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WEAK SYMMETRIC INTEGRALS WITH RESPECT TO THE FRACTIONAL BROWNIAN MOTION

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Abstract. The aim of this paper is to establish the weak convergence, in the topology of the Skorohod space, of the ν-symmetric Riemann sums for functionals of the fractional Brownian motion when the Hurst parameter takes the critical value $H = (4\ell+2)^{-1}$, where $\ell = \ell(\nu) \geq 1$ is the largest natural number satisfying $\int_0^1 \alpha^{2j}\nu(d\alpha) = \frac{1}{2j+1}$ for all $j = 0, \dots, \ell-1$. As a consequence, we derive a change-of-variable formula in distribution, where the correction term is a stochastic integral with respect to a Brownian motion that is independent of the fractional Brownian motion.

1. Introduction

Suppose that $B^H = \{B^H_t, t \geq 0\}$ is a fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$, that is, B^H is a centered Gaussian process with covariance given by

$$R(s,t) := \mathbb{E}[B^H_s B^H_t] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \tag{1.1}$$

for any $s,t \geq 0$. When $H < \frac{1}{2}$, it is well-known that the integral $\int_0^t g(B^H_s)dB^H_s$ does not exist in general as a path-wise Riemann-Stieltjes integral. In the pioneering work [5], Gradinaru, Nourdin, Russo and Vallois proved that this integral can be defined as the limit in probability of suitable symmetric Riemann sums if the Hurst parameter is not too small. Let us briefly describe the main contribution of [5].

Let ν be a symmetric probability measure on $[0,1]$, meaning that $\nu(A) = \nu(1-A)$ for any Borel set $A \subset [0,1]$. Given a continuous function $g : \mathbb{R} \to \mathbb{R}$, consider the ν-symmetric Riemann sums of $g(B^H_s)$ in the interval $[0,t]$ given by

$$S^\nu_n(g,t) = \sum_{j=0}^{\lfloor nt \rfloor - 1} (B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}) \int_0^1 g\left(B^H_{\frac{j}{n}} + \alpha(B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}})\right)\nu(d\alpha),$$

where $n \geq 1$ is an integer and $\lfloor x \rfloor$ denotes the integer part of x for any $x \geq 0$. Then, following [5] and providing the limit exists, the ν-symmetric integral is defined as the limit

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in probability of the ν-symmetric Riemann sums as n tends to infinity, namely,

$$\int_0^t g(B_s^H) d^\nu B_s^H = \lim_{n\to\infty} S_n^\nu(g,t).$$

It is proved in [5] that this integral exists for $g = f'$ with $f \in \mathcal{C}^{4\ell(\nu)+2}(\mathbb{R})$, if the Hurst parameter satisfies $H > \frac{1}{4\ell(\nu)+2}$. Here we denote by $\ell(\nu) \geq 1$ the largest positive natural number such that

$$\int_0^1 \alpha^{2j} \nu(d\alpha) = \frac{1}{2j+1} \qquad \forall j = 0, 1, \dots, \ell(\nu) - 1. \tag{1.2}$$

Moreover, in this case the integral $\int_0^t f'(B_s^H) d^\nu B_s^H$ satisfies the chain rule

$$f(B_t^H) = f(0) + \int_0^t f'(B_s^H) d^\nu B_s^H.$$

Basic examples of ν-symmetric Riemann sums and integrals are the following:

(i) If $\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$, then S_n^ν are the trapezoidal Riemann sums. In this case $\ell(\nu) = 1$ and ν-symmetric integrals exist for $H > \frac{1}{6}$.
(ii) If $\nu = \frac{1}{6}\delta_0 + \frac{2}{3}\delta_{1/2} + \frac{1}{6}\delta_1$, then $\ell(\nu) = 2$. In this case, S_n^ν are the Simpson Riemann sums and ν-symmetric integrals exist for $H > \frac{1}{10}$.
(iii) If ν is the Lebesgue measure, then $\ell(\nu) = \infty$, and ν-symmetric integrals exist for any $H \in (0,1)$.

The lower bound $\frac{1}{4\ell(\nu)+2}$ for the Hurst parameter is sharp, in the sense that for $H = \frac{1}{4\ell(\nu)+2}$ the ν-symmetric integral diverges in $L^2(\Omega)$ for $f(x) = x^2$. This has been proved, for the example (i) above, in the references [2] and [5]. The goal of this paper is to show that when $H = \frac{1}{4\ell(\nu)+2}$, the ν-symmetric Riemann sums converge in distribution and, as a consequence, we obtain a change-of-variable formula in law with a correction term which is an Itô stochastic integral with respect to a Brownian motion which is independent of B^H. More precisely, the main result is the following theorem.

We say that a function $f : \mathbb{R} \to \mathbb{R}$ has *moderate growth* if there exist positive constants A, B and $\alpha < 2$ such that $|f(x)| \leq Ae^{B|x|^\alpha}$ for all $x \in \mathbb{R}$.

Theorem 1.1. *Fix a symmetric probability measure ν on $[0,1]$ with $\ell := \ell(\nu) < \infty$ and let $B^H = \{B_t^H, t \geq 0\}$ be a fractional Brownian motion with Hurst parameter $H = \frac{1}{4\ell+2}$. Consider a function $f \in \mathcal{C}^{20\ell+5}(\mathbb{R})$ such that f and its derivatives up to the order $20\ell + 5$ have moderate growth. Then,*

$$S_n^\nu(f',t) \xrightarrow[n\to\infty]{\mathcal{L}} f(B_t^H) - f(0) - c_\nu \int_0^t f^{(2\ell+1)}(B_s^H)\, dW_s, \tag{1.3}$$

where $W = \{W_t, t \geq 0\}$ is a Brownian motion independent of B^H, c_ν is a constant depending only on ν, and the convergence holds in the topology of the Skorohod space $D([0,\infty))$.

The value of the constant c_ν in (1.3) is $c_\nu = k_{\nu,\ell}\sigma_\ell$, where $k_{\nu,\ell}$ is defined in (3.2) and σ_ℓ is given by

$$\sigma_\ell^2 = \mathbb{E}[X_1^{4\ell+2}] + 2\sum_{j=1}^\infty \mathbb{E}\left[(X_1 X_{1+j})^{2\ell+1}\right], \tag{1.4}$$

where $X_j = B_j^{1/(4\ell+2)} - B_{j-1}^{1/(4\ell+2)}$ for $j \geq 1$ (see, for instance, [15, Theorem 10]).

The statement of Theorem 1.1 can be interpreted as a change-of-variable formula in law. Indeed, although the sequence of ν-symmetric Riemann sums $S_n^\nu(f', t)$ fails in general to converge in probability and the ν-symmetric integral $\int_0^t f'(B_s^H) d^\nu B_s^H$ does not exist in the sense introduced above, this sequence converges in law and we can still call the limit (which is defined only in law) the ν-symmetric integral, and denote it by $\int_0^t f'(B_s^H) d^\nu B_s^H$. In this way, we can write

$$f(B_t^H) = f(0) + \int_0^t f'(B_s^H) d^\nu B_s^H + c_\nu \int_0^t f^{(2\ell+1)}(B_s^H)\, dW_s,$$

where this formula has to be understood in the sense that the random variables $\int_0^t f'(B_s^H) d^\nu B_s^H$ and $f(B_t^H) - f(0) - c_\nu \int_0^t f^{(2\ell+1)}(B_s^H)\, dW_s$ have the same law.

Some particular cases have already been addressed recently in the literature. In the case $\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ (trapezoidal Riemann sums), the critical value is $H = \frac{1}{6}$, and the corresponding version of Theorem 1.1 was proved by Nourdin, Réveillac and Swanson in [13]. The convergence results for trapezoidal Riemann sums were extended to a general class of Gaussian processes by Harnett and Nualart in [6]. If $\nu = \frac{1}{6}\delta_0 + \frac{2}{3}\delta_{\frac{1}{2}} + \frac{1}{6}\delta_1$ (Simpson Riemann sums), the critical point is $H = \frac{1}{10}$ and the convergence in law for any fixed time $t \geq 0$ was proved by Harnett and Nualart in [8].

For related results in the case of midpoint Riemann sums, we refer to the works by Burdzy and Swanson [1], Nourdin and Réveillac [12] and Harnett and Nualart [7]. In this case the critical value of the Hurst parameter is $H = \frac{1}{4}$, and the complementary term in the Itô formula involves a second derivative. See also Nourdin [9].

Let us briefly describe the strategy we will follow for the proof of Theorem 1.1. First, using Taylor's formula and the properties of the symmetric measure ν derived in [5], we determine the following decomposition for ν-symmetric Riemann sums

$$S_n^\nu(f', t) = f(B_{\lfloor nt \rfloor}^H) - f(0) - \sum_{h=\ell}^{2\ell} \Phi_n^h(t) - R_n(t),$$

where, for each $h = \ell, \ldots, 2\ell$,

$$\Phi_n^h(t) = \sum_{j=0}^{\lfloor nt \rfloor - 1} k_{\nu,h}\, f^{(2h+1)}(\widetilde{B}_{\frac{j}{n}}^H)(\Delta_j^n B^H)^{2h+1},$$

the constants $k_{\nu,h}$ are defined in (3.2) and we use the notation $\widetilde{B}_{\frac{j}{n}}^H = \frac{1}{2}\big(B_{\frac{j}{n}}^H + B_{\frac{j+1}{n}}^H\big)$ and $\Delta_j^n B^H = B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H$. The residual term $R_n(t)$ is a weighted sum of the powers $(\Delta_j^n B^H)^{4\ell+2}$ with coefficients that converge to zero as n tends to infinity. Taking into account that $H = \frac{1}{4\ell+2}$, it is not difficult to show that $R_n(t)$ converges to zero in probability, uniformly in compact sets.

In Lemma 4.1, proved in the Appendix, we show that for any $h = \ell, \dots, 2\ell$, the moment of order four $\mathbb{E}\left[|\Phi_n^h(t) - \Phi_n^h(s)|^4\right]$, for any $0 \le s \le t \le T$, can be estimated by

$$C_T \sum_{N=2}^{4} n^{\frac{2(\ell-h)N}{2\ell+1}} \left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right)^N .$$

This lemma is proved by expressing the product of increments $\prod_{i=1}^4 (\Delta_{j_i}^n B^H)^{2h+1}$ as a linear combination of multiple stochastic integrals and applying the duality relationship between multiple stochastic integrals and the iterated Malliavin derivative. As a consequence, for $h = \ell+1, \dots, 2\ell$, the terms $\Phi_n^h(t)$ converge to zero in the topology of the Skorohod space $D([0,\infty))$, and for $h = \ell$, the sequence $\Phi_n^\ell(t)$ is tight.

