Pathwise Uniqueness of the Stochastic Heat Equation with Spatially Inhomogeneous White Noise

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Abstract. We study the solutions of the stochastic heat equation driven by spatially inhomogeneous multiplicative white noise based on a fractal measure. We prove pathwise uniqueness for solutions of this equation when the noise coefficient is Hölder continuous of index $\gamma > 1 - \frac{\eta}{2(\eta+1)}$. Here $\eta \in (0,1)$ is a constant that defines the spatial regularity of the noise.

1 Introduction and Main Results

We study solutions of the stochastic heat equation with spatially inhomogeneous white noise which is given by

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\Delta u(t,x) + \sigma(t,x,u(t,x))\dot{W}, \quad t \geq 0, \quad x \in \mathbb{R}. \tag{1.1}$$

Here Δ denotes the Laplacian and $\sigma(t,x,u) : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with at most linear growth in the u variable. We assume that the noise $\dot{W}$ is white noise on $\mathbb{R}_+ \times \mathbb{R}$ based on a σ-finite measure $\mu(dx)dt$. Equations like (1.1) may arise from scaling limits of critical interacting branching particle systems. For example, in the case where $\sigma(t,x,u) = \sqrt{u}$ and $\mu(dx) := dx$, the equation describes the evolution in time and space of the density of the classical *super-Brownian motion* (see e.g. [21]). If μ is any finite measure and $\sigma(u) = \sqrt{u}$, then (1.1) describes the evolution of the density of *catalytic super-Brownian motion* with catalyst $\mu(dx)$ (see e.g. [26]).

In this work we consider the problem of pathwise uniqueness for solutions of (1.1) where $\sigma(\cdot,\cdot,u)$ is Hölder continuous in u and $\dot{W}$ is a spatially inhomogeneous Gaussian white noise based on a measure $\mu(dx)dt$. More precisely W is a mean zero Gaussian process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, where $(\mathcal{F}_t)$ satisfies the usual hypothesis and we assume that W has the following properties. We denote by

$$W_t(\phi) = \int_0^t \int_{\mathbb{R}} \phi(s,y) W(dyds), \quad t \geq 0,$$

the stochastic integral of a function ϕ with respect to W. We denote by $\mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ the space of compactly supported infinitely differentiable functions on $\mathbb{R}_+ \times \mathbb{R}$. We assume that W has the following covariance structure

$$E(W_t(\phi)W_t(\psi)) = \int_0^t \int_{\mathbb{R}} \phi(s,y)\psi(s,y)\mu(dy)ds, \quad t \geq 0,$$

for $\phi, \psi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$. Assume that the measure μ satisfies the following conditions:

(i) There exists $\eta \in (0,1)$ such that

$$\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |x-y|^{-\eta+\varepsilon} \mu(dy) < \infty, \quad \forall \varepsilon > 0, \tag{1.2}$$

(ii)

$$\mathrm{cardim}(\mu) = \eta.$$

Note that (ii) means that there exists a Borel set $A \subset \mathbb{R}$ of Minkowski dimension η such that $\mu(A^c) = 0$, and this fails for $\eta' < \eta$ (see Definition 1.3).

In what follows, if a white noise is based on the measure $dx \times dt$ (that is $\mu(dx)$ is the Lebesgue measure), we will call it a homogenous white noise. The stochastic heat equation with homogeneous white noise was studied among many others, by Cabaña [2], Dawson [5], [6], Krylov and Rozovskii [12], [14], [13], Funaki [11], [10] and Walsh [23]. Pathwise uniqueness of the solutions for the stochastic heat equation, when the white noise coefficient σ is Lipschitz continuous was derived in [23].

In recent years there has been substantial work on the pathwise uniqueness of the stochastic heat equations with Hölder continuous noise coefficients (see e.g. [1], [3], [15], [16], [17], [19]). The pathwise uniqueness for the solutions of the stochastic heat equation, when the white noise coefficient $\sigma(\cdot,\cdot,u)$ is Hölder continuous in u of index $\gamma > 3/4$ was established in [16]. Pathwise uniqueness for the solutions of the d-dimensional stochastic heat equation driven by colored noise, with Hölder continuous noise coefficients, was studied in [17]. The result in [17] was later improved by Rippl and Sturm in [19]. Note that in both [16] and [19] the critical Hölder exponent of the noise coefficient which is required for uniqueness is $1 - c/2$, where c is the Hölder continuity index in space of the solutions. In this work we obtain weaker conditions for the pathwise uniqueness of (1.1) driven by spatially inhomogeneous white noise. More precisely, we show that in our case, the linear relation between the spatial regularity of the solutions and the condition on Hölder continuity of σ can be improved (see Theorem 1.5 and Remark 1.9). The main reason for the improvement arises from the fact that W is based on a fractal measure μ. This allows us to give a more efficient covering argument than the one in [16] (see Remark 3.1).

The method of proof in [16], [17] and [19] is the Yamada-Watanabe argument for stochastic PDEs (see Section 2 of [16]). The argument relies on the regularly of the difference between two solutions of (1.1) near their zero set. In the case where (1.1) is driven by spatially inhomogeneous white noise, this method does not go through. The singular nature of the noise causes the solutions to (1.1) to be rougher than the solutions in the white noise case, and drives us to change the argument which was developed in [16] (see Remark 4.1 for a more accurate explanation).

Zähle in [26] studied (1.1) driven by spatially inhomogeneous white noise and $\sigma(t,x,u) = \sqrt{u}$. In this setting he proved that (1.1) arises as scaling limits of a critical branching particle system which is known as a catalytic super Brownian motion with catalyst μ. Zähle's work contributed to our main motivation to choose μ that satisfies (1.2).

Before we describe in more detail the known uniqueness results for the case of equations driven by homogeneous and inhomogeneous white noises, we introduce additional notation and definitions.

Notation. For every $E \subset \mathbb{R}$, we denote by $\mathcal{C}(E)$ the space of continuous functions on E. In addition, a superscript k (respectively, ∞) indicates that functions are k times (respectively, infinite times), continuously differentiable. A subscript b (respectively, c) indicates that they are also bounded (respectively, have compact support). For $f \in \mathcal{C}(\mathbb{R})$ set

$$\|f\|_\lambda = \sup_{x\in\mathbb{R}} |f(x)|e^{-\lambda|x|}, \tag{1.3}$$

and define

$$\mathcal{C}_{tem} := \{f \in \mathcal{C}(\mathbb{R}), \|f\|_\lambda < \infty \ \text{ for every } \ \lambda > 0\}.$$

The topology on this space is induced by the norms $\|\cdot\|_\lambda$ for $\lambda > 0$.

For $I \subset \mathbb{R}_+$ let $\mathcal{C}(I,\mathcal{E})$ be the space of all continuous functions on I taking values in a topological space $\mathcal{E}$ endowed with the topology of uniform convergence on compact subsets of I. $u \in \mathcal{C}(\mathbb{R}_+,\mathcal{C}_{tem})$ means that u is a continuous function on $\mathbb{R}_+ \times \mathbb{R}$ and

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} |u(t,x)|e^{-\lambda|x|} < \infty, \ \ \forall\ \lambda > 0,\ T > 0.$$

In many cases it is possible to show that solutions to (1.1) are in $\mathcal{C}(\mathbb{R}_+,\mathcal{C}_{tem})$ (see for example: Theorem 1.1 in [16] and Theorem 2.2 in [22]).

We set

$$G_t(x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}},\ x \in \mathbb{R},\ t > 0. \tag{1.4}$$

We extend the definitions of strong solutions and pathwise uniqueness of solutions to (1.1), which are given in Section 1 of [16].

Definition 1.1 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a probability space and let W be a white noise process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Let $\mathcal{F}_t^W \subset \mathcal{F}_t$ be the filtration generated by W. A stochastic process $u : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ which is jointly measurable and $(\mathcal{F}_t^W)$-adapted, is said to be a stochastically strong solution to (1.1) with initial condition u_0 on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, if for all $t \geq 0$ and $x \in \mathbb{R}$,*

$$u(t,x) \quad = \quad G_t u_0(x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(s,y,u(s,y))W(ds,dy), \quad P-\text{a.s.} \tag{1.5}$$

Here $G_t f(x) = \int_{\mathbb{R}} G_t(x-y)f(y)dy$, for all f such that the integral exists.

In this work we study uniqueness for (1.1) in the sense of pathwise uniqueness. The definition of pathwise uniqueness is given below.

Definition 1.2 *We say that pathwise uniqueness holds for solutions of (1.1) in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem})$ if for every deterministic initial condition, $u_0 \in \mathcal{C}_{tem}$, any two solutions to (1.1) with sample paths a.s. in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem})$ are equal with probability 1.*

Convention. Constants whose values are unimportant and may change from line to line are denoted by C_i, M_i, $i = 1, 2, ..$, while constants whose values will be referred to later and appear initially in say, Lemma $i.j$ (respectively, Equation $(i.j)$) are denoted by $C_{i.j}$ (respectively, $C_{(i.j)}$).

Next we present in more detail some results on pathwise uniqueness for the solutions of (1.1) driven by homogeneous white noise which are relevant to our context. If σ is Lipschitz continuous in the u-variable, the existence and uniqueness of a strong solution to (1.1) in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem})$ is well known (see e.g. [22]). The proof uses the standard tools that were developed in [23] for solutions to SPDEs. In [16], Lipschitz assumptions on σ were relaxed and the following conditions were introduced: for every $T > 0$, there exists a constant $C_{(1.6)}(T) > 0$ such that for all $(t, x, u) \in [0, T] \times \mathbb{R}^2$,

$$|\sigma(t,x,u)| \leq C_{(1.6)}(T)(1 + |u|); \tag{1.6}$$

for some $\gamma > 3/4$ there are $\bar{R}_1, \bar{R}_2 > 0$ and for all $T > 0$ there is an $\bar{R}_0(T)$ so that for all $t \in [0, T]$ and all $(x, u, u') \in \mathbb{R}^3$,

$$|\sigma(t,x,u) - \sigma(t,x,u')| \leq \bar{R}_0(T) e^{\bar{R}_1|x|}(1 + |u| + |u'|)^{\bar{R}_2}|u - u'|^{\gamma}. \tag{1.7}$$

Mytnik and Perkins in [16] proved that if $u_0 \in \mathcal{C}_{tem}$, $\mu(dx) = dx$, and $\sigma : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ satisfy (1.6), (1.7) then there exists a unique strong solution of (1.1) in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem})$. It was also shown in [16] that addition of a Lipschitz continuous drift term to the right-hand side of (1.1) does not affect the uniqueness result.

Before we introduce our results, let us define some spaces of measures that will be used in the definition of spatially inhomogeneous white noise.

Notation: Let $\eta \in (0, 1)$. For a measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, let us define

$$\phi_{\eta,\mu}(x) \quad := \quad \int_{\mathbb{R}} |x - y|^{-\eta}\mu(dy). \tag{1.8}$$

Let $\dim_B(A)$ be the Minkowski dimension (also known as the box-counting dimension) of any set $A \subset \mathcal{B}(\mathbb{R})$.

Definition 1.3 *A measure μ is said to have carrying dimension cardim$(\mu) = l$, if there exists a Borel set A such that $\mu(A^c) = 0$ and dim$_B(A) = l$, and this fails for any $l' < l$.*

Loosely speaking, Definition 1.3 implies that for arbitrary small $\delta > 0$, the set A can be covered by δ^{-l} balls of diameter δ. However, it would be impossible to cover A with $\delta^{-l+\varepsilon}$ amount of δ-balls for any $\varepsilon > 0$. This means that the measure μ is singular and concentrated on a fractal set. The choice of a fractal μ is motivated by the study of catalytic reaction diffusion systems. In several biological and chemical reactions, the catalyst is concentrated on a zero Lebesgue measure set. In some cases these systems formally correspond to the stochastic heat equations driven by white noise which is based on a fractal measure (see [8] and references therein).

Denote by $M_f(\mathbb{R})$ the space of finite measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We introduce the following subset of $M_f(\mathbb{R})$:

$$M_f^\eta(\mathbb{R}) := \left\{ \mu \in M_f(\mathbb{R}) \middle| \sup_{x\in\mathbb{R}} \phi_{\eta-\varepsilon,\mu}(x) < \infty, \ \forall \varepsilon > 0, \text{ and } \mathrm{cardim}(\mu) = \eta \right\}. \tag{1.9}$$

Next we define the inhomogeneous white noise that we are going to work with.

Definition 1.4 *A white noise W based on the measure $\mu(dx) \times dt$, where $\mu \in M_f^\eta(\mathbb{R})$, is called a spatially inhomogeneous white noise based on μ. The corresponding white noise process $W_t(A) := W((0,t] \times A)$, where $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R})$, is called a spatially inhomogeneous white noise process based on μ.*

Now we are ready to state the main result of the paper: the pathwise uniqueness to the stochastic heat equation (1.1) with spatially inhomogeneous white noise holds for a class of Hölder continuous noise coefficients. The existence of a weak solution to this equation under similar assumptions on μ and less restrictive assumptions on σ was proved by Zähle in [26].

Theorem 1.5 *Let $\dot{W}$ be a spatially inhomogeneous white noise based on a measure $\mu \in M_f^\eta(\mathbb{R})$, for some $\eta \in (0,1)$. Let $u(0,\cdot) \in \mathcal{C}_{tem}(\mathbb{R})$. Assume that $\sigma : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ satisfies (1.6) and (1.7), for some γ satisfying*

$$\gamma > 1 - \frac{\eta}{2(\eta+1)}; \tag{1.10}$$

then pathwise uniqueness holds for the solutions to (1.1) with sample paths a.s. in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem}(\mathbb{R}))$.

A few remarks about the theorem are in order.

Remark 1.6 *The result of Theorem 1.5 still applies if we add a drift term $b(t,x,u(t,x))$ to the right-hand side of (1.1), assuming that b is Lipschitz continuous in u and has at most linear growth (as in (1.2) and (1.4) of [16]). The changes in the proof of Theorem (1.5) when (1.1) has an additional drift term follows the same lines as the proof in Section 8 of [16].*

Remark 1.7 *Assume that the condition cardim$(\mu) = \eta$ is omitted from the assumptions of Theorem 1.5. Then from the proof of Theorem 1.5 one can obtain that, in this case, pathwise uniqueness holds for the solutions of (1.1) if $\gamma > 1 - \eta/4$. Note that assumptions on cardim(μ) and $\phi_{\eta-\varepsilon,\mu}(\cdot)$ in (1.9) are related in the following manner. Let*

$$I_\eta(\mu) \quad := \quad \int_{\mathbb{R}} \phi_{\eta,\mu}(x)\mu(dx).$$

$\phi_{\eta,\mu}(\cdot)$ (in (1.8)) and $I_{\eta,\mu}(\cdot)$ are often called the η-potential and η-energy of the measure μ respectively. In Section 4.3 of [9], the connection between the sets of measures above and the Hausdorff dimension of sets that contain their support is introduced. Theorem 4.13 in [9] states that if a mass distribution μ on a set $F \subset \mathbb{R}$ has finite η-energy, that is,

$$I_\eta(\mu) < \infty,$$

then the Hausdorff dimension of F is at least η. Recall that a measure μ is called a mass distribution on a set $F \subset \mathbb{R}$, if the support of μ is contained in F and $0 < \mu(F) < \infty$ (see definition in Section 1.3 of [9]). For the relation between Hausdorff dimension and Minkowski dimension we refer to Proposition 4.1 in [9].

Remark 1.8 *Let us discuss the connection between Theorem 1.5 and Theorem 1.2 in [16]. The case of $\eta = 1$ formally corresponds to the "homogeneous" white noise case that was studied in [16]. We see that in Theorem 1.5, our lower bound on γ coincides with the bound 3/4 obtained in [16]. Note that it was shown in [1] and [15], that the 3/4 bound is optimal in the homogeneous case (i.e. counterexamples for any $0 \leq \gamma < 1/2$ and $1/2 \leq \gamma < 3/4$ were constructed, respectively). We mentioned earlier that when (1.1) is driven by inhomogeneous* *white noise, it introduces a different connection between the spatial regularity of the solutions and the Hölder index of σ which ensures uniqueness (see more details in Remark 1.9). Therefore, it would be very interesting to investigate in the future if the Yamada-Watanabe argument for the stochastic heat equation derives an optimal condition for uniqueness as it did in [16].*

In our proof we use the Yamada-Watanabe argument for the stochastic heat equation that was carried out in [16] for equations driven by homogeneous white noise. We describe very briefly the main idea of the argument. Let $\tilde{u} \equiv u^1 - u^2$ be the difference between two solutions to (1.1). The proof of uniqueness relies on the regularity of $\tilde{u}$ at the points x_0 where $\tilde{u}(t, x_0)$ is "small". To be more precise, we need to show that there exists a certain ξ, such that for points x_0 where $\tilde{u}(t, x_0) \approx 0$ and for points x nearby, we have

$$|\tilde{u}(t,x) - C_1(\omega)(x - x_0)| \leq C_2(\omega)|x - x_0|^{\xi}, \tag{1.11}$$

for some (random) constants $C_1,\ C_2$. Moreover, we will show that in our case, for any ξ such that

$$\xi < \frac{\eta}{2(1-\gamma)} \wedge (1+\eta),$$

(1.11) holds for x_0 such that $\tilde{u}(t, x_0) \approx 0$. This will allow us to derive the following condition for the pathwise uniqueness

$$\gamma > 1 - \frac{\eta}{2(1+\eta)}.$$

Note that in [16] the corresponding condition was $\xi < 2$, which is what one gets by setting $\eta = 1$ and $\gamma = 3/4$.

Remark 1.9 *Recall that in Theorem 1.2 of [16] it was proved that when the Hölder exponent of the noise coefficient γ is larger than $3/4$, then pathwise uniqueness holds for (1.1) with space-time white noise. In this case, the spatial Hölder continuity index of the solutions is $1/2 - \varepsilon$, for any arbitrary small $\varepsilon > 0$. In Theorem 1.2 of [19], pathwise uniqueness for the solutions of (1.1) with coloured noise was proved when $\gamma > (2 + \alpha)/4$. The constant $\alpha \in (0, 1)$ in [19] controls the spatial correlation of the noise. In the setting of [19], the solutions have spatial Hölder continuity index $1 - \alpha/2 - \varepsilon$, for any small $\varepsilon > 0$ (see Proposition 1.8(b) in [17]). Note that in both [16] and [19] the Hölder exponent of the noise coefficient which is required for uniqueness is larger than $1 - c/2$, where the solutions are Hölder continuous in space, with any exponent less than c. It was proved in Theorem 2.5 in [25] that when μ satisfies (1.2), the solution to (1.1) is Hölder continuous with any exponent $\xi < \eta/2$ in space. Note that Theorem 1.5 proves that the Hölder index of σ which is required for uniqueness is strictly smaller than $1 - \eta/4$. Therefore, the linear connection between the spatial regularity of the solution and the Hölder continuity of σ which is implied by the results in [16] and [19] do not apply here.*

One of the by-products of the proof of Theorem 1.5 is the following theorem. We prove under milder assumptions that the difference of two solutions of (1.1) is Hölder continuous in the spatial variable with any exponent $\xi < 1$ at the points of the zero set.

Theorem 1.10 *Assume the hypotheses of Theorem 1.5. However instead of (1.10) suppose*

$$\gamma > 1 - \frac{\eta}{2}. \tag{1.12}$$

Let u^1 and u^2 be two solutions of (1.1) with sample paths in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem})$ a.s. and with the same initial condition $u^1(0) = u^2(0) = u_0 \in C_{tem}$. Let $u \equiv u^1 - u^2$,

$$T_K = \inf\left\{s \geq 0 : \sup_{y\in\mathbb{R}}(|u^1(s,y)| \vee |u^2(s,y)|)e^{-|y|} > K\right\} \wedge K, \tag{1.13}$$

for some constant $K > 0$ and

$$S_0(\omega) = \{(t, x) \in [0, T_K] \times \mathbb{R} : u(t, x) = 0\}.$$

Then at every $(t_0, x_0) \in S_0$, u is Hölder continuous with exponent ξ in space and with exponent $\xi/2$ in time, for any $\xi < 1$.

Remark 1.11 *Theorem 1.10 corresponds to Theorem 2.3 in [16]. It was proved in [16] that when (1.1) is driven by homogeneous space-time white noise and $\gamma > 1/2$, then u is Hölder continuous with exponent ξ for any $\xi < 1$, for points in S_0. In Theorem 1.10 we show that when $\gamma > 1 - \frac{\eta}{2}$, then u is Hölder continuous with exponent*

$\xi < 1$ for points in S_0. Note that these two conditions on γ coincide when $\eta = 1$, that is when the inhomogeneous space-time white noise formally corresponds to homogeneous white noise. Another version of Theorem 1.10 for (1.1) driven by coloured noise was proved in [17]. It follows from Theorem 4.1 in [17] that if $\gamma > \alpha/2$, then u is Hölder continuous with exponent ξ for any $\xi < 1$, where $\alpha \in (0,1)$ controls the spatial correlation of the noise (see (8) in [17]). Recall that when $\alpha = 1$ coloured noise formally corresponds to homogeneous white noise. We observe that for the above regularity results, when $\alpha = 1$ the conditions on γ for the homogeneous white noise and for coloured noise are similar.

Zähle in [25] considered (1.1) when $\dot{W}$ is an inhomogeneous white noise based on $\mu(dx) \times dt$, where μ satisfies conditions which are slightly more general than (1.2). The existence and uniqueness of a strong $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem})$ solution to (1.1) when σ is continuous on $\mathbb{R}^2 \times \mathbb{R}_+$, Lipschitz continuous in u and satisfies (1.6) was proved in [25]. The Hölder continuity of the solutions to (1.1) with inhomogeneous white noise based on $\mu(dx) \times dt$ as above was also derived under some more relaxed assumptions on σ. In fact, it was proved in [25] that the solution to (1.1), when μ satisfies (1.2), is Hölder continuous with any exponent $\xi < \eta/2$ in space and $\xi < \eta/4$ in time.

Organization of the paper: The rest of this paper is devoted to the proofs of Theorems 1.5 and 1.10. In Section 2 we prove Theorem 1.5 under the hypothesis of Proposition 2.3. Since the verification of this hypothesis is rather long, in Section 3 we discuss the heuristics of the proof. In Section 4 we prove the hypothesis of Proposition 2.3 under certain regularity assumptions on the difference between solutions to (1.1) (see Proposition 4.8). In Section 5 we introduce some integral bounds for the heat kernel which will be essential for the proof of Proposition 4.8. In Section 6 we prove some local bounds on the difference of solutions, which are needed for the proof of Proposition 4.8. Section 7 is dedicated to the proof of Proposition 4.8. In Section 8 we prove Proposition 7.1, which is one of the ingredients for the proofs of Proposition 4.8 and Theorem 1.10. Finally, Section 9 is devoted to the proofs of Theorems 6.5 and Theorem 1.10. The proofs of Lemmas 2.4, 4.6 and Lemmas 5.6–5.8 are available in the Appendix. A list of notation appears at the end of this paper.

2 Proof of Theorem 1.5

Let us introduce the following useful proposition.

Proposition 2.1 *Let $u_0 \in C_{tem}$. Let σ be a continuous function satisfying (1.6). Then any solution $u \in C(\mathbb{R}_+, C_{tem})$ to (1.1) satisfies the following property. For any $T, \lambda > 0$ and $p \in (0, \infty)$,*

$$E\left(\sup_{0 \le t \le T} \sup_{x \in \mathbb{R}} |u(t,x)|^p e^{-\lambda |x|} \right) < \infty.$$

The proof of Proposition 2.1 follows the same lines as the proof of Proposition 1.8 in [17]. It uses the factorization method developed by Da Prato, Kwapien and Zabczyk in [4]. In fact, in our case, the calculations become simpler because of the orthogonality of the white noise. Since the proof of Proposition 2.1 is straightforward and technical, it is omitted.

Proof of Theorem 1.5 The proof follows the similar lines as the proof of Theorem 1.2 in [16]. In what follows let u^1 and u^2 be two solutions of (1.1) on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with sample paths in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem})$ P-a.s., with the same initial condition $u^1(0) = u^2(0) = u_0 \in C_{tem}$ and the same white noise. By Proposition 4.4 in [25], (1.5) is equivalent to the distributional form of (1.1). That is, for $i = 1, 2$ and for every $\phi \in C_c^\infty(\mathbb{R})$:

$$\begin{aligned} \langle u^i(t), \phi \rangle &= \langle u_0, \phi \rangle + \frac{1}{2} \int_0^t \langle u^i(s), \Delta \phi \rangle ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \sigma(s, x, u^i(s,x)) \phi(x) W(ds, dx), \quad \forall t \ge 0, \ P-\text{a.s.} \end{aligned} \tag{2.1}$$

Let $R_0 = \bar{R}_0(K)$ and $R_1 = \bar{R}_1 + \bar{R}_2$. By the same truncation argument as in Section 2 of [16], it is enough to prove Theorem 1.5 with the following condition instead of (1.7):

There are $R_0, R_1 > 1$ so that for all $t > 0$ and all $(x, u, u') \in \mathbb{R}^3$,

$$|\sigma(t,x,u) - \sigma(t,x,u')| \le R_0 e^{R_1 |x|} |u - u'|^\gamma. \tag{2.2}$$

We use the following definitions and notations from [17]. Let

$$a_n = e^{-n(n+1)/2} \tag{2.3}$$

so that

$$a_{n+1} = a_n e^{-n-1} = a_n a_n^{2/n}. \tag{2.4}$$

Define functions $\psi_n \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\mathrm{supp}(\psi_n) \subset (a_n/2, a_{n-1}/2)$,

$$0 \le \psi_n(x) \le \frac{2}{nx}, \quad \forall x \in \mathbb{R}, \tag{2.5}$$

and

$$\int_{a_n/2}^{a_{n-1}/2} \psi_n(x)dx = 1. \tag{2.6}$$

Finally, set

$$\phi_n(x) = \int_0^{|x|} \int_0^y \psi_n(z)dzdy. \tag{2.7}$$

Note that $\phi_n(x) \uparrow |x|$ uniformly in x and $\phi_n(x) \in \mathcal{C}^\infty(\mathbb{R})$. We also have

$$\phi_n'(x) = \mathrm{sign}(x) \int_0^{|x|} \psi_n(y)dy, \tag{2.8}$$

$$\phi_n''(x) = \psi_n(|x|). \tag{2.9}$$

Thus,

$$|\phi_n'(x)| \le 1, \ \forall x \in \mathbb{R}, \ n \in \mathbb{N}, \tag{2.10}$$

and for any function h which is continuous at zero

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi_n''(x)h(x)dx = h(0).$$

Define

$$u \equiv u^1 - u^2.$$

Let $\langle \cdot, \cdot \rangle$ denote the scalar product on $L^2(\mathbb{R})$. Let $m \in \mathbb{N}$ and recall that $G_t(x)$ denotes the heat kernel. We also use the notation $G_t^{''}(x) = \frac{d^2}{dx^2} G_t(x)$. Apply Itô's formula to the semimartingales $\langle u_t^i, G_{m^{-2}}(x-\cdot)\rangle = G_{m^{-2}} u_t^i(x)$, $i = 1, 2$ in (2.1) to get

$$\begin{aligned} &\phi_n(G_{m^{-2}} u_t(x)) \\ &= \int_0^t \int_{\mathbb{R}} \phi_n'(G_{m^{-2}} u_s(x))(\sigma(s,y,u^1(s,y)) - \sigma(s,y,u^2(s,y)))G_{m^{-2}}(x-y)W(ds,dy) \\ &+ \int_0^t \phi_n'(G_{m^{-2}} u_s(x))\langle u_s, \frac{1}{2} G_{m^{-2}}^{''}(x-\cdot)\rangle ds \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{R}} \psi_n(|G_{m^{-2}} u_s(x)|)(\sigma(s,y,u^1(s,y)) - \sigma(s,y,u^2(s,y)))^2 G_{m^{-2}}(x-y)^2 \mu(dy)ds. \end{aligned} \tag{2.11}$$

Remark 2.2 *In Section 2 of [16], after equation (2.35), the following mollifier was used for the same purpose that we use $G_{m^{-2}}$ in this proof. Let $\Phi \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfy $0 \le \Phi \le 1$, $\mathrm{supp}(\Phi) \subset (-1,1)$, $\int_{\mathbb{R}} \Phi(x)dx = 1$, and set $\Phi_x^m(y) = m\Phi\big(m(x-y)\big)$. We would like to emphasize here that Φ^m from [16] has a compact support, while $G_{m^{-2}}$ is the heat kernel which has an unbounded support. This choice of mollifier will help us later on (see Remark 4.1).*

Fix $t_0 \in (0,\infty)$ and let us integrate (2.11) with respect to the x-variable, against another nonnegative test function $\Psi \in \mathcal{C}_c([0,t_0] \times \mathbb{R})$. Choose $K_1 \in \mathbb{N}$ large enough so that for $\lambda = 1$,

$$\|u_0\|_\lambda < K_1 \ \text{ and } \Gamma \equiv \{x : \Psi_s(x) > 0 \text{ for some } s \le t_0\} \in (-K_1, K_1). \tag{2.12}$$

Now apply the stochastic Fubini Theorem (see Theorem 2.6 in [23]) and Proposition 2.5.7 in [18] to $\langle \phi_n(G_{m^{-2}}u_t(\cdot)), \Psi_t\rangle$ as in [16] to get,

$$
\begin{aligned}
&\langle \phi_n(G_{m^{-2}}u_t(\cdot)), \Psi_t\rangle \\
&= \int_0^t \int_{\mathbb{R}} \langle \phi_n'(G_{m^{-2}}u_s(\cdot))G_{m^{-2}}(\cdot - y), \Psi_s\rangle(\sigma(s,y,u^1(s,y)) - \sigma(s,y,u^2(s,y)))W(ds,dy) \\
&\quad + \int_0^t \int_{\mathbb{R}} \phi_n'(G_{m^{-2}}u_s(x))\langle u_s, \frac{1}{2}G''_{m^{-2}}(x-\cdot)\rangle \Psi_s(x)dxds \\
&\quad + \frac{1}{2}\int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} \psi_n(|G_{m^{-2}}u_s(x)|)(\sigma(s,y,u^1(s,y)) - \sigma(s,y,u^2(s,y)))^2 G_{m^{-2}}(x-y)^2\mu(dy)\Psi_s(x)dxds \\
&\quad + \int_0^t \langle \phi_n(G_{m^{-2}}u_s(\cdot)), \dot{\Psi}_s\rangle ds \\
&=: I_1^{m,n}(t) + I_2^{m,n}(t) + I_3^{m,n}(t) + I_4^{m,n}(t).
\end{aligned} \tag{2.13}
$$

Note that for $i = 2, 4$, $I_i^{m,n}(t)$ look exactly like the terms considered in [16]. The only difference is that we chose here the heat kernel as a mollifier instead of the compact support mollifier that was chosen in [16]. For $I_1^{m,n}, I_3^{m,n}$, the expressions are different since here we use the inhomogeneous white noise, which is based on the measure $\mu(dy)ds$. The main difficulty in this work, is to show that $I_3^{m,n}(t)$ converges to 0 when $n, m \to \infty$. The convergence of $I_3^{m,n}(t)$ relies on the regularity of the difference between two solutions of (1.1) near their zero set (see Section 3 for the heuristic explanation). As mentioned earlier, since the solutions to (1.1) are rougher than the solutions in the white noise case, we needed to change the argument which was developed in [16] (see Remark 4.1).

Notation. We fix the following positive constants $\varepsilon_1, \varepsilon_0$ satisfying

$$
0 < \varepsilon_1 < \frac{1}{100}\big(\gamma - 1 + \frac{\eta}{2(\eta+1)}\big), \quad 0 < \varepsilon_0 < \frac{\eta^2\varepsilon_1}{100}. \tag{2.14}
$$

Let

$$
\kappa_0 = \frac{1}{\eta+1}. \tag{2.15}
$$

Note that by our assumption on η, $\kappa_0 \in [1/2, 1)$. The choice of κ_0 will become clear in Section 3. Set $m_n = a_{n-1}^{-\kappa_0-\varepsilon_0} = \exp\{(\kappa_0+\varepsilon_0)(n-1)n/2\}$, for $n \in \mathbb{N}$.

From this point on, we focus on $I_i^{m,n+1}$, where m is set to $m = m_{n+1}$. Note that for $I_3^{m_{n+1},n+1}$ we may assume $|x| \le K_1$ by (2.12). Recall that T_K was defined in (1.13). If $s \le T_K$, then we have

$$
|u^i(s,y)| \le Ke^{|y|}, \quad i = 1, 2. \tag{2.16}
$$

By (2.2), (2.5) and $\mathrm{supp}(\psi_n) \subset (a_n/2, a_{n-1}/2)$, we get for $t \in [0, t_0]$,

$$
\begin{aligned}
&I_3^{m_{n+1},n+1}(t\wedge T_K) \\
&\le \frac{C(R_0)}{(n+1)}\int_0^{t\wedge T_K}\int_{\mathbb{R}}\int_{\mathbb{R}} |\langle u_s, G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-\cdot)\rangle|^{-1} \mathbb{1}_{\{a_{n+1}/2 \le |\langle u_s, G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-\cdot)\rangle| \le a_n/2\}} e^{2R_1|y|}|u(s,y)|^{2\gamma} \\
&\quad \times G^2_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-y)\mu(dy)\Psi_s(x)dxds.
\end{aligned} \tag{2.17}
$$

Moreover, since $\sup_{x\in\mathbb{R}}|G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x)| \le m_{n+1}/\sqrt{2\pi}$ it follows that for all $t\in[0,t_0]$,

$$\begin{aligned}
&I_3^{m_{n+1},n+1}(t\wedge T_K)\\
&\le C(R_0)\frac{m_{n+1}}{(n+1)}\int_0^{t\wedge T_K}\int_{\mathbb{R}}\int_{\mathbb{R}}|\langle u_s, G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-\cdot)\rangle|^{-1}\mathbb{1}_{\{a_{n+1}/2\le|\langle u_s,G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-\cdot)\rangle|\le a_n/2\}}e^{2R_1|y|}|u(s,y)|^{2\gamma}\\
&\quad\times G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-y)\mu(dy)\Psi_s(x)dxds\\
&\le C(R_0)a_n^{-\kappa_0-\varepsilon_0}2a_{n+1}^{-1}\int_0^{t\wedge T_K}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{1}_{\{a_{n+1}/2\le|\langle u_s,G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-\cdot)\rangle|\le a_n/2\}}e^{2R_1|y|}|u(s,y)|^{2\gamma}\\
&\quad\times G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-y)\mu(dy)\Psi_s(x)dxds\\
&\le C(R_0)a_n^{-1-\kappa_0-\varepsilon_0-2/n}\int_0^{t\wedge T_K}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{1}_{\{a_{n+1}/2\le|\langle u_s,G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-\cdot)\rangle|\le a_n/2\}}e^{2R_1|y|}|u(s,y)|^{2\gamma}\\
&\quad\times G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-y)\mu(dy)\Psi_s(x)dxds, \qquad (2.18)
\end{aligned}$$

where we used (2.4) in the last inequality. Define

$$\begin{aligned}
&I^n(t)\\
&:= a_n^{-1-\kappa_0-\varepsilon_0-2/n}\int_0^{t\wedge T_K}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{1}_{|\langle u_s,G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-\cdot)\rangle|\le a_n/2\}}e^{2R_1|y|}|u(s,y)|^{2\gamma}G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-y)\mu(dy)\Psi_s(x)dxds.
\end{aligned} \tag{2.19}$$

The following proposition is crucial for the proof of Theorem 1.5. Recall that t_0 was fixed before (2.12).

Notation: Let $\mathbb{N}^{\ge K_1} = \{K_1, K_1+1, ...\}$.

Proposition 2.3 *Suppose $\{U_{M,n,K} : M,n,K\in\mathbb{N}, K\ge K_1\}$ are $(\mathcal{F}_t)$-stopping times such that for each $K\in\mathbb{N}^{\ge K_1}$,*

$$\begin{aligned}
U_{M,n,K}\le T_K,\ U_{M,n,K}\uparrow T_K \ \textit{as } M\to\infty \textit{ for each } n \textit{ and}\\
\lim_{M\to\infty}\sup_n P(U_{M,n,K}<T_K)=0.
\end{aligned} \tag{2.20}$$

Also

$$\textit{for all } M\in\mathbb{N},\ \lim_{n\to\infty}E(I^n(t_0\wedge U_{M,n,K}))=0. \tag{2.21}$$

Then, the conclusion of Theorem 1.5 holds.

Proof: The proof of Proposition 2.3 follows the same lines as the proof of Proposition 2.1 in [16]. Fix an arbitrary $K\in\mathbb{N}^{\ge K_1}$, and $t\in[0,t_0]$. Let

$$Z_n(t)=\int_{\mathbb{R}}\phi_n(\langle u_t, G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-\cdot)\rangle)\Psi_t(x)dx. \tag{2.22}$$

From Lemma 6.2(ii) in [22] we have

$$\sup_{t\in(0,T]}\sup_{y\in\mathbb{R}}e^{-\lambda|y|}\int_{\mathbb{R}}G_t(y-z)e^{\lambda|z|}dy<C(\lambda),\ \text{for all } \lambda\in\mathbb{R}. \tag{2.23}$$

By similar lines as in the proof of Proposition 2.1 in [16] and with (2.23) we have

$$0\le Z_n(t\wedge T_K)\le 2Ke^{K_1+1}C_{2.24}(\Psi). \tag{2.24}$$

Let $g_{m_{n+1},n}(s,y)=\langle\phi_n'(G_{a_n^{2(\kappa_0+\varepsilon_0)}}u_s(\cdot))G_{a_n^{2(\kappa_0+\varepsilon_0)}}(\cdot-y),\Psi_s\rangle$. Recall that Ψ is a nonnegative function, then from (2.10) we have

$$|g_{m_{n+1},n}(s,y)|\le\langle G_{a_n^{2(\kappa_0+\varepsilon_0)}}(\cdot-y),\Psi_s\rangle. \tag{2.25}$$

We will use the following lemma to bound $E(\langle I_1^{m_n,n}\rangle_{t\wedge T_K})$.

Lemma 2.4 *Let $\mu \in M_f^\eta(\mathbb{R})$. Then for every $K>0$ we have*

$$\sup_{\varepsilon\in(0,1]}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{|y|}G_\varepsilon(z-y)\mathbb{1}_{\{|z|\le K\}}\mu(dy)dz<\infty. \tag{2.26}$$

The proof of Lemma 2.4 is given in the Appendix.

By (1.6), (2.25), Jensen's inequality, Proposition 2.1, and Lemma 2.4 we have

$$\begin{aligned} E(\langle I_1^{m_{n+1},n}\rangle_{t\wedge T_K}) &= E\Big(\int_0^{t\wedge T_K}\int_{\mathbb{R}}(g_{m_{n+1},n}(s,y))^2(\sigma(s,y,u^1(s,y))-\sigma(s,y,u^2(s,y)))^2\mu(dy)ds\Big)\\ &\le C(T)\int_0^t\int_{\mathbb{R}}(\langle G_{a_n^{2(\kappa_0+\varepsilon_0)}}(\cdot-y),\Psi_s\rangle)^2e^{|y|}E(e^{-|y|}(1+|u^1(s,y)|^2+|u^2(s,y)|^2))\mu(dy)ds\\ &\le C(T)\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}e^{|y|}G_{a_n^{2(\kappa_0+\varepsilon_0)}}(z-y)\Psi_s^2(z)\mu(dy)dzds\\ &\le C(T)\|\Psi\|_\infty^2\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}e^{|y|}G_{a_n^{2(\kappa_0+\varepsilon_0)}}(z-y)\mathbb{1}_{\{|z|\le K_1+1\}}\mu(dy)dzds\\ &\le C(T,\|\Psi\|_\infty,K_1). \end{aligned} \tag{2.27}$$

Since $T_K\wedge t\to t$, as $K\to\infty$, it follows that

$$\{I_1^{m_{n+1},n}(s):\ s\le t_0\}\text{ is an }\mathbb{L}^2-\text{bounded sequence of }\mathbb{L}^2-\text{martingales.}$$

We can handle $I_2^{m_{n+1},n}$ similarly to (2.46)–(2.47) in [16] (see also Lemma 2.2(b) in [17]), and get for any stopping time T,

$$I_2^{m_{n+1},n}(t\wedge T)\to\int_0^{t\wedge T}\int_{\mathbb{R}}|u(s,x)|\frac{1}{2}\Delta\Psi_s(x)dxds,\quad\text{in }\mathbb{L}^1\text{ as }n\to\infty. \tag{2.28}$$

By applying the same steps as in the proof of Lemma 2.2(c) in [17], we get that for any stopping time T

$$I_4^{m_{n+1},n}(t\wedge T)\to\int_0^{t\wedge T}\int_{\mathbb{R}}|u(s,x)|\dot{\Psi}_s(x)dxds,\quad\text{in }\mathbb{L}^1\text{ as }n\to\infty.$$

The rest of the proof is identical to the proof of Proposition 2.1 in [16], and hence it is omitted. ■

The rest of this work is devoted to the verification of the hypothesis of Proposition 2.3. Verifying this hypothesis is long and involved. In the next section we provide short heuristics for this proof.

3 Heuristics for the Verification of the Hypothesis of Proposition 2.3

We have shown in Section 2 that the proof of Theorem 1.5 depends only on the verification of the hypothesis of Proposition 2.3. In this section we give a heuristic explanation for this proof. We adapt the argument from Section 2 of [16] to the case of inhomogeneous white noise.

