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COMPONENT SIZES FOR LARGE QUANTUM ERDŐS-RÉNYI GRAPH NEAR CRITICALITY

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The N vertices of a quantum random graph are each a circle independently punctured at Poisson points of arrivals, with parallel connections derived through for each pair of these punctured circles by yet another independent Poisson process. Considering these graphs at their critical parameters, we show that the joint law of the re-scaled by $N^{2/3}$ and ordered sizes of their connected components, converges to that of the ordered lengths of excursions above zero for a reflected Brownian motion with drift. Thereby, this work forms the first example of an inhomogeneous random graph, beyond the case of effectively rank-1 models, which is rigorously shown to be in the Erdős-Rényi graphs universality class in terms of Aldous's results.

1. Introduction. The Erdős-Rényi random graph [8] is the simplest and most studied example of a random graph ensemble. Such a graph, denoted by $G(N,p)$, has N vertices, with each pair of vertices connected with probability p, independently of all other pairs. Its phase transition phenomena are well understood. In particular, for $p = \frac{c}{N}$ with $c > 1$, the largest component in $G(N,p)$ has $\Theta(N)$ vertices and the second largest $\mathrm{O}(\ln N)$ vertices (as $N \to \infty$, with probability 1), for $p = \frac{c}{N}$ with $c < 1$ the largest component has $\mathrm{O}(\ln N)$ vertices and when $p = \frac{1}{N}$, the largest component of $G(N,p)$ has $\Theta(N^{2/3})$ vertices (c.f. [8, 5, 18]).

Aldous [2] considered the asymptotic behavior of $G(N,p)$ inside the "scaling window" of this phase transition, namely for $N \to \infty$ and $|p-1/N|$ small enough, showing that the ordered set of component sizes rescaled by $N^{2/3}$ then converges to an ordered set of excursion lengths of reflected inhomogeneous Brownian motion with a certain drift. Various other random graph models exhibit a phase transition phenomenon similar to the Erdős-Rényi random graph. While some further follow the same behaviour as $G(N,p)$ in their near-critical regime, the near-critical regime of others falls into different universality classes.

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For example, Nachmias and Peres [21] prove that the random graph ensemble obtained by performing percolation on a random d-regular ($d \geq 3$) graph on N vertices with percolation probability $p = 1/(d-1) + aN^{-1/3}$ for $a \in \mathbb{R}$ fixed, falls into the same universality class as the Erdős-Rényi random graph. The random multi-graph whose N vertices are constructed using the configuration model, with its vertex degrees being i.i.d. variables, each having the distribution ν, has a richer behavior. Indeed, Joseph [15] shows that when ν has a finite third moment, the near-critical regime of this model falls into the Erdős-Rényi random graph's universality class, whereas if $\nu_k \sim ck^{-\tau}$ as $k \to \infty$, $c > 0$, $\tau \in (3,4)$ then the relevant scaling changes to $N^{-(\tau-2)/(\tau-1)}$ and the limit is an ordered set of the excursion lengths of some other drifted process with independent increments above past minima (near-critical regime of the Erdős-Rényi universality class, is also obtained in [22] for more general class of degree distributions of finite third moment). A similar behavior has been found in the near critical regime of the Rank-1 model (a special case of the general in-homogeneous random graph studied in [6], which has received much attention recently). Such graph has random i.i.d. weights $\{x_i\}$ associated to its vertices, and edges chosen independently, with the edge (i,j) chosen with probability $p_{i,j} = \min\{c\frac{x_i x_j}{N}, 1\}$, for some positive constant $c = c(N)$ (c.f. [23]). For x_i having finite third moment, the near-critical regime corresponds to $c(N) = 1 + aN^{-1/3}$, in which case [2] shows that this model (formulated slightly differently), falls into the Erdős-Rényi graph's universality class (similar results have been later proved in [3, 24]). In contrast, for Rank-1 model with power-law degrees of exponent $\tau \in (3,4)$, [4] show that the sizes of the components, re-scaled by $N^{-(\tau-2)/(\tau-1)}$, converge to hitting times of certain thinned Lévy process.

Our aim here is to study the near-critical behavior of the so-called quantum version of Erdős-Rényi random graph (QRG). We note in passing, that both the motivation and terminology come from the stochastic geometric (Fortuin–Kasteleyn type) representation of the quantum Curie-Weiss model at inverse temperature $\beta > 0$ (we exclude here the ground state case of $\beta = \infty$), in transverse magnetic field of strength $\lambda > 0$ (at $\lambda = 0$ it reduces to the Erdős-Rényi ensemble, see Remark 1.1). We refer the reader to [12] for more information on such stochastic geometric representations (that were originally developed in [1, 7] for the general ferromagnetic context), moving on instead, to the precise description of the QRG (as in [13]).

The model:. With $G_N = \{1, \ldots, N\}$ and $\mathbb{S}_\beta$ denoting the circle of length β, let $\mathcal{G}_N^\beta = G_N \times \mathbb{S}_\beta$, associating to each site $i \in G_N$ the copy $\mathbb{S}_\beta^i = i \times \mathbb{S}_\beta$ of $\mathbb{S}_\beta$, so a point in $\mathcal{G}_N^\beta$ has two coordinates, its site (in G_N) and time (in

$\mathbb{S}_\beta$) coordinates. The QRG is then the following random subset $\mathcal{G}_N^\beta \setminus \mathcal{H}$ of $\mathcal{G}_N^\beta$, equipped with random links $\bigcup_{i,j} \mathcal{L}_{i,j}$ between pairs of points of the type $\{(i,t)$ and (j,t), for $i \neq j\}$. To construct the QRG we first punch within each $\mathbb{S}_\beta^i$ finitely many holes, according to independent Poisson point processes $\mathcal{H}_i$, $i \in G_N$, of intensity $\lambda > 0$, so each resulting punctured circle $\mathbb{S}_\beta^i \setminus \mathcal{H}_i$ consists of m_i disjoint connected intervals

$$\mathbb{S}_\beta^i \setminus \mathcal{H}_i = \bigcup_{l=1}^{m_i} I_i^l \tag{1.1}$$

(the number of holes $\#\mathcal{H}_i = m_i$, except when $\#\mathcal{H}_i = 0$, in which case $m_i = 1$). We next add links between pairs of points in $\mathcal{G}_N^\beta$ of the same time coordinates (i.e. between points (i,t) and (j,t) where $i \neq j$ and $t \in \mathbb{S}_\beta$), as follows. With each (unordered) pair of sites $i,j \in G_N$ we associate a copy $\mathbb{S}_\beta^{i,j}$ of $\mathbb{S}_\beta$ and a Poisson point process of links $\mathcal{L}_{i,j}$ on $\mathbb{S}_\beta^{i,j}$ with intensity $\frac{1}{N}$. The processes $\mathcal{L}_{i,j} = \mathcal{L}_{j,i}$ are assumed to be independent for different (i,j) and also independent of the processes of holes $\mathcal{H}_i$. Two intervals I_i^l and I_j^k of the decomposition (1.1) are then considered to be directly connected if there exists some $t \in \mathcal{L}_{i,j}$ such that both $(i,t) \in I_i^l$ and $(j,t) \in I_j^k$. Setting $\mathcal{H} := \cup_i \mathcal{H}_i$ (a *finite* collection of points), the decomposition

$$\mathcal{G}_N^\beta \setminus \mathcal{H} = \mathcal{C}_1 \vee \cdots \vee \mathcal{C}_\ell \tag{1.2}$$

of $\mathcal{G}_N^\beta \setminus \mathcal{H}$ into maximal connected components is, thereby, well defined (see Figure 1 for an example with $N = 4$). Further, each fixed $x \in \mathcal{G}_N^\beta$ is a.s. not in $\mathcal{H}$, hence the notion of the connected component $\mathcal{C}(x)$ containing x in the decomposition (1.2), is also well defined, and hereafter the *size* of a connected component $\mathcal{C}_j$ (or $\mathcal{C}(x)$), means the number of intervals it contains, and $\mathcal{P}(\mathcal{C}(x)) = \sum_I |I| \, 1_{I \in \mathcal{C}(x)}$ denotes the cumulative length of intervals constituting the component $\mathcal{C}(x)$.

REMARK 1.1. *For $\lambda = 0$ there are no holes, so each $\mathbb{S}_\beta^i \setminus \mathcal{H}_i$ consists of one connected component, which equals to $\mathbb{S}_\beta^i$ itself. We are then back to the Erdős-Rényi random graph $G(N,p)$ with $p = 1 - e^{-\frac{\beta}{N}}$ (the probability that $\mathbb{S}_\beta^i$ and $\mathbb{S}_\beta^j$ are directly connected).*

Treating each interval I_i^k as a vertex, Janson in [14] notices that the QRG is an instance of the general in-homogeneous model of [6]. However, the probability of direct connection between two intervals depends on the size of their overlap and not only on the individual lengths of these intervals.

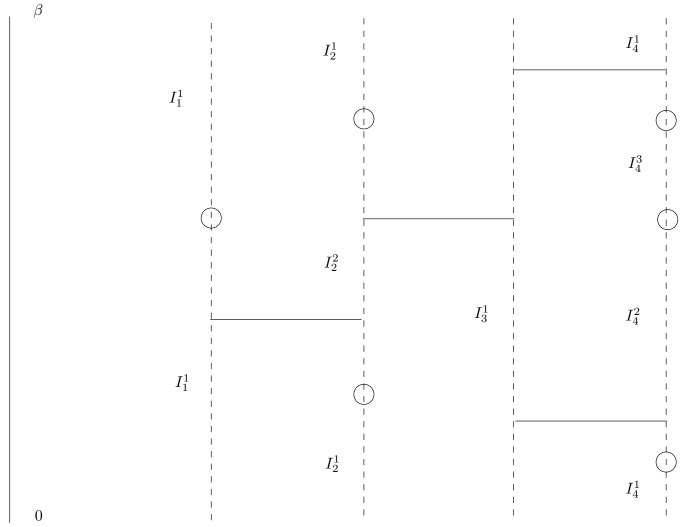

FIG 1. *An example of the decomposition of $\mathcal{G}_N^\beta$ after all the holes are punched and the links are drawn: $\mathcal{G}_N^\beta \setminus \mathcal{H} = \mathcal{C}_1 \vee \mathcal{C}_2 \vee \mathcal{C}_3$, where $\mathcal{C}_1 = I_1^1 \cup I_2^2 \cup I_3^1 \cup I_4^1 \cup I_4^2$, $\mathcal{C}_2 = I_2^1$ and $\mathcal{C}_3 = I_4^3$.*

Beyond separating our model from the class of rank-1 models, this property makes it inherently different from the other models we have mentioned thus far (all of whom mimic the idea of rank-1 random graphs, in the sense that certain *vertex related weights* determine the probabilities in which edges are present in the graph).

An equivalent description of the QRG in case $\lambda > 0$, which we adopt hereafter, has N circles of length $\theta \triangleq \lambda\beta$ with a unit intensity Poisson process of holes on them, using now i.i.d. Poisson processes of intensity $1/(\lambda N)$ for creating links between each pair of (punched) circles. The critical curve for the QRG model in the (β, λ)-parameter space, is obtained in [13] by comparisons with a critical branching process whose offspring distribution is the *cut-gamma* distribution $\Gamma_\theta(2,1)$ (namely, the law of $J := (U+V) \wedge \theta$ for U, V i.i.d. Exp(1) variables). Using the preceding parametrization, the resulting curve $\beta = \beta_c(\lambda)$ corresponds to

$$\beta_c = \frac{\theta}{F(\theta)}, \quad \lambda = F(\theta), \quad \text{for} \quad F(\theta) = 2(1 - e^{-\theta}) - \theta e^{-\theta} \tag{1.3}$$

(where $F(\theta)$ is precisely the expected length J of the interval I in the QRG upon our rescaling by λ). It is easy to check that $\lambda(\theta) : [0, \infty) \mapsto [0, 2)$ is concave, increasing and $\beta_c(\theta) : [0, \infty) \mapsto [1, \infty)$ is strictly increasing, such that the curve $\beta_c(\lambda) : [0,2) \mapsto [1,\infty)$ is strictly increasing. The critical curve is alternatively given by

$$F(\beta,\lambda) := \lambda^{-1} F(\lambda\beta) = 1\,, \tag{1.4}$$

and it is further shown in [13] that taking $F(\beta,\lambda) > 1$ (or equivalently, $\beta > \beta_c(\lambda)$), yields the emergence of an $\Theta(N)$-giant connected component in the disjoint decomposition (1.2), whereas when $F(\beta,\lambda) < 1$ (equivalently, $\beta < \beta_c(\lambda)$), all connected components are typically of order $\mathrm{O}(\ln N)$. Our first result complements [13] by proving that at criticality the largest component is of size $\Theta(N^{2/3})$ (so the QRG admits a version of the Erdős-Rényi phase transition).

THEOREM 1.2. *Suppose (β,λ) is a critical point, namely $F(\beta,\lambda) = 1$. Then, for the largest component $\mathcal{C}_{\max}$ of the QRG, we have that:*

(a) There exist c_*, N_0 and A_0 finite, such that for all $N > N_0$ and $A > A_0$,

$$\mathbb{P}\left(\mathcal{P}(\mathcal{C}_{\max}) > AN^{2/3}\right) \le c_* A^{-3/2}. \tag{1.5}$$

(b) There exists N_1 finite such that for all $N > N_1$ and $\delta > 0$,

$$\mathbb{P}\left(\mathcal{P}(\mathcal{C}_{\max}) < \lfloor \delta N^{2/3} \rfloor\right) \le (6+4\beta^2)\delta^{3/5}. \tag{1.6}$$

Our primary objective is to further analyze the QRG model, and in particular its component sizes near criticality, thereby confirming that the QRG is in the same universality class as the Erdős-Rényi random graph. Whereas our proofs also rely on an exploration process for estimating the connected components sizes, in contrast to all cases dealt with before (i.e. [2, 3, 15, 20, 21]), here we may have many intervals sharing the same vertex (that is, a site $i \in G_N$, or alternatively, the corresponding circle $\mathbb{S}^i_\theta$). Thus, our exploration process (or breadth first walk), may re-visit an already visited vertex (circle), as many times as the number of intervals sharing such vertex. The latter is an *unbounded* random variable, thereby posing a serious challenge to our analysis. While Theorem 1.2 is rough enough that we can surpass this difficulty by showing that multiple returns to same vertex are rare enough to not matter, this is no longer true for our main result, Theorem 1.3, about the scaling limits of ordered component sizes. Indeed, our limiting process drift differs from that of [2] by additional quadratic factor representing the already explored portion of the relevant circle. Indeed, the question of convergence of such quantum random graphs, as metric spaces, is completely open due to this precise problem of multiple visits to the same vertex.