Thus, the only nonzero contribution to the limit in law of the ν-symmetric Riemann sums $S_n^\nu(f', t)$ is the term

$$k_{\nu,\ell} \sum_{j=0}^{\lfloor nt \rfloor -1} f^{(2\ell+1)}(\widetilde{B}^H_{\frac{j}{n}})(\Delta_j^n B^H)^{2\ell+1}, \tag{1.5}$$

and it suffices to show that the finite dimensional distributions of this process converge to those of $c_\nu \int_0^t f^{(2\ell+1)}(B_s^H)\, dW_s$. Notice that the process appearing in (1.5) is a weighted sum of the odd powers $(\Delta_j^n B^H)^{2\ell+1}$ of the fBm. It is well-known that for $H = \frac{1}{4\ell+2}$, the sums of these odd powers converge in law to a Gaussian random variable. More precisely, the following stable convergence holds

$$\left(\sum_{j=0}^{\lfloor nt \rfloor -1} \left(\Delta_j^n B^H\right)^{2\ell+1}, B_t^H, t \ge 0 \right) \xrightarrow[n\to\infty]{\mathcal{L}} \left(\sigma_\ell W_t, B_t^H, t \ge 0\right), \tag{1.6}$$

where σ_ℓ is defined in (1.4) and in the right-hand side, the process W is a Brownian motion independent of B^H. The proof of the convergence for a fixed t follows from the Breuer-Major Theorem (we refer to [11, Chapter 7] and [4] for a proof of this result based on the Fourth Moment theorem). Then the convergence of the weighted sums (1.5) follows from the methodology of small blocks/big blocks used, for instance in the works [3] and [4]. The basic ingredient in this approach is the proof that the reminder term converges to zero and this follows from Lemma 4.2, proved in the Appendix. However, unlike the above references, the convergence to zero of the reminder term cannot be established using fractional calculus techniques because $H < \frac{1}{2}$, and it requires the application of integration-by-parts formulas from Malliavin calculus.

The paper is organized as follows. Section 2 contains some preliminaries on the Malliavin calculus and the fractional Brownian motion. Section 3 is devoted to the proof of Theorem 1.1 and in Section 4 we show two basic technical lemmas.

2. Preliminaries

In the next two subsections, we discuss some notions of Malliavin calculus and fractional Brownian motion. Throughout the paper C_T and C will denote any positive constants depending or not on T respectively; they may change from one expression to another.

2.1. **Elements of Malliavin Calculus.** Let $\mathfrak{H}$ be a real separable infinite-dimensional Hilbert space and let $X = \{X(h), h \in \mathfrak{H}\}$ be an isonormal Gaussian process over $\mathfrak{H}$. This means that X is a centered Gaussian family, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with a covariance structure given by

$$\mathbb{E}[X(h)X(g)] = \langle h, g\rangle_{\mathfrak{H}}, \quad h, g \in \mathfrak{H}.$$

We assume that $\mathcal{F}$ is the σ-algebra generated by X.

For any integer $q \geq 1$, let $\mathfrak{H}^{\otimes q}$ and $\mathfrak{H}^{\odot q}$ denote, respectively, the qth tensor product and the qth symmetric tensor product of $\mathfrak{H}$.

Let $\{e_n, n \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$, $g \in \mathfrak{H}^{\odot q}$ and $r \in \{0, \dots, p \wedge q\}$, the rth-*order contraction* of f and g is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,\dots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \ldots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \ldots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}}, \tag{2.1}$$

where $f \otimes_0 g = f \otimes g$ and, for $p = q$, $f \otimes_q g = \langle f, g\rangle_{\mathfrak{H}^{\otimes q}}$. Notice that $f \otimes_r g$ is not necessarily symmetric. We denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$.

Let $\mathcal{H}_q$ denote the qth *Wiener chaos* of X, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the qth Hermite polynomial defined by

$$H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q}\big(e^{-x^2/2}\big). \tag{2.2}$$

For $q \geq 1$, let $I_q(\cdot)$ denote the generalized Wiener-Itô multiple stochastic integral. It is known that the map

$$I_q(h^{\otimes q}) = H_q(X(h)) \tag{2.3}$$

provides a linear isometry between $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!}\,\|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and $\mathcal{H}_q$ (equipped with the $L^2(\Omega)$ norm). For $q = 0$, we set by convention $\mathcal{H}_0 = \mathbb{R}$ and I_0 equal to the identity map.

Multiple stochastic integrals satisfy the following product formula. Let $p, q \geq 1$ positive integers. Let $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$. Then,

$$I_p(f)I_q(g) = \sum_{z=0}^{p\wedge q} z! \binom{p}{z}\binom{q}{z} I_{p+q-2z}(f \widetilde{\otimes}_z g), \tag{2.4}$$

where $\otimes_z$ is the contraction operator defined in (2.1).

From the hypercontractivity property of the Ornstein-Uhlenbeck semigroup, it is well-known that all $L^r(\Omega)$-norms, $r > 1$, are equivalent on each Wiener chaos. In particular, for any real number $r \geq 2$, any integer $p \geq 2$ and any $f \in \mathfrak{H}^{\odot p}$, we have

$$\|I_p(f)\|_{L^r(\Omega)} \leq C_{r,p}\|I_p(f)\|_{L^2(\Omega)} = C_{r,p}\sqrt{p!}\|f\|_{\mathfrak{H}^{\otimes p}}. \tag{2.5}$$

Let $\mathcal{S}$ be the set of all smooth and cylindrical random variables of the form

$$F = g\left(X(\phi_1), \dots, X(\phi_n)\right),$$

where $n \geq 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function with compact support, and $\phi_i \in \mathfrak{H}$. The *Malliavin derivative* of F with respect to X is the element of $L^2(\Omega; \mathfrak{H})$ defined as

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \left(X(\phi_1), \ldots, X(\phi_n) \right) \phi_i.$$

By iteration, we can define the qth derivative $D^q F$ for every $q \geq 2$, which is an element of $L^2(\Omega; \mathfrak{H}^{\odot q})$.

For any integer $q \geq 1$ and any real number $p \geq 1$, let $\mathbb{D}^{q,p}$ denote the closure of $\mathcal{S}$ with respect to the norm $\| \cdot \|_{\mathbb{D}^{q,p}}$, defined as

$$\|F\|_{\mathbb{D}^{q,p}}^p = \mathbb{E}\big[|F|^p\big] + \sum_{i=1}^{q} \mathbb{E}\left(\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p \right).$$

More generally, for any Hilbert space V, we denote by $\mathbb{D}^{q,p}(V)$ the corresponding Sobolev space of V-valued random variables.

The Malliavin derivative D fulfills the following chain rule. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $F = (F_1, \ldots, F_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F) DF_i.$$

We denote by δ the *Skorohod integral*, also called the *divergence operator*, which is the adjoint of the operator D. More precisely, a random element $u \in L^2(\Omega; \mathfrak{H})$ belongs to the domain of δ, denoted by Dom δ, if and only if, for any $F \in \mathbb{D}^{1,2}$, we have

$$\big| \mathbb{E}\big(\langle DF, u \rangle_{\mathfrak{H}} \big) \big| \leq c_u \, \|F\|_{L^2(\Omega)},$$

where c_u is a constant depending only on u. If $u \in$ Dom δ, then the random variable $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}(F\delta(u)) = \mathbb{E}\big(\langle DF, u \rangle_{\mathfrak{H}} \big).$$

This is called the Malliavin integration by parts formula and it holds for every $F \in \mathbb{D}^{1,2}$. For $q \geq 1$, the multiple Skorohod integral is defined iteratively as $\delta^q(u) = \delta(\delta^{q-1}(u))$, with $\delta^0(u) = u$. From this definition, we have

$$\mathbb{E}\big(F\delta^q(u) \big) = \mathbb{E}\big(\langle D^q F, u \rangle_{\mathfrak{H}^{\otimes q}} \big), \tag{2.6}$$

for any $u \in$ Dom δ^q and any $F \in \mathbb{D}^{q,2}$. Moreover, $\delta^q(h) = I_q(h)$ for any $h \in \mathfrak{H}^{\odot q}$. We refer to [14] for a detailed account on the Malliavin calculus for an arbitrary isonormal Gaussian process.

2.2. **Fractional Brownian motion.** Let $B^H = \{B_t^H, t \geq 0\}$ denote a fractional Brownian motion with Hurst parameter H. Namely, B^H is a centered Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with covariance given by (1.1). We assume that $\mathcal{F}$ is generated by B^H. Along the paper we suppose that $H < \frac{1}{2}$.

We denote by $\mathcal{E}$ the set of $\mathbb{R}$-valued step functions on $[0,\infty)$. Let $\mathfrak{H}$ be the Hilbert space defined as the completion of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathfrak{H}} = R(s,t).$$

The mapping $\mathbb{1}_{[0,t]} \to B_t^H$ can be extended to a linear isometry between the Hilbert space $\mathfrak{H}$ and the Gaussian space spanned by B^H. In this way $\{B^H(h), h \in \mathfrak{H}\}$ is an isonormal Gaussian process as in Section 2.1.

Recall the notation $\widetilde{B}^H_{\frac{j}{n}} = \frac{1}{2}\big(B^H_{\frac{j}{n}} + B^H_{\frac{j+1}{n}}\big)$ and $\Delta_j^n B^H = B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}$. Moreover, we set

$$\begin{aligned} \partial_{\frac{j}{n}} &= \mathbb{1}_{[\frac{j}{n},\frac{j+1}{n}]}, \\ \varepsilon_t &= \mathbb{1}_{[0,t]} \end{aligned}$$

and

$$\widetilde{\varepsilon}_{\frac{j}{n}} = \frac{1}{2}\big(\varepsilon_{\frac{j}{n}} + \varepsilon_{\frac{j+1}{n}}\big) = \frac{1}{2}\big(\mathbb{1}_{[0,\frac{j}{n}]} + \mathbb{1}_{[0,\frac{j+1}{n}]}\big).$$

The fractional Brownian motion with Hurst parameter H satifies

$$\mathbb{E}\big[(\Delta_j^n B^H)^2\big] = \langle \partial_{\frac{j}{n}}, \partial_{\frac{j}{n}} \rangle_{\mathfrak{H}} = n^{-2H}. \tag{2.7}$$

Moreover, using the fact that the function $x \to x^{2H}$ is concave for $H < \frac{1}{2}$, for any $t \geq 0$ and any integer $j \geq 0$, we obtain

$$\big|\mathbb{E}\big[(\Delta_j^n B^H) B_t^H\big]\big| = \big|\big\langle \partial_{\frac{j}{n}}, \varepsilon_t \big\rangle_{\mathfrak{H}}\big| \leq n^{-2H}. \tag{2.8}$$

The following lemma has been proved in [8, Lemma 2.6].