For the sake of simplicity we omit ε_0, $2/n$ and the exponent $e^{2R_1|y|}$ from I^n, in the calculations of this section. We also replace $a_n/2$ with a_n in the indicator function in I^n. For the same reasons we replace the mollifier function in I^n from $G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-\cdot)$ to the following compact support mollifier, which was used in Section 2 of [16]. Let $\Phi\in\mathcal{C}_c^\infty(\mathbb{R})$ satisfy $0\le\Phi\le1$ supp$(\Phi)\in(-1,1)$ and $\int_{\mathbb{R}}\Phi(x)dx=1$, and set $\Phi_x^m(y)=m\Phi(m(x-y))$. Our goal is to show that

$$\begin{aligned} &I^n(t)\\ &\approx a_n^{-1-\kappa_0}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{1}_{|\langle u_s,\Phi_x^{m_n}\rangle|\le a_n\}}|u(s,y)|^{2\gamma}\Phi_x^{m_{n+1}}(y)\Psi_s(x)\mu(dy)dxds\\ &\to 0\text{ as }n\to\infty. \end{aligned} \tag{3.1}$$

Note that the term I^n corresponds to the local time term in the proof of the Yamada-Watanabe argument for SDEs (that is I_3 in the proof of Theorem 1 in [24]). Just as in the SDEs case, the uniqueness of (1.1) depends on the convergence to 0 of the "local time" term I^n when $n \to \infty$.

The following discussion is purely formal. To simplify the exposition we assume that u' (the spatial derivative of u) exists. The key to derive (3.1) is to control u near its zero set. We will show in Section 6 a rigorous analog for the following statement:

$$\gamma > 1 - \frac{\eta}{2(\eta+1)} \Rightarrow \qquad u'(s,\cdot) \text{ is } \zeta - \text{ Hölder continuous on } \{x : u(s,x) \approx u'(s,x) \approx 0\}, \quad \forall \zeta < \eta. \tag{3.2}$$

For the rest of this section we assume that (3.2) holds.

Let us expand I^n in (3.1) to get,

$$\begin{aligned} & I^n(t) \\ & \approx a_n^{-1-\kappa_0} \sum_\beta \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\{|u(s,x)| \le a_n, u'(s,x) \approx \pm a_n^\beta\}} |u(s,y)|^{2\gamma} \Phi_x^{m_{n+1}}(y) \Psi_s(x) \mu(dy) dx ds \\ & \equiv \sum_\beta I_\beta^n(t), \end{aligned} \tag{3.3}$$

where $\sum_\beta$ is a summation over a finite grid $\beta_i \in [0, \bar\beta]$ where $\bar\beta$ will be specified later. The notation $u'(s, x) \approx \pm a_n^{\beta_i}$ refers to a partition of the space-time to sets where $|u'(s,x)| \in [a_n^{\beta_{i+1}}, a_n^{\beta_i}]$. In what follows, assume that $u'(s,x) \approx a_n^{\beta_i}$ (the negative values are handled in the same way). Let us expand $u(s,y)$ in (3.3) to a Taylor series. In this step we will also motivate the choice of κ_0 in (2.15). The fact that

$$\text{supp}\,(\Phi_x^{m_{n+1}}) \subset [x - a_n^{\kappa_0}, x + a_n^{\kappa_0}] \tag{3.4}$$

and (3.2) imply that for any $y \in \text{supp}\,(\Phi_x^{m_{n+1}})$,

$$\begin{aligned} |u(s,y)| &\le |u(s,x)| + (|u'(s,x)| + M|x-y|^\zeta)|x-y| \\ &\le a_n + a_n^{\beta+\kappa_0} + M a_n^{(\zeta+1)\kappa_0} \\ &\le C a_n^{(\beta \wedge \kappa_0 \zeta) + \kappa_0}. \end{aligned} \tag{3.5}$$

In order to get an optimal upper bound on $|u(s,y)|$, we compared the first and third summands in the second line of (3.5), and obtained the condition $(\zeta+1)\kappa_0 = 1$. This condition together with (3.2) leads to (2.15). From now on we assume (2.15).

From (3.3) and (3.5) we have

$$I_\beta^n(t) \le C a_n^{-1-\kappa_0+2\gamma[(\beta\wedge\kappa_0\zeta)+\kappa_0]} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\{|u(s,x)| \le a_n, u'(s,x) \approx \pm a_n^\beta\}} \Phi_x^{m_{n+1}}(y) \Psi_s(x) \mu(dy) dx ds. \tag{3.6}$$

Recall that $\text{cardim}(\mu) = \eta$. Hence by Definition 1.3, there exists a set $A \subset \mathbb{R}$, such that $\mu(A^c) = 0$ and $\dim_B(A) = \eta$. From this, (3.4), the fact that $\text{supp}(\Psi_s) \subset [-K_1, K_1]$, and the definition of the Minkowski dimension (box-counting dimension) in Section 3.1 of [9], we can construct a set $\mathbb{V}^\beta$ which is a union of at most $N(\beta) \le C(K_1) a_n^{-\beta-\varepsilon}$ balls of diameter $3a_n^{\beta/\eta}$, such that it contains $A \cap [-K_1, K_1]$. Here $\varepsilon > 0$ is arbitrarily small. We conclude that the integration with respect to the x-axis on the right-hand side of (3.6) could be done over the set $\mathbb{V}^\beta$, instead of the whole real line.

Notation: For a set $B \subset \mathbb{R}$, let $|B|$ denote the Lebesgue of B.

From the discussion above, we have

$$|\mathbb{V}^\beta| \le 3C(K_1) a_n^{-\beta+\beta/\eta-\varepsilon}. \tag{3.7}$$

Integration over $\mathbb{V}^\beta$ will help us to improve the argument from [16] for the case of inhomogeneous noise, since it takes into account that the noise "lives" on a "smaller" set.

Note that by the definitions of $\Phi_x^{m_{n+1}}$ and the fact that $\mu \in M_f^\eta(\mathbb{R})$ we have

$$\begin{aligned}\int_{\mathbb{R}} \Phi_x^{m_{n+1}}(y)\mu(dy) &\leq a_n^{-\kappa_0} \int_{\mathbb{R}} \mathbb{1}_{\{|x-y|\leq a_n^{\kappa_0}\}} |x-y|^{\eta-\varepsilon}|x-y|^{-\eta+\varepsilon}\mu(dy) \\ &\leq Ca_n^{-\kappa_0(1-\eta+\varepsilon)}. \end{aligned} \tag{3.8}$$

From (3.6) and (3.8) we have

$$I_\beta^n(t) \leq Ca_n^{-1-\kappa_0+2\gamma[(\beta\wedge\kappa_0\zeta)+\kappa_0]-\kappa_0(1-\eta+\varepsilon)} \int_0^t \int_{\mathbb{V}^\beta} \mathbb{1}_{\{|u(s,x)|\leq a_n, u'(s,x)\approx\pm a_n^\beta\}} \Psi_s(x)dxds. \tag{3.9}$$

Note that from (3.3) we get that the convergence of $I_\beta^n(t)$ to 0 when $n \to \infty$ ensures the convergence of $I^n(t)$. Our goal is therefore to show that the right-hand side of (3.9) goes to zero as $n \to \infty$, for any $\beta \geq 0$. The value $\beta = 0$ is a little bit special, so we will concentrate here on $\beta > 0$. Fix some $\bar{\beta} > 0$ whose value will be verified at the end of the section. For $\beta \geq \bar{\beta}$, we get from (3.7) and (3.9),

$$\begin{aligned} I_\beta^n(t) &\leq Cta_n^{-1-\kappa_0+2\gamma[(\beta\wedge\kappa_0\zeta)+\kappa_0]-\kappa_0(1-\eta+\varepsilon)}|\mathbb{V}^\beta| \\ &\leq Cta_n^{-1-\kappa_0+2\gamma[(\bar{\beta}\wedge\kappa_0\zeta)+\kappa_0]-\kappa_0(1-\eta+\varepsilon)}|\mathbb{V}^{\bar{\beta}}| \\ &\leq C(K_1)ta_n^{-1-\kappa_0+2\gamma[(\bar{\beta}\wedge\kappa_0\zeta)+\kappa_0]-\kappa_0(1-\eta+\varepsilon)-\bar{\beta}+\bar{\beta}/\eta-\varepsilon}. \end{aligned} \tag{3.10}$$

Consider the case where $0 < \beta < \bar{\beta}$. Let

$$S_n(s) = \{x \in [-K_1, K_1] : |u(s,x)| < a_n, u'(s,x) \geq a_n^\beta\}. \tag{3.11}$$

From (3.2) we have that if n is large enough, then for every $x \in S_n(s)$, $u'(s,y) \geq a_n^\beta/2$ if $|y-x| \leq L^{-1}a_n^{\beta/\zeta}$ (where $L = (2M)^{1/\zeta}$ and M is from (3.5)). By the Fundamental Theorem of Calculus we get,

$$u(s,y) > a_n, \text{ for all } 4a_n^{1-\beta} < |y-x| \leq L^{-1}a_n^{\beta/\zeta}. \tag{3.12}$$

The covering argument in Section 2 of [16] (above equation (2.60)) suggests that $|S_n(s)|$ can be covered by $4K_1La_n^{-\beta/\zeta}$ disjoint balls of diameter $8a_n^{1-\beta}$. Since we are interested in bounding $I_\beta^n(t)$, from (3.9) it is sufficient to cover $S_n(s) \cap \mathbb{V}^\beta$. We will assume for now that $a_n^{\beta/\eta} \geq a_n^{1-\beta}$ for $\beta \in [0, \bar{\beta}]$. This inequality will be verified when we fix all our constants at the end of this section. From the discussion above and the construction of $\mathbb{V}^\beta$ and $S_n(s)$, we get that $S_n(s) \cap \mathbb{V}^\beta$ can be covered with $C(M,K_1)a_n^{-\beta-\varepsilon}$ balls of diameter $8a_n^{1-\beta}$. From the discussion above we have for $0 < \beta < \bar{\beta}$,

$$|S_n(s) \cap \mathbb{V}^\beta| \leq C(M,K_1)a_n^{-\beta-\varepsilon}a_n^{1-\beta}. \tag{3.13}$$

From (3.9) and (3.13), we get,

$$I_\beta^n(t) \leq C(M,K_1,t)a_n^{-1-\kappa_0+2\gamma[(\beta\wedge\kappa_0\zeta)+\kappa_0]-\kappa_0(1-\eta+\varepsilon)-\beta-\varepsilon+1-\beta}. \tag{3.14}$$

Recall that $\varepsilon > 0$ is arbitrarily small. Hence from (3.10) and (3.14) we get that $\lim_{n\to\infty} I_\beta^n = 0$ if

$$\gamma[(\beta \wedge \kappa_0\eta) + \kappa_0] > \frac{(2-\eta)\kappa_0 + 2\beta}{2}, \ \forall \beta \leq \bar{\beta}, \tag{3.15}$$

and

$$\gamma[(\bar{\beta} \wedge \kappa_0\eta) + \kappa_0] > \frac{(2-\eta)\kappa_0 + \bar{\beta} + 1 - \bar{\beta}/\eta}{2}. \tag{3.16}$$

Now recall that $\gamma > 1 - \frac{\eta}{2(\eta+1)}$. In this case the choice $\bar{\beta} = 1 - \bar{\beta}/\eta$ together with $\bar{\beta} = \kappa_0\eta$ are optimal for (3.15) and (3.16). Since $\kappa_0 = \frac{1}{1+\eta}$ was fixed, we get that $\bar{\beta} = \frac{\eta}{\eta+1}$ and then by substituting $\bar{\beta} = \frac{\eta}{\eta+1}$, $\kappa_0 = \frac{1}{\eta+1}$ in (3.15) and (3.16) we get $\lim_{n\to\infty} I_\beta^n = 0$, if (1.10) holds.

Remark 3.1 *Note that (3.9) corresponds to (2.58) in Section 2 of [16]. In the case of (1.1) driven by homogeneous white noise, the integral in (2.58) is bounded by a constant times the Lebesgue measure of the covering of S_n. In (3.9) we improved the bound on I_β^n by using a covering of a smaller set $S_n(s) \cap \mathbb{V}^\beta$. The improved upper bound allowed us to get (1.10) as a condition for pathwise uniqueness for the solutions of (1.1) driven by inhomogeneous white noise. A direct implementation of the cover from Section 2 of [16] to our case would give us the condition $\gamma > 1 - \eta/4$ which is more restrictive than (1.10).*

4 Verification of the Hypotheses of Proposition 2.3

This section is devoted to the verification of the hypothesis of Proposition 2.3. The proof follows the same lines as the proof of Proposition 2.1 in Section 3 of [16]. Let u^1 and u^2 be as in Section 2. We assume also the hypothesis of Theorem 1.5 and (2.2).

Let

$$D(s,y) = \sigma(s,y,u^1(s,y)) - \sigma(s,y,u^2(s,y)). \tag{4.1}$$

From (1.5) and (4.1) we have

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(y-x) D(s,y) W(ds,dy), \quad P-\text{a.s. for all } (t,x). \tag{4.2}$$

By (2.2) we have

$$|D(s,y)| \le R_0(T) e^{R_1|y|} |u(s,y)|^{\gamma}. \tag{4.3}$$

Let $\delta \in (0,1]$. Recall that $G_t(\cdot)$ was defined as the heat kernel. Let

$$u_{1,\delta}(t,x) = G_\delta(u_{(t-\delta)_+})(x) \tag{4.4}$$

and

$$u_{2,\delta}(t,x) = u(t,x) - u_{1,\delta}(t,x). \tag{4.5}$$

Note that by the same argument as in Section 3 of [16], both $u_{1,\delta}$ and $u_{2,\delta}$ have sample paths in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem})$. From (4.2) and (4.4) we have

$$u_{1,\delta}(t,x) = \int_{\mathbb{R}} \left(\int_0^{(t-\delta)_+} \int_{\mathbb{R}} G_{(t-\delta)_+ - s}(y-z) D(s,y) W(ds,dy) \right) G_\delta(z-x) dz. \tag{4.6}$$

By the stochastic Fubini theorem we get

$$u_{1,\delta}(t,x) = \int_0^{(t-\delta)_+} \int_{\mathbb{R}} G_{t-s}(y-x) D(s,y) W(ds,dy), \quad P-\text{a.s. for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{4.7}$$

From (4.5) and (4.7) we have

$$u_{2,\delta}(t,x) = \int_{(t-\delta)_+}^t \int_{\mathbb{R}} G_{t-s}(y-x) D(s,y) W(ds,dy), \quad P-\text{a.s. for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{4.8}$$

Let $\epsilon \in (0,\delta)$ and

$$\tilde{u}_{2,\delta}(t,\epsilon,x) = \int_{(t+\epsilon-\delta)_+}^t \int_{\mathbb{R}} G_{t+\epsilon-s}(y-x) D(s,y) W(ds,dy), \quad P-\text{a.s. for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{4.9}$$

From (4.2), (4.7) and Fubini's Theorem we have

$$\langle u_t, G_\epsilon(x-\cdot) \rangle = u_{1,\delta}(t+\epsilon, x) + \tilde{u}_{2,\delta}(t,\epsilon,x). \tag{4.10}$$

Remark 4.1 *The verification of the hypothesis of Proposition 2.3 relies on the regularity of u at space-time points (t,x) where $G_\epsilon u_t(x) \le a_n$ (see Section 3 for heuristics). In [16] a sufficient regularity for u was obtained by the following decomposition*

$$u(t,x) = u_{1,\delta}(t,\cdot) + u_{2,\delta}(t,\cdot), \tag{4.11}$$

and by deriving the regularity of $u_{1,\delta_n}(t,\cdot)$ and $u_{2,\delta_n}(t,\cdot)$ (see Lemmas 6.5 and 6.7 in [16]). In our case we could not show that $u_{2,\delta_n}(t,\cdot)$ is "regular enough" to get the criterion (1.10) for uniqueness. Therefore, we had to change the argument from [16]. The crucial observation is that the integral on the right-hand side of (2.19) is taken over points where $\langle u_t, G_{a_n^{2(\kappa_0+\varepsilon_0)}}(x-\cdot)\rangle \le a_n$. Therefore by the decomposition in (4.10), to bound (2.19) it is enough to use the regularity of $u_{1,\delta_n}(s+\varepsilon_n, x)$, $\tilde{u}_{2,\delta_n}(s,\varepsilon_n,x)$ (for particular δ_n, ε_n). It turns out that $\tilde{u}_{2,\delta}(t,\epsilon,x)$ is "regular enough" so that the proof of Proposition 2.3 goes through, and as a result we get the condition (1.10) for the pathwise uniqueness.

We adopt the following notation from Section 3 of [16].

Notation. If $s,t,\delta \geq 0$ and $x \in \mathbb{R}$, let $\mathbb{G}_\delta(s,t,x) = G_{(t-s)_+ + \delta}(u_{(s-\delta)_+})(x)$ and $F_\delta(s,t,x) = -\frac{d}{dx}\mathbb{G}_\delta(s,t,x) \equiv -\mathbb{G}'_\delta(s,t,x)$, if the derivative exists.

We will need the following Lemma.

Lemma 4.2 *$\mathbb{G}'_\delta(s,t,x)$ exists for all $(s,t,x) \in \mathbb{R}_+^2 \times \mathbb{R}$, is jointly continuous in (s,t,x), and satisfies*

$$F_\delta(s,t,x) = \int_0^{(s-\delta)_+} \int_{\mathbb{R}} G'_{(t\vee s)-r}(y-x)D(r,y)W(dr,dy), \text{ for all } s \in \mathbb{R}_+,$$
$$P - a.s. \text{ for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{4.12}$$

The proof of Lemma 4.2 is similar to the proof of Lemma 3.1 in [16] and hence is omitted.

Remark 4.3 *Since $\mathbb{G}_\delta(t,t,x) = u_{1,\delta}(t,x)$, as a special case of Lemma 4.2 we get that $u'_{1,\delta}(t,x)$ is a.s. jointly continuous and satisfies*

$$u'_{1,\delta}(t,x) = -\int_0^{(t-\delta)_+} \int_{\mathbb{R}} G'_{t-s}(y-x)D(s,y)W(ds,dy), \quad P - a.s. \text{ for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{4.13}$$

Recall that ε_0 and ε_1 were defined in (2.14). Next we define some constants that are used repeatedly throughout the proof.

Notation. We introduce the following grid of β. Let

$$L = L(\varepsilon_0, \varepsilon_1) = \left\lfloor \left(\frac{\eta}{\eta+1} - 6\eta\varepsilon_1\right)\frac{1}{\varepsilon_0} \right\rfloor, \tag{4.14}$$

and

$$\begin{aligned} \beta_i &= i\varepsilon_0 \in \left[0, \frac{\eta}{\eta+1} - 6\eta\varepsilon_1\right], \quad \alpha_i = 2\left(\frac{\beta_i}{\eta} + \varepsilon_1\right) \in \left[0, \frac{2}{\eta+1}\right], \quad i = 0, ..., L, \\ \beta_{L+1} &= \frac{\eta}{\eta+1} - \eta\varepsilon_1. \end{aligned} \tag{4.15}$$

Note that $\beta = \beta_i, \quad i = 0, ...L+1$, satisfies

$$0 \leq \beta \leq \frac{\eta}{\eta+1} - \eta\varepsilon_1. \tag{4.16}$$

Definition 4.4 *For $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,*

$$\begin{aligned} \hat{x}_n(t,x) &= \inf\{y \in [x - a_n^{\kappa_0}, x + a_n^{\kappa_0}] : |u(t,y)| = \inf\{|u(t,z)| : |z - x| \leq a_n^{\kappa_0}\}\} \\ &\in [x - a_n^{\kappa_0}, x + a_n^{\kappa_0}]. \end{aligned} \tag{4.17}$$

In what follows we introduce some notation which is relevant to the support of the measure μ. Recall that K_1 was defined in (2.12) and that κ_0 was defined in (2.15).

Notation We fix $K_0 \in \mathbb{N}^{\geq K_1}$. Let $\rho \in (0, 1-\kappa_0]$. Since cardim$(\mu) = \eta$, then, by Definition 1.3, there exists a set $A \subset \mathbb{R}$ such that $\mu(A^c) = 0$ and $\dim_B(A) = \eta$. Then for all n sufficiently large, there exists $N(n, K_0, \eta, \rho, \varepsilon_0) \in \mathbb{N}$ such that

$$N(n, K_0, \eta, \rho, \varepsilon_0) \leq 2K_0 a_n^{-\rho-\varepsilon_0}, \tag{4.18}$$

and a covering, $\{\tilde{V}_i^{n,\rho}\}_{i=1}^{N(n,K_0,\eta,\rho,\varepsilon_0)}$, such that $\tilde{V}_i^{n,\rho}$ is an open interval of length bounded by $|\tilde{V}_i^{n,\rho}| \leq a_n^{\rho/\eta}$, for all $i = 1, \ldots, N(n, K_0, \eta, \rho, \varepsilon_0)$ and

$$(A \cap [-K_0, K_0]) \subset \cup_{i=1}^{N(n,K_0,\eta,\rho,\varepsilon_0)} \tilde{V}_i^{n,\rho}. \tag{4.19}$$

We refer to Section 2.1 in [9] for the definition of the Minkowski dimension. Note that (4.18) and (4.19) follow immediately from those definitions.

Remark 4.5 *Note that* $1-\kappa_0=\frac{\eta}{1+\eta}=\bar\beta$, *hence the range of* $\rho\in(0,\bar\beta]$ *is similar to the range of the* β_i*'s in (3.3).*

We define the following extension of the covering above. Let $V_i^{n,\rho}=\{x:|x-y|\le a_n^{\rho/\eta}\text{ for some }y\in\tilde V_i^{n,\rho}\}$. Note that $|V_i^{n,\rho}|\le 3a_n^{\rho/\eta}$, for all $i=1,\dots,N(n,K_0,\eta,\rho,\varepsilon_0)$. From (4.18) and (4.19) we get

$$\begin{aligned}|V_i^{n,\rho}|N(n,K_0,\eta,\rho,\varepsilon_0) &\le 6K_0a_n^{\rho/\eta}a_n^{-\rho-\varepsilon_0}\\ &= 6K_0a_n^{(1/\eta-1)\rho-\varepsilon_0}.\end{aligned}\tag{4.20}$$

Let $\mathbb{V}^{n,\eta,\rho,\varepsilon_0}=\cup_{i=1}^{N(n,K_0,\eta,\rho,\varepsilon_0)}V_i^{n,\rho}$. We will use the set $\mathbb{V}^{n,\eta,\rho,\varepsilon_0}$ repeatedly throughout this proof.

We define the following sets. If $s\ge0$ set

$$\begin{aligned}J_{n,0}(s) &= \big\{x:|x|\le K_0,|\langle u_s,G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-\cdot)\rangle|\le\frac{a_n}{2},u'_{1,a_n^{2\kappa_0}}(s,\hat x_n(s,x))\ge\frac{a_n^{\varepsilon_0}}{4}\big\}\\ J_{n,L}(s) &= \big\{x:|x|\le K_0,|\langle u_s,G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-\cdot)\rangle|\le\frac{a_n}{2},u'_{1,a_n^{2\kappa_0}}(s,\hat x_n(s,x))\in[0,\frac{a_n^{\beta_L}}{4}]\big\}.\end{aligned}\tag{4.21}$$

For $i=1,...,L-1$ set

$$J_{n,i}(s)=\big\{x\in\mathbb{V}^{n,\eta,\beta_i,\varepsilon_0}:|\langle u_s,G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-\cdot)\rangle|\le\frac{a_n}{2},u'_{1,a_n^{2\kappa_0}}(s,\hat x_n(s,x))\in[\frac{a_n^{\beta_{i+1}}}{4},\frac{a_n^{\beta_i}}{4}]\big\}.\tag{4.22}$$

For $i=0,...,L$, define

$$J_{n,i}=\big\{(s,x):0\le s,\quad x\in J_{n,i}(s)\big\}.\tag{4.23}$$

We also define

$$\hat J_n(s)=\big\{x:|x|\le K_0,\text{ and }x\notin\cup_{i=1}^{L-1}\mathbb{V}^{n,\eta,\beta_i,\varepsilon_0}\big\}.\tag{4.24}$$

Note that

$$\big\{x:|x|\le K_0,\ u'_{1,a_n^{2\kappa_0}}(s,\hat x_n(s,x))\ge0,\ |\langle u_s,G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-\cdot)\rangle|\le\frac{a_n}{2}\big\}\subset\big(\cup_{i=0}^{L}J_{n,i}(s)\big)\cup\hat J_n(s),\ \forall s\ge0.$$

Recall that t_0 was fixed before (2.12). If $0\le t\le t_0$, let

$$\hat I^n(t)=a_n^{-1-\kappa_0-\varepsilon_0-\frac{2}{n}}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{1}_{\hat J_n(s)}(x)e^{2R_1|y|}|u(s,y)|^{2\gamma}G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\Psi_s(x)\mu(dy)dxds,\tag{4.25}$$

and

$$I_i^n(t)=a_n^{-1-\kappa_0-\varepsilon_0-\frac{2}{n}}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{1}_{J_{n,i}(s)}(x)e^{2R_1|y|}|u(s,y)|^{2\gamma}G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\Psi_s(x)\mu(dy)dxds.\tag{4.26}$$

The following lemma gives us an essential bound on the heat kernel.

Lemma 4.6 *Let* $T>0$. *Let* $\mu\in M_f^\eta(\mathbb{R})$ *for some* $\eta\in(0,1)$. *For any* $\nu_1\in(0,1/2)$, *there is a* $C_{4.6}(\eta,\nu_1,\lambda,T)>0$ *so that*

$$\int_{\mathbb{R}}e^{\lambda|x-y|}G_t(x-y)\mathbb{1}_{\{|x-y|>t^{1/2-\nu_1}\}}\mu(dy)\le C_{4.6}(\eta,\nu_1,\lambda,T)\exp\{-t^{-2\nu_1}/8\},\ \forall x\in\mathbb{R},\ t\in(0,T],\ \lambda>0.$$

The proof of Lemma 4.6 is given in the Appendix

The following lemma is one of the ingredients in the proof of (2.21) in Proposition 2.3. Recall that t_0 is a positive constant which was fixed before (2.12).

Lemma 4.7 *For any* $K\in\mathbb{N}^{\ge K_1}$ *we have*

$$\lim_{n\to\infty}\hat I^n(t\wedge T_K)=0,\quad\forall\ t\le t_0.\tag{4.27}$$

Proof: Let $y \in A \cap [-K_1, K_1]$, where K_1 was defined in (2.12) and A was defined before (4.18). Note that by (2.15), (4.15) and the construction of $\mathbb{V}^{n,\eta,\beta_i,\varepsilon_0}$, if $x \in \hat{J}_n(s)$ then $|x-y| > a_n^{\kappa_0}$. Use this and Lemma 4.6 to get

$$\begin{aligned}
\hat{I}^n(t\wedge T_K) &\leq C(K) a_n^{-1-\kappa_0-\varepsilon_0-\frac{2}{n}} \int_0^{t_0}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\hat{J}_n(s)}(x) e^{2R_1|y|} e^{2|y|} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\Psi_s(x)\mu(dy)dxds \\
&\leq C(K) a_n^{-1-\kappa_0-\varepsilon_0-\frac{2}{n}} \int_0^{t_0}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{|x-y|>a_n^{\kappa_0}\}} e^{2(R_1+1)|y|} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\Psi_s(x)\mu(dy)dxds \\
&\leq C_{4.6}(K,\eta,\varepsilon_0,R_1,t_0) a_n^{-1-\kappa_0-\varepsilon_0-\frac{2}{n}} \int_0^{t_0}\int_{\mathbb{R}} e^{-a_n^{-\varepsilon_0}/8}\Psi_s(x)dxds \\
&\leq C(K,\eta,\varepsilon_0,R_1,t_0,K_1,\|\Psi\|_\infty) a_n^{-1-\kappa_0-\varepsilon_0-\frac{2}{n}} e^{-a_n^{-\varepsilon_0}/8}. \qquad (4.28)
\end{aligned}$$

From (2.3) and (4.28) we get (4.27). ■

Let

$$\begin{aligned}
&I_+^n \qquad (4.29)\\
&= a_n^{-1-\kappa_0-\varepsilon_0-\frac{2}{n}} \int_0^{t_0}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{u'_{1,a_n^{2\kappa_0}}(s,\hat{x}_n(s,x))\geq 0\}} \mathbb{1}_{\{|\langle u_s, G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-\cdot)\rangle|\leq a_n/2\}} e^{2R_1|y|}|u(s,y)|^{2\gamma} \\
&\quad\times G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\Psi(x)\mu(dy)dxds.
\end{aligned}$$

Then, to verify the hypothesis of Proposition 2.3, it suffices to construct the sequence of stopping times $\{U_{M,n} \equiv U_{M,n,K_0} : M, n \in \mathbb{N}\}$ satisfying (2.20) and

$$\text{for each } M \in \mathbb{N}, \quad \lim_{n\to\infty} E(I_+^n(t_0 \wedge U_{M,n})) = 0. \qquad (4.30)$$

Note that (4.30) implies (2.21) by symmetry. By (4.24)–(4.26) and (4.29) we have

$$I_+^n \leq \sum_{i=0}^{L} I_i^n(t) + \hat{I}^n(t), \quad \forall\, t \leq t_0. \qquad (4.31)$$

Therefore, from (4.31) and Lemma 4.7, to prove (2.21) it is enough to show that for $i = 0, ..., L$,

$$\text{for all } M \in \mathbb{N}, \quad \lim_{n\to\infty} E(I_i^n(t_0 \wedge U_{M,n})) = 0. \qquad (4.32)$$

Notation. Let $\bar{l}(\beta) = a_n^{\beta/\eta+5\varepsilon_1}$.

We introduce the sets $\{\tilde{J}_{n,i}(s)\}_{i=1,...,L}$ to be defined as follows. In each of these sets we consider the points $x \in [-K_0, K_0]$ such that $|\langle u_s, G_{a_n^{2\kappa_0+2\varepsilon_0}}(\cdot - x)\rangle| \leq \frac{a_n}{2}$ as in $\{J_{n,i}(s)\}_{i=1,...,L}$. Motivated by (3.3) and (4.10), in each of these sets we restrict $u'_{1,a_n^{\alpha_0}}(s + a_n^{2\kappa_0+2\varepsilon_0}, x')$ to certain intervals, for $x' \in [x - 5\bar{l}(\beta_0), x + 5\bar{l}(\beta_0)]$, and bound the increment of $\tilde{u}_{2,a_n^{\alpha_0}}(s, a_n^{2\kappa_0+2\varepsilon_0}, \cdot)$. The bound on $\int_{\mathbb{R}} e^{2R_1|y|}|u(s,y)|^{2\gamma} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\mu(dy)$ in the definition of $\{\tilde{J}_{n,i}(s)\}_{i=1,...,L}$ will be used for technical reasons (see (4.42) and (4.43)).

$$\begin{aligned}
\tilde{J}_{n,0}(s) = \Big\{ &x \in [-K_0, K_0] : |\langle u_s, G_{a_n^{2\kappa_0+2\varepsilon_0}}(\cdot - x)\rangle| \leq \frac{a_n}{2}, u'_{1,a_n^{\alpha_0}}(s + a_n^{2\kappa_0+2\varepsilon_0}, x') \geq \frac{a_n^{\beta_1}}{16}, \\
&\forall x' \in [x - 5\bar{l}(\beta_0), x + 5\bar{l}(\beta_0)], \\
&|\tilde{u}_{2,a_n^{\alpha_0}}(s, a_n^{2\kappa_0+2\varepsilon_0}, x') - \tilde{u}_{2,a_n^{\alpha_0}}(s, a_n^{2\kappa_0+2\varepsilon_0}, x'')| \leq 2^{-75} a_n^{\beta_1}(|x'-x''| \vee a_n), \\
&\forall x' \in [x - 4a_n^{\kappa_0}, x + 4a_n^{\kappa_0}],\ x'' \in [x' - \bar{l}(\beta_0), x' + \bar{l}(\beta_0)],\ \text{and} \\
&\int_{\mathbb{R}} e^{2R_1|y|}|u(s,y)|^{2\gamma} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\mu(dy) \leq C(R_1, K_0, \eta, \varepsilon_0) a_n^{2\gamma\kappa_0 - \kappa_0(1-\eta) - 2\varepsilon_0} \Big\},
\end{aligned}$$

$$\begin{aligned}
\tilde{J}_{n,L}(s) = \Big\{ &x \in [-K_0, K_0] : |\langle u_s, G_{a_n^{2\kappa_0+2\varepsilon_0}}(\cdot - x)\rangle| \leq \frac{a_n}{2}, u'_{1,a_n^{\alpha_L}}(s + a_n^{2\kappa_0+2\varepsilon_0}, x') \leq a_n^{\beta_L}, \qquad (4.33)\\
&\forall x' \in [x - 5\bar{l}(\beta_L), x + 5\bar{l}(\beta_L)], \\
&|\tilde{u}_{2,a_n^{\alpha_L}}(s, a_n^{2\kappa_0+2\varepsilon_0}, x') - \tilde{u}_{2,a_n^{\alpha_L}}(s, a_n^{2\kappa_0+2\varepsilon_0}, x'')| \leq 2^{-75} a_n^{\beta_{L+1}}(|x'-x''| \vee a_n), \\
&\forall x' \in [x - 4a_n^{\kappa_0}, x + 4a_n^{\kappa_0}],\ x'' \in [x' - \bar{l}(\beta_L), x' + \bar{l}(\beta_L)],\ \text{and} \\
&\int_{\mathbb{R}} e^{2R_1|y|}|u(s,y)|^{2\gamma} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\mu(dy) \leq C(R_1, K_0, \eta, \varepsilon_0) a_n^{2\gamma(\kappa_0+\beta_L) - \kappa_0(1-\eta) - 2\varepsilon_0} \Big\},
\end{aligned}$$

and for $i \in \{1, ..., L-1\}$,

$$\begin{aligned}\tilde{J}_{n,i}(s) \;=\; & \{x \in \mathbb{V}^{\eta,\beta_i,\varepsilon_0} : |\langle u_s, G_{a_n^{2\kappa_0+2\varepsilon_0}}(\cdot - x)\rangle| \le \frac{a_n}{2}, u'_{1,a_n^{\alpha_i}}(s + a_n^{2\kappa_0+2\varepsilon_0}, x') \in [a_n^{\beta_{i+1}}/16, a_n^{\beta_i}], \\ & \forall x' \in [x - 5\bar{l}(\beta_i), x + 5\bar{l}(\beta_i)], \\ & |\tilde{u}_{2,a_n^{\alpha_i}}(s, a_n^{2\kappa_0+2\varepsilon_0}, x') - \tilde{u}_{2,a_n^{\alpha_i}}(s, a_n^{2\kappa_0+2\varepsilon_0}, x'')| \le 2^{-75} a_n^{\beta_{i+1}}(|x' - x''| \vee a_n), \\ & \forall x' \in [x - 4a_n^{\kappa_0}, x + 4a_n^{\kappa_0}],\ x'' \in [x' - \bar{l}(\beta_i), x' + \bar{l}(\beta_i)],\ \text{and} \\ & \int_{\mathbb{R}} e^{2R_1|y|}|u(s,y)|^{2\gamma} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\mu(dy) \le C(R_1, K_0, \eta, \varepsilon_0) a_n^{2\gamma(\kappa_0+\beta_i)-\kappa_0(1-\eta)-2\varepsilon_0}\}. \end{aligned} \tag{4.34}$$

Finally for $0 \le i \le L$, set

$$\tilde{J}_{n,i} = \{(s,x) : s \ge 0, x \in \tilde{J}_{n,i}(s)\}. \tag{4.35}$$

Notation Let $n_M(\varepsilon_1) = \inf\{n \in \mathbb{N} : a_n^{\varepsilon_1} \le 2^{-M-8}\}$, $n_0(\varepsilon_1, \varepsilon_0) = \sup\{n \in \mathbb{N} : a_n^{\kappa_0} < 2^{-a_n^{-\varepsilon_0\varepsilon_1/4}}\}$, where $\sup\emptyset = 1$ and

$$n_1(\varepsilon_0, K_0) = \inf\left\{n \in \mathbb{N} : a_n \int_{-a_n^{-\varepsilon_0}}^{a_n^{-\varepsilon_0}} G_1(y)dy - 2K_0 e^{K_0} \int_{a_n^{-\varepsilon_0}}^{\infty} e^{|y|} G_1(y)dy > \frac{a_n}{2}\right\}. \tag{4.36}$$

The following proposition corresponds to Proposition 3.3 in [16]. We will prove this proposition in Section 7.

Proposition 4.8 *$\tilde{J}_{n,i}(s)$ is a compact set for all $s \ge 0$. There exist stopping times $\{U_{M,n} = U_{M,n,K_0} : M, n \in \mathbb{N}\}$, satisfying (2.20) and $n_2(\varepsilon_0, \kappa_0, \gamma, \eta, K_0, R_1) \in \mathbb{N}$ such that for $i \in \{0, 1, ..., L\}$, $\tilde{J}_{n,i}(s)$ contains $J_{n,i}(s)$ for all $0 \le s \le U_{M,n}$ and*

$$n > n_M(\varepsilon_1) \vee n_0(\varepsilon_0, \varepsilon_1) \vee n_1(\varepsilon_0, K_0) \vee n_2(\varepsilon_0, \kappa_0, \gamma, \eta, K_0, R_1). \tag{4.37}$$

Throughout the rest of this section we assume that the parameters $M, n \in \mathbb{N}$ satisfy (4.37).

The following Lemma corresponds to Lemma 3.4 in [16].

Lemma 4.9 *Assume $i \in \{0, ..., L\}$, $x \in \tilde{J}_{n,i}(s)$ and $|x - x'| \le 4a_n^{\kappa_0}$.*

(a) *If $i > 0$, then*

$$|\langle u_s, G_{a_n^{2\kappa_0+2\varepsilon_0}}(\cdot - x'')\rangle - \langle u_s, G_{a_n^{2\kappa_0+2\varepsilon_0}}(\cdot - x')\rangle| \le 2a_n^{\beta_i}(|x' - x''| \vee a_n), \quad \forall |x'' - x'| \le \bar{l}_n(\beta_i).$$

(b) *If $i < L$, and $a_n \le |x'' - x'| \le \bar{l}_n(\beta_i)$, then*

$$\langle u_s, G_{a_n^{2\kappa_0+2\varepsilon_0}}(\cdot - x'')\rangle - \langle u_s, G_{a_n^{2\kappa_0+2\varepsilon_0}}(\cdot - x')\rangle \begin{cases} \ge 2^{-5} a_n^{\beta_{i+1}}(x'' - x') & \text{if } x'' \ge x', \\ \le 2^{-5} a_n^{\beta_{i+1}}(x'' - x') & \text{if } x' \ge x''. \end{cases}$$

The proof follows the same lines as the proof of Lemma 3.4 in [16], hence it is omitted.

Notation. Let $l_n(\beta_i) = 65 a_n^{1-\beta_{i+1}}$. The following Lemma corresponds to Lemma 3.6 in [16].

Lemma 4.10 *If $i \in \{0, ..., L\}$, then*

$$l_n(\beta_i) < a_n^{\kappa_0} < \frac{1}{2}\bar{l}_n(\beta_i). \tag{4.38}$$

The proof of Lemma 4.10 is similar to the proof of Lemma 3.6 in [16], hence it is omitted.

