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ERDŐS-FELLER-KOLMOGOROV-PETROWSKY LAW OF THE ITERATED LOGARITHM FOR SELF-NORMALIZED MARTINGALES: A GAME-THEORETIC APPROACH

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We prove an Erdős-Feller-Kolmogorov-Petrowsky law of the iterated logarithm for self-normalized martingales. Our proof is given in the framework of the game-theoretic probability of Shafer and Vovk. Like many other game-theoretic proofs, our proof is self-contained and explicit.

1. Main Result. Let S_n be a martingale with respect to a filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$, and let $x_n = S_n - S_{n-1}$ be the martingale difference. Various versions of the law of the iterated logarithm (LIL), assuming different regularity conditions on the growth of $|x_n|$, have been given in literature. The Erdős-Feller-Kolmogorov-Petrowsky law of the iterated logarithm (EFKP-LIL [17, Chapter 5.2]) is an important one.

Lévy stated the EFKP-LIL for symmetric Bernoulli random variables without proving it [13]. Kolmogorov seems to be the first to give a proof. Later, Ville [21] proved the validity part of EFKP-LIL and Erdős [6] proved both the validity part and the sharpness part, with a complete proof. EFKP-LIL has been generalized by Feller [7] for bounded and independent random variables and [8] (see also Bai [1]) for the i.i.d. case. Further, EFKP-LIL has been generalized for martingales by Strassen [20], Jain, Jogdeo and Stout [10], Philipp and Stout [16], Einmahl and Mason [5] and Berkes, Hörmann and Weber [2]. In particular, Einmahl and Mason [5] proved a martingale analogue of Feller's result in [7], just as Stout [19] obtained a martingale analogue of Kolmogorov's result in [11].

For self-normalized processes, EFKP-LIL was derived by [9, 3] in the i.i.d. case. However EFKP-LIL has not been derived in the martingale case, even though de la Peña, Klass and Lai [4] obtained the usual LIL. The purpose of this paper is to prove EFKP-LIL for self-normalized martingales.

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For a positive non-decreasing continuous function $\psi(\lambda)$ let

$$I(\psi) := \int_1^\infty \psi(\lambda) e^{-\psi(\lambda)^2/2} \frac{d\lambda}{\lambda}. \tag{1.1}$$

We state our main theorem.

THEOREM 1.1. *Let S_n, $n = 1, 2, \ldots,$ be a martingale with $S_0 = 0$ and $x_n = S_n - S_{n-1}$ be the corresponding martingale difference with respect to a filtration $\{\mathcal{F}_n\}_{n=0}^\infty$ such that*

$$|x_n| \le c_n \ a.s.$$

for some $\mathcal{F}_{n-1}$-measurable random variable c_n. Set

$$A_n^2 := \sum_{i=1}^n x_i^2$$

and suppose ψ is a positive non-decreasing continuous function.

If $I(\psi) < \infty$, then

$$\mathrm{P}\left(S_n < A_n \psi(A_n^2) \ a.a. \mid \lim A_n = \infty, \limsup c_n \frac{\psi(A_n^2)^3}{A_n} < \infty \right) = 1, \tag{1.2}$$

where a.a. (almost always) means "except for a finite number of n".

If $I(\psi) = \infty$, then

$$\mathrm{P}\left(S_n \ge A_n \psi(A_n^2) \ i.o. \mid \lim A_n = \infty, \limsup c_n \frac{\psi(A_n^2)^3}{A_n} < \infty \right) = 1, \tag{1.3}$$

where i.o. (infinitely often) means "for infinitely many n".

This theorem is a self-normalized version of the result in Einmahl and Mason [5] and a EFKP-LIL version of the result in de la Peña, Klass and Lai [4]. Note that our result is not a direct generalization of these results, because the assumptions in our theorem are somewhat different from those of the previous results. We are not assuming the existence of the second moment of x_n. In Appendix we give examples of martingales which satisfy the assumptions of Theorem 1.1 but do not possess finite second moments. The order of growth $A_n/(\psi(A_n^2))^3$ for c_n is currently the best known order for EFKP-LIL even in the independent case ([2]). We call (1.2) the *validity* and (1.3) the *sharpness* of EFKP-LIL.

Implicit in the statements (1.2) and (1.3) is the assumption that

$$\mathrm{P}\left(\lim A_n = \infty, \limsup c_n \frac{\psi(A_n^2)^3}{A_n} < \infty\right) > 0,$$

but we are not assuming

$$\mathrm{P}\left(\lim A_n = \infty, \limsup c_n \frac{\psi(A_n^2)^3}{A_n} < \infty\right) = 1.$$

Thus (1.2) is equivalent to

$$\mathrm{P}\left(\lim A_n = \infty, \limsup c_n \frac{\psi(A_n^2)^3}{A_n} < \infty, S_n \geq A_n \psi(A_n^2) \ i.o.\right) = 0. \tag{1.4}$$

For our proof we adopt Shafer and Vovk's framework of game-theoretic probability [18]. To prove (1.2), for example, we explicitly construct a nonnegative martingale diverging to infinity on the event of (1.4).

We use the following notation throughout the paper:

$$\ln_k n := \underbrace{\ln \ln \ldots \ln}_{k\text{times}} n.$$

We also fix a small positive δ for the rest of this paper, e.g., $\delta = 0.01$. For our proof, as is often seen in the upper-lower class theory (cf. Feller [8, Lemma 1]), we can restrict our attention to ψ such that

$$\psi^L(n) \leq \psi(n) \leq \psi^U(n) \text{ for all sufficiently large } n, \tag{1.5}$$

where

$$\psi^L(n) := \sqrt{2\ln_2 n + 3\ln_3 n}, \quad \psi^U(n) := \sqrt{2\ln_2 n + 4\ln_3 n}.$$

Here L means the lower class and U means the upper class. It can be verified that $I(\psi^U) < \infty$ and $I(\psi^L) = \infty$.

The rest of this paper is organized as follows. In Section 2 we give a game-theoretic statement corresponding to our main theorem. In Section 3 we prove validity, and in Section 4 we prove sharpness.

2. Preliminaries on Game-Theoretic Probability. Before setting up a game-theoretic framework for our result, we give some general discussion on how game-theoretic proofs are constructed. The game-theoretic probability initiated by Shafer and Vovk [18] provides a foundation of probability theory alternative to the standard measure-theoretic probability. Game-theoretic proofs of standard results, such as the strong law of large numbers, are self-contained and explicit. Also game-theoretic results are often stronger than measure-theoretic results, because game-theoretic results can be immediately translated to measure-theoretic results by replacing moves of a player, called Reality, by measure-theoretic random variables. This is discussed in Chapter 8 of [18]. In this paper, our main result is in fact Theorem 2.1 below, which can be translated to Theorem 1.1 by replacing x_n in the game SPUFG below by realizations of x_n in Theorem 1.1.

Our aim is to prove (1.4) by a game-theoretic argument. Let E denote the event in (1.4). In order to prove $\mathrm{P}(E) = 0$ in the measure-theoretic sense, we construct a non-negative martingale $\mathcal{K}_n$ which diverges to infinity on E, more precisely $\limsup \mathcal{K}_n = \infty$ on E. Then by the martingale convergence theorem for non-negative martingales we have $\mathrm{P}(E) = 0$. This can be accomplished by setting up an appropriate game and constructing a betting strategy, such that its capital process $\mathcal{K}_n$ is always non-negative and $\mathcal{K}_n$ diverges to infinity for *every* path in E. A game-theoretic proof often looks very different from a measure-theoretic one, because a game-theoretic proof is based on a path-wise argument and in this sense it is deterministic. Verifying a game-theoretic proof consists of checking series of inequalities for $\mathcal{K}_n$ for a fixed path. Self-normalization is natural for game-theoretic probability, because the normalization is given by the path itself. In order to deal with conditional variance, a game should include a setup for pricing the quadratic variation of a martingale.

In order to state a game-theoretic version of Theorem 1.1, consider the following simplified predictably unbounded forecasting game (SPUFG, Section 5.1 of [18]) with the initial capital $\alpha > 0$.

Simplified Predictably Unbounded Forecasting Game
Players: Forecaster, Skeptic, Reality
Protocol:
  $\mathcal{K}_0 := \alpha$.
  FOR $n = 1, 2, \ldots$:
    Forecaster announces $c_n \geq 0$.
    Skeptic announces $M_n \in \mathbb{R}$.
    Reality announces $x_n \in [-c_n, c_n]$.
    $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.
Collateral Duties: Skeptic must keep $\mathcal{K}_n$ non-negative. Reality must keep $\mathcal{K}_n$ from tending to infinity.

Usually α is taken to be 1, but in Section 4 we use $\alpha \neq 1$ for notational simplicity.

We prove the following theorem, which implies Theorem 1.1 by Chapter 8 of [18].

Theorem 2.1. *Consider SPUFG. Let ψ be a positive non-decreasing* *continuous function. If* $I(\psi) < \infty$*, Skeptic can force*

$$A_n^2 \to \infty \text{ and } \limsup c_n \frac{\psi(A_n^2)^3}{A_n} < \infty \ \Rightarrow\ S_n < A_n \psi(A_n^2) \quad a.a. \tag{2.1}$$

and if $I(\psi) = \infty$*, Skeptic can force*

$$A_n^2 \to \infty \text{ and } \limsup c_n \frac{\psi(A_n^2)^3}{A_n} < \infty \ \Rightarrow\ S_n \geq A_n \psi(A_n^2) \quad i.o. \tag{2.2}$$

An advantage of the game-theoretic statement in this theorem is that no assumption is needed on the probability of the conditioning event "$A_n^2 \to \infty$ and $\limsup c_n \psi(A_n^2)^3/A_n < \infty$".