Lemma 2.1. *Let $H < \frac{1}{2}$ and let $n \geq 2$ be an integer. Then, there exists a constant C not depending on T such that:*
a) For any $t \in [0,T]$,

$$\sum_{j=0}^{\lfloor nT \rfloor - 1} \big|\big\langle \partial_{\frac{j}{n}}, \varepsilon_t \big\rangle_{\mathfrak{H}}\big| \leq C \lfloor nT \rfloor^{2H} n^{-2H}.$$

b) For any integers $r \geq 1$ and $0 \leq i \leq \lfloor nT \rfloor - 1$,

$$\sum_{j=0}^{\lfloor nT \rfloor - 1} \big|\big\langle \partial_{\frac{j}{n}}, \partial_{\frac{i}{n}} \big\rangle_{\mathfrak{H}}^r\big| \leq C n^{-2rH}, \tag{2.9}$$

and consequently

$$\sum_{j,i=0}^{\lfloor nT \rfloor - 1} \big|\big\langle \partial_{\frac{j}{n}}, \partial_{\frac{i}{n}} \big\rangle_{\mathfrak{H}}^r\big| \leq C \lfloor nT \rfloor n^{-2rH}. \tag{2.10}$$

The next result provides useful estimates when we compare two partitions. Its proof is based on computing telescopic sums.

Lemma 2.2. *We fix two integers $n > m \geq 2$, and for any $j \geq 0$, we define $k := k(j) = \sup\{i \geq 0 : \frac{i}{m} \leq \frac{j}{n}\}$. The following inequalities hold true for some constant C_T depending only on T:*

$$\sum_{j=0}^{\lfloor nT\rfloor -1} \left|\left\langle \partial_{\frac{j}{n}}, \varepsilon_{\frac{k(j)}{m}} \right\rangle_{\mathfrak{H}}\right| \leq C_T m^{1-2H}, \tag{2.11}$$

$$\sum_{j=0}^{\lfloor nT\rfloor -1} \left|\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{j}{n}} - \varepsilon_{\frac{k(j)}{m}} \right\rangle_{\mathfrak{H}}\right| \leq C_T m^{1-2H} \tag{2.12}$$

and, for any $0 \leq i \leq \lfloor nT\rfloor - 1$,

$$\sum_{j=0}^{\lfloor nT\rfloor -1} \left|\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{i}{n}} - \varepsilon_{\frac{k(i)}{m}} \right\rangle_{\mathfrak{H}}\right| \leq C_T m^{-2H}. \tag{2.13}$$

Proof. Let us first show (2.11). We can write

$$\begin{aligned}
\sum_{j=0}^{\lfloor nT\rfloor -1} \left|\left\langle \partial_{\frac{j}{n}}, \varepsilon_{\frac{k(j)}{m}} \right\rangle_{\mathfrak{H}}\right| &= \sum_{j=0}^{\lfloor nT\rfloor -1} \left|\mathbb{E}\left[\left(B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}\right) B^H_{\frac{k(j)}{m}}\right]\right| \\
&= \frac{1}{2}\sum_{j=0}^{\lfloor nT\rfloor -1} \left| \left(\frac{j+1}{n}\right)^{2H} - \left(\frac{j}{n}\right)^{2H} - \left|\frac{j+1}{n} - \frac{k(j)}{m}\right|^{2H} + \left|\frac{j}{n} - \frac{k(j)}{m}\right|^{2H} \right| \\
&\leq \frac{1}{2} n^{-2H} \sum_{j=0}^{\lfloor nT\rfloor -1} \left[(j+1)^{2H} - j^{2H}\right] \\
&\qquad + \frac{1}{2}\sum_{j=0}^{\lfloor nT\rfloor -1} \left[\left(\frac{j+1}{n} - \frac{k(j)}{m}\right)^{2H} - \left(\frac{j}{n} - \frac{k(j)}{m}\right)^{2H}\right].
\end{aligned}$$

The first term is a telescopic sum and it is easy to show that

$$\frac{1}{2} n^{-2H} \sum_{j=0}^{\lfloor nT\rfloor -1} \left[(j+1)^{2H} - j^{2H}\right] = \frac{1}{2} n^{-2H} (\lfloor nT\rfloor)^{2H} \leq C_T \leq C_T m^{1-2H}.$$

For the second term, observe that, for a fixed $k = 0, \dots, \lfloor mT\rfloor + 1$, the sum of the terms for which $k(j) = k$ is telescopic and is bounded by a constant times m^{-2H}. Summing over all possible values of k, we obtain the desired bound $C_T m^{1-2H}$.

The inequality (2.12) is an immediate consequence of (2.11) and the following easy fact:

$$\begin{aligned}
\sum_{j=0}^{\lfloor nT\rfloor -1} \left|\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}\right| &= \frac{1}{2}\sum_{j=0}^{\lfloor nT\rfloor -1} \left|\mathbb{E}\left[\left(B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}\right)\left(B^H_{\frac{j+1}{n}} + B^H_{\frac{j}{n}}\right)\right]\right| \\
&= \frac{n^{-2H}}{2} \sum_{j=0}^{\lfloor nT\rfloor -1} \left[(j+1)^{2H} - j^{2H}\right] = \frac{1}{2} n^{-2H} (\lfloor nT\rfloor)^{2H} \leq C_T \leq C_T m^{1-2H}.
\end{aligned}$$

Let us now proceed with the proof of (2.13). We can write

$$\begin{aligned}
&\sum_{j=0}^{\lfloor nT\rfloor-1} \left|\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{i}{n}} - \varepsilon_{\frac{k(i)}{m}}\rangle_{\mathfrak{H}}\right| \\
&\quad= \frac{1}{2}\sum_{j=0}^{\lfloor nT\rfloor-1} \left|\mathbb{E}\Big[\big(B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}\big)\big(B^H_{\frac{i}{n}} + B^H_{\frac{i+1}{n}} - 2\,B^H_{\frac{k(i)}{m}}\big)\Big]\right| \\
&\quad\le \frac{1}{2}\sum_{j=0}^{\lfloor nT\rfloor-1} \left|\mathbb{E}\Big[\big(B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}\big)\big(B^H_{\frac{i}{n}} - B^H_{\frac{k(i)}{m}}\big)\Big]\right| \\
&\qquad\qquad + \frac{1}{2}\sum_{j=0}^{\lfloor nT\rfloor-1} \left|\mathbb{E}\Big[\big(B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}\big)\big(B^H_{\frac{i+1}{n}} - B^H_{\frac{k(i)}{m}}\big)\Big]\right| \\
&\quad=: A_1 + A_2.
\end{aligned}$$

Let us first consider the term A_1. The main idea to estimate this term is to use the fact that the covariance between the increments $B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}$ and $B^H_{\frac{i}{n}} - B^H_{\frac{k(i)}{m}}$ is nonpositive if $j \ge i$ or $j \le j_0$, for some index j_0 depending on i. Then the sums with $j \ge i$ or $j \le j_0$ are telescopic and can be easily estimated. Finally, it suffices to consider the remaining summands. Proceeding in this way, we write

$$A_1 = \frac{1}{2}\sum_{j=0}^{\lfloor nT\rfloor-1} |H_j|,$$

where

$$H_j = \left|\frac{i}{n} - \frac{j}{n}\right|^{2H} - \left|\frac{i}{n} - \frac{j+1}{n}\right|^{2H} + \left|\frac{k(i)}{m} - \frac{j+1}{n}\right|^{2H} - \left|\frac{k(i)}{m} - \frac{j}{n}\right|^{2H}.$$

Taking into account that $\frac{k(i)}{m} \le \frac{i}{n}$, it follows that, for $j \ge i$

$$\begin{aligned}
H_j &= \left(\frac{j}{n} - \frac{i}{n}\right)^{2H} - \left(\frac{j+1}{n} - \frac{i}{n}\right)^{2H} + \left(\frac{j+1}{n} - \frac{k(i)}{m}\right)^{2H} - \left(\frac{j}{n} - \frac{k(i)}{m}\right)^{2H} \\
&= 2H\int_0^{\frac{1}{n}} \left[\left(\frac{j}{n} + x - \frac{k(i)}{m}\right)^{2H-1} - \left(\frac{j}{n} + x - \frac{i}{n}\right)^{2H-1}\right] dx \le 0.
\end{aligned}$$

On the other hand, if j_0 is the largest integer $j \ge 0$ such that $\frac{j+1}{n} \le \frac{k(i)}{m}$, then, for $j \le j_0$,

$$\begin{aligned}
H_j &= \left(\frac{i}{n} - \frac{j}{n}\right)^{2H} - \left(\frac{i}{n} - \frac{j+1}{n}\right)^{2H} + \left(\frac{k(i)}{m} - \frac{j+1}{n}\right)^{2H} - \left(\frac{k(i)}{m} - \frac{j}{n}\right)^{2H} \\
&= -2H\int_0^{\frac{1}{n}} \left[\left(\frac{k(i)}{m} - \frac{j}{n} - x\right)^{2H-1} - \left(\frac{i}{n} - \frac{j}{n} - x\right)^{2H-1}\right] dx \le 0.
\end{aligned}$$

Consider the decomposition

$$A_1 = \frac{1}{2}\left(\sum_{j=0}^{j_0} |H_j| + \sum_{j=j_0+1}^{i-1} |H_j| + \sum_{j=i}^{\lfloor nT\rfloor-1} |H_j|\right) =: \frac{1}{2}\left(A_{11} + A_{12} + A_{13}\right).$$

