Isoperimetric lower bounds for critical exponents for long-range percolation

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Abstract. We study independent long-range percolation on $\mathbb{Z}^d$ where the vertices $x$ and $y$ are connected with probability $1 - e^{-\beta ||x-y||^{-\alpha}}$ for $\alpha > 0$. Provided the critical exponents $\delta$ and $2 - \eta$ defined by $\delta = \lim_{n \to \infty} \frac{-\log(n)}{\log(||x||)}$ and $2 - \eta = \lim_{x \to \infty} \frac{\log(\mathbb{P}_\beta(||x||))}{\log(||x||)} + d$ exist, where $K_0$ is the cluster containing the origin, we show that

$$\delta \geq \frac{d + (\alpha \wedge 1)}{d - (\alpha \wedge 1)} \quad \text{and} \quad 2 - \eta \geq \alpha \wedge 1.$$ 

The lower bound on $\delta$ is believed to be sharp for $d = 1, \alpha \in \left[\frac{3}{4}, 1\right)$ and for $d = 2, \alpha \in \left[\frac{2}{4}, 1\right]$, whereas the lower bound on $2 - \eta$ is sharp for $d = 1, \alpha \in (0, 1)$, and for $\alpha < 1$ for $d > 1$, and is not believed to be sharp otherwise. Our main tool is a connection between the critical exponents and the isoperimetry of cubes inside $\mathbb{Z}^d$.

Résumé. Nous étudions la percolation de longue portée sur $\mathbb{Z}^d$ dont les sommets indépendants $x$ et $y$ sont connectés avec probabilité $1 - e^{-\beta ||x-y||^{-\alpha}}$ pour $\alpha > 0$. En supposant que les exposants critiques $\delta$ et $2 - \eta$ définis par $\delta = \lim_{n \to \infty} \frac{-\log(n)}{\log(||x||)}$ et $2 - \eta = \lim_{x \to \infty} \frac{\log(\mathbb{P}_\beta(||x||))}{\log(||x||)} + d$ existent, où $K_0$ est l’amas contenant l’origine, nous démontrons

$$\delta \geq \frac{d + (\alpha \wedge 1)}{d - (\alpha \wedge 1)} \quad \text{et} \quad 2 - \eta \geq \alpha \wedge 1.$$ 

La borne inférieure de $\delta$ est censée être précise pour $d = 1, \alpha \in \left(\frac{3}{4}, 1\right)$ et pour $d = 2, \alpha \in \left(\frac{2}{4}, 1\right)$. Alors que la borne inférieure de $2 - \eta$ est précise pour $d = 1, \alpha \in (0, 1)$, et pour $\alpha < 1$ pour $d > 1$, elle est probablement inexacte dans les autres cas. Notre outil principal est une relation entre les exposants critiques et l’isopérimétrie des cubes dans $\mathbb{Z}^d$.

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1. Introduction

Consider Bernoulli bond percolation on $\mathbb{Z}^d$ where we include an edge between the vertices $x, y \in \mathbb{Z}^d$ with probability $1 - e^{-\beta J(x, y)}$ and independent of all other edges. The function $J : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ is a kernel that is symmetric, i.e., $J(x, y) = J(y, x)$ for all $x, y \in \mathbb{Z}^d$. We denote the resulting probability measure by $\mathbb{P}_\beta$ and its expectation by $\mathbb{E}_\beta$. Edges that are included are also referred to as open. We are interested in the case where the kernel is also translation invariant and integrable, meaning that $J(x, y) = J(0, y - x)$ for all $x, y \in \mathbb{Z}^d$ and $\sum_{x \in \mathbb{Z}^d} J(0, x) < \infty$. The integrability condition guarantees that the resulting graph is almost surely locally finite. This procedure creates certain clusters, which are the connected components in the resulting random graph. Write $K_x$ for the cluster containing the vertex $x \in \mathbb{Z}^d$. A major question in percolation theory is the emergence of infinite clusters, for which we define the critical parameter $\beta_c$ by