The following quantities are required for our main result. First, let $\widehat{F}(x) = 2 - \frac{2}{x}(1 - e^{-x})$ and

$$\sigma^2 = \mathbb{E}[(J/\lambda)] + \mathrm{var}(J/\lambda) =: F_2(\beta, \lambda)\,, \qquad \gamma(\theta) = \frac{\mathbb{E}[\widehat{F}(\theta - J)]}{F(\theta)}\,. \tag{1.7}$$

Then, for standard Brownian motion $\{W(s), s \geq 0\}$ and any $a \in \mathbb{R}$, consider the process

$$W^{a,\beta,\sigma}(s) := \sigma W(s) + \rho^{a,\beta}(s)\,, \quad \text{for} \quad \rho^{a,\beta}(s) = as - \frac{s^2}{2}\Big(1 + \gamma(\lambda\beta)\big(1 - \frac{1}{\beta}\big)\Big) \tag{1.8}$$

and the associated process of non-negative excursions

$$B^{a,\beta,\sigma}(s) = W^{a,\beta,\sigma}(s) - \min_{0 \leq u \leq s} W^{a,\beta,\sigma}(u). \tag{1.9}$$

THEOREM 1.3. *Fix $a \in \mathbb{R}$ and (β, λ) a point on the critical curve of (1.4). Consider parameters $(\beta_N, \lambda_N) \to (\beta, \lambda)$ such that $F(\beta_N, \lambda_N) = 1 + aN^{-1/3}$. Then, denoting the ordered sizes of components of the graph by $|\mathcal{C}_1^{a,N}|$, $|\mathcal{C}_2^{a,N}|, \ldots$, we have when $N \to \infty$ that*

$$\Big(N^{-2/3}|\mathcal{C}_1^{a,N}|,\, N^{-2/3}|\mathcal{C}_2^{a,N}|, \ldots\Big) \stackrel{d}{\Rightarrow} (\gamma_1, \gamma_2, \ldots)\,,$$

where $\{\gamma_j\}$ denote the ordered lengths of the excursions of the process $B^{a,\beta,\sigma}$ above zero, and the convergence of component sizes holds with respect to the $l^2_\searrow$ topology (as defined in [2]).

In Section 2 we prove Theorem 1.2 by adapting to our context the ideas set forth in [20]. Specifically, the main task here is to construct a pair of manageable auxiliary counting processes, which are not too far apart, while stochastically dominating (from above and below, respectively), the counting process that determines the size of our components (thereby circumventing much of the difficulty associated with the precise counting).

Section 3 is devoted to the proof of Theorem 1.3 which requires finer estimates and thereby some new ideas. What sets our analysis apart of all those mentioned before, is Proposition 3.4 which provides rough estimates on the number of sites of QRG visited twice, or more, during the first $k = \mathrm{O}(N^{2/3})$ steps of the exploration process. It shows in particular that only the *first return to a site* plays a crucial role, with subsequent returns playing no role when the relevant limit is considered. Combined with further rough estimate on the number of sites visited exactly once during the first k steps, it thus allows us to thereafter adapt the program of [2] to the QRG setting. Specifically, Subsection 3.1 deals with weak convergence of the law induced by the rescaled breadth first walk to the law of $W^{a,\beta,\sigma}$ defined on the space of RCLL functions $D([0,\infty))$, equipped with the topology of uniform convergence on finite intervals. Finally, in Subsection 3.2 we collate all the above results into a proof of Theorem 1.3.

A further insight gained from our proof is that the QRG model is in the Erdös-Rényi universality class by the confluence of two reasons: first the small probability of many returns to the same vertex (circle); second and more crucial is the relatively fast relaxation of its exploration process, which thereby behaves approximately as a Markov process. One may examine the latter feature in many other inhomogeneous random graph models, and where it is present, proceed to try proving that they too belong to the Erdös-Rényi universality class.

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2. Proof of Theorem 1.2.

We shall first prove that $|\mathcal{C}_{\max}| = \Theta_{\mathbb{P}}(N^{2/3})$, then to conclude the result of Theorem 1.2, we shall use natural bounds arising from the arguments used to prove the former.

In particular, our first step towards proving Theorem 1.2 will be the following proposition.

PROPOSITION 2.1. *Suppose (β,λ) is a critical point, namely $F(\beta,\lambda)=1$. Then, for the largest component $\mathcal{C}_{\max}$ of the QRG, we have that:*

(a) There exist c_*, N_0 and A_0 finite, such that for all $N > N_0$ and $A > A_0$,

$$\mathbb{P}\left(|\mathcal{C}_{\max}| > AN^{2/3}\right) \le c_* A^{-3/2}. \tag{2.1}$$

(b) There exists N_1 finite such that for all $N > N_1$ and $\delta > 0$,

$$\mathbb{P}\left(|\mathcal{C}_{\max}| < \lfloor \delta N^{2/3} \rfloor\right) \le (6+4\beta^2)\delta^{3/5}. \tag{2.2}$$

In proving $|\mathcal{C}_{\max}| = \Theta_{\mathbb{P}}(N^{2/3})$, to bypass the problem of multiple visits of the same vertex by our exploration process (as described in Subsection 2.1), we stochastically sandwich it between the over-counting process of Subsection 2.2, and the under-counting process of Subsection 2.3. By combining the upper and lower bounds provided by these two auxiliary processes, we complete the proof of Theorem 1.2.

2.1. *Exploration process.* Taking advantage of conditional independence properties of Poisson processes, we start with an algorithmic definition of the exploration process for our model (following [13], who used it for examining a single component). This algorithm allows us to sequentially construct (or sample), the rescaled QRG, interval by interval. In this description, our vertices (circles) have first been labeled $\{1, 2 \ldots, N\}$, and after k steps of the algorithm, we fully explore k intervals, having $A_k \geq 0$ active points (unexplored ends of connections with the already explored intervals, or the point around which a new component starts), while the rest of the space is declared to be neutral (note that we start exploring a new component of the graph upon arriving at $A_{k-1} = 0$, but not before).

Initial stage: We fix the vertex $w_0 = 1$ and choose a point t uniformly at random on this vertex. At end of step $k = 0$ we have $A_0 = 1$, with (w_0, t) as our sole active point and the whole space considered neutral.

At step $k \geq 1$:

(a) If $A_{k-1} > 0$ we choose an active point (w_k, t) whose vertex has the smallest index among all active points. In case of a tie, choose the active point which chronologically appeared earlier than the others on the same vertex.
(b) If $A_{k-1} = 0$ and there exists at least one neutral circle, we choose w_k to be the neutral vertex with the smallest index and uniformly at random mark a new active point (w_k, t) on this vertex.
(c) If $A_{k-1} = 0$ and there is no neutral circle, we choose w_k to be the vertex of smallest index among the vertices having some neutral part, marking new active point (w_k, t) uniformly at random on the neutral part of $\mathbb{S}_\theta^{w_k}$.
(d) If $A_{k-1} = 0$ and there is no neutral part available on any circle, then this ends the exploration process.

Using i.i.d. Exp(1) variables U, V, we carve out of the maximal neutral interval $\{w_k\} \times (t_1, t_2)$ around (w_k, t) the sub-interval $I_k := \{w_k\} \times \widetilde{I}$ for $\widetilde{I} = (t_1 \vee (t - V), t_2 \wedge (t + U))$. For $\mathbb{S}_\theta^{w_k}$ completely neutral we take $t_2 = -t_1 = \infty$ and $\widetilde{I} = \mathbb{S}_\theta$ whenever $U + V \geq \theta$ (resulting with the length of $\widetilde{I}$ having the $\Gamma_\theta(2, 1)$ law). We then remove from the list of active points all those points which got encompassed by the interval I_k, including the base point (w_k, t). The links in the graph connected to all such points other than (w_k, t), are considered to be surplus edges.

Connections of I_k: With $I_k = \{w_k\} \times \widetilde{I}_k$, for each $i \neq w_k$ we view $\widetilde{I}_k$ as a subset of $\mathbb{S}^{w_k,i}$ and sequentially for $i = 1, 2, \dots, N$, sample the process of links $\mathcal{L}_{w_k,i}$ for $t' \geq 0$ restricted to $\widetilde{I}_k$. We erase all links between I_k and points on already explored intervals, and register each link end (i, t') on the neutral space as an active point, labeled with the time (order) of its registration.
When done exploring all the connections from I_k, we change its status from neutral to that of an explored interval and increase k by one, continuing with this procedure till no neutral space remains (which happens after finitely many steps, since the number of intervals in the QRG is finite). To recover the resulting QRG we need only to keep track of the explored intervals (end-points), and the ζ_k new links that have been formed in each step.

Now, let $\eta_k = \zeta_k - (\mathrm{sur}(k) - \mathrm{sur}(k-1))$, where sur(k) count all the surplus edges found by the end of each of the first k steps of exploration. Then, by definition

$$A_k = \begin{cases} A_{k-1} + \eta_k - 1, & \text{if } A_{k-1} > 0 \\ \eta_k, & \text{if } A_{k-1} = 0\,. \end{cases} \tag{2.3}$$

As mentioned before, the exploration of the first component containing the point $(1, t)$ sampled at the initial stage ends at $\tau_1 = \min\{k \geq 1 : A_k = 0\}$, with its size $|\mathcal{C}(1, t)|$ being τ_1 (the number of explored intervals thus far). A new component whose size is $\tau_2 - \tau_1$ is then explored from step τ_1 till the end of step $\tau_2 = \min\{k > \tau_1 : A_k = 0\}$, and so on.

2.2. *Overcounting.* Let $m_i^{(T)} \leq m_i$ count the intervals in vertex i which belong to components of $\mathcal{G}_N^\beta$ whose sizes exceed T and $\mathcal{C}(i, *)$ denotes the connected component of $\mathcal{G}_N^\beta$ containing $\mathbb{S}_\theta^i$, *after erasing* all the holes punched in $\mathbb{S}_\theta^i$ by $\mathcal{H}_i$. Since the size of the component containing interval I_i^l of (1.1) is at most $|\mathcal{C}(i, *)| + m_i - 1$, it follows that $m_i^{(2T)} = 0$ whenever both $m_i \leq T$ and $|\mathcal{C}(i, *)| \leq T$. Hence, by Markov's inequality

$$\begin{aligned} \mathbb{P}\left(|\mathcal{C}_{\max}| \geq 2T\right) &\leq \mathbb{P}\Big(\sum_{i=1}^N m_i^{(2T)} \geq 2T\Big) \\ &\leq \frac{1}{2T}\sum_{i=1}^N \mathbb{E}[m_i^{(2T)}] \leq \frac{1}{2T}\sum_{i=1}^N \mathbb{E}\Big[m_i(\mathbf{1}_{|\mathcal{C}(i,*)|>T} + \mathbf{1}_{m_i>T})\Big]\,. \end{aligned}$$

Further, $|\mathcal{C}(i, *)|$, $i \in G_N$, are identically distributed random variables, each of which is independent of the corresponding variable m_i which in turn has the Poisson$(\theta) \vee 1$ distribution. Consequently,

$$\mathbb{P}\left(|\mathcal{C}_{\max}| \geq 2T\right) \leq \frac{N(\theta + \mathrm{e}^{-\theta})}{2T}\Big[\mathbb{P}(|\mathcal{C}(1, *)| > T) + \mathbb{P}(m_1 \geq T)\Big]\,. \tag{2.4}$$

Taking $T = (A/2)N^{2/3} = H^2$ we thus establish part (a) of Theorem 1.2, upon showing that for some c finite and all H and N large enough

$$\mathbb{P}(|\mathcal{C}(1,*)| > H^2) \le \frac{c}{H}\,. \tag{2.5}$$

Since $\mathcal{C}(1,*)$ corresponds to the exploration process starting at $I_1 = \{1\} \times \mathbb{S}_\theta$, it suffices to consider the value of τ_1 when the corresponding $\{\eta_k\}_{k\ge1}$ are replaced in (2.3) by another, simpler to analyze, collection $\{\xi_k\}$ that stochastically dominate them.

For us to be able to estimate tail probabilities of τ_1 using the i.i.d. sequence $\{\xi_k\}$, we must define appropriate coupling between $\{\eta_k\}$ and $\{\xi_k\}$. To this end, let us define

$$S_n = S_1 + \sum_{k=2}^{n} (\xi_k - 1)\,, \tag{2.6}$$

and consider the following monotone coupling between A_k associated to $\mathcal{C}(1,*)$ and S_k up to time τ_1.

Like A_0, we begin with setting $S_0 = 1$. Since $\mathcal{C}(1,*)$ corresponds to the exploration process starting at $I_1 = \{1\} \times \mathbb{S}_\theta$, η_1 follows Poisson$\Big(\frac{(N-1)\theta}{N\lambda}\Big)$ distribution. Let ξ_1 be η_1 together with *self-links* of the interval to itself. Therefore, ξ_1 follows Poisson$\big(\frac{\theta}{\lambda}\big)$. Since the coupling is up to time τ_1 we only have to consider the $A_{k-1} \ge 0$ case at step $k \ge 2$. As explained above, we choose an active point (w_k, t) and sample links included in the counting towards ζ_k. In order to define ξ_k recall the procedure of sampling connections and consider the following addition to it. In addition to carving out of the maximal neutral interval I_k around (w_k, t), consider also the full interval I_k^S around (w_k, t) having the length law $\Gamma_\theta(2, 1)$. In addition to the links sampled by $\mathcal{L}_{w_k,i}$-s $\forall i \ne w_k$ restricted to $\widetilde{I}_k$, run a unit intensity Poisson process on $I_k^S \cap I_k^c$, and another independent Poisson process of $\big(\frac{1}{N}\big)$ intensity on the interval $\widetilde{I}_k$ counting self-links to the same interval. Let υ_k count the arrival points of these additional Poisson processes. Define $\xi_k = \zeta_k^S + \upsilon_k$ with ζ_k^S counting all the links created by $\mathcal{L}_{w_k,i}$-s restricted to $\widetilde{I}_k$ without erasing those whose end points fall on already explored intervals. Obviously, $\xi_k \ge \eta_k$ for all $k \ge 1$. Note also that for $k \ge 2$, the random variables ξ_k are i.i.d. each following Poisson$\Big(\frac{J_k}{\lambda}\Big)$ conditioned on J_k, which are i.i.d. $\Gamma_\theta(2, 1)$, and independent of the ξ_1.

Defining $\tau = \min\{n \ge 1 : S_n = 0\}$ as the first hitting time of zero by the process S_k, and using the monotone coupling argument, the inequality in (2.5) follows from the bound

$$\mathbb{P}\left(\tau > H^2\right) \le \frac{c}{H}. \tag{2.7}$$

Having $\{-1, 0, 1, 2, \ldots\}$-valued increments, recall Kemperman's formula for such a random walk, stating that for any $\ell \ge 0$ and $n \ge 1$,

$$\mathbb{P}(\tau = n+1 | S_1 = \ell) = \frac{\ell}{n}\mathbb{P}(S_{n+1} = 0 |\, S_1 = \ell)$$

(see [10, Theorem 7, p.165]). Then, we can write

$$\begin{aligned}\mathbb{P}\left(\tau = n+1\right) &= \sum_{\ell=0}^{n} \frac{\ell}{n}\mathbb{P}(S_{n+1} - S_1 = -\ell)\mathbb{P}\left(S_1 = \ell\right) \\ &\le \frac{\mathbb{E}(S_1)}{n} \sup_{\ell}\{\mathbb{P}(S_{n+1} - S_1 = -\ell)\}\,. \end{aligned} \tag{2.8}$$

Our assumption that $F(\beta, \lambda) = 1$, implies that $\mathbb{E}\xi_2 = \lambda^{-1}\mathbb{E}J_2 = 1$, so $\{S_n\}_{n\ge 2}$ has zero-mean i.i.d. increments of finite exponential tails. Thus, applying the local CLT for the lattice random walk $S_{n+1} - S_1$ (see [16, Proposition 2.4.4]), we deduce from (2.8) that

$$\mathbb{P}(\tau = n+1) \le cn^{-3/2}\,, \tag{2.9}$$

for some c finite and all n, which together with (2.7), proves that

$$\mathbb{P}\left(|\mathcal{C}_{\max}| > AN^{2/3}\right) \le c_* A^{-3/2}. \tag{2.10}$$

2.3. *Undercounting.* To bound the lower tail of $|\mathcal{C}_{\max}|$, we construct a stochastic lower bound for all component sizes by following a more restrictive exploration process, which after forming the first active point on each vertex $w \in G_N$, voids all space on that same vertex beyond the relevant interval around this active point (thus sequentially producing components with no more intervals than does the original exploration process). Specifically, after the initial stage, at each step $k \ge 1$ the restrictive exploration considers for I_k only active intervals or completely neutral circles (as in parts (a) and (b) of the original exploration process defined in Section 2.1), till none such are left. It also keeps at most one connection from I_k to any, as of yet, never visited (in particular, completely neutral) circle $\mathbb{S}^i_\theta$, ignoring (erasing) all the other links which are being formed in step k by the original exploration process.