Lemma 4.11 **(a)** *For all $s \ge 0$,*

$$|\tilde{J}_{n,0}(s)| \le 10K_0 \bar{l}_n(\beta_0)^{-1} l_n(\beta_0).$$

(b) *For all $i \in \{1, ..., L-1\}$ and $s \ge 0$,*

$$|\tilde{J}_{n,i}(s)| \le 10K_0 \bar{l}_n(\beta_i)^{-\eta} l_n(\beta_i).$$

Proof: The proof of (a) follows the same lines as the proof of Lemma 3.7 in [16] for the case where $i = 0$. The proof (b) also follows the same lines as the proof of Lemma 3.7 in [16] for the case where $i = \{1, ..., L-1\}$. The major difference is that in our case $\tilde{J}_{n,i}(s) \subset \mathbb{V}^{n,\eta,\beta_i,\varepsilon_0}$, so instead of covering $\tilde{J}_{n,i}(s)$ with $\bar{l}_n(\beta_i)^{-1}$ balls as in [16], we can cover it with a smaller number of balls which is proportional to $\bar{l}_n(\beta_i)^{-\eta}$. ∎

Proof of (2.21) in Proposition 2.3. Recall that to prove (2.21) in Proposition 2.3, it is enough to show (4.32), for $i = 0, ..., L$. In fact we will prove a stronger result. Recall that n_M, n_0, n_1, n_2 were defined before and in Proposition 4.8. We will show that for

$$n > n_M(\varepsilon_1) \vee n_0(\varepsilon_0, \varepsilon_1) \vee \frac{2}{\varepsilon_1} \vee n_1(\varepsilon_0, K_0) \vee n_2(\varepsilon_0, \kappa_0, \gamma, \eta, K_0, R_1), \tag{4.39}$$

$$I_i^n(t_0 \wedge U_{M,n}) \leq C(\eta, K_0, t_0, \|\Psi\|_\infty) a_n^{\gamma - 1 + \frac{\eta}{2(\eta+1)} - 13\varepsilon_1}, \tag{4.40}$$

which implies (4.32) since $\gamma > 1 - \frac{\eta}{2(\eta+1)} + 100\varepsilon_1$. By Proposition 4.8, $\frac{2}{n} < \varepsilon_1$ (by (4.39)) and (2.14) we have

$$\begin{aligned} & I_i^n(t_0 \wedge U_{M,n}) \qquad (4.41)\\ &= a_n^{-1-\kappa_0-\varepsilon_0-\frac{2}{n}} \int_0^{t_0 \wedge U_{M,n}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2R_1|y|} \mathbb{1}_{J_{n,i}(s)}(x) |u(s,y)|^{2\gamma} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y) \Psi_s(x) \mu(dy) dx ds \\ &\leq a_n^{-1-\kappa_0-2\varepsilon_1} \int_0^{t_0} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2R_1|y|} \mathbb{1}_{\{s<U_{M,n}\}} \mathbb{1}_{\tilde{J}_{n,i}(s)}(x) G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y) |u(s,y)|^{2\gamma} \Psi_s(x) \mu(dy) dx ds. \end{aligned}$$

Consider first the case where $i = 0$. For $x \in \tilde{J}_{n,0}(s)$ we have

$$\int_{\mathbb{R}} e^{2R_1|y|} |u(s,y)|^{2\gamma} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y) \mu(dy) \leq C(R_1, K_0, \eta, \varepsilon_0) a_n^{2\gamma\kappa_0 - \kappa_0(1-\eta) - 2\varepsilon_0}. \tag{4.42}$$

We get from (4.41), (4.42) and Lemma 4.11(a),

$$\begin{aligned} & I_0^n(t_0 \wedge U_{M,n}) \qquad (4.43)\\ &\leq C(R_1, K_0, \eta, \varepsilon_0) a_n^{-1-\kappa_0-2\varepsilon_1} a_n^{2\gamma\kappa_0-\kappa_0(1-\eta)-2\varepsilon_0} \|\Psi\|_\infty \int_0^{t_0} \int_{\mathbb{R}} \mathbb{1}_{\tilde{J}_{n,0}(s)}(x) \mathbb{1}_{[-K_0,K_0]}(x) dx ds \\ &\leq C(R_1, K_0, \eta, \varepsilon_0, t_0, \|\Psi\|_\infty) a_n^{-1-\kappa_0-2\varepsilon_1} a_n^{2\gamma\kappa_0-\kappa_0(1-\eta)-2\varepsilon_0} 10 K_0 \bar{l}_n(\beta_0)^{-1} l_n(\beta_0). \end{aligned}$$

From the definitions of $\bar{l}_n(\beta_i), l_n(\beta_i)$ we get,

$$\begin{aligned} I_0^n(t_0 \wedge U_{M,n}) &\leq C(R_1, K_0, \eta, \varepsilon_0, t_0, \|\Psi\|_\infty) a_n^{-1-\kappa_0-2\varepsilon_1} a_n^{2\gamma\kappa_0-\kappa_0(1-\eta)-2\varepsilon_0} a_n^{-5\varepsilon_1} 65 a_n^{1-\varepsilon_0} \\ &\leq C(R_1, K_0, \eta, \varepsilon_0, t_0, \|\Psi\|_\infty) a_n^{\rho_0}. \qquad (4.44) \end{aligned}$$

From (2.14) , (2.15) and (1.10) we have

$$\begin{aligned} \rho_0 &= -1 - \kappa_0 - 2\varepsilon_1 + 2\gamma\kappa_0 + \kappa_0(\eta - 1) - 5\varepsilon_1 + 1 - 3\varepsilon_0 \\ &\geq 2\kappa_0\gamma - 2\kappa_0 + \kappa_0\eta - 8\varepsilon_1 \\ &\geq \gamma - 1 + \frac{\eta}{2} - 8\varepsilon_1. \qquad (4.45) \end{aligned}$$

Consider now $i = \{1, ..., L\}$. Assume $x \in \tilde{J}_{n,i}(s)$. Repeat the same steps as in (4.43) to get,

$$\begin{aligned} & I_i^n(t_0 \wedge U_{M,n}) \qquad (4.46)\\ &\leq C(R_1, K_0, \eta, \varepsilon_0, t_0, \|\Psi\|_\infty) a_n^{-1-\kappa_0-2\varepsilon_1} a_n^{2\gamma(\beta_i+\kappa_0)+\kappa_0(\eta-1)-2\varepsilon_0} \int_0^{t_0} |\tilde{J}_{n,i}(s)| ds. \end{aligned}$$

For $i = \{1, \cdots, L-1\}$ apply Lemma 4.11(b) to (4.46) to get

$$\begin{aligned} & I_i^n(t_0 \wedge U_{M,n}) \\ &\leq C(K_0, \eta, \varepsilon_0, t_0, \|\Psi\|_\infty, \omega) a_n^{-1-\kappa_0-2\varepsilon_1+2\gamma(\beta_i+\kappa_0)+\kappa_0(\eta-1)-2\varepsilon_0} 10 K_0 \bar{l}_n(\beta_i)^{-\eta} l_n(\beta_i) \\ &\leq C(K_0, \eta, \varepsilon_0, t_0, \|\Psi\|_\infty, \omega) a_n^{-1-\kappa_0-2\varepsilon_1+2\gamma(\beta_i+\kappa_0)+\kappa_0(\eta-1)-2\varepsilon_0} a_n^{-\beta_i-5\varepsilon_1} 65 a_n^{1-\beta_{i+1}} \\ &\leq C(K_0, \eta, \varepsilon_0, t_0, \|\Psi\|_\infty, \omega) a_n^{\rho_{1,i}}. \qquad (4.47) \end{aligned}$$

Use (2.15), (2.14), (1.10) and (4.15) to get $\beta_i < 1 - \kappa_0$, and

$$\begin{aligned} \rho_{1,i} &= 2\kappa_0\gamma - 2\beta_i(1-\gamma) - (2-\eta)\kappa_0 - 3\varepsilon_0 - 7\varepsilon_1 \\ &> 2\kappa_0\gamma - 2(1-\kappa_0)(1-\gamma) - (2-\eta)\kappa_0 - 3\varepsilon_0 - 7\varepsilon_1 \\ &\geq 2\gamma - \frac{2\eta}{\eta+1} - \frac{2-\eta}{\eta+1} - 15\varepsilon_1 \\ &\geq \gamma - 1 + \frac{\eta}{2(\eta+1)} - 8\varepsilon_1. \qquad (4.48) \end{aligned}$$

For $i = L$, we repeat the same steps as in (4.46). We also use $\mathbb{V}^{n,\eta,\rho,\varepsilon_0}$ with $\rho = 1 - \beta_{L+1} = \eta\kappa_0 + \varepsilon_1\eta$, to cover the integration region. Then we use (4.20), (2.14), (4.16), (2.15) and (1.10) to get

$$\begin{aligned} & I^n_L(t_0 \wedge U_{M,n}) \\ & \leq C(R_1, K_0, \eta, \varepsilon_0, \|\Psi\|_\infty) t_0 a_n^{(\eta-2)\kappa_0-1-3\varepsilon_1+2\gamma(\beta_L+\kappa_0)} \sum_{i=1}^{N(n,K_0,\eta,\eta\kappa_0,\varepsilon_0)} |V_i^{n,\kappa_0}| \\ & \leq C(R_1, K_0, \eta, \varepsilon_0, t_0, \|\Psi\|_\infty) a_n^{-1-\kappa_0-3\varepsilon_1+2\gamma-12\varepsilon_1-\varepsilon_0} \\ & \leq C(R_1, K_0, \eta, \varepsilon_0, t_0, \|\Psi\|_\infty) a_n^{\gamma-1+\frac{\eta}{2(\eta+1)}-13\varepsilon_1}. \end{aligned} \tag{4.49}$$

From (4.44), (4.45), (4.47), (4.48) and (4.49), it follows that,

$$I^n_i(t_0 \wedge U_{M,n}) \leq C(R_1, K_0, \eta, \varepsilon_0, t_0, \|\Psi\|_\infty) a_n^{\gamma-1+\frac{\eta}{2(1+\eta)}-13\varepsilon_1}, \ \forall\, i = 0, ..., L, \tag{4.50}$$

and we proved (4.40). ■

5 Some Integral Bounds for the Heat Kernel

In this section we introduce some integral bounds for the heat kernel. These bounds will be useful for the proofs in the following sections. First let us recall some useful lemmas from [16], [20] and [25].

For $0 \leq p \leq 1$, $q \in \mathbb{R}$ and $0 \leq \Delta_2 \leq \Delta_1 \leq t$, define

$$J_{p,q}(\Delta_1, \Delta_2, \Delta) = \int_{t-\Delta_1}^{t-\Delta_2} (t-s)^q \big(1 \wedge \frac{\Delta}{t-s}\big)^p ds.$$

We will use Lemmas 4.1, 4.2 from [16].

Lemma 5.1 **(a)** *If $q > p - 1$, then*

$$J_{p,q}(\Delta_1, \Delta_2, \Delta) \leq \frac{2}{q+1-p} (\Delta \wedge \Delta_1)^p \Delta_1^{q+1-p}.$$

(b) *If $-1 < q < p - 1$, then*

$$\begin{aligned} J_{p,q}(\Delta_1, \Delta_2, \Delta) \quad & \leq \quad ((p-1-q)^{-1} + (q+1)^{-1}) \big[(\Delta \wedge \Delta_1)^{q+1} \mathbb{1}_{\{\Delta_2 \leq \Delta\}} \\ & \quad + (\Delta \wedge \Delta_1)^p \Delta_2^{q-p+1} \mathbb{1}_{\{\Delta_2 > \Delta\}}\big] \\ & \leq \quad ((p-1-q)^{-1} + (q+1)^{-1}) \Delta^p (\Delta \vee \Delta_2)^{q-p+1}. \end{aligned}$$

(c) *If $q < -1$, then*

$$J_{p,q}(\Delta_1, \Delta_2, \Delta) \leq 2|q+1|^{-1} (\Delta \wedge \Delta_2)^p \Delta_2^{q+1-p}.$$

Let

$$G'_t(x) = \frac{\partial G_t(x)}{\partial x}.$$

Lemma 5.2

$$|G'_t(z)| \leq C_{5.2} t^{-1/2} G_{2t}(z).$$

The inequalities in the following lemma were introduced as equations (2.4e) and (2.4f) in Section 2 of [20].

Lemma 5.3 *For any $0 \leq \delta \leq 1$, there exists a constant $C_{5.3} > 0$ such that*

(a)

$$|G_t(x-y) - G_t(x'-y)| \leq C_{5.3} |x - x'|^\delta t^{-(1+\delta)/2} \big(e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x'-y)^2}{2t}}\big), \tag{5.1}$$

for all $t \geq 0$, $x, x' \in \mathbb{R}$.

(b)

$$|G'_t(x-y) - G'_t(x'-y)| \leq C_{5.3}|x-x'|^\delta t^{-(2+\delta)/2}\big(e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x'-y)^2}{2t}}\big), \tag{5.2}$$

for all $t \geq 0$, $x, x' \in \mathbb{R}$.

We will use the following upper bound on the exponential function.

Lemma 5.4 *Let* $a > 0$, *then for any* $\delta > 0$, *there exists a constant* $C_{5.4}(a,\delta) > 0$ *such that*

$$e^{-\frac{x^2}{at}} \leq C_{5.4}(a,\delta)t^{\delta/2}|x|^{-\delta},$$

for all $t > 0$, $x \in \mathbb{R}$.

The proof of Lemma 5.4 is trivial, hence it is omitted.

The following lemma puts together the results of Lemma 3.4(c) and Lemma 3.7 from [25].

Lemma 5.5 *Let* $\mu \in M_f^\eta(\mathbb{R})$ *for some* $\eta \in (0,1)$. *Let* $\varpi \in (0,\eta)$ *and* $T > 0$. *Then for every* $\lambda \geq 0$, $r \in (0, 2+\eta-\varpi)$ *and* $t \in [0,T]$,

(a) *There exists a constant* $C_{(5.3)}(r,\lambda,T,\eta,\varpi) > 0$ *such that*

$$e^{-\lambda|x|}\int_{\mathbb{R}} e^{\lambda|y|}G^r_{t-s}(x-y)\mu(dy) < \frac{C_{(5.3)}(r,\lambda,T,\eta,\varpi)}{(t-s)^{(r-\eta+\varpi)/2}}, \ \forall s \in [0,t), \quad x \in \mathbb{R}. \tag{5.3}$$

(b) *There exists a constant* $C_{(5.4)}(r,\lambda,T,\eta,\varpi) > 0$ *such that*

$$\sup_{x\in\mathbb{R}} e^{-\lambda|x|}\int_0^t\int_{\mathbb{R}} e^{\lambda|y|}G^r_{t-s}(x-y)\mu(dy)ds < C_{(5.4)}(r,\lambda,T,\eta,\varpi), \ \forall t \in [0,T]. \tag{5.4}$$

(c) *For every* $\delta \in (0,\eta-\varpi)$, *there exists a constant* $C_{(5.5)}(\delta,T,\eta,\varpi,\lambda) > 0$ *such that*

$$\int_0^{t\vee t'}\int_{\mathbb{R}} e^{\lambda|y|}(G_{t-s}(x-y) - G_{t'-s}(x'-y))^2\mu(dy)ds$$
$$\leq C_{(5.5)}(\delta,T,\eta,\varpi,\lambda)(|t-t'|^{\delta/2} + |x-x'|^\delta)e^{\lambda|x|}e^{\lambda|x-x'|}, \ \forall t,t' \in [0,T], \ x,x' \in \mathbb{R}, \ \lambda > 0. \tag{5.5}$$

(d) *For every* $a \in [0,2]$, $\varepsilon \in (0,\eta/2)$ *and* $\theta \geq 0$, *there exists a constant* $C_{(5.6)}(\theta,a,\eta,T,\varepsilon) > 0$ *such that*

$$\int_{\mathbb{R}} e^{\theta|x-y|}(x-y)^{2\alpha}G_t(y-x)^2\mu(dy) \leq \frac{C_{(5.6)}(\theta,a,\eta,T,\varepsilon)}{t^{1-\alpha-\eta/2+\varepsilon}}, \ \forall t \in (0,T], \ x \in \mathbb{R}. \tag{5.6}$$

Proof: (a) follows immediately from Lemma 3.4(c) in [25]. (b) follows from (a) by integration. (c) was introduced as equation 8 in Lemma 3.7 of [25]. The proof of (d) follows from Lemma 5.4 with $\delta = 2\alpha+\eta-\varepsilon$ along with (1.2). ■

Notation Let

$$d((t,x),(t',x')) = |t-t'|^{1/2} + |x'-x|. \tag{5.7}$$

The following lemma is a modification of Lemma 4.3 from [16].

Lemma 5.6 *Let* $\mu \in M_f^\eta(\mathbb{R})$ *for some* $\eta \in (0,1)$ *and let* $\varepsilon \in (0,\eta/2)$.

(a) *Then, there exists a constant* $C_{5.6}(\eta,\varepsilon) > 0$ *such that* $0 \leq s < t \leq t'$, $x,x' \in \mathbb{R}$,

$$\int_{\mathbb{R}} (G_{t'-s}(x'-y) - G_{t-s}(x-y))^2\mu(dy) \leq C_{5.6}(\eta,\varepsilon)(t-s)^{\eta/2-1-\varepsilon}\left[1 \wedge \frac{d((t,x),(t',x'))^2}{t-s}\right].$$

(b) *For any $R>2$ there is a $C_{5.6}(R,\eta,\varepsilon,\nu_0,\nu_1)>0$ so that for any $0\le p,r\le R$, $\nu_0,\nu_1\in(1/R,1/2)$, $0\le s<t\le t'\le R$, $x,x'\in\mathbb{R}$,*

$$\begin{aligned}&\int_{\mathbb{R}} e^{r|x-y|}|x-y|^p(G_{t'-s}(x'-y)-G_{t-s}(x-y))^2 1\!\!1_{\{|x-y|>(t'-s)^{1/2-\nu_0}\vee 2|x'-x|\}}\mu(dy)\\ &\le C_{5.6}(R,\eta,\varepsilon,\nu_0,\nu_1)|t-s|^{\eta/2-1-\varepsilon}\exp\{-\nu_1(t'-s)^{-2\nu_0}/32\}\\ &\times\left[1\wedge\frac{d((t,x),(t',x'))^2}{t-s}\right]^{1-(\nu_1/2)}.\end{aligned}$$

The following Lemma is a modification of Lemma 4.4 from [16].

Lemma 5.7 *Let $\mu\in M_f^\eta(\mathbb{R})$ for some $\eta\in(0,1)$ and let $\varepsilon\in(0,\eta/2)$.*

(a) *Then, there exists a constant $C_{5.7}(\eta,\varepsilon)>0$ such that $s<t\le t'$, $x,x'\in\mathbb{R}$,*

$$\int_{\mathbb{R}}(G'_{t'-s}(x'-y)-G'_{t-s}(x-y))^2\mu(dy)\le C_{5.7}(\eta,\varepsilon)(t-s)^{\eta/2-\varepsilon-2}\left[1\wedge\frac{d((t,x),(t',x'))^2}{t-s}\right].$$

(b) *For any $R>2$ there is a $C_{5.7}(R,\nu_0,\nu_1,\eta,\varepsilon)>0$ so that for any $0\le p,r\le R$, $\nu_0,\nu_1\in(1/R,1/2)$, $0\le s<t\le t'\le R$, $x,x'\in\mathbb{R}$,*

$$\begin{aligned}&\int_{\mathbb{R}} e^{r|x-y|}|x-y|^p(G'_{t'-s}(x'-y)-G'_{t-s}(x-y))^2 1\!\!1_{\{|x-y|>(t'-s)^{1/2-\nu_0}\vee 2|x'-x|\}}\mu(dy)\\ &\le C_{5.7}(R,\nu_0,\nu_1,\eta,\varepsilon)(t-s)^{\eta/2-2-\varepsilon}\exp\{-\nu_1(t'-s)^{-2\nu_0}/64\}\\ &\times\left[1\wedge\frac{d((t,x),(t',x'))^2}{t-s}\right]^{1-(\nu_1/2)}.\end{aligned}$$

The following lemma follows from Lemmas 5.5 and 5.6(a).

Lemma 5.8 *Let $\mu\in M_f^\eta(\mathbb{R})$ for some $\eta\in(0,1)$. Let $\lambda>0$ and $\varepsilon\in(0,\eta/2)$. There is a $C_{5.8}(\eta,\varepsilon,\lambda)>0$ such that for any $0\le s<t\le t'$, $x,x'\in\mathbb{R}$,*

$$\int_{\mathbb{R}} e^{\lambda|y|}\big(G_{t-s}(x'-y)-G_{t'-s}(x-y)\big)^2\mu(dy)\ \le\ C_{5.8}(\eta,\varepsilon,\lambda)(t-s)^{\eta/2-\varepsilon-2}d((t,x),(t',x'))^2e^{2\lambda|x|}e^{2\lambda|x'-x|}.$$

The proofs of Lemmas 5.6–5.8 are rather long and technical, so they are given in the Appendix.

6 Local Bounds on the Difference of Solutions

This section is devoted to establishing local bounds on the difference of two solutions of (1.1). These bounds are crucial for the construction of the stopping times in Proposition 4.8. In this section we assume again that u_1,u_2 are two solutions of (1.1). We denote $u=u^1-u^2$ and we assume the hypotheses of Theorem 1.5 and (2.2). Our argument follows the same lines as the argument in Section 5 of [16].

Notation For all $K,N,n\in\mathbb{N}$, $\xi\in(0,1)$ and $\beta\in(0,\frac{\eta}{\eta+1}]$, let

$$\begin{aligned}Z(K,N,\xi)(\omega)\ &=\ \{(t,x)\in[0,T_K]\times[-K,K]:\ \text{there is a }(\hat t_0,\hat x_0)\in[0,T_K]\times\mathbb{R},\\ &\qquad\text{such that } d((t,x),(\hat t_0,\hat x_0))\le 2^{-N}\text{ and }|u(\hat t_0,\hat x_0)|\le 2^{-N\xi}\},\end{aligned}\tag{6.1}$$

$$\begin{aligned}Z(N,n,K,\beta)\ &=\ \{(t,x)\in[0,T_K]\times[-K,K]:\text{there is a }(\hat t_0,\hat x_0)\in[0,T_K]\times\mathbb{R}\\ &\qquad\text{such that } d((\hat t_0,\hat x_0),(t,x))\le 2^{-N},\\ &\qquad |u(\hat t_0,\hat x_0)|\le a_n\wedge(a_n^{1-\kappa_0}2^{-N}),\ \text{and } |u'_{1,a_n^{2\kappa_0}}(\hat t_0,\hat x_0)|\le a_n^\beta\},\end{aligned}$$

and for $\beta=0$ define $Z(N,n,K,0)(\omega)=Z(N,n,K)(\omega)$ as above, but with the condition on $|u'_{1,a_n}(\hat t_0,\hat x_0)|\le a_n^\beta$ omitted.

Recall that we fixed $\eta \in (0,1)$ and $\gamma \in \left(1 - \frac{\eta}{2(\eta+1)}, 1\right)$ in (1.10). Let

$$\gamma_m = \frac{(\gamma - 1 + \eta/2)(1-\gamma^m)}{1-\gamma} + 1, \quad \tilde{\gamma}_m = \gamma_m \wedge (1+\eta). \tag{6.2}$$

From (6.2) we get

$$\gamma_{m+1} = \gamma\gamma_m + \eta/2, \quad \gamma_0 = 1. \tag{6.3}$$

Note that γ_m increases to $\gamma_\infty = \frac{(\gamma-1+\eta/2)}{1-\gamma} + 1 = \frac{\eta}{2(1-\gamma)} > \eta + 1$ (the last inequality follows by (1.10)) and therefore we can define a finite natural number, $\bar{m} > 1$, by

$$\bar{m} = \min\{m : \gamma_{m+1} > \eta + 1\} = \min\{m : \gamma\gamma_m > 1 + \frac{\eta}{2}\}. \tag{6.4}$$

Note that

$$\tilde{\gamma}_{\bar{m}+1} = \eta + 1. \tag{6.5}$$

Remark 6.1 *In this section we often use the constraint $\gamma > 1 - \eta/2$. Note that it holds trivially for γ satisfying (1.10).*

Definition 6.2 *A collection of $[0,\infty]$-valued random variables, $\{N(\alpha) : \alpha \in A\}$, is stochastically bounded uniformly in α if*

$$\lim_{M\to\infty} \sup_{\alpha \in A} P(N(\alpha) \geq M) = 0. \tag{6.6}$$

Recall that K_1 was chosen to satisfy (2.12).

Property (P_m). For $m \in \mathbb{Z}_+$ we let (P_m) denote the following property:

$$\begin{aligned}
&\text{For any } n \in \mathbb{N}, \xi, \varepsilon_0 \in (0,1), K \in \mathbb{N}^{\geq K_1} \text{ and } \beta \in [0, \frac{\eta}{\eta+1}], \text{ there is an} \\
&N_1(\omega) = N_1(m,n,\xi,\varepsilon_0,K,\beta,\eta) \text{ in } \mathbb{N} \text{ a.s. such that for all } N \geq N_1, \\
&\text{if } (t,x) \in Z(N,n,K,\beta),\ t^{'} \leq T_K \text{ and } d((t,x),(t',x')) \leq 2^{-N}, \\
&\text{then } |u(x',t')| \leq a_n^{-\varepsilon_0} 2^{-N\xi}\big[(a_n^{\kappa_0} \vee 2^{-N})^{\tilde{\gamma}_m - 1} + a_n^{\beta} \mathbb{1}_{\{m>0\}}\big]. \\
&\text{Moreover } N_1 \text{ is stochastically bounded uniformly in } (n,\beta).
\end{aligned} \tag{6.7}$$

The goal of this section is to prove the following proposition.

Proposition 6.3 *For any $m \leq \bar{m} + 1$, (P_m) holds.*

Remark 6.4 *We will prove Proposition 6.3 by induction.*

We introduce the following theorem, that will help us to prove (P_0).

Theorem 6.5 *Assume the same assumptions as in Theorem 1.5 except now allow $\gamma \geq 1 - \eta/2$. For each $K \in \mathbb{N}$ and $\xi \in (0,1)$ there is an $N_0(\xi, K, \omega) \in \mathbb{N}$ a.s. such that for all natural numbers $N \geq N_0$ and all $(t,x) \in Z(K,N,\xi)$,*

$$d((t',x'),(t,x)) \leq 2^{-N} \text{ and } t' \leq T_K \text{ implies } |u(t',x') - u(t,x)| \leq 2^{-N\xi}.$$

The proof of Theorem 6.5 is given in Section 9. Theorem 6.5 was proved in [16] for the case of homogeneous white noise (see Theorem 2.3 therein).

Proof of (P_0) The proof of (P_0) is similar to the proof of (P_0) in Section 5 of [16]: just replace Theorem 2.3 in [16] with Theorem 6.5. Exactly as in the proof of (P_0) in Section 5 of [16], we get that $N_1 = N_1(0, \xi, K, \eta)$. That is, N_1 does not depend on (n, β). ■

To carry out the induction, we first use (P_m) to get a local modulus of continuity for F_δ (as in Section 5 of [16]).

Recall that F_δ was given by (4.12) where

$$D(r, y) = \sigma(r, y, u^1(r, y)) - \sigma(r, y, u^2(r, y)).$$

From (4.12) we get for $s \le t \le t'$ and $s \le s' \le t'$

$$\begin{aligned} |F_\delta(s,t,x) - F_\delta(s',t',x')| &\le |F_\delta(s,t',x') - F_\delta(s',t',x')| + |F_\delta(s,t',x') - F_\delta(s,t,x)| \\ &= \left| \int_{(s-\delta)_+}^{(s'-\delta)_+} \int_{\mathbb{R}} G'_{t'-r}(y-x') D(r,y) W(dr,dy) \right| \\ &\quad + \left| \int_0^{(s-\delta)_+} \int_{\mathbb{R}} \big(G'_{t'-r}(y-x') - G'_{t-r}(y-x) \big) D(r,y) W(dr,dy) \right|. \end{aligned} \tag{6.8}$$

From (6.8) and (4.3) we realize that to get the bound on $|F_\delta(s,t,x) - F_\delta(s',t',x')|$ we may use the bounds on the following square functions

$$\begin{aligned} Q_{T,\delta}(s,s',t',x') &= \int_{(s\wedge s'-\delta)_+}^{(s\vee s'-\delta)_+} \int_{\mathbb{R}} G'_{t'-r}(y-x')^2 e^{2R_1|y|} |u(r,y)|^{2\gamma} \mu(dy) dr, \\ Q_{S,1,\delta,\nu_0}(s,t,x,t',x') &= \int_0^{(s-\delta)_+} \int_{\mathbb{R}} \mathbb{1}_{\{|x-y|>(t'-r)^{1/2-\nu_0} \vee 2|x'-x|\}} \\ &\quad \times (G'_{t'-r}(x'-y) - G'_{t-r}(x-y))^2 e^{2R_1|y|} |u(r,y)|^{2\gamma} \mu(dy) dr, \\ Q_{S,2,\delta,\nu_0}(s,t,x,t',x') &= \int_0^{(s-\delta)_+} \int_{\mathbb{R}} \mathbb{1}_{\{|x-y|\le(t'-r)^{1/2-\nu_0} \vee 2|x'-x|\}} \\ &\quad \times (G'_{t'-r}(x'-y) - G'_{t-r}(x-y))^2 e^{2R_1|y|} |u(r,y)|^{2\gamma} \mu(dy) dr, \end{aligned} \tag{6.9}$$

for $\nu_0 \in (0, 1/2)$, $\delta \in (0, 1]$ and $s \le t \le t', s' \le t'$.

We will use the following lemmas to bound the terms in (6.9).

Lemma 6.6 *Let $0 \le m \le \bar{m} + 1$ and assume (P_m). For any n, ξ, ε_0, K and β as in (P_m), if $\bar{d}_N = d((s,y),(t,x)) \vee 2^{-N}$ and $\sqrt{C_{6.6}(\omega)} = (4a_n^{-\varepsilon_0} + 2^{2N_1(\omega)} 2Ke^K)$, then for any $N \in \mathbb{N}$, on*

$$\{\omega : (t,x) \in Z(N,n,K,\beta), N > N_1(m,n,\xi,\varepsilon_0,K,\beta)\}, \tag{6.10}$$

we have

$$|u(s,y)| \le \sqrt{C_{6.6}(\omega)} e^{|y-x|} \bar{d}_N^{\xi} \left[(a_n^{\kappa_0} \vee \bar{d}_N)^{\tilde{\gamma}_m - 1} + \mathbb{1}_{\{m>0\}} a_n^{\beta} \right], \ \forall s \le T_K, \ y \in \mathbb{R}. \tag{6.11}$$

The proof of Lemma 6.6 is similar to the proof of Lemma 5.2 in [16], hence it is omitted.

Remark 6.7 *If $m = 0$ we may set $\varepsilon_0 = 0$ in the above and N_1 does not depend on $(n, \varepsilon_0, \beta)$ by the proof of (P_0).*

Lemma 6.8 *For all $K \in \mathbb{N}^{\ge K_1}, R > 2/\eta$ there exists $C_{6.8}(K, R, R_1, \nu_0, \eta) > 0$ and an $N_{6.8} = N_2(K, \omega) \in \mathbb{N}$ a.s. such that for all $\nu_0, \nu_1 \in (1/R, 1/2), \delta \in (0,1], \beta \in [0, \frac{\eta}{1+\eta}]$ and $N, n \in \mathbb{N}$, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, on*

$$\{\omega : (t,x) \in Z(N,n,K,\beta), N > N_{6.8}\},$$

$$Q_{S,1,\delta,\nu_0}(s,t,x,t',x') \le C_{6.8} 2^{4N_{6.8}} \left[d^{2-\nu_1} + (d \wedge \sqrt{\delta})^{2-\nu_1} \delta^{-2+\eta/2-\varepsilon_0} (d \wedge 1)^{4\gamma} \right], \quad \forall s \le t \le t', \ x' \in \mathbb{R}.$$

Here $d = d((t',x'),(t,x))$.

Proof: The proof is almost similar to the proof of Lemma 5.4 in [16]. The only difference is that we use Lemma 5.7(b) instead of Lemma 4.4(b) from [16]. Therefore, we get the exponent $-2+\eta/2-\varepsilon_0$ instead of $-3/2$ for δ. ■

Lemma 6.9 *Let $0 \le m \le \bar m+1$ and assume (P_m). For any $K \in \mathbb{N}^{\ge K_1}, R > 2/\eta, n \in \mathbb{N}, \varepsilon_0 \in (0,1)$, and $\beta \in [0, \frac{\eta}{\eta+1}]$, there exists $C_{6.9}(K,R,R_1,\nu_0,\eta) > 0$ and an $N_{6.9} = N_{6.9}(m,n,\varepsilon_0,K,\beta,\eta,\omega) \in \mathbb{N}$ a.s. such that for any $\nu_1 \in (1/R,1/2), \nu_0 \in (0,\eta_1/32), \delta \in [a_n^{2\kappa_0},1], N \in \mathbb{N}$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, on*

$$\{\omega : (t,x) \in Z(N,n,K,\beta), N > N_{6.9}\},$$

$$\begin{aligned} Q_{S,2,\delta,\nu_0}(s,t,x,t',x') \quad &\le \quad C_{6.9}\big[a_n^{-2\varepsilon_0} + 2^{4N_{6.9}}\big]\big[d^{2-\nu_1}(\bar\delta_N^{(\gamma\tilde\gamma_m-2+\eta/2-\varepsilon_0)\wedge 0} + a_n^{2\beta\gamma}\bar\delta_N^{\gamma-2+\eta/2-\varepsilon_0}) \\ &\quad + (d\wedge\sqrt{\delta})^{2-\nu_1}\delta^{-2+\eta/2-\varepsilon_0}(\bar d_N^{2\gamma\tilde\gamma_m} + a_n^{2\beta\gamma}\bar d_N^{2\gamma})\big], \\ &\quad \forall s \le t \le t',\ |x'| \le K+1. \end{aligned}$$

Here $d = d((t',x'),(t,x)), \bar d_N = d \vee 2^{-N}$ and $\bar\delta_N = \delta \vee \bar d_N^2$. Moreover, $N_{6.9}$ is stochastically bounded uniformly in (n,β).

Proof: The proof is similar to the proof of Lemma 5.5 in [16]. First we use Lemma 6.6 to bound $|u(r,y)|$ in the integral defining $Q_{S,2,\delta,\nu_0}$. Since we may assume $s \ge \delta$, and from the assumptions we know that $\delta \ge a_n^{2\kappa_0}$, therefore we have $d((r,y),(t,x)) \ge a_n^{\kappa_0}$. Hence, when we use (6.11) to bound $|u(r,y)|$, we may drop the max with $a_n^{\kappa_0}$ in (6.11). Then we proceed as in Lemma 5.5 in [16]. The only difference is that we use Lemma 5.7(a) instead of Lemma 4.4(a) from [16]. Therefore, we get the power $-2+\eta/2-\varepsilon_0$ instead of $-3/2$ for δ. ■

Lemma 6.10 *Let $0 \le m \le \bar m+1$ and assume (P_m). For any $K \in \mathbb{N}^{\ge K_1}$, $R > 2/\eta$, $n \in \mathbb{N}, \varepsilon_0 \in (0,1)$, and $\beta \in [0, \frac{\eta}{\eta+1}]$, there is a $C_{6.10}(K,R_1,\mu(\mathbb{R}),\eta,\varepsilon_1) > 0$ and $N_{6.10} = N_{6.10}(m,n,R,\varepsilon_0,K,\beta)(\omega) \in \mathbb{N}$ a.s. such that for any $\nu_1 \in (1/R,1/2), \delta \in [a_n^{2\kappa_0},1], N \in \mathbb{N}$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, on*

$$\{\omega : (t,x) \in Z(N,n,K,\beta), N > N_{6.10}\}, \tag{6.12}$$

$$\begin{aligned} Q_{T,\delta}(s,s',t',x') \quad &\le \quad C_{6.10}\big[a_n^{-2\varepsilon_0} + 2^{4N_{6.10}}\big]|s-s'|^{1-\nu_1/2}\big[\bar\delta_N^{(\gamma\tilde\gamma_m-2+\eta/2-\varepsilon_0)\wedge 0} + a_n^{2\beta\gamma}\bar\delta_N^{\gamma-2+\eta/2-\varepsilon_0} \\ &\quad + 1\!\!1_{\{\delta<\bar d_N^2\}}\delta^{-2+\eta/2-\varepsilon_0}(\bar d_N^{2\gamma\tilde\gamma_m} + a_n^{2\beta\gamma}\bar d_N^{2\gamma})\big], \\ &\quad \forall s \le t \le t',\ s' \le t' \le T_K,\ |x'| \le K+1. \end{aligned} \tag{6.13}$$

Here $d = d((t',x'),(t,x)), \bar d_N = d \vee 2^{-N}$ and $\bar\delta_N = \delta \vee \bar d_N^2$. Moreover, $N_{6.10}$ is stochastically bounded uniformly in (n,β).

The proof of Lemma 6.10 follows the same lines as the proof of Lemma 5.6 in [16].

Proof: Let $\xi = 1 - ((2R)^{-1} \wedge \frac{\eta^2}{2(\eta+1)})$ and define $N_{6.10} = N_1(m,n,\xi,\varepsilon_0,K,\beta)$ so that $N_{6.10}$ is stochastically bounded uniformly in (n,β), immediately from (P_m). Assume that $s \vee s' \equiv \bar s \ge \delta$; otherwise, $Q_{T,\delta}(s,s',t',x') \equiv 0$. Let $\underline{s} = s \wedge s'$. We use Lemma 6.6 to bound $|u(r,y)|$ in the integrand of $Q_{T,\delta}$ and the maximum with $a_n^{\kappa_0}$ can be ignored since $a_n^{\kappa_0} \le \sqrt{t'-r}$ in the calculations below. We argue as in the proof of Lemma 5.6 in [16], that for ω as in (6.12) and s,t,s',t',x' as in (6.13) we have

$$\begin{aligned} Q_{T,\delta}(s,s',t',x') \quad &\le \quad C_{6.6}\int_{(\underline{s}-\delta)_+}^{\bar s-\delta}\int_{\mathbb{R}} G'_{t'-r}(y-x')^2 e^{2R_1K}e^{2(R_1+1)|x-y|}[2^{-N} \vee (\sqrt{t'-r}+|y-x|)]^{2\gamma\xi} \\ &\quad \times\big[2^{-N} \vee (\sqrt{t'-r}+|y-x|)^{\tilde\gamma_m-1} + a_n^{\beta}\big]^{2\gamma}\mu(dy)dr. \end{aligned} \tag{6.14}$$

Note that

$$\begin{aligned} 2^{-N} \vee (\sqrt{t'-r}+|y-x|) \quad &\le \quad \big(2^{-N} \vee |x-x'|\big) + \sqrt{t'-r} + |y-x'| \\ &\le \quad \bar d_N + \sqrt{t'-r} + |y-x'|, \end{aligned} \tag{6.15}$$

and

$$e^{(2R_1+1)|y-x|} \le C(K,R_1)e^{(2R_1+1)|y-x'|}. \tag{6.16}$$

Apply Lemma 5.2, (6.15) and (6.16) to (6.14) to get

$$
\begin{aligned}
&Q_{T,\delta}(s,s',t',x')\\
&\le C_{6.6}C(K,R_1)\int_{(\underline{s}-\delta)_+}^{\bar{s}-\delta}\int_{\mathbb{R}}(t'-r)^{-1}G_{2(t'-r)}(y-x')^2e^{2(R_1+1)|y-x'|}[\bar{d}_N^{2\gamma\xi}+(t'-r)^{\gamma\xi}+|y-x'|^{2\gamma\xi}]\\
&\quad\times\big[\bar{d}_N^{2\gamma(\tilde{\gamma}_m-1)}+(t'-r)^{\gamma(\tilde{\gamma}_m-1)}+|y-x'|^{2\gamma(\tilde{\gamma}_m-1)}+a_n^{2\beta\gamma}\big]\mu(dy)dr. \qquad (6.17)
\end{aligned}
$$

Apply Lemma 5.5(d) with $a=\gamma\xi$ and $a=\gamma(\tilde{\gamma}_m-1)$ to (6.17) to get

$$
\begin{aligned}
&Q_{T,\delta}(s,s',t',x')\\
&\le C(K,R_1,\eta)\int_{(\underline{s}-\delta)_+}^{\bar{s}-\delta}(t'-r)^{-2+\eta/2-\varepsilon_0}[\bar{d}_N^{2\gamma\xi}+(t'-r)^{\gamma\xi}]\\
&\quad\times\big[\bar{d}_N^{2\gamma(\tilde{\gamma}_m-1)}+(t'-r)^{\gamma(\tilde{\gamma}_m-1)}+a_n^{2\beta\gamma}\big]dr\\
&\le C(K,R_1,\eta)\int_{(\underline{s}-\delta)_+}^{\bar{s}-\delta}\mathbb{1}_{\{r\le t'-\bar{d}_N^2\}}(t'-r)^{-2+\eta/2-\varepsilon_0}\big[(t'-r)^{\gamma(\tilde{\gamma}_m+\xi-1)}+a_n^{2\beta\gamma}(t'-r)^{\gamma\xi}\big]dr\\
&\quad+C(K,R_1,\eta)\int_{(\underline{s}-\delta)_+}^{\bar{s}-\delta}\mathbb{1}_{\{r>t'-\bar{d}_N^2\}}(t'-r)^{-2+\eta/2-\varepsilon_0}dr[\bar{d}_N^{2\gamma(\tilde{\gamma}_m+\xi-1)}+a_n^{2\beta\gamma}\bar{d}_N^{2\gamma\xi}]\\
&=: C(K,R_1,\eta)(J_1+J_2). \qquad (6.18)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_{(\underline{s}-\delta)_+}^{\bar{s}-\delta}\mathbb{1}_{\{r>t'-\bar{d}_N^2\}}(t'-r)^{-2+\eta/2-\varepsilon_0}dr\\
&\le \mathbb{1}_{\{\delta<\bar{d}_N^2\}}[(t'-\bar{s}+\delta)^{-2+\eta/2-\varepsilon_0}|s'-s|\wedge\frac{1}{1-\eta/2+\varepsilon_0}(t'-\bar{s}+\delta)^{-1+\eta/2-\varepsilon_0}]\\
&\le \frac{1}{1-\eta/2-\varepsilon_0}\mathbb{1}_{\{\delta<\bar{d}_N^2\}}\delta^{-2+\eta/2-\varepsilon_0}(|s'-s|\wedge\delta). \qquad (6.19)
\end{aligned}
$$

From (6.18) and (6.19) we get

$$
\begin{aligned}
J_2 &\le C(\eta)\mathbb{1}_{\{\delta<\bar{d}_N^2\}}\delta^{-2+\eta/2-\varepsilon_0}(|s'-s|\wedge\delta)\bar{d}_N^{(-2\gamma(1-\xi))}[\bar{d}_N^{2\gamma\tilde{\gamma}_m}+a_n^{2\beta\gamma}\bar{d}_N^{2\gamma}]\\
&\le C(\eta)\mathbb{1}_{\{\delta<\bar{d}_N^2\}}\delta^{-2+\eta/2-\varepsilon_0}(|s'-s|\wedge\delta)^{1-\nu_1/2}[\bar{d}_N^{2\gamma\tilde{\gamma}_m}+a_n^{2\beta\gamma}\bar{d}_N^{2\gamma}], \qquad (6.20)
\end{aligned}
$$

where we have used the fact that $\gamma(1-\xi)\le 1-\xi\le(2R)^{-1}\le\nu_1/2$.