For the terms A_{11} and A_{13}, we obtain, respectively

$$\begin{aligned} A_{11} &= \sum_{j=0}^{j_0}(-H_j) \\ &= \left(\frac{i}{n}-\frac{j_0+1}{n}\right)^{2H} - \left(\frac{k(i)}{m}-\frac{j_0+1}{n}\right)^{2H} - \left(\frac{i}{n}\right)^{2H} + \left(\frac{k(i)}{m}\right)^{2H} \\ &\leq 2\left(\frac{i}{n}-\frac{k(i)}{m}\right)^{2H} \leq Cm^{-2H} \end{aligned}$$

and

$$\begin{aligned} A_{13} &= \sum_{j=i}^{\lfloor nT\rfloor -1}(-H_j) \\ &= \left(\frac{\lfloor nT\rfloor}{n}-\frac{i}{n}\right)^{2H} - \left(\frac{\lfloor nT\rfloor}{n}-\frac{k(i)}{m}\right)^{2H} + \left(\frac{i}{n}-\frac{k(i)}{m}\right)^{2H} \\ &\leq \left(\frac{i}{n}-\frac{k(i)}{m}\right)^{2H} \leq m^{-2H}. \end{aligned}$$

Finally, for the term A_{12}, we have

$$\begin{aligned} A_{12} &\leq \sum_{j=j_0+1}^{i-1}\left|\left(\frac{i}{n}-\frac{j}{n}\right)^{2H} - \left(\frac{i}{n}-\frac{j+1}{n}\right)^{2H}\right| \\ &\quad + \sum_{j=j_0+1}^{i-1}\left|\left(\frac{j+1}{n}-\frac{k(i)}{m}\right)^{2H} - \left(\frac{j}{n}-\frac{k(i)}{m}\right)^{2H}\right| \\ &=: A_{121}+A_{122}. \end{aligned}$$

The term A_{121} is a telescopic sum which produces a contribution of the form

$$\left(\frac{i-j_0-1}{n}\right)^{2H} \leq C_T m^{-2H}$$

and the term A_{122} can be bounded as follows

$$\begin{aligned} A_{122} &\leq \left|\frac{j_0+2}{n}-\frac{k(i)}{m}\right|^{2H} + \left|\frac{j_0+1}{n}-\frac{k(i)}{m}\right|^{2H} + \sum_{j=j_0+2}^{i-1}\left[\left(\frac{j+1}{n}-\frac{k(i)}{m}\right)^{2H} - \left(\frac{j}{n}-\frac{k(i)}{m}\right)^{2H}\right] \\ &\leq C_T m^{-2H} + \left(\frac{i}{n}-\frac{k(i)}{m}\right)^{2H} - \left(\frac{j_0+2}{n}-\frac{k(i)}{m}\right)^{2H} \leq C_T m^{-2H}. \end{aligned}$$

The term A_2 can be treated in a similar way. This completes the proof. □

We will use the following lemma.

Lemma 2.3. *For any odd integer $r \geq 1$, we have*

$$(\Delta_j^n B^H)^r = \sum_{u=0}^{\lfloor \frac{r}{2}\rfloor} C_{r,u} n^{-2uH}\, I_{r-2u}(\partial_{\frac{j}{n}}^{\otimes r-2u}),$$

where $C_{r,u}$ are some integers.

Proof. By (2.7), we have $\|\Delta_j^n B^H\|_{L^2(\Omega)} = n^{-H}$. For any integer $q \geq 1$, we recall (see (2.2)) that $H_q(x)$ denotes the Hermite polynomial of degree q. Using an inductive argument coming from the relation $H_{q+1}(x) = xH_q(x) - qH_{q-1}(x)$, it follows that

$$x^r = \sum_{u=0}^{\lfloor \frac{r}{2} \rfloor} C_{r,u} H_{r-2u}(x), \tag{2.14}$$

where $C_{r,u}$ is an integer. Applying (2.3) to $h = n^H \partial_{\frac{j}{n}}$, that is, $X(h) = \Delta_j^n B^H / \|\Delta_j^n B^H\|_{L^2(\Omega)} = n^H \Delta_j^n B^H$, we can write

$$H_r(n^H \Delta_j^n B^H) = I_r(n^{rH} \partial_{\frac{j}{n}}^{\otimes r}). \tag{2.15}$$

Substituting (2.15) into (2.14), yields

$$n^{rH} \left(\Delta_j^n B^H\right)^r = \sum_{u=0}^{\lfloor \frac{r}{2} \rfloor} C_{r,u} I_{r-2u}(n^{(r-2u)H} \partial_{\frac{j}{n}}^{\otimes r-2u}),$$

which implies the desired result. □

3. Proof of Theorem 1.1

We recall that $\ell = \ell(\nu)$ is defined by (1.2). The first ingredient of the proof is the following expansion, established in [5], based on Taylor's formula and the properties of the measure ν

$$\begin{aligned} f(b) &= f(a) + (b-a) \int_0^1 f'(a + \alpha(b-a)) \nu(d\alpha) \\ &\quad + \sum_{h=\ell}^{2\ell} k_{\nu,h} f^{(2h+1)} \left(\frac{a+b}{2} \right) (b-a)^{2h+1} + (b-a)^{4\ell+2} C(a,b), \end{aligned} \tag{3.1}$$

where $a, b \in \mathbb{R}$ and $C(a,b)$ is a continuous function such that $C(a,a) = 0$. The constants $k_{\nu,h}$ are given by

$$k_{\nu,h} = \frac{1}{(2h)!} \left[\frac{1}{(2h+1)4^h} - \int_0^1 \left(\alpha - \frac{1}{2} \right)^{2h} \nu(d\alpha) \right]. \tag{3.2}$$

Applying equality (3.1) to $a = B^H_{\frac{j}{n}}$ and $b = B^H_{\frac{j+1}{n}}$ and using the notation $\widetilde{B}^H_{\frac{j}{n}} = \frac{1}{2}\big(B^H_{\frac{j}{n}} + B^H_{\frac{j+1}{n}}\big)$ and $\Delta^n_j B^H = B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}$, yields

$$\begin{aligned} f(B^H_{\lfloor nt \rfloor}) - f(0) &= \sum_{j=0}^{\lfloor nt \rfloor -1} \Delta^n_j B^H \int_0^1 f'\big(B^H_{\frac{j}{n}} + \alpha \Delta^n_j B^H\big)\nu(d\alpha) \\ &\quad + \sum_{h=\ell}^{2\ell} \sum_{j=0}^{\lfloor nt \rfloor -1} k_{\nu,h}\, f^{(2h+1)}(\widetilde{B}^H_{\frac{j}{n}})(\Delta^n_j B^H)^{2h+1} \\ &\quad + \sum_{j=0}^{\lfloor nt \rfloor -1} C(B^H_{\frac{j}{n}}, B^H_{\frac{j+1}{n}})(\Delta^n_j B^H)^{4\ell+2}, \end{aligned}$$

which can be written as

$$f(B^H_{\lfloor nt \rfloor}) - f(0) = S^\nu_n(f', t) + \sum_{h=\ell}^{2\ell} \Phi^h_n(t) + R_n(t), \tag{3.3}$$

where, for each $h = \ell, \dots, 2\ell$,

$$\Phi^h_n(t) = \sum_{j=0}^{\lfloor nt \rfloor -1} k_{\nu,h}\, f^{(2h+1)}(\widetilde{B}^H_{\frac{j}{n}})(\Delta^n_j B^H)^{2h+1} \tag{3.4}$$

and

$$R_n(t) = \sum_{j=0}^{\lfloor nt \rfloor -1} C(B^H_{\frac{j}{n}}, B^H_{\frac{j+1}{n}})(\Delta^n_j B^H)^{4\ell+2}.$$

Let us consider the convergence of each term in the decomposition (3.3). First we will show that the term $R_n(t)$ converges to zero in probability, uniformly in compact sets. In fact, for any $T > 0$, $K, \epsilon > 0$, we can write

$$P\left(\sup_{0\le t\le T} |R_n(t)| > \epsilon\right) \le P\left(\sup_{\substack{s,t\in[0,T]\\ |t-s|\le \frac{1}{n}}} |C(B^H_s, B^H_t)| > \frac{1}{K}\right) + P\left(\sum_{j=0}^{\lfloor nT \rfloor -1} (\Delta^n_j B^H)^{4\ell+2} > K\epsilon\right). \tag{3.5}$$

Taking into account that $H = \frac{1}{4\ell+2}$ and using (2.7), we can write, with μ_k denoting the kth moment of the standard Gaussian distribution,

$$P\left(\sum_{j=0}^{\lfloor nT \rfloor -1} (\Delta^n_j B^H)^{4\ell+2} > K\epsilon\right) \le \frac{\mu_{4\ell+2}}{K\epsilon}\frac{\lfloor nT \rfloor}{n} \le \frac{T\mu_{4\ell+2}}{K\epsilon}. \tag{3.6}$$

From (3.5) and (3.6), letting first $n \to \infty$ and then $K \to \infty$ it follows that for any $\epsilon > 0$ and $T > 0$,

$$\lim_{n\to\infty} P\left(\sup_{0\le t\le T} |R_n(t)| > \epsilon\right) = 0.$$

On the other hand, by Lemma 4.1, the terms Φ_n^h with $h = \ell+1, \dots, 2\ell$ converge to zero in the topology of $D([0,\infty))$ and do not contribute to the limit. As a consequence, the proof of Theorem 1.1 follows from the next proposition.

Proposition 3.1. *Under the assumptions of Theorem 1.1, one has*

$$\Phi_n^\ell(t) = \sum_{j=0}^{\lfloor nt \rfloor -1} f^{(2\ell+1)}(\widetilde{B}^H_{\frac{j}{n}})(\Delta_j^n B^H)^{2\ell+1} \xrightarrow[\text{n}\to\infty]{\mathcal{L}} \sigma_\ell \int_0^t f^{(2\ell+1)}(B_s^H)\, dW_s, \tag{3.7}$$

where $W = \{W_t, t \geq 0\}$ is a Brownian motion independent of B^H, σ_ℓ is the constant defined in (1.4), and the convergence holds in the topology of the Skorohod space $D([0,\infty))$.

