Hausdorff dimension of the uniform measure of Galton–Watson trees without the $X\log X$ condition

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Summary. We consider a Galton–Watson tree with offspring distribution $\nu$ of finite mean. The uniform measure on the boundary of the tree is obtained by putting mass 1 on each vertex of the $n$-th generation and taking the limit $n \to \infty$. In the case $E[\nu \log(\nu)] < \infty$, this measure has been well studied, and it is known that the Hausdorff dimension of the measure is equal to $\log(m)$ ([3], [14]). When $E[\nu \log(\nu)] = \infty$, we show that the dimension drops to 0. This answers a question of Lyons, Pemantle and Peres [15].

Résumé. Nous considérons un arbre de Galton–Watson dont le nombre d’enfants $\nu$ a une moyenne finie. La mesure uniforme sur la frontière de l’arbre s’obtient en chargeant chaque sommet de la $n$-ième génération avec une masse 1, puis en prenant la limite $n \to \infty$. Dans le cas $E[\nu \log(\nu)] < \infty$, cette mesure est bien étudiée, et l’on sait que la dimension de Hausdorff de la mesure est égale à $\log(m)$ ([3], [14]). Lorsque $E[\nu \log(\nu)] = \infty$, nous montrons que la dimension est 0. Cela répond à une question posée par Lyons, Pemantle et Peres [15].

Keywords: Galton–Watson tree, Hausdorff dimension.

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1 Introduction

Let $T$ be a Galton–Watson tree of root $e$, associated to the offspring distribution $q := (q_k, k \geq 0)$. We denote by $GW$ the distribution of $T$ on the space of rooted trees, and $\nu$ a generic random variable on $\mathbb{N}$ with distribution $q$. We suppose that $m := \sum_{k \geq 0} kq_k \in (1, \infty)$: the tree has a positive probability of survival, denoted by $\rho$. We let $GW^*$ be the Galton–Watson

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measure conditionally on $T$ being infinite. For any vertex $u$, we write $|u|$ for the height of vertex $u$ ($|e| = 0$), $\nu(u)$ for the number of children of $u$, and $Z_n$ is the population at height $n$. We define $\partial T$ as the set of all infinite self-avoiding paths of $T$ starting from the root and we define a metric on $\partial T$ by $d(r, r') := e^{-|r \wedge r'|}$ where $r \wedge r'$ is the highest vertex belonging to $r$ and $r'$. The space $\partial T$ is called boundary of the tree, and elements of $\partial T$ are called rays. The metric space $(\partial T, d)$ is a random compact ultra-metric space.

When $E[\nu \log(\nu)] < \infty$ (with $0 \log(0) := 0$), it is well-known that the martingale $m^{-n} Z_n$ converges in $L^1$ and almost surely to a limit which is positive GW*-a.s. ([6]). Seneta [17] and Heyde [4] proved that in the general case (i.e allowing $E[\nu \log(\nu)]$ to be infinite), there exist constants $(c_n)_{n \geq 0}$ such that

(a) $W_\infty := \lim_{n \to \infty} \frac{Z_n}{c_n}$ exists a.s.

(b) $W_\infty > 0$ GW*-a.s.

(c) $\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = m$.

In particular, for each vertex $u \in T$, if $Z_k(u)$ stands for the number of descendants $v$ of $u$ such that $|v| = |u| + k$, we can define

$$W_\infty(u) := \lim_{k \to \infty} \frac{Z_k(u)}{c_k}$$

and we notice that $m^{-n} \sum_{|v|=n} W_\infty(u) = W_\infty(e)$. Notice that the variables $W_\infty(u)$ depend on the choice of the constants $(c_n)_n$. We refer to Section 2 for our choice of the constants $(c_n)_n$.

**Definition.** On the event $\partial T \neq \emptyset$, the uniform measure (also called branching measure) is the unique Borel measure on $\partial T$ such that

$$\text{UNIF}(\{r \in \partial T, r_n = u}\}) := \frac{m^{-n} W_\infty(u)}{W_\infty(e)}$$

for any integer $n$ and any vertex $u$ of height $n$.

