Global contractivity for Langevin dynamics with distribution-dependent forces and uniform in time propagation of chaos

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\textbf{Abstract.} We study the long-time behaviour of both the classical second-order Langevin dynamics and the nonlinear second-order Langevin dynamics of McKean-Vlasov type. By a coupling approach, we establish global contraction in an $L^2$ Wasserstein distance with an explicit dimension-free rate for pairwise weak interactions. For external forces corresponding to a $\kappa$-strongly convex potential a contraction rate of order $O(\sqrt{t})$ is obtained in certain cases. But the result is not restricted to these forces. It rather includes multi-well potentials and non-gradient-type external forces as well as non-gradient-type repulsive and attractive interaction forces. The proof is based on a novel distance function which combines two contraction results for large and small distances and uses a coupling approach adjusted to the distance. By applying a componentwise adaptation of the coupling we provide uniform in time propagation of chaos bounds for the corresponding mean-field particle system.

\textbf{Résumé.} Nous étudions le comportement en temps long de la dynamique de second ordre de Langevin ainsi que sa version non linéaire de type McKean-Vlasov. Par une approche par couplage, nous établissons la contraction globale en distance de Wasserstein $L_1$ avec un taux explicite indépendant de la dimension dans le cas d’une faible interaction par paire. Lorsque la force de confinement correspond à un potentiel $\kappa$-fortement convexe, un taux de contraction de l’ordre de $O(\sqrt{t})$ est obtenu dans certains cas. Ce résultat de contraction ne se limite pas à ce type de forces. Il est possible de considérer également des confinements de type non gradient et multi-terrains ainsi que des interactions non gradients attractives ou répulsives. Notre preuve repose sur une nouvelle fonction distance qui combine deux résultats de contraction pour les petites et grandes distances et utilise ainsi un couplage adapté. En utilisant une adaptation coordonnée par coordonnées du couplage, cela permet d’obtenir la propagation du chaos uniforme en temps pour le système de particules à champ moyen associé.

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1. Introduction

In this paper, we are interested in the long-time behaviour of the Langevin diffusion $(X_t,Y_t)_{t \geq 0}$ of McKean-Vlasov type on $\mathbb{R}^{2d}$ given by the stochastic differential equation

\begin{equation}
\begin{aligned}
\mathrm{d}X_t &= Y_t \mathrm{d}t \\
\mathrm{d}Y_t &= (ub(X_t) + u \int_{\mathbb{R}^d} \tilde{b}(X_t,z) \mu_t^x(\mathrm{d}z) - \gamma Y_t) \mathrm{d}t + \sqrt{2\gamma \mu_t^x} \mathrm{d}B_t, \\
\end{aligned}
\end{equation}

where $b: \mathbb{R}^d \to \mathbb{R}^d$ and $\tilde{b}: \mathbb{R}^{2d} \to \mathbb{R}^d$ are two Lipschitz continuous functions, $u, \gamma > 0$ are two positive constants and $(B_t)_{t \geq 0}$ is a $d$-dimensional standard Brownian motion. The functions $b$ and $\tilde{b}$ denote the \textit{external force} and the \textit{interaction force}, respectively. If $\tilde{b} \equiv 0$, (1) corresponds to the classical Langevin dynamics, which is also of particular interest and whose long-time behaviour will separately be studied in detail. Existence of a solution and uniqueness in law hold provided the initial conditions have bounded second moments and $b$ and $\tilde{b}$ are Lipschitz continuous [38, Theorem 2.2].

Equation (1) is the probabilistic description of the Vlasov-Fokker-Planck equation given by

\begin{equation}
\partial_t f_t(x,y) = \nabla_x \cdot \left[ \gamma \nabla_y f_t(x,y) + \gamma y f_t(x,y) + u \left( \tilde{b}(x) + \int_{\mathbb{R}^d} \tilde{b}(x,z) \mu_t^x(\mathrm{d}z) \right) f_t(x,y) \right] - u \nabla_x \cdot \left[ y f_t(x,y) \right],
\end{equation}
where \( f_t \) is the time dependent density function on \( \mathbb{R}^{2d} \) and \( \bar{\mu}_t^N \) is the marginal distribution in the first component of \( \bar{\mu}_t (dx dy) = f_t (x, y) dx dy \). The solution \( (\bar{\mu}_t)_{t \geq 0} \) of (2) describes the density function of the process \( (\bar{X}_t, \bar{Y}_t)_{t \geq 0} \) which moves according to (1). Often, \( b \) and \( \tilde{b} \) are of the form \( b(x) = -\nabla V(x) \) and \( \tilde{b}(x, x') = -\nabla_x W(x, x') \) for all \( x, x' \in \mathbb{R}^d \) and for some functions \( V \in \mathcal{C}^1(\mathbb{R}^d) \) and \( W \in \mathcal{C}^1(\mathbb{R}^{2d}) \), which are called confinement potential and interaction potential, respectively.

Besides the long-time behaviour of (1), we study the mean-field particle system corresponding to (1) with \( N \in \mathbb{N} \) particles which is given by

\[
\begin{align*}
\begin{cases}
    dX^{i,N}_t = Y^{i,N}_t dt \\
    dY^{i,N}_t = (ub(X^{i,N}_t) + N^{-1} \sum_{j=1}^{N} \tilde{b}(X^{i,N}_t, X^{j,N}_t) - \gamma Y^{i,N}_t) dt + \sqrt{2\gamma u} dB^i_t, & i = 1, \ldots, N.
\end{cases}
\end{align*}
\]

We are interested in establishing conditions on \( b \) and \( \tilde{b} \) such that for all \( t \geq 0 \) for \( N \to \infty \) the law of the particles converges to the law of \( (\bar{X}_t, \bar{Y}_t) \). This phenomenon was stated under the name propagation of chaos and was first introduced by Kac for the Boltzmann equation in [32]. For finite time horizon, bounds on the difference between the law of the particle system and the law of \( N \) independent solutions to (1) are established by McKean [36] provided \( b \) and \( \tilde{b} \) are Lipschitz continuous and bounded. This result is further developed in e.g. [38, 43], see [13, 14] for an overview and the references therein.

The equations (1), (2), (3) and its variants have various applications in physics. If \( \tilde{b} \equiv 0 \), the solution of (1) can be interpreted as a particle having a position \( \bar{X}_t \) and a velocity \( \bar{Y}_t \) and which moves according to the external force. The constant \( \gamma > 0 \) corresponds to the friction parameter and \( u > 0 \) denotes the inverse of the mass per particle. Equation (3) describes many particles whose moves are additionally determined by pairwise interactions given by the interaction force. Equation (2) describes the limit distribution as the number of particles tends to infinity.

In the deep learning community, Langevin dynamics with a mean-field interaction provide a tool to prove trainability of neural networks [37, 42]. Algorithms using Langevin dynamics have a better long-time behaviour compared to the overdamped Langevin dynamics [15, 16], which forms a degenerated special case of the Langevin dynamics, where the limit for \( \gamma \) to infinity is taken [41, Section 6.5.1]. Therefore, nonlinear Langevin dynamics became recently popular for training networks as the Generative Adversarial Network (GAN) [33].

If \( \bar{b} \equiv 0 \) and \( b = -\nabla V \), then under some mild conditions on \( V \) the unique invariant measure is given by the Boltzmann-Gibbs distribution

\[
\mu_\infty (dx \ dy) \propto \exp(-V(x) - |y|^2/(2u)),
\]

see e.g. [41, Proposition 6.1]. Otherwise, i.e., if \( b \) is not of gradient-type or \( \tilde{b} \neq 0 \), it is often not clear if uniqueness of an invariant probability measure holds (see [19]) and how fast the marginal law of a solution of (1) converges towards it.

Getting a clear picture of the long-time behaviour of processes given by stochastic differential equations with and without nonlinear forces of McKean-Vlasov type is of wide interest and the objective of many works. For the overdamped Langevin dynamics forming a first-order equation, the long-time behaviour is studied using both analytic approaches as functional inequalities (e.g. [3, 5]) and probabilistic approaches as coupling techniques. Via a reflection coupling, Eberle [23] established contraction in \( L^1 \) Wasserstein distance with respect to a carefully aligned distance function with explicit rates for locally non-convex potentials. For the dynamics with an additional nonlinear drift term, which appears to model for example granular media (see [4]), exponential convergence rates have been investigated for uniformly convex potentials in [10] using gradient flow structure, Logarithmic Sobolev inequalities and transportation cost inequalities (see [11, 12, 34] for relaxations to certain non-uniformly convex potentials). Further, [12, 34] provide uniform in time propagation of chaos estimates for the corresponding particle system. Based on a coupling approach consisting of a mixture of a synchronous and a reflection coupling, uniform in time propagation of chaos is shown in [22] for possibly non-strongly convex confinement potentials and possibly non-convex interaction potentials. For the unconfined dynamics (i.e., \( \bar{b} = 0 \)) exponential convergence is studied in [6, 12] for convex interaction potentials applying analytic tools. If the convexity assumption on the interaction potential is removed, exponential convergence and propagation of chaos can still be established for unconfined overdamped Langevin dynamics via a sticky coupling approach (see [21]) for a class of interaction forces that split in a linear term and a perturbation part.

Proving contraction rates for second-order SDEs given by (1) is more delicate as additionally one has to deal with the hypoellipticity of the diffusion. In the case of the classical Langevin dynamics with a gradient-type force, i.e., when \( b = \nabla V \) and \( \bar{b} \equiv 0 \) hold, exponential convergence is studied in e.g. [1, 17, 18, 28–30, 45] using analytic methods including the Witten Laplacian, semigroups, functional inequalities and hypocoercivity. To our knowledge, the best-known contraction rate is obtained for \( \kappa \)-strongly convex potentials \( V \) in [9], where contraction in \( L^2 \) distance is shown with a rate of order \( O(\sqrt{\kappa}) \) via a Poincaré type inequality. Harris type theorems, involving a Lyapunov drift condition, provide
a probabilistic technique to analyse the long-time behaviour of Langevin dynamics, see [2, 35, 44, 46]. An alternative powerful probabilistic approach, which provides quantitative rates, is based on couplings. Via a synchronous coupling approach, Dalalyan and Riou-Durand [16] showed contraction in Wasserstein distance with rate of order $O(\kappa/\sqrt{t})$ for $\kappa$-strongly convex potentials with $L$-Lipschitz continuous gradients if $L\gamma^{-2}\alpha \leq 1$ holds. In [24], Eberle, Guillin and Zimmer introduced a coupling for the Langevin dynamics including non-convex confinement potentials and showed exponential convergence with explicit rates. There, contraction is shown in a specific $L^1$ Wasserstein distance with respect to a semimetric involving a Lyapunov function. More precisely, for large distances, a synchronous coupling is considered and the Lyapunov function in the semimetric yields contraction. For small distances, the noise is synchronized on a line, where contraction for the position is observed, and reflected otherwise to force the dynamics to return to that line. Combining the results of the different areas, contraction in average is obtained for a carefully aligned semimetric. Due to the Lyapunov function, the contraction rate depends on the dimension and the semimetric is not applicable for nonlinear Langevin dynamics, which suggests getting rid of the Lyapunov function and treating the area of large distances differently.

