INHOMOGENEOUS SPATIO-TEMPORAL POINT PROCESSES ON LINEAR NETWORKS FOR VISITORS’ STOPS DATA

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We analyse the spatio-temporal distribution of visitors’ stops by touristic attractions in Palermo (Italy) using theory of stochastic point processes living on linear networks. We first propose an inhomogeneous Poisson point process model, with a separable parametric spatio-temporal first-order intensity. We account for the spatial interaction among points on the given network, fitting a Gibbs point process model with mixed effects for the purely spatial component. This allows us to study first-order and second-order properties of the point pattern, accounting both for the spatio-temporal clustering and interaction and for the spatio-temporal scale at which they operate. Due to the strong degree of clustering in the data, we then formulate a more complex model, fitting a spatio-temporal Log-Gaussian Cox process to the point process on the linear network, addressing the problem of the choice of the most appropriate distance metric.

1. Introduction. Global Positioning System (GPS) devices can record units with their travel times and spatial coordinates of locations to a high degree of temporal precision. Nowadays, GPS devices are compact, equipped with significant autonomy and, of the utmost importance, they can store the geographical coordinates of a statistical unit at a given time. The widespread availability of GPS data allows the analysis of social behaviour across time and geographic scales (Butz and Torrey, 2006). Given the wide interest in understanding the human mobility, applications of GPS tracking technologies in social science research is widespread (see e.g., Elgethun et al. (2003); Zenk et al. (2011)).

In this paper, we focus on GPS data recording places that tourists are often visiting while moving in a constrained space (linear network) given by the topological structure of the study area (i.e. streets, squares, etc.). In particular, data concerns cruisers’ stoppings in Palermo (Italy) in a day, whose spatial and temporal positions along the given network are recorded. Although original data actually constitutes individual trajectories of visitors, we summarise the behaviour of aggregated units. We assume that the visitors’ stops may represent a spatio-temporal point pattern and therefore, we propose a modelling framework for inhomogeneous spatio-temporal point processes. In particular, we analyse first-order characteristics of the process through the estimation of the spatio-temporal intensity, given the linear network. Then, we consider Log-Gaussian Cox processes to analyse the spatio-temporal structure inherent in the data. The obtained results may be useful for identifying both the most visited places and the relationships among the various points of interest, with interesting implications for policy makers management.

We thus aim at describing visitors’ stops proposing new parametric approaches in the context of spatio-temporal point processes on linear networks, contributing to this framework.

Keywords and phrases: Gibbs point processes, Global Positioning System, Intensity estimation, Linear networks, Log-Gaussian Cox processes, Spatio-temporal point processes.
with different aspects. We first consider parametric estimation of the spatio-temporal first-order intensity on the given network. Previous papers have estimated parametric separable spatio-temporal intensities, such as Tamayo-Uria, Mateu and Diggle (2014), but, to the authors knowledge, our approach represents a novelty in the context of point processes on linear networks, since most of the literature about point processes on networks is concerned with non-parametric estimation of the spatio-temporal intensity (Moradi and Mateu, 2020; Moradi et al., 2019; Mateu, Moradi and Cronie, 2020). Of course, even if first-order characteristics allow to describe the spatio-temporal displacement of point patterns, a dependence modelling approach allows to account for effects that might affect the spatio-temporal structure (Baddeley, Rubak and Turner, 2015), both related to the characteristics of individuals, referring to the marked point processes methodology, and to the characteristics of the linear network, represented by spatial covariates. Indeed, conversely to the spatial context, spatio-temporal dependence models on linear networks represent a new research field to investigate.

In second place, and to take into account the spatial interaction among points, since visitors stop several times during the day, we consider Gibbs point process models with mixed effects (Illian and Hendrichsen, 2010; Baddeley, Møller and Waagepetersen, 2000) on the networks. Indeed, since trajectories of each tourist are available, we can track all the individual movements, studying the subject-specific behaviour. We consider the Berman-Turner device for maximum pseudolikelihood (Baddeley, Møller and Waagepetersen, 2000), using a quadrature scheme generated on the network. Then, we fit a parametric model incorporating both spatial inhomogeneity and interpoint interactions.

Finally, a first proposal in fitting a Log-Gaussian Cox process to spatio-temporal point processes on linear networks is also provided. Our analyses have been carried out using the spatio-temporal package stlnpp (Moradi, Cronie and Mateu, 2020) of the software R (R Core Team, 2021) for point pattern analysis on linear networks, and also the packages mgcv, stpp and CompRandFld (Wood, 2003; Gabriel et al., 2021; Padoan and Bevilacqua, 2015) for intensity estimation and spatio-temporal simulations.

The structure of the paper is as follows. Section 2 presents the data and the motivating problem. Section 3 provides an overview of spatio-temporal point processes on linear networks, referring to both first- and second-order characteristics, and the methods used throughout the rest of the paper are detailed in Section 4. Section 5 presents the data analysis, and Section 6 is devoted to conclusions.

2. Data and the motivating problem. The available data refer to a survey carried out in Palermo (Italy) in 2014 for cruiser passengers disembarking in the harbour of Palermo. GPS was used to record the location of the participant tourists and their arrival and departure times. In particular, right before the embarkation of visitors, a survey is carried out (Abbruzzo, Ferrante and De Cantis, 2021), collecting information regarding individual characteristics also related to their stay, such as the visit of any specific touristic site. The purpose of this survey is to study the behaviour of cruise passengers in urban contexts, defining profiles of visitors depending on the city and the touristic attractions. Procedures and other survey characteristics may be found in De Cantis et al. (2016) and Shoval et al. (2020).

We consider the GPS trajectories at an aggregated level by analysing individual-level points of interest. Indeed, the analysed data are obtained by using the density-based spatial clustering with noise (dbscan) algorithm, that is a well-known non-parametric data clustering algorithm proposed by Ester et al. (1996). Given a set of points in some space, it clusters points that are closely packed together (points with many nearby neighbours), marking as outliers those points lying alone in low-density regions (whose nearest neighbours are too far away). This, applied to a set of trajectories, produces a set of clusters (whose centroids can be seen as stops) and a set of noise points (that can be removed from the analysis) (?). Each
cluster is therefore identified by a spatial and a temporal location, and this information is used to build the point pattern. Since the aim is to study the spatio-temporal displacement of the visitors’ stops, we here ignore the trajectories and time is discretised in hours in order to identify the densest hours during a day. Indeed, we assume that the stops identify the points of interest visited by tourists. The complete spatio-temporal point pattern is displayed in Figure 1. Information about the spatial support, that is the road network $L$, has been obtained from the OpenStreetMap data.

Due to interpretational and computational reasons, we focus our analysis on a subset of the original data. As the complete dataset actually include some noise clusters and most of the visitors’ stops occur in the historical downtown, the analysis is restricted to the spatio-temporal point pattern consisting of 159 stops of 44 visitors, stopping 4 times on average during the day, in the downtown of Palermo city on the 18th April 2014. The linear network has 990 vertices and 1180 lines. The data consist, for each visitor stop, of the spatial location on the road network and the occurrence time during the day, with $t \in T = [8 \text{ AM}, 6 \text{ PM}]$.

In the left panel of Figure 2, the locations of the events are displayed, as obtained from the dbscan algorithm. On the right panel, the locations considered in this analysis are displayed. Indeed, when computing the stlnpp object in R, points that lay outside the linear support are relocated to the closest location from the original one, but on the network.