For J_1, let $p=\gamma(\tilde{\gamma}_m+\xi-1)-(2-\eta/2+\varepsilon_0)$ or $\gamma\xi-(2-\eta/2+\varepsilon_0)$ for $0\le m-1\le\bar{m}$. Recall that $\tilde{\gamma}_m\in[1,1+\eta]$, and $\eta\in(0,1)$. By our choice of ξ and from the bounds on $\gamma,\tilde{\gamma}_m,\eta$ we get

$$
\begin{aligned}
p &\ge \gamma\xi-(2-\frac{\eta}{2}+\varepsilon_0)\\
&\ge -2+\frac{\eta}{2}-\varepsilon_0+\gamma\big(1-\frac{\eta^2}{2(\eta+1)}\big)\\
&\ge -2+\frac{\eta}{2}-\varepsilon_0+\gamma-\frac{\eta^2}{2(\eta+1)}\\
&> -1+10\varepsilon_1, \qquad (6.21)
\end{aligned}
$$

where we have used (2.14) in the last inequality. From the same bounds on $\gamma,\tilde{\gamma}_m,\eta,\xi$, we also get

$$
\begin{aligned}
p &\le \gamma(\tilde{\gamma}_m+\xi-1)-(2-\frac{\eta}{2}+\varepsilon_0)\\
&\le \gamma(1+\eta)-2+\frac{\eta}{2}-\varepsilon_0\\
&\le -1+\frac{3}{2}\eta-\varepsilon_0\\
&< \frac{1}{2}. \qquad (6.22)
\end{aligned}
$$

From (6.21) and (6.22) we get

$$p \in \left(10\varepsilon_1 - 1, \frac{1}{2}\right).$$

Let $p' = p \wedge 0$ and let $0 \le \varepsilon \le -p'$. Since $t' \le K$ and $p' \in [0, 1-10\varepsilon_1)$, we get similarly to (5.25) in [16],

$$\begin{aligned} I(p) &:= \int_{(\underline{s}-\delta)_+}^{\bar{s}-\delta} \mathbb{1}_{\{r \le t' - \bar{d}_N^2\}} (t'-r)^p dr \\ &\le C(K,\varepsilon_1)|s'-s|^{1-\varepsilon}(\bar{\delta}_N)^{\varepsilon+p'}. \end{aligned} \tag{6.23}$$

Define $q = p + \gamma(1-\xi)$, so that $q = \gamma\tilde{\gamma}_m - 2 + \eta/2 - \varepsilon_0$ or $q = \gamma - (2 - \eta/2 + \varepsilon_0)$. We distinguish between two cases as follows.

The first case is $q \le 0$. Then $p' = p < 0$. Choose $\varepsilon = \gamma(1-\xi) \le (2R)^{-1} < \nu_1/2$, then $\varepsilon + p' = q \le 0$. Thus we can use (6.23) with this ε to get

$$\begin{aligned} I(p) &\le C(K,\varepsilon_1)|s'-s|^{1-\varepsilon}(\bar{\delta}_N)^q \\ &\le C(K,\varepsilon_1)|s'-s|^{1-\nu_1/2}(\bar{\delta}_N)^q. \end{aligned} \tag{6.24}$$

The second case is $q > 0$. Then $p' = (q - \gamma(1-\xi)) \wedge 0 \ge -\gamma(1-\xi)$. Choose $\varepsilon = -p' \le \gamma(1-\xi) \le (2R)^{-1} < \nu_1/2$ and again we can apply (6.23) to get

$$\begin{aligned} I(p) &\le C(K,\varepsilon_1)|s'-s|^{1-\varepsilon} \\ &\le C(K,\varepsilon_1)|s'-s|^{1-\nu_1/2}. \end{aligned} \tag{6.25}$$

From (6.24) and (6.25) we get

$$I(p) \le C(K,\varepsilon_1)|s'-s|^{1-\nu_1/2}(\bar{\delta}_N)^{q\wedge 0}. \tag{6.26}$$

From (6.18) and (6.26) we get

$$J_1 \le C(K,\varepsilon_1)|s'-s|^{1-\nu_1/2}\left[\bar{\delta}_N^{(\gamma\tilde{\gamma}_m-2+\eta/2-\varepsilon_0)\wedge 0} + a_n^{2\beta\gamma}\bar{\delta}_N^{(\gamma-2+\eta/2-\varepsilon_0)\wedge 0}\right]. \tag{6.27}$$

From (6.18), (6.20) and (6.27), (6.13) follows. ∎

Notation: Let

$$d((s,t,x),(s',t',x')) = \sqrt{|s-s'|} + \sqrt{|t-t'|} + |x-x'|$$

and

$$\begin{aligned} &\bar{\Delta}_{u_1'}(m,n,\alpha,\varepsilon_0,2^{-N}) \\ &= a_n^{-2\varepsilon_0}\left[a_n^{-\alpha(1-\eta/4)}2^{-N\gamma\tilde{\gamma}_m} + (a_n^{\alpha/2} \vee 2^{-N})^{(\gamma_m\gamma-2+\eta/2)\wedge 0} + a_n^{-\alpha(1-\eta/4)+\beta\gamma}(a_n^{\alpha/2} \vee 2^{-N})^{\gamma}\right]. \end{aligned} \tag{6.28}$$

Proposition 6.11 *Let $0 \le m \le \bar{m}+1$ and assume (P_m). For any $n \in \mathbb{N}, \nu_1 \in (0,\eta/2), \varepsilon_0 \in (0,1)$, $K \in \mathbb{N}^{\ge K_1}, \alpha \in [0, 2\kappa_0]$, and $\beta \in [0, \frac{\eta}{\eta+1}]$, there exists $N_{6.11} = N_{6.11}(m,n,\nu_1,\varepsilon_0,K,\alpha,\beta,\eta,\mu(\mathbb{R}))(\omega)$ in $\mathbb{N}^{\ge 2}$ a.s. such that for all $N \ge N_{6.11}, (t,x) \in Z(N,n,K,\beta)$, $s \le t$, $s' \le t' \le T_K$,*

$$d((s,t,x),(s',t',x')) \le 2^{-N},$$

implies that,

$$|F_{a_n^\alpha}(s,t,x) - F_{a_n^\alpha}(s',t',x')| \le 2^{-86} d((s,t,x),(s',t',x'))^{1-\nu_1}\bar{\Delta}_{u_1'}(m,n,\alpha,\varepsilon_0,2^{-N}).$$

Moreover $N_{6.11}$ is stochastically bounded uniformly in (n,α,β).

Proof: The proof of Proposition 6.11 is similar to the proof of Proposition 5.8 in [16]. Let $R = 33/(\nu_1\eta)$ and choose $\nu_0 \in (R^{-1}, \nu_1/32)$. Let $d = d((t,x),(t',x'))$, $\bar{d}_N = d \vee 2^{-N}$ and

$$Q_{a_n^\alpha}(s,t,x,s',x',t') = Q_{T,a_n^\alpha}(s,s',t',x') + \sum_{i=1}^{2} Q_{S,i,a_n^\alpha,\nu_0}(s,t,x,t',x'). \tag{6.29}$$

We use Lemmas 6.8–6.10 to bound $Q_{a_n^\alpha}(s,t,x,s',x',t')$. We get that there exists $C(K,\nu_1)$ and $N_2(m,n,\nu_1,\varepsilon_0, K,\beta,\eta)$ stochastically bounded uniformly in (n,β), such that for all $N\in\mathbb{N}$ and (t,x), on

$$\{\omega:(t,x)\in Z(N,n,K+1,\beta),N\geq N_2\},$$

$$\begin{aligned}&R_0^\gamma Q_{a_n^\alpha}(s,t,x,s',x',t')^{1/2}\\&\leq C(K,\nu_1)\big[a_n^{-\varepsilon_0}+2^{2N_2}\big](d+\sqrt{|s'-s|})^{1-\nu_1/2}\big\{(a_n^{\alpha/2}\vee\bar d_N)^{(\gamma\tilde\gamma_m-2+\eta/2-\varepsilon_0)\wedge 0}+a_n^{\beta\gamma}(a_n^{\alpha/2}\vee\bar d_N)^{\gamma-2+\eta/2-\varepsilon_0})\\&\quad+a_n^{-1+\eta/4-\varepsilon_0/2}(\bar d_N^{\gamma\tilde\gamma_m}+a_n^{\beta\gamma}\bar d_N^{\gamma})\big\},\ \forall s\leq t\leq t',\ s'\leq t'\leq T_K,\ |x'|\leq K+2.\end{aligned}$$

The rest of the proof follows the proof of Proposition 5.8 in [16] after (5.37) there. Briefly, we use the Dubins-Schwarz theorem to bound $|F_{a_n^\alpha}(s,t,x)-F_{a_n^\alpha}(s',t',x')|$. We get that the power of α_n in $\bar\Delta_{u_1'}(m,n,\alpha,\varepsilon_0,2^{-N})$ changes from $-3/4\alpha$ in [16] to $-(1-\eta/4)\alpha$ and the power of $(a_n^{\alpha/2}\vee 2^{-N})$ changes to $(\gamma\gamma_m-2+\eta/2)\wedge 0$ instead of $(\gamma_{m+1}-2)\wedge 0$ in [16]. ∎

From Remark 4.3 we have $F_\delta(t,t,x)=-u'_{1,\delta}(t,x)$. Hence the following corollary follows immediately from Proposition 6.11.

Corollary 6.12 *Let $0\leq m\leq\bar m+1$ and assume (P_m). Let $n,\nu_1,\varepsilon_0,K,\alpha$ and β as in Proposition 6.11. For all $N\geq N_{6.11}$, $(t,x)\in Z(N,n,K,\beta)$ and $t'\leq T_K$, if $d((t,x),(t',x'))\leq 2^{-N}$ then*

$$|u'_{1,a_n^\alpha}(t,x)-u'_{1,a_n^\alpha}(t',x')|\leq 2^{-85}d((t,x),(t',x'))^{1-\nu_1}\bar\Delta_{u_1'}(m,n,\alpha,\varepsilon_0,2^{-N}).\tag{6.30}$$

We would like to bound $|u'_{1,\delta}-u'_{1,a_n^{2\kappa_0}}|$. Let $\delta\geq a_n^{2\kappa_0}$ and $s=t-\delta+a_n^{2\kappa_0}$. It is easy to check that

$$u'_{1,\delta}(t,x)=-F_{a_n^{2\kappa_0}}(t-\delta+a_n^{2\kappa_0},t,x).\tag{6.31}$$

The following lemmas will be used to bound $|F_{a_n^{2\kappa_0}}(s,t,x)-F_{a_n^{2\kappa_0}}(t,t,x)|$, where the hypothesis $\sqrt{t-s}\leq 2^{-N}$ from Proposition 6.11 is weakened considerably (see Proposition 6.14).

Lemma 6.13 *Let $0\leq m\leq\bar m+1$ and assume (P_m). For any $K\in\mathbb{N}^{\geq K_1}$, $R>2/\eta$, $n\in\mathbb{N},\varepsilon_0\in(0,1)$, and $\beta\in[0,\frac{\eta}{\eta+1}]$, there is a $C_{6.13}(K,R_1,\mu(\mathbb{R}),\eta)>0$ and $N_{6.13}=N_{6.13}(m,n,R,\varepsilon_0,K,\beta)(\omega)\in\mathbb{N}$ a.s. such that for any $\nu_1\in(1/R,\eta/2),N\in\mathbb{N}$ and $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, on*

$$\{\omega:(t,x)\in Z(N,n,K,\beta),N>N_{6.13}\},$$

$$\begin{aligned}Q_{T,a_n^{2\kappa_0}}(s,t,t,x)\quad\leq\quad & C_{6.13}\big[a_n^{-2\varepsilon_0}+2^{4N_{6.13}}\big]\big\{|t-s|^{1-\nu_1/4}\big[((t-s)\vee a_n^{2\kappa_0})^{\gamma\tilde\gamma_m-2+\eta/2-\varepsilon_0}\\&+a_n^{2\beta\gamma}((t-s)\vee a_n^{2\kappa_0})^{\gamma-2+\eta/2-\varepsilon_0}\big]\\&+\mathbb{1}_{\{a_n^{2\kappa_0}<2^{-2N}\}}((t-s)\wedge a_n^{2\kappa_0})a_n^{-2+\eta/2-\varepsilon_0}2^{N\nu_1/2}(2^{-2N\gamma\tilde\gamma_m}+a_n^{2\beta\gamma}2^{-2N\gamma})\big\},\\&\forall s\leq t.\end{aligned}\tag{6.32}$$

Moreover $N_{6.13}$ is stochastically bounded uniformly in (n,β).

Proof: The proof follows the same lines as the proof of Lemma 5.10 in [16]. Fix $\theta\in(0,\eta/2-\varepsilon_0)$ such that $\gamma-\theta>1-\eta/2-\varepsilon_0$. Let $\xi=1-((4\gamma R)^{-1}\wedge\theta)$ and define $N_{6.13}=N_1(m,n,\xi(R),\varepsilon_0,K,\beta)$. Then, we get from (P_m) that $N_{6.13}$ is stochastically bounded uniformly in (n,β). Assume that $t\geq a_n^{2\kappa_0}$; otherwise $Q_{T,a_n^{2\kappa_0}}(s,t,t, x)\equiv 0$. Now repeat the same steps as in Lemma 6.10. Here we have $x'=x$, $s'=t'=t$ and $\delta=a_n^{2\kappa_0}$. We get that

$$\begin{aligned}J_1&=\int_{(s-a_n^{2\kappa_0})_+}^{t-a_n^{2\kappa_0}}\mathbb{1}_{\{r<t-2^{-2N}\}}\big[(t-r)^{\gamma(\tilde\gamma_m+\xi-1)-2+\eta/2-\varepsilon_0}+a_n^{2\beta\gamma}(t-r)^{\gamma\xi-2+\eta/2-\varepsilon_0}\big]dr,\\J_2&=\int_{(s-a_n^{2\kappa_0})_+}^{t-a_n^{2\kappa_0}}\mathbb{1}_{\{r\geq t-2^{-2N}\}}(t-r)^{-2+\eta/2-\varepsilon_0}dr2^{-2N\gamma\xi}[2^{-2N\gamma(\tilde\gamma_m-1)}+a_n^{2\beta\gamma}],\end{aligned}\tag{6.33}$$

and

$$Q_{T,a_n}(s,t,t,x)\leq C_{6.6}C(K,R_1,\eta)[J_1+J_2].\tag{6.34}$$

Repeat the same steps as in (6.19) and (6.20) with $d_N = 2^{-N}, \delta = a_n^{2\kappa_0}, s' = t$, and use the fact that $\gamma(1-\xi(R)) \le (4R)^{-1} \wedge \theta \le \nu_1/4$ to get,

$$\begin{aligned} J_2 &\le C(\eta)1\!\!1_{\{a_n^{2\kappa_0}<2^{-2N}\}} a_n^{-2+\eta/2-\varepsilon_0}(|t-s|\wedge a_n^{2\kappa_0})2^{2N\gamma(1-\xi)}[2^{-2N\gamma\tilde\gamma_m} + a_n^{2\beta\gamma}2^{-2N\gamma}] \\ &\le C(\eta)1\!\!1_{\{a_n^{2\kappa_0}<2^{-2N}\}} a_n^{-2+\eta/2-\varepsilon_0}(|t-s|\wedge a_n^{2\kappa_0})2^{N\nu_1/2}[2^{-2N\gamma\tilde\gamma_m} + a_n^{2\beta\gamma}2^{-2N\gamma}]. \end{aligned} \tag{6.35}$$

For J_1, let $p = \gamma(\tilde\gamma_m + \xi - 1) - 2 + \eta/2 - \varepsilon_0$ or $p = \gamma\xi - 2 + \eta/2 - \varepsilon_0$. Recall that $\eta \in (0,1), \gamma \in (0, 1-\eta/2-\varepsilon_0+\theta)$ and $R > 2/\eta$. Therefore $\xi = 1 - ((4\gamma R)^{-1} \wedge \theta) \ge 1-\theta$ and

$$p \in \left(\theta(\eta/2 - \varepsilon_0 - \theta) - 1, \frac{1}{2}\right). \tag{6.36}$$

If we consider the case of $p \ge 0$ and $p < 0$ separately we get, as in the proof of Lemma 5.10 in [16], that

$$\int_{(s-a_n^{2\kappa_0})_+}^{(t-a_n^{2\kappa_0})} (t-r)^p dr \le C(\eta)(t-s)\big((t-s)\vee a_n^{2\kappa_0}\big)^p. \tag{6.37}$$

Use (6.37) and the fact that $\gamma(1-\xi(R)) \le (4R)^{-1} \le \nu_1/4$ to get

$$\begin{aligned} J_1 &\le C(\eta)(t-s)\big[((t-s)\vee a_n^{2\kappa_0})^{\gamma(\tilde\gamma_m+\xi-1-\varepsilon_0)-2+\eta/2} + a_n^{2\beta\gamma}((t-s)\vee a_n^{2\kappa_0})^{\gamma\xi-2+\eta/2-\varepsilon_0}\big] \\ &\le C(\eta)(t-s)^{1-\gamma(1-\xi)}\big[((t-s)\vee a_n^{2\kappa_0})^{\gamma\tilde\gamma_m-2+\eta/2-\varepsilon_0} + a_n^{2\beta\gamma}((t-s)\vee a_n^{2\kappa_0})^{\gamma-2+\eta/2-\varepsilon_0}\big] \\ &\le C(\eta,K)(t-s)^{1-\nu_1/4}\big[((t-s)\vee a_n^{2\kappa_0})^{\gamma\tilde\gamma_m-2+\eta/2-\varepsilon_0} + a_n^{2\beta\gamma}((t-s)\vee a_n^{2\kappa_0})^{\gamma-2+\eta/2-\varepsilon_0}\big]. \end{aligned} \tag{6.38}$$

From (6.33), (6.34), (6.35) and (6.38) we get (6.32). ■

Proposition 6.14 *Let $0 \le m \le \bar m + 1$ and assume (P_m). For any $n \in \mathbb{N}, \nu_1 \in (0,\eta/2), \varepsilon_0 \in (0,1), K \in \mathbb{N}^{\ge K_1}$ and $\beta \in [0, \frac{\eta}{\eta+1}]$, there is an $N_{6.14} = N_{6.14}(m.n,\nu_1,\varepsilon_0,K,\beta,\eta,\mu(\mathbb{R}))(\omega) \in \mathbb{N}$ a.s. such that for all $N \ge N_{6.14}$, $(t,x) \in Z(N,n,K,\beta), s \le t$ and $\sqrt{t-s} \le N^{-4/\nu_1}$ implies that*

$$\begin{aligned} |F_{a_n^{2\kappa_0}}(s,t,x) - F_{a_n^{2\kappa_0}}(t,t,x)| \le\ & 2^{-81}a_n^{-3\varepsilon_0}\Big\{2^{-N(1-\nu_1)}(a_n^{\kappa_0}\vee 2^{-N})^{(\gamma\gamma_m-2+\eta/2)\wedge 0} \\ & + 2^{N\nu_1}a_n^{-1+\eta/4+\kappa_0}\left(\frac{2^{-N}}{a_n^{3\kappa_0-\kappa_0\eta/2-1+\eta/4}}+1\right)\big(2^{-N\gamma\tilde\gamma_m} + a_n^{\beta\gamma}(a_n^{\kappa_0}\vee 2^{-N})^\gamma\big) \\ & + (t-s)^{(1-\nu_1)/2}\big((\sqrt{t-s}\vee a_n^{\kappa_0})^{\gamma\tilde\gamma_m-2+\eta/2} \\ & + a_n^{\beta\gamma}(\sqrt{t-s}\vee a_n^{\kappa_0})^{\gamma-2+\eta/2}\big)\Big\}. \end{aligned} \tag{6.39}$$

Moreover $N_{6.14}$ is stochastically bounded, uniformly in (n,β).

Proof: The proof is similar to the proof of Proposition 5.11 in [16]. First we bound $R_0^\gamma Q_{T,a_n^{2\kappa_0}}(s,t,t,x)^{1/2}$ as follows. Let $N_2(m,n,\nu_1,\varepsilon_0,K,\beta) = \frac{8}{\nu_1}[N_{6.13} + \tilde N_0(K)]$, where $\tilde N_0(K) \in \mathbb{N}$ is large enough such that

$$\begin{aligned} C_{6.13}R_0^\gamma\big[a_n^{-\varepsilon_0} + 2^{2N_{6.13}}\big]2^{-\nu_1 N_2/4} &\le C_{6.13}R_0^\gamma\big[a_n^{-\varepsilon_0} + 2^{2N_{6.13}}\big]2^{-2N_{6.13}-2\tilde N_0(K)} \\ &\le 2^{-100}a_n^{-\varepsilon_0}. \end{aligned} \tag{6.40}$$

On

$$\{\omega : (t,x) \in Z(N,n,K,\beta), N \ge N_{6.13}(m,n,2/\nu_1,\varepsilon_0,K,\beta)\},$$

we have

$$\begin{aligned} R_0^\gamma Q_{T,a_n^{2\kappa_0}}(s,t,t,x)^{1/2} &\le (\sqrt{t-s})^{1-\nu_1/2}\Delta_1(m,n,\sqrt{t-s}\vee a_n^{\kappa_0}) + 2^{N\nu_1/2}\Delta_2(m,n,2^{-N}), \\ & \quad s \le t,\ \sqrt{t-s} \le 2^{-N_2}, \end{aligned} \tag{6.41}$$

where

$$\Delta_1(m,n,\sqrt{t-s}\vee a_n^{\kappa_0}) \quad := \quad 2^{-100}a_n^{-3\varepsilon_0}\big\{(\sqrt{t-s}\vee a_n^{\kappa_0})^{\gamma\tilde\gamma_m-2+\eta/2}+a_n^{\beta\gamma}(\sqrt{t-s}\vee a_n^{\kappa_0})^{\gamma-2+\eta/2}\big\},$$

$$\Delta_2(m,n,2^{-N}) \quad := \quad 2^{-100}a_n^{-3\varepsilon_0-1+\eta/4+\kappa_0}\big(2^{-N\gamma\tilde\gamma_m}+a_n^{\beta\gamma}2^{-N\gamma}\big).$$

The only difference from the proof in [16] is that we use Lemma 6.13 instead of Lemma 5.10 in [16]. This gives the values Δ_1,Δ_2 above. The rest of the proof is similar to the proof of Proposition 5.11 in [16]. We use the Dubins-Schwarz theorem and Proposition 6.11 with

$$\begin{aligned}&\bar\Delta_{u_1'}(m,n,2\kappa_0,\varepsilon_0,2^{-(N-1)})\\&=a_n^{-2\varepsilon_0}\big[a_n^{-2\kappa_0(1-\eta/4)}2^{-(N-1)\gamma\tilde\gamma_m}+(a_n^{\kappa_0}\vee 2^{(N-1)})^{(\tilde\gamma_m\gamma-2+\eta/2)\wedge 0}+a_n^{-2\kappa_0(1-\eta/4)+\beta\gamma}(a_n^{\kappa_0}\vee 2^{-(N-1)})^{\gamma}\big].\end{aligned}$$

to get

$$\begin{aligned}&|F_{a_n^{2\kappa_0}}(s,t,x)-F_{a_n^{2\kappa_0}}(t,t,x)|\\&\le 2^{-81}2^{-N(1-\nu_1)}a_n^{-3\varepsilon_0}\big[a_n^{-2\kappa_0(1-\eta/4)}2^{-N\gamma\tilde\gamma_m}+(a_n^{\kappa_0}\vee 2^{-N})^{(\tilde\gamma_m\gamma-2+\eta/2)\wedge 0}+a_n^{-2\kappa_0(1-\eta/4)+\beta\gamma}(a_n^{\kappa_0}\vee 2^{-N})^{\gamma}\big].\\&\quad+2^{-99}a_n^{-3\varepsilon_0}(t-s)^{1-\nu_1}\big\{(\sqrt{t-s}\vee a_n^{\kappa_0})^{\gamma\tilde\gamma_m-2+\eta/2}+a_n^{\beta\gamma}(\sqrt{t-s}\vee a_n^{\kappa_0})^{\gamma-2+\eta/2}\big\}\\&\quad+2^{-98}a_n^{-3\varepsilon_0-1+\eta/4+\kappa_0}2^{N\nu_1}\big(2^{-N\gamma\tilde\gamma_m}+a_n^{\beta\gamma}2^{-N\gamma}\big).\end{aligned}\tag{6.42}$$

Note that

$$2^{-N(1-\nu_1)}a_n^{-2\kappa_0(1-\eta/4)}+2^{N\nu_1}a_n^{-1+\eta/4+\kappa_0}\le 2^{N\nu_1}a_n^{-1+\eta/4+\kappa_0}\left(\frac{2^{-N}}{a_n^{3\kappa_0-\kappa_0\eta/2-1+\eta/4}}+1\right).\tag{6.43}$$

From (6.42) and (6.43) we get (6.39). ∎

We would like to bound the increment of $\mathbb{G}_{\alpha_n}$ with $\alpha\in[0,2\kappa_0]$. As in Lemma 4.2, for F_δ, we get

$$\mathbb{G}_\delta(s,t,x)=\int_0^{(s-\delta)_+}\int_{\mathbb{R}}G_{(t\vee s)-r}(y-x)D(r,y)W(dr,dy),\quad\text{for all } s \text{ a.s. for all } (t,x).$$

We need to bound $\mathbb{G}_{a_n^\alpha}(s,t,x)-\mathbb{G}_{a_n^\alpha}(t,t,x)$ with $\alpha\in[0,2\kappa_0]$, so we repeat the same process that led to Proposition 6.11. The difference is that now we deal with the Gaussian densities G_{t-r} instead of the derivatives G'_{t-r}. Recall that Proposition 6.11 followed from Lemmas 6.8–6.10. In order to bound the increment of $G_{a_n^{\kappa_0}}$, one needs analogues to the above lemmas with Gaussian densities G_{t-r} instead of the derivative G'_{t-r}. The proofs of these lemmas and the proposition follow the same lines as the proof of Lemmas 6.8–6.10 and Proposition 6.11; therefore they are omitted. Here is the final statement.

Proposition 6.15 *Let $0\le m\le\bar m+1$ and assume (P_m). For any $n\in\mathbb{N},\nu_1\in(0,\eta/2)$, $\varepsilon_0\in(0,1),K\in\mathbb{N}^{\ge K_1},\alpha\in[0,2\kappa_0]$, and $\beta\in[0,\frac{\eta}{\eta+1}]$, there is an $N_{6.15}=N_{6.15}(m,n,\nu_1,\varepsilon_0,K,\alpha,\beta,\eta,\mu(\mathbb{R}))(\omega)$ in $\mathbb{N}$ a.s. such that for all $N\ge N_{6.15}$, $(t,x)\in Z(N,n,K,\beta), s\le t$ and $\sqrt{t-s}\le 2^{-N}$,*

$$\begin{aligned}|\mathbb{G}_{a_n^\alpha}(s,t,x)-\mathbb{G}_{a_n^\alpha}(t,t,x)| \quad &\le \quad 2^{-92}(t-s)^{(1-\nu_1)/2}a_n^{-3\varepsilon_0}a_n^{-\alpha(1/2-\eta/4)}\\&\quad\times\big[(a_n^{\alpha/2}\vee 2^{-N})^{\gamma\tilde\gamma_m}+a_n^{\beta\gamma}(a_n^{\alpha/2}\vee 2^{-N})^{\gamma}\big].\end{aligned}$$

We will use the modulus of continuity of u'_{1,a_n^α} from Corollary 6.12 to get the modulus of continuity for u_{1,a_n^α}.

Notation. Define

$$\begin{aligned}\bar\Delta_{u_1}(m,n,\alpha,\varepsilon_0,2^{-N},\eta) \quad = \quad &a_n^{-3\varepsilon_0-(1-\eta/4)\alpha}\big[a_n^{\beta}a_n^{(1-\eta/4)\alpha}+a_n^{\beta\gamma}(a_n^{\alpha/2}\vee 2^{-N})^{\gamma+1}\\&+(a_n^{\alpha/2}\vee 2^{-N})^{\gamma\tilde\gamma_m+1}+\mathbb{1}_{\{m\ge\bar m\}}a_n^{\alpha(1-\eta/4)}(a_n^{\alpha/2}\vee 2^{-N})^{\eta}\big].\end{aligned}$$

If $\nu>0$ let $N'_{6.16}(\nu)$ be the smallest natural number such that $2^{1-N}\le N^{-4/\nu}$ whenever $N>N'_{6.16}(\nu)$.

Proposition 6.16 *Let $0 \le m \le \bar{m}+1$ and assume (P_m). For any $n \in \mathbb{N}, \nu_1 \in (0, \gamma - 1 + \eta/2)$, $\varepsilon_0, \varepsilon_1 \in (0,1), K \in \mathbb{N}^{\ge K_1}, \alpha \in [0, 2\kappa_0]$, and $\beta \in [0, \frac{\eta}{\eta+1}]$, there is an $N_{6.16} = N_{6.16}(m,n,\nu_1,\varepsilon_0,K,\alpha,\beta,\eta,\mu(\mathbb{R}))(\omega)$ in $\mathbb{N}$ a.s. such that for all $N \ge N_{6.16}$, n, α satisfying*

$$a_n^{2\kappa_0} \le 2^{-2(N_{6.14}(m,n,\nu_1/2,\varepsilon_0,K,\beta,\eta,\mu(\mathbb{R}))+1)} \wedge 2^{-2(N'_{6.16}(\frac{\nu_1\varepsilon_1}{2\kappa_0})+1)}, \text{ and } \alpha \ge \varepsilon_1, \tag{6.44}$$

$(t,x) \in Z(N,n,K,\beta),\ t' \le T_K$,

$$d((t,x),(t',x')) \le 2^{-N},$$

implies,

$$|u_{1,a_n^\alpha}(t,x) - u_{1,a_n^\alpha}(t',x')| \le 2^{-90} d((t,x),(t',x'))^{1-\nu_1} \bar{\Delta}_{u_1}(m,n,\alpha,\varepsilon_0,2^{-N},\eta). \tag{6.45}$$

Moreover $N_{6.16}$ is stochastically bounded uniformly in $n \in \mathbb{N}, \alpha \in [0,2\kappa_0]$ and $\beta \in [0, \frac{\eta}{\eta+1}]$.

Remark 6.17 *Although n appears in both sides of (6.44), the fact that $N_{6.14}$ is stochastically bounded ensures that (6.44) holds for infinitely many n.*

Proof: The proof of Proposition 6.16 follows the same lines as the proof of Proposition 5.13 in [16]. Let

$$\begin{aligned} N''_{6.16}(m,n,\nu_1,\varepsilon_0,K,\alpha,\beta,\eta,\mu(\mathbb{R})) &= ((2N_{6.11})(m,n,\nu_1/2,\varepsilon_0,K+1,\alpha,\beta,\eta,\mu(\mathbb{R})) \\ &\quad \vee N_{6.15}(m,n,\nu_1,\varepsilon_0,K+1,\alpha,\beta,\eta,\mu(\mathbb{R})) + 1. \end{aligned} \tag{6.46}$$

Hence $N''_{6.16}$ is stochastically bounded in (n,α,β). Assume (6.44) and

$$N \ge N''_{6.16}, (t,x) \in Z(N,n,K,\beta),\ t' \le T_K, \text{ and } d((t,x),(t',x')) \le 2^{-N}. \tag{6.47}$$

As in the proof of Proposition 5.13 in [16], $(t',x') \in Z(N-1,n,K+1,\beta)$ and we may assume $t' \le t$.

Recall that

$$\mathbb{G}_{a_n^\alpha}(t',t,x) = G_{t-t'+a_n^\alpha}(u(t'-a_n^\alpha,\cdot))(x) = G_{t-t'}(u_{1,a_n^\alpha}(t',\cdot))(x). \tag{6.48}$$

From (6.48) we get that the increment of u_{1,a_n^α} can be bounded by

$$\begin{aligned} |u_{1,a_n^\alpha}(t',x') - u_{1,a_n^\alpha}(t,x)| &\le |u_{1,a_n^\alpha}(t',x') - u_{1,a_n^\alpha}(t',x)| \\ &\quad + |u_{1,a_n^\alpha}(t',x) - G_{t-t'}(u_{1,a_n^\alpha}(t',\cdot))(x)| \\ &\quad + |\mathbb{G}_{a_n^\alpha}(t',t,x) - \mathbb{G}_{a_n^\alpha}(t,t,x)| \\ &\equiv T_1 + T_2 + T_3. \end{aligned} \tag{6.49}$$

For T_1 let $(\hat{t}_0,\hat{x}_0)$ be as in the definition of $(t,x) \in Z(N,n,K,\beta)$. Let y be between x' and x. Therefore, $d((t',y),(t,x)) \le 2^{-N}$ and $d((\hat{t}_0,\hat{x}_0),(t,x)) \le 2^{-N}$. Use Corollary 6.12 twice, with $\nu_1/2$ in place of ν_1, (6.31) and the definition of $(\hat{t}_0,\hat{x}_0)$ in $Z(N,n,K,\beta)$, to get

$$\begin{aligned} |u'_{1,a_n^\alpha}(t',y)| &\le |u'_{1,a_n^\alpha}(t',y) - u'_{1,a_n^\alpha}(t,x)| \\ &\quad + |u'_{1,a_n^\alpha}(t,x) - u'_{1,a_n^\alpha}(\hat{t}_0,\hat{x}_0)| \\ &\quad + |u'_{1,a_n^\alpha}(\hat{t}_0,\hat{x}_0) - u'_{1,a_n^{2\kappa_0}}(\hat{t}_0,\hat{x}_0)| \\ &\quad + |u'_{1,a_n^{2\kappa_0}}(\hat{t}_0,\hat{x}_0)| \\ &\le 2^{-84} 2^{-N(1-\nu_1/2)} \bar{\Delta}_{u'_1}(m,n,\alpha,\varepsilon_0,2^{-N}) \\ &\quad + |F_{a_n^{2\kappa_0}}(\hat{t}_0 - a_n^\alpha + a_n^{2\kappa_0},\hat{t}_0,\hat{x}_0) - F_{a_n^{2\kappa_0}}(\hat{t}_0,\hat{t}_0,\hat{x}_0)| \\ &\quad + a_n^\beta. \end{aligned} \tag{6.50}$$

We would like to bound the increment of $F_{a_n^{2\kappa_0}}$ in (6.50) with the use of Proposition 6.14, but we need some adjustments first. Choose N' such that

$$2^{-N'-1} \le a_n^{\kappa_0} \le 2^{-N'}. \tag{6.51}$$

From (6.44) we have

$$a_n^{\kappa_0} \le 2^{-N_{6.14}(m,n,\nu_1/2,\varepsilon_0,K,\beta,\eta,\mu(\mathbb{R}))-1}. \tag{6.52}$$

From (6.51), (6.52) we get

$$N' \ge N_{6.14}(m,n,\nu_1/2,\varepsilon_0,K,\beta,\eta,\mu(\mathbb{R})). \tag{6.53}$$

Also from (6.44) and (6.51) we have

$$2^{-N'-1} \le a_n^{\kappa_0} \le 2^{-N'_{6.16}(\frac{\nu_1\varepsilon_1}{2\kappa_0})-1}. \tag{6.54}$$

Hence $N' \ge N'_{6.16}(\frac{\nu_1\varepsilon_1}{2\kappa_0})$ and by (6.51), our choice of $\alpha \ge \varepsilon_1$ and the previously introduced notation for $N'_{6.16}(\frac{\nu_1\varepsilon_1}{2\kappa_0})$ we have

$$a_n^{\alpha/2} \le 2^{-\frac{N'\alpha}{2\kappa_0}} \le 2^{-\frac{N'\varepsilon_1}{2\kappa_0}} \le N'^{-\frac{4\varepsilon_1}{2\kappa_0}\frac{2\kappa_0}{\nu_1\varepsilon_1}} = N'^{-\frac{4}{\nu_1}}. \tag{6.55}$$

By the definition of $(\hat{t}_0,\hat{x}_0)$ in $Z(N,n,K,\beta)$ and (6.51) we have

$$|u(\hat{t}_0,\hat{x}_0)| \le a_n = a_n \wedge (a_n^{1-\kappa_0}2^{-N'}), \tag{6.56}$$

and therefore, $(\hat{t}_0,\hat{x}_0) \in Z(N',n,K,\beta)$. From (6.53) and (6.55) we get that the assumptions of Proposition 6.14 hold with N' instead of N, $(\hat{t}_0,\hat{x}_0)$ instead of (t,x), $\nu_1/2$ instead of ν_1 and $s = \hat{t}_0 - a_n^{\alpha} + a_n^{2\kappa_0}$. Hence from Proposition 6.14 and the simple inequality $a_n^{\kappa_0} \le a_n^{\alpha/2}$, one easily gets

$$\begin{aligned}
&|F_{a_n^{2\kappa_0}}(\hat{t}_0 - a_n^{\alpha} + a_n^{2\kappa_0},\hat{t}_0,\hat{x}_0) - F_{a_n^{2\kappa_0}}(\hat{t}_0,\hat{t}_0,\hat{x}_0)| \\
&\le 2^{-78}a_n^{-3\varepsilon_0}\big\{a_n^{\kappa_0(1-\frac{\nu_1}{2})}a_n^{\kappa_0[(\gamma\gamma_m-2+\eta/2)\wedge 0]} + a_n^{-\kappa_0(1-\eta/2+\nu_1/2)}\big(a_n^{\kappa_0\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma}a_n^{\kappa_0\gamma}\big) \\
&\quad + a_n^{\frac{\alpha}{2}(1-\frac{\nu_1}{2})}\big(a_n^{\frac{\alpha}{2}(\gamma\tilde{\gamma}_m-2+\frac{\eta}{2})} + a_n^{\beta\gamma}a_n^{\frac{\alpha}{2}(\gamma-2+\frac{\eta}{2})}\big)\big\}.
\end{aligned} \tag{6.57}$$

Since $a_n^{\kappa_0} \le a_n^{\alpha/2}$, the middle term in (6.57) is bounded by the third term and we get

$$\begin{aligned}
&|F_{a_n^{2\kappa_0}}(\hat{t}_0 - a_n^{\alpha} + a_n^{2\kappa_0},\hat{t}_0,\hat{x}_0) - F_{a_n^{2\kappa_0}}(\hat{t}_0,\hat{t}_0,\hat{x}_0)| \\
&= 2^{-77}a_n^{-3\varepsilon_0}\big\{a_n^{\kappa_0(1-\frac{\nu_1}{2})}a_n^{\kappa_0[(\gamma\gamma_m-2+\frac{\eta}{2})\wedge 0]} + a_n^{-\frac{\alpha}{2}(1-\eta/2+\nu_1/2)}\big(a_n^{\frac{\alpha}{2}\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma}a_n^{\frac{\alpha}{2}\gamma}\big)\big\}.
\end{aligned} \tag{6.58}$$

Recall from (6.3) that $\gamma_m \ge 1$. Hence by our assumptions on ν_1 ($\nu_1 \in (0,\gamma-1+\eta/2)$) and (6.3) we get that

$$\gamma\gamma_m \ge 1 - \frac{\eta}{2} + \nu_1. \tag{6.59}$$

From (6.59) we immediately get

$$1 - \frac{\nu_1}{2} + \gamma\gamma_m - 2 + \frac{\eta}{2} \ge \frac{\nu_1}{2} > 0. \tag{6.60}$$

From (6.60) we have

$$a_n^{\kappa_0[1-\frac{\nu_1}{2}+(\gamma\gamma_m-2+\frac{\eta}{2})\wedge 0]} \le (a_n^{\alpha/2} \vee 2^{-N})^{1-\frac{\nu_1}{2}+(\gamma\gamma_m-2+\eta/2)\wedge 0}. \tag{6.61}$$

Recall the definition of $\bar{\Delta}_{u'_1}(m,n,\alpha,\varepsilon_0,2^{-N})$ given in (6.28). Apply (6.58) and (6.61) to (6.50), and repeat the same steps as in (5.79) in [16] to get

$$\begin{aligned}
|u'_{1,a_n^{\alpha}}(t',y)| &\le 2^{-84}2^{-N(1-\nu_1/2)}a_n^{-2\varepsilon_0}\big[a_n^{-\alpha(1-\eta/4)}2^{-N\gamma\tilde{\gamma}_m} \\
&\quad + (a_n^{\alpha/2} \vee 2^{-N})^{(\gamma\gamma_m-2+\eta/2)\wedge 0} + a_n^{-\alpha(1-\eta/4)+\beta\gamma}(a_n^{\alpha/2} \vee 2^{-N})^{\gamma}\big] \\
&\quad + 2^{-77}a_n^{-3\varepsilon_0}\big\{(a_n^{\alpha/2} \vee 2^{-N})^{1-\frac{\nu_1}{2}+(\gamma\gamma_m-2+\eta/2)\wedge 0} + a_n^{-\frac{\alpha}{2}(1-\eta/2+\nu_1/2)}\big(a_n^{\frac{\alpha}{2}\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma}a_n^{\frac{\alpha}{2}\gamma}\big)\big\} \\
&\quad + a_n^{\beta} \\
&\equiv 2^{-76}\tilde{\Delta}_{u_1}(m,n,\alpha,\varepsilon_0,\nu_1,a_n^{\alpha/2} \vee 2^{-N},\eta) + a_n^{\beta}.
\end{aligned} \tag{6.62}$$

Since $\gamma \in (0,1)$ and by the assumption $\nu_1 < \gamma - 1 + \eta/2$, we get that $1 - \nu_1/2 > 0$. Together with (6.60) it follows that $\tilde{\Delta}_{u_1}$ is monotone increasing in $a_n^{\alpha/2} \vee 2^{-N}$. From the Mean Value Theorem and (6.62) we get

$$T_1 \le \big[2^{-76}\tilde{\Delta}_{u_1}(m,n,\alpha,\varepsilon_0,\nu_1,a_n^{\alpha/2} \vee 2^{-N},\eta) + a_n^{\beta}\big]|x-x'|. \tag{6.63}$$

Recall that $t' \le t$. From (6.46) and (6.47) we get that $N > N_{6.15}$ and $\sqrt{t'-t} \le 2^{-N}$. Apply Proposition 6.15 to T_3 to get

$$T_3 \le 2^{-92}(t'-t)^{(1-\nu_1)/2} a_n^{-3\varepsilon_0} a_n^{-\alpha(1/2-\eta/4)}\big[(a_n^{\alpha/2} \vee 2^{-N})^{\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma}(a_n^{\alpha/2} \vee 2^{-N})^{\gamma}\big]. \tag{6.64}$$

The last term that we have to bound is T_2. By repeating the same steps as in the proof of Proposition 5.13 in [16] we get

$$T_2 \le C(K)\sqrt{t-t'}[a_n^{\beta} + \tilde{\Delta}_{u_1}(m,n,\alpha,\varepsilon_0,\nu_1,a_n^{\alpha/2} \vee 2^{-N},\eta)]. \tag{6.65}$$

Use (6.63), (6.64) and (6.65) in (6.49), then use $d((t,x),(t',x')) \le 2^{-N}$, to get

$$\begin{aligned}
&|u_{1,a_n^{\alpha}}(t',x') - u_{1,a_n^{\alpha}}(t,x)| \\
&\le (C_{6.66}(K)2^{-N\nu_1/2} + 2^{-92})d((t,x),(t',x'))^{1-\nu_1} a_n^{-\alpha(1-\eta/4)-3\varepsilon_0}\Big[a_n^{\alpha(1-\eta/4)}a_n^{\beta} \\
&\quad +(a_n^{\alpha/2} \vee 2^{-N})\big\{a_n^{\alpha(1-\eta/4)}(a_n^{\alpha/2} \vee 2^{-N})^{(\gamma\tilde{\gamma}_m-2+\eta/2)\wedge 0} \\
&\quad +(a_n^{\alpha/2} \vee 2^{-N})^{\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma}(a_n^{\alpha/2} \vee 2^{-N})^{\gamma}\big\}\Big].
\end{aligned} \tag{6.66}$$

Choose $N_1(K,\nu_1)$ such that

$$2^{-N_1\nu_1/2}C_{6.66}(K) \le 2^{-92}, \tag{6.67}$$

and define $N_{6.16} = N''_{6.16} \vee N_1$ which is stochastically bounded uniformly in $(n,\alpha,\beta) \in \mathbb{N} \times [0,2\kappa_0] \times [0,\frac{\eta}{\eta+1}]$. Assume $N \ge N_{6.16}$. Note that if $m < \bar{m}$, from (6.3) and (6.4) we get

$$\begin{aligned}
a_n^{\alpha(1-\eta/4)}(a_n^{\alpha/2} \vee 2^{-N})^{(\gamma\tilde{\gamma}_m-2+\eta/2)\wedge 0} &= a_n^{\alpha(1-\eta/4)}(a_n^{\alpha/2} \vee 2^{-N})^{\gamma\gamma_m-2+\eta/2} \\
&\le (a_n^{\alpha/2} \vee 2^{-N})^{\gamma\gamma_m} \\
&\le (a_n^{\alpha/2} \vee 2^{-N})^{\gamma\tilde{\gamma}_m}.
\end{aligned} \tag{6.68}$$

If $m \ge \bar{m}$ we have

$$a_n^{\alpha(1-\eta/4)}(a_n^{\alpha/2} \vee 2^{-N})^{(\gamma\tilde{\gamma}_m-2+\eta/2)\wedge 0} = a_n^{\alpha(1-\eta/4)}(a_n^{\alpha/2} \vee 2^{-N})^{\eta-1}. \tag{6.69}$$

From (6.66)–(6.69) we get

$$\begin{aligned}
&|u_{1,a_n^{\alpha}}(t',x') - u_{1,a_n^{\alpha}}(t,x)| \\
&\le 2^{-90}d((t,x),(t',x'))^{1-\nu_1} a_n^{-\alpha(1-\eta/4)-3\varepsilon_0}\Big[a_n^{\alpha(1-\eta/4)+\beta} + \mathbb{1}_{\{m\ge\bar{m}\}}(a_n^{\alpha/2} \vee 2^{-N})^{\eta}a_n^{\alpha(1-\eta/4)} \\
&\quad +(a_n^{\alpha/2} \vee 2^{-N})^{\gamma\tilde{\gamma}_m+1} + a_n^{\beta\gamma}(a_n^{\alpha/2} \vee 2^{-N})^{\gamma+1}\Big] \\
&= 2^{-90}d((t,x),(t',x'))^{1-\nu_1}\bar{\Delta}_{u_1}(m,n,\alpha,\varepsilon_0,2^{-N},\eta).
\end{aligned}$$

■

We need to bound the increments of $u_{2,a_n^{\alpha}}$ as we did for $u_{1,a_n^{\alpha}}$ in Proposition 6.16. First we introduce the following notation.