Note that this restrictive process has no surplus edges and its number of active points A^f_k, starts at $A^f_0 = 1$ and follows the recursion

$$A^f_k = \begin{cases} A^f_{k-1} + \eta^f_k - 1, & \text{if } A^f_{k-1} > 0 \\ \eta^f_k, & \text{if } A^f_{k-1} = 0\,. \end{cases} \tag{2.11}$$

Here, conditioned on A^f_{k-1} and J_k, the variables η^f_k are independent $\text{Bin}(N^f_{k-1}, 1 - e^{-J_k/(\lambda N)})$ variables for i.i.d. $\Gamma_\theta(2,1)$-distributed collection $\{J_k\}$ and $N^f_k := N - k - (A^f_k \vee 1)$. As before, the component sizes are given by $\tau^f_r - \tau^f_{r-1}$, for successive returns to zero $\tau^f_r = \min\{k > \tau^f_{r-1} : A^f_k = 0\}$, starting at $\tau^f_0 = 0$.

Had we replaced J_k by $\mathbb{E}J_k = \lambda$, it would have resulted in the exploration process for the (effectively) critical Erdős-Rényi random graph $G(N, 1 - e^{-1/N})$, for which (2.2) is well-known, for example, see [20, Theorem 2]. As we are not aware of a study of component sizes for our inhomogeneous graph, we next adapt the proof of [20, Theorem 2] to our context.

First, from the recursion (2.11) conditioned on the event $\{A^f_{k-1} > 0\}$, then

$$(A^f_k)^2 - (A^f_{k-1})^2 = (\eta^f_k - 1)^2 + 2(\eta^f_k - 1)A^f_{k-1}.$$

Conditioned on the event $\{0 < A^f_{k-1} \le h\}$ for some arbitrary $h > 0$, which we shall specify later, we observe that

$$\mathbb{E}\left[(A^f_k)^2 - (A^f_{k-1})^2 \middle| A^f_{k-1}\right] \ge \frac{(N-h-k)^2}{(\lambda N)^2}\mathbb{E}[J_k^2] - 2\frac{h}{N}(h+k) + \mathrm{O}\left(\frac{1}{N}\right). \tag{2.12}$$

Further, since $\mathbb{E}[J_k^2] > \lambda^2$, so for $h = c_h N^{1/3}$, for all $k \le T_h$ with $T_h = c_1 N^{2/3}$, and large enough N

$$\mathbb{E}\left[(A^f_k)^2 - (A^f_{k-1})^2 \middle| A^f_{k-1}\right] \ge 1 - 2c_h c_1\,, \tag{2.13}$$

where c_h and c_1 are arbitrary positive constants to be chosen later.

The latter bound applies also when $A^f_{k-1} = 0$, as then $A^f_k = \eta^f_k$. Now, taking $b := 1 - 2c_h c_1 > 0$, we consider the stopping time

$$\tau_h = T_h \wedge \min\{k \ge 0 : A^f_k \ge h\}$$

noting that by the preceding calculation, $L_k := (A^f_{k\wedge\tau_h})^2 - b(k \wedge \tau_h)$ is a sub-martingale starting at $L_0 = 1$. Further, it is shown in [20, Proof of Lemma 5] that for ξ a $\text{Bin}(n,p)$ variable, and any $n \le N$, the distribution of $\xi - r$, conditioned on the event $\{\xi \ge r\}$, is stochastically dominated by $\text{Bin}(N,p)$. Thus, in our setting, given $\tau_h = k \le T_h$, $N^f_{k-1} = n \le N$,

$A^f_{k-1} = \ell$ and $p_N = 1 - e^{-J_k/(\lambda N)}$, with $\{J_k\}$ i.i.d $\Gamma_\theta(2,1)$, we have that conditioned on the event $\{A^f_k \geq h\}$, the distribution of $(A^f_k - h)$ is stochastically dominated by $\mathrm{Bin}(N, p_N)$ conditioned on the same p_N. Averaging over all possible n, k, ℓ, p_N values, we deduce that conditioned on $\left\{A^f_{\tau_h} \geq h\right\}$ the over shoot $\left(A^f_{\tau_h} - h\right)$ is stochastically dominated by $\mathrm{Bin}(N, p_N)$ conditioned on $p_N = 1 - e^{-J/(\lambda N)}$ with $J \sim \Gamma_\theta(2,1)$. Consequently,

$$\mathbb{E}[(A^f_{\tau_h})^2] \leq h^2 + 2h\mathbb{E}[Np_N] + \mathbb{E}[(Np_N)^2] + \mathbb{E}[Np_N(1-p_N)] \leq (h+2)^2\,,$$

for $h = c_h N^{1/3}$ and all N large enough. With $\tau_h \leq T_h$, upon applying the optional stopping theorem for the L^2-bounded sub-martingale $\{L_k\}$, we find that $\mathbb{E}[(A^f_{\tau_h})^2] \geq 1 + b\mathbb{E}(\tau_h)$. Thus, by Markov's inequality,

$$\mathbb{P}(\tau_h = T_h) \leq \frac{\mathbb{E}[\tau_h]}{T_h} \leq \frac{(h+2)^2}{bT_h} = \frac{c_h^2}{bc_1}(1 + o(1))\,. \tag{2.14}$$

Fixing $T_0 = \delta N^{2/3}$ note that $|\mathcal{C}_{\max}|$ exceeds the value of the stopping time

$$\tau_0 = T_0 \wedge \min\{s \geq 0 : A^f_{\tau_h + s} = 0\}\,,$$

so the stated bound (2.2) follows once we show that for any $b' > \lambda^{-2}\mathbb{E}[J^2] + 2c_h(c_1 + \delta)$,

$$\mathbb{P}\left(\tau_0 < T_0 \,|\, A^f_{\tau_h} \geq h\right) \leq \frac{b'\delta}{c_h^2}\,. \tag{2.15}$$

Indeed, we then choose $c_h = \frac{1}{2}\delta^{1/5}$ and $c_1 = 1/(4c_h)$, so $b = 1/2$ and $b' = \beta^2 + 1$ works whenever $\delta \leq 1/2$.

To derive (2.15) consider the uniformly bounded, non-negative process $M_k = \max\{h - A^f_{\tau_h + k}, 0\}$. If $0 < M_{k-1} < h$ then by (2.11)

$$M_k^2 - M_{k-1}^2 \leq \left(\eta^f_{\tau_h + k} - 1\right)^2 + 2\left(1 - \eta^f_{\tau_h + k}\right) M_{k-1}\,.$$

The same inequality applies when $M_{k-1} = 0$ (i.e. $A^f_{\tau_h + k - 1} \geq h$, so $M_k \leq \max\{1 - \eta^f_{\tau_h + k}, 0\}$). By definition, $A^f_{\tau_h + k - 1} \leq h$ whenever $M_{k-1} \neq 0$, hence for N large enough and all $k \leq T_0$, $\tau_h \leq T_h$ we find, as in the derivation of (2.13) that

$$\mathbb{E}\Big[M_k^2 - M_{k-1}^2 \,|\, M_{k-1} < h\Big] \leq \lambda^{-2}\mathbb{E}[J^2] + 2\frac{h}{N}(T_h + T_0 + h) + o(1) \leq b'\,.$$

Thus, conditioned on the event $A^f_{\tau_h} \geq h$ the process $\{M^2_{k\wedge\tau_0} - b'(k\wedge\tau_0)\}$ is a super-martingale which starts at zero. Noting that $\{\tau_0 < T_0\} \subseteq \{M_{\tau_0} = h\}$, upon applying the optional stopping theorem for this process at τ_0, we conclude that

$$\mathbb{P}\left(\tau_0 < T_0 \,|\, A^f_{\tau_h} \geq h\right) \leq h^{-2}\mathbb{E}\big[M^2_{\tau_0} \,|\, A^f_{\tau_h} \geq h\big] \leq b' h^{-2}\mathbb{E}[\tau_0 \,|\, A^f_{\tau_h} \geq h] \leq b'\frac{T_0}{h^2}\,,$$

as stated. □

Proof of Theorem 1.2: Observe that $\mathcal{P}(\mathcal{C}_{\max})$ is stochastically dominated by $\sum_{i=1}^{|\mathcal{C}_{\max}|}\xi_i$, where $\{\xi_i\}$ are i.i.d. cut-Gamma random variables. Repeating the arguments set forth in section 2.2 we can easily conclude the upper bound as in equation (2.1).

Similarly, we shall propose a process using the same undercounting algorithm such that it is stochastically dominated by $\mathcal{P}(\mathcal{C}_{\max})$. In particular, instead of accounting only the number of once visited vertices, let us associate a cut gamma $\Gamma_\theta(2,1)$ random variable with every such vertex visited. Then consider the sum of all such random variables, which clearly is dominated by $\mathcal{P}(\mathcal{C}_{\max})$. Thereafter, again following the same steps as in section 2.3 we conclude the required result of Theorem 1.2. □

3. Proof of Theorem 1.3.

3.1. *Exploration process and Brownian excursions.* Recall the length of a sampled interval of the QRG being $J = \min(U+V,\theta)$ for i.i.d. standard Exponential variables U, V (a distribution we denote by $\Gamma_\theta(2,1)$). With $\mathbb{E}(J) = F(\theta)$ of (1.3), the critical curve has the explicit expression $\lambda = F(\theta)$. Further, the critical window around some $\lambda_\star = F(\theta_\star)$ for $\theta_\star > 0$, corresponds to fixing $a \in \mathbb{R}$ and considering

$$\theta_N \to \theta_\star\,, \qquad F(\theta_N) = \lambda_N\big(1 + aN^{-1/3}\big)\,. \tag{3.1}$$

Let $\widetilde{N} = \sum_{i=1}^N m_i$ denote the total number of steps in the exploration process of section 2.1, and $(Y_N^{a,\theta_N}(k), k \leq \widetilde{N})$ be the breadth-first walk associated with the QRG on $\mathcal{G}^N_{\theta_N}$, where (θ_N, λ_N) satisfy (3.1). That is,

$$Y_N^{a,\theta_N}(k) = Y_N^{a,\theta_N}(k-1) + \eta_k - 1, \qquad Y_N^{a,\theta_N}(0) = 1, \tag{3.2}$$

for η_k of recursion (2.3). Thus, $Y_N^{a,\theta_N}(k)$ (which may well become negative as k grows), counts the number of active points at the end of step k, minus the number of explored components before step k.

As in [2], observe that lengths of excursions of the process $Y_N^{a,\theta_N,+}(k) = Y_N^{a,\theta_N}(k) - \min_{l\le k} Y_N^{a,\theta_N}(k)$ above zero correspond to size of the connected component containing the vertex where the process $Y_N^{a,\theta_N}(k)$ started.

Setting

$$\beta_\star = \frac{\theta_\star}{\lambda_\star}, \qquad \sigma_\star^2 = F_2(\beta_\star, \lambda_\star) = \frac{1}{\lambda_*^2}\mathbb{E}\Big[\big((U+V)\wedge\theta_\star\big)^2\Big], \tag{3.3}$$

our goal in this subsection is to prove the following proposition.

Proposition 3.1. *For (θ_N, λ_N) that satisfy (3.1), as $N \to \infty$, the processes*

$$\bar{Y}_N^{a,\theta_N}(s) = N^{-1/3} Y_N^{a,\theta_N}(\lfloor N^{2/3}s\rfloor \wedge \widetilde{N}), \tag{3.4}$$

converge in law to $W^{a,\beta_\star,\sigma_\star}$ of (1.8) (on the space $D([0,\infty))$ equipped with the topology of uniform convergence on compacts).

Recall that ζ_k links are generated at step k of the exploration process and let

$$Z_N^{a,\theta_N}(l) = Z_N^{a,\theta_N}(l-1) + \zeta_l - 1, \qquad Z_N^{a,\theta_N}(0) = 1, \tag{3.5}$$

be the corresponding breadth-first walk. Since we sample intervals only when they are to be explored, the walk Z_N does not distinguish between active points that end as intervals of the QRG and those that are later found to be on surplus edges. Nevertheless, our next proposition controls the number of active points, which as seen in Remark 3.3, yields having at most $O_{\mathbb{P}}(1)$ surplus edges till step $sN^{2/3}$.

Proposition 3.2. *Fixing a, recall the count A_k of active points at the end of step k of the exploration for (θ_N, λ_N) satisfying (3.1). Then, for some $K = K(a,s) < \infty$ and all $L, N \ge L_0(a,s)$,*

$$\mathbb{P}\left(N^{-1/3} \max_{k\le sN^{2/3}} \{A_k\} > L\right) \le KL^{-2}. \tag{3.6}$$

The proof of above proposition involves elaborate, but crude, bounds on functionals of Z_N, and thus we defer it to the appendix.

Remark 3.3. *Recall $sur(l)$ counting the surplus links detected in part (a) of the exploration process during its first l steps. The order of exploring active points is such that the first active point formed on any given vertex* *never contributes to* $sur(l)$*. Hence, conditional on the state of the process at the start of its* k*-th step, the number of active points registered during that step that contribute to any future* $sur(l)$*, is stochastically dominated by a Poisson*$(A_{k-1}\theta_N(\lambda_N, N)^{-1})$ *random variable. In particular, for some* κ *finite and all* N*,* l*,*

$$\mathbb{E}[sur(l)] \le \frac{\kappa}{N}\mathbb{E}\Big[\sum_{k=0}^{l-1} A_k\Big] \le \frac{\kappa l}{N}\mathbb{E}\Big[\max_{k\le l}\{A_k\}\Big]\,. \tag{3.7}$$

From Proposition 3.2 we thus have that $\mathbb{E}[sur(sN^{2/3})]$ *is uniformly bounded in* N*.*

The control on the number of vertices which the exploration process visits at least twice by the end of the k-th step (for $k = sN^{2/3}$), is crucial for the success of our analysis. To this end, we define hereafter *the number of visits to vertex* $v \in G_N$ *by the end of the* k*-th exploration step*, as the total number of active points formed on $\mathbb{S}^v_{\theta_N}$ by that time, i.e., we count past active points which were removed from the list and also those which are currently active by the end of time k.