To get results on the long-time behaviour for nonlinear Langevin diffusions given by (1), we have to handle both the difficulties coming from the nonlinearity and the hypoellipticity of the equation. Beginning with the analytic approaches, let us mention the work by Villani [45], where the hypocoercivity is extended to the framework on the torus with small interactions, see also the work by Bouchut and Dolbeault [8]. Using a free energy approach, convergence to equilibrium is studied in [20] for specific non-convex confining potentials and convex polynomial interaction potentials. Applying functional inequalities for mean-field models, established in [26] to prove convergence to equilibrium in weighted Sobolev norm, Monmarché and Guillin proved propagation of chaos for (3) in [27, 39]. There, they considered both strongly convex confinement potentials and more general confinement potentials and attractive interaction potentials with at most quadratic growth.

Coupling techniques are also employed in the study of the nonlinear dynamics (1). In [7], convergence to equilibrium is shown via a synchronous coupling for small Lipschitz interactions and a quadratic-like friction term. The combination of the coupling approach of [24] and a Lyapunov function is used in [33] to prove exponential contraction in the case of certain small mean-field potentials of non-convolution-type. There, the results are applied to the numerical discretized versions of the dynamics corresponding to the Hamiltonian Stochastic Gradient Descent, and the connection to the analysis of deep neural networks is drawn, see [31] for further references on the connection to deep learning. Very closely related to this work is the recent preprint [25] by Guillin, Le Bris and Monmarché, which has been prepared independently in parallel. They considered non-globally convex confinement potentials and Lipschitz continuous even interaction potentials and extended the approach by [24]. More precisely, they modified the semimetric by a sophisticated Lyapunov function to treat the nonlinear Langevin dynamics and to obtain propagation of chaos bounds. The main differences between this work and [25] are that here we include forces that are not necessarily of gradient type and that we establish global contractivity with dimension-free rates by constructing a novel distance function and modifying the coupling approach of [24] appropriately. In particular, we consider two separate metrics $r_l$ and $r_s$ for large and small distances instead of a semimetric involving a Lyapunov function and establish contraction for both metrics separately. For small distances we make use of the results by [24], whereas for large distances we consider a twisted 2-norm structure for the metric $r_l$ of the form $(x \cdot (Ax) + x \cdot (By) + y \cdot (C y))$ with positive definite matrices $A, B, C \in \mathbb{R}^{d \times d}$. This structure is similar to the structure appearing in the Lyapunov function in [35, 44] and to the norm used in e.g. [1] to prove contraction for certain strongly convex potentials.

Then, our first main contribution is a global contraction result in Wasserstein distance with respect to a distance $\rho$ that is carefully glued of $r_s$ and $r_l$ and that is equivalent to the Euclidean distance. More precisely, we impose $b$ to be a sum of a linear function $-K x$, where $K \in \mathbb{R}^{d \times d}$ is a positive definite matrix with smallest eigenvalue $\kappa$, and a certain Lipschitz continuous function $g(x)$ with Lipschitz constant $L_g$ which is such that $b$ includes gradients of asymptotically strongly convex potentials. If the friction parameter $\gamma$ is sufficiently large, i.e., $\gamma^2 > 2L_g^2\alpha/\kappa$, and if the Lipschitz constant $L$ of the interaction force $b$ is sufficiently small, we prove for two probability measures $\bar{\mu}_0$ and $\bar{\nu}_0$ on $\mathbb{R}^{2d}$ with finite second moment,

\begin{equation}
\mathcal{W}_\rho(\bar{\mu}_t, \bar{\nu}_t) \leq e^{-ct} \mathcal{W}_\rho(\bar{\mu}_0, \bar{\nu}_0), \quad \text{and} \quad \mathcal{W}_1(\bar{\mu}_t, \bar{\nu}_t) \leq M_1 e^{-ct} \mathcal{W}_1(\bar{\mu}_0, \bar{\nu}_0),
\end{equation}

where $\bar{\mu}_t$ and $\bar{\nu}_t$ are the laws of the solutions $(\bar{X}_t, \bar{Y}_t)$ and $(\bar{X}_t', \bar{Y}_t')$ to (1) with initial distribution $\bar{\mu}_0$ and $\bar{\nu}_0$, respectively. The dimension-free constants $c$ and $M_1$ depend on $\kappa, \gamma, \alpha$, on the largest eigenvalue of $K$ and on properties of $g$. Note that the additional constant $M_1$ in the second bound measures the difference between the distance $\rho$ and the Euclidean distance.

These bounds are established using a modification of the coupling introduced in [24], which is a synchronous coupling for large distances and mainly a reflection coupling for small distances except on one line the noise is synchronized. In this
work, we adjust the transition from synchronous coupling for large distances to reflection coupling for small distances to suit the underlying distance function. Namely, the synchronous coupling is applied when \( r_l \) is considered and the coupling approach of [24] when \( r_s \) is considered.

This approach which does not rely on a Lyapunov function has the advantage that the upper bound in (4) depends only on the Wasserstein distance between the two initial distributions and is independent of the two distributions themselves (cf. [24, 25, 33]). Further, the metric \( r_l \) is chosen such that the rate of the contraction result for large distances is optimized up to a constant. We emphasize that these bounds give also global contractivity for the classical Langevin dynamics and improve the result obtained in [24].

Moreover, using the ansatz for large distances, we contribute to the analysis of the optimal contraction rate for strongly convex potentials and improve the results of [16]. If the drift corresponds to a \( \kappa \)-strongly convex potential, we can split \( V \) in a linear part \( x \cdot (Kx) \), where \( K \) is a positive definite matrix with smallest eigenvalue \( \kappa \), and a convex function \( G \) with \( L_G \)-Lipschitz continuous gradients. We prove contraction in Wasserstein distance with respect to a distance function of the same form as \( r_l \) with rate \( c = \gamma / 2 \min(1/4, \kappa \gamma ^{-2}) \) provided \( L_G u \gamma ^{-2} \leq 3/4 \) holds. If the perturbation \( G \) is sufficiently small, i.e., \( L_G \leq 3\kappa \), we obtain for optimized \( \gamma \) a rate of order \( O(\sqrt{\kappa}) \), that coincides with the order given in the \( L^2 \) contraction result in [9], and otherwise we obtain a rate of the same order as in [16].

Finally, applying a componentwise version of the preceding coupling we establish a uniform in time propagation of chaos bound for the corresponding particle system (3), i.e., we show for a probability measure \( \mu_0 \) on \( \mathbb{R}^{2d} \) with finite second moment,

\[
W_{1,\ell^1_N}(\tilde{\mu}_t^N, \mu_t^N) \leq C_1 c^{-1} N^{-1/2},
\]

where \( \mu_t^N \) is the law of the particles driven by (3) with initial distribution \( \mu_0^N = \mu_0^\otimes N \) and \( \tilde{\mu}_t^N \) is the product law of \( N \) independent solutions to (1) with initial distribution \( \mu_0 \). Here, \( C_1 \) is a constant depending on \( \kappa \), \( \gamma \), \( u \), \( d \), on properties of \( g \), and on the second moment of \( \mu_0 \). The normalized \( \ell^1 \)-distance \( \ell^1_N \) is given by

\[
\ell^1_N((x,y), (\tilde{x}, \tilde{y})) = N^{-1} \sum_{i=1}^N (|x^i - \tilde{x}^i| + |y^i - \tilde{y}^i|), \quad \text{for all } x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d,
\]

where \( | \cdot | \) denotes the Euclidean metric.

Eventually, we note that the construction of the metric for large distance can be applied to prove contraction to specific unconfined cases, where \( b \equiv 0 \) and \( \bar{b} \) is a small perturbation of a linear force.

**Notation:** For some space \( \mathbb{X} \), which is here either \( \mathbb{R}^{2d} \) or \( \mathbb{R}^{2N^d} \), we denote its Borel \( \sigma \)-algebra by \( B(\mathbb{X}) \). The space of all probability measures on \( (\mathbb{X}, B(\mathbb{X})) \) is denoted by \( \mathcal{P}(\mathbb{X}) \). Let \( \mu, \nu \in \mathcal{P}(\mathbb{X}) \). A coupling \( \omega \) of \( \mu \) and \( \nu \) is a probability measure on \( (\mathbb{X} \times \mathbb{X}, B(\mathbb{X}) \otimes B(\mathbb{X})) \) with marginals \( \mu \) and \( \nu \). The \( L^p \) Wasserstein distance with respect to a distance function \( d: \mathbb{X} \times \mathbb{X} \to \mathbb{R} \) is defined by

\[
W_{p,d}(\mu, \nu) = \inf_{\omega \in \Pi(\mu, \nu)} \left( \int_{\mathbb{X} \times \mathbb{X}} d(x,y)^p \omega(dx) \omega(dy) \right)^{1/p},
\]

where \( \Pi(\mu, \nu) \) denotes the set of all couplings of \( \mu \) and \( \nu \). We write \( W_p \) if the underlying distance function is the Euclidean distance.

**Outline of the paper:** In Section 2, we state the contraction results for the classical Langevin dynamics and give an informal construction of the coupling and the metric. In Section 3, we state the framework and the contraction results for Langevin dynamics of McKean-Vlasov type before defining rigorously the metric and the coupling approach in Section 4. Uniform in time propagation of chaos is established in Section 5. The proofs are postponed to Section 6.

## 2. Contraction for classical Langevin dynamics

### 2.1. Contraction for Langevin dynamics with strongly convex confinement potential

First, we consider the Langevin dynamics without a non-linear drift and with confinement potential \( V \) given by the stochastic differential equation

\[
\begin{align*}
\d X_t &= Y_t \, dt, \\
\d Y_t &= (-\gamma Y_t - u \nabla V(X_t)) \, dt + \sqrt{2\gamma} u \, dB_t,
\end{align*}
\]

where \( \gamma \) and \( u \) are positive constants.
with initial condition \((X_0, Y_0) = (x, y) \in \mathbb{R}^d\) and with \(d\)-dimensional standard Brownian motion \((B_t)_{t \geq 0}\). We impose for \(V \in C^2(\mathbb{R}^d)\):

**Assumption 1.** There exist a positive definite matrix \(K \in \mathbb{R}^{d \times d}\) with smallest eigenvalue \(\kappa > 0\) and a convex function \(G : \mathbb{R}^d \to \mathbb{R}\) with \(L_G\)-Lipschitz continuous gradients, i.e.,

\[
\langle \nabla G(x) - \nabla G(\bar{x}), x - \bar{x} \rangle \geq 0 \quad \text{and} \quad |\nabla G(x) - \nabla G(\bar{x})| \leq L_G|x - \bar{x}| \quad \text{for all } x, \bar{x} \in \mathbb{R}^d,
\]

such that

\[
V(x) = x \cdot (Kx)/2 + G(x) \quad \text{for any } x \in \mathbb{R}^d.
\]

We note that Assumption 1 is satisfied for all \(\kappa\)-strongly convex functions \(V\) with \(L_V\)-Lipschitz continuous gradients, i.e.,

\[
\langle \nabla V(x) - \nabla V(y), x - y \rangle \geq \kappa|x - y|^2 \quad \text{and} \quad |\nabla V(x) - \nabla V(y)| \leq L_V|x - y| \quad \text{for all } x, y \in \mathbb{R}^d.
\]

Note that the splitting of \(V\) in \(K\) and \(G\) is in general not unique. A natural choice is given by \(K = \kappa \text{Id}\) and \(G(x) = V(x) - (\kappa/2)|x|^2\), where \(\text{Id}\) is the \(d \times d\) identity matrix. As we see later, we often want a splitting of \(V\) such that the Lipschitz constant \(L_G\) is minimized.