We observe that, for any vertex $u$ of height $n$,

$$\text{UNIF}(\{r \in \partial T, r_n = u\}) = \lim_{k \to \infty} \frac{Z_k(u)}{Z_{n+k}}.$$
In particular, it does not depend on the particular choice of the constants \((c_n)_n\). The uniform measure can be seen informally as the probability distribution of a ray taken uniformly in the boundary. This paper is interested in the Hausdorff dimension of \(\text{UNIF} \), defined by

\[
\dim(\text{UNIF}) := \min\{\dim(E), \text{UNIF}(E) = 1\}
\]

where the minimum is taken over all subsets \(E \subset \partial T\) of \(\text{UNIF}\)-measure 1 and \(\dim(E)\) is the Hausdorff dimension of set \(E\). The case \(E[\nu \log(\nu)] < \infty\) has been well studied. In the seminal but flawed paper [5], then in [3] and in [14] for a simple proof, it is shown that \(\dim(\text{UNIF}) = \log(m)\) \(GW^*\)-almost surely. Exact Hausdorff and packing measures are given in [7],[8],[19],[20],[2]. A description of the multifractal spectrum is available in [9],[16],[18]. The case \(E[\nu \log(\nu)] = \infty\) presented as Question 3.1 in [15] was left open. This case is proved to display an extreme behaviour.

**Theorem 1.1.** If \(E[\nu \log(\nu)] = \infty\), then \(\dim(\text{UNIF}) = 0\) for \(GW^*-a.e\) tree \(T\).

It is known (see [11]) that the Hausdorff dimension of \(\partial T\) is \(\log m\) as soon as \(m \in (1, \infty)\). Therefore the theorem tells us that the uniform measure is concentrated on a thin part of the boundary. Indeed we will see that for \(\text{UNIF}\)-a.e. ray \(r\), the number of children of \(r_n\) is greater than \((m - o(1))^n\) for infinitely many \(n\), and it is easy to see that the Hausdorff dimension of the set of such rays is zero. The proof relies on the construction of a particular measure \(Q\), under which the distribution of the number of children of a uniformly chosen ray is more tractable. Using a size-biased measure to gain information on the uniform ray already appears in the paper of Duquesne [2], see Lemma 3.1 there.

Section 2 contains the description of the new measure in terms of a spine decomposition. Then we prove Theorem 1.1 in Section 3. Section 4 contains some open questions.

## 2 A spine decomposition

For \(k \geq 1\) and \(s \in (0,1)\), we call \(\phi_k(s)\) the probability generating function of \(Z_k\)

\[
\phi_k(s) := E[s^{Z_k}].
\]

We denote by \(\phi_k^{-1}(s)\) the inverse map on \((q_0,1)\). Recall that \(\rho\) is the survival probability hence the unique solution in \([0,1)\) of \(\phi(s) = s\). Let \(s \in (\rho,1)\). Then \(M_n := \phi_n^{-1}(s)Z_n\) defines
a martingale and converges in $L^1$ to some $M_\infty > 0$ a.s ([4]). Therefore we can take for the Seneta-Heyde norming in (a)
\[ c_n := \frac{-1}{\log(\phi_n^{-1}(s))} \]
which we will do from now on. Hence we can rewrite equivalently $M_n = e^{-Z_n/c_n}$ and $M_\infty = e^{-W_\infty(c)}$. Notice that $M_n$ depends on our choice of $s$. In [10], Lynch introduces the so-called derivative martingale (because it is the martingale obtained by differentiating with respect to $s$)
\[ D_n := e^{1/c_n} \frac{Z_n}{\phi_n'(\phi_n^{-1}(s))} M_n. \]
(2.1)
The fact that it is a martingale is straightforward by differentiation. It is nonnegative hence converges almost surely to some $D_\infty$. Let us follow the argument of Lynch (restricted to our case $E[\nu] < \infty$) to give some properties of this martingale. Since $(M_n)_n$ is a martingale, $(-M_n \log(M_n))_n$ defines a (nonnegative) supermartingale. Therefore its expectation is nonincreasing. Notice that
\[ -M_n \log(M_n) = \phi_n'(\phi_n^{-1}(s)) \frac{1}{c_n} e^{-1/c_n} D_n \]
Taking the expectation implies that $\phi_n'(\phi_n^{-1}(s)) \frac{1}{c_n} e^{-1/c_n}$ is nonincreasing hence converges. We show now that the limit is nonzero. In view of (2.1), since $D_n$ and $M_n$ converge almost surely (and $c_n$ goes to $+\infty$), $\frac{Z_n}{\phi_n'(\phi_n^{-1}(s))}$ converges almost surely. Since $Z_n/c_n$ converges almost surely to a nondegenerate random variable, the limit of $\phi_n'(\phi_n^{-1}(s))/c_n$ is nonzero indeed. From (2.2), we see that $D_\infty$ is positive $GW^*$-almost surely and $(D_n)_n$ is bounded. In particular, it converges in $L^1$.

