$$

where $\mathsf{E}K(\omega) \leqslant C_2$. Define for $\kappa > 0$ an event

$$B(\kappa) := \{K(\omega) \leqslant \kappa\}.$$

The Chebyshev inequality implies that $\mathsf{P}(B(\kappa)) \geqslant 1 - C_2\kappa^{-1}$. Therefore

$$\mathsf{P}(A_m(\varepsilon) \cap B(\kappa)) \geqslant 1 - C_1 M N^2 2^{14m} \sqrt{\delta}\varepsilon^{-1} - C_2\kappa^{-1}. \tag{7.3}$$

We choose now large κ such that $C_2\kappa^{-1} \leqslant \varepsilon/2$.

It follows from the above definitions and a change of variables $t' := t + s$ in the integral that on event $A_m(\varepsilon) \cap B(\kappa)$ we have

$$\int_{k2^{-m}}^{(k+1)2^{-m}} \mathbb{1}_U(V(t+s,z)+x)\,dt \leqslant 2 \cdot 2^{-m}(\varepsilon + \kappa 2^{-m}) \tag{7.4}$$

for all $x \in [-2MN, 2MN]$, $s \in [0,1]$, $z \in [-N,N]$, $0 \leqslant k \leqslant 2^m - 1$.

Now we fix $r \in \mathbb{N}$, $s \in [0,1]$, $z \in [-N,N]$, $f_1, f_2 \in \mathcal{C}^h(1,1,M)$. Put

$$f(t,z) := f_1(t,z) + \lambda_r(f_2)(t,z).$$

It follows from Fatou's lemma and the fact that the set U is open

$$\begin{aligned}\int_0^1 \mathbb{1}_U(V(t+s,z)+f(t,z))\,dt &\leqslant \liminf_{n\to\infty}\int_0^1 \mathbb{1}_U(V(t+s,z)+\lambda_n(f)(t,z))\,dt \\ &=: \liminf_{n\to\infty} I_n.\end{aligned} \tag{7.5}$$

On the other hand, for any $n \geqslant m$ we have

$$I_n \leqslant I_m + \sum_{l=m}^{n-1} |I_{l+1} - I_l|. \tag{7.6}$$

The function $\lambda_m(f)$ is constant on time intervals $[k2^{-m}, (k+1)2^{-m})$. Moreover, since $f_1, f_2 \in \mathcal{C}^h(1,1,M)$, we see that $|\lambda_m(f)(t,z)| \leqslant 2NM$ for $t \in [0,1]$, $z \in [-N,N]$. Therefore, (7.4) yields that on $A_m(\varepsilon) \cap B(\kappa)$

$$\int_0^1 \mathbb{1}_U(V(t+s,z)+\lambda_m(f)(t,z))\,dt \leqslant 2\varepsilon + \kappa 2^{-m+1}.$$

In a similar way we estimate the second term in the right-hand side of (7.6). It follows from the definition of the approximation operator λ that if $l \geqslant r$, then $\lambda_r(f_2)((i+1)2^{-l}, z) = \lambda_r(f_2)((i+1/2)2^{-l}, z)$ for any $i = 0, 1, \ldots, 2^l - 1$; further, if $l < r$, then $\lambda_r(f_2)((i+1)2^{-l}, z) = f_2((i+1)2^{-l}, z)$ and we have

$\lambda_r(f_2)((i+1/2)2^{-l},z) = f_2((i+1/2)2^{-l},z)$. This observation, the definition of the set $B(\kappa)$ and a change of variables $t' := t+s$ in the integral imply that for $l \in \mathbb{Z}_+$ on $A_m(\varepsilon) \cap B(\kappa)$

$$
\begin{aligned}
|I_{l+1} - I_l| \leqslant & \kappa 2^{-l(3/4-\rho/4-\theta)} \sum_{i=0}^{2^l-1} |f((i+1)2^{-l},z) - f((i+1/2)2^{-l},z)|^\rho \\
\leqslant & \kappa M N 2^{-l(3/4-5\rho/4+h\rho-\theta)} \sum_{i=0}^{2^l-1} (i+1/2)^{-\rho} \\
\leqslant & 2(1-\rho)^{-1}\kappa M N 2^{-l(\rho(h-1/4)-1/4-\theta)} \\
\leqslant & 2(1-\rho)^{-1}\kappa M N 2^{-l\theta},
\end{aligned}
$$

where in the last inequality we took into the account that ρ and θ were chosen according to (7.2). Combining this with the previous estimate and (7.6), we finally get on $A_m(\varepsilon) \cap B(\kappa)$ for any $n \geqslant m$

$$
\begin{aligned}
&\int_0^1 \mathbb{1}_U(V(t+s,z) + \lambda_n(f)(t,z))\,dt \\
&\qquad\qquad \leqslant 2\varepsilon + \kappa 2^{-m+1} + 2(1-\rho)^{-1}\kappa M N 2^{-m\theta}(1-2^{-\theta})^{-1}.
\end{aligned}
$$

Recall that we have already chosen κ, ρ, θ. Now we choose large m such that the right-hand side of the above inequality is less than 3ε. Finally, we choose small δ such than the right-hand side of (7.3) is bigger than $1-\varepsilon$.

Thus, we got that on the set $D := A_m(\varepsilon) \cap B(\kappa)$ we have for any $n \geqslant m$

$$
\int_0^1 \mathbb{1}_U(V(t+s,z) + \lambda_n(f)(t,z))\,dt \leqslant 3\varepsilon
$$

and $\mathsf{P}(D) \geqslant 1-\varepsilon$. This inequality together with (7.5) yields the statement of the lemma. □

Proof of Lemma 3.3. The lemma is proved by a straightforward application of Lemma 4.9. □

SUPPLEMENTARY MATERIAL

Supplement to "Regularization by noise and flows of solutions for a stochastic heat equation".
(doi: COMPLETED BY THE TYPESETTER; .pdf). The supplementary material provides proofs of auxiliary results related to the properties of the heat kernel.

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