We use the same line of argument as in [15] and Chapter 5 of Shafer and Vovk [18]. We employ a Bayesian mixture of constant-proportion betting strategies. Here we give basic properties of constant-proportion betting strategies.

A constant-proportion betting strategy with betting proportion $\gamma > 0$ sets

$$M_n = \gamma \mathcal{K}_{n-1}.$$

However, $\mathcal{K}_n$ becomes negative if $\gamma x_n < -1$. For simplicity we consider applying the strategy ("keep the account open") as long as $\gamma c_n \leq \delta$ and set $M_n = 0$ once $\gamma c_n > \delta$ happens ("freeze the account"). Define a stopping time

$$\sigma_\gamma := \min\{n \mid \gamma c_n > \delta\}. \tag{2.3}$$

Note the monotonicity of σ_γ, i.e., $\sigma_{\gamma'} \geq \sigma_\gamma$ if $\gamma' \leq \gamma$. We denote the capital process of the constant-proportion betting strategy with this stopping time by $\mathcal{K}_n^\gamma$. With the initial capital of $\mathcal{K}_0^\gamma = \alpha$, the value of $\mathcal{K}_n^\gamma$ is written as

$$\mathcal{K}_n^\gamma = \alpha \prod_{i=1}^{\min(n,\sigma_\gamma - 1)} (1 + \gamma x_i).$$

We have

$$t - \frac{t^2}{2} - t^2 \times |t| \leq \ln(1+t) \leq t - \frac{t^2}{2} + t^2 \times |t|$$

when $|t| \leq \delta$. Then by taking the logarithm of $\prod_{i=1}^n (1 + \gamma x_i)$, for $n < \sigma_\gamma$, we have

$$\gamma S_n - \frac{\gamma^2 A_n^2}{2} - \gamma^3 A_n^2 \bar{c}_n \leq \ln\left(\mathcal{K}_n^\gamma / \alpha\right) \leq \gamma S_n - \frac{\gamma^2 A_n^2}{2} + \gamma^3 A_n^2 \bar{c}_n$$

and

$$e^{-\gamma^3 A_n^2 \bar{c}_n} e^{\gamma S_n - \gamma^2 A_n^2/2} \le \mathcal{K}_n^\gamma/\alpha \le e^{\gamma^3 A_n^2 \bar{c}_n} e^{\gamma S_n - \gamma^2 A_n^2/2}, \tag{2.4}$$

where

$$\bar{c}_n := \max_{1\le i\le n} c_i.$$

We also set up some notation for expressing the condition in (2.1) and (2.2). An infinite sequence of Forecaster's and Reality's announcements $\omega = (c_1, x_1, c_2, x_2, \ldots)$ is called a *path* and the set of paths $\Omega = \{\omega\}$ is called the *sample space*. Define a subset $\Omega_{<\infty}$ of Ω by

$$\Omega_{<\infty} := \left\{\omega \mid A_n^2 \to \infty, \limsup_n c_n \frac{\psi(A_n^2)^3}{A_n} < \infty\right\}.$$

For an arbitrary path $\omega \in \Omega_{<\infty}$ we have

$$\exists C(\omega) < \infty, \exists n_1(\omega), \forall n > n_1(\omega),\ c_n < C(\omega)\frac{A_n}{\psi(A_n^2)^3},\ \ \psi(A_n^2) \ge 1. \tag{2.5}$$

The last inequality holds by the lower bound in (1.5).

3. Validity. We prove the validity in (2.1) of Theorem 2.1. In this section we let $\alpha = 1$. We discretize the integral in (1.1) as

$$\sum_{k=1}^{\infty} \frac{\psi(k)}{k} e^{-\psi(k)^2/2} < \infty. \tag{3.1}$$

Since $xe^{-x^2/2}$ is decreasing for $x \ge 1$, the function $\lambda \mapsto \frac{\psi(\lambda)}{\lambda} e^{-\psi(\lambda)^2/2}$ is decreasing for λ such that $\psi(\lambda) \ge 1$, and convergence of the integral in (1.1) is equivalent to convergence of the the sum in (3.1).

Convergence of the infinite series in (3.1) implies the existence of a non-decreasing sequence of positive reals a_k diverging to infinity ($a_k \uparrow \infty$), such that the series multiplied term by term by a_k is still convergent:

$$Z := \sum_{k=1}^{\infty} a_k \frac{\psi(k)}{k} e^{-\psi(k)^2/2} < \infty.$$

This is easily seen by dividing the infinite series into blocks of sums less than or equal to $1/2^k$ and multiplying the k-th block by k (see also [14, Lemma 4.15]).

For $k \ge 1$ let

$$p_k := \frac{1}{Z} a_k \frac{\psi(k)}{k} e^{-\psi(k)^2/2}$$

and consider the capital process of a countable mixture of constant-proportion strategies

$$\mathcal{K}_n := \sum_{k=1}^{\infty} p_k \mathcal{K}_n^{\gamma_k}, \quad \text{where} \quad \gamma_k := \frac{\psi(k)}{\sqrt{k}}. \tag{3.2}$$

Note that $\mathcal{K}_n$ is never negative. By the upper bound in (1.5), as $k \to \infty$ we have

$$\gamma_k \le \frac{\psi^U(k)}{\sqrt{k}} = \sqrt{\frac{2\ln_2 k + 4\ln_3 k}{k}} \to 0. \tag{3.3}$$

We will show that $\limsup_n \mathcal{K}_n = \infty$ on any path $\omega \in \Omega_{<\infty}$ satisfying $S_n \ge A_n \psi(A_n^2)$ *i.o.* We bound $Z\mathcal{K}_n$ as

$$Z\mathcal{K}_n \ge \sum_{k=\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}^{\lfloor A_n^2 \rfloor} p_k \mathcal{K}_n^{\gamma_k}. \tag{3.4}$$

We first check that all accounts on the right-hand side of (3.4) are open for sufficiently large n and that the lower bound in (2.4) can be applied to each term of (3.4) for $\omega \in \Omega_{<\infty}$.

Lemma 3.1. *Let $\omega \in \Omega_{<\infty}$. Let $C = C(\omega)$ in (2.5). For sufficiently large n*

$$\bar{c}_n = \max_{1 \le i \le n} c_i < (1+\delta) C \frac{A_n}{\psi(A_n^2)^3}. \tag{3.5}$$

Proof. Note that the first $n_1(\omega)$ c's i.e., $c_1, \dots, c_{n_1(\omega)}$, do not matter since $\lim_{n\to\infty} A_n/\psi(A_n^2)^3 = \infty$. For $l > n_1(\omega)$, by (2.5) we have

$$c_l \le C \frac{A_l}{\psi(A_l^2)^3} \le C A_l.$$

Hence c_l such that $A_l \le A_n/\psi(A_n^2)^3$ do not matter in $\bar{c}_n$.

For c_l such that $A_l > A_n/\psi(A_n^2)^3$ we have

$$c_l \le C \frac{A_l}{\psi\big(A_n^2/\psi(A_n^2)^6\big)^3} \le C \frac{A_n}{\psi\big(A_n^2/\psi(A_n^2)^6\big)^3} = C \frac{A_n}{\psi(A_n^2)^3} \frac{\psi(A_n^2)^3}{\psi\big(A_n^2/\psi(A_n^2)^6\big)^3}.$$

But by (1.5), both $\psi(A_n^2)$ and $\psi\big(A_n^2/\psi(A_n^2)^6\big)$ are of the order $\sqrt{2\ln_2 A_n^2}(1+o(1))$ and $\psi(A_n^2)/\psi\big(A_n^2/\psi(A_n^2)^6\big) \to 1$ as $n \to \infty$. Hence (3.5) holds. □

LEMMA 3.2. *Let $\omega \in \Omega_{<\infty}$. For sufficiently large n, $\sigma_{\gamma_k} > n$ for all $k = \lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor, \dots, \lfloor A_n^2\rfloor$.*

PROOF. By the monotonicity of ψ, we have $\gamma_k \le \psi(A_n^2)/\sqrt{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}$ for $k = \lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor, \dots, \lfloor A_n^2\rfloor$. Then by the monotonicity of σ_γ, it suffices to show

$$\frac{\psi(A_n^2)}{\sqrt{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}}\bar{c}_n \le \delta$$

for sufficiently large n. By (3.5), the left-hand side is bounded from above by

$$\frac{\psi(A_n^2)}{\sqrt{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}}\times(1+\delta)C\frac{A_n}{\psi(A_n^2)^3} = (1+\delta)C\frac{A_n}{\sqrt{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}}\frac{1}{\psi(A_n^2)^2}.$$

But this converges to 0 as $n \to \infty$. □

By Lemma 3.2 and the lower bound in (2.4), for sufficiently large n, we have

$$\mathcal{K}_n^{\gamma_k} \ge e^{-\gamma_k^3 A_n^2 \bar{c}_n} e^{\gamma_k S_n - \gamma_k^2 A_n^2/2}, \quad k = \lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor, \dots, \lfloor A_n^2\rfloor$$

and $Z\mathcal{K}_n$ can be bounded from below as

$$\begin{aligned} Z\mathcal{K}_n &\ge Z \sum_{k=\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}^{\lfloor A_n^2\rfloor} p_k \exp(\gamma_k S_n - \frac{\gamma_k^2 A_n^2}{2} - \gamma_k^3 A_n^2 \bar{c}_n) \\ &= \sum_{k=\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}^{\lfloor A_n^2\rfloor} a_k \frac{\psi(k)}{k} \exp(-\frac{\psi(k)^2}{2} + \gamma_k S_n - \frac{\gamma_k^2 A_n^2}{2} - \gamma_k^3 A_n^2 \bar{c}_n). \end{aligned}$$