We establish a global contraction result for (6) in \(L^p\) Wasserstein distance with respect to the distance function \(r : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to [0, \infty)\) given by

\[
r((x, y), (\bar{x}, \bar{y}))^2 = \gamma^2 u(x - \bar{x}) \cdot (K(x - \bar{x})) + \frac{1}{2}(1 - 2\lambda)(x - \bar{x}) + \gamma^{-1}(y - \bar{y})^2 + \frac{1}{2} \gamma^{-2}|y - \bar{y}|^2
\]

for \((x, y), (\bar{x}, \bar{y}) \in \mathbb{R}^{2d}\) with

\[
\lambda = \min(1/8, \kappa u \gamma^{-2}/2).
\]

**Theorem 1** (Contractivity for strongly convex potentials). For \(t \geq 0\), let \(\mu_t\) and \(\nu_t\) be the law at time \(t\) of the processes \((X_t, Y_t)\) and \((X_t', Y_t')\), respectively, where \((X_s, Y_s)_{s \geq 0}\) and \((X_s', Y_s')_{s \geq 0}\) are solutions to (6) with initial distributions \(\mu_0\) and \(\nu_0\) on \(\mathbb{R}^{2d}\), respectively. Suppose Assumption 1 holds and

\[
L_G u \gamma^{-2} \leq 3/4.
\]

Then, for any \(1 \leq p < \infty\)

\[
W_{p, r}(\mu_t, \nu_t) \leq e^{-ct}W_{p, r}(\mu_0, \nu_0) \quad \text{and} \quad W_p(\mu_t, \nu_t) \leq Me^{-ct}W_p(\mu_0, \nu_0),
\]

where the contraction rate \(c\) is given by

\[
c = \gamma \lambda = \min(\gamma / 8, \kappa u \gamma^{-1}/2).
\]

The constant \(M\) is given by

\[
M = \sqrt{\max(u L_K + \gamma^2/2, \max((u \kappa)^{-1}, 2))},
\]

where \(L_K\) denotes the largest eigenvalue of \(K\).

**Proof.** The proof is based on a synchronous coupling and is postponed to Section 6.1. \(\square\)

**Remark 2.** If \(V\) is a quadratic function, then \(L_G = 0\) and the restriction on \(\gamma\) vanishes. In this case, the \(L^2\) spectral gap of the corresponding generator is given by

\[
c_{\text{gap}} = (1 - \sqrt{(1 - 4\kappa u \gamma^{-2})^+}) \gamma/2,
\]

cf. [41, Section 6.3]. More precisely, \(c_{\text{gap}} = \gamma/2\) if \(4\kappa u \gamma^{-2} \geq 1\), and \(\kappa u \gamma^{-1} \leq c_{\text{gap}} \leq 2\kappa u \gamma^{-1}\) if \(4\kappa u \gamma^{-2} < 1\). Hence, the contraction rate is of the same order as the spectral gap. In particular for \(\gamma = 2\sqrt{\kappa u}\) the optimal contraction rate \(c = \sqrt{8u}/8\) is obtained. If \(L_G \leq 3\kappa\), \(\gamma = 2\sqrt{\kappa u}\) satisfies condition (10) and yields the optimal contraction rate of order \(O(\sqrt{\gamma})\). Otherwise, for \(\gamma = \sqrt{(4/3)L_G u}\) the contraction rate is optimized and of order \(O(\kappa/\sqrt{L_G})\).
2.2. Framework for classical Langevin dynamics with general external forces

Next, we consider the classical Langevin dynamics \((X_t, Y_t)_{t \geq 0}\) with a general external drift given by the stochastic differential equation

\[
\begin{aligned}
dX_t &= Y_t dt, \\
dY_t &= (-\gamma Y_t + ub(X_t)) dt + \sqrt{2\gamma} \, dB_t,
\end{aligned}
\]

with initial condition \((X_0, Y_0) = (x, y) \in \mathbb{R}^{2d}\).

We impose the following assumption on the force \(b\):

**Assumption 2.** The function \(b : \mathbb{R}^d \to \mathbb{R}^d\) is Lipschitz continuous and there exist a positive definite matrix \(K \in \mathbb{R}^{d \times d}\) with smallest eigenvalue \(\kappa \in (0, \infty)\) and largest eigenvalue \(L_K \in (0, \infty)\), a constant \(\bar{R} \in (0, \infty)\) and a function \(g : \mathbb{R}^d \to \mathbb{R}^d\) with Lipschitz constant \(L_g \in (0, \infty)\) such that

\[
b(x) = -Kx + g(x)
\]

and

\[
g(x) - g(x), x - \bar{x} \leq 0 \quad \text{for all } x, \bar{x} \in \mathbb{R}^d \text{ such that } |x - \bar{x}| \geq \bar{R}.
\]

**Remark 3.** Suppose that \(b = -\nabla V\) where \(V\) is a potential function with an \(L_V\)-Lipschitz continuous gradient and that is \(k\)-strongly convex outside a Euclidean ball of radius \(\bar{R}\), i.e.,

\[
\langle \nabla V(x) - \nabla V(\bar{x}), x - \bar{x} \rangle \geq k |x - \bar{x}|^2 \quad \text{for all } x, \bar{x} \in \mathbb{R}^d \text{ such that } |x|, |\bar{x}| \geq \bar{R}.
\]

Note that \(\nabla V\) can be split in \(\nabla V(x) = kx + h(x)\) where \(h : \mathbb{R}^d \to \mathbb{R}^d\) is an \(L_h\)-Lipschitz continuous function with \(L_h \leq L_V + k\) and \(\langle h(x) - h(x), x - \bar{x} \rangle \geq 0\) for all \(x, \bar{x} \in \mathbb{R}^d\) such that \(|x|, |\bar{x}| \geq \bar{R}\). Then for \(l \leq \frac{1}{2} \min(1, \frac{\bar{R}}{k})\), \(b = -\nabla V\) satisfies Assumption 2 with \(L_g \leq L_V + (1 - l)k\), \(\kappa = (1 - l)k \geq \max(\frac{1}{2}k, k - \frac{\bar{R}^2}{2})\) and \(\bar{R} = 2R \frac{L_g}{L_V}\).

**Example 4** (Double-well potential). For \(\beta > 0\), we consider the double-well potential \(V \in \mathcal{C}_1(\mathbb{R})\) defined by

\[
V(x) = \begin{cases} 
\beta \left( \frac{|x|^4}{4} - |x|^2 \right) & \text{for } |x| \leq 2, \\
\beta \left( \frac{3|x|^2}{2} - 4 \right) & \text{for } |x| > 2.
\end{cases}
\]

This potential has a Lipschitz continuous gradient and is strongly convex with convexity constant \(k = 3\beta\) outside a Euclidean ball with radius \(\bar{R} = 2\). We consider the splitting \(\nabla V(x) = -\kappa x + g(x)\) with \(\kappa = (2/3)k = 2\beta\) and

\[
g(x) = \begin{cases} 
-\beta(x^3 - 3x) & \text{for } |x| \leq 2, \\
-\beta x & \text{for } |x| > 2.
\end{cases}
\]

Then, the function \(g\) is Lipschitz continuous with Lipschitz constant \(L_g = 9\beta\) and (15) is satisfied for sufficiently large \(\bar{R}\).

2.3. Construction of the metric and the coupling

We provide an informal construction of the coupling and the complementary metric. Given two Brownian motions \((B_t)_{t \geq 0}, (B'_t)_{t \geq 0}\) and \((x, y), (x', y') \in \mathbb{R}^{2d}\), let \(((X_t, Y_t), (X'_t, Y'_t))_{t \geq 0}\) be an arbitrary coupling of two solutions to (13). It holds for the difference process \((Z_t, W_t)_{t \geq 0} = (X_t - X'_t, Y_t - Y'_t)_{t \geq 0}\),

\[
\begin{aligned}
dZ_t &= W_t \, dt, \\
dW_t &= (-\gamma W_t + ub(X_t) - ub(X'_t)) \, dt + \sqrt{2\gamma} \, dB_t - B_t - B'_t.
\end{aligned}
\]

Adapting the idea of the coupling construction from [24], the process \((Z_t, Q_t)_{t \geq 0} = (Z_t, Z_t + \gamma^{-1} W_t)_{t \geq 0}\) satisfies the stochastic differential equation

\[
\begin{aligned}
dZ_t &= -\gamma Z_t \, dt + \gamma Q_t \, dt, \\
dQ_t &= \gamma^{-1} u(b(X_t) - b(X'_t)) \, dt + \sqrt{2\gamma^{-1}} \, dB_t - B_t - B'_t.
\end{aligned}
\]
As in [24], we apply a synchronous coupling for \( Q_t = 0 \), since in this case the first equation of (17) is contractive and the absence of the noise ensures that the dynamics is not driven away from this area by random fluctuations. Apart from \( Q_t = 0 \), we want to apply a reflection coupling, which guarantees that the dynamics returns to the line \( Q_t = 0 \). Note that this construction leads to a coupling that is sticky on the hyperplane \( \{(x, y),(x', y')\} \in \mathbb{R}^{2d}: x - x' + \gamma^{-1}(y - y') = 0\}. However, since it is technically hard to construct this sticky coupling, we consider approximations of the coupling, which are rigorously stated in Section 4.2 and which suffice for our purpose. Similarly as in [24], we show for \( r_s(t) = \alpha|Z_t| + |Q_t| < R_1 \) with appropriately chosen constants \( \alpha, R_1 \) that there exists a concave increasing function \( f \) depending on \( \alpha \) and \( R_1 \) such that \( f(r_s(t)) \) is contractive on average. Note that the application of a concave function has the effect that a decrease in \( r_s \) has a larger impact than an increase in \( r_s \).

On the other hand, if the difference process \( (Z_{t_i}, W_{t_i})_{i \geq 0} \) is sufficiently far away from the origin, we obtain under Assumption 2 for the force \( b \) contractivity for the process \( r_1(t) = (\gamma^{-2}uZ_t \cdot (KZ_t) + (1/2)[(1 - 2\tau)Z_t + \gamma^{-1}W_t]^2 + (1/2)|\gamma^{-1}W_t|^{1/2}, \tau > 0 \) is a constant depending on \( \kappa, \gamma, \alpha, L_\alpha, L_g \). More precisely, we obtain local contractivity with contraction rate \( \gamma \tau \) for \( r_1(t)^2 > \mathcal{R} \) for some \( \mathcal{R} > 0 \) depending on \( \alpha, R_1, \kappa, \gamma, \alpha, L_\alpha, L_g \). The process \( r_1(t) \) is designed such that the local contraction rate is optimized up to some constant, see Lemma 19.

We construct a metric which is globally contractive on average using the previously established coupling. The key idea lies in combining \( r_s \) and \( r_1 \) in such a way, that the two local contraction results imply global contractivity in the new metric. Note that for simplicity, we write \( r_1 \) and \( r_s \) both for the norm \( r_1(z, w) \) (respectively \( r_s(z, w) \)) of \( (z, w) \in \mathbb{R}^{2d} \) and for the distance \( r_1((x, y), (x', y')) \) (respectively \( r_s((x, y), (x', y')) \)) of \( (x, y), (x', y') \in \mathbb{R}^{2d} \).