$$\beta_c = \inf \{ \beta \geq 0 : \mathbb{P}_\beta (|K_0| = \infty) > 0 \}.$$ 

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A comparison with a Galton-Watson tree shows that there are no infinite clusters for \( \beta < \left( \sum_{x \in \mathbb{Z}^d} J(0, x) \right)^{-1} \), which shows \( \beta_c > 0 \). For \( d > 1 \) and \( J \neq 0 \) it is well-known that \( \beta_c < \infty \), whereas for \( d = 1 \) it is known that \( \beta_c < \infty \) in the case where \( J(x, y) \approx \|x - y\|^{-1-\alpha} \) for \( \alpha < 1 \) [19, 36], whereas \( \beta_c = \infty \) for \( \alpha > 1 \). Long-range percolation mostly deals with the case where \( J(x, y) \approx \|x - y\|^{-\alpha} \) for some \( \alpha > 0 \), where we write \( J(x, y) \approx \|x - y\|^{-\alpha} \) if the ratio between them satisfies \( \varepsilon < \frac{J(x, y)}{\|x - y\|^{-\alpha}} < \varepsilon^{-1} \) for a small enough \( \varepsilon > 0 \) and \( \|x - y\| \) large enough. In general it is expected that for \( \alpha > d \) the resulting graph looks similar to nearest-neighbor percolation, is very well connected for \( \alpha < d \), and shows a self-similar behavior for \( \alpha = d \). See [3, 4, 6, 8–10, 14] for results pointing in this direction. From the definition of \( \beta_c \) and the standard Harris coupling [25] we see that \( \mathbb{P}_\beta (|K_0| = \infty) > 0 \) for \( \beta > \beta_c \) and \( \mathbb{P}_\beta (|K_0| = \infty) = 0 \) for \( \beta < \beta_c \), but it is not clear what happens at \( \beta = \beta_c \). For \( J(x, y) \approx \|x - y\|^{-\alpha} \) with \( \alpha \in (0, d) \) and all \( \delta \in \mathbb{N}_{>0} \) the second author showed that \( \mathbb{P}_{\beta_c} (|K_0| = \infty) = 0 \) [7, Theorem 1.5], whereas for \( d = 1 \) and \( J(x, y) \approx \|x - y\|^{-2} \) it is a result by Aizenman and Newman that \( \mathbb{P}_{\beta_c} (|K_0| = \infty) > 0 \) [2]. For \( d \geq 2 \) and \( \alpha \geq d \) it is also expected that \( \mathbb{P}_{\beta_c} (|K_0| = \infty) = 0 \), but there is no full proof known at the moment. Whenever there is no infinite cluster at the critical value, it is a central question how fast the tail of the cluster at criticality \( \mathbb{P}_{\beta_c} (|K_0| \geq n) \) and the two-point function \( \mathbb{P}_{\beta_c} (0 \leftrightarrow x) \) tend to 0 as \( n \), respectively \( \|x\| \), grow. Here we write \( x \leftrightarrow y \) if there exists an open path from \( x \) to \( y \). It is conjectured that

\[
\begin{align*}
\mathbb{P}_{\beta_c} (|K_0| \geq n) &\approx n^{-1/\delta} \quad \text{as } n \to \infty, \\
\mathbb{P}_{\beta_c} (0 \leftrightarrow x) &\approx \|x\|^{-d+2-\eta} \quad \text{as } \|x\| \to \infty
\end{align*}
\]

for certain numbers \( \eta, \delta \) depending on \( d \) and \( \alpha \), but not on the precise details of the kernel \( J \). Here, we write \( f(n) \approx \sum\alpha c^{n} \) if \( f(n) = n^{\alpha + o(1)} \). Even the existence of the exponents is not clear and it is still open, whether the limits \( \lim_{n \to \infty} \frac{\log \mathbb{P}_{\beta_c} (|K_0| \geq n)}{\log(n)} \) and \( \lim_{\|x\| \to \infty} \frac{\log \mathbb{P}_{\beta_c} (0 \leftrightarrow x)}{\log(\|x\|)} \) exist. The widely accepted conjecture is that they exist. This has been for example proven for other models of percolation like two-dimensional percolation on the triangular lattice [34, 37, 38] or percolation for high enough dimension \( d \), or for small enough \( \alpha \) [24]. Recently, Hutchcroft proved the upper bounds \( \delta \leq \frac{d-\alpha}{d+\alpha} \) and \( 2-\eta \leq \alpha \) [31], improving his previous result \( \delta \leq \frac{d}{d+\alpha} \) [29] which is, to our knowledge, the first rigorous proof of a power-law decay of \( \mathbb{P}_{\beta_c} (|K_0| \geq n) \) for long-range percolation.

Our results In this paper, we give lower bounds on the exponents \( \delta \) and \( 2-\eta \). We will always assume an upper bound on the kernel \( J \) of the form \( J(x, y) \leq C_1 \|x-y\|^{-d-\alpha} \) for some constant \( C_1 < \infty \).

Theorem 1.1. Let \( \alpha \in (0, 1) \) for \( d = 1 \), respectively \( \alpha > 0 \) for \( d > 1 \). Suppose that \( J(x, y) \leq C_1 \|x-y\|^{-d-\alpha} \) and the exponent \( \delta \) defined in (1) exists. Then

\[
\delta \geq \frac{d + (\alpha \wedge 1)}{d - (\alpha \wedge 1)}.
\]
Theorem 1.1 is an immediate consequence of Proposition 2.7. It is only of interest in dimension \( d \in \{1, 2\} \) and for \( \alpha > \frac{d}{2} \), as it is known in wider generality that \( \delta \geq 2 \) [1, 20, Proposition 10.29]. For the case where \( d = 1 \) and \( \alpha \in [\frac{1}{2}, 1) \), respectively where \( d = 2 \) and \( \alpha \in [\frac{2}{3}, 1) \), our lower bound coincides with the conjectured true value of \( \delta \).

In particular, Theorem 1.1 shows that for \( d \in \{1, 2\} \) and \( \alpha > \frac{d}{2} \) the model does not exhibit the so called 'mean-field behavior'. The notion of 'mean-field behavior' is a notion that comes from physics, and roughly means that all the critical exponents are the same as in models of infinite dimension, such as Erdős-Rényi graphs (in the \( n \to \infty \) limit) or the binary tree. There are several ways of precisely defining this notion, but applied to our case all of them imply, among other things, that the exponents \( \delta \) and \( 2 - \eta \) exist and take the values \( \delta = 2 \) and \( 2 - \eta = 2 \land \alpha \). In a major breakthrough by Hara and Slade [21] mean-field behavior was established for high dimensional nearest-neighbour percolation. It was later also established for long-range percolation with \( d > 3 \) or \( \alpha < \frac{d}{2} \) [24]. The lower bounds in Theorem 1.1 rule out the mean-field behavior for \( d \in \{1, 2\} \) and \( \alpha > \frac{d}{2} \), as they imply that \( \delta > 2 \) in this regime.