Proof. In order to show Proposition 3.1, we will first prove that the sequence of processes $\{\Phi_n^\ell(t), t \geq 0\}$ is tight in $D([0,\infty))$, and then that their finite dimensional distributions converge to those of

$$\left\{ \sigma_\ell \int_0^t f^{(2\ell+1)}(B_s^H)\, dW_s, t \geq 0 \right\}.$$

Notice that the tightness of the sequence Φ_n^ℓ is a consequence of Lemma 4.1. Indeed, this lemma implies that for any $0 \leq s < t \leq T$, there exist a constant C_T depending on T, such that

$$\mathbb{E}\left[|\Phi_n^\ell(t) - \Phi_n^\ell(s)|^4\right] \leq C_T \sum_{N=2}^{4} \left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n} \right)^N.$$

It remains to show the convergence of the finite-dimensional distributions. Fix a finite set of points $0 \leq t_1 < \cdots \leq t_d \leq T$. We want to show the following convergence in law, as n tends to infinity:

$$(\Phi_n^\ell(t_1), \dots, \Phi_n^\ell(t_d)) \xrightarrow[\text{n}\to\infty]{\mathcal{L}} (Y_1, \dots, Y_d), \tag{3.8}$$

where

$$Y_i = \sigma_\ell \int_0^{t_i} f^{(2\ell+1)}(B_s^H)\, dW_s, \quad i = 1, \dots, d,$$

$W = \{W_t, t \geq 0\}$ is a Brownian motion independent of B^H, and σ_ℓ is the constant defined in (1.4).

Taking into account the convergence (1.6), the main ingredient in the proof of the convergence (3.8) is the methodology based on the small blocks/big blocks (see, for instance, [4]). This method consists in considering two integers $2 \leq m < n$ and let first n tend to infinity and later m tend to infinity. For any $k \geq 0$ we define the set

$$I_k = \left\{ j \in \{0, \dots, \lfloor nt_i \rfloor - 1\} : \frac{k}{m} \leq \frac{j}{n} < \frac{k+1}{m} \right\}.$$

The basic ingredient in this approach is the decomposition

$$\begin{aligned}\Phi_n^\ell(t_i) &= \sum_{k=0}^{\lfloor mt_i\rfloor}\sum_{j\in I_k} f^{(2\ell+1)}(B^H_{\frac{k}{m}})\big(\Delta_j^n B^H\big)^{2\ell+1}\\ &\quad+\sum_{k=0}^{\lfloor mt_i\rfloor}\sum_{j\in I_k}\Big[f^{(2\ell+1)}(\widetilde{B}^H_{\frac{j}{n}})-f^{(2\ell+1)}(B^H_{\frac{k}{m}})\Big]\big(\Delta_j^n B^H\big)^{2\ell+1}\\ &=: A^{(1,i)}_{n,m}+A^{(2,i)}_{n,m}.\end{aligned}$$

From Lemma 4.2 with $r=2\ell+1$ and $\phi=f^{(2\ell+1)}$, we can write, for any $q>2$,

$$\begin{aligned}\mathbb{E}[(A^{(2,i)}_{n,m})^2] &\le C_T \sup_{0\le w\le 3(2\ell+1)}\sup_{0\le j\le\lfloor nT\rfloor-1}\big\|f^{(w)}(\widetilde{B}^H_{\frac{j}{n}})-f^{(w)}(B^H_{\frac{k(j)}{m}})\big\|^2_{L^q(\Omega)}\\ &\quad+C_T \sup_{0\le w\le 3(2\ell+1)}\sup_{0\le j\le\lfloor nT\rfloor-1}\big\|f^{(w)}(\widetilde{B}^H_{\frac{j}{n}})\big\|^2_{L^2(\Omega)}(m^{-2H}+n^{2H-1}m^{2-4H})\\ &\quad+C_T \sup_{0\le w\le 3(2\ell+1)}\sup_{0\le i,j\le\lfloor nT\rfloor-1}\big\|f^{(w)}(\widetilde{B}^H_{\frac{i}{n}})\big\|_{L^2(\Omega)}\big\|f^{(w)}(\widetilde{B}^H_{\frac{j}{n}})-f^{(w)}(B^H_{\frac{k(j)}{m}})\big\|_{L^2(\Omega)}\\ &\quad\times(1+n^{2H-1}m^{2-4H})\\ &\le C_T\Bigg[m^{-2H}+n^{2H-1}m^{2-4H}\\ &\quad\times\Bigg(1+\sup_{0\le w\le 3(2\ell+1)}\sup_{\substack{s,t\in[0,T]\\ |t-s|\le\frac{1}{m}}}\Bigg\|f^{(w)}\Big(\frac{B^H_s+B^H_{s+\frac{1}{n}}}{2}\Big)-f^{(w)}(B^H_t)\Bigg\|^2_{L^q(\Omega)}\Bigg)\Bigg],\end{aligned}$$

where $k:=k(j)=\sup\{i\ge 0:\frac{i}{m}\le\frac{j}{n}\}$. This implies

$$\limsup_{n\to\infty}\mathbb{E}[(A^{(2,i)}_{n,m})^2]\le C_T\Big(m^{-2H}+\sup_{0\le w\le 3(2\ell+1)}\sup_{\substack{s,t\in[0,T]\\ |t-s|\le\frac{1}{m}}}\big\|f^{(w)}(B^H_s)-f^{(w)}(B^H_t)\big\|^2_{L^2(\Omega)}\Big),$$

which converges to zero as m tends to infinity.

On the other hand, from (1.6) we deduce that the vector $(A^{(1,1)}_{n,m},\dots A^{(1,d)}_{n,m})$ converges in law, as n tends to infinity, to the vector with components

$$\sigma_\ell\sum_{k=0}^{\lfloor mt_i\rfloor} f^{(2\ell+1)}(B^H_{\frac{k}{m}})(W_{\frac{k+1}{m}}-W_{\frac{k}{m}}),$$

$i=1,\dots,d$, where W is a Brownian motion independent of B^H. Each of these components converges in $L^2(\Omega)$ to the stochastic integral $\sigma_\ell\int_0^{t_i} f^{(2\ell+1)}(B^H_s)\,dW_s$, as m tends to infinity. This completes the proof of the theorem. □

4. Appendix

This section is devoted to state and prove a couple of technical lemmas. The first lemma is the basic ingredient to show that the sequence of processes Φ_n^ℓ is tight and the processes Φ_n^h for $h=\ell+1,\dots,2\ell$ converge to zero in $D([0,\infty))$. For this we need to estimate the fourth moment of the increments of the processes Φ_n^h.

Lemma 4.1. *Consider the processes Φ_n^h, $h = \ell, \dots, 2\ell$ defined in (3.4). Then, for any $0 \le s < t \le T$ we have*

$$\mathbb{E}\left[|\Phi_n^h(t) - \Phi_n^h(s)|^4\right] \le C_T \sum_{N=2}^{4} \left(\lfloor nt \rfloor - \lfloor ns \rfloor\right)^N n^{-2NH(2h+1)},$$

where the constant C_T depends only on T.

Proof. For any $0 \le s < t \le T$ we can write

$$\mathbb{E}\left[|\Phi_n(t) - \Phi_n(s)|^4\right] = \sum_{j_1,j_2,j_3,j_4=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} \mathbb{E}\left[\prod_{i=1}^{4} \left(f^{(2h+1)}(\widetilde{B}^H_{\frac{j_i}{n}})(\Delta^n_{j_i} B^H)^{2h+1}\right)\right].$$

By Lemma 2.3 we obtain

$$(\Delta^n_{j_i} B^H)^{2h+1} = \sum_{u=0}^{h} C_{2h+1,u} n^{-2uH}\, I_{2h+1-2u}(\partial_{\frac{j_i}{n}}^{\otimes(2h+1-2u)}),$$

which leads to

$$\prod_{i=1}^{4} (\Delta^n_{j_i} B^H)^{2h+1} = \sum_{u_1,u_2,u_3,u_4=0}^{h} C_{h,\mathbf{u}} n^{-2|\mathbf{u}|H} \prod_{i=1}^{4} \left(I_{2h+1-2u_i}(\partial_{\frac{j_i}{n}}^{\otimes(2h+1-2u_i)}) \right),$$

where $C_{h,\mathbf{u}}$ is a constant depending on h and the vector $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and we use the notation $|\mathbf{u}| = u_1 + u_2 + u_3 + u_4$. To simplify the notation we write $2h + 1 - u_i = v_i$ for $i = 1, 2, 3, 4$. The product formula for multiple stochastic integrals (2.4) allows us to write

$$\prod_{i=1}^{4} \left(I_{v_i}(\partial_{\frac{j_i}{n}}^{\otimes v_i}) \right) = \sum_{\alpha \in \Lambda} C_\alpha \prod_{1 \le i < k \le 4} \langle \partial_{\frac{j_i}{n}}, \partial_{\frac{j_k}{n}} \rangle_{\mathcal{H}}^{\alpha_{ik}}$$
$$\times I_{|\mathbf{v}|-2|\alpha|} \left(\partial_{\frac{j_1}{n}}^{\otimes v_1-\alpha_{12}-\alpha_{13}-\alpha_{14}} \otimes \partial_{\frac{j_2}{n}}^{\otimes v_2-\alpha_{12}-\alpha_{23}-\alpha_{24}} \otimes \partial_{\frac{j_3}{n}}^{\otimes v_3-\alpha_{13}-\alpha_{23}-\alpha_{34}} \otimes \partial_{\frac{j_4}{n}}^{\otimes v_4-\alpha_{14}-\alpha_{24}-\alpha_{34}} \right),$$

where $|\mathbf{v}| = v_1 + v_2 + v_3 + v_4 = 8h + 4 - |\mathbf{u}|$, Λ is the set of all multiindices $\alpha = (\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34})$ with $\alpha_{ik} \ge 0$, such that

$$\begin{aligned} \alpha_{12} + \alpha_{13} + \alpha_{14} &\le v_1 \\ \alpha_{12} + \alpha_{23} + \alpha_{24} &\le v_2 \\ \alpha_{13} + \alpha_{23} + \alpha_{34} &\le v_3 \\ \alpha_{14} + \alpha_{24} + \alpha_{34} &\le v_4. \end{aligned}$$

For any $\mathbf{j} = (j_1, j_2, j_3, j_4)$, $\lfloor ns \rfloor \le j_i \le \lfloor nt \rfloor - 1$, we set

$$Y_{\mathbf{j}} = \prod_{i=1}^{4} f^{(2h+1)}(\widetilde{B}^H_{\frac{j_i}{n}}),$$

and

$$h_{\mathbf{j},\alpha,\mathbf{v}} = \partial_{\frac{j_1}{n}}^{\otimes v_1-\alpha_{12}-\alpha_{13}-\alpha_{14}} \otimes \partial_{\frac{j_2}{n}}^{\otimes v_2-\alpha_{12}-\alpha_{23}-\alpha_{24}} \otimes \partial_{\frac{j_3}{n}}^{\otimes v_3-\alpha_{13}-\alpha_{23}-\alpha_{34}} \otimes \partial_{\frac{j_4}{n}}^{\otimes v_4-\alpha_{14}-\alpha_{24}-\alpha_{34}}.$$