Proposition 3.4. *For the exploration of the* QRG *at parameters satisfying (3.1), let* $\nu^l_{\ge m}$ *count the total number of visits by the end of its* l*-th step, to sites (circles), having at least* m *such visits each. Then, for some finite* κ*, all positive* A, s *and* N *large enough, we have that for* $m = 1, 2, 3$ *and any* $l \in [1, sN^{2/3}]$*,*

$$\mathbb{P}(\nu^l_{\ge m} \ \ge\ Al^m N^{1-m}) \le \frac{\kappa^m}{A}\,. \tag{3.8}$$

Proof. Let $\mathcal{F}_t$ denote the filtration generated by the state of the exploration process of Subsection 2.1 and $\tau(v)$ the stage in which it first visits $v \in G_N$ (so $\mathcal{F}_{\tau(v)}$ records the state of the exploration process immediately after selecting its first active point on $\mathbb{S}^v_\theta$). For any $l \ge 1$ and v let $L_v(l)$ count the links whose end points are on the neutral part of v[1] till the end of the l-th step of that process, with $L'_v(l)$ counting only such links made after $\tau(v)$ (setting $L'_v(l) = 0$ in case $\tau(v) \ge l+1$). Only one interval is explored in each step, hence

$$\mathbb{E}[\nu^l_{\ge 1}] \le l + \sum_{v=1}^{N} \mathbb{E}[L_v(l)]\,. \tag{3.9}$$

[1] We note here that $L_v(l)$ is different from the total number of visits to the vertex v by time l because according to our description, the point where we start the exploration process, and the points where we restart our exploration process after $A_k = 0$ are indeed counted as visits but these are not identified as end points of links.

Note that $L_v(l)$ increases when considering the over counting process, so the l explored intervals are complete circles (of length θ_N), other than the circle at v, which remains completely neutral, even after links to it are formed and intervals are sampled around the links. Thus, $L_v(l)$ is stochastically dominated by a Poisson random variable with parameter

$$\frac{\theta_N\, l}{\lambda_N\, N} \le \frac{\kappa' l}{N}\,, \tag{3.10}$$

for some κ' finite and all N large enough. In particular, by (3.9) $\mathbb{E}[\nu^l_{\ge 1}] \le (\kappa'+1)l$, which in combination with Markov's inequality establishes (3.8) for $m = 1$ and any $\kappa \ge \kappa' + 1$.

Next recall that for $k \le N$, either $A_k > 0$ so part (a) of the exploration process applies at the k-th step, or else part (b) applies for it (since at most $k-1 < N$ vertices have been explored before). Consequently, assuming hereafter that $sN^{2/3} \le N$, part (c) of the exploration does not occur throughout its first l steps. Further, all active points chosen in part (b) or the initial stage of the process result with a *first* visit of new vertex. Hence we have in analogy with (3.9) that

$$\mathbb{E}[\nu^l_{\ge 2}\,] \le 2\sum_{v=1}^{N} \mathbb{E}[L'_v(l)]\,. \tag{3.11}$$

As argued before, conditional on $\mathcal{F}_{\tau(v)}$ the value of $L'_v(l)$ increases if from time $\tau(v)$ onward we modify the process to have all explored intervals be complete circles (of length θ_N), on vertices other than v, while keeping the circle at v completely neutral. That is, conditionally on $\mathcal{F}_{\tau(v)}$ the variable $L'_v(l)$ is stochastically dominated by a Poisson variable of parameter $\kappa' l N^{-1}$ times the indicator on the event $\{\tau(v) < l+1\}$. Hence, for any $l \ge 1$ and $v \in G_N$,

$$\mathbb{E}\big[L'_v(l)\,\big|\,\mathcal{F}_{\tau(v)}\big] \le \Big(\frac{\kappa' l}{N}\Big)\,\mathbf{1}_{\{\tau(v)<l+1\}}\,. \tag{3.12}$$

Summing over v the expected value of (3.12), we deduce from (3.11) that

$$\mathbb{E}[\nu^l_{\ge 2}\,] \le 2\Big(\frac{\kappa' l}{N}\Big)\,\mathbb{E}[\nu^l_{\ge 1}] \le 2(\kappa'+1)\kappa' l^2 N^{-1} \tag{3.13}$$

from which we recover (3.8) for $m = 2$ and $\kappa = 2\kappa' + 1$ (by Markov's inequality).

Finally, repeating this argument, now with $\tau(v)$ the time at which the second active point on $v \in G_N$ is selected, we deduce that

$$\mathbb{E}[\nu^l_{\ge 3}\,] \le 3\Big(\frac{\kappa' l}{N}\Big)\,\mathbb{E}[\nu^l_{\ge 2}\,],$$

which upon suitably increasing the value of κ, yields (3.8) for $m = 3$. □

Proof of Proposition 3.1. Equipping $D([0,\infty))$ with the topology of uniform convergence on compacts, let

$$\bar{Z}_N^{a,\theta_N}(s) = N^{-1/3} Z_N^{a,\theta_N}(\lfloor N^{2/3} s \rfloor \wedge \widetilde{N}) , \tag{3.14}$$

for the breadth first walk $Z_N^{a,\theta_N}(\cdot)$ of (3.5). Recall that

$$\bar{Z}_N^{a,\theta_N}(s) - \bar{Y}_N^{a,\theta_N}(s) = N^{-1/3} \text{sur}(\lfloor N^{2/3} s \rfloor \wedge \widetilde{N})$$

is non-decreasing in s and so by Remark 3.3, as $N \to \infty$,

$$\sup_{s \le s_0} \left| \bar{Z}_N^{a,\theta_N}(s) - \bar{Y}_N^{a,\theta_N}(s) \right| \to 0 \qquad \text{in } \mathbb{P} . \tag{3.15}$$

It thus suffices to prove that $\bar{Z}_N^{a,\theta_N}$ converges in law to the desired limit W^{a,β_*}. To this end, by Doob's decomposition with respect to the canonical filtration $\mathcal{F}_k$ associated with the exploration process, we get that

$$Z_N^{a,\theta_N} = M_N^{a,\theta_N} + B_N^{a,\theta_N} , \qquad \left(M_N^{a,\theta_N}\right)^2 = Q_N^{a,\theta_N} + D_N^{a,\theta_N} , \tag{3.16}$$

with martinagles M_N^{a,θ_N}, Q_N^{a,θ_N} (null at $k = 0$), and predictable processes B_N^{a,θ_N} and D_N^{a,θ_N}. Adopting the notation $\bar{M}_N$, $\bar{B}_N$, in accordance with (3.14), and $\bar{Q}_N$, $\bar{D}_N$ similarly scaled by extra factor $N^{-1/3}$ in accordance to the RHS of (3.16), we show in Lemmas 3.5 and 3.6, respectively, that for $\sigma_\star$ of (3.3) and any finite s_0, as $N \to \infty$,

$$\bar{D}_N^{a,\theta_N}(s_0) \xrightarrow{\mathbb{P}} \sigma_\star^2 s_0 , \tag{3.17}$$

$$\mathbb{E}\Big[\sup_{s \le s_0} \big| \bar{M}_N^{a,\theta_N}(s) - \bar{M}_N^{a,\theta_N}(s^-) \big|^2 \Big] \to 0 . \tag{3.18}$$

Combining (3.17) and (3.18), it then follows from [9, Theorem 7.1.4] that the martingales $\{\bar{M}_N^{a,\theta_N}\}$ converge weakly in $D([0,\infty))$ to $\sigma_\star W$ for a standard Brownian motion W. Further, we show in Proposition 3.7 that

$$\sup_{s \le s_0} \left| \bar{B}_N^{a,\theta_N}(s) - \rho^{a,\beta_\star}(s) \right| \to_{\mathbb{P}} 0 . \tag{3.19}$$

That is, the sequence of predictable processes $\bar{B}_N^{a,\theta_N}$ converges in probability in $D([0,\infty))$ to the non-random $\rho^{a,\beta_\star}$ of (1.8), hence $\bar{Z}_N^{a,\theta_N}$ converges in law to $W^{a,\beta_\star,\sigma_\star}$. □

Proceeding with the proof of (3.17)–(3.19), we often drop the indices (a, θ_N) from Z_N^{a,θ_N} and related random variables. We start by establishing (3.17). That is, LEMMA 3.5. *For (θ_N, λ_N) that satisfy (3.1), $\sigma_\star$ of (3.3), any $\delta > 0$ and s_0 finite,*

$$\lim_{N\to\infty} \sup_{l \le N^{2/3} s_0} \mathbb{P}(|D_N^{a,\theta_N}(l) - \sigma_\star^2 l| \ge 2\delta N^{2/3}) = 0\,. \tag{3.20}$$

Proof. Recall that $D_N(k) - D_N(k-1) = \text{var}(\zeta_k|\mathcal{F}_{k-1})$ (starting at $D_N(0) = 0$). Augmenting $\mathcal{F}_{k-1}$ by $I_k = \{w\} \times \widetilde{I}_k$, the conditional law of ζ_k is Poisson(φ_k), with φ_k denoting the aggregate over circles other than $\mathbb{S}^w_{\theta_N}$, of the length of their neutral space restricted to $\widetilde{I}_k$ and divided by $\lambda_N N$. As such, both its conditional mean and conditional variance are given by φ_k, hence by the variance conditioning decomposition,

$$\text{var}(\zeta_k|\mathcal{F}_{k-1}) = \mathbb{E}[\varphi_k|\mathcal{F}_{k-1}] + \text{var}[\varphi_k|\mathcal{F}_{k-1}]\,.$$

Further, at the start of the k-th step there are at least $N-k$ completely neutral circles beyond the vertex on which the k-th explored interval lies. Hence, for $J_k = |\widetilde{I}_k|$ and any $k \le s_0 N^{2/3}$,

$$0 \le \frac{J_k}{\lambda_N} - \varphi_k \le \frac{J_k k}{\lambda_N N} \le \kappa' N^{-1/3}$$

(for $\kappa' = s_0 \sup_N\{\theta_N/\lambda_N\}$ finite). It thus suffices to prove (3.20) for $\widehat{D}_N(l)$ instead of $D_N(l)$, where

$$\widehat{D}_N(k) - \widehat{D}_N(k-1) = \mathbb{E}[J_k/\lambda_N \,|\, \mathcal{F}_{k-1}] + \text{var}[J_k/\lambda_N \,|\, \mathcal{F}_{k-1}] =: \Delta_k\,.$$

The non-negative Δ_k are uniformly bounded by $\overline{\Delta} := \sup_N\{(\theta_N/\lambda_N) + (\theta_N/\lambda_N)^2\}$. Moreover, whenever $\mathcal{F}_{k-1}$ dictates that the k-th step explores the first interval on a given vertex $w \in G_N$, it yields a conditionally independent J_k that follow the $\Gamma_{\theta_N}(2,1)$ distribution. We consequently have that $\Delta_k = F_2(\beta_N, \lambda_N)$ (for $F_2(\beta, \lambda)$ of (1.7)), in any such step. This applies to all but at most $\nu^l_{\ge 2}$ of the first l steps, hence

$$|\widehat{D}_N(l) - F_2(\beta_N, \lambda_N) l| \le \overline{\Delta}\, \nu^l_{\ge 2}\,.$$

From Proposition 3.4 we know that $\mathbb{P}(\nu^l_{\ge 2} \ge \delta N^{2/3}) \to 0$ for $l = N^{2/3} s_0$ and we thus get (3.20) for $\widehat{D}_N(l)$ upon noting that $F_2(\beta_N, \lambda_N) \to \sigma_\star^2$ of (3.3). □

We next establish (3.18), thereby moving closer to completing the proof of Proposition 3.1.

LEMMA 3.6. *For (θ_N, λ_N) that satisfy (3.1) and any s_0 finite,*

$$\lim_{N\to\infty} N^{-2/3} \mathbb{E}\big[\max_{1 \le l \le s_0 N^{2/3}} |M_N^{a,\theta_N}(l) - M_N^{a,\theta_N}(l-1)|^2\big] = 0. \tag{3.21}$$

Proof. Recall that $M_N(\cdot)$ is the martingale part of $Z_N(\cdot)$. Hence, from (3.5)

$$M_N(l) - M_N(l-1) = \zeta_l - \mathbb{E}(\zeta_l|\mathcal{F}_{l-1})$$

and (3.21) amounts to showing that

$$N^{-2/3}\mathbb{E}\big[\max_{l\le s_0 N^{2/3}} (\zeta_l - \mathbb{E}(\zeta_l|\mathcal{F}_{l-1}))^2\big] \to 0.$$

Clearly, $(\zeta_l - \mathbb{E}(\zeta_l|\mathcal{F}_{l-1}))^2 \le 2\zeta_l^2 + 2\mathbb{E}(\zeta_l|\mathcal{F}_{l-1})^2$. Further, in Section 2.2 we saw that conditionally on $\mathcal{F}_{l-1}$ the variable ζ_l is stochastically dominated by the independent $\xi_l \ge 0$ whose mean $F(\beta_N, \lambda_N)$ is uniformly bounded (in N). Hence, $\sup_l \mathbb{E}(\zeta_l|\mathcal{F}_{l-1})^2 \le \sup_N F(\beta_N,\lambda_N)^2$ is finite and it suffices to show that for i.i.d. (ξ_l),

$$N^{-2/3}\mathbb{E}[\max_{l\le s_0 N^{2/3}} \zeta_l^2] \le N^{-2/3}\mathbb{E}[\max_{l\le s_0 N^{2/3}} \xi_l^2] \to 0\,. \tag{3.22}$$

Finally, recall [11, equation (6')] that the expected maximum of n i.i.d. variables of zero-mean and unit variance is at most $(n-1)/\sqrt{2n-1}$. Consequently, the expectation on the right side of (3.22) grows at most at rate $\mathrm{O}(N^{1/3})$, which proves (3.21) (and thereby (3.18) as well). □

For the remainder of Section 3.1 we complete the proof of Proposition 3.1 by establishing (3.19). Indeed, upon rearranging the expression (1.8) for $\rho^{a,\beta_\star}$, this is precisely the statement of our next proposition.

PROPOSITION 3.7. *For (θ_N, λ_N) that satisfy (3.1) and any s_0 finite, as $N\to\infty$,*

$$\sup_{s\le s_0}\Big|\bar{B}_N^{a,\theta_N}(s) - \big(a\,s-\frac{s^2}{2}\big) + \frac{s^2}{2}\big(1-\frac{1}{\beta_\star}\big)(1-\gamma(\theta_\star)) - \frac{s^2}{2}\big(1-\frac{1}{\beta_\star}\big)\Big| \xrightarrow{\mathbb{P}} 0\,. \tag{3.23}$$

Turning to the proof of Proposition 3.7, we first express $B_N(l)$ as the sum of the terms (3.24)-(3.26), which for $l = sN^{2/3}$ upon further scaling by $N^{-1/3}$ converge to the three limit expressions on the LHS of (3.23), respectively.

LEMMA 3.8. *For (θ_N, λ_N) that satisfy (3.1), let*

$$\Theta_m^k = \{v\in G_N : v \textit{ has exactly } m \textit{ explored intervals after } k \textit{ steps}\}\,,$$

and E_m^k denote the event that the interval $\widetilde{I}_{k+1}$ explored during the $(k+1)$-st step is on some vertex $w\in\Theta_m^k$. Then, for fixed s_0, all N large enough and *uniformly over* $l \in [1, s_0 N^{2/3}]$,

$$B_N(l) = \mathrm{O}_{\mathbb{P}}(1) - l + \frac{1}{\lambda_N N} \sum_{k=0}^{l-1} (N-k) F(\theta_N) \tag{3.24}$$

$$- \frac{1}{\lambda_N} \sum_{k=0}^{l-1} \Big(F(\theta_N) - \mathbb{E}\big(|\widetilde{I}_{k+1}| \big| \mathcal{F}_k\big) \Big) \mathbf{1}_{E_1^k} \tag{3.25}$$

$$+ \frac{1}{\lambda_N N} \sum_{k=0}^{l-1} \mathbf{1}_{E_0^k} \sum_{v \in \Theta_1^k} \mathbb{E}\big(|\widetilde{I}_{k+1} \cap \widetilde{I}^n(v)| \big| \mathcal{F}_k\big)\,, \tag{3.26}$$

where $\widetilde{I}^n(v)$ *denotes the neutral part of the vertex* v.