**Theorem 1.2.** Let \( \alpha \in (0, 1) \) for \( d = 1 \), respectively \( \alpha > 0 \) for \( d > 1 \). Suppose that \( J(x, y) \leq C_1 \|x - y\|^{-d-\alpha} \) and the exponent \( 2 - \eta \) defined in (2) exists. Then

\[
2 - \eta \geq \alpha \land 1.
\]

Theorem 1.2 is an immediate consequence of Proposition 2.6. In the case where \( J(x, y) \simeq \|x - y\|^{-d-\alpha} \), Theorem 1.2 shows together with Hutchcroft’s result [31] that \( 2 - \eta = \alpha \) for \( \alpha \leq 1 \), respectively \( \alpha < 1 \) for \( d = 1 \), provided the exponent \( 2 - \eta \) defined in (2) exists. This also gives a partial solution to [31, Problem 4.1], which asks for conditions under which the upper bound \( 2 - \eta \leq \alpha \) has a matching lower bound. Provided that the conjectured picture described in (3) below holds, our proof also shows that the crossover value \( \delta_c(d) \) defined in (3) below satisfies \( \delta_c(d) \geq 1 \) for all dimensions \( d \geq 2 \). We could alternatively define the exponent \( 2 - \eta \) by \( \sum_{x \in \Lambda_n} \mathbb{P}_{\beta_c}(0 \leftrightarrow x) \approx n^{2-\eta} \). For \( \alpha < 1 \) the results of [31] together with Proposition 2.6 show that the exponent \( 2 - \eta \) defined like this exists and equals \( \alpha \). See also the discussion after Proposition 2.6 for more details.

Our proofs only assume an upper bound on the kernel \( J \), so in particular the results are still valid for nearest-neighbor percolation. However, the bound \( 2 - \eta \geq 1 \) observed in this situation already follows from the proof of sharpness of the phase transition of Duminil-Copin and Tassion [11], and the lower bound \( \delta \geq 3 \) observed for \( d = 2 \) follows from \( 2 - \eta \geq 1 \) and the hyperscaling inequality \((2-\eta)(\delta+1) \leq d(\delta-1)\) proven by Hutchcroft [29]. This hyperscaling inequality can be rearranged to \( \delta \geq \frac{2+2-\eta}{2-\eta} \) and using \( d = 2, 2 - \eta \geq 1 \) shows \( \delta \geq 3 \). But our proof still shows \( \delta \geq 3 \) without this machinery and without assuming the existence of the exponent \( 2 - \eta \). Our main tool for the proofs of Theorem 1.1 and Theorem 1.2 (respectively Proposition 2.7 and Proposition 2.6) is a connection between the critical exponents and the isoperimetry of the boxes \( \Lambda_n = \{-n, \ldots, n\}^d \) in section 2.2.

**Related work** The critical behavior of percolating systems is typically a difficult problem. There has been considerable progress on the understanding of percolation on various graphs at and near criticality over the last years, see for example [13, 15–18, 22, 23, 26–31, 35]. The physics prediction for the critical exponent \( 2 - \eta \) is given by

\[
2 - \eta(d, \alpha) = \begin{cases} 
\alpha & \text{for } \alpha \leq 2 - \eta_{SR}(d), \\
2 - \eta_{SR} & \text{for } \alpha > 2 - \eta_{SR}(d),
\end{cases}
\]

where \( 2 - \eta_{SR}(d) \) is the corresponding exponent for short-range percolation on \( \mathbb{Z}^d \). The prediction for the exponent \( \delta \) is given by

\[
\delta(d, \alpha) = \begin{cases} 
\frac{2}{d} & \text{for } \alpha \leq \frac{d}{2}, \\
\frac{d+2-\eta}{d-\alpha} & \text{for } \alpha \in [\frac{d}{2}, \delta_{SR}(d)], \\
\delta_{SR}(d) & \text{for } \alpha \geq \delta_{SR}(d)
\end{cases}
\]

where \( \delta_{SR}(d) \) is the corresponding exponent for short-range percolation and \( \delta_{SR}(d) \) and the crossover value \( \delta_c(d) \) are such that the function \( \delta(d, \alpha) \) is continuous in \( \alpha \). See also [29, section 1.3] or [20, section 9 and 10] for a broader overview of these predictions and references to the physics literature. The critical exponents are typically better understood in high dimension or for \( \alpha < \frac{d}{2} \), where the triangle condition holds and methods involving the lace expansion can be used [5, 11, 12, 21, 24]. Also for dimension \( d = 2 \), and in particular for the triangular lattice, the situation is much better understood, due to works of Kesten, Smirnov and Werner [32, 34, 37, 38]. Here one knows that \( \delta_{SR}(2) = \frac{91}{18} \). This also explains the conjectured pictures in Figure 2 and shows that the crossover value \( \delta_{SR}(2) \) is expected to be \( \frac{44}{9} \). Also for the
Fig 2: The critical exponents $2 - \eta$ and $\delta$ for $d = 2$. On the left: The blue line is our lower bound, the yellow line is the upper bound proven in [31], and the red line is the conjectured true value. The part where all three of them agree ($\alpha \in (0, 1]$) is purple and the part where the upper bound and the conjectured true value agree ($\alpha \in (1, \frac{3}{24}]$) is orange. On the right: The yellow curve is the upper bound on $\delta$ shown in [31], the red curve is the conjectured true value of $\delta$, and the blue curve is our lower bound. The part where the lower bound and the conjectured true value agree ($\alpha \in \left[\frac{2}{3}, 1\right]$) is purple.

hierarchical lattice the phase transition is better understood, due to recent results of Hutchcroft [30]. The lower bound $\delta \geq \frac{d + \alpha}{d - \alpha}$ proven for the hierarchical lattice is similar to our lower bound for $d = 1$ and also shows absence of mean-field behavior for $\alpha > \frac{d}{4}$ on the hierarchical lattice.