Now we assume that $S_n \ge A_n \psi(A_n^2)$ *i.o.* for the path $\omega \in \Omega_{<\infty}$. Then for sufficiently large n such that $S_n \ge A_n\psi(A_n^2)$, $\psi(A_n^2)/(\psi(A_n^2)-1) \le 1+\delta$ and $A_n/\left(\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor\right)^{1/2} \le 1+\delta$, we bound the exponent part by (2.4) as

$$\begin{aligned} -\frac{\psi(k)^2}{2} + \gamma_k S_n - \frac{\gamma_k^2 A_n^2}{2} &\ge -\frac{\psi(k)^2}{2} + A_n\psi(A_n^2)\frac{\psi(k)}{\sqrt{k}} - \frac{\psi(k)^2}{k}\frac{A_n^2}{2} \\ &= \psi(k)\left(-\frac{1}{2}\left(1+\frac{A_n^2}{k}\right)\psi(k) + \sqrt{\frac{A_n^2}{k}}\psi(A_n^2)\right) \\ &\ge -\frac{\psi(A_n^2)^2}{2}\left(\sqrt{\frac{A_n^2}{k}}-1\right)^2 \ge -\frac{\psi(A_n^2)^2}{2}\left(\frac{A_n^2}{k}-1\right)^2 \end{aligned}$$

$$\geq -\frac{1}{2}\left(\frac{\psi(A_n^2)}{\psi(A_n^2)-1}\right)^2 \geq -\frac{1}{2} - 2\delta$$

and by Lemma 3.1

$$\begin{aligned}
\gamma_k^3 A_n^2 \bar{c}_n &\leq \frac{\psi(A_n^2)^3}{(\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor)^{3/2}} A_n^2(1+\delta)C\frac{A_n}{\psi(A_n^2)^3} \\
&\leq (1+\delta)C\left(\frac{A_n}{(\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor)^{1/2}}\right)^3 \\
&\leq C(1+\delta)^4. \qquad (3.6)
\end{aligned}$$

For sufficiently large n, we have

$$\begin{aligned}
\psi(A_n^2) \leq \psi^U(A_n^2) < \psi^U(2k) &= \sqrt{2\ln_2 2k + 4\ln_3 2k} \\
&< 2\sqrt{2\ln_2 k + 3\ln_2 k} = 2\psi^L(k) \leq 2\psi(k).
\end{aligned}$$

Thus by (3.6),

$$\begin{aligned}
Z\mathcal{K}_n &\geq \sum_{k=\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}^{\lfloor A_n^2\rfloor} a_k \frac{\psi(k)}{k} \exp\left(-\frac{1}{2} - 2\delta - C(1+\delta)^4\right) \\
&\geq a_{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor} \frac{\psi(A_n^2)}{2A_n^2} \sum_{k=\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}^{\lfloor A_n^2\rfloor} \exp\left(-\frac{1}{2} - 2\delta - C(1+\delta)^4\right) \\
&\geq a_{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor} \frac{\psi(A_n^2)}{2A_n^2}\left(\frac{A_n^2}{\psi(A_n^2)} - 1\right) \exp\left(-\frac{1}{2} - 2\delta - C(1+\delta)^4\right) \\
&= a_{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}\left(\frac{1}{2} - \frac{\psi(A_n^2)}{2A_n^2}\right) \exp\left(-\frac{1}{2} - 2\delta - C(1+\delta)^4\right).
\end{aligned}$$

Since $a_{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor} \to \infty$ as $n \to \infty$, we have shown

$$\omega \in \Omega_{<\infty},\ S_n \geq A_n\psi(A_n^2)\ i.o.\ \Rightarrow\ \limsup_{n\to\infty} \mathcal{K}_n = \infty.$$

4. Sharpness. We prove the sharpness in (2.2) of Theorem 2.1. As in Section 4.2 of [18] and in [14], in order to prove the sharpness, it suffices to show the following proposition.

PROPOSITION 4.1. *Consider SPUFG. Let ψ be a positive non-decreasing continuous function. If $I(\psi) = \infty$, then for each $C > 0$, Skeptic can force*

$$A_n^2 \to \infty, \limsup_n c_n \frac{\psi(A_n^2)^3}{A_n} \leq C \ \Rightarrow\ S_n \geq A_n\psi(A_n^2) \ \ i.o. \qquad (4.1)$$

Once we prove this proposition, we can take the mixture over $C = 1, 2, \ldots$. Then the sharpness follows, because for each $\omega \in \Omega_{<\infty}$, there exists $C(\omega)$ satisfying (2.5). We denote

$$\Omega_C := \left\{ \omega \in \Omega \mid A_n^2 \to \infty, \limsup_n c_n \frac{\psi(A_n^2)^3}{A_n} < (1-\delta)C \right\},$$
$$\Omega_0 := \left\{ \omega \in \Omega \mid \lim_{n\to\infty} A_n^2 < \infty \right\},$$
$$\Omega_{=\infty} := \left\{ \omega \in \Omega \mid A_n^2 \to \infty, \limsup_n c_n \frac{\psi(A_n^2)^3}{A_n} = \infty \right\}.$$

We divide our proof of Proposition 4.1 into several subsections. For notational simplicity we use the initial capital of $\alpha = 1 - 2/e = (e-2)/e$ in this section. In Sections 4.1 and 4.2 we only consider γ and n with $n < \sigma_\gamma$. As in Lemma 3.2 for the validity, this condition will be satisfied for sufficiently small γ and relevant n.

4.1. *Uniform mixture of constant-proportion betting strategies.* We consider a continuous uniform mixture of constant-proportion strategies with the betting proportion $u\gamma$, $2/e \le u \le 1$. This is a Bayesian strategy, a similar one to which has been considered in [12].

Define

$$\mathcal{L}_n^\gamma := \int_{2/e}^1 \prod_{i=1}^{\min(n,\sigma_\gamma - 1)} (1 + u\gamma x_i) du, \qquad \mathcal{L}_0^\gamma = \alpha = 1 - e/2.$$

At round $n < \sigma_\gamma$ this strategy bets $M_n = \int_{2/e}^1 u\gamma \prod_{i=1}^{n-1}(1 + u\gamma x_i) du$. Then by induction on $n < \sigma_\gamma$ the capital process is indeed written as

$$\mathcal{L}_n^\gamma = \mathcal{L}_{n-1}^\gamma + M_n x_n = \int_{2/e}^1 \prod_{i=1}^{n-1}(1 + u\gamma x_i)du + x_n \int_{2/e}^1 u\gamma \prod_{i=1}^{n-1}(1 + u\gamma x_i)du$$
$$= \int_{2/e}^1 \prod_{i=1}^{n}(1 + u\gamma x_i)du.$$

Applying (2.4), we have

$$e^{-\gamma^3 A_n^2 \bar{c}_n} \int_{2/e}^1 e^{u\gamma S_n - u^2\gamma^2 A_n^2/2} du \le \mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n} \int_{2/e}^1 e^{u\gamma S_n - u^2\gamma^2 A_n^2/2} du,$$

for $n < \sigma_\gamma$. We further bound the integral in the following lemma.

LEMMA 4.2. *For $n < \sigma_\gamma$,*

(4.2)
$$\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n} e^{2\gamma(S_n/e - \gamma A_n^2/e^2)} \quad \text{if} \quad S_n \le 2\gamma A_n^2/e,$$

(4.3)
$$\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n} \min\left\{ e^{S_n^2/(2A_n^2)} \frac{\sqrt{2\pi}}{\gamma A_n}, e^{\gamma S_n/2} \right\} \quad \text{if} \quad 2\gamma A_n^2/e < S_n < \gamma A_n^2,$$

(4.4)
$$\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n} \min\left\{ e^{S_n^2/(2A_n^2)} \frac{\sqrt{2\pi}}{\gamma A_n}, e^{\gamma S_n - \gamma^2 A_n^2/2} \right\} \quad \text{if} \quad S_n \ge \gamma A_n^2.$$

PROOF. Completing the square we have

$$-\frac{1}{2}u^2\gamma^2 A_n^2 + u\gamma S_n = -\frac{\gamma^2 A_n^2}{2}\left(u - \frac{S_n}{\gamma A_n^2}\right)^2 + \frac{S_n^2}{2A_n^2}.$$

Hence by the change of variables

$$v = \gamma A_n \left(u - \frac{S_n}{\gamma A_n^2}\right), \qquad du = \frac{dv}{\gamma A_n},$$

we obtain

$$\begin{aligned}\int_{2/e}^1 e^{u\gamma S_n - u^2\gamma^2 A_n^2/2} du &= e^{S_n^2/(2A_n^2)} \int_{2/e}^1 \exp\left(-\frac{\gamma^2 A_n^2}{2}\left(u - \frac{S_n}{\gamma A_n^2}\right)^2\right) du \\ &= e^{S_n^2/(2A_n^2)} \frac{1}{\gamma A_n} \int_{2\gamma A_n/e - S_n/A_n}^{\gamma A_n - S_n/A_n} e^{-v^2/2} dv.\end{aligned}$$

Then for all cases we can bound $\mathcal{L}_n^\gamma$ from above as

$$\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n + S_n^2/(2A_n^2)} \frac{\sqrt{2\pi}}{\gamma A_n}. \tag{4.5}$$

Without change of variables, we can also bound the integral $\int_{2/e}^1 g(u)du$, $g(u) := e^{u\gamma S_n - u^2\gamma^2 A_n^2/2}$, directly as

$$\int_{2/e}^1 g(u)du \le \max_{2/e \le u \le 1} g(u).$$

Note that

$$g(2/e) = e^{2\gamma(S_n/e - \gamma A_n/e^2)}, \quad g(1) = e^{\gamma S_n - \gamma^2 A_n^2/2}. \tag{4.6}$$

We now consider the following three cases.