Applying the duality formula (2.6) we obtain

$$\mathbb{E}\left[Y_{\mathbf{j}} I_{|\mathbf{v}|-2|\alpha|}(h_{\mathbf{j},\alpha,\mathbf{v}})\right] = \mathbb{E}\left[\langle D^{|\mathbf{v}|-2|\alpha|} Y_{\mathbf{j}}, h_{\mathbf{j},\alpha,\mathbf{v}}\rangle_{\mathcal{H}^{\otimes |\mathbf{v}|-2|\alpha|}}\right].$$

Therefore, we have shown the following formula

$$\begin{aligned}\mathbb{E}\left[|\Phi_n^h(t)-\Phi_n^h(s)|^4\right] &= \sum_{\mathbf{j}}\sum_{\mathbf{u}} C_{h,\mathbf{u}} n^{-2|\mathbf{u}|H} \sum_{\alpha\in\Lambda} C_\alpha \\ &\times\left(\prod_{1\le i<k\le 4}\langle \partial_{\frac{j_i}{n}}, \partial_{\frac{j_k}{n}}\rangle_{\mathcal{H}}^{\alpha_{ik}}\right)\mathbb{E}\left[\langle D^{|\mathbf{v}|-2|\alpha|} Y_{\mathbf{j}}, h_{\mathbf{j},\alpha,\mathbf{v}}\rangle_{\mathcal{H}^{\otimes |\mathbf{v}|-2|\alpha|}}\right],\end{aligned}$$

where the components of $\mathbf{j}$ satisfy $\lfloor ns\rfloor \le j_i \le \lfloor nt\rfloor - 1$ and $0\le u_i\le h$. Finally, the inner product $\langle D^{|\mathbf{v}|-2|\alpha|} Y_{\mathbf{j}}, h_{\mathbf{j},\alpha,\mathbf{v}}\rangle_{\mathcal{H}^{\otimes |\mathbf{v}|-2|\alpha|}}$ can be expressed in the form

$$\sum_{\beta\in\Gamma}\Phi_\beta \prod_{1\le i,k\le 4}\langle \partial_{\frac{j_i}{n}}, \widetilde{\varepsilon}_{\frac{j_k}{n}}\rangle_{\mathcal{H}}^{\beta_{ik}},$$

where $\beta = (\beta_{ik})_{1\le i,k\le 4}$ is a matrix with nonnegative entries such that

$$\begin{aligned}\sum_{k=1}^4 \beta_{1k} &= v_1 - \alpha_{12} - \alpha_{13} - \alpha_{14}\\ \sum_{k=1}^4 \beta_{2k} &= v_2 - \alpha_{12} - \alpha_{23} - \alpha_{24}\\ \sum_{k=1}^4 \beta_{3k} &= v_3 - \alpha_{13} - \alpha_{23} - \alpha_{34}\\ \sum_{k=1}^4 \beta_{4k} &= v_4 - \alpha_{14} - \alpha_{24} - \alpha_{34}.\end{aligned}$$

Notice that $|\beta| = \sum_{i,k=1}^4 \beta_{ik} = |\mathbf{v}| - 2|\alpha|$. Moreover, the random variables Φ_β are linear combinations of products of the form $\prod_{i=1}^4 f^{(w_i)}(B^H_{\widetilde{\varepsilon}_{\frac{j_i}{n}}})$, with $2h+1\le w_i\le 2h+1+|\mathbf{v}|-2|\alpha|$. This leads to the following estimate

$$\mathbb{E}\left[|\Phi_n^h(t)-\Phi_n^h(s)|^4\right] \le C_T \sum_{\mathbf{j}}\sum_{\mathbf{u}} n^{-2|\mathbf{u}|H}\sum_{\alpha\in\Lambda}\sum_{\beta\in\Gamma}\prod_{1\le i<k\le 4}\left|\langle \partial_{\frac{j_i}{n}}, \partial_{\frac{j_k}{n}}\rangle_{\mathcal{H}}^{\alpha_{ik}}\right|\prod_{1\le i,k\le 4}\left|\langle \partial_{\frac{j_i}{n}}, \widetilde{\varepsilon}_{\frac{j_k}{n}}\rangle_{\mathcal{H}}^{\beta_{ik}}\right|.$$

Consider the decomposition of the above sum as follows

$$\mathbb{E}\left[|\Phi_n(t)-\Phi_n(s)|^4\right] \le C_T\left(A_n^{(1)} + A_n^{(2)} + A_n^{(3)}\right),$$

where $A_n^{(1)}$ contains all the terms such that at least two components of α are nonzero, $A_n^{(2)}$ contains all the terms such that one component of α is nonzero and the others vanish, and $A_n^{(3)}$ contains all the terms such that all the components of α are zero.

Step 1. Let us first estimate $A_n^{(1)}$. Without any loss of generality, we can assume that $\alpha_{12} \geq 1$ and $\alpha_{13} \geq 1$. From (2.9) with $r = 1$, we obtain

$$\sum_{j_1=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} \left| \langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_2}{n}} \rangle_{\mathcal{H}} \right| \leq C n^{-2H} \tag{4.1}$$

and

$$\sum_{j_3=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} \left| \langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_3}{n}} \rangle_{\mathcal{H}} \right| \leq C n^{-2H}.$$

We estimate each of the remaining factors by n^{-2H}. In this way, we obtain a bound of the form

$$A_n^{(1)} \leq C(\lfloor nt \rfloor - \lfloor ns \rfloor)^2 n^{-2H(|\mathbf{u}|+|\alpha|+|\beta|)}.$$

Taking into account that $|\alpha| \leq \frac{1}{2}|\mathbf{v}|$, we can write

$$\begin{aligned} |\mathbf{u}| + |\alpha| + |\beta| &= |\mathbf{u}| + |\alpha| + |\mathbf{v}| - 2|\alpha| \\ &= |\mathbf{u}| + |\mathbf{v}| - |\alpha| \geq |\mathbf{u}| + \frac{|\mathbf{v}|}{2} = 4h + 2 - \frac{|\mathbf{u}|}{2} \geq 4h + 2, \end{aligned}$$

and, as a consequence,

$$A_n^{(1)} \leq C(\lfloor nt \rfloor - \lfloor ns \rfloor)^2 n^{-4H(2h+1)}. \tag{4.2}$$

Step 2. For the term $A_n^{(2)}$, we can assume that $\alpha_{12} \geq 1$ and all the other components of α vanish. In this case, we still have the inequality (4.1). Then, we estimate each of the remaining factors by n^{-2H}. In this way, we obtain a bound of the form

$$A_n^{(2)} \leq C(\lfloor nt \rfloor - \lfloor ns \rfloor)^3 n^{-2H(|\mathbf{u}|+|\alpha|+|\beta|)}.$$

Taking into account that $|\alpha| = \alpha_{12} \leq v_1 = 2h + 1 - u_1 \leq 2h + 1$, we can write

$$\begin{aligned} |\mathbf{u}| + |\alpha| + |\beta| &= |\mathbf{u}| + |\alpha| + |\mathbf{v}| - 2|\alpha| \\ &= |\mathbf{u}| + |\mathbf{v}| - |\alpha| \geq |\mathbf{u}| + |\mathbf{v}| - 2h - 1 = 6h + 3, \end{aligned}$$

and, as a consequence, we obtain

$$A_n^{(2)} \leq C(\lfloor nt \rfloor - \lfloor ns \rfloor)^3 n^{-6H(2h+1)}. \tag{4.3}$$

Step 3. Estimating all terms by n^{-2H}, we get

$$A_n^{(3)} \leq C(\lfloor nt \rfloor - \lfloor ns \rfloor)^4 n^{-2H(|\mathbf{u}|+|\beta|)}.$$

We have

$$|\mathbf{u}| + |\beta| = |\mathbf{u}| + |\mathbf{v}| = 8h + 4,$$

and, as a consequence, we obtain

$$A_n^{(3)} \leq C(\lfloor nt \rfloor - \lfloor ns \rfloor)^4 n^{-8H(2h+1)}. \tag{4.4}$$

In conclusion, from (4.2), (4.3) and (4.4), we obtain the desired estimate. This completes the proof of the lemma. ∎

The second lemma provides a bound for the residual term in the application of the small blocks/big blocks technique and it is a variation of [8, Lemma 3.2]. Its proof is based on the techniques of Malliavin calculus. As before for two integers $n > m \geq 2$, for any $j \geq 0$ we define $k := k(j) = \sup\{i \geq 0 : \frac{i}{m} \leq \frac{j}{n}\}$.