Notation.

$$\begin{aligned}\bar{\Delta}_{1,u_2}(m,n,\varepsilon_0,2^{-N},\eta) &= a_n^{-3\varepsilon_0}2^{-N\gamma}\big[(2^{-N}\vee a_n^{\alpha})^{\gamma(\tilde{\gamma}_m-1)}+a_n^{\gamma\beta}\big],\\ \bar{\Delta}_{2,u_2}(m,n,\varepsilon_0,\eta) &= a_n^{-3\varepsilon_0}\big[a_n^{(\alpha/2)(\gamma\tilde{\gamma}_m-1+\eta/2)}+a_n^{(\alpha/2)(\gamma-1+\eta/2)}a_n^{\gamma\beta}\big].\end{aligned}\tag{6.70}$$

Proposition 6.18 *Let $0\le m\le \bar{m}+1$ and assume (P_m). Let $\theta\in(0,\gamma-1+\eta/2)$. Then for any $n\in\mathbb{N},\nu_1\in(0,\theta),\varepsilon_0\in(0,1),K\in\mathbb{N}^{\ge K_1},\alpha\in[0,2\kappa_0]$, and $\beta\in[0,\frac{\eta}{\eta+1}]$, there is an $N_{6.18}=N_{6.18}(m,n,\nu_1,\varepsilon_0,K,\alpha,\beta,\eta)\in\mathbb{N}$ a.s. such that for all $N\ge N_{6.18},(t,x)\in Z(N,n,K,\beta)$, and $t'\le T_K$,*

$$\begin{aligned}&d\equiv d((t,x),(t',x'))\le 2^{-N} \text{ implies that}\\ &|u_{2,a_n^\alpha}(t,x)-u_{2,a_n^\alpha}(t',x')|\le 2^{-89}\left[d^{\frac{\eta-\nu_1}{2}}\bar{\Delta}_{1,u_2}(m,n,\varepsilon_0,2^{-N},\eta)+d^{1-\nu_1}\bar{\Delta}_{2,u_2}(m,n,\varepsilon_0,\eta)\right].\end{aligned}$$

Moreover, $N_{6.18}$ is stochastically bounded, uniformly in (n,α,β).

The proof of Proposition 6.18 follows the same lines as the proof of Proposition 6.16, and hence we omit it. Also see Section 7 in [16].

The following lemma is crucial to the proof of Proposition 6.3.

Lemma 6.19 *Let γ satisfy (1.10), that is $\gamma>1-\frac{\eta}{2(\eta+1)}$. For all $n,m\in\mathbb{N},0\le\beta\le\frac{\eta}{\eta+1}$ and $0<d\le 1$,*

$$a_n^{\beta\gamma}(a_n^{\kappa_0}\vee d)^{\gamma_1-1}\le a_n^{\beta}+(d\vee a_n^{\kappa_0})^{\tilde{\gamma}_{m+1}-1}.\tag{6.71}$$

Proof: The proof of Lemma 6.19 follows the same lines of Lemma 5.15 in [16].

From (6.3) we have

$$\gamma_1-1=\gamma-1+\eta/2.\tag{6.72}$$

From (1.10) we have

$$\gamma+\frac{\eta}{2}-1>\frac{\eta^2}{2(\eta+1)}.\tag{6.73}$$

From (2.15), (6.73), (1.10) and (4.16) we have

$$\begin{aligned}\beta(\gamma-1)+\kappa_0(\gamma+\eta/2-1) &\ge \beta(\gamma-1)+\frac{1}{\eta+1}\left(\frac{\eta^2}{2(\eta+1)}\right)\\ &\ge -\beta\frac{\eta}{2(\eta+1)}+\frac{1}{\eta+1}\left(\frac{\eta^2}{2(\eta+1)}\right)\\ &\ge 0.\end{aligned}\tag{6.74}$$

From (6.74) we have

$$\beta\gamma+\kappa_0(\gamma+\eta/2-1)\quad\ge\quad\beta.\tag{6.75}$$

Case 1. $d\le a_n^{\kappa_0}$. From (6.72) and (6.75) we get

$$\begin{aligned}a_n^{\beta\gamma}(a_n^{\kappa_0}\vee d)^{\gamma_1-1} &= a_n^{\beta\gamma+\kappa_0(\gamma+\eta/2-1)}\\ &\le a_n^{\beta}.\end{aligned}\tag{6.76}$$

Case 2. $a_n^{\kappa_0}<d\le a_n^{\beta}$ and $a_n^{\beta}\le d^{\eta}$. From (1.10) and (6.72) we have,

$$\begin{aligned}a_n^{\beta\gamma}(a_n^{\kappa_0}\vee d)^{\gamma_1-1} &\le d^{(1+\eta)\gamma+\eta/2-1}\\ &\le d^{\eta}\\ &= d^{\tilde{\gamma}_{\bar{m}+1}-1}\\ &\le d^{\tilde{\gamma}_{m+1}-1},\end{aligned}\tag{6.77}$$

where we have used (6.2) and (6.5) in the last two lines.

Case 3. $a_n^{\kappa_0} < d \le a_n^\beta$ and $d^\eta < a_n^\beta$. From (1.10) we have

$$\begin{aligned}\beta\gamma + (\gamma+\eta/2-1)\frac{\beta}{\eta} &\ge \beta\Big(\gamma\frac{\eta+1}{\eta}+\frac{1}{2}-\frac{1}{\eta}\Big)\\ &\ge \beta. \end{aligned}\tag{6.78}$$

From (6.78), (6.72) and the assumption of this case we have

$$\begin{aligned}a_n^{\beta\gamma}(a_n^{\kappa_0}\vee d)^{\gamma_1-1} &\le a_n^{\beta\gamma+(\gamma+\eta/2-1)\beta/\eta}\\ &\le a_n^\beta. \end{aligned}\tag{6.79}$$

Case 4. $a_n^\beta < d$. The following inequity follows directly from (1.10),

$$2\gamma+\frac{\eta}{2}-1\ge\eta. \tag{6.80}$$

Note that by (4.16) and (2.15), $\beta < \kappa_0$. In this case we have in particular $a_n^{\kappa_0} < d$. This, (1.10), (6.72) and (6.80) imply

$$\begin{aligned}a_n^{\beta\gamma}(a_n^{\kappa_0}\vee d)^{\gamma_1-1} &\le d^{2\gamma+\eta/2-1}\\ &\le d^\eta\\ &\le d^{\tilde\gamma_{m+1}-1}, \end{aligned}\tag{6.81}$$

where for the last inequality we have used the same argument as in (6.77).

From (6.76), (6.77), (6.79) and (6.81), (6.71) follows. ∎

Proof of Proposition 6.3 The proof follows the same lines as the proof of Proposition 5.1 in [16].

Let $0\le m\le\bar m$ and assume (P_m). Our goal is to derive (P_{m+1}). Let $\varepsilon_0\in(0,1), M=\lceil\frac{2}{\eta\varepsilon_0}\rceil, \varepsilon_2=\frac{2\kappa_0}{M}\le\kappa_0\varepsilon_0\eta$ and set $\alpha_i = i\varepsilon_2$ for $i=0,1,...,M$, so that $\alpha_i\in[\varepsilon_2,2\kappa_0]$ for $i\ge1$. Recall that γ satisfies (1.10) is fixed. From (1.10) we get that $\gamma>1-\eta/2$. Let n,ξ,K and β be as in (P_m) where we may assume $\xi>2-\gamma-\eta/2$ without loss of generality. Define $\nu_1=1-\xi\in(0,\gamma-1+\eta/2)$, $\xi'=\xi+(1-\xi)/2\in(\xi,1)$,

$$\begin{aligned}N_2(m,n,\xi,\varepsilon_0,K,\beta,\eta)(\omega) &= \vee_{i=1}^M N_{6.16}(m,n,\nu_1,\varepsilon_0/6,K+1,\alpha_i,\beta,\eta)(\omega),\\ N_3(m,n,\xi,\varepsilon_0,K,\beta,\eta)(\omega) &= \vee_{i=1}^M N_{6.18}(m,n,\nu_1,\varepsilon_0/6,K+1,\alpha_i,\beta,\eta)(\omega),\end{aligned}$$

$$\begin{aligned}N_4(m,n,\xi,\varepsilon_0,K,\beta) &= \lceil\frac{2}{1-\xi}((N_{6.14}(m,n,\nu_1/2,\varepsilon_0/6,K+1,\beta,\eta)\vee N'_{6.16}(\nu_1\varepsilon_2/(2\kappa_0)))+1)\rceil\\ &\equiv \lceil\frac{1}{1-\xi}N_5(m,n,\nu_1,\varepsilon_0,K,\beta,\eta)\rceil. \end{aligned}\tag{6.82}$$

Recall that in the verification of (P_0), we chose $\varepsilon_0=0$, and $N_1=N_1(0,\xi',K,\eta)$ was independent of n and β. Let

$$N_6(\xi,K,\beta)=N_1(0,\xi',K,\eta),$$

and

$$N_1(m,n,\xi,\varepsilon_0,K,\beta,\eta)=(N_2\vee N_3\vee N_4(m,n,\xi,\varepsilon_0,K,\beta,\eta))\vee N_6(\xi,K,\beta)+1\in\mathbb{N},\ P-\text{a.s.}.$$

In what follows we often omit the dependence of $N_1(m,n,\xi,\varepsilon_0,K,\beta,\eta)$ in $m,n,\xi,\varepsilon_0,K,\beta,\eta$ and write N_1. Because $N_{6.14},N_{6.16},N'_{6.16},N_{6.18}$ are stochastically bounded uniformly in (n,β), therefore, so is N_1.

Assume $N\ge N_1,\ (t,x)\in Z(N,n,K,\beta),\ t'\le T_K$ and $d((t,x),(t',x'))\le 2^{-N}$. Consider first the case of

$$a_n^{2\kappa_0\eta} > 2^{-N_5(m,n,\nu_1,\varepsilon_0,K,\beta,\eta)}. \tag{6.83}$$

Recall that $\tilde\gamma_{m+1}-1\le\eta$. Since $N\ge N_1(0,\xi',K,\eta)$, we get from (P_0) with $\varepsilon_0=0$ and ξ' in place of ξ and then, (6.83) and our choice of ξ',

$$\begin{aligned}|u(t',x')| &\le 2^{-N\xi'}\\ &\le 2^{-N\xi'}\big[(a_n^{\kappa_0}\vee 2^{-N})^{\tilde\gamma_{m+1}-1}\big]2^{N_5/2}\\ &\le 2^{-N(1-\xi)/2}2^{N_5/2}2^{-N\xi}\big[(a_n^{\kappa_0}\vee 2^{-N})^{\tilde\gamma_{m+1}-1}+a_n^{\beta}\big]\\ &\le 2^{-N\xi}\big[(a_n^{\kappa_0}\vee 2^{-N})^{\tilde\gamma_{m+1}-1}+a_n^{\beta}\big], \end{aligned}\tag{6.84}$$

where we have used $N\ge N_4\ge(1-\xi)^{-1}N_5$ in the last line. From (6.84), (P_{m+1}) follows.

Next, we deal with the complement of (6.83). Assume that

$$a_n^{2\kappa_0\eta}\le 2^{-N_5(m,n,\nu_1,\varepsilon_0,K,\beta,\eta)}.\tag{6.85}$$

Let $N'=N-1\ge N_2\vee N_3$. Note that $(\hat t_0,\hat x_0)$, which is defined as the point near (t,x) in the set $Z(N,n,K,\beta)$, satisfies $(\hat t_0,\hat x_0)\in Z(N,n,K+1,\beta)\subset Z(N',n,K+1,\beta)$. We also get from the triangle inequality that $d((\hat t_0,\hat x_0),(t',x'))\le 2^{-N'}$. From (6.82) and (6.85) we get that (6.44) holds with $(\varepsilon_0/6,K+1)$ instead of (ε_0,K). Therefore, from inequality $N'\ge N_2$ we can apply Proposition 6.16 to $\alpha=\alpha_i\ge\varepsilon_2,\ i=1,...,M$, with $(\hat t_0,\hat x_0)$ instead of (t,x), $\varepsilon_0/6$ instead of ε_0, and N' instead of N. Since $N'\ge N_3$ we may apply Proposition 6.18 with the same parameters.

Choose $i\in\{1,...,M\}$ such that

$$\begin{aligned}&(i)\text{ if } 2^{-N'}>a_n^{\kappa_0},\ \text{ then } a_n^{\alpha_i/2}<2^{-N'}\le a_n^{\alpha_{i-1}/2}=a_n^{\alpha_i/2}a_n^{-\varepsilon_2/2},\\ &(ii)\text{ if } 2^{-N'}\le a_n^{\kappa_0},\ \text{ then } i=M \text{ and so } a_n^{\alpha_i/2}=a_n^{\kappa_0}\le 2^{-N'}.\end{aligned}\tag{6.86}$$

In either case we have

$$a_n^{\alpha_i/2}\vee 2^{-N'}\le a_n^{\kappa_0}\vee 2^{-N'},\tag{6.87}$$

and

$$a_n^{-(1-\eta/4)\alpha_i}(a_n^{\kappa_0}\vee 2^{-N'})^{2-\eta/2}\le a_n^{-(1-\eta/4)\varepsilon_2}.\tag{6.88}$$

Apply Proposition 6.16 as described above with $1-\nu_1=\xi$ and use the facts that $d((\hat t_0,\hat x_0)),(t',x')\le 2^{-N'}$, $\tilde\gamma_m=\gamma_m$ for $m\le\bar m$ and $\gamma_{m+1}=\gamma\gamma_m+\eta/2$. Then use (6.87), (6.88) to get

$$\begin{aligned}&|u_{1,a_n^{\alpha_i}}(\hat t_0,\hat x_0)-u_{1,a_n^{\alpha_i}}(t',x')|\\ &\le 2^{-90}d((\hat t_0,\hat x_0),(t',x'))^{\xi}a_n^{-\varepsilon_0/2-(1-\eta/4)\alpha_i}\big[a_n^{\beta}a_n^{(1-\eta/4)\alpha_i}+a_n^{\beta\gamma}(a_n^{\alpha_i/2}\vee 2^{-N'})^{\gamma+1}\\ &\quad+(a_n^{\alpha_i/2}\vee 2^{-N'})^{\gamma\tilde\gamma_m+1}+1\!\!1_{\{m\ge\bar m\}}a_n^{\alpha_i(1-\eta/4)}(a_n^{\alpha_i/2}\vee 2^{-N'})^{\eta}\big]\\ &\le 2^{-89}2^{-N'\xi}a_n^{-\varepsilon_0/2}\big[a_n^{\beta}+a_n^{-(1-\eta/4)\varepsilon_2}a_n^{\beta\gamma}(a_n^{\kappa_0}\vee 2^{-N'})^{\gamma-1+\eta/2}\\ &\quad+a_n^{-(1-\eta/4)\varepsilon_2}(a_n^{\kappa_0}\vee 2^{-N'})^{\gamma\gamma_m-1+\eta/2}+1\!\!1_{\{m\ge\bar m\}}(a_n^{\kappa_0}\vee 2^{-N'})^{\eta}\big],\ i=1,...,M.\end{aligned}\tag{6.89}$$

Apply Proposition 6.18 with $\alpha=\alpha_i\ge\varepsilon_2$, $(\hat t_0,\hat x_0)$ instead of (t,x), $\varepsilon_0/6$ instead of ε_0, N' instead of N, $1-\nu_1=\xi$. Use the facts that $d((\hat t_0,\hat x_0),(t',x'))\le 2^{-N'}$, $\tilde\gamma_m=\gamma_m$ for $m\le\bar m$ and $\gamma_{m+1}=\gamma\gamma_m+\eta/2$, and use (6.87) to get

$$\begin{aligned}&|u_{2,a_n^{\alpha_i}}(\hat t_0,\hat x_0)-u_{2,a_n^{\alpha_i}}(t',x')|\\ &\le 2^{-89}\Bigg[d((\hat t_0,\hat x_0),(t',x'))^{\xi/2-(1-\eta)/2}a_n^{-\varepsilon_0/2}2^{-N'\gamma}\big[(a_n^{\kappa_0}\vee 2^{-N'})^{\gamma(\tilde\gamma_m-1)}+a_n^{\beta\gamma}\big]\\ &\quad+d((\hat t_0,\hat x_0),(t',x'))^{\xi}a_n^{-\varepsilon_0/2}\big[a_n^{\frac{\kappa_0}{2}(\gamma\tilde\gamma_m-1+\frac{\eta}{2})}+a_n^{\beta\gamma}a_n^{\frac{\kappa_0}{2}(\gamma-1+\frac{\eta}{2})}\big]\Bigg]\\ &\le 2^{-89}a_n^{-\varepsilon_0/2}\Bigg[2^{-N'(\xi/2+1/2)}2^{-N'(\gamma-1+\eta/2)}\big[(a_n^{\kappa_0}\vee 2^{-N'})^{\gamma(\gamma_m-1)}+a_n^{\beta\gamma}\big]\\ &\quad+2^{-N'\xi}\big[(a_n^{\kappa_0}\vee 2^{-N'})^{(\gamma\gamma_m-1+\eta/2)}+a_n^{\beta\gamma}(a_n^{\kappa_0}\vee 2^{-N'})^{(\gamma-1+\frac{\eta}{2})}\big]\Bigg]\\ &\le 2^{-88}a_n^{-\varepsilon_0/2}2^{-N'\xi}\big[(a_n^{\kappa_0}\vee 2^{-N'})^{\gamma\gamma_m-1+\eta/2}+a_n^{\beta\gamma}(a_n^{\kappa_0}\vee 2^{-N'})^{\gamma-1+\eta/2}\big],\quad i=1,...,M.\end{aligned}$$

From (6.89), (6.90) we get

$$\begin{aligned}
&|u(\hat{t}_0,\hat{x}_0)-u(t',x')| \\
&\le |u_{2,a_n^{\alpha_i}}(\hat{t}_0,\hat{x}_0)-u_{2,a_n^{\alpha_i}}(t',x')|+|u_{1,a_n^{\alpha_i}}(\hat{t}_0,\hat{x}_0)-u_{1,a_n^{\alpha_i}}(t',x')| \\
&\le 2^{-87}2^{-N'\xi}a_n^{-\varepsilon_0/2}a_n^{-(1-\eta/4)\varepsilon_2}\big[a_n^{\beta}+a_n^{\beta\gamma}(a_n^{\kappa_0}\vee 2^{-N'})^{\gamma-1+\eta/2} \\
&\quad +(a_n^{\kappa_0}\vee 2^{-N'})^{\gamma\gamma_m-1+\eta/2}+\mathbb{1}_{\{m=\bar{m}\}}(a_n^{\kappa_0}\vee 2^{-N'})^{\eta}\big].
\end{aligned} \tag{6.90}$$

Consider $m=\bar{m}$ and $m<\bar{m}$ separately to get

$$(a_n^{\kappa_0}\vee 2^{-N'})^{\gamma\gamma_m-1+\eta/2}+\mathbb{1}_{\{m=\bar{m}\}}(a_n^{\kappa_0}\vee 2^{-N'})^{\eta}\le 2(a_n^{\kappa_0}\vee 2^{-N'})^{\tilde{\gamma}_{m+1}-1}. \tag{6.91}$$

Recall that $\varepsilon_2\le\eta\kappa_0\varepsilon_0$ and therefore $(1-\eta/4)\varepsilon_2\le\varepsilon_0/2$. Use this and (6.91) on (6.90) to get

$$\begin{aligned}
&|u(\hat{t}_0,\hat{x}_0)-u(t',x')| \\
&\le 2^{-84}2^{-N\xi}a_n^{-\varepsilon_0}\big[a_n^{\beta}+a_n^{\beta\gamma}(a_n^{\kappa_0}\vee 2^{-N})^{\gamma-1+\eta/2}+(a_n^{\kappa_0}\vee 2^{-N})^{\tilde{\gamma}_{m+1}-1}\big].
\end{aligned} \tag{6.92}$$

From (6.92) and the facts that $|u(\hat{t}_0,\hat{x}_0)|<a_n^{1-\kappa_0}2^{-N}$ and $\gamma_0=1,\ \gamma_1=\gamma+\eta/2$, we get

$$\begin{aligned}
&|u(t',x')| \\
&\le a_n^{-\varepsilon_0}2^{-N\xi}\big[a_n^{1-\kappa_0}2^{-N(1-\xi)}+2^{-84}\big[a_n^{\beta}+a_n^{\beta\gamma}(a_n^{\kappa_0}\vee 2^{-N})^{\gamma_1-1}+(a_n^{\kappa_0}\vee 2^{-N})^{\tilde{\gamma}_{m+1}-1}\big]\big].
\end{aligned} \tag{6.93}$$

By our choice of N_1 and N_4 we have $N(1-\xi)\ge 1$. Recall that $\beta\in[0,\frac{\eta}{1+\eta}]=[0,1-\kappa_0]$. Therefore

$$a_n^{1-\kappa_0}2^{-N(1-\xi)}\le\frac{a_n^{1-\kappa_0}}{2}\le\frac{a_n^{\beta}}{2}. \tag{6.94}$$

From Lemma 6.19 we have

$$a_n^{\beta\gamma}(a_n^{\kappa_0}\vee 2^{-N})^{\gamma_1-1}\le a_n^{\beta}+(2^{-N}\vee a_n^{\kappa_0})^{\tilde{\gamma}_{m+1}-1}. \tag{6.95}$$

Apply (6.94), (6.95) to (6.93) to get

$$|u(t',x')|\quad\le\quad a_n^{-\varepsilon_0}2^{-N\xi}\big[a_n^{\beta}+(2^{-N}\vee a_n^{\kappa_0})^{\tilde{\gamma}_{m+1}-1}\big], \tag{6.96}$$

which implies (P_{m+1}). ■

7 Proof of Proposition 4.8

This section is dedicated to the proof of Proposition 4.8. The proof follows the same lines as the proof of Proposition 3.3 in [16]. Before we start with the proof we will need the following Proposition.

Proposition 7.1 *Let $0\le m\le\bar{m}+1$ and assume (P_m). Then for any $n\in\mathbb{N}$, $\varepsilon_0\in(0,\eta/2),\nu_1\in(0,\varepsilon_0],K\in\mathbb{N}^{\ge K_1},\alpha\in[0,2\kappa_0]$, and $\beta\in[0,\frac{\eta}{\eta+1}]$, there is an $N_{7.1}=N_{7.1}(m,n,\nu_1,\varepsilon_0,K,\alpha,\beta,\eta)$ a.s. such that for all $N\ge N_{7.1},(t,x)\in Z(N,n,K,\beta)$,*

$$\begin{aligned}
&d\equiv d((t,x),(t,x'))\le 2^{-N}\ \text{implies that} \\
&|\tilde{u}_{2,a_n^{\alpha}}(t,a_n^{2\kappa_0+2\varepsilon_0},x)-\tilde{u}_{2,a_n^{\alpha}}(t,a_n^{2\kappa_0+2\varepsilon_0},x')| \\
&\le 2^{-89}a_n^{-\varepsilon_0}\big[(a_n^{-\kappa_0(1-\eta/2)-3\varepsilon_0}d)\wedge d^{\eta/2-\varepsilon_0}\big]\bar{\Delta}_{1,u_2}(m,n,\varepsilon,2^{-N},\eta)+d^{1-\nu_1}\bar{\Delta}_{2,u_2}(m,n,\varepsilon,\eta),
\end{aligned} \tag{7.1}$$

where

$$\begin{aligned}
\bar{\Delta}_{1,u_2}(m,n,\varepsilon_0,2^{-N},\eta) &= 2^{-N\gamma(1-\varepsilon_0)}\big[(2^{-N}\vee a_n^{\kappa_0})^{\gamma(\tilde{\gamma}_m-1)}+a_n^{\gamma\beta}\big], \\
\bar{\Delta}_{2,u_2}(m,n,\varepsilon_0,\eta) &= a_n^{\alpha(1+\eta/2-\varepsilon_0)/2}.
\end{aligned} \tag{7.2}$$

Moreover, $N_{6.18}$ is stochastically bounded, uniformly in (n,α,β).

The proof of Proposition 7.1 is given in Section 8.

In Proposition 6.3 we established the property $(P_{\bar{m}+1})$. Therefore, we can use the conclusions of Corollary 6.12 and Proposition 6.14 along with Proposition 7.1, with $m = \bar{m}+1$. We will use these conclusions to derive the modulus of continuity for u_{1,a_n^α} and $\tilde{u}_{2,a_n^\alpha}$.

We will construct a few sequences of stopping times $\{U^{(i)}_{M,n,\beta_j}\}_{M,n}$, $i = 1,2,\ldots,4$, $j = 1,2,\ldots$ and, roughly speaking, we will show that their minimum is the required sequence of stopping times $\{U_{M,n}\}_{M,n}$. We fix $K_0 \in \mathbb{N}^{\geq K_1}$ as before (4.18) and the positive constants $\varepsilon_0, \varepsilon_1$ as in (2.14). For $0 < \beta \leq \frac{\eta}{\eta+1} - \eta\varepsilon_1$, define

$$\alpha = \alpha(\beta) = 2(\beta/\eta + \varepsilon_1) \in [0, 2\kappa_0], \tag{7.3}$$

and

$$\begin{aligned} U^{(1)}_{M,n,\beta} \;=\; \inf\Big\{ & t: \text{ there are } \varepsilon \in [a_n^{\kappa_0+\varepsilon_0}, 2^{-M}], |x| \leq K_0+1,\ \hat{x}_0, x' \in \mathbb{R},\ \text{s.t. } |x-x'| \leq 2^{-M}, \\ & |x-\hat{x}_0| \leq \varepsilon,\ |u(t,\hat{x}_0)| \leq a_n \wedge (a_n^{1-\kappa_0}\varepsilon),\ |u'_{1,a_n^{2\kappa_0}}(t,\hat{x}_0)| \leq a_n^{\beta}, \\ & \text{and } |u'_{1,a_n^\alpha}(t+a_n^{2\kappa_0+2\varepsilon_0}, x) - u'_{1,a_n^\alpha}(t+a_n^{2\kappa_0+2\varepsilon_0}, x')| > 2^{-76} a_n^{-\varepsilon_0-(2-\eta/2)\varepsilon_1}(|x-x'| \vee a_n^{\kappa_0+\varepsilon_0})^{1-\varepsilon_0} \\ & \times\big[a_n^{-(2-\eta/2)\beta/\eta}(\varepsilon \vee |x'-x|)^{(\eta+1)\gamma} + a_n^{\beta(\eta-1)/\eta} + a_n^{\beta\gamma-\beta(2-\eta/2)/\eta}(\varepsilon \vee |x'-x|)^{\gamma}\big]\Big\} \wedge T_{K_0}. \end{aligned} \tag{7.4}$$

Define $U^{(1)}_{M,n,0}$ by the same expression with $\beta = 0$, but without the condition on $|u'_{1,a_n^{2\kappa_0}}(t,\hat{x}_0)|$. Just as in Section 6 of [16], $U^{(1)}_{M,n,\beta}$ is an $(\mathcal{F}_t)$-stopping time.

Lemma 7.2 *For each $n \in \mathbb{N}$ and β as in (4.16), $U^{(1)}_{M,n,\beta} \uparrow T_{K_0}$ as $M \uparrow \infty$ and*

$$\lim_{M\to\infty} \sup_{n, 0\leq\beta\leq\frac{\eta}{\eta+1}-\eta\varepsilon_1} P(U^{(1)}_{M,n,\beta} < T_{K_0}) = 0. \tag{7.5}$$

Proof: From the monotonicity in M and (7.5) the first assertion is trivial. Let us consider the second assertion. Recall that n_M was defined before (4.36). By Proposition 6.3 we can use Corollary 6.12 with $\varepsilon_0/2$ instead of ε_0, $m = \bar{m}+1$, $\nu_1 = \varepsilon_0, K = K_0+1$, and α, β as in (7.3) and (4.16) respectively. Therefore, there exists $N_0(n,\varepsilon_0,\varepsilon_1,K_0+1,\beta) \in \mathbb{N}$ a.s., stochastically bounded uniformly in (n,β) (where β is as in (4.16)), such that if

$$N \geq N_0(\omega),\ (t,x) \in Z(N,n,K_0+1,\beta),\ a_n^{\kappa_0+\varepsilon_0} + |x-x'| \leq 2^{-N}, \tag{7.6}$$

then,

$$\begin{aligned} & |u'_{1,a_n^\alpha}(t+a_n^{2\kappa_0+2\varepsilon_0}, x) - u'_{1,a_n^\alpha}(t+a_n^{2\kappa_0+2\varepsilon_0}, x')| \\ & \leq |u'_{1,a_n^\alpha}(t+a_n^{2\kappa_0+2\varepsilon_0}, x) - u'_{1,a_n^\alpha}(t,x)| + |u'_{1,a_n^\alpha}(t+a_n^{2\kappa_0+2\varepsilon_0}, x') - u'_{1,a_n^\alpha}(t,x)| \\ & \leq 2^{-83}(|x-x'| \vee a_n^{\kappa_0+\varepsilon_0})^{1-\varepsilon_0} a_n^{-\varepsilon_0}\big[a_n^{-2(\beta/\eta+\varepsilon_1)(1-\eta/4)} 2^{-N\gamma(\eta+1)} + (2^{-N} \vee a_n^{(\beta/\eta+\varepsilon_1)})^{((\eta+1)\gamma-2+\eta/2)\wedge 0} \\ & \quad + a_n^{-2(\beta/\eta+\varepsilon_1)(1-\eta/4)+\beta\gamma}(a_n^{\beta/\eta+\varepsilon_1} \vee 2^{-N})^{\gamma}\big]. \end{aligned} \tag{7.7}$$

Since $\gamma > 1 - \frac{\eta}{2(\eta+1)} + 100\varepsilon_1$ (see (2.14)) we have

$$(\eta+1)\gamma - 2 + \eta/2 \geq \eta - 1. \tag{7.8}$$

and

$$\begin{aligned} a_n^{-\beta(2-\eta/2)/\eta+\beta\gamma+(\beta/\eta+\varepsilon_1)\gamma} & \leq a_n^{\beta(\gamma(\eta+1)/\eta-2/\eta+1/2)} \\ & \leq a_n^{(\eta-1)\beta/\eta}. \end{aligned} \tag{7.9}$$

Note that

$$a_n^{(\beta/\eta+\varepsilon_1)(\eta-1)} \leq a_n^{\beta(\eta-1)/\eta} a_n^{-(2-\eta/2)\varepsilon_1}. \tag{7.10}$$

From (7.7), (7.8), (7.9) and (7.10) we have

$$\begin{aligned}&|u'_{1,a_n^\alpha}(t+a_n^{2\kappa_0+2\varepsilon_0},x)-u'_{1,a_n^\alpha}(t+a_n^{2\kappa_0+2\varepsilon_0},x')|\\&=2^{-83}(|x-x'|\vee a_n^{\kappa_0+\varepsilon_0})^{1-\varepsilon_0}a_n^{-\varepsilon_0-(2-\eta/2)\varepsilon_1}\big[a_n^{-\beta(2-\eta/2)/\eta}2^{-N\gamma(\eta+1)}+(2^{-N}\vee a_n^{\beta/\eta})^{\eta-1}\\&\quad+a_n^{-\beta(2-\eta/2)/\eta+\beta\gamma}2^{-\gamma N}\big].\end{aligned}\tag{7.11}$$

Assume that $\beta>0$ (if $\beta=0$ we can omit the bound on $|u'_{1,a_n^\alpha}(t,\hat{x}_0)|$ in what follows). Assume that $M\geq N_0(n,\varepsilon_0,\varepsilon_1,K_0+1,\beta)+2$. Suppose that for some $t<T_{K_0}(\leq T_{K_0+1})$ there are $\varepsilon\in[a_n^{\kappa_0+\varepsilon_0},2^{-M}]$, $|x|\leq K_0+1,\hat{x}_0,x'\in\mathbb{R}$ satisfying $|x-x'|\leq 2^{-M}$, $|\hat{x}_0-x|\leq\varepsilon$, $|u(t,\hat{x}_0)|\leq a_n\wedge(a_n^{1-\kappa_0}\varepsilon)$ and $|u'_{1,a_n^\alpha}(t+a_n^{2\kappa_0},\hat{x}_0)|\leq a_n^\beta$. If $x\neq x'$, then

$$0<(|x-x'|+a_n^{\kappa_0+\varepsilon})\vee\varepsilon\leq 2^{-M+1}\leq 2^{-N_0-1}.\tag{7.12}$$

By (7.12) we may choose $N>N_0$ such that $2^{-N-1}<\varepsilon\vee(a_n^{\kappa_0+\varepsilon_0}+|x-x'|)\leq 2^{-N}$. Then (7.6) holds and we get from (7.11)

$$\begin{aligned}&|u'_{1,a_n^\alpha}(t+a_n^{2\kappa_0+2\varepsilon_0},x)-u'_{1,a_n^\alpha}(t+a_n^{2\kappa_0+2\varepsilon_0},x')|\\&\leq 2^{-78}(|x-x'|\vee a_n^{\kappa_0+\varepsilon_0})^{1-\varepsilon_0}a_n^{-\varepsilon_0-(2-\eta/2)\varepsilon_1}\big[a_n^{-\beta(2-\eta/2)/\eta}(|x-x'|\vee a_n^{\kappa_0+\varepsilon_0}\vee\varepsilon)^{\gamma(\eta+1)}+a_n^{\beta(\eta-1)/\eta}\\&\quad+a_n^{-\beta(2-\eta/2)/\eta+\beta\gamma}(|x-x'|\vee a_n^{\kappa_0+\varepsilon_0}\vee\varepsilon)^{\gamma}\big]\\&\leq 2^{-76}(|x-x'|\vee a_n^{\kappa_0+\varepsilon_0})^{1-\varepsilon_0}a_n^{-\varepsilon_0-(2-\eta/2)\varepsilon_1}\big[a_n^{-\beta(2-\eta/2)/\eta}(|x-x'|\vee\varepsilon)^{\gamma(\eta+1)}+a_n^{\beta(\eta-1)/\eta}\\&\quad+a_n^{-\beta(2-\eta/2)/\eta+\beta\gamma}(|x-x'|\vee\varepsilon)^{\gamma}\big].\end{aligned}\tag{7.13}$$

If $x=x'$ the bound in (7.13) is trivial. From (7.13) we get that if $M\geq N_0(n,\varepsilon_0,\varepsilon_1,K_0+1,\beta)+2$, then $U^{(1)}_{M,n,\beta}=T_{K_0}$. Therefore, we have shown that

$$P(U^{(1)}_{M,n,\beta}<T_{K_0})=P(M<N_0+2).$$

This completes the proof because $N_0(n,\varepsilon_0,\varepsilon_1,K_0+1,\beta)$ is stochastically bounded uniformly in (n,β), where β satisfies (4.16). ■

Next we define a stopping time related to the increment of u_{2,a_n^α}. For $0<\beta<\frac{\eta}{\eta+1}-\varepsilon_1$ define

$$\begin{aligned}U^{(2)}_{M,n,\beta}\;=\;\inf\Big\{&t:\text{ there are }\varepsilon\in[0,2^{-M}],|x|\leq K_0+1,\ \hat{x}_0,x'\in\mathbb{R},\text{ s.t. }|x-x'|\leq 2^{-M},\\&|x-\hat{x}_0|\leq\varepsilon,\ |u(t,\hat{x}_0)|\leq a_n\wedge(a_n^{1-\kappa_0}\varepsilon),\\&|u'_{1,a_n^{2\kappa_0}}(t,\hat{x}_0)|\leq a_n^\beta,\text{ and }|\tilde{u}_{2,a_n^\alpha}(t,a_n^{2\kappa_0+2\varepsilon_0},x)-\tilde{u}_{2,a_n^\alpha}(t,a_n^{2\kappa_0+2\varepsilon_0},x')|\\&>2^{-87}a_n^{-\varepsilon_0}\Big[((a_n^{-\kappa_0(1-\eta/2)-3\varepsilon_0}|x-x'|)\wedge|x-x'|^{\eta/2-\varepsilon_0})(\varepsilon\vee|x'-x|)^{\gamma(1-\varepsilon_0)}\\&\quad\times\big[((a_n^{\kappa_0}\vee\varepsilon\vee|x'-x|)^{\eta\gamma}+a_n^{\beta\gamma}\big]+|x-x'|^{1-\varepsilon_0}a_n^{\frac{3\beta}{2}+\varepsilon_1-\varepsilon_0})\Big]\Big\}\wedge T_{K_0}.\end{aligned}\tag{7.14}$$

Define $U^{(2)}_{M,n,0}$ by the same expression with $\beta=0$, but without the condition on $|u'_{1,a_n^{2\kappa_0}}(t,\hat{x}_0)|$. Just as in Section 6 of [16], $U^{(2)}_{M,n,\beta}$ is an $(\mathcal{F}_t)$-stopping time.