As we see in Section 6.2, the lower bound \( \mathcal{R} \) in the contraction result for large distances is fixed due to the dependence on the drift assumptions, whereas the upper bound \( R_1 \) in the result for small distances is flexible with the drawback that the contraction rate gets smaller for larger \( R_1 \). To benefit from the local contraction results, we want for all \((z, w) \in \mathbb{R}^{2d}\) that \( r_s(z, w) \leq R_1 \) or \( r_1(z, w)^2 \geq \mathcal{R} \) holds, which we achieve by choosing \( R_1 \) sufficiently large. We construct a continuous transition between \( r_s \) and \( r_1 \) by considering \( r_s \wedge (D_K + \varepsilon r_1) \), where the constant \( \varepsilon \) satisfies \( 2\varepsilon r_1 \leq r_s \) and the constant \( D_K \) is given such that \( r_s(z, w) \wedge (D_K + \varepsilon r_1(z, w)) = r_s(z, w) \) for \( (z, w) \) with \( r_1(z, w)^2 \leq \mathcal{R} \). Then, we set \( R_1 \) such that \( r_s(z, w) \wedge (D_K + \varepsilon r_1(z, w)) = D_K + \varepsilon r_1(z, w) \) is guaranteed for \( (z, w) \) with \( r_s(z, w) \leq R_1 \).

In particular, in this construction the level set \( r_s(z, w) = D_K \) is optimally encompassed by the level set \( r_s(z, w) = R_1 \) and \( r_1(z, w)^2 = \mathcal{R} \), as illustrated in Figure 1, and \( r_s(z, w) \leq R_1 \) or \( r_1(z, w)^2 \geq \mathcal{R} \) is ensured. We define the metric \( \rho((x, y), (x', y')) = f(r_s((x, y), (x', y')) \wedge (D_K + \varepsilon r_1((x, y), (x', y')))) \). As illustrated in Figure 2, we obtain \( f(r_s) \) for small distances and \( f(D_K + \varepsilon r_1((x, y), (x', y')))) \) for large distances. A detailed rigorous construction and a proof showing that \( \rho \) defines a metric are given in Section 4.
2.4. A global contraction result for the classical Langevin dynamics with general external force

We establish the main contraction result for the classical Langevin dynamics given by (13).

Theorem 5. For \( t \geq 0 \), let \( \mu_t \) and \( \nu_t \) be the law at time \( t \) of the processes \((X_t, Y_t)\) and \((X'_t, Y'_t)\), respectively, where \((X_s, Y_s)_{s \geq 0}\) and \((X'_s, Y'_s)_{s \geq 0}\) are solutions to (13) with initial distributions \( \mu_0 \) and \( \nu_0 \) on \( \mathbb{R}^2 \), respectively. Suppose Assumption 2 holds and

\[
L_g u \gamma^{-2} < \frac{\kappa}{2L_g}.
\]

Then,

\[
W_1,\rho(\mu_t, \nu_t) \leq e^{-ct}W_1,\rho(\mu_0, \nu_0) \quad \text{and} \quad W_1(\mu_t, \nu_t) \leq M_1 e^{-ct}W_1(\mu_0, \nu_0),
\]

where the distance \( \rho \) is defined precisely in (35) below and the contraction rate \( c \) is given by

\[
c = \gamma \exp(-\Lambda) \min \left( \frac{(L_K + L_g)u\gamma^{-2}}{4} \Lambda^{1/2} \frac{1}{\sqrt{8}} \frac{\tau E}{2} \right) \quad \text{with}
\]

\[
\Lambda = \frac{L_K + L_g}{4} R_1^2,
\]

\[
\tau := \min(1/8, \gamma^{-2}u\kappa/2 - \gamma^{-4}L_g^2u^2),
\]

\[
E := \frac{1}{6} \min \left( 1, \frac{\sqrt{8} \gamma}{\sqrt{8u(L_K + L_g)}}, \frac{\kappa u}{2} \gamma^{-1}, 2(L_K + L_g)u\gamma^{-2} \right).
\]

The constants \( R_1 \) satisfies

\[
\frac{2}{3} \min(1, 2(L_K + L_g)u\gamma^{-2}) \sqrt{\frac{8u \mathbb{1}_{\{R>0\}} + L_g u R^2}{\gamma^2}} \leq R_1 \leq 4 \max \left( \frac{\sqrt{8} (L_K + L_g)u}{\gamma \sqrt{2}}, \frac{\mathbb{1}_{\{R>0\}} + L_g u R^2}{\gamma^2} \right),
\]

and is explicitly stated in (38). The constant \( M_1 \) is given by

\[
M_1 = \max(2(L_K + L_g)u\gamma^{-1} + \gamma, 1) \frac{1}{2} \exp(\Lambda) \max \left( 3, \frac{3\gamma^2}{2(L_K + L_g)u} \right) \max(\sqrt{2/(\kappa u)}, 2).
\]
Proof. The proof is postponed to Section 6.2.

Remark 6. Compared to the contraction result obtained in [24, Theorem 2.3], global contractivity in Wasserstein distance is obtained with rate $c$ given in (19) which is independent of the dimension $d$.

Remark 7 (Kinetic behaviour). If $\gamma$ is chosen such that $\kappa u\gamma^{-2}$, $L_g u\gamma^{-2}$ and $L_K u\gamma^{-2}$ are fixed and further $L_K R^2$ and $L_g R^2$ are fixed, we obtain similarly to [24, Corollary 2.9] that the contraction rate is of order $\Omega(R^{-1})$.

Remark 8. If $R = 0$, the metric $\rho$ defined in (35) reduces to $\rho((x, y), (\bar{x}, \bar{y})) = (\gamma^{-2}(x - \bar{x}) \cdot (K(x - \bar{x}) + (1/2))(1 - 2\tau)(x - \bar{x}) + \gamma^{-1}(y - \bar{y})^2 + (1/2)\gamma^{-1}(y - \bar{y})^2)^{1/2}$ and the coupling given in Section 4.2 becomes the synchronous coupling. This metric differs from $\tau$ defined in (8) by the constant $\tau$, since here the drift $b$ is not necessarily of gradient-type and we can not make use of the co-coercivity property as in the proof of Theorem 1. Following the proof given in Section 6.2, we obtain contraction in $L^1$ Wasserstein distance, with contraction rate $c = \min(\gamma/16, \kappa\gamma^{-1}/4 - 8\gamma^{-3}L_g^2 u^2)$. We remark that the constant $E$ vanishes in the contraction rate, which measures the difference between the two metrics that are considered in general for $\rho$. If $L_g \leq 2\kappa$, the contraction rate is maximized for $\gamma = u^{1/2}(2\kappa + (4\kappa^2 - 8L_g^2)1/2)^{1/2}$ and satisfies $c = u^{1/2}(2\kappa + (4\kappa^2 - 8L_g^2)1/2)^{1/2}/16$, i.e., in this case the rate is of order $O(\sqrt{\kappa})$.

Example 9 (Double-well potential). For the model given in Example 4, we obtain contraction with respect to the designed Wasserstein distance if $\gamma > 9\sqrt{3}$ is satisfied.

3. Contraction for nonlinear Langevin dynamics of McKean-Vlasov type

Consider the Langevin dynamics of McKean-Vlasov type given in (1). We require Assumption 2 for the function $b: \mathbb{R}^d \to \mathbb{R}^d$. For the function $\tilde{b}: \mathbb{R}^{2d} \to \mathbb{R}^d$ we impose:

Assumption 3. The function $\tilde{b}: \mathbb{R}^{2d} \to \mathbb{R}^d$ is $\tilde{L}$-Lipschitz continuous.

Example 10 (Quadratic interaction potential). Consider $\tilde{b}(x, y) = ky$ with $k \in \mathbb{R}$. Then $\tilde{L} = |k|$ and $\tilde{b}$ corresponds to the interaction potential $W(x, y) = -kx \cdot y$. This potential is attractive for $k > 0$ and repulsive for $k < 0$.

Example 11 (Mollified Coulomb, Newtonian and logarithmic potentials). The gradients of the Coulomb potential and of the Newtonian potential, which describe charged and self-gravitating particles [8], are not Lipschitz continuous. However, the gradient of a mollified version (see [25]) given by

$$W(x, y) = \frac{\tilde{k}}{|x - y|^p + q^p}^{1/p} \quad \text{for } p \geq 2, q \in \mathbb{R}_+ \text{ and } \tilde{k} \in \mathbb{R}$$

satisfies Assumption 3, since $|\text{Hess } W| < \infty$, and therefore $\nabla_x W$ is Lipschitz continuous. In the same line, the gradient of the mollified version of the logarithmic potential given by

$$W(x, y) = -2\log((|x - y|^p + q^p)^{1/p}) \quad \text{for } p \geq 2, q \in \mathbb{R}_+$$

satisfies Assumption 3.

Under the above conditions, we establish contraction in an $L^1$ Wasserstein distance.

Theorem 12 (Contraction for nonlinear Langevin dynamics). Let $\tilde{\mu}_0$ and $\tilde{\nu}_0$ be two probability distributions on $\mathbb{R}^{2d}$ with finite second moment. For $t \geq 0$, let $\tilde{\mu}_t$ and $\tilde{\nu}_t$ be the law at time $t$ of the processes $(\tilde{X}_t, \tilde{Y}_t)$ and $(\tilde{X}'_t, \tilde{Y}'_t)$, respectively, where $(\tilde{X}_s, \tilde{Y}_s)_{s \geq 0}$ and $(\tilde{X}'_s, \tilde{Y}'_s)_{s \geq 0}$ are solutions to (1) with initial distribution $\tilde{\mu}_0$ and $\tilde{\nu}_0$, respectively. Suppose Assumption 2, Assumption 3 and (18) hold. Let $\tilde{L}$ satisfy

$$\tilde{L} \leq \exp(-\Lambda) \min \left\{ \frac{\gamma \tau}{12} \sqrt{\frac{\kappa}{u}} \min(1, 2(L_K + L_g) u \gamma^{-2}), \frac{L_K + L_g}{4} \right\} + \frac{2}{\sqrt{12}},$$

where $\Lambda$ and $\tau$ are given in (20) and (21), respectively. Then

$$W_{1, \rho}(\tilde{\mu}_t, \tilde{\nu}_t) \leq c^{-ct} W_{1, \rho}(\tilde{\mu}_0, \tilde{\nu}_0) \quad \text{and} \quad W_1(\tilde{\mu}_t, \tilde{\nu}_t) \leq M_1 e^{-ct} W_1(\tilde{\mu}_0, \tilde{\nu}_0),$$

where the distance $\rho$ is given in (35) and $c = c/2$ with $c$ given in (19). The constant $M_1$ is given in (24). Moreover, there exists a unique invariant probability measure $\tilde{\mu}_\infty$ for (1) and convergence in $L^1$ Wasserstein distance to $\tilde{\mu}_\infty$ holds.

Proof. The proof is based on the coupling approach and the metric construction given in Section 4.1 and Section 4.2, respectively, and is postponed to Section 6.2.
Remark 13. In comparison to [25, Theorem 3.1], global contractivity is established with a contraction rate and a restriction on the Lipschitz constant $\tilde{L}$ that are independent of the dimension $d$.

Remark 14. Compared to the contraction result in Theorem 5 for classical Langevin dynamics, the contraction rate deteriorates by a factor of 2 to compensate for the nonlinear interaction terms.

If $R = 0$, (25) reduces to $L \leq \tau \sqrt{\bar{k}_{u}/8}$ and contraction holds with rate $\tilde{c} = \min(\gamma/32, \kappa u \gamma^{-1}/8 - L_{g}^{2} u^{2} \gamma^{-3}/2)$ by Lemma 19 and (67). If $L_{g} \leq \sqrt{2k}$, the contraction rate is maximized for $\gamma = \sqrt{u(2\kappa + (4\kappa^{2} - 8L_{g}^{2})^{1/2})^{1/2}}$ yielding $\tilde{c} = \sqrt{u(2\kappa + (4\kappa^{2} - 8L_{g}^{2})^{1/2})^{1/2}/32}$. If the drift is additionally of gradient-type, we can adapt the proof of Theorem 1 and use the co-coercivity property to obtain a contraction rate of order $O(\sqrt{\bar{k}})$ for $L_{g} \leq 3\bar{k}$ and a rate of order $O(\kappa / \sqrt{L_{g}})$ for $L_{g} > 3\bar{k}$.