Case 1 $S_n \le 2\gamma A_n^2/e$. In this case $S_n/(\gamma A_n^2) \le 2/e$ and by the unimodality of $g(u)$ we have $\max_{2/e \le u \le 1} g(u) = g(2/e)$. Hence (4.2) follows from (4.6).

Case 2 $2\gamma A_n^2/e < S_n < \gamma A_n^2$. In this case $\max_{2/e\le u\le 1} g(u) = g(S_n/(\gamma A_n^2)) = e^{S_n^2/(2A_n^2)}$ and $\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n} e^{S_n^2/(2A_n^2)}$. Furthermore in this case $S_n^2 < \gamma A_n^2 S_n$ implies $S_n^2/(2A_n^2) < \gamma S_n/2$ and we also have

$$\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n} e^{\gamma S_n/2}. \tag{4.7}$$

By (4.5) and (4.7), we have (4.3).

Case 3 $S_n \ge \gamma A_n^2$. Then $S_n/(\gamma A_n^2) \ge 1$ and $\max_{2/e\le u\le 1} g(u) = g(1)$. Hence

$$\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n} e^{\gamma S_n - \gamma^2 A_n^2/2}. \tag{4.8}$$

By (4.5) and (4.8), we have (4.4).

□

4.2. *Buying a process and selling a process.* Next we consider the following capital process.

$$\mathcal{Q}_n^\gamma := 2\mathcal{L}_n^\gamma - \mathcal{K}_n^{\gamma e}. \tag{4.9}$$

This capital process consists of buying two units of $\mathcal{L}_n^\gamma$ and selling one unit of $\mathcal{K}_n^{\gamma e}$. As we show in Lemma 4.3, $\mathcal{K}_n^{\gamma e}$ cuts off the growth of $\mathcal{L}_n^\gamma$ in $S_n \ge e\gamma A_n^2$. This combination of selling and buying is essential in the game-theoretic proof of LIL in Chapter 5 of [18] and [15].

We want to bound $\mathcal{Q}_n^\gamma$ from above.

Lemma 4.3. *Let*

$$C_1 := 2e^{\gamma^3 A_n^2 \bar{c}_n} \exp\left(\frac{(2e-1)((1+e^3)\gamma^3 A_n^2 \bar{c}_n + \ln 2)}{(e-1)^2}\right). \tag{4.10}$$

Then for $n < \sigma_{\gamma e}$,

(4.11)

$$\mathcal{Q}_n^\gamma \le C_1 \quad \text{if} \quad S_n \le \gamma A_n^2/e,$$

(4.12)

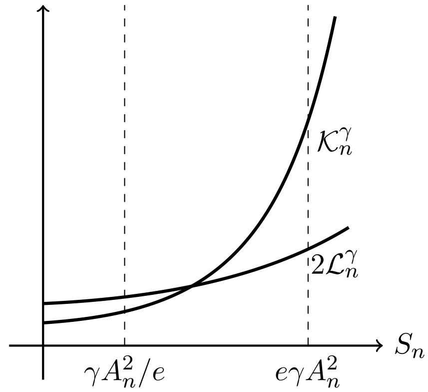
$$\mathcal{Q}_n^\gamma \le 2e^{\gamma^3 A_n^2 \bar{c}_n} \min\left\{ e^{S_n^2/(2A_n^2)} \frac{\sqrt{2\pi}}{\gamma A_n}, e^{\gamma S_n} \right\} \quad \text{if} \quad \gamma A_n^2/e < S_n < e\gamma A_n^2,$$

(4.13)

$$\mathcal{Q}_n^\gamma \le C_1 \quad \text{if} \quad S_n \ge e\gamma A_n^2.$$

Remark 4.4. In this lemma, C_1 depends on $\bar{c}_n$, γ and A_n through $\gamma^3 A_n^2 \bar{c}_n$. However from Section 4.5 on, we bound $\gamma^3 A_n^2 \bar{c}_n$ from above by a constant. Hence, C_1 can be also taken to be a constant (cf. (4.34)) not depending on γ and A_n. Also note that the interval for S_n in (4.12) is larger than the interval in (4.3).

Remark 4.5. As shown in the following figure, $2\mathcal{L}_n^\gamma$ increases more slowly with increasing S_n than $\mathcal{K}_n^\gamma$.



In Section 4.5, we introduce another capital process $\mathcal{N}_n^{\gamma_k,D}$ which contains many $\mathcal{Q}_n^\gamma$ with various betting ratios γ and we complete the proof of sharpness by the strategy based on $\mathcal{N}_n^{\gamma_k,D}$. This slow increase of $\mathcal{Q}_n^\gamma$ or $2\mathcal{L}_n^\gamma$ enables us to derive the bound (4.43) and this fact is crucial for our proof of EFKP-LIL.

Proof. We bound $\mathcal{Q}_n^\gamma = 2\mathcal{L}_n^\gamma - \mathcal{K}_n^{\gamma e}$ from above in the following three cases:

$$\text{(i) } S_n \le \gamma A_n^2/e, \quad \text{(ii) } \gamma A_n^2/e < S_n < e\gamma A_n^2, \quad \text{(iii) } S_n \ge e\gamma A_n^2,$$

Case (i) In this case $S_n/e - \gamma A_n^2/e^2 \le 0$. Hence (4.11) follows from (4.2) and $\mathcal{Q}_n^\gamma \le 2\mathcal{L}_n^\gamma$.

Case (ii) We again use $\mathcal{Q}_n^\gamma \le 2\mathcal{L}_n^\gamma$. If $\gamma A_n^2/e < S_n \le 2\gamma A_n^2/e$, then

$$\frac{S_n}{e} - \frac{\gamma A_n^2}{e^2} \le \frac{\gamma A_n^2}{e^2} \le \frac{S_n}{e}$$

and $\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n} e^{2\gamma S_n/e} \le e^{\gamma^3 A_n^2 \bar{c}_n} e^{\gamma S_n}$ from (4.2). Otherwise (4.12) follows from (4.3) and (4.4).

Case (iii) Since $S_n \geq eA_n^2\gamma > A_n^2\gamma$, by (4.8) we have $\mathcal{L}_n^\gamma \leq e^{\gamma^3 A_n^2 \bar{c}_n} e^{\gamma S_n - \gamma^2 A_n^2/2}$ and

$$\begin{aligned}\mathcal{Q}_n^\gamma &\leq 2\mathcal{L}_n^\gamma - \mathcal{K}_n^{\gamma e} \leq 2e^{\gamma^3 A_n^2 \bar{c}_n} e^{\gamma S_n - \gamma^2 A_n^2/2} - e^{-\gamma^3 e^3 A_n^2 \bar{c}_n} e^{\gamma e S_n - \gamma^2 e^2 A_n^2/2} \\ &= 2e^{\gamma^3 A_n^2 \bar{c}_n} e^{\gamma S_n - \gamma^2 A_n^2/2} \left(1 - \frac{1}{2} e^{-(1+e^3)\gamma^3 A_n^2 \bar{c}_n} e^{\gamma(e-1)S_n - (e^2-1)\gamma^2 A_n^2/2}\right).\end{aligned}$$

Hence if the right-hand side is non-positive we have $\mathcal{Q}_n^\gamma \leq 0$:

$$\begin{aligned}&S_n \geq eA_n^2\gamma \quad \text{and} \\ &\quad -(1+e^3)\gamma^3 A_n^2 \bar{c}_n - \ln 2 + \gamma(e-1)S_n - \frac{1}{2}(e^2-1)\gamma^2 A_n^2 \geq 0 \\ &\qquad \Rightarrow \quad \mathcal{Q}_n^\gamma \leq 0.\end{aligned} \tag{4.14}$$

Otherwise, write $B_n := (1+e^3)\gamma^3 A_n^2 \bar{c}_n + \ln 2$ and consider the case

$$\gamma(e-1)S_n - \frac{1}{2}(e^2-1)\gamma^2 A_n^2 \leq B_n.$$

Dividing this by $e-1$ and also considering $S_n \geq eA_n^2\gamma$, we have

$$\gamma S_n - \frac{1}{2}(e+1)\gamma^2 A_n^2 \leq \frac{B_n}{e-1}, \tag{4.15}$$

$$-S_n + eA_n^2\gamma \leq 0. \tag{4.16}$$

$\gamma \times$ (4.16) + (4.15) gives

$$\frac{1}{2}(e-1)\gamma^2 A_n^2 \leq \frac{B_n}{e-1} \quad \text{or} \quad \frac{1}{2}\gamma^2 A_n^2 \leq \frac{B_n}{(e-1)^2}.$$

Then by (4.15)

$$\gamma S_n - \frac{1}{2}\gamma^2 A_n^2 \leq \frac{B_n}{e-1} + \frac{e}{2}\gamma^2 A_n^2 \leq \frac{B_n}{e-1} + \frac{eB_n}{(e-1)^2} = \frac{(2e-1)B_n}{(e-1)^2}.$$

Hence just using $\mathcal{Q}_n^\gamma \leq 2\mathcal{L}_n^\gamma$ and (4.8) in this case, we obtain

$$\mathcal{Q}_n^\gamma \leq 2e^{\gamma^3 A_n^2 \bar{c}_n} \exp\left(\frac{(2e-1)((1+e^3)\gamma^3 A_n^2 \bar{c}_n + \ln 2)}{(e-1)^2}\right) = C_1. \tag{4.17}$$

This also covers (4.14) and we have (4.17) for the whole case (iii).