Lemma 7.3 *For each $n\in\mathbb{N}$ and β as in (4.16), $U^{(2)}_{M,n,\beta}\uparrow T_{K_0}$ as $M\uparrow\infty$ and*

$$\lim_{M\to\infty}\sup_{n,0\leq\beta\leq\frac{\eta}{\eta+1}-\eta\varepsilon_1}P(U^{(2)}_{M,n,\beta}<T_{K_0})=0.$$

Proof: As before, we only need to show the second assertion. By Proposition 6.3 we can use Proposition 7.1 with $m=\bar{m}+1$, $\nu_1=\varepsilon_0,K=K_0+1$ and α,β as in (7.3) and (4.16) respectively. Therefore, there exists $N_0(n,\varepsilon_0,\varepsilon_1,K_0+1,\beta)\in\mathbb{N}$ a.s., stochastically bounded uniformly in (n,β) as in (4.16), such that if

$$N\geq N_0(\omega),\ (t,x)\in Z(N,n,K_0+1,\beta),\ |x-x'|\leq 2^{-N},\tag{7.15}$$

$$
\begin{aligned}
&|\tilde{u}_{2,a_n^\alpha}(t,a_n^{2\kappa_0+2\varepsilon_0},x)-\tilde{u}_{2,a_n^\alpha}(t,a_n^{2\kappa_0+2\varepsilon_0},x')| \\
&\le 2^{-89}a_n^{-\varepsilon_0}\Big[((|x-x'|a_n^{-\kappa_0(1-\eta/2)-3\varepsilon_0})\wedge|x-x'|^{\eta/2-\varepsilon_0})2^{-N\gamma(1-\varepsilon_0)}\big[(2^{-N}\vee a_n^{\kappa_0})^{\gamma\eta}+a_n^{\gamma\beta}\big] \\
&\quad +|x-x'|^{1-\varepsilon_0}\big[a_n^{(\beta/\eta+\varepsilon_1)(1+\eta/2-\varepsilon_0)}\big]\Big].
\end{aligned}
\tag{7.16}
$$

Recall that $\eta\in(0,1)$, then from (4.16) and (2.15) we have

$$
\big(\frac{\beta}{\eta}+\varepsilon_1\big)\big(1+\frac{\eta}{2}-\varepsilon_0\big)\ge\frac{3}{2}\beta+\varepsilon_1-2\varepsilon_0. \tag{7.17}
$$

Apply (7.17) to (7.16) to get

$$
\begin{aligned}
&|\tilde{u}_{2,a_n^\alpha}(t,a_n^{2\kappa_0+2\varepsilon_0},x)-\tilde{u}_{2,a_n^\alpha}(t,a_n^{2\kappa_0+2\varepsilon_0},x')| \\
&\le 2^{-89}a_n^{-\varepsilon_0}\Big[((|x-x'|a_n^{-\kappa_0(1-\eta/2)-3\varepsilon_0})\wedge|x-x'|^{\eta/2-\varepsilon_0})2^{-\gamma N(1-\varepsilon_0)}\big[(2^{-N}\vee a_n^{\kappa_0})^{\eta\gamma}+a_n^{\gamma\beta}\big] \\
&\quad +|x-x'|^{1-\varepsilon_0}a_n^{3\beta/2+\varepsilon_1-2\varepsilon_0}\Big].
\end{aligned}
\tag{7.18}
$$

The rest of the proof is similar to the proof of Lemma 7.2, where (7.18) is used instead of (7.13). ■

Notation.

$$
\tilde{\Delta}_{u_1'}(n,\varepsilon,\varepsilon_0,\beta,\eta)=a_n^{-\varepsilon_0}\varepsilon^{-\varepsilon_0}\big\{\varepsilon(\varepsilon\vee a_n^{\kappa_0})^{\eta-1}+(\varepsilon a_n^{-\kappa_0(2-\eta/2)}+a_n^{-1+\eta/4+\kappa_0})\big(\varepsilon^{(\eta+1)\gamma}+a_n^{\beta\gamma}(\varepsilon\vee a_n^{\kappa_0})^{\gamma}\big)\big\}. \tag{7.19}
$$

For $0<\beta<\frac{\eta}{\eta+1}-\varepsilon_1$ define

$$
\begin{aligned}
U^{(3)}_{M,n,\beta} \;=\; \inf\Big\{ & t:\ \text{there are } \varepsilon\in[2^{-a_n^{-(\beta/\eta+\varepsilon_1)\varepsilon_0/4}},2^{-M}], |x|\le K_0+1,\ \hat{x}_0\in\mathbb{R},\ \text{s.t. } |x-\hat{x}_0|\le\varepsilon, \\
& |u(t,\hat{x}_0)|\le a_n\wedge(a_n^{1-\kappa_0}\varepsilon),\ |u'_{1,a_n^{2\kappa_0}}(t,\hat{x}_0)|\le a_n^{\beta},\ \text{and} \\
& |u'_{1,a_n^{2\kappa_0}}(t,x)-u'_{1,a_n^\alpha}(t+a_n^{2\kappa_0+2\varepsilon_0},x)|>2^{-78}(\tilde{\Delta}_{u_1'}(n,\varepsilon,\varepsilon_0,\beta,\eta)+a_n^{\beta+\varepsilon_1\eta^2/4})\Big\}\wedge T_{K_0}.
\end{aligned}
\tag{7.20}
$$

Define $U^{(3)}_{M,n,0}$ by the same expression with $\beta=0$, but without the condition on $|u'_{1,a_n^{2\kappa_0}}(t,\hat{x}_0)|$. Just as in Section 6 of [16], $U^{(3)}_{M,n,\beta}$ is an $(\mathcal{F}_t)$-stopping time.

Lemma 7.4 *For each $n\in\mathbb{N}$ and β as in (4.16), $U^{(3)}_{M,n,\beta}\uparrow T_{K_0}$ as $M\uparrow\infty$ and*

$$
\lim_{M\to\infty}\sup_{n,0\le\beta\le\frac{\eta}{\eta+1}-\eta\varepsilon_1}P(U^{(3)}_{M,n,\beta}<T_{K_0})=0.
$$

Proof: The proof of Lemma 7.4 follows the same lines as the proof of Lemma 6.3 in [16]; hence we omit the details. The main difference in the proofs is that we use Proposition 6.14 instead of Proposition 5.11 in [16]. ■

For $0<\beta<\frac{\eta}{\eta+1}-\varepsilon_1$ define

$$
\begin{aligned}
U^{(4)}_{M,n,\beta} \;=\; \inf\Big\{ & t:\ \text{there are } \varepsilon\in[0,2^{-M}], |x|\le K_0+1,\ \hat{x}_0,\, x'\in\mathbb{R},\ \text{s.t. } |x-x'|\le 2^{-M}, \\
& |x-\hat{x}_0|\le\varepsilon,\ |u(t,\hat{x}_0)|\le a_n\wedge(a_n^{1-\kappa_0}\varepsilon),\ |u'_{1,a_n^{2\kappa_0}}(t,\hat{x}_0)|\le a_n^{\beta}, \\
& \text{and } |u(s,x')|>4a_n^{-\varepsilon_0/8}\varepsilon^{1-\varepsilon_0/8}\big[(a_n^{\kappa_0}\vee\varepsilon)^{\eta}+a_n^{\beta}\big]\Big\}\wedge T_{K_0}.
\end{aligned}
\tag{7.21}
$$

Define $U^{(4)}_{M,n,0}$ by the same expression with $\beta=0$, but without the condition on $|u'_{1,a_n^{2\kappa_0+2\varepsilon_0}}(t,\hat{x}_0)|$. Just as in Section 6 of [16], $U^{(4)}_{M,n,\beta}$ is an $(\mathcal{F}_t)$-stopping time.

Lemma 7.5 *For each $n \in \mathbb{N}$ and β as in (4.16), $U^{(4)}_{M,n,\beta} \uparrow T_{K_0}$ as $M \uparrow \infty$ and*

$$\lim_{M\to\infty} \sup_{n, 0\le\beta\le\frac{\eta}{\eta+1}-\eta\varepsilon_1} P(U^{(4)}_{M,n,\beta} < T_{K_0}) = 0. \tag{7.22}$$

Proof: From the monotonicity in M and (7.22) the first assertion is trivial. Let us consider the second assertion. By Proposition 6.3 with $\varepsilon_0/8$ instead of ε_0, $1-\varepsilon_0/8$ instead of ξ, $m = \bar{m}+1$, $K = K_0+1$, and β, α as in (4.16) and (7.3), respectively, there exists $N_1(n, \varepsilon_0/8, \varepsilon_1, K_0+1, \beta, \eta) \in \mathbb{N}$ a.s., stochastically bounded uniformly in (n,β) as in (4.16), such that if

$$N \ge N_1(\omega),\ (t,x) \in Z(N, n, K_0+1, \beta), \tag{7.23}$$

then,

$$|u(t,x')| \le a_n^{-\varepsilon_0/8} 2^{-N(1-\varepsilon_0/8)} \big[(a_n^{\kappa_0} \vee 2^{-N}])^{\eta} + a_n^{\beta}\big],\ \forall |x-x'| \le 2^{-N}. \tag{7.24}$$

Assume that $\beta > 0$ (if $\beta = 0$ we can omit the bound $|u'_{1,a_n^{\alpha}}(t,\hat{x}_0)|$ in what follows). Assume also that $M \ge N_1(n, \varepsilon_0/8, \varepsilon_1, K_0+1, \beta, \eta)$. Suppose for some $t < T_{K_0} (\le T_{K_0+1})$ there are $\varepsilon \in [0, 2^{-M}], |x| \le K_0+1$, $\hat{x}_0 \in \mathbb{R}$ satisfying $|x-\hat{x}_0| \le \varepsilon, |u(t,\hat{x}_0)| \le a_n \wedge (a_n^{1-\kappa_0}\varepsilon)$ and $|u'_{1,a_n^{2\kappa_0}}(t,\hat{x}_0)| \le a_n^{\beta}$. We may choose $N \ge N_1$ such that $2^{-N-1} < \varepsilon \le 2^{-N}$. Then (7.23) holds and we get from (7.24)

$$|u(t,x')| \le 4 a_n^{-\varepsilon_0/8} \varepsilon^{1-\varepsilon_0/8} \big[(a_n^{\kappa_0} \vee \varepsilon)^{\eta} + a_n^{\beta}\big],\ \forall |x-x'| \le 2^{-N}. \tag{7.25}$$

We have shown that

$$P(U^{(4)}_{M,n,\beta} < T_{K_0}) = P(M < N_1).$$

This completes the proof because $N_1(n, \varepsilon_0/8, \varepsilon_1, K_0+1, \beta)$ is stochastically bounded uniformly in (n,β), where β satisfies (4.16). ∎

Let

$$U_{M,n,\beta} = \wedge_{j=1}^{4} U^{(j)}_{M,n,\beta} \tag{7.26}$$

and

$$U_{M,n} = \wedge_{i=0}^{L(\varepsilon_0,\varepsilon_1)} U_{M,n,\beta_i}, \tag{7.27}$$

where $\{\beta_i\}_{i=0}^{L(\varepsilon_0,\varepsilon_1)}$ are defined in (4.15). Recall that $U_{M,n}$ depends on the fixed values of $K_0, \varepsilon_0, \varepsilon_1$. Note that $\beta_i \in [0, \frac{\eta}{\eta+1} - \varepsilon_1]$, for $i = 0, ..., L$ by (4.16) and $\alpha_i = \alpha(\beta_i)$ are given by (7.3). By Lemmas 7.2–7.5, $\{U_{M,n}\}$ satisfy (2.20) in Proposition 2.3.

To complete the proof of Proposition 4.8, we need to prove that the sets $\tilde{J}_{n,i}(s)$ are compact for all $s \ge 0$ and to show that $\tilde{J}_{n,i}(s)$ contains $J_{n,i}(s)$, for all $0 \le s \le U_{M,n}$ and $i = 0, ..., L$. In the following lemmas we will prove the inclusion $J_{n,i}(s) \subset \tilde{J}_{n,i}(s)$. We assume that M, n satisfy (4.37) throughout the rest of the section.

We will also need the following lemma. Recall that n_1 is defined in (4.36).

Lemma 7.6 *Let $s \in [0, T_{K_0}]$ and $x \in \mathbb{R}$. Assume that $n \ge n_1(\varepsilon_0, K_0)$. If*

$$|\langle u(s,\cdot), G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-\cdot)\rangle| \le \frac{a_n}{2}, \tag{7.28}$$

then,

$$|u(s, \hat{x}_n(s,x))| \le a_n.$$

Proof: Let $(s,x) \in [0, T_{K_0}] \times \mathbb{R}$. Suppose that (7.28) is satisfied for some $n > n_1(\varepsilon_0, K_0)$. By a simple change of variable we get

$$\int_{-\infty}^{x-a_n^{\kappa_0}} e^{|y|} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)dy + \int_{x+a_n^{\kappa_0}}^{\infty} e^{|y|} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)dy \le 2e^{K_0} \int_{a_n^{-\varepsilon_0}}^{\infty} e^{|y|} G_1(y)dy. \tag{7.29}$$

Assume that $|u(s,\hat{x}_n(s,x))| > a_n$. From (7.29) and the continuity of u and our choice of $s \le T_{K_0}$ it follows that

$$\begin{aligned} \left|\int_{\mathbb{R}} u(s,x) G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)dy\right| &\ge a_n \int_{x-a_n^{\kappa_0}}^{x+a_n^{\kappa_0}} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)dy \\ &\quad -K_0 \int_{-\infty}^{x-a_n^{\kappa_0}} e^{|y|} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)dy \\ &\quad -K_0 \int_{x+a_n^{\kappa_0}}^{\infty} e^{|y|} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)dy \\ &\ge a_n \int_{-a_n^{-\varepsilon_0}}^{a_n^{-\varepsilon_0}} G_1(y)dy - 2K_0 e^{K_0} \int_{a_n^{-\varepsilon_0}}^{\infty} e^{|y|} G_1(y)dy \\ &> \frac{a_n}{2}. \end{aligned} \tag{7.30}$$

The last inequality follows from the assumption that $n > n_1$. We get the contradiction with (7.28) and the result follows. ■

Notation. Let

$$n_2(\varepsilon_0,\kappa_0,\gamma,\eta,K_0,R_1) = \inf\left\{ n \in \mathbb{N} : \frac{32 C_{(5.3)}(1,2R_1,\eta,\varepsilon_0/4)}{C_{4.6}(\eta,\varepsilon_0,2R_1+2)K_0^2 e^{2K_0}} \ge a_n^{-2\gamma(\kappa_0+\frac{\eta}{\eta+1})+\kappa_0(1-\eta)+2\varepsilon_0} e^{-\frac{a_n^{-2\varepsilon_0}}{32}} \right\}.$$

Lemma 7.7 *If $i \in \{0,...,L\}$, $0 \le s \le U_{M,n}$, and $x \in J_{n,i}(s)$, then for all $n \ge n_2(\varepsilon_0,\kappa_0,\gamma,\eta,K_0,R_1)$*

$$\int_{\mathbb{R}} e^{2R_1|y|} |u(s,y)|^{2\gamma} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\mu(dy) \le 128 C_{(5.3)}(1,2R_1,\eta,\varepsilon_0/4) e^{2R_1K_0} a_n^{2\gamma(\kappa_0+\beta_i)-\kappa_0(1-\eta)-2\varepsilon_0}.$$

Proof: Assume that (n,i,s,x) are as above and set $\varepsilon = a_n^{\kappa_0}$. We have $|\langle u_s, G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-\cdot)\rangle| \le a_n/2$. By Lemma 7.6 we get

$$|u(s,\hat{x}_n(s,x))| \le a_n \le a_n \wedge (a_n^{1-\kappa_0}\varepsilon),\ |\hat{x}_n(s,x) - x| \le \varepsilon. \tag{7.31}$$

The definition of $J_{n,i}$ implies that for $i = 1,...,L$,

$$|u'_{1,a_n^{2\kappa_0}}(s,\hat{x}_n(s,x))| \le a_n^{\beta_i}/4. \tag{7.32}$$

From (2.14) and (2.15) we note that $\varepsilon_1 < \kappa_0$. From (4.37) we have $n \ge n_M(\varepsilon_1)$ and therefore

$$a_n^{\kappa_0} \le 2^{-M}. \tag{7.33}$$

Use (7.31)–(7.33), with $|\hat{x}_n(s,x)| \le K_0+1$ and $s < U_{M,n} \le U^{(4)}_{M,n,\beta_i}$, and take $x = \hat{x}_0 = \hat{x}_n(s,x)$ in the definition of $U^{(4)}_{M,n,\beta_i}$ to get for $i = 0,...,L$

$$\begin{aligned} |u(s,y)| &\le 4 a_n^{-\varepsilon_0/8} a_n^{\kappa_0(1-\varepsilon_0/8)} \left[a_n^{\eta\kappa_0} + a_n^{\beta_i}\right] \\ &\le 8 a_n^{-\varepsilon_0/8} a_n^{\kappa_0(1-\varepsilon_0/8)+\beta_i} \\ &\le 8 a_n^{\kappa_0+\beta_i-\varepsilon_0/4}, \quad \forall y \in [x-a_n^{\kappa_0}, x+a_n^{\kappa_0}], \end{aligned} \tag{7.34}$$

where we have used (2.15) and (4.16) to get $\beta_i \le \kappa_0\eta$ in the second inequity.

From Lemma 5.5(a) with $r = 1$, $\lambda = 2R_1$, $s = 0$, $t = a_n^{2\kappa_0+2\varepsilon_0}$ and $\varpi = \varepsilon_0/4$ we have

$$\begin{aligned} \int_{\mathbb{R}} e^{2R_1|y|} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\mu(dy) &\le C_{(5.3)}(1,2R_1,\eta,\varepsilon_0/4) e^{2R_1K_0} a_n^{-(\kappa_0+\varepsilon_0)(1-\eta+\varepsilon_0/4)} \\ &\le C_{(5.3)}(1,2R_1,\eta,\varepsilon_0/4) e^{2R_1K_0} a_n^{-\kappa_0(1-\eta)-5\varepsilon_0/4},\ \forall x \in [-K_0,K_0]. \end{aligned} \tag{7.35}$$

From Lemma 4.6 with $t = a_n^{2\kappa_0+2\varepsilon_0}$ and $\nu_1 = \varepsilon_0$ we have

$$\begin{aligned} &\int_{\mathbb{R}} e^{(2R_1+2)|y|} G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y) 1_{\{|x-y| \ge a_n^{\kappa_0}\}} \mu(dy) \\ &\le 2C_{4.6}(\eta,\varepsilon_0,2R_1+2) e^{(2R_1+2)K_0} e^{-\frac{a_n^{-2\varepsilon_0}}{32}},\ \forall x \in [-K_0,K_0]. \end{aligned} \tag{7.36}$$

Recall that T_K was defined in (1.13). From (7.34)–(7.36) and the fact that $(x,s)\in[-K_0,K_0]\times[0,T_{K_0}]$ we have

$$\begin{aligned}\int_{\mathbb{R}} e^{2R_1|y|}|u(s,y)|^{2\gamma}G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\mu(dy) &= \int_{x-a_n^{\kappa_0}}^{x+a_n^{\kappa_0}} e^{2R_1|y|}|u(s,y)|^{2\gamma}G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\mu(dy) \\ &\quad+\int_{\mathbb{R}} e^{2R_1|y|}|u(s,y)|^{2\gamma}G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\mathbb{1}_{\{|x-y|\ge a_n^{\kappa_0}\}}\mu(dy) \\ &\le 64a_n^{2\gamma(\kappa_0+\beta_i)-\varepsilon_0/2}\int_{x-a_n^{\kappa_0}}^{x+a_n^{\kappa_0}} e^{2R_1|y|}G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\mu(dy) \\ &\quad+K_0^2\int_{\mathbb{R}} e^{(2R_1+2)|y|}G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-y)\mathbb{1}_{\{|x-y|\ge a_n^{\kappa_0}\}}\mu(dy) \\ &\le 64C_{(5.3)}(1,2R_1,\eta,\varepsilon_0/4)e^{2R_1K_0}a_n^{2\gamma(\kappa_0+\beta_i)-\varepsilon_0/2}a_n^{-\kappa_0(1-\eta)-5\varepsilon_0/4} \\ &\quad+2K_0^2C_{4.6}(\eta,\varepsilon_0,2R_1+2)e^{(2R_1+2)K_0}e^{-\frac{a_n^{-2\varepsilon_0}}{32}} \\ &\le 128C_{(5.3)}(1,2R_1,\eta,\varepsilon_0/4)e^{2R_1K_0}a_n^{2\gamma(\kappa_0+\beta_i)-\kappa_0(1-\eta)-2\varepsilon_0},\ \forall i=0,...,L,\end{aligned}\tag{7.37}$$

where the last inequality follows from our choice n_2. ∎

Lemma 7.8 *If* $i\in\{0,...,L\},\ 0\le s\le U_{M,n},$ *and* $x\in J_{n,i}(s)$, *then*

- **(a)** $|u'_{1,a_n^{2\kappa_0}}(s,\hat{x}_n(s,x))-u'_{1,a_n^{\alpha_i}}(s+a_n^{2\kappa_0+2\varepsilon_0},\hat{x}_n(s,x))|\le 2^{-73}a_n^{\beta_i+\varepsilon_1\eta^2/4}$,
- **(b)** *for* $i>0$, $|u'_{1,a_n^{\alpha_i}}(s+a_n^{2\kappa_0+2\varepsilon_0},\hat{x}_n(s,x))|\le a_n^{\beta_i}/2$,
- **(c)** *for* $i<L$, $u'_{1,a_n^{\alpha_i}}(s+a_n^{2\kappa_0+2\varepsilon_0},\hat{x}_n(s,x))>a_n^{\beta_{i+1}}/8$.

Proof: Assume that (n,i,s,x) are as above and set $\varepsilon=a_n^{\kappa_0}$. We have $|\langle u_s,G_{a_n^{2\kappa_0+2\varepsilon_0}}(x-\cdot)\rangle|\le a_n/2$. By Lemma 7.6 we get

$$|u(s,\hat{x}_n(s,x))|\le a_n\le a_n\wedge(a_n^{1-\kappa_0}\varepsilon),\ |\hat{x}_n(s,x)-x|\le\varepsilon.\tag{7.38}$$

The definition of $J_{n,i}$ implies for $i=1,...,L$

$$|u'_{1,a_n^{2\kappa_0}}(s,\hat{x}_n(s,x))|\le a_n^{\beta_i}/4.\tag{7.39}$$

From (2.14) and (2.15) we note that $\varepsilon_1<\kappa_0$. From (4.37) it follows that $n>n_0(\varepsilon_1,\varepsilon_0)\vee n_M(\varepsilon_0)$, therefore

$$2^{-M}\ge a_n^{\kappa_0}=\varepsilon\ge 2^{-a_n^{-\varepsilon_0\varepsilon_1/4}}.\tag{7.40}$$

Use (7.38), (7.40), $|\hat{x}_n(s,x)|\le K_0+1$ and $s<U_{M,n}\le U^{(3)}_{M,n,\beta_i}$, and take $x=\hat{x}_0=\hat{x}_n(s,x)$ in the definition of $U^{(3)}_{M,n,\beta_i}$ to get

$$\begin{aligned}&|u'_{1,a_n^{2\kappa_0}}(s,\hat{x}_n(s,x))-u'_{1,a_n^{\alpha_i}}(s+a_n^{2\kappa_0+2\varepsilon_0},\hat{x}_n(s,x))|\\ &\le 2^{-78}(\tilde{\Delta}_{u_1'}(n,a_n^{\kappa_0},\varepsilon_0,\beta_i,\eta)+a_n^{\beta_i+\varepsilon_1\eta^2/4}).\end{aligned}\tag{7.41}$$

Recall that $\eta\in(0,1),\ \kappa_0=\frac{1}{\eta+1}$ are fixed. From (4.15) we have

$$\beta_i+\varepsilon_1\eta\le\eta\kappa_0,\ i=0,...,L+1.\tag{7.42}$$

Note that for $\eta\in(0,1)$,

$$\frac{1}{\eta+1}-1+\frac{\eta}{4}+\left(1-\frac{\eta}{2(\eta+1)}\right)\ge\frac{\eta}{\eta+1}.\tag{7.43}$$

Use (2.15), (2.14), (4.15), (4.16) and (7.43) to get

$$\begin{aligned}&\kappa_0-1+\frac{\eta}{4}+\gamma\big(\kappa_0+\beta_i\big)-\beta_i\\ &\ge\frac{1}{\eta+1}-1+\frac{\eta}{4}+\left(1-\frac{\eta}{2(\eta+1)}+100\varepsilon_1\right)\big(\frac{1}{\eta+1}+\beta_i\big)-\beta_i\\ &=\frac{1}{\eta+1}-1+\frac{\eta}{4}+\left(1-\frac{\eta}{2(\eta+1)}\right)-\frac{\eta}{\eta+1}+50\varepsilon_1\\ &\ge 50\varepsilon_1,\end{aligned}\tag{7.44}$$

Use (2.15), (2.14), (4.15) and (4.16) to get

$$\begin{aligned}
&-\kappa_0+\kappa_0\frac{\eta}{2}+\gamma\big(\kappa_0+\beta_i\big)-\beta_i\\
&\geq -\frac{1}{\eta+1}+\frac{\eta}{2(\eta+1)}+\Big(1-\frac{\eta}{2(\eta+1)}+100\varepsilon_1\Big)\big(\frac{1}{\eta+1}+\beta_i\big)-\beta_i\\
&= -\frac{1}{\eta+1}+\frac{\eta}{2(\eta+1)}+\Big(1-\frac{\eta}{2(\eta+1)}\Big)-\frac{\eta}{\eta+1}+50\varepsilon_1\\
&\geq 50\varepsilon_1.
\end{aligned}\tag{7.45}$$

From (7.44) and (7.45) we immediately get

$$\kappa_0-1+\frac{\eta}{4}+\gamma\big(\kappa_0+\beta_i\big)\quad\geq\quad\beta_i+\varepsilon_1\tag{7.46}$$

and

$$-\kappa_0\big(1-\frac{\eta}{2}\big)+\gamma\big(\kappa_0+\beta_i\big)\quad\geq\quad\beta_i+\varepsilon_1.\tag{7.47}$$

From (7.42), (7.46), (7.47) and (2.14) we get,

$$\begin{aligned}
&\tilde{\Delta}_{u_1'}(n,a_n^{\kappa_0},\varepsilon_0,\beta_i,\eta)\\
&=a_n^{-\varepsilon_0(1+\kappa_0)}\big\{a_n^{\eta\kappa_0}+\big(a_n^{\kappa_0-1+\eta/4}+a_n^{-\kappa_0(1-\eta/2)}\big)\big(a_n^{\kappa_0(\eta+1)\gamma}+a_n^{\beta_i\gamma}a_n^{\gamma\kappa_0}\big)\big\}\\
&\leq 4\big\{a_n^{\beta_i+\varepsilon_1\eta-2\varepsilon_0}+a_n^{\beta_i+\varepsilon_1/2}\big\}\\
&\leq 8a_n^{\beta_i+\varepsilon_1\eta/2}.
\end{aligned}\tag{7.48}$$

From (7.41) and (7.48) we get

$$\begin{aligned}
&|u'_{1,a_n^{2\kappa_0+2\varepsilon_0}}(s,\hat{x}_n(s,x))-u'_{1,a_n^{\alpha_i}}(s+a_n^{2\kappa_0+2\varepsilon_0},\hat{x}_n(s,x))|\\
&\leq 2^{-73}a_n^{\beta_i+\varepsilon_1\eta^2/4},
\end{aligned}\tag{7.49}$$

and we are done with claim (a) of the lemma. (b) is immediate from (a) and the fact that $|u'_{1,a_n^{2\kappa_0}}(s,\hat{x}_n(s,x))|\leq a_n^{\beta_i}/4$ (by the definition of $J_{n,i}$ for $i>0$).

(c) Since $\varepsilon_0\leq\varepsilon_1\eta^2/100$ by (2.14), $a_n^{\beta_i+\varepsilon_1\eta^2/4}\leq a_n^{\beta_{i+1}}$ by (4.15). For $i\leq L$ we have by the definition of $J_{n,i}$, $u'_{1,a_n^{2\kappa_0}}(s,\hat{x}_n(s,x))\geq a_n^{\beta_{i+1}}$. Then (c) follows from (a) and the triangle inequality. ■

Lemma 7.9 *If $i\in\{0,...,L\}$, $0\leq s\leq U_{M,n}$, $x\in J_{n,i}(s)$, and $|x-x'|\leq 5\bar{l}_n(\beta_i)$, then*

(a) *for $i>0$, $|u'_{1,a_n^{\alpha_i}}(s+a_n^{2\kappa_0+2\varepsilon_0},x')|\leq a_n^{\beta_i}$,*

(b) *for $i<L$, $u'_{1,a_n^{\alpha_i}}(s+a_n^{2\kappa_0+2\varepsilon_0},x')>a_n^{\beta_{i+1}}/16$.*

Proof: Let (n,i,s,x,x') as above and set $\varepsilon=|x-x'|+a_n^{\kappa_0}$. Then from (4.16) and (4.37) we have

$$\varepsilon\leq 5\bar{l}_n(\beta_i)+a_n^{\kappa_0}=5a_n^{\beta_i/\eta+5\varepsilon_1}+a_n^{\kappa_0}\leq 5a_n^{5\varepsilon_1}+a_n^{\kappa_0}\leq 2^{-M}.\tag{7.50}$$

From Definition 4.4 we have

$$|\hat{x}_n(s,x)-x'|\leq a_n^{\kappa_0}+|x-x'|\leq\varepsilon,\ |\hat{x}_n(s,x)|\leq|x|+1\leq K_0+1.\tag{7.51}$$

By (7.51), $s<U_{M,n}\leq U^{(1)}_{M,n,\beta_i}$ and the definition of $U^{(1)}_{M,n,\beta_i}$, with $\hat{x}_n(s,x)$ in the role of x we conclude that

$$\begin{aligned}
&|u'_{1,a_n^{\alpha_i}}(s+a_n^{2\kappa_0+2\varepsilon_0},x')-u'_{1,a_n^{\alpha_i}}(s+a_n^{2\kappa_0+2\varepsilon_0},\hat{x}_n(s,x))|\leq 2^{-76}a_n^{-\varepsilon_0-(2-\eta/2)\varepsilon_1}(|x-x'|+a_n^{\kappa_0})^{1-\varepsilon_0}\\
&\times\big[a_n^{-(2-\eta/2)\beta_i/\eta}(|x'-x|+a_n^{\kappa_0})^{(\eta+1)\gamma}+a_n^{\beta_i(\eta-1)/\eta}+a_n^{\beta_i(\gamma\eta-2+\eta/2)/\eta}(|x'-x|+a_n^{\kappa_0})^{\gamma}\big].
\end{aligned}\tag{7.52}$$

From (2.14) and (4.15) we have

$$\beta_{i+1}=\beta_i+\varepsilon_0\leq\beta_i-2\varepsilon_0-(2-\eta/2)\varepsilon_1+5\varepsilon_1(1-\varepsilon_0),\ i=0,...,L.\tag{7.53}$$

Use (2.15) and (4.15) to deduce $\beta_i/\eta + 5\varepsilon_1 \le \kappa_0$, for $i = 0, ..., L$. Since $n > n_M(\varepsilon_1)$ by (4.37) we get

$$|x - x'| + a_n^{\kappa_0} \le 6 a_n^{\beta_i/\eta + 5\varepsilon_1} \le a_n^{\beta_i/\eta}. \tag{7.54}$$

From (2.15) and (4.15) we have $\beta_i/\eta \le \kappa_0 < 1$, for $i = 0, \dots L$. Using this and (7.8), (7.53), (7.54) on (7.52) we get,

$$\begin{aligned}
&|u'_{1,a_n^{\alpha_i}}(t + a_n^{2\kappa_0+2\varepsilon_0}, x') - u'_{1,a_n^{\alpha_i}}(t + a_n^{2\kappa_0+2\varepsilon_0}, \hat{x}_n(s,x))| \\
&\le 2^{-76} a_n^{-\varepsilon_0-(2-\eta/2)\varepsilon_1} a_n^{(\beta_i/\eta+5\varepsilon_1)(1-\varepsilon_0)} \big[a_n^{(\eta-1)\beta_i/\eta} + 2a_n^{(\gamma(\eta+1)-2+\eta/2)\beta_i/\eta}\big] \\
&\le 2^{-74} a_n^{-\varepsilon_0-(2-\eta/2)\varepsilon_1} a_n^{5\varepsilon_1(1-\varepsilon_0)} a_n^{(1-\varepsilon_0)\beta_i/\eta} a_n^{(\eta-1)\beta_i/\eta} \\
&\le 2^{-74} a_n^{-\varepsilon_0-(2-\eta/2)\varepsilon_1+5\varepsilon_1(1-\varepsilon_0)} a_n^{\beta_i-\varepsilon_0} \\
&= 2^{-74} a_n^{\beta_{i+1}}.
\end{aligned} \tag{7.55}$$

From (7.55) and Lemma 7.8(b) and (c) we immediately get claims (a) and (b) of this lemma. ■

Lemma 7.10 *If $i \in \{0, ..., L\}$, $0 \le s \le U_{M,n}$, $x \in J_{n,i}(s)$, and $|x - x'| \le 4a_n^{\kappa_0}$, then*

$$|\tilde{u}_{2,a_n^{\alpha_i}}(s, a_n^{2\kappa_0+2\varepsilon_0}, x') - \tilde{u}_{2,a_n^{\alpha_i}}(s, a_n^{2\kappa_0+2\varepsilon_0}, x'')| \quad \le \quad 2^{-75} a_n^{\beta_{i+1}}(|x' - x''| \vee a_n), \ \forall |x' - x''| < \bar{l}_n(\beta_i). \tag{7.56}$$

Proof: The proof uses the ideas from the proofs of Lemma 6.7 in [16].

Let (n, i, s, x, x') as above and set $\varepsilon = 5a_n^{\kappa_0} \le 2^{-M}$, by (4.37). Then

$$|x' - \hat{x}_n(s,x)| \le |x' - x| + a_n^{\kappa_0} \le \varepsilon, \ |x'| \le |x| + 1 \le K_0 + 1. \tag{7.57}$$

From Lemma 7.6 we have

$$|u(s, \hat{x}_n(s,x))| \le a_n = a_n \wedge (a_n^{1-\kappa_0}\varepsilon). \tag{7.58}$$

From the definition of $(s,x) \in J_{n,i}$ we have for $i > 0$,

$$|u'_{1,a_n^{2\kappa_0}}(s, \hat{x}_n(s,x))| \le a_n^{\beta_i}/4 \le a_n^{\beta_i}. \tag{7.59}$$

Let

$$Q(n, \varepsilon_0, \beta_i, r) = a_n^{-\varepsilon_0}\big[\big(a_n^{-\kappa_0(1-\eta/2)-3\varepsilon_0} r\big) \wedge \big(r^{(\eta-2\varepsilon_0)/2}\big)\big]\big[(r \vee a_n^{\kappa_0})^{(1+\eta-\varepsilon_0)\gamma} + (r \vee a_n^{\kappa_0})^{\gamma(1-\varepsilon_0)} a_n^{\beta_i\gamma}\big].$$

Assume that $|x' - x''| \le \bar{l}_n(\beta_i) \le 2^{-M}$ where the last inequality is by (4.37). From $s < U_{M,n} \le U^{(2)}_{M,n,\beta_i}$, and the definition of $U^{(2)}_{M,n,\beta_i}$, with (x', x'') replacing (x, x') we get

$$\begin{aligned}
&|\tilde{u}_{2,a_n^{\alpha_i}}(s, a_n^{2\kappa_0+2\varepsilon_0}, x'') - \tilde{u}_{2,a_n^{\alpha_i}}(s, a_n^{2\kappa_0+2\varepsilon_0}, x')| \\
&\le 2^{-87} a_n^{-\varepsilon_0}\big[\big(|x'' - x'| a_n^{-\kappa_0(1-\eta/2)-3\varepsilon_0}\big) \wedge \big(|x'' - x'|^{(\eta-2\varepsilon_0)/2}\big)\big]\big[((5a_n^{\kappa_0}) \vee |x'' - x'|)^{(\eta+1-\varepsilon_0)\gamma} \\
&\quad + ((5a_n^{\kappa_0}) \vee |x'' - x'|)^{\gamma(1-\varepsilon_0)} a_n^{\beta_i\gamma}\big] + |x'' - x'|^{1-\varepsilon_0} a_n^{3\beta_i/2+\varepsilon_1-\varepsilon_0}\big). \\
&\le 2^{-82}\big(Q(n, \varepsilon_0, \beta_i, |x'' - x'|) + |x'' - x'|^{1-\varepsilon_0} a_n^{3\beta_i/2+\varepsilon_1-\varepsilon_0}\big).
\end{aligned} \tag{7.60}$$

We show that

$$Q(n, \varepsilon_0, \beta_i, r) \le 2a_n^{\beta_{i+1}} r, \quad \forall\ 0 \le r \le \bar{l}_n(\beta_i). \tag{7.61}$$

Case 1. $a_n^{\kappa_0} \le r \le \bar{l}_n(\beta_i)$.

$$\begin{aligned}
Q(n, \varepsilon_0, \beta_i, r) \quad &\le \quad a_n^{-\varepsilon_0} r^{(\eta-2\varepsilon_0)/2}\big[r^{(\eta+1)\gamma-\varepsilon_0} + a_n^{\beta_i\gamma} r^{\gamma-\varepsilon_0}\big] \\
&= \quad a_n^{-\varepsilon_0}\big[r^{(\eta+1)\gamma+\eta/2-2\varepsilon_0} + a_n^{\beta_i\gamma} r^{\gamma+\eta/2-2\varepsilon_0}\big].
\end{aligned} \tag{7.62}$$

Therefore, (7.61) holds if

$$r^{(\eta+1)\gamma+\eta/2-1-2\varepsilon_0} \le a_n^{\beta_{i+1}+\varepsilon_0}, \tag{7.63}$$

and

$$a_n^{\beta_i\gamma} r^{\gamma+\eta/2-1-2\varepsilon_0} \le a_n^{\beta_{i+1}+\varepsilon_0}. \tag{7.64}$$

From (2.14) we have

$$(1+\eta)\gamma + \eta/2 - 1 - 2\varepsilon_0 > \eta + \varepsilon_1, \tag{7.65}$$

and hence

$$r^{(\eta+1)\gamma+\eta/2-1-2\varepsilon_0} \le r^{\eta}.$$

Hence by the upper bound on r in this case, it suffices to show that $a_n^{\eta(\beta_i/\eta+5\varepsilon_1)} \le a_n^{\beta_{i+1}+\varepsilon_0}$, which is clear from (2.14) and (4.15). Hence, (7.63) follows. Turning to (7.64), from the upper bound on r and (4.15) we have

$$\begin{aligned} a_n^{\beta_i\gamma} r^{\gamma+\eta/2-1-2\varepsilon_0} a_n^{-\beta_{i+1}-\varepsilon_0} &\le a_n^{\beta_i\gamma+(\gamma+\eta/2-1-2\varepsilon_0)(\beta_i/\eta+5\varepsilon_1)-\beta_{i+1}-\varepsilon_0} \\ &\le a_n^{\frac{\beta_i}{\eta}(\gamma(\eta+1)-1+\eta/2)-\beta_i+5\varepsilon_1(\gamma-1+\eta/2-2\varepsilon_0)-2\varepsilon_0-2\varepsilon_0\frac{\beta_i}{\eta}} \\ &\le a_n^{5\varepsilon_1(\gamma-1+\eta/2-2\varepsilon_0)-2\varepsilon_0} \\ &\le 1, \end{aligned} \tag{7.66}$$

where we used (2.14) and (7.65) in the last two inequalities. From (7.66) we get (7.64). This proves (7.61) in the first case.

Case 2. $0 \le r < a_n^{\kappa_0}$.
Now, let us show that (7.61) is satisfied in this case. Recall that in this case we get from (2.15),

$$\begin{aligned} Q(n,\varepsilon_0,\beta_i,r) &\le a_n^{-\varepsilon_0-\kappa_0(1-\eta/2)-3\varepsilon_0} r\big[a_n^{\kappa_0(\eta+1-\varepsilon_0)\gamma} + a_n^{\gamma(1-\varepsilon_0)\kappa_0+\beta_i\gamma}\big] \\ &= a_n^{-5\varepsilon_0-\kappa_0(1-\eta/2)} r\big[a_n^{\gamma(1-\varepsilon_0)} + a_n^{\gamma\kappa_0+\beta_i\gamma}\big]. \end{aligned} \tag{7.67}$$

From (4.16) and (2.15) we have

$$a_n^{\gamma} \le a_n^{\gamma\kappa_0+\beta_i\gamma}. \tag{7.68}$$

From (7.67) and (7.68) we conclude that (7.61) holds if

$$r a_n^{\beta_{i+1}} \ge r a_n^{\gamma\kappa_0+\beta_i\gamma-5\varepsilon_0-\kappa_0(1-\eta/2)}. \tag{7.69}$$

Note that from (2.14) we have

$$1-\gamma \le \frac{\eta}{2(\eta+1)} - 10\varepsilon_1, \tag{7.70}$$

and therefore,

$$\frac{\eta}{\eta+1}(1-\gamma) \le \big(\gamma - 1 + \frac{\eta}{2}\big)\frac{1}{\eta+1} - 10\varepsilon_1. \tag{7.71}$$

From (7.71), (2.15) and (4.16) we have

$$\beta_{i+1}(1-\gamma) \quad \le \quad \big(\gamma - 1 + \frac{\eta}{2}\big)\kappa_0 - 10\varepsilon_1. \tag{7.72}$$

From (7.72) and (4.15) we get

$$\beta_{i+1} \le \gamma\beta_i + \big(\gamma - 1 + \frac{\eta}{2}\big)\kappa_0 - 5\varepsilon_0, \tag{7.73}$$

and therefore (7.69) is satisfied. From (7.67)–(7.69), we get (7.61) for Case 2.