2. Proofs

Before going to the proofs, we want to introduce a theorem that deals with the universal tightness of the maximum open cluster inside a random graph. It is a subset of [29, Theorem 2.2], which turned out to be extremely useful in various models of random graphs. We write $|K_{\max}(\Lambda)|$ for the cardinality of the largest open cluster in $\Lambda$. Note that $K_{\max}(\Lambda)$ is in general not well-defined as a subset of $\Lambda$, since there can be distinct clusters with the same cardinality. But this will not cause any problems in the following. We define the typical value of $|K_{\max}(\Lambda)|$ by

$$M_\beta(\Lambda) = \min \left\{ n \geq 0 : \mathbb{P}_\beta (|K_{\max}(\Lambda)| \geq n) \leq e^{-1} \right\}. \quad (4)$$

The theorem deals with general weighted graphs $G = (V, E, J)$, where $J : E \to [0, \infty)$ is a function that gives weights to the edges. Now edges are open or closed independent of each other and an edge $e \in E$ is open with probability $1 - e^{-\beta J(e)}$, where $\beta \geq 0$ is a parameter. In particular, long-range percolation on the integer lattice can be modelled as a weighted random graph with the weight function $J(\{x, y\}) = J(x - y)$.

**Theorem 2.1** (Universal tightness of the maximum cluster size). *Let $G = (V, E, J)$ be a countable weighted graph and let $\Lambda \subseteq V$ be finite and non-empty. Then the inequalities*

$$\mathbb{P}_\beta (|K_{\max}(\Lambda)| \geq \alpha M_\beta(\Lambda)) \leq e^{-\frac{\alpha}{\theta}} \quad (5)$$

and

$$\mathbb{P}_\beta (|K_u \cap \Lambda| \geq \alpha M_\beta(\Lambda)) \leq e \cdot \mathbb{P}_\beta (|K_u \cap \Lambda| \geq M_\beta(\Lambda)) e^{-\frac{\alpha}{\theta}} \quad (6)$$

*hold for every $\beta \geq 0, \alpha \geq 1, and u \in V$.*

We will use this theorem at many points in this paper. For the lower bound on $\delta$ we define $\theta := \frac{1}{4}$. In the following we will always assume that

$$\sum_{k=1}^{n} \mathbb{P}_\beta (|K_0| \geq k) \leq C n^{1-\theta} \quad (7)$$
holds for some constant $C < \infty$. Note that this already implies that $\mathbb{P}_\beta (|K_0| \geq n) \leq n^{-1} \sum_{k=1}^n \mathbb{P}_\beta (|K_0| \geq k) \leq C n^{-\theta}$. Furthermore, for $\theta < 1$ the bound $\mathbb{P}_\beta (|K_0| \geq k) \leq C k^{-\theta}$ for all $k \in \{1, \ldots, n\}$ also implies (7) with a different constant $C'$ depending on $C$ and $\theta$.

For the lower bound on the exponent of the two-point function $2 - \eta$ we define $\Lambda_n = \{-n, \ldots, n\}^d$ and assume that

$$
\frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{P}_\beta (0 \leftrightarrow x) \leq C n^{-d+2-\eta}
$$

holds for some constant $C < \infty$. From this definition we directly see that we can always assume that $-d + 2 - \eta \leq 0$, as the statement is trivially true otherwise.

### 2.1. Moments of the cluster size inside boxes

In this section, we give bounds on the expected size of the cluster inside boxes, i.e., $E_\beta [\{ |K_0(\Lambda_n)\} ]$, given the upper bounds on the tail of the cluster (7) or the two-point function (8). For $\Lambda \subset \mathbb{Z}^d$ and $x \in \Lambda$ we use the notation $K_x(\Lambda)$ for the set of vertices $y \in \Lambda$ that are connected to $x$ through an open path that lies entirely within $\Lambda$. The next lemma translates bounds of the tail of the cluster size into bounds of the typical largest cluster inside boxes of size $n$.

**Lemma 2.2.** Assume that (7) holds for some constant $1 \leq C < \infty$. Let $\Lambda \subset \mathbb{Z}^d$ be a finite set of size $n$. Then one has

$$
M_\beta(\Lambda) \leq 3 C n^{\frac{1-\theta}{\theta}}
$$

**Proof.** For $x \in \Lambda$, let $K_x(\Lambda)$ be the cluster of $x$ inside $\Lambda$. We use the notation $\tilde{C} = 3C$ and get that

$$
E_\beta \left[ \left\{ x \in \Lambda : |K_x(\Lambda)| \geq \tilde{C} n^{\frac{1-\theta}{\theta}} \right\} \right] = \sum_{x \in \Lambda} \mathbb{P}_\beta \left( |K_x(\Lambda)| \geq \tilde{C} n^{\frac{1-\theta}{\theta}} \right)
$$

$$
\leq \sum_{x \in \Lambda} \mathbb{P}_\beta \left( |K_x| \geq \tilde{C} n^{\frac{1-\theta}{\theta}} \right) \leq \sum_{x \in \Lambda} C \tilde{C}^{-\theta} n^{-\frac{1-\theta}{\theta}} = C \tilde{C}^{-\theta} n^{-\frac{1-\theta}{\theta}} = C \tilde{C}^{-\theta} n^{\frac{1-\theta}{\theta}}.
$$