Since we expect that people stops more often near well-known touristic attractions, we computed $Z(u)$ as the distance from the nearest touristic attraction, where the metric used was the shortest-path distance on the network from the location of these known attractions:

- Quattro Canti
- Cathedral
- Teatro Massimo
- Fontana Pretoria
- Teatro Politeama
- Piazza Bellini
- Chiesa Valdese
- Via Principe di Belmonte

Note that $Z(u)$ is defined on a support that is slightly smaller than the original considered network, as the latter is not completely connected. The location of the touristic attractions on the linear network and the values of the resulting distance variable are displayed in Figure 3.

3. Spatio-temporal point processes on linear networks. Point processes on linear networks are recently considered to analyse events occurring on particular network structures (e.g. traffic accidents). They were firstly introduced in the spatial context and then extended to the spatio-temporal case, focusing on the analysis of first- and second-order summary statistics (Ang, Baddeley and Nair, 2012; McSwiggan, Baddeley and Nair, 2017; Rakshit, Nair and Baddeley, 2017; Moradi et al., 2019; Rakshit, Baddeley and Nair, 2019; Moradi and Mateu, 2020). For a general treatment of spatial and spatio-temporal point processes on the Euclidean space, the reader is referred to Diggle (2013).

A linear network $L = \bigcup_{i=1}^{n} l_i \subset \mathbb{R}^2$ is commonly taken as a finite union of line segments $l_i \subset \mathbb{R}^2$ of positive length. A line segment is defined as $l_i = [u_i, v_i] = \{ku_i + (1-k)v_i : 0 \leq k \leq 1\}$, where $u_i, v_i \in \mathbb{R}^2$ are the endpoints of $l_i$. For any $i \neq j$ the intersection of $l_i$ and $l_j$ is either empty or an endpoint of both segments.

A spatio-temporal linear network point process is a point process on the product space $L \times T$, where $L$ is a linear network and $T$ is a subset (interval) of $\mathbb{R}^+$. We hereafter focus on a spatio-temporal point process $X$ on a linear network $L$ with no overlapping points $(u, t)$ where $u \in L$ is the location of an event and $t \in T(T \subseteq \mathbb{R}^+)$ is the corresponding time.
Fig 1: The grey segments represent the streets of Palermo city, obtained from OpenStreetMap. In black, the points representing the visitors’ stops.

occurrence of \( u \). Note that the temporal state-space \( T \) might be either a continuous or a discrete set. A realisation of \( X \) with \( n \) points is represented by \( x = (u_i, t_i), i = 1, \ldots, n \) where \((u_i, t_i) \in L \times T\). A spatio-temporal disc with centre \((u, t) \in L \times T\), network radius \( r > 0 \) and
Fig 2: Original locations of points and the locations relocated on the network support.

Fig 3: **Left**: Location of the points of interest, on the network. **Right**: Values of the variable distance from the nearest touristic attraction.

temporal radius \( h > 0 \) is defined as

\[
b((u, t), r, h) = \{(v, s) \in L \times T : d_L(u, v) \leq r \quad \text{and} \quad |t - s| \leq h\},
\]

where \( |\cdot| \) is a numerical distance, and \( d_L(\cdot, \cdot) \) stands for the appropriate distance in the network. The cardinality of any subset \( A \subseteq L \times T \), \( N(X \cap A) \in 0, 1, \ldots \), is the number of points of \( X \) restricted to \( A \), whose expected number is denoted by

\[
\nu(A) = \mathbb{E}[N(X \cap A)], \quad A \subseteq L \times T,
\]

where \( \nu \), the intensity measure of \( X \), is a locally finite product measure on \( L \times T \) (Baddeley, Bárány and Schneider, 2006). Assume that \( X \) has an intensity function \( \lambda(\cdot) \) and a second-
order product density function $\lambda_2(\cdot, \cdot)$, hence,
\[
\mathbb{E}[N(X \cap A)] = \int_A \nu(d(u, t)) = \int_A \lambda(u, t)d_2(u, t), A \subseteq L \times T,
\]
where $d_2(u, t)$ corresponds to integration over $L \times T$, and
\[
\mathbb{E}[N(X \cap A)N(X \cap B)] = \int_A \int_B \lambda_2((u, t), (v, s))d_2(u, t)d_2(v, s), \quad A, B \subseteq L \times T.
\]

3.1. **Spatio-temporal Poisson processes.** The Poisson point process model is crucial for defining other more complicated models and represents a benchmark model in exploratory data analysis. For a spatio-temporal Poisson point process $X$, the points $(u, t)$ on $L \times T$ have to satisfy the following conditions:

- In any bounded set $A \subseteq L \times T$, $N(X \cap A)$ follows a Poisson distribution with expected number $\int_A \lambda(u, t)d_2(u, t)$. For instance, if we assume $A = A_L \times A_T \subseteq L \times T$, then the expected value is the expected number of points in the sub-network $A_L \subseteq L$ and within the time interval $A_T \subseteq T$.
- For any $k$ arbitrary disjoint subsets of $L \times T$, say $A_1, \ldots, A_k$, their cardinality $N(X \cap A_1), \ldots, N(X \cap A_k)$ are independent variables.

Moreover, for any time interval $S = [s_1, s_2] \subset T$, the projection of $X$ onto $L$ defines a spatial Poisson process on $L$ with intensity function $\int_{s_1}^{s_2} \lambda(u, t)dt$. Similarly, for any set of segments $A \subseteq L$, the projection of $X$ onto $T$ defines a temporal Poisson process with intensity function $\int_A \lambda(u, t)dt$ (see Illian et al., 2008, Chapter 6; Diggle, 2013, Chapter 10).

Throughout the paper we assume that the only information we have is the point locations and their corresponding time of occurrence, that is marks are neglected.

3.2. **Inhomogeneous Poisson processes.** The description of the observed point pattern intensity is a crucial issue dealing with spatio-temporal data. For linear network point patterns, non-parametric estimators of the intensity function $\lambda(\cdot, \cdot)$ have been proposed (Mateu, Moradi and Cronie, 2020), suggesting any variation of the distribution of the process over its state-space $L \times T$. When dealing with intensity estimation for spatio-temporal point processes, it is quite common to assume that the intensity function $\lambda(u, t)$ is separable (Diggle, 2013; Gabriel and Diggle, 2009). Under this assumption, the intensity function is given by the product $\lambda(u, t) = \lambda(u)\lambda(t)$ where $\lambda(u)$ and $\lambda(t)$ are non-negative functions on $L$ and $T$, respectively (Gabriel and Diggle, 2009). A kernel-based intensity estimator for spatio-temporal linear network point processes, based on the first-order separability assumption, considered in Moradi and Mateu (2020), is given by
\[
\hat{\lambda}(u, t) = \frac{\hat{\lambda}(u)\hat{\lambda}(t)}{n}, \quad (u, t) \in L \times T
\]
that is an unbiased estimator for the expected number of observed points. The spatial intensity, given the network, denoted by $\lambda(u)$, is usually estimated by non-parametric approaches mostly based on kernel-based estimator. For instance, spatial smoothing can be performed using a kernel based on the path distances in the network, using the ‘equal-split continuous’ rule described in Okabe and Sugihara (2012); computation is rapidly performed by solving the classical heat equation on the network, as described in McSwiggan, Baddeley and Nair (2017). The smoothing parameter for the kernel estimator of the temporal intensity function $\lambda(t)$ can be the usual bandwidth suggested by Silverman’s rule of thumb (Silverman, 2018). In Mateu, Moradi and Cronie (2020) a pseudo-separable estimator is proposed, accounting
for spatio-temporal interactions while relaxing the separability assumption and the specification of a metric on $L \times T$. In this paper we introduce, for the first time in the context of spatio-temporal point processes, parametric estimates of the spatial and temporal components following a dependence model with mixed effects approach. We estimate the spatial intensity through a Gibbs point process model with mixed effects, and the temporal intensity by a Poisson harmonic model. The latter can be defined as follows