Remark 15. The contraction results can be extended to unconfined Langevin dynamics. Consider $b \equiv 0$ and $\tilde{b} : \mathbb{R}^{2d} \to \mathbb{R}^{d}$ given by $\tilde{b}(x, y) = -K(x - y) + \bar{g}(x - y)$ where $K \in \mathbb{R}^{d \times d}$ is a positive definite matrix with smallest eigenvalue $\tilde{\kappa}$ and where $\bar{g} : \mathbb{R}^{d} \to \mathbb{R}^{d}$ is an anti-symmetric, $L_{\bar{g}}$-Lipschitz continuous function $\bar{g} : \mathbb{R}^{d} \to \mathbb{R}^{d}$. If $L_{\bar{g}} \leq (\gamma/2)^{\sqrt{\bar{k}}/u} \min(1/8, \bar{k} u \gamma^{-2}/2)$, contraction in an $L^{1}$ Wasserstein distance can be shown via a synchronous coupling approach. The underlying distance function in the Wasserstein distance is based on a similar twisted 2-norm structure as the distance $r_{L}$ given in (26). We note that the conditions on $L_{g}$ and $\tilde{L}$ are combined in the restrictive condition on $L_{\bar{g}}$, which implies $L_{\bar{g}} \leq \tilde{\kappa}/8$ and which gives only contraction for small perturbations of linear interaction forces. A detailed analysis of the unconfined dynamics is given in Appendix A.

4. Metric and coupling

4.1. Metric construction

Suppose Assumption 2 holds. For both the classical Langevin dynamics and the nonlinear Langevin dynamics we consider the metrics $r_{t}, r_{s} : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to [0, \infty)$ given by

\[
\begin{align*}
 r_{t}((x, y), (\bar{x}, \bar{y})) := u \frac{1}{\gamma^{2}} |x - \bar{x}| \cdot (K(x - \bar{x})) + \frac{(1 - 2\tau)^{2}}{2} |x - \bar{x}|^{2} + \gamma^{-1} (1 - 2\tau)(x - \bar{x}) \cdot (y - \bar{y}) + \gamma^{-2} |y - \bar{y}|^{2} \\
= \gamma^{-2} u (x - \bar{x}) \cdot (K(x - \bar{x})) + \frac{1}{2} |1 - 2\tau)(x - \bar{x}) + \gamma^{-1} (y - \bar{y})|^{2} + \frac{1}{2} \gamma^{-2} |y - \bar{y}|^{2},
\end{align*}
\]

and

\[
 r_{s}((x, y), (\bar{x}, \bar{y})) := \alpha |x - \bar{x}| + |x - \bar{x} + \gamma^{-1} (y - \bar{y})|,
\]

for $(x, y), (\bar{x}, \bar{y}) \in \mathbb{R}^{2d}$, where the constants $\tau$ and $\alpha$ are given by (21) and

\[
\alpha := 2(L_{K} + L_{g}) u \gamma^{-2},
\]

respectively. Next, we state the rigorous construction of the metric $\rho : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to [0, \infty)$ which is applied in Theorem 5 and Theorem 12, and that is glued together of $r_{t}$ and $r_{s}$ in an appropriate way. Note that $r_{t}$ and $r_{s}$ are equivalent metrics. More precisely, for all $(x, y), (\bar{x}, \bar{y}) \in \mathbb{R}^{2d}$ it holds $2r_{t}((x, y), (\bar{x}, \bar{y})) \leq r_{s}((x, y), (\bar{x}, \bar{y}))$ with

\[
\epsilon = (1/2) \min(1, (2/3) \alpha / (\sqrt{L_{K} u \gamma^{-1}}), \alpha).
\]

Indeed, for $(z, w) = (x - \bar{x}, y - \bar{y})$

\[
 r_{t}^{2}((x, y), (\bar{x}, \bar{y})) \leq L_{K} \gamma^{-2} u |z|^{2} + \frac{1}{2} |z + \gamma^{-1} w|^{2} + 2\gamma |z| |z + \gamma w| + 2\tau^{2} |z|^{2} + \frac{1}{2} |\gamma^{-1} w|^{2}
\]

and

\[
 r_{s}^{2}((x, y), (\bar{x}, \bar{y})) \geq \frac{1}{2} (|z|)^{2} + \frac{1}{2} \min(\alpha^{2}, 1) \gamma^{-2} |w|^{2}
\]

\[
= \frac{\alpha^{2}}{2} |z|^{2} + \alpha |z| |z + \gamma^{-1} w| + \frac{1}{2} |z + \gamma^{-1} w|^{2} + \frac{1}{2} \min(1, \alpha^{2}) \gamma^{-2} |w|^{2},
\]

and

\[
4\epsilon^{2} \leq \min \left( \frac{\alpha^{2}}{2(L_{K} u \gamma^{-2} + 2\tau L_{K} u \gamma^{-2}/2)}, (1, \alpha^{2}) \right) \leq \min \left( \frac{\alpha^{2}}{2(L_{K} u \gamma^{-2} + 2\tau^{2})}, (1, \frac{\alpha}{2\tau}, \alpha^{2}) \right),
\]
since $\alpha > \kappa \gamma^{-2}$ and $\tau \leq \min(1/8, L_{K} \gamma^{-2} u/2)$ by (28) and (21). Further, for all $(x,y), (\bar{x}, \bar{y}) \in \mathbb{R}^{d}$ it holds $E_{r_{s}}(x,y), (\bar{x}, \bar{y})) \leq r_{f}(x,y), (\bar{x}, \bar{y}))$ with
\begin{equation}
E = \min(\sqrt{\kappa u} \gamma^{-1} / (\sqrt{8} \alpha), 1/2),
\end{equation}

since
\begin{equation}
\frac{r_{s}(t)}{r_{f}(t)} \geq \left( \frac{\kappa u \gamma^{-2} |z|^{2} + (1/2)((1 - 2 \tau)z + \gamma^{-1}w|^{2})}{2(\alpha + 2 \tau)^{2} |z|^{2} + 2((1 - 2 \tau)z + \gamma^{-1}w|^{2})} \right)^{1/2} \geq \min \left( \frac{\sqrt{\kappa u} \gamma^{-1}}{\sqrt{8} \alpha}, \frac{1}{2} \right).
\end{equation}

Define
\begin{equation}
\Delta((x,y), (\bar{x}, \bar{y})) := r_{s}((x,y), (\bar{x}, \bar{y})) - \epsilon r_{f}((x,y), (\bar{x}, \bar{y}))
\end{equation}
for $(x,y), (\bar{x}, \bar{y}) \in \mathbb{R}^{d}$ and
\begin{equation}
D_{K} := \sup_{((x,y), (\bar{x}, \bar{y})) \in \mathbb{R}^{d} : (x-y, (\bar{x}, \bar{y}) \in K} \Delta((x,y), (\bar{x}, \bar{y}))
\end{equation}
where the compact set $K \subset \mathbb{R}^{d}$ is given by
\begin{equation}
K := \{(z,w) \in \mathbb{R}^{d} : \gamma^{-2} uz \cdot (Kz) + (1/2)((1 - 2 \tau)z + \gamma^{-1}w|^{2} + (1/2)|\gamma^{-1}w|^{2} \leq R}.
\end{equation}
with
\begin{equation}
R = (1/\tau)(8u \mathbb{1}_{\{R > 0} + L_{g}uR^{2}) \gamma^{-2}.
\end{equation}

We define the metric $\rho : \mathbb{R}^{d} \times \mathbb{R}^{d} \to [0, \infty)$ by
\begin{equation}
\rho((x,y), (\bar{x}, \bar{y})) := f(\Delta((x,y), (\bar{x}, \bar{y})) \wedge D_{K}) + \epsilon r_{f}((x,y), (\bar{x}, \bar{y}))
\end{equation}
for $(x,y), (\bar{x}, \bar{y}) \in \mathbb{R}^{d}$, where $\Delta$ and $D_{K}$ are given in (31) and (32). The function $f$ is an increasing concave function defined by
\begin{equation}
f(r) := \int_{0}^{r} \phi(s) \psi(s) ds,
\end{equation}
where
\begin{equation}
\phi(s) := \exp \left( - \frac{\alpha \gamma^{2} (s \wedge R_{1})^{2}}{4u} \right), \quad \Phi(s) = \int_{0}^{s} \phi(x) dx,
\end{equation}
\begin{equation}
\psi(s) := 1 - \frac{\hat{c}}{2} \gamma u^{-1} \int_{0}^{s \wedge R_{1}} \Phi(x) \phi(x)^{-1} dx, \quad \hat{c} = \frac{1}{\gamma u^{-1} \int_{0}^{R_{1}} \Phi(s) \phi(s)^{-1} ds},
\end{equation}
and where $R_{1}$ is given by
\begin{equation}
R_{1} := \sup_{((x,y), (\bar{x}, \bar{y})) \in \mathbb{R}^{d} : \Delta((x,y), (\bar{x}, \bar{y})) \leq D_{K}} r_{s}((x,y), (\bar{x}, \bar{y})).
\end{equation}
The construction of the function $f$ is adapted from [23]. Since $\psi(s) \in [1/2, 1]$, it holds for $r \geq 0$
\begin{equation}
f'(R_{1}) r = (\phi(R_{1})/2)r \leq \Phi(r)/2 \leq f(r) \leq \Phi(r) \leq r.
\end{equation}
Note that $R_{1}$ is finite and $R_{1} \leq \sup_{\Delta((x,y), (\bar{x}, \bar{y})) \leq D_{K}} 2\Delta((x,y), (\bar{x}, \bar{y})) \leq 2D_{K}$ holds, since $\Delta((x,y), (\bar{x}, \bar{y})) = r_{s}((x,y), (\bar{x}, \bar{y})) - \epsilon r_{f}((x,y), (\bar{x}, \bar{y})) \geq (1/2)r_{s}((x,y), (\bar{x}, \bar{y}))$ for any $(x,y), (\bar{x}, \bar{y}) \in \mathbb{R}^{d}$ by (29). Hence, $\hat{c}$ given in (37) and $f$ are well-defined. Further, $R_{1} \leq 2D_{K} \leq 2 \sup_{((x,y), (\bar{x}, \bar{y})) \in \mathbb{R}^{d} : (x-y, (\bar{x}, \bar{y}) \in K} (\mathcal{E}^{-1} - 2\epsilon)r_{f}((x,y), (\bar{x}, \bar{y})) \leq 2(\mathcal{E}^{-1} - 2\epsilon)\sqrt{R}$.
The constant $R_1$ is also bounded from below by
\[
R_1 \geq \sup_{((x,y), (\bar{x}, \bar{y})): \Delta((x,y), (\bar{x}, \bar{y})) \leq D_K} 2\epsilon r_1((x,y), (\bar{x}, \bar{y})) \geq 2\epsilon \sqrt{R},
\]

since $\Delta((x,y), (\bar{x}, \bar{y})) \leq D_K$ for all $(x,y), (\bar{x}, \bar{y}) \in \mathbb{R}^{2d}$ such that $r_1((x,y), (\bar{x}, \bar{y}))^2 = R$. By (34), (29), (30), the two bounds on $R_1$ imply the relation (23) of $R$ and $R_1$ given in Theorem 5.