□

4.3. *Change of time scale and dividing the rounds into cycles.* For proving the sharpness we consider the change of time scale from λ to k:

$$\lambda = e^{5k\ln k} = k^{5k}.$$

By taking the derivative of $\ln\lambda = 5k\ln k$, we have $d\lambda/\lambda = 5(\ln k+1)dk$. Since $\ln k$ and $\ln k+1$ coincide within a constant factor, the integrability condition is written as

$$\int_1^\infty \psi(\lambda)e^{-\psi(\lambda)^2/2}\frac{d\lambda}{\lambda} = \infty \;\Leftrightarrow\; \int_1^\infty (\ln k)\psi(e^{5k\ln k})e^{-\psi(e^{5k\ln k})^2/2}dk = \infty.$$

Let $f(x) := \psi(e^{5x\ln x})e^{-\psi(e^{5x\ln x})^2/2}$. Since $xe^{-x^2/2}$ is decreasing for $x \ge 1$, the function $f(x)$ is decreasing for x such that $\psi(e^{5x\ln x}) \ge 1$. Thus, for sufficiently large k and x such that $k \le x \le k+1$, we have

$$\frac{1}{2}\ln(k+1)f(k+1) \le \ln k f(x+1) \le \ln x f(x) \le \ln(k+1)f(x) \le 2\ln k f(k).$$

Hence, we have

$$\int_1^\infty (\ln k)\psi(e^{5k\ln k})e^{-\psi(e^{5k\ln k})^2/2}dk = \infty$$
$$\Leftrightarrow \sum_{k=1}^\infty (\ln k)\psi(e^{5k\ln k})e^{-\psi(e^{5k\ln k})^2/2} = \infty.$$

Then, it suffices to show (4.1) if $\sum_{k=1}^\infty(\ln k)\psi(e^{5k\ln k})e^{-\psi(e^{5k\ln k})^2/2} = \infty$.

As in Chapter 5 of [18] and [15], we divide the time axis into "cycles". However, unlike in Chapter 5 of [18] and [15], our cycles are based on stopping times. Let

$$n_k := k^{5k}, \quad k = 1, 2, \ldots, \tag{4.18}$$

and define a family of stopping times

$$\tau_k := \min\left\{n \mid A_n^2 \ge n_k\right\}. \tag{4.19}$$

We define the k-th cycle by $[\tau_k, \tau_{k+1}]$, $k \ge 1$. Note that τ_k is finite for all k if and only if $A_n^2 \to \infty$. Betting strategy for the k-th cycle is based on the following betting proportion:

$$\gamma_k := \frac{\psi(n_{k+1})}{\sqrt{n_{k+1}}}k^2. \tag{4.20}$$

Note that γ_k in (4.20) is slightly different from (3.2).

For the rest of this section, we check the growth of various quantities along the cycles. Let $\omega \in \Omega_C$. For sufficiently large n,

$$|x_n| \le c_n \le C\frac{A_n}{\psi(A_n^2)^3}. \tag{4.21}$$

Furthermore $A_n^2 = A_{n-1}^2 + x_n^2$. This allows us to bound x_n^2 and A_n^2 in terms of A_{n-1}^2. By squaring (4.21) we have

$$x_n^2 \le C^2\frac{A_{n-1}^2}{\psi(A_n^2)^6 - C^2} \tag{4.22}$$

and

$$A_n^2 = A_{n-1}^2 + x_n^2 \le A_{n-1}^2(1+\frac{C^2}{\psi(A_n^2)^6 - C^2}) = A_{n-1}^2\frac{\psi(A_n^2)^6}{\psi(A_n^2)^6 - C^2}. \tag{4.23}$$

Since $\psi(A_n^2)^6/(\psi(A_n^2)^6 - C^2) \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\frac{A_n^2}{A_{n-1}^2} = 1.$$

Note that $A_{\tau_k-1}^2 < n_k \le A_{\tau_k}^2$ by the definition of τ_k. Hence for $\omega \in \Omega_C$ we also have

$$\lim_{k\to\infty}\frac{A_{\tau_k}^2}{n_k} = 1. \tag{4.24}$$

The limits in the following lemma will be useful for our argument.

LEMMA 4.6. *For $\omega \in \Omega_C$*

$$\lim_{k\to\infty}\frac{\psi^U(n_k)}{\psi(n_{k+1})} = 1, \quad \lim_{k\to\infty}\frac{k^5A_{\tau_k}^2}{n_{k+1}} = e^{-5}, \quad \lim_{k\to\infty}\gamma_kA_{\tau_k}\psi(n_{k+1}) = 0. \tag{4.25}$$

PROOF. All of $\psi^U(n_k)$, $\psi^U(n_{k+1})$, $\psi^L(n_k)$, $\psi^L(n_{k+1})$, $\psi(n_{k+1})$, $\psi(n_{k+1}/k^4)$ are of the order

$$\sqrt{2\ln\ln e^{5k\ln k}}(1+o(1)) = \sqrt{2\ln k}(1+o(1)) \tag{4.26}$$

as $k \to \infty$ and the first equality holds by (1.5). The second equality holds by (4.24) and

$$\lim_{k\to\infty}\frac{k^5n_k}{n_{k+1}} = \lim_{k\to\infty}\frac{k^{5(k+1)}}{(k+1)^{5(k+1)}} = \lim_{k\to\infty}\left(1-\frac{1}{k+1}\right)^{5(k+1)} = e^{-5}.$$

Then $A_{\tau_k}^2/n_{k+1} = (1+o(1))n_k/n_{k+1} = O(k^{-5})$ and the third equality holds by

$$\gamma_k A_{\tau_k}\psi(n_{k+1}) \le \psi(n_{k+1})^2 k^2((1+\delta)n_k/n_{k+1})^{1/2} \to 0 \qquad (k\to\infty).$$

□

4.4. *Stopping times for aborting and sequential freezing for each cycle.* In (4.32) of the next section we will introduce another capital process $\mathcal{M}_n^{\gamma_k,k}$, which will be employed in each cycle. Here we introduce some stopping times for aborting the cycle and for sequential freezing of accounts in $\mathcal{M}_n^{\gamma_k,k}$.

We say that we *abort* the k-th cycle, when we freeze all accounts in the k-th cycle and wait for the $(k+1)$-th cycle. There are two cases for aborting the k-th cycle. The first case is when some c_n is too large for $\omega \in \Omega_C$. Define

$$\sigma_{k,C} := \min\left\{n \ge \tau_k \mid c_n\psi(A_{\tau_k}^2)^3 > (1+\delta)CA_{n-1}\right\}. \tag{4.27}$$

We will abort the k-th cycle if $\sigma_{k,C} < \tau_{k+1}$. Note that for $\omega \in \Omega_C$, there exists $k_1(\omega)$ such that

$$\sigma_{k,C} = \infty, \quad \text{for} \quad k \ge k_1(\omega). \tag{4.28}$$

Another case is when S_n is too large. Define

$$\nu_k := \min\{n \ge \tau_k \mid A_n\psi(A_n^2) < S_n\}. \tag{4.29}$$

If $\nu_k < \tau_{k+1}$, then Skeptic is happy to abort the k-th cycle, because he wants to force $S_n \ge A_n\psi(A_n^2)$ *i.o.* The above two stopping times will be used in the final construction of a dynamic strategy in Section 4.6.

For each cycle, we define another family of stopping times indexed by $w = 1,\dots,\lceil \ln k\rceil$, by

$$\tau_{k,w} := \min\left\{n \mid A_n^2 \ge e^{2(w+2)}\frac{n_{k+1}}{k^4}\right\}. \tag{4.30}$$

for sequential freezing of accounts of $\mathcal{M}_n^{\gamma_k,k}$ in (4.32). We have $\tau_k \le \tau_{k,w}$ for $k \ge 1$ and $w \ge 1$, because

$$\frac{n_{k+1}}{k^4} = \frac{(k+1)^{5(k+1)}}{k^4} > k^{5k} = n_k.$$

LEMMA 4.7. *Let $\omega \in \Omega_C$. $\tau_{k,\lceil \ln k\rceil} \le \tau_{k+1}$ for sufficiently large k.*

PROOF. By $A^2_{\tau_{k,w}-1} \le e^{2(w+2)} n_{k+1}/k^4$ and by (4.22), for sufficiently large k we have

$$x^2_{\tau_{k,w}} \le (1+\delta) C^2 \frac{A^2_{\tau_{k,w}-1}}{\psi(A^2_{\tau_k})^6} \le \frac{(1+\delta)C^2}{\psi(A^2_{\tau_k})^6} \times \frac{e^{2(w+2)} n_{k+1}}{k^4}$$

and

$$A^2_{\tau_{k,w}} \le A^2_{\tau_{k,w}-1} + x^2_{\tau_{k,w}} \le (1+\delta) e^{2(w+2)} \frac{n_{k+1}}{k^4}. \tag{4.31}$$

Then

$$A^2_{\tau_{k,\lceil \ln k \rceil}} \le (1+\delta)\left(e^{2(\ln k + 2)} \frac{n_{k+1}}{k^4}\right) = (1+\delta) e^4 \frac{n_{k+1}}{k^2} \le n_{k+1} \le A^2_{\tau_{k+1}}.$$

□

We also compare $\tau_{k,w}$ to $\sigma_{\gamma_k e^{-w+1}}$ defined in (2.3). This is needed for applying the bounds derived in previous sections to $\mathcal{M}_n^{\gamma_k,k}$ in the next section.