If there is one $x \in \Lambda$ such that $|K_x(\Lambda)| \geq \tilde{C} n^{\frac{1-\theta}{\theta}}$, then there are at least $\tilde{C} n^{\frac{1-\theta}{\theta}}$ many such $x \in \Lambda$. So in particular, if $|K_{\max}(\Lambda)| \geq \tilde{C} n^{\frac{1-\theta}{\theta}}$, then there are at least $\tilde{C} n^{\frac{1-\theta}{\theta}}$ many vertices $x \in \Lambda$ with $|K_x(\Lambda)| \geq \tilde{C} n^{\frac{1-\theta}{\theta}}$. This implies that

$$
\mathbb{P}_\beta \left( |K_{\max}(\Lambda)| \geq \tilde{C} n^{\frac{1-\theta}{\theta}} \right) \leq \frac{1}{\tilde{C} n^{\frac{1-\theta}{\theta}}} \left\{ x \in \Lambda : |K_x(\Lambda)| \geq \tilde{C} n^{\frac{1-\theta}{\theta}} \right\}
$$

and taking expectations on both sides yields that

$$
\mathbb{P}_\beta \left( |K_{\max}(\Lambda)| \geq \tilde{C} n^{\frac{1-\theta}{\theta}} \right) \leq \frac{1}{\tilde{C} n^{\frac{1-\theta}{\theta}}} E_\beta \left[ \left\{ x \in \Lambda : |K_x(\Lambda)| \geq \tilde{C} n^{\frac{1-\theta}{\theta}} \right\} \right]
$$

$$
\leq \frac{1}{\tilde{C} n^{\frac{1-\theta}{\theta}}} C \tilde{C}^{-\theta} n^{\frac{1-\theta}{\theta}} = C \tilde{C}^{-1-\theta} = C(3C)^{-1-\theta} < \frac{1}{3} < \frac{1}{\epsilon}
$$

which shows that $M_\beta(\Lambda) \leq 3 C n^{\frac{1-\theta}{\theta}}$.

**Lemma 2.3.** Assume that (7) holds. Let $\Lambda \subset \mathbb{Z}^d$ be a finite set of size $n$. Then there exists a constant $C_2 = C_2(C, \theta)$ such that

$$
E_\beta [||K_0(\Lambda)||] \leq C_2 n^{\frac{1-\theta}{\theta}}.
$$

**Proof.** The proof is heavily based on the use of Theorem 2.1. For abbreviation, we simply write $M = M_\beta(\Lambda)$. Thus we get that

$$
E_\beta [||K_0(\Lambda)||] = \sum_{k=1}^{\infty} \mathbb{P}_\beta (|K_0(\Lambda)| \geq k) = \sum_{l=0}^{\infty} \sum_{k=1}^{M} \mathbb{P}_\beta (|K_0(\Lambda)| \geq lM + k)
$$
= \sum_{k=1}^{M} \mathbb{P}_\beta (|K_0(\Lambda)| \geq k) + \sum_{l=1}^{\infty} \sum_{k=1}^{M} \mathbb{P}_\beta (|K_x(\Lambda)| \geq lM + k)

\leq CM^{1-\theta} + \sum_{l=1}^{\infty} \sum_{k=1}^{M} \mathbb{P}_\beta (|K_0(\Lambda)| \geq lM)

\leq CM^{1-\theta} + M \sum_{l=1}^{\infty} e \mathbb{P}_\beta (|K_0(\Lambda)| \geq M) e^{-\frac{l}{\beta}}

\leq CM^{1-\theta} + eCM^{1-\theta} \sum_{l=1}^{\infty} e^{-\frac{l}{\beta}} \leq C' M^{1-\theta} \leq C_2 n^{1-\theta}

for some constants $C', C_2 < \infty$. Here we used the result of Lemma 2.2 for the last inequality. \qed

The next Lemma translates the average bound on the two-point function (8) into bounds on the restricted cluster size.

For two sets $A, B \subset \mathbb{Z}^d$ we introduce the notation $A \xleftarrow{\Lambda_n} B$, meaning that there exists a path from $A$ to $B$ that uses edges with both endpoints in $\Lambda_n$ only.

**Lemma 2.4.** Assume that (8) holds. Then one has

$$\mathbb{E}_\beta ||K_0(\Lambda_n)|| \leq 3^d C n^{2-\eta}.$$  

for all $x \in \Lambda_n$.