Proof of Lemma 3.8. Let $\zeta_{k+1}(v)$ counts the links to $v \in G_N$ formed during the $(k+1)$ exploration step, which conditionally on $\mathcal{F}_k$, follow the Poisson distribution with parameter $|\widetilde{I}_{k+1} \cap \widetilde{I}^n(v)|/(\lambda_N N)$, bounded by the non-random $\theta_N/(\lambda_N N)$. Recalling that $B_N(l)$ is the predictable process in Doob's decomposition of $Z_N(l)$ we get from (3.5) that

$$B_N(l) = -(l-1) + \sum_{k=0}^{l-1} \mathbb{E}(\zeta_{k+1}|\mathcal{F}_k) = -(l-1) + \sum_{k=0}^{l-1} \sum_{v=1}^{N} \mathbb{E}(\zeta_{k+1}(v)|\mathcal{F}_k)$$

$$= -(l-1) + \frac{1}{\lambda_N N} \sum_{k=0}^{l-1} \sum_{v=1}^{N} \mathbb{E}\big(|\widetilde{I}_{k+1} \cap \widetilde{I}^n(v)| \,\big|\mathcal{F}_k\big)\,. \tag{3.27}$$

We shall re-group the sum in (3.27) over $v \in \Theta_0^k$ (yielding the terms (3.24) and (3.25)), over $v \in \Theta_1^k$ (from which we get (3.26)), and over v in the remainder of G_N, denoted hereafter by $\Theta_{\geq 2}^k$. Indeed, if $v \in \Theta_0^k$ then

$$\mathbb{E}\big(|\widetilde{I}_{k+1} \cap \widetilde{I}^n(v)| \,\big|\mathcal{F}_k\big) = \mathbb{E}\big(|\widetilde{I}_{k+1}| \,\big|\mathcal{F}_k\big)\,. \tag{3.28}$$

Further simplification occurs under the event E_0^k, in which case the RHS of (3.28) is precisely $F(\theta_N)$ (since $\widetilde{I}_{k+1}$ is the first explored interval on its vertex). In contrast, under the event E_1^k the interval $\widetilde{I}_{k+1}$ is the second to be explored on its vertex, hence the RHS of (3.28) is smaller than $F(\theta_N)$, yielding the correction term (3.25).
Now, by definition $\sum_m |\Theta_m^k| = N$ and $\sum_m m|\Theta_m^k| = k$, hence

$$|\Theta_0^k| - (N-k) = \sum_{m \geq 2} (m-1)|\Theta_m^k| \leq \nu_{\geq 2}^k\,. \tag{3.29}$$

In particular, also $|\Theta^k_{\geq 2}| \leq \nu^k_{\geq 2}$ and from Proposition 3.4 we deduce that

$$\sum_{k=0}^{l-1}\Big([|\Theta^k_0|-(N-k)]\mathbb{E}(|\widetilde{I}_{k+1}|\,|\mathcal{F}_k)+\sum_{v\in\Theta^k_{\geq 2}}\mathbb{E}\big(|\widetilde{I}_{k+1}\cap\widetilde{I}^n(v)|\,|\mathcal{F}_k\big)\Big) \leq 2l\theta_N\nu^l_{\geq 2} = \mathrm{O}_{\mathbb{P}}\left(l^3\,N^{-1}\right)\,,$$

so dividing both sides by $(\lambda_N N)$, this is included for $l \leq s_0 N^{2/3}$ within the $\mathrm{O}_{\mathbb{P}}(1)$ term in (3.24). Three other contributions are included in that same $\mathrm{O}_{\mathbb{P}}(1)$. First, by restricting to E^k_1 in (3.25) we introduce a uniformly bounded (in l, N) total error in the sum over $v \in \Theta^k_0$, at each of the at most $\nu^l_{\geq 3} = \mathrm{O}_{\mathbb{P}}(1)$ steps corresponding to events $E^k_{\geq 2}$. Next, the sum (3.25) of non-negative bounded terms has at most $\nu^l_{\geq 2}$ non-zero summands. Hence by eliminating there the factors $(1-k/N)$ we introduced only an additional error of order $(l/N)\nu^l_{\geq 2} = \mathrm{O}_{\mathbb{P}}(1)$. Since $|\Theta^k_1| \leq l$ and $E^k_{\geq 1}$ occurs for at most $\nu^l_{\geq 2}$ values of $k < l$, our restriction of the sum (3.26) to steps in which E^k_0 holds, introduced yet another error of order at most $(l/N)\nu^l_{\geq 2} = \mathrm{O}_{\mathbb{P}}(1)$. Since all error terms are non-decreasing in l, the $\mathrm{O}_{\mathbb{P}}(1)$ we obtain for $l = s_0 N^{2/3}$, also applies for smaller l. □

Proof of Proposition 3.7. Plugging $F(\theta_N) = \lambda_N\left(1 + a\,N^{-1/3}\right)$, we find that the RHS of (3.24) is

$$\mathrm{O}_{\mathbb{P}}(1) - l + (1 + a\,N^{-1/3})(l - \frac{l^2}{2N})\,,$$

which upon setting $l = sN^{2/3}$ and scaling by $N^{-1/3}$, converges to $(as - \frac{s^2}{2})$ when $N \to \infty$, uniformly over $s \leq s_0$. We complete the proof of the proposition by way of Lemmas 3.9 and 3.10, which show that upon scaling by $N^{-1/3}$, for $l = sN^{2/3}$ the terms in (3.26) and (3.25) converge in probability as $N \to \infty$, uniformly over $s \leq s_0$, to the appropriate non-random limits, respectively. Indeed, by the preceding we conclude that the processes $s \mapsto \bar{B}_N(s)$ converge to the deterministic path $s \mapsto \rho^{a,\beta_\star}(s)$ of (1.8), uniformly over $s \leq s_0$. □

LEMMA 3.9. *For (θ_N, λ_N) satisfying (3.1), under the notations of Lemma 3.8 we have that uniformly over $s \leq s_0$,*

(3.30)
$$\lim_{N\to\infty}\frac{1}{\lambda_N\,N^{4/3}}\sum_{k=0}^{s\,N^{2/3}-1}\mathbf{1}_{E^k_0}\sum_{v\in\Theta^k_1}\mathbb{E}(|\tilde{I}_{k+1}\cap\tilde{I}^n(v)||\mathcal{F}_k) = \frac{s^2}{2}\Big(1-\frac{1}{\beta_\star}\Big) \quad in\ \mathbb{P}\,.$$

LEMMA 3.10. *In the setting of Lemmas 3.8 and 3.9, uniformly over $s \le s_0$,*

(3.31)
$$\lim_{N\to\infty} \frac{1}{\lambda_N\, N^{1/3}} \sum_{k=0}^{sN^{2/3}-1} \Big(F(\theta_N) - \mathbb{E}\big(|\widetilde{I}_{k+1}|\big|\mathcal{F}_k\big)\Big) \mathbf{1}_{E_1^k} = \frac{s^2}{2}\big(1-\frac{1}{\beta_\star}\big)\big(1-\gamma(\theta_\star)\big) \quad \text{in } \mathbb{P}.$$

REMARK 3.11. *Recall that for two arcs $\mathcal{I}$ and $\mathcal{I}'$ in $\mathbb{S}_\theta$ of uniformly chosen relative shift U, the expected length of $(\mathcal{I}'+U)\cap\mathcal{I}$ is the product of arc lengths divided by θ. For i.i.d. $\mathcal{I}$ and $\mathcal{I}'$ of length $\Gamma_\theta(2,1)$ we thus get that $\mathbb{E}[|(\mathcal{I}'+U)\cap\mathcal{I}|] = F(\theta)^2/\theta$, which at $\lambda_\star = F(\theta_\star)$ equals $\lambda_\star/\beta_\star$. The heuristic behind (3.30) is thus that $|\Theta_1^k| = k(1+o(1))$ for large k and further, for most $v\in\Theta_1^k$ the complement in $\mathbb{S}_{\theta_N}$ of $\widetilde{I^n}(v)$ and the interval $\widetilde{I}_{k+1}$ can be well approximated by such i.i.d. pair of intervals $(\mathcal{I}'+U)$ and $\mathcal{I}$. Similarly, the heuristic behind (3.31) is that up to $l = sN^{2/3}$ the process explores the second interval on some vertex $\{\mathbb{S}^w_{\theta_N}, w\in G_N\}$ about $N^{1/3}$ times the expression on the* LHS *of (3.30), whereas such interval is well approximated by $\mathcal{I}'_U \subseteq \mathcal{I}^n$ randomly carved around active point chosen uniformly* within *$\mathcal{I}^n$. Such $\mathcal{I}'_U$ has conditional expected length $\widehat{F}(|\mathcal{I}^n|)$, whose expectation for $\Gamma_\theta(2,1)$ variable $|\mathcal{I}|$ is $\gamma(\theta)F(\theta)$.*

Turning next to the proof of Lemma 3.9, consider the collection of explored, labeled trees, where we assign labeled node k to the interval explored in step k, while ignoring the surplus edges (see [2, Fig.1] for an illustration of this tree). We further mark each node of these trees with the vertex $w\in G_N$ on which the interval lies and the process of points $X_k \in [0,\theta_N]$ around which the explored intervals are constructed. Then, for any $k\ge 0$, writing 0_* as the root of the tree containing node k, we define $d(k) = d_{\text{tree}}(0_*, k)$ for the tree distance $d_{\text{tree}}(\cdot,\cdot)$ (which count only the links on the unique path between any pair of connected nodes).

Consider the process $(X_k)_{k\ge 0}$ restricted to a branch of an exploration tree, such that the initial location X_0 is distributed as ν on $\mathbb{S}_{\theta_N}$, where ν is Lebesgue measure on $\mathbb{S}_{\theta_N}$ (choice of measure ν is arbitrary, and does not affect the result). Taking k and k' such that $d(k') < d(k)$, introduce $X_k|_{(\mathcal{F}_{k'},\widetilde{I}_{k'})}$ as the random variable X_k conditioned on $\mathcal{F}_{k'}$ and the interval $\widetilde{I}_{k'}$ built around $X_{k'}$. Let $P(k',k)$ be the unique path connecting two explorations times k' and k, which includes both k' and k. Then define

(3.32)
$$\varphi(k',k) = \max\{j\in P(k',k)\backslash\{k\} : \widetilde{I}_j \text{ is not the first interval on its vertex}\}$$

whenever the above set is nonempty, and we set $\varphi(k',k) = k$, otherwise.

LEMMA 3.12. *Let us fix two epochs k and k' from the exploration process such that they both belong to the same branch and that $d(k') < d(k)$. Then, writing $\pi_{k',k,N}$ for the distribution of X_k conditioned on $\left(\mathcal{F}_{k'}, \widetilde{I}_{k'}\right)$ on $\mathbb{S}_{\theta_N}$, and π_N for the normalised Lebesgue measure on $\mathbb{S}_{\theta_N}$, there exists a deterministic $\rho < 1$ such that*

$$d_{TV}\left(\pi_{k',k,N}, \pi_N\right) \leq \mathbb{E}^{(k',k)}\left(\rho^{f(k',k)}\right)$$

where d_{TV} is the total variation distance, $\mathbb{E}^{(k',k)}$ is expectation with respect to the measure induced by $\left(X_{\varphi(k',k)}, \varphi(k',k)\middle|\right)$ conditioned on $\left(\mathcal{F}_{k'}, \widetilde{I}_{k'}\right)$ and $f(k',k) = d_{tree}(\varphi(k',k), k)$.

Proof. First, notice that the result is trivially true when $\varphi(k',k) = k$. Next, let us assume that $(\varphi(k',k) \neq k)$, and note that conditioned on $X_{\varphi(k',k)}$ the distribution of X_k does not depend on the past prior to $\varphi(k',k)$. Thus for $j \neq k$

(3.33)

$$\mathbb{P}\left(X_k \in A \middle| \mathcal{F}_{k'}, \widetilde{I}_{k'}, X_j, \varphi(k',k) = j\right) = \mathbb{P}\left(X_k \in A \middle| X_j, \varphi(k',k) = j\right).$$

Additionally, the collection of random variables $X_j | (X_{\varphi(k',k)}, \varphi \neq k)$ for $j \geq \varphi(k',k)$ forms an irreducible and aperiodic Markov chain on a compact state space. Notice that the transition probability kernel of this Markov chain satisfies the minoration condition (iv) of Theorem 16.0.2 in [19] with $m = 1$ (borrowing notation from the same reference).

Then writing π for the uniform measure on $\mathbb{S}_{\theta_N}$, and using the equivalence of statements (iv) and $(viii)$ of Theorem 16.0.2 in [19] there exists positive constant $\rho < 1$ such that

$$d_{TV}\left(\pi'_{\varphi(k',k),k,N}, \pi_N\right) < \rho^{d_{\text{tree}}(\varphi(k',k),k)} \tag{3.34}$$

Note here that ρ depends only on the transition kernel of the Markov chain, and thus does not depend on the value of $\varphi(k',k)$.

Therefore, in view of the above equation,

$$d_{TV}\left(\pi_{k',k,N}, \pi_N\right) = \sup_A \left|\mathbb{E}^{(k',k)}\left(\pi_{\varphi(k,k'),k,N}(A)\right) - \pi_N(A)\right|. \tag{3.35}$$

Then, applying Jensen's inequality

$$d_{TV}\left(\pi_{k',k,N}, \pi_N\right) \leq \mathbb{E}^{(k',k)}\left(\sup_A \left|\pi_{\varphi(k',k),k,N}(A) - \pi_N(A)\right|\right) \tag{3.36}$$

Thereafter, using (3.34)

$$d_{TV}\left(\pi_{k',k,N},\pi_N\right) \le \mathbb{E}^{(k',k)}\left(\rho^{d_{\text{tree}}(\varphi(k',k),k)}\right) \tag{3.37}$$

which concludes the proof.