LEMMA 4.8. *Let $\omega \in \Omega_C$. $\tau_{k,w} \le \sigma_{\gamma_k e^{-w+1}}$ for sufficiently large k.*

PROOF. By (4.31) and by Lemma 3.1, for sufficiently large k

$$\begin{aligned} \gamma_k e^{-w+1} \bar{c}_{\tau_{k,w}} &\le \frac{\psi(n_{k+1})}{\sqrt{n_{k+1}}} k^2 e^{-w+1} \times (1+\delta)^2 C \frac{e^{w+2}\sqrt{n_{k+1}}}{k^2 \psi(A^2_{\tau_k})^3} \\ &\le (1+\delta)^2 C e^3 \frac{\psi(n_{k+1})}{\psi(A^2_{\tau_k})^3} \le \delta, \end{aligned}$$

because $\psi(n_{k+1})/\psi(A^2_{\tau_k})^3 \to 0$ as $k \to \infty$ by (4.26). □

4.5. *Further discrete mixture of processes for each cycle with sequential freezing.* We introduce another discrete mixture of capital processes for the k-th cycle. Define

$$\mathcal{M}_n^{\gamma_k,k} := \frac{1}{\lceil \ln k \rceil} \sum_{w=1}^{\lceil \ln k \rceil} \mathcal{Q}^{\gamma_k e^{-w}}_{\min(n,\tau_{k,w})} = \frac{1}{\lceil \ln k \rceil} \sum_{w=1}^{\lceil \ln k \rceil} (2\mathcal{L}^{\gamma_k e^{-w}}_{\min(n,\tau_{k,w})} - \mathcal{K}^{\gamma_k e^{-w+1}}_{\min(n,\tau_{k,w})}). \tag{4.32}$$

As we show in Lemma 4.10, the growth of $\mathcal{M}_n^{\gamma_k,k}$ can be bounded from above because of splitting the initial capital into $\lceil \ln k \rceil$ accounts and applying $\mathcal{Q}^{\gamma_k e^{-w}}_{\min(n,\tau_{k,w})}$ to each account. This boundedness of $\mathcal{M}_n^{\gamma_k,k}$ is important because we use $\mathcal{M}_n^{\gamma_k,k}$ in the form of (4.37) below. Note that the w-th account in the sum of $\mathcal{M}_n^{\gamma_k,k}$is frozen at the stopping time $\tau_{k,w}$. This is needed since the bound for c_n is growing even during the k-th cycle.

In order to bound $\mathcal{M}_n^{\gamma_k,k}$, we first bound C_1 in (4.10) for each w in the sum of (4.32) by a constant independent of n. Note that we only need to consider $n \le \tau_{k,w}$ for the w-th account.

LEMMA 4.9. *Let $\omega \in \Omega_C$. $(\gamma_k e^{-w})^3 A_n^2 \bar{c}_n$ and hence C_1 are bounded from above by*

$$(\gamma_k e^{-w})^3 A_n^2 \bar{c}_n \le (1+\delta)^5 C e^6, \tag{4.33}$$

$$C_1 \le 2e^{(1+\delta)^5 C e^6} \exp\left(\frac{(2e-1)((1+\delta)^5 C e^6(1+e^3)+\ln 2)}{(e-1)^2}\right) =: \bar{C}_1, \tag{4.34}$$

for sufficiently large k.

PROOF. By (4.26), for sufficiently large k

$$\frac{\psi(n_{k+1})}{\psi(A^2_{\tau_{k,w}})} \le \frac{\psi(n_{k+1})}{\psi(n_k)} \le 1+\delta. \tag{4.35}$$

Thus

$$\begin{aligned}
\gamma_k^3 e^{-3w} A^2_{\min(n,\tau_{k,w})} \bar{c}_{\min(n,\tau_{k,w})} &\le \gamma_k^3 e^{-3w} \times A^2_{\tau_{k,w}} \times \bar{c}_{\min(n,\tau_{k,w})} \\
&\le \frac{\psi(n_{k+1})^3}{n_{k+1}^{3/2}} k^6 e^{-3w} \times A^2_{\tau_{k,w}} \times (1+\delta) C \frac{A_{\tau_{k,w}}}{\psi(A^2_{\tau_k})^3} \\
&\le (1+\delta) C \frac{\psi(n_{k+1})^3}{\psi(A^2_{\tau_k})^3} k^6 e^{-3w} \frac{A^3_{\tau_{k,w}}}{n_{k+1}^{3/2}} \le (1+\delta)^5 C e^6.
\end{aligned}$$

□

LEMMA 4.10. *Let $\omega \in \Omega_C$. For sufficiently large k,*

$$\mathcal{M}_n^{\gamma_k,k} \le \bar{C}_1 + \frac{2}{\lceil \ln k \rceil} e^{(1+\delta)^5 C e^6} \max_{\gamma \in [\gamma_k/k, \gamma_k]} \left(\min\{ e^{S_n^2/(2n)} \frac{\sqrt{2\pi}}{\gamma A_n}, e^{\gamma S_n} \} \right), \quad n \in [\tau_k, \tau_{k+1}], \tag{4.36}$$

where $\bar{C}_1$ is given by the right-hand side of (4.34).

PROOF. We have $|\gamma_k e^{-w}\bar{c}_{\min(n,\tau_{k,w})}| \le |\gamma_k e^{-w+1}\bar{c}_{\min(n,\tau_{k,w})}| \le \delta$ by Lemma 4.8. Then we can complete the proof of (4.36) by Lemma 4.3 and Lemma 4.8 because the length of the interval

$$\left\{ w \mid \frac{S_n}{ne} < \gamma e^{-w} < \frac{S_n e}{n} \right\}$$

is equal to 2. □

As in Chapter 5 of Shafer and Vovk [18], we use $\mathcal{M}_n^{\gamma_k,k}$ in the following form.

$$\mathcal{N}_n^{\gamma_k,D} := \alpha + \frac{1}{D}\lceil \ln k \rceil \psi(n_{k+1}) e^{-\psi(n_{k+1})^2/2}(\alpha - \mathcal{M}_{n-\tau_k}^{\gamma_k,k}),$$
$$\alpha = 1 - \frac{2}{e},\ D = \frac{24\sqrt{2\pi}e^{(1+\delta)^5 e^6 C} + 4\bar{C}_1}{\alpha}. \tag{4.37}$$

Here we give a specific value of D for definiteness, but from the proof below it will be clear that any sufficiently large D can be used. Since the strategy for $\mathcal{M}_{n-\tau_k}^{\gamma_k,k}$ is applied only to x_n's in the cycle, $\alpha = \mathcal{N}_{\tau_k}^{\gamma_k,D} = \mathcal{M}_0^{\gamma_k}$. Concerning $\mathcal{N}_n^{\gamma_k,D}$ we prove the following two propositions.

PROPOSITION 4.11. *Let $\omega \in \Omega_C$. Suppose that*

$$-A_n\psi^U(A_n^2) \le S_n \le A_n\psi(A_n^2), \qquad \forall n \in [\tau_k, \tau_{k+1}]. \tag{4.38}$$

and $\tau_{k+1} < \sigma_{k,C}$. Then for sufficiently large k

$$\mathcal{N}_n^{\gamma_k,D} \ge \frac{\alpha}{2}, \qquad \forall n \in [\tau_k, \tau_{k+1}], \tag{4.39}$$

and

$$\mathcal{N}_{\tau_{k+1}}^{\gamma_k,D} \ge \alpha\left(1 + \frac{1-\delta}{D}\lceil \ln k \rceil \psi(n_{k+1}) e^{-\psi(n_{k+1})^2/2}\right). \tag{4.40}$$

PROOF. In our proof we denote $t = n - \tau_k$, $S_t = S_n - S_{\tau_k}$ and $A_t^2 = A_n^2 - A_{\tau_k}^2$ for $n > \tau_k$. For proving (4.39), we use (4.36) for S_t. We bound $\mathcal{M}_t^{\gamma_k,k}$ from above. By the term $\dfrac{2}{\lceil \ln k \rceil}$ on the right-hand side of (4.36), it suffices to show

$$S_t \le A_{\tau_k}\psi^U(A_{\tau_k}^2) + \sqrt{A_{\tau_k}^2 + A_t^2}\,\psi(A_{\tau_k}^2 + A_t^2)$$
$$\Rightarrow\ \psi(n_{k+1})e^{-\psi(n_{k+1})^2/2} 2e^{(1+\delta)^5 e^6 C}\min\{e^{S_t^2/(2A_t^2)}\frac{\sqrt{2\pi}}{\gamma A_t}, e^{\gamma S_t}\} \le \frac{D\alpha}{4},$$

$$\forall\gamma \in [\gamma_k/k, \gamma_k],\ \forall t \in [0, \tau_{k+1} - \tau_k]$$

for sufficient large k. Let

$$c_1 = \frac{9}{(1+2\delta)^2} \quad \text{s.t.} \quad \frac{1}{2} - \frac{1}{\sqrt{c_1}} - \delta > 0. \tag{4.41}$$

We distinguish two cases:

$$\text{(a) } A_t^2 \le \frac{\psi(n_{k+1})^2}{c_1\gamma^2}, \quad \text{(b) } \frac{\psi(n_{k+1})^2}{c_1\gamma^2} < A_t^2 \le A_{\tau_{k+1}}^2 - A_{\tau_k}^2.$$

For case (a), $A_{\tau_k}\psi^U(A_{\tau_k}^2) \le (1+\delta)A_{\tau_k}\psi(n_{k+1})$ by the first equality in Lemma 4.6 for sufficiently large k. Also $\psi(A_{\tau_k}^2 + A_t^2) \le \psi(n_{k+1})$. Hence in this case

$$\gamma S_t \le \left((1+\delta)\gamma A_{\tau_k} + \sqrt{\gamma^2 A_{\tau_k}^2 + \psi(n_{k+1})^2/c_1}\right)\psi(n_{k+1}).$$

Then for $\gamma \le \gamma_k$ by the third equality in Lemma 4.6

$$\begin{aligned}\gamma S_t &\le \left((1+\delta)\gamma_k A_{\tau_k} + \sqrt{\gamma_k^2 A_{\tau_k}^2 + \psi(n_{k+1})^2/c_1}\right)\psi(n_{k+1}) \\ &= \psi(n_{k+1})^2\left(\frac{1}{\sqrt{c_1}} + \delta\right)\end{aligned} \tag{4.42}$$

for sufficiently large k. Since

$$\begin{aligned}&\psi(n_{k+1})e^{-\psi(n_{k+1})^2/2}2e^{(1+\delta)^5e^6C}e^{\gamma S_t} \\ &\quad\le \psi(n_{k+1})\exp\left(-\psi(n_{k+1})^2\big(\frac{1}{2} - \frac{1}{\sqrt{c_1}} - \delta\big)\right)2e^{(1+\delta)^5e^6C} \\ &\quad\to 0 \quad (k\to\infty),\end{aligned}$$

we have $\mathcal{N}_n^{\gamma_k,D} \ge \alpha/2$ uniformly in $\gamma \in [\gamma_k/k, \gamma_k]$.