We proved that $\phi_n'(\phi_n^{-1}(s))/c_n$ converges to some positive constant. It follows from (c) that
\[ \lim_{n \to \infty} \frac{\phi_{n+1}'(\phi_n^{-1}(s))}{\phi_n'(\phi_n^{-1}(s))} = m. \]

We are interested in the probability measure $Q$ on the space of rooted trees defined by
\[ \frac{dQ}{dGW} := D_\infty. \]
Let us describe this change of measure. We call a marked tree a couple $(T, r)$ where $T$ is a rooted tree and $r$ a ray of the tree $T$. Let $(\mathbb{T}, \xi)$ be a random variable in the space of all marked trees (equipped with some probability $\mathbb{P}(\cdot)$), whose distribution is given by the following rules. Conditionally on the tree up to level $k$ and on the location of the ray at level $k$, (which we denote respectively by $\mathbb{T}_k$ and $\xi_k$),
• the number of children of the vertices at generation \( k \) are independent

• the vertex \( \xi_k \) has a number \( \nu(\xi_k) \) of children such that for any \( \ell \)

\[{2.4} \quad \mathbb{P}(\nu(\xi_k) = \ell) = \tilde{q}_\ell := q_\ell \ell \exp \left( -\frac{\ell - 1}{c_{k+1}} \right) \frac{\phi_k'(\phi_k^{-1}(s))}{\phi_{k+1}'(\phi_k^{-1}(s))} \]

• the number of children of a vertex \( u \neq \xi_k \) at generation \( k \) verifies for any \( \ell \)

\[{2.5} \quad \mathbb{P}(\nu(u) = \ell) = \tilde{q}_\ell := q_\ell e^{1/c_k} \exp \left( -\frac{\ell}{c_{k+1}} \right) \]

• the vertex \( \xi_{k+1} \) is chosen uniformly among the children of \( \xi_k \)

We call the ray \( \xi \) the spine. We refer to [13], [12] for motivation on spine decompositions. In our case, we can see \( T \) as a Galton-Watson tree in varying environment and with immigration.

The fact that (2.4) and (2.5) define probabilities come from the equations (remember that by definition \( e^{-1/c_k} = \phi_k^{-1}(s) \))

\[
E \left[ (\phi_{k+1}^{-1}(s))^\nu \right] = \phi_k^{-1}(s), \quad E \left[ \nu(\phi_{k+1}^{-1}(s))^\nu-1 \right] = \frac{\phi_{k+1}'(\phi_k^{-1}(s))}{\phi_k'(\phi_k^{-1}(s))}. 
\]

When \( s = 1 \), equations (2.4) and (2.5) become \( \mathbb{P}(\nu(\xi_k) = \ell) = \ell q_\ell / m \) and \( \mathbb{P}(\nu(u) = \ell) = q_\ell \), which is the size-biased Galton–Watson tree of [13].

**Proposition 2.1.** Under \( Q \), the tree \( T \) has the distribution of \( T \). Besides, for \( \mathbb{P} \)-almost every tree \( T \), the distribution of \( \xi \) conditionally on \( T \) is the uniform measure \( \text{UNIF} \). In other words, for any nonnegative measurable function \( F \),

\[
E \left[ D_{\infty} \int_{\partial T} \text{UNIF}(dr)F(T, r) \right] = E_{\phi} \left[ F(T, \xi) \right].
\]