$$
\log \lambda(t) = \delta + \sum_{k \in K_\kappa} \{\alpha_k \cos(k\omega t) + \beta_k \sin(k\omega t)\}
$$

where $\omega = 2\pi\kappa$, with $\kappa$ depending on the unit of analysis, e.g. it is 4 for quarterly data, 12 for monthly data, 52.25 for weekly data, and 365.25 for daily data, if considering a 1-year period. Moreover, $K_\kappa$ is the subset of time instants we use to split our observation period. In our application, we chose $\kappa = 24$, corresponding to the hours in a day, and $K_\kappa = \{1, 2, 4, 8\}$, determining temporal cycles up to 3 hours. Finally, $\delta$, $\alpha_k$ and $\beta_k$ ($\forall k \in K_\kappa$) are parameters to be estimated.

While the temporal component can be fitted without any alteration arising for the accounting of the network, the spatial component is the one needing careful attention, and should be changed in order to take into account the structure of the network. In the next section we address this problem.

4. New modelling approaches on linear networks. In this section the new modelling approaches proposed in this paper are introduced. In particular, Section 4.1 deals with a spatial point process model, that is a Gibbs point process model with mixed effects, and its estimation on a linear network is introduced. Then, Section 4.2 develops Log-Gaussian Cox processes on a linear network. Finally, in Section 4.3 the diagnostic tools used in the paper are presented.

4.1. Gibbs point process models with mixed effects. We here introduce a new modelling approach for describing the spatio-temporal behaviour of the visitors. The aim is to fit a parametric model to the points (representing the visitors’ stops) accounting for the individual choices during the day by introducing a random subject-specific effect.

At this aim we refer to Gibbs point process models with mixed effects (Illian and Hendrichsen, 2010), conforming the procedure to the linear networks context. More specifically, we focus on a pairwise term interaction process, where the conditional intensity is a log-linear model and the interaction terms are obtained as the sum of fixed and random effects, modelling the variation within each individual. The considered model reflects both the intergroup differences and interaction strength, proposing a flexible interaction function. The proposed approach refers to the application of the Berman-Turner device (Baddeley, Møller and Waagepetersen, 2000) to a generalised linear mixed model with log link and Poisson outcome for large datasets. The model is fitted using the functions of the package mgcv of R.

Let $M$ be the number of visitors on a linear network $L$, each generating the point patterns $x_1, \ldots, x_M$ that can be thought as the individual pattern of stops. This flexible procedure allows to account for the individual characteristics information both by suitable random and fixed factors, and by external covariates.

Here we only assume that visitors’ stops interact within each individual’s path and that the intensity of the pattern varies among the individuals. Then, the ID of the visitor is introduced as a random effect in the model. This resolves in assuming, for each $x_m$ with $m = 1, \ldots, M$, a pairwise interaction process (Van Lieshout, 2000), with conditional intensity (Kallenberg, 1984) given by

$$
\lambda_{\theta, \phi_m}(u; x_m) = b_{\theta, \phi_m}(u) \prod_{i=1, x_m \neq u}^{n(x_m)} h_{\theta, \phi_m}(u, x_{mi})
$$

(4.1)
where \(n(x_m)\) is the number of points in \(x_m\), that is the number of stops per visitor, and \(b_{\theta,\phi_m}(u)\) and \(h_{\theta,\phi_m}(u,v)\) are

\[
b_{\theta,\phi_m}(u) = \exp(\theta^T B_1(u) + \phi_m^T B_2(u))
\]

and

\[
h_{\theta,\phi_m}(u,v) = \exp(\theta^T H_1(u,v) + \phi_m^T H_2(u,v))
\]

i.e., two functions that model the intensity and the interaction, respectively.

Each of these functions consist of the sum of a term describing the general intensity (or interaction) and another term describing the intensity (or interaction) specific to this point, i.e., two functions that model the intensity and the interaction, respectively.

Typically, \(H\) would be a pairwise interaction function, while \(B(u) = (B_1(u), B_2(u))^T\) would be a vector of convenient scalar functions such as polynomials or orthonormal functions of coordinates. Furthermore, this model can accommodate various kind of interpoint interactions. For example, in many applications, interaction strength can be assumed to gradually decrease with distance. In situations where simple models like the Strauss process might be too simplistic and inappropriate, an interaction function which can be used to model either attraction or repulsion is preferable (Illian et al., 2008).

Therefore, Gibbs point processes with mixed effects considered in this paper have an intensity of the form

\[
\lambda_{\theta,\phi_m}(u,x_m) = \exp(\theta^T S_1(u,x_m) + \phi_m^T S_2(u,x_m))
\]

where

\[
S_k(u,x_m) = B_k(u) + \sum_{i=1,x_m,\neq u}^{n(x_m)} H_k(u,x_{mi}), \quad \text{for} \quad k = 1, 2
\]

is a vector of spatial covariates defined at each point \(u\) in \(L\). It is easy to introduce spatial trend or dependence on spatial covariates, by simply adding more terms to the linear predictor \(S\) in the associated Poisson log-linear regression model.
Finally, the conditional intensity for \( x_m \) in (4.1) becomes

\[
\lambda_{\theta, \phi_m}(u, x_m) = \exp \left( \theta^T B_1(u) + \phi_m^T B_2(u) + \sum_{i=1}^{n(x_m)} H_1(u, x_{mi}) + \phi_m^T \sum_{i=1}^{n(x_m)} H_2(u, x_{mi}) \right)
\]

and the density with respect to the homogeneous Poisson process is then

\[
f(x_m; \theta, \phi_m) = \alpha(\theta, \phi_m) \prod_{i=1}^{n(x_m)} b_{\theta, \phi_m}(x_{mi}) \prod_{i<j} h_{\theta, \phi_m}(x_{mi}, x_{mj})
\]

where \( \alpha(\theta, \phi_m) \) is a normalising constant.

Assuming independence between the patterns observed for each individual \( m \) we can derive a density of a model of the overall pattern as

\[
f(x_1, \ldots, x_M; \theta, \phi_m) = \tilde{\alpha}(\theta, \phi) \prod_{m=1}^{M} \prod_{i=1}^{n(x_m)} b_{\theta, \phi_m}(x_{mi}) \prod_{i<j} h_{\theta, \phi_m}(x_{mi}, x_{mj})
\]

where \( \phi = (\phi_1, \ldots, \phi_M) \) and \( \tilde{\alpha}(\theta, \phi) \) is a further normalising constant.