By this construction for the metric $\rho$, it holds $\Delta((x,y), (\bar{x}, \bar{y})) \wedge D_K + \epsilon r_1((x,y), (\bar{x}, \bar{y})) = r_s((x,y), (\bar{x}, \bar{y}))$ for $\Delta((x,y), (\bar{x}, \bar{y})) \leq D_K$, and in particular for $r_s((x,y), (\bar{x}, \bar{y})) \leq R$.

Further, $(\Delta((x,y), (\bar{x}, \bar{y})) \wedge D_K) + \epsilon r_1((x,y), (\bar{x}, \bar{y})) = D_K + \epsilon r_1((x,y), (\bar{x}, \bar{y}))$ for $\Delta((x,y), (\bar{x}, \bar{y})) > D_K$ and in particular for $r_s((x,y), (\bar{x}, \bar{y})) > R_1$.

If $R = 0$, then $K = \{(0,0)\}$ and hence $D_K = R_1 = 0$ and $f(r) = r$. In this case, we can omit the factor $\epsilon$ in (35) and (46) and set $\rho((x,y), (\bar{x}, \bar{y})) = r_1((x,y), (\bar{x}, \bar{y}))$.

**Lemma 16.** The function $\rho$ given in (35) defines a metric on $\mathbb{R}^{2d}$ and is equivalent to the Euclidean distance on $\mathbb{R}^{2d}$.

**Proof.** Symmetry and positive definiteness hold directly. Hence, $\rho$ is a semimetric. To prove the triangle inequality, we note that for $(x,y), (\bar{x}, \bar{y}), (\tilde{x}, \tilde{y}) \in \mathbb{R}^{2d}$,
\[
(\Delta((x,y), (\bar{x}, \bar{y})) \wedge D_K + \epsilon r_1((x,y), (\bar{x}, \bar{y})))
= r_s((x,y), (\bar{x}, \bar{y})) \wedge (D_K + \epsilon r_1((x,y), (\bar{x}, \bar{y})))
\leq (r_s((x,y), (\bar{x}, \bar{y}))\wedge r_s((\tilde{x}, \tilde{y}), (\bar{x}, \bar{y}))\wedge (D_K + \epsilon r_1((x,y), (\bar{x}, \bar{y})))
\]

and
\[
\Delta((x,y), (\bar{x}, \bar{y})) \wedge D_K + \epsilon r_1((x,y), (\bar{x}, \bar{y})) \leq r_s((x,y), (\bar{x}, \bar{y})) \leq \max(\alpha + 1, \gamma^{-1}) \max(|x - \bar{x}| + |y - \bar{y}|)
\]

(40)

and
\[
\Delta((x,y), (\bar{x}, \bar{y})) \wedge D_K + \epsilon r_1((x,y), (\bar{x}, \bar{y})) \geq r_s((x,y), (\bar{x}, \bar{y})) \geq \epsilon r_1((x,y), (\bar{x}, \bar{y})) \geq \epsilon(\kappa u \gamma^{-2}|x - \bar{x}|^2 + \frac{1}{2} \gamma^{-2}|y - \bar{y}|^2)^{1/2}
\]

(41)

Then, by (39),
\[
C_1(|x,y| - (\bar{x}, \bar{y})| \leq \rho((x,y), (\bar{x}, \bar{y})) \leq C_2(|x,y| - (\bar{x}, \bar{y})|)
\]

(42)

with $C_1 = f'(R_1)\epsilon r_1 \min(\sqrt{\kappa u}, 1/\sqrt{2})$ and $C_2 = \sqrt{2} \max(\alpha + 1, \gamma^{-1})$.

\[\Box\]

4.2. Coupling for Langevin dynamics

To prove Theorem 5 and Theorem 12 we construct a coupling of two solutions to (1). The construction is partially adapted from the coupling approach introduced in [24]. Recall that $\bar{b} \equiv 0$ in Theorem 5.

Let $\xi$ be a positive constant, which we take finally to the limit $\xi \rightarrow 0$. Let $(B^e_t)_{t \geq 0}$ and $(\nu^e_t)_{t \geq 0}$ be two independent $d$-dimensional Brownian motions and let $\mu_0, \nu_0$ be two probability measures on $\mathbb{R}^{2d}$. The coupling $((\bar{X}_t, \bar{Y}_t), (\bar{Z}_t, \bar{Y}_t))_{t \geq 0}$...
of two copies of solutions to (1) is a solution to the SDE on \( \mathbb{R}^{2d} \times \mathbb{R}^{2d} \) given by

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dX_t}{dt} = Y_t dt \\
\frac{dY_t}{dt} = (-\gamma Y_t + ub(X_t)) + u \int_{\mathbb{R}^d} b(X_t,z)\hat{\mu}^d_t(dz))dt + \sqrt{2}u\alpha \sigma(Z_t, W_t) dB^{dc}_t + \sqrt{2}\gamma \sigma(Z_t, W_t) dB^{rc}_t \\
\end{array} \right.
\end{align*}
\]

(43)

where \( \hat{\mu}^d_t = \text{Law}(X_t) \) and \( \hat{\sigma}^d_t = \text{Law}(Y_t) \). Further, \( Z_t = X_t - X'_t, W_t = Y_t - Y'_t, Q_t = Z_t + \gamma^{-1} W_t \) and \( e_t = Q_t/Q_t \) if \( Q_t \neq 0 \) and \( e_t = 0 \) otherwise. The functions \( rc, sc : \mathbb{R}^2 \to [0, 1] \) are Lipschitz continuous and satisfy \( rc^2 + sc^2 \equiv 1 \) and

\[
\begin{align*}
rc(z, w) = 0 & \quad \text{if } |z + \gamma^{-1} w| = 0 \quad \text{or } (r_s(z, w)) \neq \epsilon(rl(z, w)) \geq D_K + \xi \cdot 1_{\{D_K > 0\}}, \\
rc(z, w) = 1 & \quad \text{if } |z + \gamma^{-1} w| \geq \xi \quad \text{and } (r_s(z, w)) \neq \epsilon(rl(z, w)) \leq D_K \quad \text{and } D_K > 0
\end{align*}
\]

(44)

for \((z, w) \in \mathbb{R}^2 \), where \( \epsilon \) is given in (29). Analogously to (26) and (27), \( r_i(z, w) = \gamma^{-1} u \alpha \cdot (K z + (1/2)(1 - 2\tau)z + \gamma^{-1} w| + (1/2)\gamma^{-1} w^2 \) and \( r_s(z, w) = \alpha |z| + |z + \gamma^{-1} w| \).

We note that by Levy’s characterization, for any solution to (74) the processes

\[
B_t := \int_0^t sc(Z_s, W_s) dB^{sc}_s + \int_0^t rc(Z_s, W_s) dB^{rc}_s
\]

\[
\tilde{B}_t := \int_0^t sc(Z_s, W_s) dB^{sc}_s + \int_0^t rc(Z_s, W_s)(Id - e_s e_s^T) dB^{rc}_s
\]

are \( d \)-dimensional Brownian motions. Therefore, (43) defines a coupling between two solutions to (1). The constructed coupling denotes a reflection coupling for \( rc \equiv 1 \) and \( sc \equiv 0 \) and a synchronous coupling for \( sc \equiv 1 \) and \( rc \equiv 0 \). Note that we obtain a synchronous coupling if \( D_K = 0 \).

The processes \((Z_t)_{t \geq 0}, (W_t)_{t \geq 0}\) and \((Q_t)_{t \geq 0}\) satisfy the following SDEs:

\[
\begin{align*}
dZ_t = W_t dt &= (Q_t - \gamma Z_t) dt, \\
dW_t = -\gamma W_t dt + u \int_{\mathbb{R}^d} b(X_t, z)\hat{\mu}^d_t(dz) - \int_{\mathbb{R}^d} b(\hat{X}_t, \hat{z})\hat{\sigma}^d_t(d\hat{z}) dt + \sqrt{2}\gamma \sigma(Z_t, W_t) dB^{rc}_t, \\
\end{align*}
\]

(45)

\[
\begin{align*}
dQ_t = \gamma^{-1} u \left( b(X_t) - b(\hat{X}_t) \right) dt + \int_{\mathbb{R}^d} b(\hat{X}_t, \hat{z})\hat{\mu}^d_t(dz) - \int_{\mathbb{R}^d} b(\hat{X}_t, \hat{z})\hat{\sigma}^d_t(d\hat{z}) dt + \sqrt{2}\gamma^{-1} \sigma(Z_t, W_t) e_t e_t^T dB^{rc}_t.
\end{align*}
\]

If \( Q_t = 0 \), we note that \( Z_t \) is contractive, which we exploit in the proof of Lemma 20.

5. Uniform in time propagation of chaos

We provide uniform in time propagation of chaos bounds for the mean-field particle system corresponding to the nonlinear Langevin dynamics of McKean-Vlasov type.

Fix \( N \in \mathbb{N} \). We consider the metric \( \rho_N : \mathbb{R}^{2Nd} \times \mathbb{R}^{2Nd} \to [0, \infty) \) given by

\[
\rho_N((x, y), (\bar{x}, \bar{y})) := N^{-1} \sum_{i=1}^N \rho((x^i, y^i), (\bar{x}^i, \bar{y}^i)) \quad \text{for } ((x, y), (\bar{x}, \bar{y})) \in \mathbb{R}^{2Nd} \times \mathbb{R}^{2Nd},
\]

(46)

where \( \rho \) is given in (35). Since \( \rho \) is a metric on \( \mathbb{R}^{2d} \times \mathbb{R}^{2d} \) by Lemma 16, \( \rho_N \) defines a metric on \( \mathbb{R}^{2Nd} \times \mathbb{R}^{2Nd} \). By (40) and (41), \( \rho_N \) is equivalent to \( l_N \) given in (5), i.e.,

\[
C_1/\sqrt{2}l_N((x, y), (\bar{x}, \bar{y})) \leq \rho_N((x, y), (\bar{x}, \bar{y})) \leq C_2/\sqrt{2}l_N((x, y), (\bar{x}, \bar{y}))
\]

(47)
with \( C_1 = \exp(-\Lambda) \min(1, 2(L_K + L_g)u\gamma^{-2})/3\gamma^{-1} \min(\sqrt{\kappa u}, 1/\sqrt{2}) \) and \( C_2 = \sqrt{2} \max(2(L_K + L_g)u\gamma^{-2} + 1, \gamma^{-1}) \).

For \( t \geq 0 \), we denote by \( \tilde{\mu}_t \) the law of the process \((\tilde{X}_t, \tilde{Y}_t)\), where \((\tilde{X}_s, \tilde{Y}_s)_{s \geq 0}\) is a solution to (1) with initial distribution \( \tilde{\mu}_0 \). We denote by \( \mu^N_t \) the law of \( \{X^{i,N}_t, Y^{i,N}_t\}_{i=1}^N \), where \( \{X^{i,N}_t, Y^{i,N}_t\}_{i=1}^N \) is a solution to (3) with initial distribution \( \mu^{0,N}_0 = \mu^{0,N}_0 \).