**Proof.** For any tree \( T \), we define \( T_n \) the tree \( T \) obtained by keeping only the \( n \) first generations. Let \( T \) be a tree. We will prove by induction that, for any integer \( n \) and any vertex \( u \) at generation \( n \),

\[{2.6} \quad \mathbb{P}(T_n = T_n, \xi_n = u) = \frac{D_n}{Z_n} \text{GW}(T_n = T_n). \]
For \( n = 0 \), it is straightforward since \( T_0 \) and \( T_n \) are reduced to the root. We suppose that this is true for \( n - 1 \), and we prove it for \( n \). Let \( \hat{u} \) denote the parent of \( u \), and, for any vertex \( v \) at height \( n - 1 \), let \( k(v) \) denote the number of children of \( v \) in the tree \( T \). We have

\[
\begin{align*}
\mathbb{P}(T_n = T_n, \xi_n = u | T_{n-1} = T_{n-1}, \xi_{n-1} = \hat{u}) &= \frac{1}{k(u)} \prod_{v=\hat{u}} \tilde{q}_k(v) \\
&= \frac{e^{1/c_n} \phi_n'(\phi_n^{-1}(s)) e^{Z_{n-1}/c_n}}{e^{1/c_{n-1}} \phi_n'(\phi_n^{-1}(s)) e^{Z_{n}/c_n}} \prod_{v=\hat{u}} q_k(v) \\
&= \frac{e^{1/c_n} \phi_n'(\phi_n^{-1}(s)) e^{Z_{n-1}/c_n}}{e^{1/c_{n-1}} \phi_n'(\phi_n^{-1}(s)) e^{Z_{n}/c_n}} \text{GW}(T_n = T_n \mid T_{n-1} = T_{n-1}).
\end{align*}
\]

We use the induction assumption to get

\[
\mathbb{P}(T_n = T_n, \xi_n = u) = e^{1/c_n} \frac{1}{\phi_n'(\phi_n^{-1}(s))} e^{-\frac{Z_n}{c_n}} \text{GW}(T_n = T_n)
\]

which proves (2.6). Summing over the \( n \)-th generation of \( T \) gives

\[
\mathbb{P}(T_n = T_n) = D_n \text{GW}(T_n = T_n) = Q(T = T_n).
\]

This computation also shows that \( \mathbb{P}(\xi_n = u \mid T_n) = 1/Z_n \) which implies that \( \xi \) is uniformly distributed on the boundary \( \partial T \). \( \square \)

### 3 Proof of Theorem 1.1

**Proposition 3.1.** Suppose that \( E[\nu \log(\nu)] = \infty \). Then we have \( \mathbb{P} \)-a.s.

\[
\limsup_{n \to \infty} \frac{1}{n} \log(\nu(\xi_n)) = \log(m).
\]

**Proof.** Let \( 1 < a < b < m \) and \( n \geq 0 \). We get from (2.4)

\[
\begin{align*}
\mathbb{P}(\nu(\xi_n) \in (a^n, b^n)) &= \frac{\phi_n'(\phi_n^{-1}(s))}{\phi_n'(\phi_n^{-1}(s))} E\left[\nu e^{-\nu^{-1}/c_{n+1}}, \nu \in (a^n, b^n)\right] \\
&\geq \frac{\phi_n'(\phi_n^{-1}(s))}{\phi_n'(\phi_n^{-1}(s))} e^{-b^n/c_n} E\left[\nu, \nu \in (a^n, b^n)\right].
\end{align*}
\]

From (c) and (2.3), we deduce that for \( n \) large enough, we have

\[
\begin{align*}
\mathbb{P}(\nu(\xi_n) \in (a^n, b^n)) \geq \frac{1}{2m} E\left[\nu, \nu \in (a^n, b^n)\right].
\end{align*}
\]
Therefore, under the condition \( E[\nu \log(\nu)] = \infty \), we have

\[
\sum_{n \geq 0} \mathbb{P}(\nu(\xi_n) \in (a^n, b^n)) = \infty.
\]  

(3.1)

We use the standard converse of the Borel-Cantelli lemma to see that \( \nu(\xi_n) > a^n \) infinitely often. Then let \( a \) go to \( m \) to prove the lower bound of the proposition. The upper bound is easy since \( \nu(\xi_m) \leq Z_{m+1} \), and we know that \( \frac{Z_n}{m^n} \) goes to 0 almost surely. \( \square \)

We turn to the proof of the theorem.