4.1.1. Model estimation through pseudolikelihood. Adapting the pseudolikelihood over the subset \( A \subseteq W \) into the network, where \( W \) is the observation window, we define the pseudolikelihood over the subset \( A \subseteq L \) as

\[
\text{PL}_A(\theta, \phi; x_1, \ldots, x_M) = \prod_{m=1}^{M} \left( \prod_{x_{mi} \in A} \lambda_{\theta, \phi_m} \exp \left( - \int_A \lambda_{\theta, \phi_m}(u; x_m) du \right) \right)
\]

that is,

\[
\text{PL}_A(\theta, \phi; x_1, \ldots, x_M) = \prod_{m=1}^{M} \left( \prod_{i=1}^{n(x_m)} b_{\theta, \phi_m}(x_{mi}) \prod_{i \neq j} h_{\theta, \phi_m}(x_{mi}, x_{mj}) \right)
\]

\[
\times \exp \left\{ - \int_A b_{\theta, \phi_m}(u) \prod_{i=1}^{n(x_m)} h_{\theta, \phi_m}(u, x_{mi}) du \right\}
\]

(4.3)

Following Baddeley, Møller and Waagepetersen (2000), we apply Berman-Turner device. Based on some quadrature rule, we approximate the integral in (4.3) for each \( m \) by a finite sum over a set of points \( u_{mj} \) with \( j = 1, \ldots, l_m \), containing all the data points. It is shown that the log-pseudolikelihood is formally equivalent to the log-likelihood of independent weighted Poisson variables, i.e.

\[
\log \text{PL}_A(\theta, \phi; x_1, \ldots, x_M) \approx \sum_{m=1}^{M} \sum_{j=1}^{l_m} y_{mj} \log \lambda_{\theta, \phi_m}(u_{mj}) - \lambda_{\theta, \phi_m}(u_{mj}) w_{mj}
\]

where \( w_{mj} \) are the quadrature weights, \( y_{mj} = \frac{z_{mj}}{w_{mj}} \) and

\[
z_{mj} = \begin{cases} 1 & \text{if } u_{mj} \in x_{m1}, \ldots, x_{mn(x_m)} \\ 0 & \text{if } u_{mj} \notin x_{m1}, \ldots, x_{mn(x_m)} \end{cases}
\]

(4.4)

In general, the procedure for the Berman-Turner device starts with generating a set of dummy points and superimposing it with the data points to form the set of quadrature points. In this paper, as we are analysing a point pattern on a linear network, the generation of
the dummy points is restricted to the considered network segments. Furthermore, as we are dealing with a marked point pattern (the ID of the visitors being the mark), the procedure is slightly modified. The first task in fitting the proposed spatial model is to create the quadrature scheme. We then replicate the dummy points for each possible mark, and generate dummy marked points at the same locations as the data points but with different marks. Finally, the two dummy patterns and the data points are superimposed, and we compute the quadrature weights and form the indicators as in (4.4).

Having generated the quadrature points, we construct the smooth function as in (4.2) by computing

\[ v_{mj} = \sum_{i=1}^{n(x_m)} H(u_{mj}, x_{mi}). \]

We finally specify the model as a log-linear Poisson regression to be fitted to the responses \( y_{mj} \) and covariate values \( v_{mj} \) with weights \( w_{mj} \). Also any spatial covariate \( Z(\cdot) \), if included in \( b_{\theta, \phi_m}(u) \), would need to be defined on each \( u_{mj} \).

Since the conditional intensity is expressed as the sum of fixed and random factors, we refer to a generalised linear mixed model with log-link and Poisson outcome. The numerical methods are designed for datasets containing upwards of several tens of thousands of data (Wood, Goude and Shaw, 2015). The advantage is much lower memory footprint, but it can also be much faster, for large datasets. An alternative fitting approach was provided by Wood et al. (2017); Li and Wood (2020), based on discretisation of covariate values and C code level parallelisation.

4.2. Log-Gaussian Cox processes. In the Euclidean context, Log-Gaussian Cox processes are the most prominent clustering models. By specifying the intensity of the process and the moments of the underlying Gaussian Random Field, it is possible to estimate both the first and second-order characteristics of the process. Following the inhomogeneous specification in Diggle (2013), a Log-Gaussian Cox process for a generic point in space and time has the intensity

\[ \Lambda(u, t) = \lambda(u, t) \exp(S(u, t)) \]

where \( S \) is a Gaussian process with \( \mathbb{E}(S(u, t)) = \mu = -0.5\sigma^2 \) and so \( \mathbb{E}(\exp S(u, t)) = 1 \) and with variance and covariance matrix \( \mathbb{C}(S(u, t), S(v, s)) = \mathbb{C}(|u - v|, |t - s|) = \sigma^2 \gamma(r, h) \), with \( \gamma(\cdot) \) the correlation function of the GRF. Following Møller, Syversveen and Waagepetersen (1998), the first-order product density and the pair correlation function of a Log-Gaussian Cox process are \( \mathbb{E}(\Lambda(u, t)) = \lambda(u, t) \) and \( g(r, h) = \exp(\sigma^2 \gamma(r, h)) \), respectively. In this paper, we consider a separable structure for the covariance function of the Gaussian Random Field (Brix and Diggle, 2001) that has exponential form for both the spatial and the temporal components,

\[ \mathbb{C}(r, h) = \sigma^2 \exp \left( \frac{-r}{\alpha} \right) \exp \left( \frac{-h}{\tau} \right), \]

where \( \sigma^2 \) is the variance, \( \alpha \) is the scale parameter for the spatial distance and \( \tau \) is the scale parameter for the temporal one. We prefer using a separable form to simplify the underlying mathematics and looking for some pragmatic strategy. In addition, the data seem to support our election here. The exponential form is widely used in this context and nicely reflects the decaying correlation structure with distance or time.

As stated by Gabriel and Diggle (2009), a single realisation of a stationary Cox process with stochastic intensity \( \Lambda(u, t) \) is indistinguishable from a realisation of an inhomogeneous
effects are separable, meaning that \( \lambda(u, t) \) can be factorised as \( \lambda(u, t) = \lambda(u) \lambda(t) \). Under this assumption, any non-separable effects are interpreted as second-order, rather than first-order. Suitable estimates of \( \lambda(u) \) and \( \lambda(t) \) depend on the characteristics of the specific application. This formulation further allows to interpret separately the temporal and spatial components, peculiar for the description of our data. Indeed, as stated throughout the paper, we are able to identify cycles in the occurrences in time, marginally with respect to the space component. Moreover, we can identify the main determinants of the spatial displacement of points, as well as assess the significance of both fixed and random effects, regardless of the temporal component.

Nevertheless, specifying a spatio-temporal model would allow to obtain predicted intensities along the whole spatio-temporal considered domain.

Since the estimation of the LGCP model for patterns on linear networks is introduced in this paper for the first time, we face the question of choosing the appropriate distance metric. In the context of analysis on networks, Cronie, Moradi and Mateu (2020) discuss the intensity reweighted moment pseudostationary (IRMPS) Log-Gaussian Cox processes.