**Theorem 17** (Propagation of chaos for Langevin dynamics). Suppose Assumption 2 and Assumption 3 hold. Let \( \tilde{\mu}_0 \) and \( \mu_0 \) be two probability distributions on \( \mathbb{R}^{2d} \) with finite second moment. Suppose that (18) holds. If \( \tilde{L} \) satisfies (25), then

\[
W_{1,\rho_N}(\tilde{\mu}^{N}_t, \mu^{N}_t) \leq e^{-\tilde{c}t}W_{1,\rho_N}(\tilde{\mu}^{N}_0, \mu^{N}_0) + C_1 \tilde{c}^{-1}N^{-1/2} \quad \text{and} \quad W_{1,\tilde{\rho}}(\tilde{\mu}^{N}_t, \mu^{N}_t) \leq M_1 e^{-\tilde{c}t}W_{1,\tilde{\rho}}(\tilde{\mu}^{0,N}_0, \mu^{0,N}_0) + M_2 \tilde{c}^{-1}N^{-1/2},
\]

where the distance \( \rho_N \) is defined in (46) and \( \tilde{c} = c/2 \) with \( c \) given in (19). The constant \( C_1 \) depends on \( \gamma, d, u, R, \kappa, L_g, \tilde{L} \) and on the second moment of \( \mu_0 \). The constants \( M_1 \) is given in (24) and \( M_2 \) is given by

\[
M_2 = 3\exp(\Lambda) \max\left(1, \frac{\gamma^2}{2(L_K + L_g)u}\right) \max(\sqrt{2/(\kappa u)}, 2).
\]

**Proof.** The proof is postponed to Section 6.3.

**Remark 18.** For \( t \geq 0 \), let \( \mu^{N}_t \) and \( \nu^{N}_t \) be the law of \( \{X^{i,N}_t, Y^{i,N}_t\}_{i=1}^N \) and \( \{X^{'i,N}_t, Y^{'i,N}_t\}_{i=1}^N \), where the processes \( \{X^{i,N}_t, Y^{i,N}_t\}_{i=1}^N \) and \( \{X^{'i,N}_t, Y^{'i,N}_t\}_{i=1}^N \) are solutions to (3) with initial distributions \( \mu^{0,N}_0 \) and \( \nu^{0,N}_0 \), respectively. An easy adaptation of the proof of Theorem 17 shows that if Assumption 2, Assumption 3, (18) and (25) hold, then

\[
W_{1,\rho_N}(\mu^{N}_t, \nu^{N}_t) \leq e^{-\tilde{c}t}W_{1,\rho_N}(\mu^{0,N}_0, \nu^{0,N}_0) \quad \text{and} \quad W_{1,\tilde{\rho}}(\mu^{N}_t, \nu^{N}_t) \leq M_1 e^{-\tilde{c}t}W_{1,\tilde{\rho}}(\mu^{0,N}_0, \nu^{0,N}_0),
\]

where \( \rho_N \) and \( M_1 \) are given in (46), and (24), respectively, and \( \tilde{c} = c/2 \) with \( c \) given in (19). To adapt the proof, a coupling between two copies of \( N \) particle systems is applied which is constructed in the same line as (74).

### 6. Proofs

#### 6.1. Proof of Section 2.1

**Proof of Theorem 1.** Given a \( d \)-dimensional standard Brownian motion on \( (B_t)_{t \geq 0} \) and \( (x, y), (x', y') \in \mathbb{R}^{2d} \), we consider the synchronous coupling \((X_t, Y_t), (X'_t, Y'_t))_{t \geq 0} \) of two copies of solutions to (6) on \( \mathbb{R}^{2d} \times \mathbb{R}^{2d} \) given by

\[
\begin{align*}
\frac{dX_t}{dt} &= Y_t dt \\
\frac{dY_t}{dt} &= (-\gamma Y_t - u\nabla V(X_t))dt + \sqrt{2\gamma u} dB_t, \quad (X_0, Y_0) = (x, y) \\
\frac{dX'_t}{dt} &= Y'_t dt \\
\frac{dY'_t}{dt} &= (-\gamma Y'_t - u\nabla V(X'_t))dt + \sqrt{2\gamma u} dB_t, \quad (X'_0, Y'_0) = (x', y').
\end{align*}
\]

Then, the difference process \((Z_t, W_t))_{t \geq 0} = (X_t - X'_t, Y_t - Y'_t)_{t \geq 0}

satisfies

\[
\begin{align*}
\frac{dZ_t}{dt} &= W_t dt \\
\frac{dW_t}{dt} &= (-\gamma W_t - uK Z_t - u(\nabla G(X_t) - \nabla G(X'_t)))dt.
\end{align*}
\]

Since \( G \) is continuously differentiable, convex and has \( L_G \)-Lipschitz continuous gradients by Assumption 1, \( G \) is co-cocercive (see e.g. [40, Theorem 2.1.5]), i.e., it holds

\[
\|\nabla G(x) - \nabla G(x')\|^2 \leq L_G(\nabla G(x) - \nabla G(x')) \cdot (x - x') \quad \text{for all} \ x, x' \in \mathbb{R}^d.
\]

Let \( A, B, C \in \mathbb{R}^{d \times d} \) be positive definite matrices given by

\[
A = \gamma^{-2}uK + (1/2)(1 - 2\lambda)^2 I_d, \quad B = (1 - 2\lambda)\gamma^{-1} I_d, \quad C = \gamma^{-2} I_d,
\]
where $\lambda$ is given in (9) and $\text{Id}$ is the $d \times d$ identity matrix. Then by Itô’s formula and Young’s inequality, we obtain
\begin{equation}
\frac{d}{dt} (Z_t \cdot (AZ_t) + Z_t \cdot (BW_t) + W_t \cdot (CW_t)) \\
= 2W_t \cdot (AZ_t) + W_t \cdot (BW_t) + Z_t \cdot (B(-\gamma W_t - uKZ_t - u(\nabla G(X_t) - \nabla G(X'_t)))) \\
+ 2W_t \cdot (C(-\gamma W_t - uKZ_t - u(\nabla G(X_t) - \nabla G(X'_t))))
\end{equation}
\begin{equation}
\leq -\gamma^{-1}u(1 - 2\lambda)Z_t \cdot (KZ_t) - (1 - 2\lambda)\gamma^{-1}uZ_t(\nabla G(X_t) - \nabla G(X'_t)) + \gamma^{-3}u^2|\nabla G(X_t) - \nabla G(X'_t)|^2 \\
+ Z_t \cdot ((2A - \gamma B - 2uKC)W_t) + ((1 - 2\lambda)\gamma^{-1} - \gamma^{-1})|W_t|^2.
\end{equation}
By (50), (9) and (10), it holds
\begin{equation}
-(1 - 2\lambda)\gamma^{-1}uZ_t \cdot (\nabla G(X_t) - \nabla G(X'_t)) + \gamma^{-3}u^2|\nabla G(X_t) - \nabla G(X'_t)|^2 \\
\leq -((1 - 2\lambda)\gamma^{-1}u - \gamma^{-3}L_Gu^2)Z_t(\nabla G(X_t) - \nabla G(X'_t)) \leq 0.
\end{equation}
Further by (9), it holds
\[-u\gamma^{-1}(1 - 4\lambda)Z_t \cdot (KZ_t) \leq -u(\gamma^{-1}/2)Z_t \cdot (KZ_t) \leq -u(\gamma^{-1}/2)\kappa|Z_t|^2 \leq -\lambda|Z_t|^2 \leq -\lambda(1 - 2\lambda)^2|Z_t|^2.
\]
Hence, $-u\gamma^{-1}(1 - 2\lambda)Z_t \cdot (KZ_t) \leq -2\lambda Z_t \cdot (AZ_t)$.

Set $r(t) = r((X_t, Y_t), (X'_t, Y'_t))$ with $r$ defined in (8). By (51) and (52), we obtain
\begin{equation}
\frac{d}{dt} r(t)^2 = \frac{d}{dt} (Z_t \cdot (AZ_t) + Z_t \cdot (BW_t) + W_t \cdot (CW_t)) \\
\leq -2\lambda Z_t \cdot (AZ_t) + Z_t \cdot (BW_t) + W_t \cdot (CW_t) = -2\lambda r(t)^2.
\end{equation}
Taking the square root and applying Grönwall’s inequality yields
\[r(t) \leq e^{-ct}r(0)
\]
with $c$ given in (11). Then for all $p \geq 1$ it holds
\[W_{p,r}(\mu_t, \nu_t) \leq \mathbb{E}[r(t)^p]^{1/p} \leq e^{-ct}\mathbb{E}[r(0)^p]^{1/p}.
\]
We take the infimum over all couplings $\gamma \in \Pi(\mu_0, \nu_0)$ and obtain the first bound. For the second bound we note that for any $(x, y), (x', y') \in \mathbb{R}^{2d}$
\[\sqrt{\min(u\gamma^{-2}\kappa, \gamma^{-2}/2)}(|x - x'|^2 + |y - y'|^2)^{1/2} \leq r((x, y), (x', y'))
\]
\[\leq \sqrt{\max(u\gamma^{-2}L_K + 1, 3/2\gamma^{-2})}(|x - x'|^2 + |y - y'|^2)^{1/2}.
\]
Hence, the second bound in Theorem 1 holds with $M$ given in (12).
\]}

\section{Proofs of Section 2.4 and Section 3}

To show Theorem 12, we prove two local contraction results using the coupling defined in (43). We write $r_t(t) = r_t((X_t, Y_t), (X'_t, Y'_t))$, $r_s(t) = r_s((X_t, Y_t), (X'_t, Y'_t))$ and $\Delta(t) = \Delta((X_t, Y_t), (X'_t, Y'_t))$.

\begin{lemma}
Suppose Assumption 2, Assumption 3 and (18) hold. Let $((X_s, Y_s), (X'_s, Y'_s))_{s \geq 0}$ be a solution to (43). Then for $t \geq 0$ with $\Delta(t) \leq D_K$, it holds
\begin{equation}
dr_t(t) \leq -c_1 r_t(t)dt + \frac{|(1 - 2\tau)Z_t + 2\gamma^{-1}W_t|}{2\gamma r_t(t)} \tilde{L}_u(|Z_t| + |Z_t|)dt \\
+ \sqrt{8\gamma^{-1}urc(Z_t, W_t)} \frac{|(1 - 2\tau)Z_t + 2\gamma^{-1}W_t|}{2r_t(t)} \cdot e_t e_t^T dB_t^c,
\end{equation}
where $c_1 = \tau \gamma/2$ with $\tau$ given in (21).
\]
Proof. Let $A, B, C \in \mathbb{R}^{d \times d}$ be positive definite matrices given by

$$A = \gamma^{-2}uK + (1/2)(1 - 2\tau)^2\text{Id}, \quad B = (1 - 2\tau)\gamma^{-1}\text{Id}, \quad \text{and} \quad C = \gamma^{-2}\text{Id},$$

where $\tau$ is given by (21) and $\text{Id}$ is the $d \times d$ identity matrix. By (45), Itô’s formula and Young’s inequality, it holds

$$\frac{d}{dt}(Z_t \cdot (AZ_t) + Z_t \cdot (BW_t) + W_t \cdot (CW_t))$$

$$\leq 2(AZ_t) \cdot W_t dt + \left((BW_t) - (BZ_t) \cdot W_t - u(BZ_t) \cdot (KZ_t) + L_g u (1 - 2\tau) \gamma^{-1} |Z_t|^2 \cdot 1_{\{|Z_t| < R\}}\right) dt$$

$$+ \left(-2W_t \cdot (CW_t) - 2u(CW_t) \cdot (KZ_t) + 2\gamma^{-2}L_g u |W_t||Z_t| + BZ_t + 2CW_t|Lu(E||Z_t|| + |Z_t|)dt$$