□

Proof of Lemma 3.9: Under the event E_0^k the interval $\widetilde{I}_{k+1}$ lies on a completely neutral circle $\mathbb{S}^w_{\theta_N}$. Letting

$$G(x,\widetilde{I}(v)) := |(\mathcal{I}'+x)\cap \widetilde{I}^n(v)|$$

and $l_N = sN^{2/3}$ we thus rewrite (3.30) as

(3.38)

$$\lim_{N\to\infty} \frac{2}{l_N^2}\sum_{k=0}^{l_N-1} \mathbf{1}_{E_0^k} \sum_{v\in\Theta_1^k} \mathbb{E}\left(G(X_{k+1},\widetilde{I}(v))\big|\mathcal{F}_k\right) = F(\theta_\star)\Big(1-\frac{1}{\beta_\star}\Big) \quad \text{in } \mathbb{P}\,.$$

Recall Remark 3.11 that for i.i.d. copy $\mathcal{I}$ of $\mathcal{I}'$ and an independent uniform point U on $\mathbb{S}_{\theta_N}$

$$\mathbb{E}[G(U,\mathcal{I})] = F(\theta_N) - \frac{F(\theta_N)^2}{\theta_N} \to F(\theta_\star)\Big(1-\frac{1}{\beta_\star}\Big)\,, \tag{3.39}$$

as $N\to\infty$ (see (3.1) and (3.3)). Similar to (3.29) we further have that

$$0 \le k - |\Theta_1^k|\mathbf{1}_{E_0^k} \le k - |\Theta_1^k| + |\Theta_1^k|\mathbf{1}_{E^k_{\ge 1}} \le \nu^k_{\ge 2} + k\mathbf{1}_{E^k_{\ge 1}}\,,$$

and as shown at end of proof of Lemma 3.8, for $N\to\infty$

$$\frac{1}{l_N^2}\sum_{k=0}^{l_N-1} k\mathbf{1}_{E^k_{\ge 1}} \le \frac{\nu^{l_N}_{\ge 2}}{l_N} = \mathrm{O}_{\mathbb{P}}(l_N/N) \to 0\,.$$

Hence, as $N\to\infty$,

$$\frac{2}{l_N^2}\sum_{k=0}^{l_N-1}\left(k - |\Theta_1^k|\mathbf{1}_{E_0^k}\right) \xrightarrow{\mathbb{P}} 0\,,$$

which in combination with (3.39) implies that

$$\lim_{N\to\infty}\frac{2}{l_N^2}\sum_{k=0}^{l_N-1}\mathbf{1}_{E_0^k}|\Theta_1^k|\,\mathbb{E}[G(U,\mathcal{I})] = F(\theta_\star)\Big(1-\frac{1}{\beta_\star}\Big) \quad \text{in } \mathbb{P}\,.$$

Thus, to get (3.38) and thereby complete the proof, it suffices to show that as $k \to \infty$,

$$\frac{1}{k^2}\mathbb{E}\left[\left\{\sum_{v\in\Theta_1^k} \mathbf{1}_{E_0^k}\left[\mathbb{E}\left(G(X_{k+1},\widetilde{I}(v))\Big|\,\mathcal{F}_k\right)-\mathbb{E}\left(G(U,\mathcal{I})\right)\right]\right\}^2\right]\to 0. \tag{3.40}$$

However, since $|\Theta_1^k| = k(1+\mathrm{o}(1))$, the above statement reduces to showing

$$\frac{1}{k^2}\mathbb{E}\left[\sum_{\substack{v,v'\in\Theta_1^k\\ v\neq v'}} \mathbf{1}_{E_0^k}\left\{\mathbb{E}\left(G(X_{k+1},\widetilde{I}(v))\Big|\,\mathcal{F}_k\right)-\mathbb{E}\left[G(U,\mathcal{I})\right]\right\}\left\{\mathbb{E}\left(G(X_{k+1},\widetilde{I}(v'))\Big|\,\mathcal{F}_k\right)-\mathbb{E}\left[G(U,\mathcal{I})\right]\right\}\right]\to 0 \tag{3.41}$$

Let us set

$$\begin{aligned}&\mathrm{Err}(v,v',k,N)\\ =\;&\mathbb{E}\left[\left\{\mathbb{E}\left(G(X_{k+1},\widetilde{I}(v))\Big|\,\mathcal{F}_k\right)-\mathbb{E}\left(G(U,\mathcal{I})\right)\right\}\left\{\mathbb{E}\left(G(X_{k+1},\widetilde{I}(v'))\Big|\,\mathcal{F}_k\right)-\mathbb{E}\left(G(U,\mathcal{I})\right)\right\}\Big|\, v,v'\in\Theta_1^k, \ldots\right.\end{aligned}$$

Then (3.41) can be rewritten as

$$\frac{1}{k^2}\sum_{\substack{v,v'\in\{1,\ldots,N\}\\ v\neq v'}} \mathbb{P}\Big(v,v'\in\Theta_1^k, E_0^k\Big)\ \mathrm{Err}(v,v',k,N). \tag{3.42}$$

Since $\mathbb{P}\left(v,v'\in\Theta_1^k,E_0^k\right)=\mathrm{O}(kN^{-1})^2$, it suffices to show that $\mathrm{Err}(v,v',k,N)\to 0$ as $k,N\to\infty$.

Going back to the notation introduced prior to Lemma 3.12, let us define $\vartheta(v)$ as the exploration time when the first interval on vertex v was built and $\vartheta^*(v)$ as the exploration time of the common ancestor of the intervals built at $\vartheta(v)$ and exploration time k, such that its interval is the first explored interval on its vertex. Further, set $\vartheta^*(v,v') = \operatorname{argmin}\{d_{\text{tree}}(\vartheta^*(v),k), d_{\text{tree}}(\vartheta^*(v'),k)\}$. Then the above can be further reduced to proving the following:

(S) For $\vartheta(v),\vartheta(v'),k$ such that $d_{\text{tree}}(\vartheta(v),k)=\Theta(\log k)$ and $d_{\text{tree}}(\vartheta(v'),k)=\Theta(\log k)$, we need to prove that conditioned on $\left(v,v'\in\Theta_1^k,E_0^k\right)$, the following

$$\mathbb{E}\left[\mathbb{E}\left(G(X_{k+1},\widetilde{I}(w))\Big|\,\mathcal{F}_k\right)\Big|\, w\in\Theta_1^k,E_0^k\right]-\mathbb{E}\left(G(U,\mathcal{I})\right),\ \text{for } w=v,v' \tag{3.43}$$

$$\mathbb{E}\left[\mathbb{E}\left(G(X_{k+1},\widetilde{I}(v))\Big|\,\mathcal{F}_k\right)\mathbb{E}\left(G(X_{k+1},\widetilde{I}(v'))\Big|\,\mathcal{F}_k\right)\Big|\, v,v'\in\Theta_1^k,E_0^k\right]-\left[\mathbb{E}\left(G(U,\mathcal{I})\right)\right]^2 \tag{3.44}$$

converge to 0 as $k\to\infty$.

We remark here that the assumptions above concerning the order of $d_{\text{tree}}(\vartheta(v), k)$ and $d_{\text{tree}}(\vartheta(v'), k)$ are indeed valid since the number of pairs of v, v' which do not satisfy such conditions is $\mathrm{o}(k)$, and hence their contribution to (3.41) can be ignored.

Before we proceed further, let us observe the following facts which will be used in the proof of above statement (S). Since for any $j \le sN^{2/3}$ probability of the event that interval $\widetilde{I}_j$ built at time j of exploration lies on a vertex with an already explored interval is $\mathrm{O}(N^{-1/3})$, we have the following:

(F1) For intervals belonging to the same branch built at times k, k' such that $d_{\text{tree}}(k', k) = \Theta(\log k)$, probability of the event $(\varphi(k', k) \in P(k', k)\backslash\{k\})$ can be shown to be $\mathrm{O}(N^{-1/3}\log k)$, where $\varphi(k', k)$ is as defined in (3.32). By Lemma 3.12, we have for some $\rho < 1$

$$d_{TV}\left(\pi_{k',k,N}, \pi_N\right) \le \mathbb{E}\left(\rho^{f(k',k)}\right), \tag{3.45}$$

where as before $f(k', k) = d_{\text{tree}}(\varphi(k', k), k)$. However, $f(k', k) = \Theta(\log k)$ on a set of probability $\left(1 - \mathrm{O}(N^{-1/3}\log k)\right)$. Hence, writing π for uniform distribution on $\mathbb{S}_{\theta_\star}$ we have $d_{TV}\left(\pi_{k',k,N}, \pi\right) \to 0$ as $k, N \to \infty$.

(F2) For any k, writing $\pi_{k,N}$ for the distribution of X_k, $d_{TV}\left(\pi_{k,N}, \pi\right) \to 0$. Indeed, by construction we start a new cluster with sampling a point from the uniform distribution, which is invariant for the dynamics of the Markov chain running only on neutral vertices. Consequently, if $d_{\text{tree}}(0_*, k) = \mathrm{o}(N^{1/3})$, we have $d_{TV}\left(\pi_{k,N}, \pi\right) \to 0$. However, due to the tree structure $d_{\text{tree}}(0_*, k) = \mathrm{O}(\log k)$.

There are three cases to consider in order to prove the statement (S).

(A) The interval built at k belongs to a cluster different from the ones that are built at $\vartheta(v)$ and $\vartheta(v')$ one can invoke fact (F2) to conclude that $d_{TV}\left(\pi_{\vartheta(v),k,N}, \pi\right) \to 0$ and $d_{TV}\left(\pi_{\vartheta(v'),k,N}, \pi\right) \to 0$.

(B) The intervals built at k and $\vartheta(v)$ belong to the same cluster and the interval built at $\vartheta(v')$ belongs to a different cluster. Note that in this case the distribution of X_k does not depend on $\widetilde{I}(v')$. By assumption $d_{\text{tree}}(\vartheta(v), k) = \Theta(\log k)$, then either $d_{\text{tree}}(\vartheta^*(v), k) = \Theta(\log k)$ or $d_{\text{tree}}(\vartheta^*(v), \vartheta(v)) = \Theta(\log k)$ (or both). In the former case, use fact (F1) to conclude that $d_{TV}\left(\pi_{v,k,N}, \pi\right) \to 0$, where we write $\pi_{v,k,N}$ for the measure induced by X_k conditioned on $\left(\mathcal{F}_{\vartheta(v)}, \widetilde{I}(v)\right)$, which in turn implies the required result. In the latter case, note that the Markov chain running only on neutral vertices is reversible. Hence, one can use the first fact to conclude that $d_{TV}\left(\pi_{\vartheta^*(v),\vartheta(v),N}, \pi\right) \to 0$ and then by the same arguments as in the proof of the second fact conclude that $d_{TV}\left(\pi_{\vartheta(v),k,N},\pi\right)\to 0$, since $\vartheta^*(v)$ plays a role of the root in this case. Note that the roles of v and v' can be interchanged without any change in the arguments above.

(C) All three intervals built at $\vartheta(v)$, $\vartheta(v')$ and k belong to the same cluster. By assumption, both $d_{\text{tree}}(\vartheta(v),k)=\Theta(\log k)$ and $d_{\text{tree}}(\vartheta(v'),k)=\Theta(\log k)$, then either $d_{\text{tree}}(\vartheta^*(v,v'),k)=\Theta(\log k)$, or both $d_{\text{tree}}(\vartheta^*(v,v'),\vartheta(v))$ and $d_{\text{tree}}(\vartheta^*(v,v'),\vartheta(v'))$ are $\Theta(\log k)$. In the former case, use fact (F1) with $\vartheta^*(v,v')$ and k to conclude that $d_{TV}\left(\pi_{v,v',k,N},\pi\right)\to 0$, where we write $\pi_{v,v',k,N}$ for the measure induced by X_k conditioned on $\left(\mathcal{F}_{\max\{\vartheta(v),\vartheta(v')\}},\widetilde{I}(v),\widetilde{I}(v')\right)$. In the latter case, note that at least one of the intervals built at $\vartheta(v)$ and $\vartheta(v')$ lies on the same branch as the one built at $\vartheta^*(v,v')$. Without loss of generality let us assume that the interval built at $\vartheta(v)$ lies on the same branch as the one built at $\vartheta^*(v,v')$. As we already mentioned above, the Markov chain running only on neutral vertices is reversible. Hence, one can use the fact (F1) for $\vartheta^*(v,v')$ and $\vartheta(v)$ to conclude that $d_{TV}\left(\pi_{\vartheta^*(v,v'),\vartheta(v),N},\pi\right)\to 0$. Using exactly the same set of arguments as in case (B) for $\vartheta^*(v,v')$ and $\vartheta(v')$ we also conclude that $d_{TV}\left(\pi_{\vartheta^*(v,v'),\vartheta(v'),N},\pi\right)\to 0$. Consequently $d_{TV}\left(\pi_{\vartheta^*(v,v'),\vartheta(v),\vartheta(v'),N},\pi\right)\to 0$. By the same arguments as in the proof of the fact (F2) $d_{TV}\left(\pi_{\vartheta^*(v,v'),k,N},\pi\right)\to 0$, since $\vartheta^*(v,v')$ plays a role of the root in this case, which leads to the main result of this lemma.

It is clear from the discussion above that (3.43) converges to 0 for $w=v,v'$. Next to analyse (3.44) let us first note that conditioned on the event $v,v'\in\Theta_1^k$, the conditional expectation $\mathbb{E}\left(G(X_{k+1},\widetilde{I}(v))\middle|\,\mathcal{F}_k\right)$ is simply a measurable function of X_{k+1}, and the left and right ends of the interval sampled on the vertex v, which we denote by $H(X_{k+1},l_v,r_v,\varpi_v)$ where l_v and r_v represent coordinates of the left and the right ends of the interval sample around the point ϖ_v on the vertex $v\in\Theta_1^k$, respectively. Thus, we have

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}\left(G(X_{k+1},\widetilde{I}(v))\middle|\,\mathcal{F}_k\right)\mathbb{E}\left(G(X_{k+1},\widetilde{I}(v'))\middle|\,\mathcal{F}_k\right)\middle|\,v,v'\in\Theta_1^k,E_0^k\right]\\=\;&\mathbb{E}\left[H(X_{k+1},l_v,r_v,\varpi_v)H(X_{k+1},l_{v'},r_{v'},\varpi_{v'})|\,v,v'\in\Theta_1^k,E_0^k\right]\\=\;&\mathbb{E}\left[\mathbb{E}\left\{H(X_{k+1},l_v,r_v,\varpi_v)H(X_{k+1},l_{v'},r_{v'},\varpi_{v'})|\,v,v'\in\Theta_1^k,E_0^k,X_{k+1}\right\}\right]\end{aligned}$$

where the inner expectation is conditioned on $\{v,v'\in\Theta_1^k,E_0^k,X_{k+1}\}$, and the outer expectation is with respect to the measure induced by X_{k+1} conditioned on $\{v,v'\in\Theta_1^k,E_0^k\}$.

Next going through cases (A), (B) and (C) we can infer that conditioned on $\{v, v' \in \Theta_1^k, E_0^k, X_{k+1}\}$, the variables $H(X_{k+1}, l_v, r_v, \varpi_v)$ and $H(X_{k+1}, l_{v'}, r_{v'}, \varpi_{v'})$ are asymptotically independent as $k \to \infty$. Thus we can write the following approximation for the above expression

$$\mathbb{E}\Big[\mathbb{E}\Big\{ H(X_{k+1}, l_v, r_v, \varpi_v) |\, v \in \Theta_1^k, E_0^k, X_{k+1}\Big\} \\ \times \mathbb{E}\Big\{ H(X_{k+1}, l_{v'}, r_{v'}, \varpi_{v'}) |\, v' \in \Theta_1^k, E_0^k, X_{k+1}\Big\}\Big]$$

Note that the two terms appearing in the above expression are conditional means of length of intersection of interval built at time $(k+1)$ with those built at times $\vartheta(v)$ and $\vartheta(v')$, respectively. Observe now that conditioned on $v, v' \in \Theta_1^k$ and the event E_0^k, the lengths of all three intervals are identically distributed, and asymptotically independent, using the arguments set forth in cases (A), (B) and (C) discussed above. Now we invoke the same arguments as in Remark 3.11 to conclude that the two inner conditional expectations do not depend on X_{k+1}, thus finally proving the statement (3.44).