Let \( X \) be a Log-Gaussian Cox process with random intensity measure \( \Gamma(A) = \int_A \Gamma(u)d_1u = \int_A \exp\{S(u)\}d_1u, \quad A \subseteq L, \) where \( S \) is a Gaussian Random Field on \( L \) with mean function \( \mu(u) \in \mathbb{R}, u \in L, \) and covariance function \( C(u, v) = C(d_L(u, v_1), d_L(u, v_2)) \in \mathbb{R}, \ v_1, v_2 \in L, \) for any \( u \in L \) and some function \( C. \) In Cronie, Moradi and Mateu (2020) it is proved that, if \( d_L \) is an arbitrary regular distance metric, \( C(u, v) = C(d_L(u, v_1), d_L(u, v_2)) \) is a proper covariance function on \( L \) which does not depend on the choice of \( u. \)

Baddeley et al. (2020) briefly address the issue of the choice of the regular metric to be used. Baddeley et al. (2017) and Rakshit, Nair and Baddeley (2017) investigated the statistical implications of using a distance metric other than the shortest-path distance. For any given metric \( \delta, \) a point process is defined as to be \( \delta-correlated \) if its pair correlation function depends only on \( \delta-distance. \) Baddeley et al. (2017) showed there is a rich class of point process models which are \( \delta-correlated \) with respect to Euclidean distance.

Nevertheless, since the fitting of the LCGP based on metrics different from the Euclidean distance is not yet formalized, we refer to the most classical LGCP defined in the Euclidean context. This choice is based on the assumption that the geometry of the network can be ignored, considering Euclidean distances instead of the shortest-path distances, and therefore considering the observation window \( W \) rather than the linear network \( L. \) This assumption may be relevant depending on the considered application. In our particular case, the structural differences between the Euclidean and shortest-path distances are certainly small, reinforcing the choice of the Euclidean distances when fitting a LGCP.

4.2.1. Model estimation through minimum contrast. Driven by a Gaussian Random Field, controlled in turn by a specified covariance structure, the implementation of the LGCP in practice requires a proper estimate of the conditional intensity function. The most common technique is the minimum contrast (Møller, Syversveen and Waagepetersen, 1998; Siino, Adelfio and Mateu, 2018). The procedure selects those parameters that minimise the squared discrepancy between parametric and non-parametric representations of the second-order properties of the LGCP (Davies and Hazelton, 2013). Minimum contrast estimation is used because direct likelihood-based inference for the parameters of interest is generally not possible. It is important to notice that the intuitiveness of the minimum contrast procedure is offset by the numerous subjective decisions that must be made in order to implement the criterion in practice.
Let the function \( J \) represent either the pair correlation function \( g \) of the \( K \)-function, and \( \hat{J} \) stands for the corresponding non-parametric estimate. The minimum contrast estimates \( \hat{\sigma}^2 \) and \( \hat{\alpha} \) are found minimising

\[
M_J\{\sigma^2, \alpha\} = \int_{r_0}^{r_{\text{max}}} w(r) \{ v[\hat{J}(r)] - v[J(r; \sigma^2, \alpha)] \}^2 dr
\]

where \( r_0 \) and \( r_{\text{max}} \) are the lower and upper lag limits of the contrast criterion, \( w(\cdot) \) denotes some scalar weight associated with each spatial lag \( r \), \( v[\cdot] \) represents some transformation of its argument and the approximation of \( M_J \) is obtained by summing over a fine sequence of lags \( R = \{r_0, r_1, \ldots, r_{\text{max}}\} \) equally spaced, so that \( R_{\text{diff}} = r_b - r_a, b > a \)

\[
M_J\{\sigma^2, \alpha\} \approx R_{\text{diff}}^{-1} \sum_{r \in R} w(r) \{ v[\hat{J}(r)] - v[J(r; \sigma^2, \alpha)] \}^2.
\]

Estimation of \( \tau \) is performed minimising the squared discrepancy between the covariance function and the expected frequency of observations at two points in time \( C(t, t - h; \tau) \) and its natural estimator, the empirical autocovariance function \( \hat{C}(t, t - h) \), for a finite sequence of temporal lags. The contrast criterion for the temporal part is

\[
M_C\{\tau\} = \sum_{h=1}^{h_{\text{max}}} \sum_{t=h+1}^{T} [\hat{C}(t, t - h) - C(t, t - h; \tau)]^2,
\]

for some user-specified value of \( h_{\text{max}} < T \), that it is the temporal counterpart of \( r_{\text{max}} \). Inspection of the theoretical version of the temporal covariance shows that it is dependent upon the spatial correlation parameters; there must therefore be estimated first and are subsequently plugged into the temporal minimum contrast procedure.

Universally preferable options for \( r_0, r_{\text{max}}, w(\cdot) \) and \( v[\cdot] \) do not exist. In this paper \( s_0 \) is chosen as the smallest interpoint distance in the observed data, \( r_{\text{max}} \) is set to \( \min(x_W, y_W)/4 \), as a good rule of thumb supported by Diggle (2013), where \( x_W \) and \( y_W \) represent the maximum width and height of the observation region \( W \), respectively. We choose to consider \( W \) instead of \( L \) because, as already stated, we are referring to the most classical LGCPs defined in the Euclidean context. Indeed, the choice of the upper limit \( r_{\text{max}} \) can be considered less important given that we are generally not interested in the second-order properties at large distances. Furthermore, Diggle (2013) advocates setting \( v[\cdot] = \sqrt{\cdot} \) and \( w(\cdot) = 1 \) when using the \( K \)-function and \( v[\cdot] = \log[\cdot] \) for the inhomogeneous pair correlation function. The arbitrariness of the minimum contrast procedure can be criticised, however the relative computational simplicity with respect to other estimation procedures (such as likelihood or Bayesian estimation procedures) makes this method suitable for estimating Cox process parameters (Siino, Adelfio and Mateu, 2018). In this paper we have chosen \( J(r) \) as the pair correlation function (Baddeley, Rubak and Turner, 2015).

4.3. Diagnostics. For diagnostics of the proposed model, a test for spatio-temporal clustering is here proposed. First, \( Q \) realisations from inhomogeneous spatio-temporal point processes on the linear network are computed, generated from the spatio-temporal intensity \( \lambda(u, t) \). This is done following the steps:

1. Define the generating intensity function \( \lambda_0(u, t) = \hat{\lambda}(u, t) \)
2. Set an upper bound \( \lambda_{\text{max}} \) for \( \lambda_0(u, t) \)
3. Simulate a homogeneous Poisson process with intensity \( \lambda_{\text{max}} \) and denote by \( N \) the number of generated points, with coordinates \( (u', t') \).
4. Compute \( p(u', t') = \frac{\lambda(u', t')}{\max(\lambda(u, t))} \) for each point \( (u', t') \) of an homogeneous Poisson process.
5 Generate a sample \( \mathbf{x}' \) of size \( N \) from the uniform distribution on \((0, 1)\)
6 Thin the simulated homogeneous Poisson process \( \mathbf{x} \) retaining the \( n \leq N \) locations for which \( \mathbf{x} \leq p \).

Having simulated \( Q > 1 \) realisations of spatio-temporal point patterns \( \mathbf{x}_1, \ldots, \mathbf{x}_Q \), these simulated processes can be used for computing \( Q \) inhomogeneous spatio-temporal \( K \)-functions on linear networks, as proposed in Moradi and Mateu (2020),

\[
(4.9) \quad \hat{K}(r, h) = \frac{1}{|L||T|} \sum_{(u, t) \in x} \sum_{(v, s) \neq (r, s)} I\{d_L(u, v) < r, |t - s| < h\} \hat{\lambda}(u, t)\hat{\lambda}(v, s)M((u, t), d_L(u, v), |t - s|),
\]

where \( \hat{\lambda}(\cdot, \cdot) \) is the estimate of the spatio-temporal intensity function previously chosen.