**Proof.** The $\infty$-distance between different 0 and $x \in \Lambda_n$ is at most $n$. We have that $|\Lambda_n| = (2n + 1)^d$. Thus linearity of expectation gives that

$$\mathbb{E}_\beta ||K_0(\Lambda_n)|| = \sum_{x \in \Lambda_n} \mathbb{P}_\beta (0 \xleftarrow{\Lambda_n} x) \leq |\Lambda_n| \sum_{x \in \Lambda_n} \mathbb{P}_\beta (0 \leftrightarrow x)$$

$$\leq (2n + 1)^d C n^{-d+2-\eta} \leq 3^d C n^{2-\eta}.$$  

\qed

### 2.2. Isoperimetric inequalities in expectation

In this section, we use the isoperimetry of the box $\Lambda_n = \{-n, \ldots, n\}^d$ in order to bound the expected number of edges at the boundary of the box, for which the end inside the box is connected to 0. For long-range percolation with a kernel $J : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ satisfying $J(x, y) \approx |x-y|^{-d-\alpha}$ the isoperimetry of the box $\Lambda_n$ changes at $\alpha = 1$. More precisely, if we denote by $\partial \Lambda_n$ the set of open edges with exactly one endpoint in $\Lambda_n$, we have that

$$\mathbb{E}_\beta ||\partial \Lambda_n|| \approx \begin{cases} n^{d-\alpha} & \text{if } \alpha < 1 \\ n^{d-1} \log(n) & \text{if } \alpha = 1 \\ n^{d-1} & \text{if } \alpha > 1 \end{cases}.$$  

Consequently, we see that for $\alpha < 1$ long-range effects determine the isoperimetry of the box, whereas for $\alpha \geq 1$ the short-range effects dominate, with logarithmic corrections at $\alpha = 1$. In particular, a point $x \in \Lambda_n$ that is chosen uniformly at random will have of order $n^{-\alpha(1+\alpha)}$ neighbors outside of the box. This is also the reason, why the term $\alpha \land 1$ pops up in the statements of Theorem 1.1 and Theorem 1.2. In the following, for two sets $A, B \subset \mathbb{Z}^d$ we use the notation $A \sim B$ if there exists a direct edge from $A$ to $B$. We also use a statement that was shown by Duminil-Copin and Tassion in [17, 18]. There it is shown that for $\beta \geq \beta_c$ and all finite sets $S \subset \mathbb{Z}^d$ containing the origin 0 one has

$$\phi_\beta (S) := \sum_{x \in S} \sum_{y \in S \setminus \{x\}} \left(1 - e^{-\beta J(x, y)}\right) \mathbb{P}_\beta (0 \xrightarrow{S} x) \geq 1.$$  

Moreover, they also showed the reverse direction, i.e., that $\phi_\beta (S) \geq 1$ for all finite sets $S \subset \mathbb{Z}^d$ with $0 \in S$ implies $\beta \geq \beta_c$, but we will not use this statement in our proof. Similar results to the result in (11) were already shown previously, see for example [33, Lemma 3.1] or [2, Lemma 5.1].
Thus, it suffices to bound the expected value of edges with both endpoints inside $\Lambda_k$ for fixed $\Lambda$. The occupation status of edges inside $\Lambda_k$ is defined by

\[ \phi_\beta (\Lambda_k) = \sum_{x \in \Lambda_k, y \notin \Lambda_k} \left( 1 - e^{-\beta J(x,y)} \right) \mathbb{P}_\beta \left( 0 \xleftarrow{\Lambda_k} x \right) \leq C_3 \mathbb{E}_\beta \left[ |K_0(\Lambda_n)| \right] f(n, \alpha) \]

where the function $f(n, \alpha)$ is defined by

\[ f(n, \alpha) = \begin{cases} n^{-\alpha} & \text{if } \alpha < 1 \\ n^{-1} \log(n) & \text{if } \alpha = 1 \\ n^{-1} & \text{if } \alpha > 1 \end{cases} \]

Proof. For $x \in \Lambda_n$ we write $t_x := \mathbb{P}_\beta \left( x \xleftarrow{\Lambda_n} 0 \right)$ and get that

\[ \sum_{x \in \Lambda_n} t_x = \sum_{x \in \Lambda_n} \mathbb{P}_\beta \left( x \xleftarrow{\Lambda_n} 0 \right) = \mathbb{E}_\beta \left[ |K_0(\Lambda_n)| \right]. \]

Next, we define $X_k$ as the number of open edges between $\Lambda_k$ and $\Lambda_k^C$ for which one end is connected to 0 within $\Lambda_k$. Formally, we define

\[ X_k := \left| \left\{ e = \{a,b\} \text{ open : } a \in \Lambda_k, b \notin \Lambda_k, \text{ and } 0 \xleftarrow{\Lambda_k} a \right\} \right|. \]

The occupation status of edges inside $\Lambda_k$ and of edges with one end outside of $\Lambda_k$ are independent random variables. So by linearity of expectation one has

\[ \mathbb{E}_\beta [X_k] = \sum_{a \in \Lambda_k, b \notin \Lambda_k} \left( 1 - e^{-\beta J(a,b)} \right) \mathbb{P}_\beta \left( 0 \xleftarrow{\Lambda_k} a \right) = \phi_\beta (\Lambda_k). \]

Thus, it suffices to bound the expected value of $X_k$ and show that there exists a $k \in \{1, \ldots, n\}$ such that the expected value $\mathbb{E}_\beta [X_k]$ is reasonably small, as in (12). For this, let $K$ be a random variable that is uniformly distributed on $\{1, \ldots, n\}$ and is independent of the percolation configuration. We write $\mathbb{P}_\beta$ for the joint distribution of the percolation configuration and $K$, and $\mathbb{E}_\beta$ for its expectation. Thus we get

\[ \mathbb{E}_\beta [X_K] = \sum_{k=1}^{n} \mathbb{E}_\beta \left[ \left| \left\{ a,b \text{ open : } a \in \{-K, \ldots, K\}^d, b \notin \{-K, \ldots, K\}^d, \text{ and } 0 \xleftarrow{\Lambda_k} a \right\} \right| \right] \]

\[ = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}_\beta \left[ \left| \left\{ a,b \text{ open : } a \in \{-k, \ldots, k\}^d, b \notin \{-k, \ldots, k\}^d, \text{ and } 0 \xleftarrow{\Lambda_k} a \right\} \right| \right] \]

\[ = \frac{1}{n} \sum_{k=1}^{n} \sum_{a \in \Lambda_n, b \in \mathbb{Z}^d} \mathbb{E}_\beta \left[ 1_{\{a \in \Lambda_k\}} 1_{\{b \notin \Lambda_k\}} 1_{\{0 \xleftarrow{\Lambda_k} a\}} \right]. \]