□

Proof of Lemma 3.10. Writing $V_{k+1} = F(\theta_N) - \mathbb{E}\big(|\widetilde{I}_{k+1}| \,\big|\, \mathcal{F}_k\big)$, it suffices to show that as $N \to \infty$,

$$R_N := \frac{1}{\lambda_N N^{1/3}} \sum_{k=0}^{sN^{2/3}-1} V_{k+1} \mathbf{1}_{E_1^k} \to \frac{s^2}{2}\Big(1 - \frac{1}{\beta_\star}\Big)(1 - \gamma(\theta_\star)) =: R_\infty \tag{3.46}$$

in L^2. That is, to show that $\mathbb{E}R_N \to R_\infty$ and

Claim 3.13. *With the above notation var$(R_N) \to 0$ when $N \to \infty$.*

Deferring the proof of Claim 3.13 to the end of the section, with V_{k+1} uniformly bounded, we complete the proof of Lemma 3.10 by showing that

$$\lim_{N\to\infty} N^{-1/3} \sum_{k=1}^{sN^{2/3}} \mathbb{P}(E_1^k) = \frac{s^2}{2}\left(1 - \frac{1}{\beta_\star}\right), \tag{3.47}$$

and that for $k, N \to \infty$ while $k = O(N^{2/3})$,

$$\mathbb{E}\left(V_{k+1} \middle| E_1^k\right) \xrightarrow{\mathbb{P}} F(\theta_\star)(1 - \gamma(\theta_\star))\,. \tag{3.48}$$

We get (3.48) by precisely the same arguments we used in proving Lemma 3.9, whereas for (3.47) we use a sandwich argument. Observe that

$$
\begin{aligned}
N^{-1/3}\sum_{k=0}^{sN^{2/3}-1}\mathbb{P}(E_1^k) \;\geq\; & \mathbb{E}\left(\frac{1}{\lambda_N N^{4/3}}\sum_{k=0}^{sN^{2/3}-1}\sum_{v\in\Theta_1^k}\mathbb{E}\left(|\widetilde{I}_{k+1}\cap\widetilde{I}^n(v)|\big|\mathcal{F}_k\right)\right)\\
& - \frac{1}{N^{1/3}}\left(\mathbb{E}(S)+\mathbb{E}(K)+\mathbb{E}(B'_{sN^{2/3}-1})\right),
\end{aligned}
\tag{3.49}
$$

where S is the total number of surplus edges till step $(sN^{2/3}-1)$; K is the total number of active points around which the intervals are sampled on the vertices having two or more explored intervals and $B'_{sN^{2/3}-1}$ is the number of active points at step $(sN^{2/3}-1)$ which are on the vertices having an explored interval or which will be connected to a surplus edge. Recall that we already showed that the total number of surplus edges till step $sN^{2/3}$ is of order constant, and by Proposition 3.4, K is $\mathrm{O}_{\mathbb{P}}(1)$. Moreover, $B'_{sN^{2/3}-1}$ is $\mathrm{O}_{\mathbb{P}}(1)$. Indeed, the ratio of number of links created at step k connected to a vertex having an explored interval to the total number of links created at step k is proportional to $\frac{k}{N}$. So that the probability of constructing an active point at step $k=cN^{2/3}$, $c\geq 0$ which is on the vertices having an explored interval is $\mathrm{O}(N^{-2/3})$ and since, by Proposition 3.2, the total number of active points at step $sN^{2/3}-1$ is $\mathrm{O}_{\mathbb{P}}(N^{1/3})$, $B'_{sN^{2/3}-1}$ is $\mathrm{O}_{\mathbb{P}}(1)$. Consequently, the second set of terms of equation (3.49) converge to zero as N goes to infinity, and the first term converges to the desired limit by Lemma 3.9, thereby proving Lemma 3.10.

Next, for the upper bound, writing $\mathcal{A}_k$ as the set of vertices which correspond to the active points A_k, we see that

$$
\begin{aligned}
N^{-1/3}\sum_{k=0}^{sN^{2/3}-1}\mathbb{P}(E_1^k) \;\leq\; & \mathbb{E}\left(\frac{1}{\lambda_N N^{4/3}}\sum_{k=0}^{sN^{2/3}-1}\sum_{v\in\Theta_1^k}\mathbb{E}\left(|\widetilde{I}_{k+1}\cap\widetilde{I}^n(v)|\big|\mathcal{F}_k\right)\right)\\
& +\mathbb{E}\left(\frac{1}{\lambda_N N^{4/3}}\sum_{k=0}^{sN^{2/3}-1}\sum_{v\in\mathcal{A}_k}\mathbb{E}\left(|\widetilde{I}_{k+1}\cap\widetilde{I}^n(v)|\big|\mathcal{F}_k\right)\right)
\end{aligned}
\tag{3.50}
$$

However, since $A_k=\mathrm{O}_{\mathbb{P}}(N^{1/3})$ (by Proposition 3.2), the second term of the above expression converges to zero. Then, we can conclude (3.47), which in turn proves Lemma 3.10. □

Proof of Claim 3.13. The variance of R_N is the sum of individual variances and the covariances over all the pairs. For the sum of individual variances observe that, there exists $M > 0$ such that

$$\mathrm{var}\left(V_{k+1}\mathbf{1}_{E_1^k}\right) \le \mathbb{E}\left(V_{k+1}\mathbf{1}_{E_1^k}\right)^2 \le M\mathbb{P}\left(E_1^k\right) = \mathrm{O}\left(N^{-1/3}\right)$$

implying that $\frac{1}{N^{2/3}}\sum_{k=0}^{sN^{2/3}-1}\mathrm{var}\left(V_{k+1}\mathbf{1}_{E_1^k}\right) \to 0$ as $N \to \infty$.

For the sum of covariances over all pairs, notice that it suffices to show that for some $R > 0$ and $0 < q < 1$, we have

$$N^{2/3}\mathrm{cov}\left(V_{k+1}\mathbf{1}_{E_1^k}, V_{k'+1}\mathbf{1}_{E_1^{k'}}\right) \le R\,\mathbb{E}\left(q^{f(k,k')}\right)$$

where $f(k,k')$ is as defined in Lemma 3.12.

In view of

$$\mathbb{E}\left(V_{k+1}\mathbf{1}_{E_1^k}V_{k'+1}\mathbf{1}_{E_1^{k'}}\right) \;=\; \mathbb{E}\left(V_{k+1}V_{k'+1}\middle|E_1^k\cap E_1^{k'}\right)\mathbb{P}\left(E_1^k\cap E_1^{k'}\right)$$

it suffices to prove the following

(C1) $N^{2/3}\left(\mathbb{P}\left(E_1^k\cap E_1^{k'}\right) - \mathbb{P}\left(E_1^k\right)\mathbb{P}\left(E_1^{k'}\right)\right) = \mathrm{o}(1)$

(C2) $\mathbb{E}\left(V_{k+1}V_{k'+1}\middle|E_1^k\cap E_1^{k'}\right) - \mathbb{E}\left(V_{k+1}\middle|E_1^k\right)\mathbb{E}\left(V_{k'+1}\middle|E_1^{k'}\right) = 0$

Proof of (C2)*:* Notice that on the event $\left(E_1^k\cap E_1^{k'}\right)$, the intervals corresponding to the time steps k and k' cannot belong to the same vertex, which in turn implies that V_{k+1} and $V_{k'+1}$ are independent when conditioned on the event $\left(E_1^k\cap E_1^{k'}\right)$. Therefore,

$$\mathbb{E}\left(V_{k+1}V_{k'+1}\middle|E_1^k\cap E_1^{k'}\right) \;=\; \mathbb{E}\left(V_{k+1}\middle|E_1^k\cap E_1^{k'}\right)\mathbb{E}\left(V_{k'+1}\middle|E_1^k\cap E_1^{k'}\right).$$

Then, noticing that since V_{k+1} conditioned on $\left(E_1^k\cap E_1^{k'}\right)$ is related to only the length of the second interval, it will have no impact from the information provided by $E_1^{k'}$, implying

$$\mathbb{E}\left(V_{k+1}V_{k'+1}\middle|E_1^k\cap E_1^{k'}\right) = \mathbb{E}\left(V_{k+1}\middle|E_1^k\right)\mathbb{E}\left(V_{k'+1}\middle|E_1^{k'}\right).$$

Proof of (C1)*:* We shall introduce a new sequence of events $\widetilde{E}_{\ge 1}^k$, which are closely related to E_1^k, and we shall prove their asymptotic independence. More precisely, let us define $\widetilde{E}_{\ge 1}^k$ as the event that at time step $(k+1)$ at least one active point is created on an already visited vertex, then we shall prove

$$N^{2/3}\left[\mathbb{P}\left(\widetilde{E}_{\ge 1}^k\cap\widetilde{E}_{\ge 1}^{k'}\right) - \mathbb{P}\left(\widetilde{E}_{\ge 1}^k\right)\,\mathbb{P}\left(\widetilde{E}_{\ge 1}^{k'}\right)\right] \to 0. \tag{3.51}$$

Observe that

$$\mathbb{P}\left(\widetilde{E}^{k}_{\geq 1}\right) = \mathbb{E}\left(1 - \exp\left(-\frac{1}{\lambda_N N}\sum_{v\in\Theta^{k}_{\geq 1}} |\widetilde{I}_{k+1}\cap I^n(v)|\right)\right)$$

implying

$$\mathbb{P}\left(\widetilde{E}^{k}_{\geq 1}\right) = \frac{1}{\lambda_N N}\mathbb{E}\left(\sum_{v\in\Theta^{k}_{\geq 1}} |\widetilde{I}_{k+1}\cap I^n(v)|\right) + \mathrm{o}\left(N^{-1}\right).$$

Similarly,

$$\mathbb{P}\left(\widetilde{E}^{k}_{\geq 1}\cap \widetilde{E}^{k'}_{\geq 1}\right) = \frac{1}{\lambda_N^2 N^2}\mathbb{E}\left(\sum_{v\in\Theta^{k}_{\geq 1}} |\widetilde{I}_{k+1}\cap I^n(v)| \sum_{w\in\Theta^{k'}_{\geq 1}} |\widetilde{I}_{k'+1}\cap I^n(w)|\right) + \mathrm{o}\left(N^{-1}\right).$$

Notice that for our purposes we can replace $\Theta^{k}_{\geq 1}$ and $\Theta^{k'}_{\geq 1}$, by Θ^{k}_{1} and $\Theta^{k'}_{1}$, respectively, as the number of terms lost by performing this step is $\mathrm{O}_{\mathbb{P}}\left(N^{2/3}\right)$ (by Proposition 3.4). Then,

$$\begin{aligned}\mathbb{P}\left(\widetilde{E}^{k}_{\geq 1}\cap \widetilde{E}^{k'}_{\geq 1}\right) &= \frac{1}{\lambda_N^2 N^2}\mathbb{E}\left(\sum_{v\in\Theta^{k}_{1}}\sum_{w\in\Theta^{k'}_{1}} \mathbb{E}\left(|\widetilde{I}_{k+1}\cap I^n(v)|\,|\widetilde{I}_{k'+1}\cap I^n(w)|\,\Big|\,\Theta^{k}_{1},\Theta^{k'}_{1}\right)\right) \\ &\quad + \mathrm{O}_{\mathbb{P}}\left(N^{-4/3}\right) + \mathrm{o}\left(N^{-1}\right).\end{aligned}$$

Let us consider a typical term of the above product, and note that the two intersections correspond to two different Markov chains in the same way as in the proof of Lemma 3.9. Thus to conclude (3.51), we need to ensure that these two chains are asymptotically independent as the distance between k and k' grows. For the paths connecting the vertex v to $\widetilde{I}_{k+1}$ and vertex w to $\widetilde{I}_{k'+1}$, let us define $z(v,w,k,k')$ as the smallest exploration time point after which the two paths do not intersect. Now using the same arguments as set forth in the proof of Lemma 3.9, on the paths from $z(v,w,k,k')$ to $\widetilde{I}_{k+1}$ and $\widetilde{I}_{k'+1}$, respectively, and invoking similar arguments on the paths connecting $z(v,w,k,k')$ to v and w, we can conclude that

$$\begin{aligned}&\mathbb{E}\left(|\widetilde{I}_{k+1}\cap I^n(v)|\,|\widetilde{I}_{k'+1}\cap I^n(w)|\,\Big|\,\Theta^{k}_{1},\Theta^{k'}_{1}\right) \\ =\ &\mathbb{E}\left(|\widetilde{\mathcal{I}}_1\cap {\mathcal{J}_1}^c|\right)\mathbb{E}\left(|\widetilde{\mathcal{I}}_2\cap {\mathcal{J}_2}^c|\right) + \mathrm{o}(1)\end{aligned}$$

where $\widetilde{\mathcal{I}}_1$, $\widetilde{\mathcal{I}}_2$, $\mathcal{J}_1$ and $\mathcal{J}_2$ are independent and identically distributed intervals carved out of $\mathbb{S}_\theta$ distributed as $\widetilde{I}$, described earlier in the exploration process of Section 2.1.

Finally, using the fact that the number of active points at each time step is $\mathrm{O}_{\mathbb{P}}(N^{1/3})$ and noting that the way of choosing an active point (if it exists) for the next step has nothing to do with the way it has been created we can conclude (C1), thereby completing the proof of Claim 3.13. □

3.2. *Joint convergence of component sizes.* Recall the statement of Theorem 1.3. In this section, we shall conclude the proof of this theorem using results from previous sections.

As pointed out in [2], Theorem 1.3 primarily has two parts:

1. First is to prove that the excursions of the limit process are matched by the excursions of the breadth first random walk.
2. Second is to arrange these excursions in the decreasing order. This can be achieved if one can ascertain that there exists a random point after which one is sure (with high probability) not to see large excursions.

In order to settle the first issue, we shall invoke Lemmas 7 and 8 of [2], which can be applied verbatim to our case, together with Proposition 3.1 proved in previous subsection.

Thus, we only need to be concerned about the second issue, for which we shall need to prove an appropriate version of Lemma 9 of [2] suited to our case.

Like in [2], let us define

$$T(y) = \min\{s : W^{a,\lambda_*,\theta_*}(s) = -y\},$$

$$T_N(y) = \min\{i : Y_N(i) = -\lfloor yn^{1/3} \rfloor\}.$$

Notice that as a consequence of Proposition 3.1

$$N^{-2/3} T_N(y) \quad \to_d \quad T(y).$$

Therefore, the following lemma completes the proof.