Given the \( Q \) inhomogeneous \( K \)-functions, their corresponding mean and covariance, denoted by \( E_K \) and \( V_K \) respectively, are computed. The test statistics is defined as

\[
T_q = \int_{r_0}^{r_{\text{max}}} \int_{h_0}^{h_{\text{max}}} \frac{\hat{K}_q(r, h) - E_K(r, h)}{\sqrt{V_K(r, h)}},
\]

one for each \( \hat{K}_q(r, h) \), obtaining \( T_1, \ldots, T_Q \). \( r_{\text{max}} \) and \( h_{\text{max}} \) are the maximum spatial and temporal distances considered for the inhomogeneous \( K \)-functions. Then, the same test statistic is computed also for the empirical point pattern, and denoted by \( T^* \). The p-value is defined as

\[
\frac{1 + \sum_{q=1}^{Q} I(T_q > T^*)}{Q + 1},
\]

where \( I(\cdot) \) is the indicator function, such that \( I(x) = 1 \) if \( x \) is true, basically counting how many times the \( K \)-functions computed over the simulated processes are higher than the one computed on the original point process. Then, if the obtained p-value is smaller than a significance level \( \alpha \) there is evidence against the null hypothesis, that is, the analysed spatio-temporal point process still presents some clustering behaviour.

The resulting inhomogeneous \( K \)-functions can also be used to obtain upper and lower envelopes at a chosen significance level \( \alpha \), and so to visually assess the possible residual clustered structure of the analysed point pattern, unexplained by the proposed model. In particular, if the estimated \( K \)-function lays above the obtained envelopes, then there is still a clustering behaviour of points that is not completely described by the proposed model. Furthermore, by a visual assessment of the results we can also get indications of possible ranges needing more complex models, able to explain and take into account the residual spatio-temporal dependence of the data.

Dealing with Log-Gaussian Cox processes, the above diagnostics method has been slightly modified. Indeed, provided the covariance parameters, diagnostics can be carried out following the same procedure outlined above, but simulating point patterns from the intensity function of the fitted LGCP. Therefore, the generation of a point process from a spatio-temporal Log-Gaussian Cox process on a linear network is proposed and carried out, obtaining the algorithm discussed above and replacing step 1 by adding the two following additional steps:

1.a Generate a realisation from a Gaussian Random Field \( S(u, t) \), with covariance function \( \mathbb{C}((u, t), (v, s)) \) and mean function \( \mu(u, t) \).
1.b Define the generating intensity function \( \lambda_0(u, t) = \hat{\lambda}(u, t) \exp(S(u, t)) \)

The algorithm provides a point pattern with \( n \) points.

The Gaussian Random Field is generated using the function \( \text{RFSim} \) of the package \texttt{CompRandFld} (Padoan and Bevilacqua, 2015). It is important to highlight that the choice
of $\lambda_{\text{max}}$ influences the number of points generated from the homogeneous Poisson process at step 3, before the thinning procedure. The simulation of the GRF can be computationally demanding, therefore it is suggested to set this value in order to limit the number of generated points.

5. Data analysis.

5.1. Exploratory analysis. Figure 4 shows the spatial and temporal displacements of visitors’ stops. The point pattern in Figure 4 (left) displays some spatial inhomogeneity, reflecting the spatial distribution of the considered points of interest. Two clusters are evident in the bottom left and top left areas of the analysed window, corresponding to the two well-known touristic attractions, Quattro Canti and Cathedral. On the right panel of Figure 4, the cumulative number of visitors stopping at given hours of the day is shown, noting two peaks in the temporal distributions of visitors’ stops, corresponding to 10 a.m. and 2 p.m.

For assessing if the spatial intensity $\lambda(u)$ might depend on a given spatial covariate $Z(u)$, Berman’s test is carried out over each covariate. In particular, we focus on the possible dependence of the spatial intensity on the two spatial coordinates, latitude and longitude, and on the variable distance from the nearest touristic attraction. The corresponding p-values are reported in Table 1. The effect of explanatory variables on the intensity of the process can be also investigated using non-parametric curve estimation techniques. Although the techniques were developed for spatial point patterns in two dimensions, they depend only on the space of values of $Z$, so they apply to point patterns in any space (Baddeley et al., 2012), in particular to networks (Baddeley, Rubak and Turner, 2015). So, to get further information on the type of dependence of the spatial intensity on the given continuous spatial covariates, the smoothed functions are computed as $f(u) = \rho(Z(u))$, and displayed in Figure 5.

From Table 1 we see that only the distance from the nearest touristic attraction influences the spatial intensity of the process. However, by looking at Figure 5, we see that the effect of the spatial coordinates varies as a function of the scale and, therefore, in a regression model for the spatial intensity, this variable could be included non-parametrically. The distance from the nearest touristic attraction seems to have an effect on the intensity of the process only up
TABLE 1

Berman’s Test for the available continuous spatial covariates

<table>
<thead>
<tr>
<th>Spatial covariate</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Latitude</td>
<td>0.9814</td>
</tr>
<tr>
<td>Longitude</td>
<td>0.9757</td>
</tr>
<tr>
<td>Distance from the nearest touristic attraction</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Fig 5: Smoothed functions linking Latitude (a), Longitude (b) and the distance from the nearest touristic attraction (c) to the spatial intensity.

to a certain distance, about 150 meters. For these small distances, the effect is negative, that is to say, as expected, the closer the tourists to an attraction point, the more likely they are to stop.

5.2. Intensity estimation.

Spatial intensity estimation. We now fit the purely spatial model as introduced in Section 4.1. We set \( B_1(u_{im}) = I \), with \( I \) the identity function and \( B_3(u_{im}) = Z(u_{im}) \). In ad-
| Parameter | Estimate | Std. Error | z value | Pr(>|z|) |
|-----------|----------|------------|---------|----------|
| $\theta_1$ (Intercept) | -8.02371 | 0.616119 | -13.022 | 0.000 *** |
| $\theta_2$ (v) | 0.092422 | 0.020467 | 4.516 | 0.000 *** |
| $\theta_3$ (d) | -0.012348 | 0.003082 | -4.006 | 0.000 *** |

<table>
<thead>
<tr>
<th>Term</th>
<th>edf</th>
<th>Ref.df</th>
<th>Chi.sq</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(u_{im}, v_{im})$ (id)</td>
<td>23.03</td>
<td>43.00</td>
<td>71.06</td>
<td>0.000 ***</td>
</tr>
<tr>
<td>$B_4(u_{im})$ (lat,long)</td>
<td>16.92</td>
<td>21.07</td>
<td>66.00</td>
<td>0.000 ***</td>
</tr>
</tbody>
</table>

Fig 6: Data and dummy points.

In addition, $B_2(u_{im})$ denotes the ID of the tourist, included as a random effect. $B_4(u_{im})$ is a non-parametric function for $u_{im} \in L$, estimated here through thin plate regression splines with 29 knots.