Consider the second term in the last line of (7.60). From (2.14) and (4.15) we get

$$\begin{aligned} \frac{3}{2}\beta_L - \beta_{L+1} + \varepsilon_1 - \varepsilon_0 &\ge \frac{3}{2}\big(\frac{\eta}{\eta+1} - 6\eta\varepsilon_1\big) - \big(\frac{\eta}{\eta+1} - \eta\varepsilon_1\big) \\ &\ge \frac{\eta}{2(\eta+1)} - 8\eta\varepsilon_1 \\ &\ge 90\varepsilon_1. \end{aligned} \tag{7.74}$$

Use (2.14) and (4.15) again to get

$$\begin{aligned}\frac{3}{2}\beta_i - \beta_{i+1} + \varepsilon_1 - \varepsilon_0 &= \frac{\beta_i}{2} + \varepsilon_1 - 2\varepsilon_0 \\ &\geq 90\varepsilon_0 \quad \forall i = 0, ..., L-1. \end{aligned} \tag{7.75}$$

If $r \geq a_n$, then from (7.74) and (7.75) we get,

$$\begin{aligned} r^{1-\varepsilon_0} a_n^{3\beta_i/2+\varepsilon_1-\varepsilon_0}(a_n^{\beta_{i+1}} r)^{-1} &\leq r^{-\varepsilon_0} a_n^{90\varepsilon_0} \\ &\leq a_n^{89\varepsilon_0} \\ &\leq 1, \ \forall i = 0, ..., L. \end{aligned} \tag{7.76}$$

From (7.76) it follows that

$$r^{1-\varepsilon_0} a_n^{3\beta_i/2+\varepsilon_1-\varepsilon_0} \leq a_n^{\beta_{i+1}}(r \vee a_n), \ \forall i = 0, ..., L. \tag{7.77}$$

From (7.60), (7.61) and (7.77) we get (7.56). ∎

Proof of Proposition 4.8 The proof of compactness is similar to the corresponding argument in the proof of Proposition 3.3 in [16]. The inclusions $J_{n,i}(s) \subset \tilde{J}_{n,i}(s)$ for $0 \leq s \leq U_{M,n}$ follow directly from Lemmas 7.6, 7.7 and 7.9, 7.10.

8 Proof of Proposition 7.1

In this section we prove Proposition 7.1. The proof follows the same lines as the proof of Proposition 5.14 in [16]. From (4.9) we get for $\delta \in [a_n^{2\kappa_0+2\varepsilon_0}, 1]$,

$$\begin{aligned} &|\tilde{u}_{2,\delta}(t, a_n^{2\kappa_0+2\varepsilon_0}, x') - \tilde{u}_{2,\delta}(t, a_n^{2\kappa_0+2\varepsilon_0}, x)| \\ &= \left| \int_{(t-a_n^{2\kappa_0+2\varepsilon_0}-\delta)_+}^{t} \int_{\mathbb{R}} (G_{t+a_n^{2\kappa_0+2\varepsilon_0}-s}(y-x') - G_{t+a_n^{2\kappa_0+2\varepsilon_0}-s}(y-x)) D(s,y) W(ds,dy) \right|. \end{aligned} \tag{8.1}$$

From (4.3) and (8.1) we conclude that we need to bound the following quadratic variations

$$\begin{aligned} \hat{Q}_{S,1,\delta,\nu_0}(t,x,x') &= \int_{(t+a_n^{2\kappa_0+2\varepsilon_0}-\delta)_+}^{t} \int_{\mathbb{R}} \mathbb{1}_{\{|x-y|>(t+a_n^{2\kappa_0+2\varepsilon_0}-s)^{1/2-\nu_0}\vee(2|x-x'|)\}} \\ &\quad \times (G_{t+a_n^{2\kappa_0+2\varepsilon_0}-s}(y-x') - G_{t+a_n^{2\kappa_0+2\varepsilon_0}-s}(y-x))^2 e^{2R_1|y|}|u(s,y)|^{2\gamma}\mu(dy)ds \\ \hat{Q}_{S,2,\delta,\nu_0}(t,x,x') &= \int_{(t+a_n^{2\kappa_0+2\varepsilon_0}-\delta)_+}^{t} \int_{\mathbb{R}} \mathbb{1}_{\{|x-y|\leq(t+a_n^{2\kappa_0+2\varepsilon_0}-s)^{1/2-\nu_0}\vee(2|x-x'|)\}} \\ &\quad \times (G_{t+a_n^{2\kappa_0+2\varepsilon_0}-s}(y-x') - G_{t+a_n^{2\kappa_0+2\varepsilon_0}-s}(y-x))^2 e^{2R_1|y|}|u(s,y)|^{2\gamma}\mu(dy)ds. \end{aligned} \tag{8.2}$$

Recall that K_1 and ε_0 were fixed in (2.12) and (2.14) respectively. Property (P_m) was introduced in (6.7).

Lemma 8.1 *For any $K \in \mathbb{N}^{\geq K_1}$ and $R > 2$ there is a $C_{8.1}(K, R_1, \eta, \nu_0, \nu_1) > 0$ and an $N_{8.1} = N_{8.1}(K, \omega, \eta) \in \mathbb{N}$ a.s. such that for all $\nu_0, \nu_1 \in (1/R, 1/2), \delta \in (0,1], N, n \in \mathbb{N}, \beta \in [0, \frac{\eta}{\eta+1}]$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, on*

$$\{\omega : (t,x) \in Z(N,n,K,\beta), N \geq N_{8.1}\}, \tag{8.3}$$

$$\hat{Q}_{S,1,\delta,\nu_0}(t,x,x') \leq C_{8.1}(K, R_1, \eta, \nu_0, \nu_1) 2^{4N_{8.1}} [d((t,x),(t,x')) \wedge \sqrt{\delta}]^{2-\nu_1} \delta^{1+\eta/2-\varepsilon_0}, \ \forall x' \in \mathbb{R}. \tag{8.4}$$

Proof: The proof follows the same lines as the proof of Lemma 7.1 in [16]. Let $d = d((t,x),(t',x'))$ and $N_{8.1} = N_1(0, 1-\eta/4, K)$, where N_1 is as in (P_0). Then, as in (P_0), N_1 depends only on $(1-\eta/4, K, \eta)$, and then for ω as in (8.3) we can use Lemma 6.6 with $m=0$ to get

$$\begin{aligned}
&\hat{Q}_{S,1,\delta,\nu_0}(t,x,x') \\
&\le C_{6.6}(\omega)\int_{(t-a_n^{2\kappa_0+2\varepsilon_0}-\delta)_+}^{t}\int_{\mathbb{R}} \mathbb{1}_{\{|x-y|>(t+a_n^{2\kappa_0+2\varepsilon_0}-s)^{1/2-\nu_0}\vee(2|x-x'|)\}} \\
&\quad\times(G_{t+a_n^{2\kappa_0+2\varepsilon_0}-s}(y-x')-G_{t+a_n^{2\kappa_0+2\varepsilon_0}-s}(y-x))^2 e^{2R_1|y|}e^{2|y-x|}(\bar{d}_N^{1-\eta/4})^{2\gamma}\mu(dy)ds \\
&\le C_{6.6}(\omega)e^{2R_1K}\int_{(t-a_n^{2\kappa_0+2\varepsilon_0}-\delta)_+}^{t}\int_{\mathbb{R}} \mathbb{1}_{\{|x-y|>(t+a_n^{2\kappa_0+2\varepsilon_0}-s)^{1/2-\nu_0}\vee(2|x-x'|)\}} \\
&\quad\times(G_{t+a_n^{2\kappa_0+2\varepsilon_0}-s}(y-x')-G_{t+a_n^{2\kappa_0+2\varepsilon_0}-s}(y-x))^2 e^{2(R_1+1)|y-x|}(1+|x-y|)^{(2-\eta/2)\gamma}\mu(dy)ds \\
&\le C_{6.6}(\omega)C(R_1,K,\eta,\nu_0,\nu_1)\int_{t-a_n^{2\kappa_0+2\varepsilon_0}-\delta}^{t}|t+a_n^{2\kappa_0+2\varepsilon_0}-s|^{\eta/2-\varepsilon_0-1}e^{-\nu_1(t+a_n^{2\kappa_0+2\varepsilon_0}-s)^{-2\nu_0}/32} \\
&\quad\times\left[1\wedge\frac{d^2}{t+a_n^{2\kappa_0+2\varepsilon_0}-s}\right]^{1-\nu_1/2}ds, \qquad (8.5)
\end{aligned}$$

where we used Lemma 5.6(b) in the last inequality. From (8.5), change of variable and Lemma 5.1(a) we get

$$\begin{aligned}
&\hat{Q}_{S,1,\delta,\nu_0}(t,x,x') \qquad (8.6)\\
&\le C_{6.6}(\omega)C(R_1,K,\eta,\nu_0,\nu_1)\int_{t-\delta}^{t}(t-s)^{1+\eta/2-\varepsilon_0}\left[1\wedge\frac{d^2}{t-s}\right]^{1-\nu_1/2}ds \\
&\le C_{6.6}(\omega)C(R_1,K,\eta,\nu_0,\nu_1)(\delta\wedge d^2)^{1-\nu_1/2}\delta^{1+\eta/2-\varepsilon_0}.
\end{aligned}$$

By Remark 6.7 we can choose $C_{6.6}$ with $\varepsilon_0=0$ and this completes the proof. ■

Lemma 8.2 *Let $0\le m\le\bar{m}+1$ and assume (P_m). Then, for any $K\in\mathbb{N}^{\ge K_1}$, $n\in\mathbb{N}$, and $\beta\in[0,\frac{\eta}{\eta+1}]$, there exist $C_{8.2}(\eta,\varepsilon_0,K,R_1)>0$ and $N_{8.2}=N_{8.2}(m,n,\varepsilon_0,K,\beta,\eta)(\omega)\in\mathbb{N}$ a.s. such that for any $\nu_0,\nu_1\in(0,1)$, $\delta\in[a_n^{2\kappa_0},1]$, $N\in\mathbb{N}$, and $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, on*

$$\{\omega:(t,x)\in Z(N,n,K,\beta),N\ge N_{8.2}\}, \qquad (8.7)$$

$$\begin{aligned}
&\hat{Q}_{S,2,\delta,\nu_0}(t,x,x') \\
&\le C_{8.2}(\eta,\varepsilon_0,K,R_1)[a_n^{-2\varepsilon_0}+2^{4N_{8.2}}]\big[(d^2a_n^{-2\kappa_0(1-\eta/2)-3\varepsilon_0})\wedge(d^{\eta-2\varepsilon_0})\big]\bar{d}_N^{2\gamma(1-\nu_1)}\big[(\bar{d}_{n,N})^{2\gamma(\tilde{\gamma}_m-1)}+a_n^{2\gamma\beta}\big] \\
&\quad\forall|x'|\le K+1. \qquad (8.8)
\end{aligned}$$

Here $d=d((t,x),(t,x'))$, $\bar{d}_N=d\vee2^{-N}$ and $\bar{d}_{n,N}=a_n^{\kappa_0}\vee\bar{d}_N$. Moreover $N_{8.2}$ is stochastically bounded uniformly in (n,β).

Proof: The proof follows the same lines as the proof of Lemma 7.2 in [16]. Set $\xi=1-\nu_1$ and $N_{8.2}(m,n,\nu_1,\varepsilon_0,K,\beta,\eta)=N_1(m,n,1-\nu_1,\varepsilon_0,K,\beta,\eta)$, which is stochastically bounded informally in (n,β) by (P_m). For ω as in (8.7), $t\le K$, $|x'|\le K+1$ we get from Lemma 6.6 and then Lemma 5.6(a),

$$\begin{aligned}
&\hat{Q}_{S,2,\delta,\nu_0}(t,x,x') \\
&\le C_{6.6}(\omega)\int_{(t+a_n^{2\kappa_0+2\varepsilon_0}-\delta)_+}^{t}\int_{\mathbb{R}}(G_{t+a_n^{2\kappa_0+2\varepsilon_0}-s}(y-x')-G_{t+a_n^{2\kappa_0+2\varepsilon_0}-s}(y-x))^2e^{2R_1K}e^{2(R_1+1)2(K+1)} \\
&\quad\times\bar{d}_N^{2\gamma\xi}\big[(a_n^{\kappa_0}\vee\bar{d}_N)^{\bar{\gamma}_m-1}+\mathbb{1}_{\{m>0\}}a_n^{\beta}\big]^{2\gamma}\mu(dy)ds \\
&\le C_{6.6}(\omega)C(\eta,\varepsilon_0,R_1,K)\bar{d}_N^{2\gamma\xi}\big[(a_n^{\kappa_0}\vee\bar{d}_N)^{\bar{\gamma}_m-1}+\mathbb{1}_{\{m>0\}}a_n^{\beta}\big]^{2\gamma} \\
&\quad\times\int_{(t+a_n^{2\kappa_0+2\varepsilon_0}-\delta)_+}^{t}(t+a_n^{2\kappa_0+2\varepsilon_0}-s)^{\eta/2-1-\varepsilon_0}\left(1\wedge\frac{d^2}{t+a_n^{2\kappa_0+2\varepsilon_0}-s}\right)ds. \qquad (8.9)
\end{aligned}$$

The factor $e^{2R_1K}e^{2(R_1+1)2(K+1)}$ in the second line of (8.9) follows from the exponential $e^{|x-y|}$ in Lemma 6.6, $e^{R_1|y|}$ in $\hat{Q}_{S,2,\delta,\nu_0}(t,x,x')$, the bound $|y|\le|x|+|x-y|$ and the bounds on $t,x,x',|x-y|$.

From Lemma 5.1(b) with $t+a_n^{2\kappa_0+2\varepsilon_0}$ instead of t, $\Delta=d^2$, $\Delta_1=\delta$ and $\Delta_2=a_n^{2\kappa_0+2\varepsilon_0}$ we have

$$\begin{aligned}&\int_{(t+a_n^{2\kappa_0+2\varepsilon_0}-\delta)_+}^t (t+a_n^{2\kappa_0+2\varepsilon_0}-s)^{\eta/2-1-\varepsilon_0}\left(1\wedge\frac{d^2}{t+a_n^{2\kappa_0+2\varepsilon_0}-s}\right)ds\\&\le C[(d^2\wedge\delta)^{\eta/2-\varepsilon_0}+(d^2\wedge\delta)a_n^{(2\kappa_0+2\varepsilon_0)(\eta/2-1-\varepsilon_0)}].\end{aligned}\tag{8.10}$$

On the other hand, by a simple integration we have

$$\int_{(t+a_n^{2\kappa_0+2\varepsilon_0}-\delta)_+}^t (t+a_n^{2\kappa_0+2\varepsilon_0}-s)^{\eta/2-1-\varepsilon_0}\left(1\wedge\frac{d^2}{t+a_n^{2\kappa_0+2\varepsilon_0}-s}\right)ds\le Cd^2a_n^{(2\kappa_0+2\varepsilon_0)(\eta/2-1-\varepsilon_0)},\tag{8.11}$$

where we have used the bound $\delta\in[a_n^{2\kappa_0},1]$ in the last inequality. From (8.9), (8.10) and (8.11), we get (8.8). ∎

Proof of Proposition 7.1 The proof follows the same lines as the proof of Proposition 5.14 in [16]. Let $\nu_1\in(0,\varepsilon_0)$, $R=\frac{25}{\nu_1}$ and choose $\nu_0\in\left(\frac{1}{R},\frac{\nu_1}{24}\right)$. Recall the previously introduced notation $\bar d_N=d\vee 2^{-N}$. Let

$$\hat Q_{a_n^\alpha}(t,x,x')=\sum_{i=1}^2\hat Q_{S,i,a_n^\alpha,\nu_0}(t,x,x').\tag{8.12}$$

By Lemmas 8.1 and 8.2 for every $K\in\mathbb{N}$ there is a constant $C_1(K,R_1,\eta,\nu_0,\nu_1,\varepsilon_0)>0$ and $N_2(m,n,\varepsilon_0,K,\beta,\eta)\in\mathbb{N}$ a.s. stochastically bounded uniformly in (n,β), such that for all $N\in\mathbb{N}$, $(t,x)\in\mathbb{R}_+\times\mathbb{R}$,

$$\text{on }\{\omega:(t,x)\in Z(N,n,m,K+1,\beta),N\ge N_2\}$$

$$\begin{aligned}&R_0^\gamma\hat Q_{a_n^\alpha}(t,x,x')^{1/2}\\&\le C_12^{2N_2}[d\wedge a_n^{\alpha/2}]^{1-\nu_1/2}a_n^{\alpha(1+\eta/2-\varepsilon_0)/2}\\&\quad+C_1[a_n^{-\varepsilon_0}+2^{2N_2}]\big[(a_n^{-\kappa_0(1-\eta/2)-2\varepsilon_0}d)\wedge d^{\eta/2-\varepsilon_0}\big]\bar d_N^{\gamma(1-\nu_1)}\big[(\bar d_{n,N})^{\gamma(\tilde\gamma_m-1)}+a_n^{\gamma\beta}\big]\\&\quad\forall\ t\le T_K,\ |x'|\le K+2.\end{aligned}\tag{8.13}$$

Let $N_3=\frac{25}{\nu_1}[N_2+N_4(K,R_1,\eta,\nu_0,\nu_1,\varepsilon_0)]$, where $N_4(K,R_1,\eta,\nu_0,\nu_1,\varepsilon_0)$ is chosen large enough so that

$$\begin{aligned}C_1(K,R_1,\eta,\nu_0,\nu_1,\varepsilon_0)[a_n^{-\varepsilon_0}+2^{2N_2}]2^{-N_3\nu_1/8}&\le C_1(K,R_1,\eta,\nu_0,\nu_1,\varepsilon_0)[a_n^{-\varepsilon_0}+2^{2N_2}]2^{-3N_2}2^{-3N_4(K,R_1,\eta,\nu_0,\nu_1,\varepsilon_0)}\\&\le a_n^{-\varepsilon_0}2^{-104}.\end{aligned}\tag{8.14}$$

Recall the notation introduced in (7.2),

$$\begin{aligned}\bar\Delta_{1,u_2}(m,n,\varepsilon_0,2^{-N},\eta)&=2^{-N\gamma(1-\varepsilon_0)}\big[(2^{-N}\vee a_n^{\kappa_0})^{\gamma(\tilde\gamma_m-1)}+a_n^{\gamma\beta}\big],\\\bar\Delta_{2,u_2}(m,n,\varepsilon_0,\eta)&=a_n^{\alpha(1+\eta/2-\varepsilon_0)/2}.\end{aligned}$$

Let $\Delta_{i,u_2}=2^{-100}\bar\Delta_{i,u_2},\ i=1,2$. Assume that $d\le 2^{-N}$. From (8.13), (8.14) and $\nu_1\le\varepsilon_0$ we get for all (t,x) and N on

$$\{\omega:(t,x)\in Z(N,n,m,K+1,\beta),N\ge N_3\},\tag{8.15}$$

$$\begin{aligned}R_0^\gamma\hat Q_{a_n^\alpha}(t,x,x')^{1/2}&\le 2^{-104}a_n^{-\varepsilon_0}\big[(a_n^{-\kappa_0(1-\eta/2)-2\varepsilon_0}d)\wedge d^{\eta/2-\varepsilon_0}\big]2^{-N\gamma(1-\nu_1)}\big[(2^{-N}\vee a_n^{\kappa_0})^{\gamma(\tilde\gamma_m-1)}+a_n^{\gamma\beta}\big]\\&\quad+2^{-104}a_n^{-\varepsilon_0}(d\wedge a_n^{\alpha/2})^{1-5\nu_1/8}a_n^{\alpha(1+\eta/2-\varepsilon_0)/2}\\&=a_n^{-\varepsilon_0}\big[(a_n^{-\kappa_0(1-\eta/2)-2\varepsilon_0}d)\wedge d^{\eta/2-\varepsilon_0}\big]\bar\Delta_{1,u_2}(m,n,\varepsilon_0,2^{-N},\eta)/16\\&\quad+(d\wedge a_n^{\alpha/2})^{1-5\nu_1/8}\bar\Delta_{2,u_2}(m,n,\varepsilon_0,\eta)/16,\ \forall\ t\le t'\le T_K,\ |x'|\le K+2.\end{aligned}\tag{8.16}$$

The rest of the proof is identical to the proof of Proposition 5.14 in [16]. We use the Dubins-Schwartz theorem and (8.16) to bound $|\tilde u_{2,\delta}(t,a_n^{2\kappa_0+2\varepsilon_0},x')-\tilde u_{2,\delta}(t,a_n^{2\kappa_0+2\varepsilon_0},x)|$ and we get (7.1). ∎

9 Proof of Theorems 6.5 and Theorem 1.10

In this section we prove Theorem 6.5. Later in this section we prove Theorem 1.10 as a consequence of Theorem 6.5. Before we start with the proofs of these theorems, we prove a weaker auxiliary result.

Recall that η was fixed in the hypothesis of Theorem 1.5. Let $\eta' \equiv \eta - \varpi$ where $\varpi > 0$ is arbitrarily small. We also recall that the sets $Z(K, N, \xi)$ we defined in (6.1).

Theorem 9.1 *Assume the hypothesis of Theorem 1.5, except allow $\gamma \in (0,1]$. Let $u_0 \in \mathcal{C}_{tem}$ and $u = u^1 - u^2$, where u^i is a $\mathcal{C}(\mathbb{R}_+, \mathcal{C}(\mathbb{R}))$ a.s. solution of (1.1), for $i = 1, 2$. Let $\xi \in (0,1)$ satisfy*

$$\begin{aligned} &\exists N_\xi = N_\xi(K,\omega) \in \mathbb{N} \text{ a.s. such that } \forall N \geq N_\xi, (t,x) \in Z(K,N,\xi) \\ &d((t,x),(t',y)) \leq 2^{-N},\ t,t' \leq T_K \Rightarrow |u(t,x) - u(t',y)| \leq 2^{-N\xi}. \end{aligned} \tag{9.1}$$

Let $0 < \xi_1 < (\gamma\xi + \eta'/2) \wedge 1$. Then there is an $N_{\xi_1} = N_{\xi_1}(K,\omega) \in \mathbb{N}$ a.s. such that for any $N \geq N_{\xi_1} \in \mathbb{N}$ and any $(t,x) \in Z(K,N,\xi)$

$$d((t,x),(t',y)) \leq 2^{-N},\ t,t' \leq T_K \Rightarrow |u(t,x) - u(t',y)| \leq 2^{-N\xi_1}. \tag{9.2}$$

Moreover, there are strictly positive constants $R, \delta, C_{9.1.1}, C_{9.1.2}$ depending only on (ξ, ξ_1) and $N(K) \in \mathbb{N}$, such that

$$P(N_{\xi_1} \geq N) \leq C_{9.1.1}(P(N_\xi \geq N/R) + K^2 \exp(-C_{9.1.2} 2^{N\delta})),\ \forall N \geq N(K). \tag{9.3}$$

An analog of Theorem 6.5, for the case of the d-dimensional stochastic heat equation driven by colored noise, is Theorem 4.1 in [17].

Proof of Theorem 9.1 The proof follows the same lines as the proof of Theorem 4.1 in [17]. Fix arbitrary $(t,x), (t',y)$ such that $d((t,x),(t',y)) \leq \varepsilon \equiv 2^{-N}$, $(N \in \mathbb{N})$ and $t \leq t'$ (the case $t' \leq t$ works analogously). Since $\xi_1 < (\gamma\xi + \eta'/2) \wedge 1$ and $\xi\gamma \in (0,1)$, we can choose $\delta \in (0, \eta'/2)$ such that

$$1 > \gamma\xi + \eta'/2 - \delta > \xi_1. \tag{9.4}$$

We can also pick $\delta' \in (0, \delta)$ and $p \in (0, \xi\gamma)$ such that

$$1 > p + \eta'/2 - \delta > \xi_1, \tag{9.5}$$

and

$$1 > \xi\gamma + \eta'/2 - \delta' > \xi_1. \tag{9.6}$$

We consider a random $N_1 = N_1(\omega, \xi, \xi_1)$ which will be specified later. Our goal is to bound the sum of following probabilities:

$$\begin{aligned} &P\big(|u(t,x) - u(t,y)| \geq |x-y|^{\eta'/2-\delta}\varepsilon^p, (t,x) \in Z(K,N,\xi), N \geq N_1\big) \\ &+P\big(|u(t,x) - u(t',x)| \geq |t'-t|^{\eta'/4-\delta/2}\varepsilon^p, (t,x) \in Z(K,N,\xi), t' \leq T_K, N \geq N_1\big). \end{aligned} \tag{9.7}$$

Let

$$\begin{aligned} D^{x,y,t,t'}(z,s) &= [G_{t-s}(x-z) - G_{t'-s}(y-z)]^2 u^{2\gamma}(s,z), \\ D^{x,t'}(z,s) &= G^2_{t'-s}(x-z) u^{2\gamma}(s,z). \end{aligned} \tag{9.8}$$

Note that (9.7) is bounded by

$$
\begin{aligned}
&P\Big(|u(t,x)-u(t,y)| \ge |x-y|^{\eta'/2-\delta}\varepsilon^p, (t,x)\in Z(K,N,\xi), N\ge N_1,\\
&\quad \int_0^t\int_{\mathbb{R}} D^{x,y,t,t}(z,s)\mu(dz)ds \le |x-y|^{\eta'-2\delta'}\varepsilon^{2p}\Big)\\
&+P\Big(|u(t,x)-u(t',x)| \ge |t'-t|^{\eta'/4-\delta/2}\varepsilon^p, (t,x)\in Z(K,N,\xi), t'\le T_K, N\ge N_1,\\
&\quad \int_t^{t'}\int_{\mathbb{R}} D^{x,t'}(z,s)\mu(dz)ds + \int_0^t\int_{\mathbb{R}} D^{x,x,t,t'}(z,s)\mu(dz)ds \le (t'-t)^{\eta'/2-\delta'}\varepsilon^{2p}\Big)\\
&+P\Big(\int_0^t\int_{\mathbb{R}} D^{x,y,t,t}(z,s)\mu(dz)ds > |x-y|^{\eta'-2\delta'}\varepsilon^{2p}, (t,x)\in Z(K,N,\xi), N\ge N_1\Big)\\
&+P\Big(\int_t^{t'}\int_{\mathbb{R}} D^{x,t'}(z,s)\mu(dz)ds + \int_0^t\int_{\mathbb{R}} D^{x,x,t,t'}(z,s)\mu(dz)ds\\
&\qquad\qquad > (t'-t)^{\eta'/2-\delta'}\varepsilon^{2p}, (t,x)\in Z(K,N,\xi), t'\le T_K, N\ge N_1\Big)\\
&=: P_1+P_2+P_3+P_4. \qquad (9.9)
\end{aligned}
$$

The bounds on P_1, P_2 are derived by the Dubins-Schwarz theorem in a similar way as in the proof of Theorem 4.1 in [17]. If $\delta''=\delta-\delta'$, then we can show that

$$
\begin{aligned}
P_1 &\le C_{(9.10)}e^{-C'_{(9.10)}|x-y|^{-\delta''}},\\
P_2 &\le C_{(9.10)}e^{-C'_{(9.10)}|t-t'|^{-\delta''/2}}, \qquad (9.10)
\end{aligned}
$$

where the constants $C_{(9.10)}, C'_{(9.10)}$ only depend on R_0, R_1 in (2.2).

In order to bound P_3, P_4, we need to split the integrals that are related to them into several parts. Let $\delta_1\in(0,\eta'/4)$ and $t_0=0, t_1=t-\varepsilon^2, t_2=t$ and $t_3=t'$. We also define

$$
\begin{aligned}
A_1^{1,s}(x) &= \{z\in\mathbb{R}: |x-z|\le 2\sqrt{t-s}\varepsilon^{-\delta_1}\} \text{ and } A_2^{1,s}(x)=\mathbb{R}\setminus A_1^{1,s}(x),\\
A_1^{2}(x) &= \{z\in\mathbb{R}: |x-z|\le 2\varepsilon^{1-\delta_1}\} \text{ and } A_2^{2}(x)=\mathbb{R}\setminus A_1^{2}(x). \qquad (9.11)
\end{aligned}
$$

For notational convenience the index s in $A_i^{1,s}(x)$ is sometimes omitted. Define,

$$
Q^{x,y,t,t'} := \int_0^t\int_{\mathbb{R}} D^{x,y,t,t'}(z,s)\mu(dz)ds = \sum_{i,j=1,2} Q_{i,j}^{x,y,t,t'},
$$

where

$$
Q_{i,j}^{x,y,t,t'} := \int_{t_{i-1}}^{t_i}\int_{A_j^{i,s}(x)} D^{x,y,t,t'}(z,s)\mu(dz)ds, \qquad (9.12)
$$

and

$$
Q^{x,t,t'} := \int_t^{t'}\int_{\mathbb{R}} D^{x,t'}(z,s)\mu(dz)ds = \sum_{j=1,2} Q_j^{x,t,t'},
$$

where

$$
Q_j^{x,t,t'} := \int_t^{t'}\int_{A_j^2(x)} D^{x,t'}(z,s)\mu(dz)ds. \qquad (9.13)
$$

Set

$$
N_1(\omega) = \left\lceil \frac{5N_\xi(\omega)}{\delta_1}\right\rceil \ge \left\lceil \frac{N_\xi(\omega)+4}{1-\delta_1}\right\rceil \in \mathbb{N}, \qquad (9.14)
$$

where $N_\xi(\omega)$ was chosen in the hypothesis of Theorem 9.1 and $\lceil\cdot\rceil$ is the greatest integer value function. We introduce three lemmas that will give us bounds on the quadratic variation terms in (9.12) and (9.13).

Lemma 9.2 *Let $N \geq N_1$. Then on $\{\omega : (t,x) \in Z(K,N,\xi)\}$,*

$$\begin{aligned} |u(s,z)| &\leq 10\varepsilon^{(1-\delta_1)\xi}, && \forall s \in [t-\varepsilon^2, t'], z \in A_1^2(x), \\ |u(s,z)| &\leq (8+3K2^{N_\xi \xi})e^{|z|}(t-s)^{\xi/2}\varepsilon^{-\delta_1\xi}, && \forall s \in [0, t-\varepsilon^2], z \in A_1^{1,s}(x). \end{aligned} \tag{9.15}$$

The proof of Lemma 9.2 is similar to the proof of Lemma 5.4 in [17]; hence it is omitted.

Lemma 9.3 *If $0 < \vartheta < \eta'/2$, $\vartheta' \leq \gamma\xi + \eta'/2$, and $\vartheta' < 1$, then on $\{\omega : (t,x) \in Z(K,N,\xi)\}$,*

(a)

$$Q_{2,1}^{x,y,t,t} \leq C(\vartheta, K, \eta')\varepsilon^{2(1-\delta_1)\xi\gamma}|x-y|^{2\vartheta},$$

(b)

$$Q_{2,1}^{x,x,t,t'} \leq C(\vartheta, K, \eta')\varepsilon^{2(1-\delta_1)\xi\gamma}|t-t'|^{\vartheta},$$

(c)

$$Q_{1,1}^{x,y,t,t} \leq C(\gamma, K, \eta')(8+3K2^{N_\xi \xi})^{2\gamma}\varepsilon^{-2\delta_1\xi\gamma}|x-y|^{2\vartheta'},$$

(d)

$$Q_{1,1}^{x,x,t,t'} \leq C(\gamma, K, \eta')(8+3K2^{N_\xi \xi})^{2\gamma}\varepsilon^{-2\delta_1\xi\gamma}|t-t'|^{\vartheta'},$$

(e)

$$Q_1^{x,t,t'} \leq C(K,\vartheta)\varepsilon^{2\gamma\xi(1-\delta_1)}|t-t'|^{\eta'/2}.$$

Proof: (a) From Lemma 9.2(a) and Lemma 5.5(c) we get

$$\begin{aligned} Q_{2,1}^{x,y,t,t} &\leq 100^{\gamma}\varepsilon^{2(1-\delta_1)\xi\gamma}\int_{t-\varepsilon^2}^{t}\int_{A_1^2(x)}[G_{t-s}(x-z) - G_{t-s}(y-z)]^2\mu(dz)ds \\ &\leq C(\eta', K, \vartheta)\varepsilon^{2(1-\delta_1)\xi\gamma}|x-y|^{2\vartheta}, \end{aligned} \tag{9.16}$$

where we use the fact that $t \leq K$.
(b) Repeat the same steps in (9.16) to get

$$Q_{2,1}^{x,x,t,t'} \leq C(\vartheta, K, \eta')\varepsilon^{2(1-\delta_1)\xi\gamma}|t-t'|^{\vartheta}. \tag{9.17}$$

(c) From Lemma 9.2(b) we get

$$\begin{aligned} &Q_{1,1}^{x,y,t,t'} \\ &\leq (8+3K2^{N_\xi \xi})^{2\gamma}\varepsilon^{-2\gamma\delta_1\xi}\int_0^{t-\varepsilon^2}(t-s)^{\xi\gamma}\int_{\mathbb{R}}e^{2\gamma|z|}[G_{t-s}(x-z) - G_{t'-s}(y-z)]^2\mu(dz)ds. \end{aligned} \tag{9.18}$$

From Lemma 5.8 we get for all $x \in [-K+1, K+1]$, $0 \leq s < t \leq t$, $(x-y)^2 + (t'-t) \leq \varepsilon^2$,

$$\int_{\mathbb{R}}e^{2\gamma|z|}[G_{t-s}(x-z) - G_{t'-s}(y-z)]^2\mu(dz) \leq C(\gamma, K, \eta')\varepsilon^{2(1-\vartheta')}(t-s)^{-2+\eta'/2}[|x-y|^{2\vartheta'} + (t'-t)^{\vartheta'}], \tag{9.19}$$

From (9.18), (9.19) and our choice of ϑ', (c) and (d) follow.
(e) From Lemmas 9.2(a) and 5.5(a) we have

$$\begin{aligned} Q_1^{x,t,t'} &\leq Ce^{2\gamma\xi(1-\delta_1)}\int_t^{t'}\int_{\mathbb{R}}e^{2\gamma|z|}G_{t'-s}^2(x-z)\mu(dz)ds \\ &\leq C(K,\vartheta)e^{2\gamma\xi(1-\delta_1)}|t'-t|^{\eta'/2}. \end{aligned}$$

■

Next we consider the terms for which $j = 2$. We will use the fact that for $t \leq T_K$ we have

$$|u(t,x)| \leq Ke^{|x|}. \tag{9.20}$$

Lemma 9.4 *For $0<\vartheta<\eta'/2$ we have for $i=1,2$, on $\{\omega:(t,x)\in Z(K,N,\xi)\}$,*

(a)

$$Q^{x,y,t,t}_{i,2}\leq C(K,\vartheta,\eta',\gamma)e^{-\varepsilon^{-2\delta_1(1-\vartheta)/4}}|x-y|^{2\vartheta},$$

(b)

$$Q^{x,x,t,t'}_{i,2}\leq C(K,\vartheta,\eta',\gamma)e^{-\varepsilon^{-2\delta_1(1-\vartheta/2)/4}}|t-t'|^{\vartheta},$$

(c)

$$Q^{x,t,t'}_{2}\leq C(K,\vartheta,\eta',\gamma)e^{-\varepsilon^{-2\delta_1(1-\vartheta)/2}}|t-t'|^{\eta'/2}.$$

The proof of Lemma 9.4 follows the same lines as the proof of Lemma 5.6 in [17]; hence it is omitted. The main difference in our proof is that we use Lemmas 5.5(a) and 5.6(a) instead of equations (46) and (49) which were used in [17].

Let $\vartheta'=\eta'/2-\delta'+\gamma\xi$. Note that $\vartheta'<1$ (by (9.6)). Recall that N_1 was defined in (9.13). Now use Lemma 9.3(a),(c) and Lemma 9.4(a) with $\vartheta=\eta'/2-\delta'$ to get for $(t,x)\in Z(K,N,\xi)$, $|x-y|<\varepsilon=2^{-N}$ and $N>N_1$,

$$\begin{aligned}Q^{x,y,t,t} &\leq Q^{x,y,t,t}_{1,1}+Q^{x,y,t,t}_{2,1}+Q^{x,y,t,t}_{1,2}+Q^{x,y,t,t}_{2,2}\\ &\leq C(\delta',K,\eta',\gamma)(\varepsilon^{2(1-\delta_1)\xi\gamma}|x-y|^{2\vartheta}+(8+3K2^{N_\xi\xi})^{2\gamma}\varepsilon^{-2\delta_1\xi\gamma}|x-y|^{2\vartheta'}+e^{-\varepsilon^{-2\delta_1(1-\vartheta)/4}}|x-y|^{2\vartheta})\\ &\leq C(\delta',K,\eta',\gamma)|x-y|^{2(\eta'/2-\delta')}\big[2^{2N_\xi\xi\gamma}\varepsilon^{2(1-\delta_1)\xi\gamma}+e^{-\varepsilon^{-2\delta_1(2-\eta')/8}}\big].\end{aligned}\qquad(9.21)$$

Use Lemmas 9.3(b),(d),(e), 9.4(b),(c) with the same ϑ,ϑ' as in (9.21) to get for $(t,x)\in Z_{K,N,\xi}$, $|t-t'|<\varepsilon^2$ and $N\geq N_1$,

$$Q^{x,x,t,t'}+Q^{x,t,t'}\leq Q^{x,x,t,t'}_{2,1}+Q^{x,x,t,t'}_{1,1}+Q^{x,t,t'}_{1}+Q^{x,x,t,t'}_{1,2}+Q^{x,x,t,t'}_{2,2}+Q^{x,t,t'}_{2}\qquad(9.22)$$

$$\begin{aligned}&\leq C(\vartheta,K,\eta',\gamma)(\varepsilon^{2(1-\delta_1)\xi\gamma}|t-t'|^{\vartheta}+(8+3K2^{N_\xi\xi})^{2\gamma}\varepsilon^{-2\delta_1\xi\gamma}|t-t'|^{\vartheta'}\\ &\quad+\varepsilon^{2\gamma\xi(1-\delta_1)}|t-t'|^{\eta'/2}+e^{-\varepsilon^{-2\delta_1(1-\vartheta/2)/4}}|t-t'|^{\vartheta}+e^{-\varepsilon^{-2\delta_1(1-\vartheta)/2}}|t-t'|^{\eta'/2})\\ &\leq C(\delta',K,\eta',\gamma)|t-t'|^{\eta'/2-\delta'}\big[\varepsilon^{2(1-\delta_1)\xi\gamma}+(8+3K2^{N_\xi\xi})^{2\gamma}\varepsilon^{-2\delta_1\xi\gamma}|t-t'|^{\gamma\xi}\\ &\quad+\varepsilon^{2\gamma\xi(1-\delta_1)}|t-t'|^{\delta'}+e^{-\varepsilon^{-2\delta_1(1-\eta'/4+\delta'/2)/4}}+e^{-\varepsilon^{-2\delta_1(1-\eta'/2+\delta')/2}}|t-t'|^{\delta'}\big]\\ &\leq C(\delta',K,\eta',\gamma)|t-t'|^{\eta'/2-\delta'}\big[\varepsilon^{2(1-\delta_1)\xi\gamma}2^{2N_\xi\xi\gamma}+e^{-\varepsilon^{-2\delta_1(2-\eta')/16}}\big],\end{aligned}\qquad(9.23)$$

where we have used the fact that $\delta'\in(0,\eta'/2)$ in the last inequality. From (9.21) and (9.22) we conclude that $P_3=P_4=0$ in (9.9) if

$$C(\delta',K,\eta',\gamma)\big[\varepsilon^{2(1-\delta_1)\xi\gamma}2^{2N_\xi\xi\gamma}+e^{-\varepsilon^{-2\delta_1(2-\eta')/16}}\big]\leq\varepsilon^{2p}.\qquad(9.24)$$

The rest of the proof is similar to the proof of Theorem 4.1 in [17]. We find conditions on N_ξ and δ_1 so that (9.24) is satisfied. Then we use the estimates in (9.10) and the fact that $P_3=P_4=0$ to bound the probabilities in (9.7). Finally a chaining argument is used to get the hypothesis of Theorem 9.1. ■

Proof of Theorem 6.5 The proof uses ideas from the proof of Corollary 4.2 in [17]. From Theorem 2.5 in [25] we get that u is uniformly Hölder -ρ continuous on compacts in $(0,\infty)\times\mathbb{R}$ for every $\rho\in(0,\eta/4)$. Define inductively $\xi_0=\eta/4$ and $\xi_{n+1}=\big[(\xi_n\gamma+\eta/2)\wedge 1\big]\big(1-\frac{1}{n+3}\big)$ so that

$$\xi_n\uparrow\frac{\eta}{2(1-\gamma)}\wedge 1.\qquad(9.25)$$

Recall that $\eta'=\eta-\varpi$ for some arbitrarily small $\varpi\in(0,\eta)$. From the assumptions of Theorem 6.5 we can choose ϖ sufficiently small such that

$$\gamma>1-\eta'/2.\qquad(9.26)$$

From (9.25) and (9.26) we have

$$\xi_n\uparrow 1.\qquad(9.27)$$

Fix n_0 so that $\xi_{n_0}\geq\xi>\xi_{n_0-1}$. Apply Theorem 9.1 inductively n_0 times to get (9.1) for ξ_{n_0-1}; (9.2) follows with $\xi_1=\xi_{n_0}$. ■

Proof of Theorem 1.10: Let $(t_0,x_0)=(t_0,x_0)(\omega)\in S_0(\omega)$. From (6.1) it follows that

$$(t_0,x_0)\in Z(K,N,\xi)(\omega),\ \forall N\geq 0. \tag{9.28}$$

From (9.28) and Theorem 6.5 it follows that there exists $N_0(\xi,K,\omega)$ such that for all $N\geq N_0$ and $(t,x)\in[0,T_K]\times\mathbb{R}$ satisfying $d((t,x),(t_0,x_0))\leq 2^{-N}$, we have

$$|u(t,x)-u(t_0,x_0)|\leq 2^{-N\xi}. \tag{9.29}$$

Let $(t',x')\in[0,T_K]\times\mathbb{R}$ such that $d((t',x'),(t_0,x_0))\leq 2^{-N_0}$. There exists $N'\geq N_0$ such that $2^{-N'-1}\leq d((t',x'),(t_0,x_0))\leq 2^{-N'}$. We get from (9.29) that

$$|u(t',x')-u(t_0,x_0)|\leq 2^{-\xi N'}\leq 2(d((t',x'),(t_0,x_0)))^{\xi}, \tag{9.30}$$

and we are done. ■

Appendix: Proofs of Lemmas 2.4, 4.6 and 5.6–5.8

In this section we prove Lemmas 2.4, 4.6 and 5.6–5.8. We start this section with an auxiliary lemma which will help us prove Lemmas 5.6 and 5.7. After the proofs of Lemmas 5.6 and 5.7, we prove Lemmas 5.8, 4.6 and 2.4.