LEMMA 3.14. *Let us denote by $p(N, y, \delta)$ the probability that the* QRG *with the parameters (θ_N, λ_N) that satisfy (3.1), contains a component of size at least $\delta N^{2/3}$ which does not contain any vertex i with $1 \le i \le yN^{1/3}$. Then,*

$$\lim_{y\to\infty} \limsup_{N} p(N, y, \delta) = 0 \qquad \textit{for all } \delta > 0.$$

Proof. Fix $\delta > 0$. Let $v_{\mathcal{C}_i}$ be the minimal vertex of the cluster $\mathcal{C}_i$, then for an interval $\mathfrak{I} \subset \mathbb{R}_+$, define

$$q(N, \mathfrak{I}) = \mathbb{E}\left(\sum_{\mathcal{C}_i\colon \text{clusters}} 1_{\left(|\mathcal{C}_i| \geq \delta N^{2/3}; v_{\mathcal{C}_i} \in N^{1/3}\mathfrak{I}\right)}\right). \tag{3.52}$$

Then conditioned on arranging the components in a decreasing order of their sizes, the labels of the vertices of any given component are going to be in random order. For a component $\mathcal{C}_i$ of size $bN^{2/3}$, define

$$\chi_N(\mathcal{C}_i) = N^{-1/3} v_{\mathcal{C}_i},$$

and

$$\mathcal{U}_{\mathcal{C}_i} = N^{-2/3}\,(\text{number of vertices in the component } \mathcal{C}_i)\,.$$

Then note that

$$\mathbb{P}(v_{\mathcal{C}_i} > N^{1/3}x \big| \mathcal{U}_{\mathcal{C}_i}) = \left(1 - \frac{\mathcal{U}_{\mathcal{C}_i} N^{2/3}}{N}\right)^{N^{1/3}x},$$

implying

$$\mathbb{P}\left(\chi_N(\mathcal{C}_i) > y \big| \mathcal{U}_{\mathcal{C}_i}\right) = \frac{\mathrm{e}^{-\mathcal{U}_{\mathcal{C}_i} y}}{1 - \mathrm{e}^{-\mathcal{U}_{\mathcal{C}_i}}} \mathbb{P}\left(\chi_N(\mathcal{C}_i) \leq 1 \big| \mathcal{U}_{\mathcal{C}_i}\right) + \mathrm{o}_{\mathbb{P}}(1). \tag{3.53}$$

Note that

$$\mathbb{P}\Big(v_{\mathcal{C}_i} \in [yN^{1/3}, \infty)\Big) = \mathbb{E}\left(\mathbb{P}\Big(v_{\mathcal{C}_i} \in [yN^{1/3}, \infty) \big| \mathcal{U}_{\mathcal{C}_i}\Big)\right). \tag{3.54}$$

At this point, conditional on cluster sizes being $|\mathcal{C}_i| = bN^{2/3}$, we note that one can adopt the proof of Proposition 3.4 to the original exploration process restricted to the construction of $\mathcal{C}_i$ in order to derive similar results for $\nu_{\geq 3}^{\mathcal{C}_i}$, the number of explored intervals belonging to $\mathcal{C}_i$ sampled by the end of the construction of $\mathcal{C}_i$, which belong to vertices (circles), having at least three such intervals each. Then, observing that $\mathcal{U}_{\mathcal{C}_i} N^{2/3} \geq \frac{1}{2}(bN^{2/3} - \nu_{\geq 3}^{\mathcal{C}_i})$, we have for $\varepsilon > 0$,

$$\mathbb{P}\left(\mathcal{U}_{\mathcal{C}_i} \geq \frac{1}{2}b - \frac{1}{2}\frac{N^{1/3+\varepsilon}}{N^{2/3}}\right) \geq \mathbb{P}\left(\nu_{\geq 3}^{\mathcal{C}_i} \leq N^{1/3+\varepsilon}\right) = 1 - \mathrm{o}(N^{-1/3}),$$

implying that $\mathcal{U}_{\mathcal{C}_i} \in \left(\frac{b}{3}, b\right)$ with probability $\left(1 - \mathrm{o}(N^{-1/3})\right)$. Consequently,

$$\mathbb{E}\left(\mathbb{P}\left(v_{\mathcal{C}_i} \in [yN^{1/3}, \infty) \Big| \mathcal{U}_{\mathcal{C}_i}\right)\right) \quad \leq \quad \frac{\mathrm{e}^{-by/3}}{1 - \mathrm{e}^{-b/3}} \mathbb{P}\left(v_{\mathcal{C}_i} \in [0, N^{1/3}]\right) + \mathrm{o}(N^{-1/3}).$$

Recalling the definition of $q(N,\mathfrak{I})$ from (3.52), and conditioning on M_b the number of clusters of size $bN^{2/3}$, while observing that given the sizes of components the minimal vertices of various different components are identically distributed, we get

$$q(N,[y,\infty)) \quad = \quad \mathbb{E}\left(\sum_{b=\delta}^{\infty} M_b \mathbb{P}\left(v_{\mathcal{C}_i} \in [yN^{1/3},\infty) \Big| \, |\mathcal{C}_i| = bN^{2/3}\right)\right).$$

This, together with (3.53) implies that

$$q(N,[y,\infty)) \quad \le \quad \frac{\mathrm{e}^{-\delta y/3}}{1-\mathrm{e}^{-\delta/3}} q(N,[0,1]) + \mathrm{o}(1).$$

Now observe that $p(N,y,\delta) \le q(N,[y,\infty))$, then in view of the above set of inequalities, in order to prove the theorem, it suffices to prove that

$$\sup_N q(N,[0,1]) < \infty, \tag{3.55}$$

Writing $t_i(v)$ as points on the v-th vertex around which intervals are constructed and explored, and denoting $\mathcal{N}(v)$ as the number of such points we observe that

$$q(N,[0,1]) \le \sum_{v=1}^{N^{1/3}} \mathbb{E}\left(\sum_{i=1}^{\mathcal{N}(v)} 1_{\{|\mathcal{C}(t_i(v))|>\delta N^{2/3}\}}\right),$$

where $\mathcal{C}(t_i(v))$ is the maximal connected component containing $t_i(v)$.

Clearly the collection $\{t_1(v),\ldots,t_{\mathcal{N}(v)}(v)\}$ is independent and identically distributed for different $v \in G_N$. We replace the exploration by the over counting process of Section 2.2 which is coupled with the exploration process until the exploration process hits zero. Then, we restart an independent (and identical) over counting process together with restarting the exploration process. We repeat this process until the end of exploration of the complete graph. Subsequently, setting $\{t_1^*(v),\ldots,t^*_{\mathcal{N}^*(v)}\}$, $\mathcal{N}^*(v)$ and $\mathcal{C}^*(t_i^*(v))$ as the corresponding elements of the over counting process, we observe that since $|\mathcal{C}(t_i^*(v))|$ are i.i.d. we have

$$\begin{aligned} q(N,[0,1]) \quad &\le \quad \sum_{v=1}^{N^{1/3}} \mathbb{E}\left(\sum_{i=1}^{\mathcal{N}^*(v)} 1_{\{|\mathcal{C}(t_i^*(v))|>\delta N^{2/3}\}}\right) \\ &= \quad N^{1/3}\mathbb{E}(\mathcal{N}^*(v))\mathbb{P}\left(|\mathcal{C}(t_i^*(v))| > \delta N^{2/3}\right) \qquad \text{(using Wald's equality).} \end{aligned}$$

Therefore, it suffices to prove that $N^{1/3}\mathbb{P}(|\mathcal{C}_0^*| \geq \delta N^{2/3})$ is bounded by a constant where $\mathcal{C}_0^*$ is a *typical* component of over counting process.

We now define the coupled over counting process via i.i.d. random variables ξ_k^w, where each ξ_k^w represents the number of links generated at k-th time step by the over counting process with the parameters λ_N and θ_N lying in the critical window (3.1). Like in Section 2.2, we set $\xi_i^w \sim \text{Poisson}\left(\frac{\theta_N}{\lambda_N}\right)$. Then define $S_k^w = S_{k-1}^w + (\xi_k^w - 1)$, with $S_0^w = 1$. Setting $\tau^w = \min\{k \geq 1 : S_k^w = 0\}$, it suffices to show that $N^{1/3}\mathbb{P}\left(\tau^w > \delta N^{2/3}\right)$ is bounded by a universal constant. Using same arguments as used in Section 2.2, we can conclude that

$$\mathbb{P}(\tau = n+1) \;\leq\; \frac{\mathbb{E}(S_1^w)}{n} \;\sup_{\ell}\{\mathbb{P}(S_{n+1}^w - S_1^w = -\ell)\} \tag{3.56}$$

Using Proposition 2.4.4 of [16]) observe that

$$\mathbb{P}(S_{n+1}^w - S_1^w = -\ell) \leq \frac{c}{n^{1/2}}.$$

Therefore,

$$\mathbb{P}(\tau = n+1) \;\leq\; c\, n^{-3/2}.$$

Subsequently, following the same arguments as in Section 2.2, we conclude that

$$N^{1/3}\mathbb{P}(|\mathcal{C}_0^*| \geq \delta N^{2/3}) \leq c,$$

where c is some non-negative constant, thereby proving the statement of the lemma. □

4. Appendix: Proof of Proposition 3.2.

With $Y_N = Y_N^{a,\theta_N}$, $Z_N = Z_N^{a,\theta_N}$ and writing $\iota(l)$ for the number of maximal connected components in the corresponding graph completely explored before step l, we use the relations

$$A_l = Y_N(l) + \iota(l) = Z_N(l) - \mathrm{sur}(l) + \iota(l)\,, \tag{4.1}$$

$$\iota(l) = 1 - \min_{0 \leq k \leq (l-1)} \{Z_N(k) - \mathrm{sur}(k)\}\,, \tag{4.2}$$

and the fact that $k \mapsto \mathrm{sur}(k)$ is non-decreasing, to find that

$$A_l = 1 + Z_N(l) - \mathrm{sur}(l) + \max_{k \leq (l-1)}\{\mathrm{sur}(k) - Z_N(k)\} \leq 1 + \max_{k \leq l}\{Z_N(l) - Z_N(k)\},$$

which can further be simplified to write

$$A_l \;\leq\; 1 + 2\max_{k \leq l}|Z_N(k)|$$

In view of the above, in order to prove Proposition 3.2 it thus suffices to show that

$$N^{-1/3} \max_{k \le sN^{2/3}} |Z_N(k)| \quad \text{is stochastically bounded.} \tag{4.3}$$

Recall the martingale decomposition, $Z_N(k) = M_N(k) + B_N(k)$, where M_N is a martingale and B_N is the predictable process. Then, for any fixed positive K, set

$$\Upsilon_N = \min\{k : |Z_N(k)| > K N^{1/3}\} \wedge (s\, N^{2/3}).$$

In view of the above definition, (4.3) reduces to showing that

$$\mathbb{E}|Z_N(\Upsilon_N)| = \mathrm{O}(N^{1/3}) \tag{4.4}$$

Clearly, by optional sampling theorem,

$$\mathbb{E}|Z_N(\Upsilon_N)| \le \mathbb{E}|M_N(\Upsilon_N)| + \mathbb{E}|B_N(\Upsilon_N)|. \tag{4.5}$$

Using the notation introduced in (3.16), we have

$$\mathbb{E}(M_N(\Upsilon_N))^2 = \mathbb{E}(D_N(\Upsilon_N)) \le \mathbb{E}(D_N(sN^{2/3})),$$

since M_N^2 is a submartingale, and $\Upsilon_N \le sN^{2/3}$.

Now, recall

$$D_N(sN^{2/3}) = \sum_{k=1}^{sN^{2/3}} \mathrm{var}(\zeta_k |\, \mathcal{F}_{k-1})$$

which by way of arguments set forth in Lemma 3.5 can be shown to be bounded by $cN^{2/3}$ for some constant $c > 0$. Thus,

$$\mathbb{E}|M_N(\Upsilon_N)| \le c^{1/2} N^{1/3}. \tag{4.6}$$

Next, using the expression for B_N obtained in Lemma 3.8 we can write

$$\begin{aligned} B_N(l) \;\; &= \;\; \mathrm{O}_{\mathbb{P}}(1) - \frac{F(\theta_N)}{\lambda_N\, N} \frac{l(l-1)}{2} \\ &\quad - \frac{1}{\lambda_N} \sum_{k=0}^{l-1} \Big(F(\theta_N) - \mathbb{E}\big(|\widetilde{I}_{k+1}| \big| \mathcal{F}_k\big)\Big) \mathbf{1}_{E_1^k} \\ &\quad + \frac{1}{\lambda_N\, N} \sum_{k=0}^{l-1} \mathbf{1}_{E_0^k} \sum_{v \in \Theta_1^k} \mathbb{E}\big(|\widetilde{I}_{k+1} \cap \widetilde{I}^n(v)| \big| \mathcal{F}_k\big) \end{aligned}$$

Notice that $|\Theta_1^k| \leq k$ (by definition), and $\mathbb{E}\left(|\widetilde{I}_{k+1} \cap \widetilde{I}^n(v)||\mathcal{F}_k\right) \leq \beta$, thus

(4.7)
$$\frac{1}{N^{4/3}} \sum_{k=0}^{sN^{2/3}-1} 1_{E_0^k} \sum_{v \in \Theta_1^k} \mathbb{E}\left(|\widetilde{I}_{k+1} \cap \widetilde{I}^n(v)||\mathcal{F}_k\right) \leq \frac{\beta}{N^{4/3}} \frac{sN^{2/3}(sN^{2/3}-1)}{2} \leq C_1$$

for some non-negative constant C_1.

Next, using $\left|F(\theta_N) - \mathbb{E}\big(|\widetilde{I}_{k+1}||\mathcal{F}_k\big)\right| \leq 2\beta$, we get

$$\left|\frac{1}{\lambda_N N^{1/3}} \sum_{k=0}^{sN^{2/3}-1} \mathbb{E}\left(\left(F(\theta_N) - \mathbb{E}\big(|\widetilde{I}_{k+1}||\mathcal{F}_k\big)\right)\mathbf{1}_{E_1^k}\right)\right| \leq \frac{2\beta}{\lambda_N N^{1/3}} \sum_{k=0}^{sN^{2/3}-1} \mathbb{P}(E_1^k)$$

However, we have from equation (3.50)

$$\begin{aligned} N^{-1/3} \sum_{k=0}^{sN^{2/3}-1} \mathbb{P}(E_1^k) \quad &\leq \quad \mathbb{E}\left(\frac{1}{\lambda_N N^{4/3}} \sum_{k=0}^{sN^{2/3}-1} \sum_{v \in \Theta_1^k} \mathbb{E}\left(|\widetilde{I}_{k+1} \cap \widetilde{I}^n(v)||\mathcal{F}_k\right)\right) \\ &\quad + \mathbb{E}\left(\frac{1}{\lambda_N N^{4/3}} \sum_{k=0}^{sN^{2/3}-1} \sum_{v \in \mathcal{A}_k} \mathbb{E}\left(|\widetilde{I}_{k+1} \cap \widetilde{I}^n(v)||\mathcal{F}_k\right)\right) \\ &\leq \quad \mathbb{E}\left(\frac{1}{\lambda_N N^{4/3}} \sum_{k=0}^{sN^{2/3}-1} |\Theta_1^k|\beta\right) + \mathbb{E}\left(\frac{1}{\lambda_N N^{4/3}} \sum_{k=0}^{sN^{2/3}-1} \sum_{v \in \mathcal{A}_k} \beta\right) \end{aligned}$$

Since, $|\Theta_1^k| \leq k$ thus, the first part of the above expression can be shown to be bounded by a non-negative universal constant C_2. For the second part, observe that

$$\mathbb{E}\left(\frac{1}{\lambda_N N^{4/3}} \sum_{k=0}^{sN^{2/3}-1} \sum_{v \in \mathcal{A}_k} \beta\right) = \frac{\beta}{\lambda_N N^{4/3}} \sum_{k=0}^{sN^{2/3}-1} \mathbb{E}|\mathcal{A}_k| \leq \frac{\beta}{\lambda_N N^{4/3}} \sum_{k=0}^{sN^{2/3}-1} \mathbb{E}(A_k)$$

Recall the definition of A_k and the fact that $\{\eta_k\}$ are stochastically bounded by $\{\xi_k\}$, which are the Poisson links defined in the over counting process. Therefore, we can conclude that $\mathbb{E}(A_k) \leq 1$. Therefore,

$$\mathbb{E}\left(\frac{1}{\lambda_N N^{4/3}} \sum_{k=0}^{sN^{2/3}-1} \sum_{v \in \mathcal{A}_k} \beta\right) \leq \frac{\beta}{\lambda_N N^{2/3}}$$

Thus we conclude that

$$\frac{2\beta}{\lambda_N N^{1/3}} \sum_{k=0}^{sN^{2/3}-1} \mathbb{P}(E_1^k) \leq C_2 + \frac{\beta}{\lambda_N N^{2/3}} \tag{4.8}$$

Collating equations (4.6), (4.7), (4.8), and (4.5) we conclude the statement of (4.4) which proves Proposition 3.2.

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