For case (b), $\psi(n_{k+1})/\sqrt{c_1} < \gamma A_t$ and $S_t \le \left((1+\delta)A_{\tau_k} + \sqrt{A_{\tau_k}^2 + A_t^2}\right)\psi(n_{k+1})$. Hence

(4.43)

$$\begin{aligned}&\psi(n_{k+1})e^{-\psi(n_{k+1})^2/2} \times 2e^{(1+\delta)^5e^6C}e^{S_t^2/(2A_t^2)}\frac{\sqrt{2\pi}}{\gamma A_t} \\ &\le \psi(n_{k+1})e^{-\psi(n_{k+1})^2/2}\end{aligned}$$

$$\times \frac{2e^{(1+\delta)^5 e^6 C}\sqrt{2\pi}\sqrt{c_1}}{\psi(n_{k+1})} \exp\left(\frac{\left((1+\delta)A_{\tau_k} + \sqrt{A_{\tau_k}^2 + A_t^2}\right)^2}{2A_t^2}\psi(n_{k+1})^2\right)$$

$$= 2e^{(1+\delta)^5 e^6 C}\sqrt{2\pi}\sqrt{c_1} \exp\left(\frac{(1+(1+\delta)^2)A_{\tau_k}^2 + 2(1+\delta)A_{\tau_k}\sqrt{A_{\tau_k}^2 + A_t^2}}{2A_t^2}\psi(n_{k+1})^2\right).$$

For $\gamma \le \gamma_k$,

$$\frac{\psi(n_{k+1})^2}{c_1\gamma^2} < A_t^2 \;\Rightarrow\; \frac{A_{\tau_k}^2}{A_t^2}\psi(n_{k+1})^2 < c_1\gamma^2 A_{\tau_k}^2 \le c_1\gamma_k^2 A_{\tau_k}^2 = c_1\frac{A_{\tau_k}^2}{n_{k+1}}k^4\psi(n_{k+1})^2$$
$$= O(k^{-1}\ln k).$$

Hence $\psi(n_{k+1})^2 A_{\tau_k}^2/A_t^2 \to 0$ as $k \to \infty$. Similarly $\psi(n_{k+1})^2 A_{\tau_k}/A_t \to 0$ as $k \to \infty$, because $\psi(n_{k+1})^2 A_{\tau_k}/A_t = O(k^{-1/2}(\ln k)^{3/2})$. Therefore the right-hand side of (4.43) is bounded from above by $2e^{(1+\delta)^5 e^6 C}\sqrt{2\pi}\sqrt{c_1}(1+\delta)$ for sufficiently large k and

$$\psi(n_{k+1})e^{-\psi(n_{k+1})^2/2} \times 2e^{(1+\delta)^5 e^6 C} e^{S_t^2/(2A_t^2)}\frac{\sqrt{2\pi}}{\gamma A_t} \le \frac{D\alpha}{4},$$

with the choice of D in (4.37) and c_1 in (4.41). This proves (4.39).

Now we prove (4.40). We focus on the w-th account when $n \ge \tau_{k,w}$. Recall that in this proof we have been denoting $A_t^2 = A_n^2 - A_{\tau_k}^2$. Similarly we denote $A_{\tau_{k,w}}^2$ instead of $A_{\tau_{k,w}}^2 - A_{\tau_k}^2$. Thus

$$e^{2(w+2)}\frac{n_{k+1}}{k^4} - A_{\tau_k}^2 \le A_{\tau_{k,w}}^2. \tag{4.44}$$

We will show that $\limsup_{k\to\infty} \mathcal{M}_{\tau_{k+1}-\tau_k}^{\gamma_k,k} \le 0$, if

(4.45)
$$S_{\tau_{k,w}} \le A_{\tau_k}\psi(A_{\tau_k}^2) + A_{\tau_{k,w}}\psi(A_{\tau_{k,w}}^2) \le \psi(n_{k+1})\left\{A_{\tau_k} + A_{\tau_{k,w}}\right\} \le 2\psi(n_{k+1})A_{\tau_{k,w}}.$$

We bound

$$\mathcal{L}_{\tau_{k,w}}^{\gamma_k e^{-w},k} := \int_{2/e}^{1} \exp\left(u\gamma_k e^{-w}S_{\tau_{k,w}} - u^2\gamma_k^2 e^{-2w}A_{\tau_{k,w}}^2/2\right)du$$

from above. Because $u\gamma_k e^{-w}S_{\tau_{k,w}} - u^2\gamma_k^2 e^{-2w}A_{\tau_{k,w}}^2/2$ is maximized at $u = S_{\tau_{k,w}}/(\gamma_k e^{-w}A_{\tau_{k,w}}^2)$ and

$$\frac{S_{\tau_{k,w}}}{\gamma_k e^{-w}A_{\tau_{k,w}}^2} \le \frac{2\psi(n_{k+1})A_{\tau_{k,w}}}{(\psi(n_{k+1})k^2/\sqrt{n_{k+1}})e^{-w}A_{\tau_{k,w}}^2} \le \frac{2\sqrt{n_{k+1}}}{k^2 e^{-w}A_{\tau_{k,w}}} \le \frac{2}{e^2} \le \frac{2}{e},$$

the integrand in $\mathcal{L}^{\gamma_k e^{-w},k}_{\tau_{k,w}}$ is maximized at $2/e$ and we have

$$\mathcal{L}^{\gamma_k e^{-w},k}_{\tau_{k,w}} \leq \exp\left(\frac{2}{e}\gamma_k e^{-w} S_{\tau_{k,w}} - \frac{2\gamma_k^2 e^{-2w} A^2_{\tau_{k,w}}}{e^2}\right).$$

By (4.44) and (4.45), for sufficiently large k,

$$\begin{aligned}\frac{2}{e}\gamma_k e^{-w} S_{\tau_{k,w}} - \frac{2\gamma_k^2 e^{-2w} A^2_{\tau_{k,w}}}{e^2} &\leq \frac{4\gamma_k \psi(n_{k+1}) A_{\tau_{k,w}}}{e^{w+1}} - \frac{2\gamma_k^2 A^2_{\tau_{k,w}}}{e^{2(w+1)}} \\ &= \frac{\psi(n_{k+1})^2 k^2 A_{\tau_{k,w}}}{\sqrt{n_{k+1}}e^w}\left(\frac{4}{e} - \frac{2k^2 A_{\tau_{k,w}}}{e^2\sqrt{n_{k+1}}e^w}\right) \\ &\leq \frac{\psi(n_{k+1})^2 k^2 A_{\tau_{k,w}}}{\sqrt{n_{k+1}}e^w}\left(\frac{4}{e} - \frac{2}{e^2}\sqrt{e^4 - \frac{(1+\delta)k^4 n_k}{n_{k+1}e^{2w}}}\right) \\ &\leq -\psi(n_{k+1})^2 \frac{k^2}{\sqrt{n_{k+1}}e^w} \times \frac{\sqrt{n_{k+1}}e^{w+2}}{k^2} \times \frac{1}{2} \\ &= -\frac{e^2\psi(n_{k+1})^2}{2}.\end{aligned}$$

The last inequality holds because $\lim_{k\to\infty} k^4 n_k/n_{k+1} = 0$ and $4/e - 2 < -1/2$. Hence $\mathcal{L}^{\gamma_k e^{-w},k}_{\tau_{k,w}} \to 0$ uniformly in $1 \leq w \leq \lceil \ln k \rceil$. This implies $\limsup_{k\to\infty} \mathcal{M}^{\gamma_k,k}_{\tau_{k+1}-\tau_k} \leq 0$. □

PROPOSITION 4.12. *Let $\omega \in \Omega_C$. Suppose that $\nu_k \leq \min(\tau_{k+1}, \sigma_{k,C})$ and*

$$-A_n \psi^U(A_n^2) \leq S_n, \ \forall n \in [\tau_k, \nu_k].$$

Then for sufficiently large k

$$\mathcal{N}^{\gamma_k,D}_{\nu_k} \geq \frac{\alpha}{2}.$$

PROOF. As in the proof of the previous lemma, we denote $t = n - \tau_k$, $S_t = S_n - S_{\tau_k}$ and $A_t^2 = A_n^2 - A^2_{\tau_k}$. We distinguish two cases:

$$\text{(a) } A^2_{\nu_k} \leq \frac{\psi(n_{k+1})^2}{c_1\gamma^2}, \quad \text{(b) } \frac{\psi(n_{k+1})^2}{c_1\gamma^2} < A^2_{\nu_k} \leq A^2_{\tau_{k+1}} - A^2_{\tau_k}.$$

For case (a), for sufficiently large k and for any $\gamma \leq \gamma_k$, as in (4.42),

$$\gamma S_{\nu_k} \leq \gamma\left(S_{\nu_k - 1} + c_{\nu_k}\right)$$

$$\le \gamma \left(\left((1+\delta)A_{\tau_k} + \sqrt{A_{\tau_k}^2 + A_{\nu_k-1}^2} \right) \psi(n_{k+1}) + (1+\delta)C \frac{\sqrt{A_{\tau_k}^2 + A_{\nu_k-1}^2}}{\psi(A_{\tau_k}^2)^3} \right)$$
$$\le \psi(n_{k+1})^2 \left(\frac{1}{\sqrt{c_1}} + \delta \right)$$

and

$$\psi(n_{k+1}) e^{-\psi(n_{k+1})^2/2} 2e^{(1+\delta)^5 e^6 C} e^{\gamma S_{\nu_k}} \to 0 \quad (k \to \infty).$$

Hence $\mathcal{N}_{\nu_k}^{\gamma_k, D} \ge \alpha/2$ uniformly in $\gamma \in [\gamma_k/k, \gamma_k]$.