Lemma A.1 *Let $\mu\in M_f^{\eta}(\mathbb{R})$ for some $\eta\in(0,\eta)$. For every $\beta\geq 0$ and $\varepsilon\in(0,\eta)$ there is a $C(\beta,\eta,\varepsilon)>0$, such that for all $0\leq s<t\leq t',\ x,x'\in\mathbb{R}$ we have*

(a)

$$\frac{1}{(t'-s)^{1+2\beta}}\int_{\mathbb{R}}|x-y|^{2\beta}\left(e^{-\frac{(x-y)^2}{2(t'-s)}}-e^{-\frac{(x-y)^2}{2(t-s)}}\right)^2\mu(dy)\leq\frac{C(\beta,\eta,\varepsilon)}{(t-s)^{1+\beta-\eta/2+\varepsilon/2}}\left[1\wedge\frac{t'-t}{t-s}\right],$$

(b)

$$\left(\frac{1}{(t'-s)^{1/2+\beta}}-\frac{1}{(t-s)^{1/2+\beta}}\right)^2\int_{\mathbb{R}}|x-y|^{2\beta}e^{-\frac{(x-y)^2}{(t-s)}}\mu(dy)\leq\frac{C(\beta,\eta,\varepsilon)}{(t-s)^{1+\beta-\eta/2+\varepsilon/2}}\left[1\wedge\frac{t'-t}{t-s}\right],$$

(c)

$$\frac{1}{(t'-s)^{1+2\beta}}\int_{\mathbb{R}}(|x-y|^{2\beta}e^{-\frac{(x-y)^2}{(t-s)}}+|x'-y|^{2\beta}e^{-\frac{(x'-y)^2}{(t-s)}})\mu(dy)\leq\frac{C(\beta,\eta,\varepsilon)}{(t-s)^{1+\beta-\eta/2+\varepsilon/2}}.$$

Proof: (a) Let $\varepsilon\in(0,\eta)$. Recall that $0\leq s<t\leq t'$. Apply Lemma 5.4 with $a=2$ and $\delta=2+2\beta+\eta-\varepsilon$ to get

$$\begin{aligned}
I_1 &:= \frac{1}{(t'-s)^{1+2\beta}}\int_{\mathbb{R}}|x-y|^{2\beta}\left(e^{-\frac{(x-y)^2}{2(t'-s)}}-e^{-\frac{(x-y)^2}{2(t-s)}}\right)^2\mu(dy)\\
&\leq \frac{1}{(t'-s)^{1+2\beta}}\int_{\mathbb{R}}|x-y|^{2\beta}\left(e^{-\frac{(x-y)^2}{2(t'-s)}}-e^{-\frac{(x-y)^2}{2(t-s)}}\right)e^{-\frac{(x-y)^2}{2(t'-s)}}\mu(dy)\\
&\leq \frac{C_{5.4}(\beta,\eta)}{(t'-s)^{1+2\beta}}\int_{\mathbb{R}}\left(e^{-\frac{(x-y)^2}{2(t'-s)}}-e^{-\frac{(x-y)^2}{2(t-s)}}\right)(t'-s)^{1+\beta+\eta/2-\varepsilon/2}|x-y|^{-2-\eta+\varepsilon}\mu(dy)\\
&= C(\beta,\eta)(t'-s)^{-\beta+\eta/2-\varepsilon/2}\left|\frac{1}{(t'-s)}-\frac{1}{(t-s)}\right|\int_{\mathbb{R}}|x-y|^{-\eta+\varepsilon}\mu(dy)\\
&\leq C(\beta,\eta,\varepsilon)\frac{t'-t}{(t-s)^{2+\beta-\eta/2+\varepsilon/2}},
\end{aligned} \tag{A.1}$$

where we have used the fact that $|e^{-u}-e^{-w}|\leq|u-w|$ for all $u,w\geq 0$ and (1.9) in the last two lines. Note that the fact that $0\leq s<t\leq t'$ was also used in the last line.

On the other hand, use Lemma 5.4 with $a = 1$ and $\delta = 2\beta + \eta - \varepsilon$ and then (1.9) to get

$$\begin{aligned} I_1 &\leq \frac{C}{(t'-s)^{1+2\beta}} \int_{\mathbb{R}} |x-y|^{2\beta} \Big(e^{-\frac{(x-y)^2}{t'-s}} + e^{-\frac{(x-y)^2}{t-s}} \Big) \mu(dy) \\ &\leq C_{5.4}(\beta,\eta) \frac{1}{(t'-s)^{1+2\beta}} \big((t'-s)^{\beta+\eta/2-\varepsilon/2} + (t-s)^{\beta+\eta/2-\varepsilon/2} \big) \int_{\mathbb{R}} |x-y|^{-\eta+\varepsilon} \mu(dy) \\ &\leq C(\beta,\eta,\varepsilon) \frac{1}{(t-s)^{1+\beta-\eta/2+\varepsilon/2}}. \end{aligned} \tag{A.2}$$

From (A.1) and (A.2) we get

$$I_1 \leq C(\beta,\eta,\varepsilon) \frac{1}{(t-s)^{1+\beta-\eta/2+\varepsilon/2}} \left[1 \wedge \frac{t'-t}{t-s} \right]. \tag{A.3}$$

(b) Denote $f(u) = u^{-1-2\beta}$ and note that if $0 < u < u'$, $0 \leq f(u) - f(u') \leq C(\beta)(u^{-1-2\beta} \wedge [u^{-2-2\beta}(u'-u)])$. Recall that $0 < s < t \leq t'$, and therefore we have

$$\begin{aligned} I_2 &:= \left(\frac{1}{(t'-s)^{1/2+\beta}} - \frac{1}{(t-s)^{1/2+\beta}} \right)^2 \int_{\mathbb{R}} |x-y|^{2\beta} e^{-\frac{(x-y)^2}{t-s}} \mu(dy) \qquad (A.4) \\ &\leq C(\beta) \big((t-s)^{-1-2\beta} \wedge [(t-s)^{-2-2\beta}(t'-t)] \big) \int_{\mathbb{R}} |x-y|^{2\beta} e^{-\frac{(x-y)^2}{t-s}} \mu(dy) \\ &\leq C(\beta,\eta) \big((t-s)^{-1-2\beta} \wedge [(t-s)^{-2-2\beta}(t'-t)] \big) (t-s)^{\beta+\eta/2-\varepsilon/2} \int_{\mathbb{R}} |x-y|^{-\eta+\varepsilon} \mu(dy) \\ &\leq C(\eta,\varepsilon) \frac{1}{(t-s)^{1+\beta-\eta/2+\varepsilon/2}} \left[1 \wedge \frac{t'-t}{t-s} \right], \end{aligned}$$

where we have used Lemma 5.4 with $\delta = 2\beta + \eta - \varepsilon$, $a = 1$ and (1.9) in the last two lines.

(c) Let $z \in \mathbb{R}$, $u > 0$. Apply Lemma 5.4 with $\delta = 2\beta + \eta - \varepsilon$ and $a = 1$. Then use (1.9) to get,

$$\begin{aligned} \frac{1}{u^{1+2\beta}} \int_{\mathbb{R}} |z-y|^{2\beta} e^{-\frac{(z-y)^2}{u}} \mu(dy) &\leq C(\beta,\eta) \frac{1}{u^{1+\beta-\eta/2+\varepsilon/2}} \int_{\mathbb{R}} |z-y|^{-\eta+\varepsilon} \mu(dy) \\ &\leq C(\beta,\eta,\varepsilon) \frac{1}{u^{1+\beta-\eta/2+\varepsilon/2}}. \end{aligned} \tag{A.5}$$

From the above we immediately get (c). ∎

Proof of Lemma 5.6 Note that

$$\begin{aligned} \int_{\mathbb{R}} (G_{t'-s}(x'-y) - G_{t-s}(x-y))^2 \mu(dy) &\leq 2 \int_{\mathbb{R}} (G_{t'-s}(x-y) - G_{t-s}(x-y))^2 \mu(dy) \\ &\quad + 2 \int_{\mathbb{R}} (G_{t'-s}(x-y) - G_{t'-s}(x'-y))^2 \mu(dy) \\ &=: 2I_1 + 2I_2. \end{aligned} \tag{A.6}$$

For I_1 we have

$$\begin{aligned} I_1 &\leq \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{\sqrt{t'-s}} e^{-\frac{(x-y)^2}{2(t'-s)}} - \frac{1}{\sqrt{t'-s}} e^{-\frac{(x-y)^2}{2(t-s)}} \right)^2 \mu(dy) \qquad (A.7) \\ &\quad + \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{\sqrt{t'-s}} e^{-\frac{(x-y)^2}{2(t-s)}} - \frac{1}{\sqrt{t-s}} e^{-\frac{(x-y)^2}{2(t-s)}} \right)^2 \mu(dy) \\ &=: \frac{1}{\pi} (I_{1,1} + I_{1,2}). \end{aligned}$$

Let $\varepsilon \in (0,\eta)$. Recall that $0 \leq s < t \leq t'$, then from Lemma A.1(a) with $\beta = 0$ it follows that

$$I_{1,1} \leq C(\eta,\varepsilon) \frac{1}{(t-s)^{1-\eta/2+\varepsilon/2}} \left[1 \wedge \frac{|t'-t|}{t-s} \right]. \tag{A.8}$$

From Lemma A.1(b) with $\beta = 0$ it follows that

$$I_{1,2} \le C(\eta,\varepsilon)\frac{1}{(t-s)^{1-\eta/2+\varepsilon/2}}\left[1 \wedge \frac{|t'-t|}{t-s}\right]. \tag{A.9}$$

From (A.7), (A.8) and (A.9) we have

$$I_1 \le C(\eta,\varepsilon)\frac{1}{(t-s)^{1-\eta/2+\varepsilon/2}}\left[1 \wedge \frac{|t'-t|}{t-s}\right]. \tag{A.10}$$

Apply Lemma 5.3(a) with $\delta = 1$ on I_2 to get

$$I_2 \quad \le \quad C\frac{|x-x'|^2}{(t'-s)^2}\int_{\mathbb{R}}\big(e^{-\frac{(x-y)^2}{(t'-s)}} + e^{-\frac{(x'-y)^2}{(t'-s)}}\big)\mu(dy). \tag{A.11}$$

Apply Lemma A.1(c) with $\beta = 0$ to get

$$I_2 \le C(\eta,\varepsilon)\frac{|x-x'|^2}{(t-s)^{2-\eta/2+\varepsilon/2}}. \tag{A.12}$$

On the other hand we have

$$I_2 \le \frac{C}{(t'-s)}\int_{\mathbb{R}}\big(e^{-\frac{(x-y)^2}{(t-s)}} + e^{-\frac{(x'-y)^2}{(t-s)}}\big)\mu(dy).$$

Then, it follows again from Lemma A.1(c) with $\beta = 0$ that

$$I_2 \quad \le \quad C(\eta,\varepsilon)\frac{1}{(t-s)^{1-\eta/2+\varepsilon/2}}. \tag{A.13}$$

From (A.12) and (A.13) we deduce

$$I_2 \le \frac{C(\eta,\varepsilon)}{(t-s)^{1-\eta/2+\varepsilon/2}}\left[1 \wedge \frac{|x-x'|^2}{t-s}\right]. \tag{A.14}$$

From (5.7), (A.6), (A.10) and (A.14) we get (a).

(b) Assume $p, r, \nu_0, \nu_1, s, t, t'$ as in (b). Note that $|y-x| > (t'-s)^{1/2-\nu_0} \vee 2|x'-x|$ implies that

$$|y-x'| \ge |y-x| - |x'-x| \ge |y-x|/2 \ge \frac{(t'-s)^{1/2-\nu_0}}{2} \tag{A.15}$$

and in particular

$$|x-y| \le 2|y-x'|. \tag{A.16}$$

Let $d = d((t,x),(t',x'))$, $w = |x-y|$ and $w' = |x'-y|$. Use Hölder's inequality and (a) to get

$$\begin{aligned}
&\int_{\mathbb{R}} e^{r|x-y|}|x-y|^p(G_{t'-s}(x'-y) - G_{t-s}(x-y))^2 \mathbb{1}_{\{|x-y|>(t'-s)^{1/2-\nu_0}\vee 2|x'-x|\}}\mu(dy) \\
&\le \left[\int_{\mathbb{R}}(G_{t'-s}(x'-y) - G_{t-s}(x-y))^2\mu(dy)\right]^{1-\nu_1/2} \\
&\quad \times \left[\int_{\mathbb{R}} e^{\frac{2r}{\nu_1}|x-y|}|x-y|^{2p/\nu_1}(G_{t'-s}(x'-y) - G_{t-s}(x-y))^2 \mathbb{1}_{\{|x-y|>(t'-s)^{1/2-\nu_0}\vee 2|x'-x|\}}\mu(dy)\right]^{\nu_1/2} \\
&\le C(R,\eta,\varepsilon,\nu_1)(t-s)^{(\eta/2-\varepsilon-1)(1-\nu_1/2)}\left[1 \wedge \frac{d^2}{t-s}\right]^{1-\nu_1/2} \\
&\quad \times \left[\int_{\mathbb{R}} e^{\frac{4r}{\nu_1}w'}(w')^{2p/\nu_1}G_{t'-s}(w')^2 \mathbb{1}_{\{w'>(t'-s)^{1/2-\nu_0}/2\}}\mu(dy)\right. \\
&\quad \left. + \int_{\mathbb{R}} e^{\frac{4r}{\nu_1}w}w^{2p/\nu_1}G_{t-s}(w)^2 \mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}}\mu(dy)\right]^{\nu_1/2},
\end{aligned} \tag{A.17}$$

where we used (A.15), (A.16) in the last inequality. Note that if $w > \frac{1}{2}u^{1/2-\nu_0}$, then

$$G_u(w)^2 \leq C(R)G_u(w). \tag{A.18}$$

Also note that since $w > \frac{1}{2}(t-s)^{1/2-\nu_0}$ we have $\frac{w^2}{4(t-s)} > \frac{(t-s)^{-2\nu_0}}{16}$. Apply Lemma 5.4 with $\delta = 1$ and $a = 8$ to get,

$$\begin{aligned} G_{t-s}(w) &\leq \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{w^2}{8(t-s)}} e^{-\frac{w^2}{8(t-s)}} e^{-\frac{(t-s)^{-2\nu_0}}{16}} \\ &\leq C(R)w^{-1}e^{-\frac{w^2}{8(t-s)}} e^{-\frac{(t-s)^{-2\nu_0}}{16}}. \end{aligned} \tag{A.19}$$

Let $\varepsilon \in (0,\eta)$. From (A.18) and (A.19) we have

$$\begin{aligned} &J_1(t-s,w) \\ &:= \left[\int_{\mathbb{R}} e^{\frac{4r}{\nu_1}w} w^{2p/\nu_1} G_{t-s}(w)^2 \mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}}\mu(dy)\right]^{\nu_1/2} \\ &\leq C(R)\left[\int_{\mathbb{R}} e^{\frac{4r}{\nu_1}w} w^{2p/\nu_1} w^{-1} e^{-\frac{w^2}{8(t-s)}} e^{-\frac{(t-s)^{-2\nu_0}}{16}} \mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}}\mu(dy)\right]^{\nu_1/2} \\ &\leq C(R)e^{-\frac{\nu_1(t-s)^{-2\nu_0}}{32}} \sup_{w>0}\left(e^{-\frac{w^2}{8(t-s)}} e^{\frac{4r}{\nu_1}w} w^{2p/\nu_1} w^{-1+\eta-\varepsilon}\mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}}\right)^{\nu_1/2} \\ &\quad\times\left[\int_{\mathbb{R}} |x-y|^{-\eta+\varepsilon}\mu(dy)\right]^{\nu_1/2} \\ &\leq C(R,\eta,\varepsilon)e^{-\frac{\nu_1(t-s)^{-2\nu_0}}{32}} \sup_{w>0}\left(e^{-\frac{w^2}{8(t-s)}} e^{\frac{4r}{\nu_1}w} w^{2p/\nu_1} w^{-1+\eta-\varepsilon}\mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}}\right)^{\nu_1/2}, \end{aligned} \tag{A.20}$$

where we have used (1.9) in the last inequality.

Since $t-s \in (0,R)$, $\eta \in (0,1)$ and $r,p \in [0,R]$, it follows that there exists $M(R,\nu_1) > 0$ such that for all $w \geq 1$ we have

$$\begin{aligned} e^{-\frac{w^2}{8(t-s)}} e^{\frac{4r}{\nu_1}w} w^{2p/\nu_1} w^{-1+\eta-\varepsilon}\mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}} &\leq e^{-\frac{w^2}{8R}} e^{\frac{4R}{\nu_1}w} w^{2R/\nu_1} \\ &\leq M(R,\nu_1). \end{aligned} \tag{A.21}$$

Recall that $\nu_0,\nu_1 \in (1/R,1/2)$. Then, there exists $C(R,\nu_0,\nu_1,\eta,\varepsilon) > 0$ such that for all $w < 1$ we have

$$\begin{aligned} &e^{-\frac{w^2}{8(t-s)}} e^{\frac{4R}{\nu_1}w} w^{2p/\nu_1} w^{-1+\eta-\varepsilon}\mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}} \\ &\leq C(R,\nu_1)e^{-\frac{1}{32}(t-s)^{-2\nu_0}}(t-s)^{(-1+\eta-\varepsilon)(1/2-\nu_0)} \\ &\leq C(R,\nu_0,\nu_1,\eta,\varepsilon). \end{aligned} \tag{A.22}$$

It follows that there exists $C(R,\nu_0,\nu_1,\eta,\varepsilon) > 0$ such that,

$$\sup_{w>0,\ t-s\in(0,R]}\left(e^{-\frac{w^2}{8(t-s)}} e^{\frac{4R}{\nu_1}w} w^{2p/\nu_1} w^{-1+\eta-\varepsilon}\mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}}\right)^{\nu_1/2} \leq C(R,\nu_0,\nu_1,\eta,\varepsilon). \tag{A.23}$$

From (A.20) and (A.23) we get

$$\sup_{w>0} J_1(t-s,w) \leq C(R,\nu_0,\nu_1,\eta,\varepsilon)e^{-\frac{\nu_1(t-s)^{-2\nu_0}}{32}}. \tag{A.24}$$

Let

$$J_2(t'-s,w') := \left[\int_{\mathbb{R}} e^{\frac{4r}{\nu_1}w'}(w')^{2p/\nu_1} G_{t'-s}(w')^2 \mathbb{1}_{\{w'>(t'-s)^{1/2-\nu_0}/2\}}\mu(dy)\right]^{\nu_1/2};$$

replace t with t' and w with w', and then repeat (A.19)–(A.24) to get

$$\sup_{w'>0} J_2(t'-s,w') \leq C(R,\nu_0,\nu_1,\eta,\varepsilon)e^{-\frac{\nu_1(t'-s)^{-2\nu_0}}{32}}. \tag{A.25}$$

From (A.17), (A.24) and (A.25) we get (b). ■

Proof of Lemma 5.7 **(a)** Let $\varepsilon \in (0,\eta)$. Note that

$$\begin{aligned}\int_{\mathbb{R}}(G'_{t'-s}(x'-y)-G'_{t-s}(x-y))^2\mu(dy) &\leq 2\int_{\mathbb{R}}(G'_{t'-s}(x-y)-G'_{t-s}(x-y))^2\mu(dy)\\ &\quad +2\int_{\mathbb{R}}(G'_{t'-s}(x-y)-G'_{t'-s}(x'-y))^2\mu(dy)\\ &=: 2I_1+2I_2. \end{aligned} \tag{A.26}$$

For I_1 we have

$$\begin{aligned} I_1 &\leq \frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{x-y}{(t'-s)^{3/2}}e^{-\frac{(x-y)^2}{2(t'-s)}}-\frac{x-y}{(t'-s)^{3/2}}e^{-\frac{(x-y)^2}{2(t-s)}}\right)^2\mu(dy) \qquad (A.27)\\ &\quad +\frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{x-y}{(t'-s)^{3/2}}e^{-\frac{(x-y)^2}{2(t-s)}}-\frac{x-y}{(t-s)^{3/2}}e^{-\frac{(x-y)^2}{2(t-s)}}\right)^2\mu(dy)\\ &:= \frac{1}{\pi}(I_{1,1}+I_{1,2}). \end{aligned}$$

From Lemma A.1(a) with $\beta=1$ we have

$$I_{1,1}\leq C(\eta,\varepsilon)\frac{1}{(t-s)^{2-\eta/2+\varepsilon/2}}\left[1\wedge\frac{|t'-t|}{t-s}\right]. \tag{A.28}$$

From Lemma A.1(b) with $\beta=1$ it follows that

$$I_{1,2}\leq C(\eta,\varepsilon)\frac{1}{(t-s)^{2-\eta/2+\varepsilon/2}}\left[1\wedge\frac{|t'-t|}{t-s}\right], \tag{A.29}$$

From (A.27), (A.28) and (A.29) we have

$$I_{1}\leq C(\eta,\varepsilon)\frac{1}{(t-s)^{2-\eta/2+\varepsilon/2}}\left[1\wedge\frac{|t'-t|}{t-s}\right]. \tag{A.30}$$

Apply Lemma 5.3(b) with $\delta=1$ on I_2 to get

$$I_2 \leq C\frac{(x'-x)^2}{(t'-s)^3}\int_{\mathbb{R}}\left(e^{-\frac{(x-y)^2}{(t'-s)}}+e^{-\frac{(x'-y)^2}{(t'-s)}}\right)\mu(dy).$$

Apply Lemma A.1(c) with $\beta=0$ to get

$$I_2\leq C(\eta,\varepsilon)\frac{(x'-x)^2}{(t'-s)^{3-\eta/2+\varepsilon/2}}. \tag{A.31}$$

On the other hand, note that

$$I_2 \leq \frac{C}{(t'-s)^3}\int_{\mathbb{R}}\left((x-y)^2e^{-\frac{(x-y)^2}{(t-s)}}+(x'-y)^2e^{-\frac{(x'-y)^2}{(t-s)}}\right)\mu(dy).$$

Apply Lemma A.1(c) with $\beta=1$ to get,

$$I_2\leq \frac{C(\eta,\varepsilon)}{(t'-s)^{2-\eta/2+\varepsilon/2}}. \tag{A.32}$$

From (A.31) and (A.32) we deduce that

$$I_2\leq \frac{C(\eta,\varepsilon)}{(t-s)^{2-\eta/2+\varepsilon/2}}\left[1\wedge\frac{(x-x')^2}{t-s}\right]. \tag{A.33}$$

From (5.7), (A.26), (A.30) and (A.33) we get (a).

(b) Assume p,r,ν_0,ν_1,s,t,t' as in (b). Let $\varepsilon\in(0,\eta)$ be arbitrarily small. Note that $|y-x|>(t'-$

$s)^{1/2-\nu_0} \vee 2|x'-x|$ implies (A.15) and (A.16).

Let $d = d((t,x),(t',x'))$, $w = |x-y|$ and $w' = |x'-y|$. Use Hölder's inequality and (a) to get

$$\begin{aligned}
&\int_{\mathbb{R}} e^{r|x-y|}|x-y|^p (G'_{t'-s}(x'-y) - G'_{t-s}(x-y))^2 \mathbb{1}_{\{|x-y|>(t'-s)^{1/2-\nu_0}\vee 2|x'-x|\}}\mu(dy) \\
&\le \left[\int_{\mathbb{R}} (G'_{t'-s}(x'-y) - G'_{t-s}(x-y))^2 \mu(dy)\right]^{1-\nu_1/2} \\
&\quad \times \left[\int_{\mathbb{R}} e^{\frac{2r}{\nu_1}|x-y|}|x-y|^{2p/\nu_1} (G'_{t'-s}(x'-y) - G'_{t-s}(x-y))^2 \mathbb{1}_{\{|x-y|>(t'-s)^{1/2-\nu_0}\vee 2|x'-x|\}}\mu(dy)\right]^{\nu_1/2} \\
&\le C(R,\varepsilon,\eta)(t-s)^{(\eta/2-2-\varepsilon)(1-\nu_1/2)}\left[1 \wedge \frac{d^2}{t-s}\right]^{1-\nu_1/2} \\
&\quad \times \left[\int_{\mathbb{R}} e^{\frac{4r}{\nu_1}w'}(w')^{2p/\nu_1} G'_{t'-s}(w')^2 \mathbb{1}_{\{w'>(t'-s)^{1/2-\nu_0}/2\}}\mu(dy)\right. \\
&\quad \left. + \int_{\mathbb{R}} e^{\frac{4r}{\nu_1}w} w^{2p/\nu_1} G'_{t-s}(w)^2 \mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}}\mu(dy)\right]^{\nu_1/2},
\end{aligned} \tag{A.34}$$

where we used (A.15), (A.16) in the last inequality. From (4.20) in [16] it follows that if $w > \frac{1}{2}u^{1/2-\nu_0}$, then

$$G'_u(w)^2 \le C(R)G_{2u}(w). \tag{A.35}$$

Also note that since $w > \frac{1}{2}(t-s)^{1/2-\nu_0}$ we have $\frac{w^2}{8(t-s)} > \frac{(t-s)^{-2\nu_0}}{32}$. Apply Lemma 5.4 with $\delta = 1$ and $a = 16$ to get,

$$\begin{aligned}
G_{2(t-s)}(w) &\le \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{w^2}{16(t-s)}} e^{-\frac{w^2}{16(t-s)}} e^{-\frac{(t-s)^{-2\nu_0}}{32}} \\
&\le C(R)w^{-1} e^{-\frac{w^2}{16(t-s)}} e^{-\frac{(t-s)^{-2\nu_0}}{32}}.
\end{aligned} \tag{A.36}$$

From (A.35) and (A.36) we have

$$\begin{aligned}
&J_1(t-s,w) \\
&:= \left[\int_{\mathbb{R}} e^{\frac{4r}{\nu_1}w} w^{2p/\nu_1} G'_{t-s}(w)^2 \mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}}\mu(dy)\right]^{\nu_1/2} \\
&\le C(R)e^{-\frac{\nu_1(t-s)^{-2\nu_0}}{64}} \left[\int_{\mathbb{R}} e^{-\frac{w^2}{16(t-s)}} e^{\frac{4r}{\nu_1}w} w^{2p/\nu_1} w^{-1} \mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}}\mu(dy)\right]^{\nu_1/2} \\
&\le C(R)e^{-\frac{\nu_1(t-s)^{-2\nu_0}}{64}} \sup_{w>0}\left(e^{-\frac{w^2}{16(t-s)}} e^{\frac{4r}{\nu_1}w} w^{2p/\nu_1} w^{-1+\eta-\varepsilon} \mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}}\right)^{\nu_1/2} \\
&\quad \times \left[\int_{\mathbb{R}} |x-y|^{-\eta+\varepsilon}\mu(dy)\right]^{\nu_1/2}.
\end{aligned} \tag{A.37}$$

As in (A.21), since $t-s \in (0,R)$, $\eta \in (0,1)$ and $r,p \in [0,R]$, we get that there exists $M(R,\nu_1) > 0$ such that for all $w \ge 1$ we have

$$e^{-\frac{w^2}{16(t-s)}} e^{\frac{4r}{\nu_1}w} w^{2p/\nu_1} w^{-1+\eta-\varepsilon} \mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}} \le M(R,\nu_1). \tag{A.38}$$

Recall that $\nu_0, \nu_1 \in (1/R, 1/2)$. As in (A.22), we get that there exists $C(R,\nu_0,\nu_1,\eta,\varepsilon) > 0$ such that for all $w < 1$ we have

$$e^{-\frac{w^2}{16(t-s)}} e^{\frac{4R}{\nu_1}w} w^{2p/\nu_1} w^{-1+\eta-\varepsilon} \mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}} \le C(R,\nu_0,\nu_1,\eta,\varepsilon). \tag{A.39}$$

From (A.38) and (A.39), we get that there exists $C(R,\nu_0,\nu_1,\eta,\varepsilon) > 0$ such that,

$$\sup_{w>0,\ t-s\in(0,R]} \left(e^{-\frac{w^2}{16(t-s)}} e^{\frac{4R}{\nu_1}w} w^{2p/\nu_1} w^{-1+\eta-\varepsilon} \mathbb{1}_{\{w>(t-s)^{1/2-\nu_0}/2\}}\right)^{\nu_1/2} \le C(R,\nu_0,\nu_1,\eta,\varepsilon). \tag{A.40}$$

From (1.9), (A.37) and (A.40) we get

$$\sup_{w>0} J_1(t-s,w) \le C(R,\nu_0,\nu_1,\eta,\varepsilon) e^{-\frac{\nu_1 (t-s)^{-2\nu_0}}{64}}. \tag{A.41}$$

Denote

$$J_2(t'-s,w') := \left[\int_{\mathbb{R}} e^{\frac{4r}{\nu_1}w'} (w')^{2p/\nu_1} G'_{t'-s}(w')^2 \mathbb{1}_{\{w'>(t'-s)^{1/2-\nu_0}/2\}} \mu(dy)\right]^{\nu_1/2}.$$

Replace t with t' and w with w', then, repeat (A.36)–(A.41) to get

$$\sup_{w'>0} J_2(t'-s,w') \le C(R,\nu_0,\nu_1,\eta,\varepsilon) e^{-\frac{\nu_1 (t'-s)^{-2\nu_0}}{64}}. \tag{A.42}$$

From (A.34), (A.41) and (A.42) we get

$$\begin{aligned}
&\int_{\mathbb{R}} e^{r|x-y|}|x-y|^p (G'_{t'-s}(x'-y) - G'_{t-s}(x-y))^2 \mathbb{1}_{\{|x-y|>(t'-s)^{1/2-\nu_0}\vee 2|x'-x|\}} \mu(dy) \\
&\le C(R,\nu_0,\nu_1,\eta,\varepsilon)(t-s)^{(\eta/2-\varepsilon-2)(1-\nu_1/2)} \left[1 \wedge \frac{d^2}{t-s}\right]^{1-\nu_1/2} e^{-\frac{\nu_1 (t'-s)^{-2\nu_0}}{64}}.
\end{aligned}$$

Proof of Lemma 5.8 The proof of Lemma 5.8 follows the same lines as the proof of Lemmas 5.6(a). Let $\lambda > 0$ and $0 \le s < t \le t',\ x, x' \in \mathbb{R}$. Note that

$$\begin{aligned}
\int_{\mathbb{R}} e^{\lambda|y|}(G_{t'-s}(x'-y) - G_{t-s}(x-y))^2 \mu(dy) &\le 2\int_{\mathbb{R}} e^{\lambda|y|}(G_{t'-s}(x-y) - G_{t-s}(x-y))^2 \mu(dy) \\
&\quad +2\int_{\mathbb{R}} e^{\lambda|y|}(G_{t'-s}(x-y) - G'_{t'-s}(x'-y))^2 \mu(dy) \\
&=: 2I_1 + 2I_2.
\end{aligned}$$

We handle I_1 as in (A.7). Recall that in the proof of Lemma 5.6(a) the bounds on $I_{1,1}$ and $I_{1,2}$ were established by Lemma A.1(a) and (b) with $\beta = 0$. To get the corresponding bounds here, we use Lemma 5.5(d) in equation (A.1) and Lemma 5.5(a) in equation (A.2), instead of Lemma 5.4 and (1.9).

To bound I_2, we use Lemma 5.3(a) as in (A.11). Then, in order to establish a bound which corresponds to Lemma A.1(c) with $\beta = 0$, we again use Lemma 5.5(a) instead of Lemma 5.4 and (1.9) in (A.5). ■

Proof of Lemma 4.6 Let $w = |x-y|$. From Lemma 5.4 with $\delta = 1$ and $a = 8$ we get for $w > t^{1/2-\nu_1}$,

$$\begin{aligned}
G_t(w) &\le \frac{1}{\sqrt{2\pi t}} e^{-\frac{w^2}{8t}} e^{-\frac{w^2}{8t}} e^{-\frac{t^{-2\nu_1}}{8}} \\
&\le C(T) w^{-1} e^{-\frac{w^2}{8t}} e^{-\frac{t^{-2\nu_1}}{8}},
\end{aligned} \tag{A.43}$$

for all $0 \le t \le T$.

Let $\varepsilon \in (0,\eta)$. From (A.23) it follows that

$$\sup_{w>0,\ t\in(0,T]} \left(e^{\lambda w} e^{-\frac{w^2}{8t}} w^{-1+\eta-\varepsilon} \mathbb{1}_{\{w>t^{1/2-\nu_1}\}}\right) < \infty. \tag{A.44}$$

From (A.43) we get

$$\begin{aligned}
&\int_{\mathbb{R}} e^{\lambda w} G_t(w) \mathbb{1}_{\{w>(t-s)^{1/2-\nu_1}\}} \mu(dy) \\
&\le C e^{-\frac{t^{-2\nu_1}}{8}} \int_{\mathbb{R}} e^{\lambda w} w^{-1} e^{-\frac{w^2}{8t}} \mathbb{1}_{\{w>t^{1/2-\nu_1}\}} \mu(dy) \\
&\le C e^{-\frac{t^{-2\nu_1}}{8}} \sup_{w>0,\ t\in(0,T]} \left(e^{\lambda w} e^{-\frac{w^2}{8t}} w^{-1+\eta-\varepsilon} \mathbb{1}_{\{w>t^{1/2-\nu_1}\}}\right) \int_{\mathbb{R}} |x-y|^{-\eta+\varepsilon} \mu(dy) \\
&\le C(\eta,\nu_1,\lambda,T) e^{-\frac{t^{-2\nu_1}}{8}},
\end{aligned}$$

where we have used (1.9) and (A.44) in the last inequality. ■

Now we are ready to prove Lemma 2.4.

Proof of Lemma 2.4 Let $K>0$. Note that

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{|y|}G_{\varepsilon}(z-y)1\!\!1_{\{|z|\le K\}}\mu(dy)dz &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{|y|}G_{\varepsilon}(z-y)1\!\!1_{\{|z|\le K\}}1\!\!1_{\{|y-z|\le K+1\}}\mu(dy)dz \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} e^{|y|}G_{\varepsilon}(z-y)1\!\!1_{\{|z|\le K\}}1\!\!1_{\{|y-z|> K+1\}}\mu(dy)dz \\ &:= I_1(K,\varepsilon)+I_2(K,\varepsilon),\ \forall\varepsilon\in(0,1]. \end{aligned} \tag{A.45}$$

For $I_1(K,\varepsilon)$ we get that

$$\begin{aligned} I_1(K,\varepsilon) &\le C(K)\int_{\mathbb{R}}\int_{\mathbb{R}} G_{\varepsilon}(z-y)dz\mu(dy) \\ &\le C(K,\mu(\mathbb{R})),\ \forall\varepsilon\in(0,1]. \end{aligned} \tag{A.46}$$

From Lemma 4.6 we get that,

$$\begin{aligned} I_2(K,\varepsilon) &\le C(K)\int_{\mathbb{R}} 1\!\!1_{\{|z|\le K\}}\int_{\mathbb{R}} e^{|y|}G_{\varepsilon}(z-y)1\!\!1_{\{|y-z|>\varepsilon^{1/4}\}}\mu(dy)dz \\ &\le C(\eta,K)\int_{\mathbb{R}} 1\!\!1_{\{|z|\le K\}}e^{-\varepsilon^{-1/2}/8}dz \\ &\le C(\eta,K),\ \forall\varepsilon\in(0,1]. \end{aligned} \tag{A.47}$$

From (A.45)–(A.47) we get (2.26). ■

List of Notations

$\dot{W}$	spatially inhomogeneous white noise based on $\mu(dx)dt$
Δ	the Laplacian operator
$\text{cardim}(\mu)$	the carrying dimension of the measure μ.
η	constant associated to μ (see (1.2))
$\mathcal{C}(E)$	continuous functions on E
$\mathcal{C}_c(E)$	continuous functions with compact support on E
$\mathcal{C}^{\infty}(E)$	infinite time continuously differentiable functions on E
$\mathcal{C}(I,E)$	continuous functions on I taking values in E
$\\|f\\|_{\lambda}$	the norm $\sup_{x\in\mathbb{R}}\|f(x)\|e^{-\lambda\|x\|}$, for $f\in\mathcal{C}(\mathbb{R})$
$\mathcal{C}_{tem}(\mathbb{R})$	continuous tempered functions on $\mathbb{R}$
$G_t(x)$	the probability density function of a centred normal distribution with variance t
$G_tf(x)$	$\int_{\mathbb{R}}G_t(x-y)f(y)dy$, for all f's such that the integral exists
γ	the Hölder index of the noise coefficient σ (see (1.7))
$\phi_{\eta,\mu}(\cdot)$	η-potential of a measure μ (see (1.8))
$M_f(\mathbb{R})$	finite measures on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$
$\dim_B(E)$	the Minkowski dimension (box dimension) of a set $E\subset\mathbb{R}$
$M_f^{\eta}(\mathbb{R})$	finite measures with finite $\sup_{x\in\mathbb{R}}\phi_{\eta-\varepsilon,\mu}(\cdot)$ for any $\varepsilon>0$, and carrying dimension η
T_K	stopping time, defined in (1.13).
$\{\phi_n(x)\}_{n\ge1}$	a set of $\mathcal{C}^{\infty}(\mathbb{R})$ functions which converge to $\|x\|$ uniformly.
$\langle\cdot,\cdot\rangle$	the scalar product on $L^2(\mathbb{R})$
t_0	constant fixed in (2.12)
K_1	constant fixed in (2.12)
$\{a_n\}_{n\ge1}$	sequence of constants, defined in (2.3)
$\varepsilon_0,\ \varepsilon_1$	constants, defined in (2.14)
κ_0	constant which equal to $\frac{1}{\eta+1}$ (see 2.15)
$\|E\|$	the Lebesgue measure of a set $E\subset\mathbb{R}$
I^n	an integral defined in (2.19)
$\mathbb{N}^{\ge K_1}$	$\{K_1,K_1+1,...\}$
$U_{M,n,K}$	sequence of stopping times introduced in Proposition 2.3
$\bar{\beta}$	constant which equal to $\frac{\eta}{\eta+1}$
$D(s,y)$	the difference between the noise coefficients of two solutions (see (4.1))

$u_{1,\delta}$, $u_{2,\delta}$, $\tilde{u}_{2,\delta}$	ingredients in the decomposition of u, introduced in (4.4), (4.5) and (4.9)
$\{\beta_i\}_{i=1}^{L+1}$	a grid, defined in (4.15)
$\hat{x}_n(t,x)$	defined in (4.17)
$\mathbb{G}_\delta(s,t,x)$	a random function, defined before Lemma 4.2
$F_\delta(s,t,x)$	$-\frac{d}{dx}\mathbb{G}_\delta(s,t,x)$
$\mathbb{V}^{n,\eta,\rho,\varepsilon_0}$	random cover defined after (4.20)
$\{J_{n,i}\}_{i=0}^{L}$, $\hat{J}_n$, $\{\tilde{J}_{n,i}\}_{i=0}^{L}$	random sets defined in (4.23), (4.24) and (4.35), respectively
$n_M(\varepsilon_1)$, $n_0(\varepsilon_1,\varepsilon_0)$, $n_1(\varepsilon_0,K)$	constants defined before and in (4.36)
$\bar{l}(\beta)$	constant which equal to $a_n^{\beta/\eta+5\varepsilon_1}$
$\bar{l}(\beta_i)$	constant which equal to $65a_n^{1-\beta_{i+1}}$
$G'_t(x)$	$\frac{\partial}{\partial x}G_t(x)$
$d((t,x),(t',x'))$	$\|t-t'\|^{1/2}+\|x'-x\|$, defined in (5.7)
$Z(K,N,\xi)$, $Z(N,n,K,\beta)$	random sets introduced in and right after (6.1)
γ_m, $\tilde{\gamma}_m$, $\bar{m}$	constants defined in (6.2)–(6.5)
(P_m)	property, defined in (6.7)
$\bar{d}_N$	equal to $d((s,y),(t,x))\vee 2^{-N}$
$\bar{\delta}_N$	equal to $\delta\vee\bar{d}_N^2$

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