**Proof of Theorem 1.1.** By Proposition 3.1, we have

\[
\mathbb{P}
\left(\limsup_{n \to \infty} \frac{1}{n} \log(\nu(\xi_n)) = \log(m)\right) = 1.
\]

In particular, for \( \mathbb{P} \)-a.e. \( T \),

\[
\mathbb{P}
\left(\limsup_{n \to \infty} \frac{1}{n} \log(\nu(\xi_n)) = \log(m) \mid T\right) = 1.
\]

By Proposition 2.1, the distribution of \( \xi \) given \( T \) is \( \text{UNIF} \). Therefore, for \( \mathbb{P} \)-a.e. \( T \),

\[
\text{UNIF}
\left(r \in \partial T : \limsup_{n \to \infty} \frac{1}{n} \log(\nu(r_n)) = \log(m)\right) = 1.
\]  

(3.2)

Again by Proposition 2.1, the distribution of \( T \) is the one of \( \mathcal{T} \) under \( Q \). We deduce that (3.2) holds for \( Q \)-a.e. tree \( \mathcal{T} \). Since \( Q \) and \( GW^* \) are equivalent, equation (3.2) holds for \( GW^* \)-a.e. tree \( \mathcal{T} \). To finish the proof, we just need to prove that the set \( F := \{ r \in \partial \mathcal{T} : \limsup_{n \to \infty} \frac{1}{n} \log(\nu(r_n)) = \log(m)\} \) has Hausdorff dimension 0. Let \( \alpha > 0 \). For a vertex \( u \), let \( B_u \) be the set of rays \( r \) such that \( u \in r \). Then \( B_u \) is a ball in the metric space \((\partial \mathcal{T}, d)\) with diameter \( \text{diam}(B_u) = e^{-|u|} \). For any integer \( k \geq 1 \), notice that we can cover \( F \) with a countable number of balls \( B_{u_i} \), where \((u_i)_i\) are vertices of height greater than \( k \) such that \( \nu(u_i) \geq m |u_i| e^{-|u_i|\alpha/2} \). Finally, observe that

\[
\sum_{i} (\text{diam}(B_{u_i}))^\alpha = \sum_{\ell \geq 0} e^{-\ell \alpha} \sum_{i} 1_{\{|u_i| = \ell\}}.
\]

For any \( \ell \), \( Z_{\ell+1} \geq \sum_{i} 1_{\{|u_i| = \ell\}} \nu(u_i) \geq m^\ell e^{-\ell \alpha/2} \sum_{i} 1_{\{|u_i| = \ell\}} \). Therefore

\[
\sum_{i} (\text{diam}(B_{u_i}))^\alpha \leq \sum_{\ell \geq 0} e^{-\ell \alpha/2} \frac{Z_{\ell+1}}{m^\ell} < \infty.
\]

It yields that the Hausdorff dimension of \( F \) is smaller than \( \alpha \), which completes the proof by letting \( \alpha \to 0 \). \( \square \)
4 Questions

These questions concern the case $E[\nu \log(\nu)] = \infty$.

**Question 1.** What is the packing dimension of UNIF?

**Question 2.** Can we give an exact Hausdorff measure for the boundary $\partial T$? This question is answered in the case $E[\nu \log(\nu)] < \infty$ by Liu [7] and Watanabe [20] in great generality. See also Duquesne [2] for an elementary proof.

**Question 3.** We know from [11] that the Hausdorff dimension of the boundary is $\log(m)$. What would be a “natural” measure on the boundary $\partial T$ with Hausdorff dimension $\log(m)$?

We saw that the measure UNIF was putting its mass on exceptional rays. Therefore, one could thing of removing “bad” rays in order to construct a well spread out measure. This is the goal of the last question.

**Question 4.** Let $(a_n)_n$ be a sequence of integers going to $\infty$, such that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = a \in [1, m]$. Consider the Galton–Watson process in varying environment, with offspring distribution at generation $n$ given by $\nu 1_{\{\nu \leq a_n\}}$. It is associated with a uniform measure (see Theorem 3 of [1]). What is the Haudorff dimension of this measure?

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**References**