For fixed $k$, the events $\{0 \xleftarrow{\Lambda_k} a\}$ and $\{a, b\text{ is open}\}$ are independent for $b \notin \Lambda_k$, as the first event depends only on edges with both endpoints inside $\Lambda_k$. For fixed $a \in \Lambda_n$, the expression $\mathbb{P}_\beta \left( 0 \xleftarrow{\Lambda_k} a \right)$ can only be positive if $k \geq \|a\|_\infty$. Combining the two previous observations we get that

\[ \mathbb{E}_\beta [X_K] = \frac{1}{n} \sum_{k=1}^{n} \sum_{a \in \Lambda_k} \sum_{b \in \mathbb{Z}^d \setminus \Lambda_k} \mathbb{P}_\beta \left( 0 \xleftarrow{\Lambda_k} a \right) \mathbb{P}_\beta (a \sim b) \]

\[ = \frac{1}{n} \sum_{a \in \Lambda_n} \sum_{k=1}^{n} \sum_{b \in \mathbb{Z}^d \setminus \Lambda_k} \mathbb{P}_\beta \left( 0 \xleftarrow{\Lambda_k} a \right) \mathbb{P}_\beta (a \sim b) \]

\[ \leq \sum_{a \in \Lambda_n} \mathbb{P}_\beta \left( 0 \xleftarrow{\Lambda} a \right) \left( \frac{1}{n} \sum_{k=1}^{n} \sum_{b \in \mathbb{Z}^d \setminus \Lambda_k} \left( 1 - e^{-\beta J(a,b)} \right) \right) \]
Using (17) we see that

\[
C_1' = C_1'(C_1, d, \beta) < \infty \text{ and } C_1'' = C_1''(C_1, d, \alpha, \beta) < \infty \text{ such that }
\]

\[
\sum_{b \in \mathbb{Z}^d \setminus \Lambda_k} \beta C_1\|a - b\|^{-d-\alpha} \leq \sum_{l=k+1-\|a\|_\infty}^{\infty} \sum_{b \in \mathbb{Z}^d : \|b-a\|_\infty = l} \beta C_1\|a - b\|^{-d-\alpha} = \sum_{l=k+1-\|a\|_\infty}^{\infty} C_1' \|b\|^{-d-\alpha} \leq \sum_{l=k+1-\|a\|_\infty}^{\infty} C_1'' \|b\|^{-d-\alpha}
\]

\[
= C_1' \sum_{l=k+1-\|a\|_\infty}^{\infty} l^{-1-\alpha} \leq C_1'' (k + 1 - \|a\|_\infty)^{-\alpha}.
\]

Using (17) we see that

\[
\frac{1}{n} \sum_{k=1}^{n} \sum_{b \in \mathbb{Z}^d \setminus \Lambda_k} \beta C_1\|a - b\|^{-d-\alpha} \leq \frac{1}{n} \sum_{k=1}^{n} C_1'' (k + 1 - \|a\|_\infty)^{-\alpha}
\]

\[
\leq C_1'' \frac{1}{n} \sum_{k=1}^{n+1} k^{-\alpha} \approx \hat{C}_1 f(n, \alpha)
\]

for a constant \( \hat{C}_1 = \hat{C}_1(C_1', \alpha) < \infty \). Inserting this result into (16) yields

\[
\frac{1}{n} \sum_{k=1}^{n} E_{B} [X_k] = E_{B} [X_K] \leq \sum_{a \in \Lambda_n} t_a \hat{C}_1 f(n, \alpha) \overset{(14)}{=} E_{B} [\|K_0(\Lambda_n)\|^2] \hat{C}_1 f(n, \alpha).
\]

So in particular there needs to exist at least one \( k \in \{1, \ldots, n\} \) for which \( E_{B} [X_k] \leq E_{B} [\|K_0(\Lambda_n)\|^2] \hat{C}_1 f(n, \alpha) \), which finishes the proof.

2.3. The proof of Theorem 1.1 and Theorem 1.2

Now we are ready to go to the main proofs. Theorem 1.1 is an immediate consequence of Proposition 2.7 and Theorem 1.2 is an immediate consequence of Proposition 2.6. Also remember the definition of the function \( f \) defined in (13) which we will use at several points below.

**Proposition 2.6.** Let \( \alpha \in (0, 1) \) for \( d = 1 \), respectively \( \alpha > 0 \) for \( d > 1 \), and assume that there exists a constant \( C_1 < \infty \) such that \( J(x, y) \leq C_1 \|x - y\|^{-d-\alpha} \) for all \( x, y \in \mathbb{Z}^d \). Provided \( \beta_c < \infty \) one has \( \sum_{x \in \Lambda_n} P_{\beta_c} (0 \leftrightarrow x) \geq \frac{1}{C_3} f(n, \alpha)^{-1} \) where \( C_3 \) is the same constant as in Lemma 2.5.