Therefore, we have

$$b_{\theta,\phi_m}(u_{im}) = \exp(\theta_1 + \phi_1mB_2(u_{im}) + \theta_3B_3(u_{im}) + B_4(u_{im}))$$
and
\[ h_{\theta,\phi}(u_{im}, v_{im}) = \exp(\theta H(u_{im}, v_{im}) + \phi H(u_{im}, v_{im})) \]
where \( H(\cdot, \cdot) \) is defined as in Equation (4.2). In this application the interaction radius is set to \( R = 100 \) meters, since it is reasonable to assume that visitors’ stops interact with each other up to this distance. The inclusion of this smooth function in the model allows attractive or repulsive interaction indicated by positive or negative interaction parameters, respectively.

Thus, our model for the spatial intensity is given by
\[ \log \lambda_{\theta,\phi}(u_{im}) = \theta_1 + \phi_{1m} B_2(u_{im}) + \theta_2 v_{im} + \theta_3 Z(u_{im}) + B_4(u_{im}) + \phi_{2m} v_{im} \]
where
1. \( \theta_1 \) is the common intercept
2. \( \theta_2 \) is the fixed effect of the smooth function in (4.5)
3. \( \theta_3 \) is the fixed effect of the distance from the nearest point of attraction
4. \( \phi_{1m} \) is the random effect of the ID
5. \( \phi_{2m} \) represents the random effects for the interaction smooth function

The final quadrature scheme used for model fitting consists of 159 data points and 2362 dummy points, displayed in Figure 6. This leads to a dataset of 110924 observations, that is equal to the number of data points plus the number of dummy points, all replicated for the number of marks.

Since the random effects \( \phi_{2m} \) for the interaction smooth function are not significant, the interaction among points turns out to be fixed, meaning that visitors’ stops attract each other in the same way for each visitor. The chosen model is
\[ \log \lambda_{\theta,\phi}(u_{im}) = \hat{\theta}_1 + \hat{\phi}_{1m} B_2(u_{im}) + \hat{\theta}_2 v_{im} + \hat{\theta}_3 B_3(u_{im}) + B_4(u_{im}). \]

In Table 2 we report the estimates of the fixed effects and the summary of the random effects. When \( \exp(\hat{\theta}_1) \) is multiplied by the length of the network, the model estimates 8.082637 stops for each individual, higher than the original average stops. The positive interaction parameter \( \exp(\hat{\theta}_2) = 1.096828 \) indicates that overall the visitors’ stops attract each other. Therefore, visitors tend to stop in the same spots. Furthermore, \( \exp(\hat{\theta}_3) = 0.9877283 \) indicates that moving away from any touristic attraction decreases the probability of visitor stopping. We notice that only the intensity varies among visitors, and not the interaction.

In Figure 7 we represent the fitted intensity for both data and generated dummy points. The model clearly estimates higher intensities in correspondence to the most visited areas, that are the Quattro Canti and the Cathedral (zoomed in the bottom panels of Fig. 7). Therefore, the fitted purely spatial model adequately describes the intensity of the densest regions and predicts the expected number of cases.

**Temporal intensity estimation with Poisson harmonic regression.** In the left panel of Figure 8, the total counts of stops per hour of the day is displayed. The temporal intensity \( \lambda(t) \) is estimated parametrically as a Poisson Harmonic regression. This is an appealing formulation when data displays a seasonal or cyclic variation. Thus it seems to make sense to model the mean function as being periodic with period 24 hours or less, and the obvious way to do that is to use trigonometric functions. After computing \( \omega = 2\pi/24 \), we fit the following Poisson model
\[ \log \lambda(t) = \delta + \alpha_1 \cos(\omega t) + \beta_1 \sin(\omega t) + \alpha_2 \cos(2\omega t) + \beta_2 \sin(2\omega t) + \alpha_3 \cos(4\omega t) + \beta_3 \sin(4\omega t) + \alpha_4 \cos(8\omega t) + \beta_4 \sin(8\omega t) \]
that models the number of visitors’ stops per hour as a function of sinusoidal and cosinusoidal functions of different periods, namely 24, 12, 6 and 3 hours cycles.

We compare this model with all the nested ones, removing one period at a time, following a F-tests stepwise procedure. In the right panel of Figure 8, we plot the fitted values of the four nested models, starting from the one with only a 24 hour cycle, adding each time half cycle. We chose the final model based on the results of the F-tests. Indeed, the cycles corresponding to 24, 12 and 6 hours are significant in predicting the total number of visitors’ stops during a
5.3. Spatio-temporal intensity and second-order characteristics. The spatio-temporal intensity is obtained as in (3.1) by multiplying the purely spatial and purely temporal intensities previously fitted. The resulting intensity had to be normalised, in order to make the estimator unbiased, that is, giving the expected number of points
\[
E\left[\int_{L \times T} \hat{\lambda}(\mathbf{u}, t)d_2(\mathbf{u}, t)\right] = \int_{L \times T} \lambda(\mathbf{u}, t)d_2(\mathbf{u}, t) = n.
\]
Therefore, the final intensity function is actually obtained as
\[
\hat{\lambda}(\mathbf{u}, t) = \frac{\hat{\lambda}(\mathbf{u})\hat{\lambda}(t)}{\int_{L \times T} \hat{\lambda}(\mathbf{u}, t)d_2(\mathbf{u}, t)}
\]
and it will be used for simulations for model diagnostics.

To test for spatio-temporal clustering, we simulate \(Q = 99\) inhomogeneous spatio-temporal Poisson processes with intensity \(\lambda_0(\mathbf{u}, t) = \hat{\lambda}(\mathbf{u}, t)\) and we then compute estimates \(\hat{K}_q(r, h)\). In Figure 9, the inhomogeneous \(K\)-function estimated from the data is displayed in white, and the envelopes obtained from \(Q = 99\) simulations and at a significance level of \(\alpha = 0.05\) are displayed in grey. As we note, the observed \(K\)-function does not lay within the envelopes for each spatial and temporal distances. This result, together with a borderline p-value of 0.05 of the test of clustering, suggests that the proposed model cannot completely catch the features of the observed point pattern. In particular, the presence of ranges where the observed \(K\)-function lays above the envelopes suggest that there is additional clustering induced from external causes, such as unobserved covariates. The findings in Figure 10
further show that the observed pattern of visitors’ stops cannot be described only by an inhomogeneous spatio-temporal Poisson point process (see in particular Figure 10(d)). Therefore, a LGCP model is further used for describing the residual clustered structure.

5.4. LGCP model fitting and diagnostics. Having established that an inhomogenous process is not enough to accurately describe the analysed pattern, we turn to the LGCP fitting. Of course, when specifying a model for point patterns whose spatial locations are restricted to a linear network, one wishes to take into account the geometry of the network. Therefore, a crucial issue when LGCP is fitted on the network, as anticipated in Section 4.2, is the choice of the most appropriate distance metric. To assess if there are differences between the Euclidean and the shortest-path distances computed for the point pattern under study, we visually compare their distributions, shown in Figure 11. As expected, the shortest-path distances are slightly overall larger than the Euclidean ones. Anyway, we chose to neglect these differences and consider the Euclidean distances, to sustain the Gaussian Random Field and the overall fitting of the LGCP model.