Lemma 4.2. *Let $r = 1, 3, 5, \ldots$ and $n > m \geq 2$ be two integers. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^{2r}$ function such that ϕ and all derivatives up to order $2r$ have moderate growth, and let $B^H = \{B_t^H, t \geq 0\}$ be a fBm with Hurst parameter $H < \frac{1}{2}$. Then, for any $q > 2$ and any $T > 0$,*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left(\sum_{j=0}^{\lfloor nt\rfloor-1} \left(\phi(\widetilde{B}^H_{\frac{j}{n}}) - \phi(B_{\frac{k(j)}{m}})\right)(\Delta_j^n B^H)^r\right)^2\right] \leq C_T \Gamma_{m,n} n^{1-2rH},$$

where C_T is a positive constant depending on q, r, H and T, and

$$\begin{aligned}
\Gamma_{m,n} \;:=\; & \sup_{0\leq w\leq 2r}\ \sup_{0\leq j\leq \lfloor nT\rfloor-1} \left\|\phi^{(w)}(\widetilde{B}^H_{\frac{j}{n}}) - \phi^{(w)}(B^H_{\frac{k(j)}{m}})\right\|^2_{L^q(\Omega)} \\
& + \sup_{0\leq w\leq 2r}\ \sup_{0\leq j\leq \lfloor nT\rfloor-1} \left\|\phi^{(w)}(\widetilde{B}^H_{\frac{j}{n}})\right\|^2_{L^2(\Omega)} (m^{-2H} + n^{2H-1} m^{2-4H}) \\
& + \sup_{0\leq w\leq 2r}\ \sup_{0\leq i,j\leq \lfloor nT\rfloor-1} \left\|\phi^{(w)}(\widetilde{B}^H_{\frac{i}{n}})\right\|_{L^2(\Omega)} \left\|\phi^{(w)}(\widetilde{B}^H_{\frac{j}{n}}) - \phi^{(w)}(B^H_{\frac{k(j)}{n}})\right\|_{L^2(\Omega)} \\
& \quad \times (1 + n^{2H-1} m^{2-4H}).
\end{aligned}$$

Proof. The proof is based in the methodology used to show Lemma 3.2 in [8]. To simplify notation, let $Y_j(\phi) := \phi(\widetilde{B}^H_{\frac{j}{n}}) - \phi(B^H_{\frac{k(j)}{m}})$, and set

$$I_t := \mathbb{E}\left[\left(\sum_{j=0}^{\lfloor nt\rfloor-1} Y_j(\phi)\left(\Delta_j^n B^H\right)^r\right)^2\right].$$

From Lemma 2.3 we obtain

$$\begin{aligned}
I_t \;=\; & \sum_{u,v=0}^{\lfloor \frac{r}{2}\rfloor} C_{r,u} C_{r,v}\, n^{-2H(u+v)} \sum_{i,j=0}^{\lfloor nt\rfloor-1} \mathbb{E}\Big[Y_i(\phi) Y_j(\phi)\, I_{r-2u}(\partial_{\frac{i}{n}}^{\otimes r-2u})\, I_{r-2v}(\partial_{\frac{j}{n}}^{\otimes r-2v})\Big] \\
\leq\; & C \sum_{u,v=0}^{\lfloor \frac{r}{2}\rfloor} n^{-2H(u+v)} \sum_{i,j=0}^{\lfloor nT\rfloor-1} \left|\mathbb{E}\Big[Y_i(\phi) Y_j(\phi)\, I_{r-2u}(\partial_{\frac{i}{n}}^{\otimes r-2u})\, I_{r-2v}(\partial_{\frac{j}{n}}^{\otimes r-2v})\Big]\right|.
\end{aligned}$$

Then we apply the product formula (2.4) in order to develop the product of two multiple stochastic integrals and we end up with

$$\begin{aligned}
I_t &\leq C\sum_{u,v=0}^{\lfloor \frac{r}{2}\rfloor}\sum_{z=0}^{(r-2u)\wedge(r-2v)} n^{-2H(u+v)}\sum_{i,j=0}^{\lfloor nT\rfloor-1}\Big|\mathbb{E}\Big[Y_i(\phi)Y_j(\phi)\\
&\qquad \times I_{2r-2(u+v)-2z}\big(\partial_{\frac{i}{n}}^{\otimes r-2u-z}\widetilde{\otimes}\partial_{\frac{j}{n}}^{\otimes r-2v-z}\big)\big\langle\partial_{\frac{i}{n}},\partial_{\frac{j}{n}}\big\rangle_{\mathfrak{H}}^{z}\Big]\Big|\\
&= C\sum_{u,v=0}^{\lfloor \frac{r}{2}\rfloor} n^{-2H(u+v)}\sum_{i,j=0}^{\lfloor nT\rfloor-1}\Big|\mathbb{E}\Big[Y_i(\phi)Y_j(\phi)\, I_{2r-2(u+v)}\big(\partial_{\frac{i}{n}}^{\otimes r-2u}\widetilde{\otimes}\partial_{\frac{j}{n}}^{\otimes r-2v}\big)\Big]\Big|\\
&\qquad +C\sum_{u,v=0}^{\lfloor \frac{r}{2}\rfloor}\sum_{z=1}^{(r-2u)\wedge(r-2v)} n^{-2H(u+v)}\sum_{i,j=0}^{\lfloor nT\rfloor-1}\Big|\mathbb{E}\Big[Y_i(\phi)Y_j(\phi)\\
&\qquad \times I_{2r-2(u+v)-2z}\big(\partial_{\frac{i}{m}}^{\otimes r-2u-z}\widetilde{\otimes}\partial_{\frac{j}{n}}^{\otimes r-2v-z}\big)\big\langle\partial_{\frac{i}{n}},\partial_{\frac{j}{n}}\big\rangle_{\mathfrak{H}}^{z}\Big]\Big|\\
&=: C(D_1+D_2).
\end{aligned}$$

We first study term D_2, that is when $z\geq 1$. On one hand, from the estimate (2.5), we get

$$\begin{aligned}
\Big\|I_{2r-2(u+v)-2z}\big(\partial_{\frac{i}{n}}^{\otimes r-2u-z}\widetilde{\otimes}\partial_{\frac{j}{n}}^{\otimes r-2v-z}\big)\Big\|_{L^{q/(q-2)}(\Omega)} &\leq C\big(\|\partial_{\frac{i}{n}}\|_{\mathfrak{H}}^{r-2u-z}\,\|\partial_{\frac{j}{n}}\|_{\mathfrak{H}}^{r-2v-z}\big)\\
&= C\,\|\partial_{\frac{1}{n}}\|_{\mathfrak{H}}^{2r-2(u+v)-2z}\\
&= C\,n^{-2H(r-u-v-z)}. \qquad (4.5)
\end{aligned}$$

On the other hand, using (2.10), we obtain

$$\sum_{i,j=0}^{\lfloor nT\rfloor-1}\Big|\big\langle\partial_{\frac{i}{n}},\partial_{\frac{j}{n}}\big\rangle_{\mathfrak{H}}^{z}\Big|\leq C\,n^{1-2zH}. \qquad (4.6)$$

Thus, from (4.5) and (4.6) and using Hölder's inequality, we deduce that the term D_2 is bounded by

$$\begin{aligned}
D_2 &\leq C\sum_{u,v=0}^{\lfloor \frac{r}{2}\rfloor}\sum_{z=1}^{(r-2u)\wedge(r-2v)} n^{-2H(u+v)}\sup_{0\leq j\leq\lfloor nT\rfloor-1}\|Y_j(\phi)\|_{L^q(\Omega)}^2\,n^{-2H(r-u-v-z)}\,n^{1-2zH}\\
&\leq C\sup_{0\leq j\leq\lfloor nT\rfloor-1}\|Y_j(\phi)\|_{L^q(\Omega)}^2\,n^{1-2rH}.
\end{aligned}$$

Now, let us study term D_1, that is when $z=0$. By (2.6) we have

$$\begin{aligned}
&\Big|\mathbb{E}\Big[Y_i(\phi)Y_j(\phi)\,I_{2(r-u-v)}\big(\partial_{\frac{i}{n}}^{\otimes r-2u}\widetilde{\otimes}\partial_{\frac{j}{n}}^{\otimes r-2v}\big)\Big]\Big|\\
&\qquad=\Big|\mathbb{E}\Big[\big\langle D^{2(r-u-v)}(Y_i(\phi)Y_j(\phi)),\partial_{\frac{i}{n}}^{\otimes r-2u}\widetilde{\otimes}\partial_{\frac{j}{n}}^{\otimes r-2v}\big\rangle_{\mathfrak{H}^{\otimes 2(r-u-v)}}\Big]\Big|.
\end{aligned}$$

Write $s=2(r-u-v)$. By definition of Malliavin derivative and Leibniz rule, $D^s_{u_1,\dots,u_s}(Y_i(\phi)Y_j(\phi))$ consists of terms of the form $D^{|J|}_{\mathbf{u}_J}(Y_i(\phi))D^{s-|J|}_{\mathbf{u}_{J^c}}(Y_j(\phi))$, where J is a subset of $\{1,\dots,s\}$, $|J|$ denotes the cardinality of J and $\mathbf{u}_J=(u_i)_{i\in J}$. Without loss of generality, we may fix J and assume that $a = |J| \geq 1$. By our assumptions on ϕ and the definition of Malliavin derivative, we know that

$$D^a(Y_i(\phi)) \;=\; \phi^{(a)}(\widetilde{B}^H_{\frac{i}{n}})\widetilde{\varepsilon}^{\otimes a}_{\frac{i}{n}} - \phi^{(a)}(B^H_{\frac{k(i)}{m}})\varepsilon^{\otimes a}_{\frac{k(i)}{m}} = Y_i(\phi^{(a)})\varepsilon^{\otimes a}_{\frac{k(i)}{m}} + \phi^{(a)}(\widetilde{B}^H_{\frac{i}{n}})\big(\widetilde{\varepsilon}^{\otimes a}_{\frac{i}{n}} - \varepsilon^{\otimes a}_{\frac{k(i)}{m}}\big),$$

where we recall that $k = k(i) = \sup\{j : \frac{j}{m} \leq \frac{i}{n}\}$, and, for each $a \leq 2r$, we have $D^a(Y_i(\phi)) \in L^2(\Omega; \mathfrak{H}^{\otimes a})$. Setting $b = s - |J| = s - a$ and with a slight abuse of notation, it follows that