For case (b), S_{ν_k} can be bounded as

$$\begin{aligned} S_{\nu_k} &\le S_{\nu_k-1} + c_{\nu_k} \le S_{\nu_k-1} + (1+\delta)C \frac{\sqrt{A_{\tau_k}^2 + A_{\nu_k-1}^2}}{\psi(A_{\tau_k}^2)^3} \\ &\le \left((1+\delta)A_{\tau_k} + \sqrt{A_{\tau_k}^2 + A_{\nu_k}^2} \right) \psi(n_{k+1}) + (1+\delta)C \frac{\sqrt{A_{\tau_k}^2 + A_{\nu_k}^2}}{\psi(A_{\tau_k}^2)^3} \\ &\le \left((1+\delta)A_{\tau_k} + \sqrt{A_{\tau_k}^2 + A_{\nu_k}^2} \left(1 + \frac{(1+\delta)C}{\psi(A_{\tau_k}^2)^3 \psi(n_{k+1})} \right) \right) \psi(n_{k+1}) \end{aligned}$$

by (4.35). Put

$$q_k^2 := \frac{A_{\tau_k}^2}{A_{\nu_k}^2} \le \frac{c_1 \gamma_k^2}{\psi(n_{k+1})^2}, \qquad s_k := \frac{(1+\delta)C}{\psi(A_{\tau_k}^2)^3 \psi(n_{k+1})},$$

so that $\lim_k q_k \psi(n_{k+1})^2 = 0$ and $\lim_k s_k \psi(n_{k+1})^2 = 0$. Then for sufficiently large k

$$\begin{aligned} &\frac{S_{\nu_k}^2}{2A_{\nu_k}^2} \\ &\le \left((1+\delta)^2 \frac{q_k^2}{2} + (1+\delta)(1+s_k) q_k \sqrt{1+q_k^2} + (1+s_k)^2 \left(\frac{1}{2} + \frac{q_k^2}{2} \right) \right) \psi(n_{k+1})^2 \\ &\le \frac{\psi(n_{k+1})^2}{2} + \delta. \end{aligned}$$

Then

$$\begin{aligned} &\psi(n_{k+1}) e^{-\psi(n_{k+1})^2/2} \times 2e^{(1+\delta)^5 e^6 C} e^{S_{\nu_k}^2/(2A_{\nu_k}^2)} \frac{\sqrt{2\pi}}{\gamma A_{\nu_k}} \\ &\le 2e^{(1+\delta)^5 e^6 C + \delta} \sqrt{2\pi c_1} e^{\delta} \le \frac{D\alpha}{4}. \end{aligned}$$

□

4.6. *Dynamic strategy forcing the sharpness.* Finally, we prove Proposition 4.1. We assume, which we may do by the validity result, that Skeptic already employs a strategy forcing $S_n \geq -A_n\psi^U(A_n^2)$ *a.a.* for $\omega \in \Omega_C$ and we define $\vartheta_k := \min\{n \geq \tau_k \mid -A_n\psi^U(A_n^2) > S_n\}$. In addition to this strategy consider the following strategy, based on Proposition 4.11.

Start with initial capital $\mathcal{K}_0 = \alpha$.
Set $k = 1$.
Do the following repeatedly:
1. Apply the capital process $\mathcal{N}_{\nu_k}^{\gamma_k, D}$ with the strategy in Proposition 4.11 for $n \in [\tau_k, \tau_{k+1}]$.
   If $\tau_{k+1} < \min(\sigma_{k,C}, \nu_k, \vartheta_k)$, then go to 2. Otherwise go to 3.
2. Let $k = k+1$. Go to 1.
3. Freeze all accounts in the capital process in $\mathcal{N}_{\nu_k}^{\gamma_k, D}$ and wait until $k' = \min\{k' > k \mid -\sqrt{\tau_{k'}}\psi^U(\tau_{k'}) \leq S_{\tau_{k'}} \leq \sqrt{\tau_{k'}}\psi(\tau_{k'})\}$. Set $k = k'$ and go to 1.

By this strategy Skeptic keeps his capital non-negative for every path ω. For $\omega \in \Omega_0$, $\tau_k = \infty$ for some k and Skeptic stays in Step 1 forever. For $\omega \in \Omega_{=\infty}$, Step 3 is performed infinite number of times, but the overshoot of $|x_n|$ in Step 3 does not make Skeptic bankrupt by Proposition 4.12. Now consider $\omega \in \Omega_C$. Since Skeptic already employs a strategy forcing $S_n \geq -A_n\psi^U(A_n^2)$ *a.a.*, the lower bound in (4.38) violated only finite number of times. By $\omega \in \Omega_C$, $n \geq \sigma_{k,C}$ is happens only finite number of times. Hence if $S_n \leq A_n\psi(A_n^2)$ *a.a.*, then Step 3 is performed only finite number of times and there exists k_0 such that only Step 2 is repeated for all $k \geq k_0$. Now by Proposition 4.11, Skeptic multiplies his capital at least by

$$1 + \frac{1-\delta}{D}\lceil \ln k \rceil \psi(n_{k+1}) e^{-\psi(n_{k+1})^2/2}.$$

for each iteration of Step 2. Then

$$\begin{aligned} \frac{1-\delta}{D} & \sum_{k=k_0}^{\infty} \lceil \ln k \rceil \psi(n_{k+1}) e^{-\psi(n_{k+1})^2/2} \\ & \leq \prod_{k=k_0}^{\infty} \left(1 + \frac{1-\delta}{D}\lceil \ln k \rceil \psi(n_{k+1}) e^{-\psi(n_{k+1})^2/2}\right). \end{aligned} \tag{4.46}$$

Since the left-hand side diverges to infinity, the above strategy forces the sharpness.

APPENDIX A: EXAMPLES OF MARTINGALES WITHOUT THE SECOND MOMENT

As discussed after Theorem 1.1 we are not assuming the existence of the second moment of x_n. Then the process can not be normalized by a quantity based on the second moment and the self-normalization becomes essential. Here are some examples.

Let $W > 1$ be a random variable such that $E(W) < \infty$ but $E(W^2) = \infty$. Let $\alpha_n \in (0,1)$, $n = 1, 2, \dots$, be a sequence of positive reals such that $\sum_n \alpha_n < \infty$. Define x_n, $n = 1, 2, \dots$ by

$$P(x_n = W) = P(x_n = -W) = \frac{\alpha_n}{2}, \qquad P(x_n = 1) = P(x_n = -1) = \frac{1-\alpha_n}{2}.$$

Here the sign of x_n is independent of $W, x_1, \dots, x_{n-1}$. Let $\mathcal{F}_n$, $n = 1, \dots,$ be the σ-field generated by $W, x_1, \dots, x_n$. $\mathcal{F}_0$ is the σ-field generated by W. Note that $E|x_n| < \infty$, $0 = E(x_n) = E(x_n \mid \mathcal{F}_{n-1})$, but $E(x_n^2) = \infty$ and the conditional variance does not exist. We let $c_n = W$. $P(|x_n| = W \ i.o.) = 0$, because $\sum_n \alpha_n < \infty$. Hence $x_n = \pm 1$ $a.a.$ and $A_n^2 = O(n)$. Then our result holds but the result of Einmahl and Mason (1990) or the result of de la Peña, Klass and Lai (2004) can not be applied.

The above simple example can be generalized as follows.

Let $W_n, n = 1, 2, \dots,$ be a sequence of positive random variables with $E(W_n) < \infty$, $E(W_n^2) = \infty$. Assume that W_n converges to a positive random variable W almost surely. Let $\epsilon_n, n = 1, 2, \dots,$ be independently and identically distributed random variables over the interval $[-1, 1]$. We assume that $\{\epsilon_n\}$ are independent of $\{W_n\}$, $E(\epsilon_n) = 0$, $E(\epsilon_n^2) = \sigma^2 > 0$. Let $\mathcal{F}_n$ be the σ-field generated by $\{W_1, \dots, W_n, \epsilon_1, \dots, \epsilon_n\}$. Let $c_n = W_{n-1}$, $S_0 = 0$.

$$x_n = S_n - S_{n-1} = W_{n-1}\epsilon_n, \quad n = 1, 2, \dots.$$

Then $E(x_n^2) = \infty$. A_n is of order $O(n)$ by the existence of $W_\infty = \lim_n W_n$ and $\limsup_n c_n \psi(A_n^2)^3 / A_n = 0$ holds.

This example can be further generalized to the case that c_n grows polynomially in n.

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