The initial values for the spatial parameters estimated through $M_J$ (4.7) are set to $\alpha_{\text{initial}} = 8$ and $\sigma^2_{\text{initial}} = \log(n)/2 = 2.53$, where $n$ is the number of points in the point pattern. As far as the temporal interval in $M_C$ (4.8), this is set equal to $[0, 10]$. The resulting estimates of the parameters are $(\hat{\alpha}, \hat{\sigma}^2, \hat{\tau}) = (5.70, 5.17, 0.51)$, obtained using the pair correlation function as a non-parametric estimate of $J$ in $M_J$ (4.7). To evaluate these estimates, in Figure 12 we plot the empirical covariances for the purely spatial and temporal components, together with

![Figure 9](image_url)
Fig 10: Top panels: [a] Values of the observed $K$-function, [b] corresponding values under Poisson: the brighter the colour the higher the values. Bottom panels: [c] $\hat{K}(r,h) - rh$ values and [d] a close-up to the comparison between $\hat{K}(r,h) - rh$ and tolerance envelopes indicating spatio-temporal clustering. The darkest colour identifies the regions in $(r,h)$ where $\hat{K}(r,h) - rh$ is greater than $\hat{K}_{0.95}(r,h)$. The bottom right panel [d] represents detail up to half of the spatial and temporal ranges.

their exponential estimates. These represent the best estimates since the minimum contrast estimation method actually selects the covariance parameters as the ones minimising the area between the empirical covariance (straight line) and the theoretical one (dashed line). While the exponential spatial estimate seems to adequately fit the empirical one, discrepancies in the temporal counterpart are due to the cyclic behaviour of the temporal intensity, that is reflected on the empirical autocovariance function estimating also negative values. Furthermore, in Figure 13 we represent empirical and theoretical spatial covariance functions at different time lags. We have chosen to consider two time instants, namely 10 a.m and 2 p.m., that are the densest hours. For each of them, we select three temporal lags, namely 1, 2 and 3, and...
compute the spatial covariances. It is evident that the spatial covariance strongly depends on time. In particular, for the morning hours, the model seems to achieve a better fit as the temporal lag increases; conversely for the afternoon hours. For an interpretational point of view, this is a reasonable result, indicating, as expected, that the spatial covariance is higher in the afternoon when visitors are less dispersed and tend to return to their gathering site. From a theoretical point of view, this confirms our hypothesis, that the phenomenon under study could not be explained only by a purely separable first-order intensity function $\lambda(u, t)$ that does not take into account the spatio-temporal interactions, as $S(u, t)$ does.

As for the inhomogeneous Poisson case, diagnostics of the fitted LGCP is performed, considering the modified algorithm in Section 4.3, choosing a small value $\lambda_{max} = 0.001$ for the simulation of the spatio-temporal point patterns. In Figure 14, the inhomogeneous
Fig 13: Empirical and theoretical spatial covariance functions at different time lags.

The $K$-function estimated from the data is displayed in white, and the envelopes, coming from $Q = 99$ simulations of inhomogeneous LGCPs and at a significance level of $\alpha = 0.05$, are displayed in grey. We now see that the envelopes of the processes simulated from the fitted LGCP model include the empirical $K$-function, indicating a much better fit compared to that in Figure 9. This, together with a p-value of 0.19, indicates an improved fitting, and supports the LGCP model as a good one to explain the spatio-temporal clustering of the visitors’ stops.

6. Conclusions and discussion. In this paper we describe GPS referred to tourists visiting the downtown of Palermo, one of the most touristic cities in Italy, with its many touristic attractions, belonging to the UNESCO-World Heritage listing. Assuming that the visitors’ stops may represent a spatio-temporal point pattern, we have estimated a spatio-temporal first-order intensity on the network represented by the streets of the downtown of Palermo. According to this approach, we describe the phenomenon under study taking into account both the support of the location displacements of points and the temporal component. Our paper contributes to the framework of spatio-temporal point processes on linear networks in different aspects.

First, we have described visitors’ stops proposing new parametric approaches in the context of spatio-temporal point processes on the network, through a parametric spatio-temporal separable intensity function. For the purely spatial intensity, we have accounted for the characteristics of each individual considering a Gibbs process with mixed effects. We have chosen the ID of each visitor as a mark and we have fitted random effects corresponding to each individual. As the model is a pairwise interaction process, we have included a smooth interaction function depending on the shortest-path distance between pair of points, to identify the degree of interaction among points. Furthermore, we have also included in the point process description the effect of spatial covariates, e.g. the distance from the nearest point of touristic attraction. The results of the proposed analysis suggest that the random effects for the
Fig 14: In white: Estimated inhomogeneous $K$-function for visitors’ stops data. In grey: envelopes based on 99 simulations from inhomogeneous LGCP spatio-temporal point processes at a significance level of 0.05.

individuals have a significant variance, meaning that the probability of stopping at different spots in the city center depends on the subject-specific individual characteristics. Fixing the interaction radius to 100 meters, we find that the interaction among points is positive, meaning that visitors tend to stop in the same locations. Finally, the effect of the distance from the nearest touristic attraction is significant and negative, that is to say, as expected, people tend to stop closer to well-known touristic attractions. For the temporal component, we have found a cyclic behaviour during the day, with periodic peaks up to 6 hours.

After having estimated the separable spatio-temporal intensity we also assess the overall goodness-of-fit of the spatio-temporal model finding some residual clustered structure, not accounted by the inhomogeneous model. In particular, we have identified the small spatial and temporal ranges for which the points of the inhomogeneous point pattern still shows some clustering degree, and this has led us to consider a more complex model, a LGCP, driven by an underlying log Gaussian random field (log GRF), allowing the representation of point aggregation depending on the properties of the GRF. In detail, estimated the covariance parameters, we have found that the LGCP model achieves a better fitting to the analysed spatio-temporal point patterns, and correctly captures all the residual clustered structure.

A crucial issue when considering a LGCP on the network is the choice of the most appropriate distance metric. In this work, given the small difference between the observed Euclidean and shortest-path distances, we fit the LGCP model by minimum contrast, based on the intensity estimated on the whole plane, and restricting the results only on the network. Future works could deal with the development of methods able to fit LGCPs on networks.
Indeed, even if the estimation of the covariance parameters by minimum contrast can be easily adapted to linear networks considering inhomogeneous versions of the pair correlation function and the $K$-function on the networks, the generation of the Gaussian Random Field on the network remains an open question. Our approach results in the possibility of both improving the fitting to the data, as well as the possibility of identifying the main determinants of the spatial displacement of points. The main contribution of the latter to our application is to give a better knowledge of the determinants of spatial intensity of cruise passengers’ stop locations during their visit in urban contexts, that may orient destination management policy. Indeed, by knowing in advance (based on predictions) the behaviour of the visitor’s stops in terms of their spatial arrangements and temporal counts, policy makers and police can anticipate these movements of visitors throughout the city.

Finally, the proposed modelling strategies, i.e. the purely spatial Gibbs point process model and Log-Gaussian Cox processes fitted on the network, could be useful in analyzing different data settings and problems describing any phenomenon with a complex spatio-temporal dependence occurring on a linear network. Indeed, our methodology can be easily adapted to analysing crime data or public health diseases, as examples of phenomena evolving in space and time.

**Acknowledgements.** The authors want to thank Prof. Stefano De Cantis, Dr. Mauro Ferrante and Prof. Noam Shoval for data availability.

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