$$\begin{aligned}
& \mathbb{E}\Big[\big\langle D^a_{\mathbf{u}_J}(Y_i(\phi))D^b_{\mathbf{u}_{J^c}}(Y_j(\phi)), \partial^{\otimes r-2u}_{\frac{i}{n}} \otimes \partial^{\otimes r-2v}_{\frac{j}{n}}\big\rangle_{\mathfrak{H}^{\otimes 2r-2(u+v)}}\Big] \\
\leq\; & \|Y_i(\phi^{(a)})\|_{L^2(\Omega)}\|Y_j(\phi^{(b)})\|_{L^2(\Omega)}\Big|\big\langle \varepsilon^{\otimes a}_{\frac{k(i)}{m}}(\mathbf{u}_J) \otimes \varepsilon^{\otimes b}_{\frac{k(j)}{m}}(\mathbf{u}_{J^c}), \partial^{\otimes r-2u}_{\frac{i}{n}} \otimes \partial^{\otimes r-2v}_{\frac{j}{n}}\big\rangle_{\mathfrak{H}^{\otimes s}}\Big| \\
& + \|Y_i(\phi^{(a)})\|_{L^2(\Omega)}\|\phi^{(b)}(\widetilde{B}^H_{\frac{j}{n}})\|_{L^2(\Omega)} \\
& \quad \times\Big|\big\langle \varepsilon^{\otimes a}_{\frac{k(i)}{m}}(\mathbf{u}_J) \otimes \big(\widetilde{\varepsilon}^{\otimes b}_{\frac{j}{n}} - \varepsilon^{\otimes b}_{\frac{k(j)}{m}}\big)(\mathbf{u}_{J^c}), \partial^{\otimes r-2u}_{\frac{i}{n}} \otimes \partial^{\otimes r-2v}_{\frac{j}{n}}\big\rangle_{\mathfrak{H}^{\otimes s}}\Big| \\
& + \|\phi^{(a)}(\widetilde{B}^H_{\frac{i}{m}})\|_{L^2(\Omega)}\|Y_j(\phi^{(b)})\|_{L^2(\Omega)} \\
& \quad \times\Big|\big\langle \big(\widetilde{\varepsilon}^{\otimes a}_{\frac{i}{m}} - \varepsilon^{\otimes a}_{\frac{k(i)}{n}}\big)(\mathbf{u}_J) \otimes \varepsilon^{\otimes b}_{\frac{k(j)}{n}}(\mathbf{u}_{J^c}), \partial^{\otimes r-2u}_{\frac{i}{m}} \otimes \partial^{\otimes r-2v}_{\frac{j}{m}}\big\rangle_{\mathfrak{H}^{\otimes s}}\Big| \\
& + \|\phi^{(a)}(\widetilde{B}^H_{\frac{i}{n}})\|_{L^2(\Omega)}\|\phi^{(b)}(\widetilde{B}^H_{\frac{j}{n}})\|_{L^2(\Omega)} \\
& \quad \times\Big|\big\langle \big(\widetilde{\varepsilon}^{\otimes a}_{\frac{i}{n}} - \varepsilon^{\otimes a}_{\frac{k(i)}{m}}\big)(\mathbf{u}_J) \otimes \big(\widetilde{\varepsilon}^{\otimes b}_{\frac{j}{n}} - \varepsilon^{\otimes b}_{\frac{k(j)}{m}}\big)(\mathbf{u}_{J^c}), \partial^{\otimes r-2u}_{\frac{i}{n}} \otimes \partial^{\otimes r-2v}_{\frac{j}{n}}\big\rangle_{\mathfrak{H}^{\otimes s}}\Big| \\
=:\; & D_{11} + D_{12} + D_{13} + D_{14}.
\end{aligned}$$

Consider first the term D_{11}. By (2.8), we have either

$$D_{11} \leq C\Big|\big\langle \varepsilon_{\frac{k(i)}{m}}, \partial_{\frac{j}{n}}\big\rangle_{\mathfrak{H}}\Big|\; n^{-2H(a+b-1)} \sup_{0\leq w\leq 2r}\; \sup_{0\leq j\leq \lfloor nT\rfloor -1} \|Y_j(\phi^{(w)})\|^2_{L^2(\Omega)}$$

or

$$D_{11} \leq C\Big|\big\langle \varepsilon_{\frac{k(i)}{m}}, \partial_{\frac{i}{n}}\big\rangle_{\mathfrak{H}}\Big|\; n^{-2H(a+b-1)} \sup_{0\leq w\leq 2r}\; \sup_{0\leq j\leq \lfloor nT\rfloor -1} \|Y_j(\phi^{(w)})\|^2_{L^2(\Omega)}.$$

By Lemma 2.1.a

$$\sum_{j=0}^{\lfloor nT\rfloor -1} \Big|\big\langle \varepsilon_{\frac{k(i)}{m}}, \partial_{\frac{j}{n}}\big\rangle_{\mathfrak{H}}\Big| \leq C \tag{4.7}$$

and by (2.11),

$$\sum_{i=0}^{\lfloor nT\rfloor -1}\; \sum_{j=0}^{\lfloor nT\rfloor -1} \Big|\big\langle \varepsilon_{\frac{k(i)}{m}}, \partial_{\frac{i}{n}}\big\rangle_{\mathfrak{H}}\Big|\Big|\big\langle \varepsilon_{\frac{k(j)}{m}}, \partial_{\frac{j}{n}}\big\rangle_{\mathfrak{H}}\Big| \leq C_T m^{2-4H}. \tag{4.8}$$

As a consequence, inequalities (4.7) and (4.8) imply

$$\begin{aligned}
&\sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} n^{-2H(u+v)} \sum_{i,j=0}^{\lfloor nT \rfloor -1} D_{11} \\
&\quad \le C \sup_{0\le w\le 2r} \sup_{0\le j\le \lfloor nT\rfloor -1} \|Y_j(\phi^{(w)})\|^2_{L^2(\Omega)} \sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} \sum_{i=0}^{\lfloor nT \rfloor -1} n^{-2H(u+v+a+b-1)} \\
&\qquad + C_T \sup_{0\le w\le 2r} \sup_{0\le j\le \lfloor nT\rfloor -1} \|Y_j(\phi^{(w)})\|^2_{L^2(\Omega)} \sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} n^{-2H(u+v+a+b-1)}\, n^{2H}\, m^{2-4H} \\
&\quad \le C_T \sup_{0\le w\le 2r} \sup_{0\le j\le \lfloor nT\rfloor -1} \|Y_j(\phi^{(w)})\|^2_{L^2(\Omega)} \left(1 + n^{1-2H} m^{2-4H}\right) n^{1-2rH},
\end{aligned}$$

where we used that $u+v+a+b-1 = 2r-(u+v)-1 \ge r$, since $u+v+1 \le 2\left\lfloor \frac{r}{2} \right\rfloor + 1 = r$ for any odd integer r.

We apply the same calculation to D_{12} and D_{13}, and we similarly obtain that

$$\begin{aligned}
&\sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} n^{-2H(u+v)} \sum_{i,j=0}^{\lfloor nT \rfloor -1} (D_{12}+D_{13}) \\
&\quad \le C_T \sup_{0\le w\le 2r} \sup_{0\le j\le \lfloor nT\rfloor -1} \|\phi^{(w)}(\widetilde{B}_{\frac{j}{n}})\|_{L^2(\Omega)} \sup_{0\le j\le \lfloor nT\rfloor -1} \|Y_j(\phi^{(w)})\|_{L^2(\Omega)} \\
&\qquad \times (1 + n^{1-2H}\, m^{2-4H})\, n^{1-2rH}.
\end{aligned}$$

Now we study term D_{14}. Inequalities (2.12) and (2.13) state that

$$\sum_{j=0}^{\lfloor nT \rfloor -1} \left| \left\langle \widetilde{\varepsilon}_{\frac{i}{n}} - \varepsilon_{\frac{k(i)}{m}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right| \le C_T m^{-2H} \quad \text{and} \quad \sum_{i=0}^{\lfloor nT \rfloor -1} \left| \left\langle \widetilde{\varepsilon}_{\frac{i}{n}} - \varepsilon_{\frac{k(i)}{m}}, \partial_{\frac{i}{n}} \right\rangle_{\mathfrak{H}} \right| \le C_T m^{1-2H}.$$

Then, with the same arguments as those used for D_{11}, we obtain

$$\begin{aligned}
&\sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} n^{-2H(u+v)} \sum_{i,j=0}^{\lfloor nT \rfloor -1} D_{14} \\
&\quad \le C_T \sup_{0\le w\le 2r} \sup_{0\le j\le \lfloor nT\rfloor -1} \|\phi^{(w)}(\widetilde{B}^H_{\frac{j}{n}})\|^2_{L^2(\Omega)} (m^{-2H} + n^{1-2H}\, m^{2-4H})\, n^{1-2rH}.
\end{aligned}$$

The proof is now concluded. ∎

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References

[1] K. Burdzy and J. Swanson (2010): A change of variable formula with Ito correction term. *Ann. Probab.* **38**, no. 5, pp. 1817-1869.

[2] P. Cheridito and D. Nualart (2005): Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter H in $(0, 1/2)$. *Ann. Inst. Henri Poincaré* **41**, pp. 1049-1081.

[3] J. M. Corcuera, D. Nualart and M. Podolskij (2014): Asymptotics of weighted random sums. *Communications in Applied and Industrial Mathematics* **6**, no. 1, e-486.

[4] J. M. Corcuera, D. Nualart and J. Woerner (2006): Power variation of some integral fractional processes. *Bernoulli* **12**, no. 4, pp. 713-735.

[5] M. Gradinaru, I. Nourdin, F. Russo and P. Vallois (2005): m-order integrals and generalized Itô's formula; the case of a fractional Brownian motion with any Hurst index. *Ann. Inst. H. Poincaré Probab. Statist.* **41**, no. 4, pp. 781-806.

[6] D. Harnett and D. Nualart (2012): Weak convergence of the Stratonovich integral with respect to a class of Gaussian processes. *Stoch. Proc. Appl.* **122**, pp. 3460-3505.

[7] D. Harnett and D. Nualart (2013): Central limit theorem for a Stratonovich integral with Malliavin calculus. *Ann. Probab.* **41**, no. 4, pp. 2820-2879.

[8] D. Harnett and D. Nualart (2015): On Simpson's rule and fractional Brownian motion with $H = \frac{1}{10}$. *J. Theoret. Probab.* **28**, pp. 1651-1688.

[9] I. Nourdin (2009): A change of variable formula for the 2D fractional Brownian motion of Hurst index bigger or equal to $1/4$. *J. Funct. Anal.* **256**, pp. 2304-2320.

[10] I. Nourdin, D. Nualart and C.A. Tudor (2010): Central and non-central limit theorems for weighted power variations of fractional Brownian motion. *Ann. Inst. H. Poincaré Probab. Statist.* **46**, pp 1055-1079.

[11] I. Nourdin and G. Peccati (2012): *Normal approximations with Malliavin calculus: from Stein's method to universality.* Cambridge University Press.

[12] I. Nourdin and A. Réveillac (2009): Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: The critical case $H = 1/4$. *Ann. Probab.* **37**, pp. 2200-2230.

[13] I. Nourdin, A. Réveillac and J. Swanson (2010): The weak Stratonovich integral with respect to fractional Brownian motion with Hurst parameter $1/6$. *Electron. J. Probab.* **15**, pp. 2087-2116.

[14] D. Nualart (2006): *The Malliavin calculus and related topics.* Springer-Verlag, Berlin, second edition.

[15] D. Nualart and S. Ortiz-Latorre (2008): Central limit theorems for multiple stochastic integrals and Malliavin calculus. *Stoch. Proc. Appl.* **118**, pp. 614-628.

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