**Proof.** We will first show that \( E_{\beta_c} [\|K_0(\Lambda_n)\|^2] \geq \frac{1}{C_3} f(n, \alpha)^{-1} \). Assume the contrary, i.e., \( E_{\beta_c} [\|K_0(\Lambda_n)\|^2] < \frac{1}{C_3} f(n, \alpha)^{-1} \). Then by Lemma 2.5 there exists a \( k \in \{1, \ldots, n\} \) with

\[
\phi_{\beta_c}(\Lambda_k) \leq C_3 E_{\beta_c} [\|K_0(\Lambda_n)\|^2] f(n, \alpha) < 1
\]

which is a contradiction to (11). Now, by linearity of expectation we have that

\[
\sum_{x \in \Lambda_n} P_{\beta_c} (0 \leftrightarrow x) \geq \sum_{x \in \Lambda_n} P_{\beta_c} \left(0 \leftrightarrow x_\Lambda\right) = E_{\beta_c} [\|K_0(\Lambda_n)\|^2] \geq \frac{1}{C_3} f(n, \alpha)^{-1}.
\]

\( \square \)
Proposition 2.6 shows in particular that for a small enough constant \( c > 0 \) we have

\[
\frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{P}_{\beta_c} (0 \leftrightarrow x) \geq cn^{-d} f(n, \alpha)^{-1} = \begin{cases} 
\frac{cn^{-d+\alpha}}{\log(n)} & \text{for } \alpha < 1 \\
\frac{cn^{-d+1}}{\log(n)} & \text{for } \alpha = 1 \\
\frac{cn^{-d+1}}{} & \text{for } \alpha > 1
\end{cases}
\]

which shows that the exponent \( 2 - \eta \) defined in (2) satisfies \( 2 - \eta \geq \alpha \wedge 1 \), provided the exponent \( 2 - \eta \) exists. In [31] it is shown that \( \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{P}_{\beta_c} (0 \leftrightarrow x) = \mathcal{O}(n^{-d+\alpha}) \). Combining this with Proposition 2.6 we get that for \( \alpha < 1 \) and a kernel \( J \) satisfying \( J(x, y) \simeq \|x - y\|^{-d-\alpha} \) one has

\[
\frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{P}_{\beta_c} (0 \leftrightarrow x) \simeq n^{-d-\alpha}.
\]

So when we alternatively define the two-point critical exponent \( 2 - \eta \) by the averaged version \( \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{P}_{\beta_c} (0 \leftrightarrow x) \approx n^{-d+2-\eta} \), then we see that this exponent exists for \( \alpha < 1 \) and equals \( \alpha \). However, it is not clear whether this statement holds without averaging, i.e., if the exponent \( 2 - \eta \) defined in (2) also exists. See also [31, Problem 4.3] for a related problem. Next, we consider the lower bound on the exponent \( \delta \).

**Proposition 2.7.** Let \( \alpha \in (0, 1) \) for \( d = 1 \), respectively \( \alpha > 0 \) for \( d > 1 \), and assume that there exists a constant \( C_1 < \infty \) such that \( J(x, y) \leq C_1 \|x - y\|^{-d-\alpha} \) for all \( x, y \in \mathbb{Z}^d \). Suppose that \( \beta_c < \infty \) and \( \sum_{k=1}^n \mathbb{P}_{\beta_c} (|K_0| \geq k) \leq Cn^{1-\frac{1}{d}} \). Then \( \delta \geq \frac{d+o(1)}{d-\alpha} \).

**Proof.** We write \( \theta = \frac{1}{\delta} \) and get that \( \sum_{k=1}^N \mathbb{P}_{\beta_c} (|K_0| \geq k) \leq CN^{1-\theta} \) for all \( N \in \mathbb{N} \). Lemma 2.3 shows that for some constant \( C' < \infty \) we have \( \mathbb{E}_{\beta} (|K_0 (\Lambda_n)|) \leq C' n^{d \left( \frac{1}{d+\theta} \right)} \). Combining this with inequality (19) we get that

\[
C' n^{d \left( \frac{1}{d+\theta} \right)} \geq \mathbb{E}_{\beta} (|K_0 (\Lambda_n)|) \geq C^{-1} f(n, \alpha)^{-1} \approx n^{(\alpha \wedge 1) + o(1)}
\]

and this shows that \( d\frac{1-\theta}{1+\theta} \geq \alpha \wedge 1 \). As we consider \( \alpha \in (0, 1) \) only for \( d = 1 \), we always have that \( \frac{\alpha \wedge 1}{d} < 1 \). Elementary calculations show that

\[
\begin{align*}
\frac{1}{d+\theta} &= \frac{d \delta - 1}{\delta + 1} \geq \alpha \wedge 1 \Leftrightarrow \delta - 1 \geq \frac{\alpha \wedge 1}{d} \delta + \frac{\alpha \wedge 1}{d} \\
\Leftrightarrow \delta &= \frac{\alpha \wedge 1}{d} \delta = \delta \left( 1 - \frac{\alpha \wedge 1}{d} \right) \geq \frac{\alpha \wedge 1}{d} + 1 \\
\Leftrightarrow \delta &\geq \frac{d + (\alpha \wedge 1)}{d - (\alpha \wedge 1)}
\end{align*}
\]

which finishes the proof. \( \Box \)

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**References**