$$+ \gamma^{-2}8\gamma u c_r(Z_t, W_t)dt + \sqrt{8\gamma u c_r(Z_t, W_t)}(BZ_t + 2CW_t) \cdot e_t e_t^T dB_t^{rc}$$

$$\leq Z_t \cdot ((-uBK + \gamma^{-1}uL_g^2C)Z_t) dt + Z_t \cdot (2A - \gamma B - 2uKC)W_t dt + ((1 - 2\tau)\gamma^{-1} - 1) |W_t|^2 dt$$

$$+ (1 - 2\tau)\gamma^{-1}uL_g |Z_t|^2 1_{\{|Z_t| < R\}} dt + ((1 - 2\tau)\gamma^{-1}Z_t + 2\gamma^{-2}W_t|\tilde{L}u(E||Z_t|| + |Z_t|)dt$$

$$+ 8\gamma^{-1}u(rc(Z_t, W_t))^2 dt + \sqrt{8\gamma u c_r(Z_t, W_t)}((1 - 2\tau)\gamma^{-1}Z_t + 2\gamma^{-2}W_t) \cdot e_t e_t^T dB_t^{rc}$$

$$\leq -2\gamma(Z_t \cdot (AZ_t) + Z_t \cdot (BW_t) + W_t \cdot (CW_t)) dt$$

$$+ (1 - 2\tau)\gamma^{-1}uL_g |Z_t|^2 1_{\{|Z_t| < R\}} dt + ((1 - 2\tau)\gamma^{-1}Z_t + 2\gamma^{-2}W_t|\tilde{L}u(E||Z_t|| + |Z_t|)dt$$

$$+ 8\gamma^{-1}u(rc(Z_t, W_t))^2 dt + \sqrt{8\gamma u c_r(Z_t, W_t)}((1 - 2\tau)\gamma^{-1}Z_t + 2\gamma^{-2}W_t) \cdot e_t e_t^T dB_t^{rc},$$

where we used (21) in the last step. More precisely, the definition of $\tau$ implies for all $z \in \mathbb{R}^d$,

$$z \cdot ((-1 - 4\tau)\gamma^{-1}uK + \gamma^{-3}L_g^2u^2\text{Id})z \leq (-1/2)\kappa u \gamma^{-1} + \gamma^{-3}L_g^2u^2||z||^2$$

$$\leq (-\tau\gamma)||z||^2 \leq (-\tau\gamma(1 - 2\tau)||z||^2.$$

Note that $r_1(t)^2 = Z_t \cdot (AZ_t) + Z_t \cdot (BW_t) + W_t \cdot (CW_t).$ Then,

$$dr_1(t)^2 \leq -2\gamma r_1(t)^2 dt + \gamma^{-1}(1 - 2\tau)L_g u |Z_t|^2 1_{\{|Z_t| < R\}} dt + \gamma^{-1}(1 - 2\tau)Z_t + 2\gamma^{-1}W_t|\tilde{L}u(E||Z_t|| + |Z_t|)dt$$

$$+ 8\gamma^{-1}u c_r(Z_t, W_t)^2 dt + \sqrt{8\gamma^{-1}u c_r(Z_t, W_t)}((1 - 2\tau)Z_t + 2\gamma^{-1}W_t) \cdot e_t e_t^T dB_t^{rc}.$$

Since $\Delta(t) \geq D_K$, it holds $r_1(t)^2 \geq R$ by (32) and (33). By (44), $r_1(Z_t, W_t)^2 \leq 1_{\{R > 0\}},$ and hence, by (34)

$$-\tau r_1(t)^2 + \gamma^{-1}(1 - 2\tau)L_g u |Z_t|^2 1_{\{|Z_t| < R\}} + 8\gamma^{-1}u c_r(Z_t, W_t)^2 \leq -\tau R + L_g u R^2 \gamma^{-1} + 8\gamma^{-1}u 1_{\{R > 0\}} \leq 0.$$

We obtain by Itô’s formula and since the second derivative of the square root is negative,

$$dr_1(t) \leq (2r_1(t))^{-1}dr_1(t)^2 \leq -c_1 r_1(t) dt + \gamma^{-1}(1 - 2\tau)Z_t + 2\gamma^{-1}W_t|2r_1(t))^{-1}|\tilde{L}u(E||Z_t|| + |Z_t|)dt$$

$$+ \sqrt{8\gamma^{-1}u c_r(Z_t, W_t)(2r_1(t))^{-1}((1 - 2\tau)Z_t + 2\gamma^{-1}W_t) \cdot e_t e_t^T dB_t^{rc}},$$

which concludes the proof.

Lemma 20. Suppose Assumption 2 and Assumption 3 hold. Fix $\xi > 0.$ Let $(\tilde{X}_s, \tilde{Y}_s, (\tilde{X}_s', \tilde{Y}_s'))_{s \geq 0}$ be a solution to (43). Let $r_s$ be given by (27) with $\alpha$ given in (28). Then for $t \geq 0$ with $\Delta(t) < D_K,$ it holds

$$df(r_s(t)) \leq -c_2 f(r_s(t)) dt + \gamma^{-1}\tilde{L}u(E||Z_t|| + |Z_t|) dt - \frac{\gamma \alpha}{4} f'(R_1)|Z_t| dt + (1 + \alpha)\xi \gamma dt + dM_t,$$

where $f$ is given in (36), $(M_t)_{t \geq 0}$ is a martingale and $c_2$ is given by

$$c_2 := \min \left(\frac{2}{\gamma u^{-1} F^R_{t \geq 0} \Psi(s)} \frac{\gamma R_1 \Phi(R_1)}{8 \Phi'(R_1)}\right),$$
By \((45)\), Itô’s formula, Assumption 2 and Assumption 3, we obtain for \(|Z_t|\)

\[\frac{d}{dt}|Z_t| = Z_t \cdot (-\gamma Z_t + \gamma Q_t)\]

for a.e. \(t\) such that \(Z_t \neq 0\) and

\[\frac{d}{dt}|Z_t| \leq \gamma|Q_t|\]

for a.e. \(t\) such that \(Z_t = 0\).

and therefore

\[(57) \quad \frac{d}{dt}|Z_t| \leq -\gamma|Z_t| + \gamma|Q_t|\] for a.e. \(t \geq 0\).

By \((45)\), Itô’s formula, Assumption 2 and Assumption 3, we obtain for \(|Q_t|\),

\[d|Q_t| = \gamma^{-1}u \cdot \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_i}(X_t) \right) - \frac{\partial}{\partial t} f(X_t)\right) dt + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_j}(X_t) \right) dW_t^j + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_j}(X_t) \right) dW_t^j\]

Note that there is no Itô correction term, since \(\phi_{|q|} = 0\) for \(q \neq 0\) and \(rc = 0\) for \(Q_t = 0\). Combining this bound with \((57)\) yields for \(r_s(t)\),

\[dr_s(t) \leq \left(\left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_i}(X_t) \right) - \frac{\partial}{\partial t} f(X_t)\right) dt + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_j}(X_t) \right) dW_t^j + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_j}(X_t) \right) dW_t^j\right) dt + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_j}(X_t) \right) dW_t^j + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_j}(X_t) \right) dW_t^j\]

By Itô’s formula,

\[df(r_s(t)) \leq f'(r_s(t)) \left(\left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_i}(X_t) \right) - \frac{\partial}{\partial t} f(X_t)\right) dt + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_j}(X_t) \right) dW_t^j + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_j}(X_t) \right) dW_t^j\right) dt + f''(r_s(t)) 4\gamma^{-1}uralc(Z_t, W_t)^2 dt\]

**Case 1:** Consider \(\Delta(t) < D_K\) and \(|Q_t| > \xi\), then \(rc(Z_t, W_t) = 1\) and \(r_s(t) < R_1\). Hence, we obtain

\[df(r_s(t)) \leq f'(r_s(t)) \alpha r_s(t) dt + f''(r_s(t)) 4\gamma^{-1} u dt + \gamma^{-1} Lu(E[|Z_t|] + |Z_t|) dt - \frac{\alpha \gamma}{2} |Z_t| f'(r_s(t)) dt + dM_t\]

\[-2c f(r_s(t)) dt + \gamma^{-1} Lu(E[|Z_t|] + |Z_t|) dt - \frac{\alpha \gamma}{2} |Z_t| f'(r_s(t)) dt + dM_t\]

where \((M_t)_{t \geq 0}\) is a martingale and \(c\) is given in \((37)\). Note that the second step holds since by \((36)\) and \((39)\),

\[f'(r) \alpha \gamma r + f''(r) 4\gamma^{-1} u \leq -2\gamma f(r)\]

for all \(r \in [0, R_1]\).

**Case 2:** Consider \(\Delta(t) < D_K\) and \(|Q_t| \leq \xi\), then \(\alpha |Z_t| = r_s(t) - |Q_t| \geq r_s(t) - \xi\). We note that

\[((L_K + L_g) u \gamma^{-2} - \alpha)|Z_t| + \alpha |Q_t| \leq \frac{1}{2} r_s(t) + (1 + \alpha) \xi\]

Since the second derivative of \(f\) is negative and \(\psi(s) \in [1/2, 1]\), it holds

\[df(r_s(t)) \leq -\frac{\alpha r_s(t)}{2} f'(r_s(t)) dt + (1 + \alpha) \gamma \xi dt + \gamma^{-1} Lu(E[|Z_t|] + |Z_t|) dt + dM_t\]

\[\leq -\frac{\alpha}{8} \int_{r_s(t)} \Phi(r) f'(r) dt - \frac{\alpha}{4} f'(r_1) |Z_t| dt + (1 + \alpha) \gamma \xi dt + \gamma^{-1} Lu(E[|Z_t|] + |Z_t|) dt + dM_t\]

Combining the two cases, we obtain the result with \(c_2\) given in \((56)\).
Proof of Theorem 17. To prove contraction, we consider the coupling \(((\tilde{X}_t, \tilde{Y}_t), (\tilde{X}'_t, \tilde{Y}'_t))_{t \geq 0}\) given in (43) and combine the results of Lemma 19 and Lemma 20. We abbreviate \(\rho(t) = f((\Delta(t) \land D_K) + cr(t))\). We distinguish two cases:

Case 1: Consider \(\Delta(t) < D_K\). Then \(r_\gamma(t) \leq R_1\) and \(\rho(t) = f(r_\gamma(t))\). By Lemma 20, it holds for \(\xi > 0\)

\[
d\rho(t) = df(r_\gamma(t)) dt + \gamma^{-1} \tilde{L} u \mathbb{E}[|Z_t| + |Z_t|] dt - \frac{\alpha \gamma}{4} f'(R_1)|Z_t| dt + (1 + \alpha) \gamma \xi dt + dM_t
\]

(60)

\[
\leq -c_2 f(r_\gamma(t)) dt + \gamma^{-1} \tilde{L} u \mathbb{E}[|Z_t|] dt - \frac{\alpha \gamma}{8} f'(R_1)|Z_t| dt + (1 + \alpha) \gamma \xi dt + dM_t,
\]

where \(c_2\) is given by (56) and \((M_t)_{t \geq 0}\) is a martingale. The second step holds by (25).

Case 2: Consider \(\Delta(t) \geq D_K\). We obtain by Lemma 19,

\[
d r_\gamma(t) \leq -c_1 r_\gamma(t) dt + \frac{|(1 - 2r)| Z_t + 2\gamma^{-1} W_t| \tilde{L} u \mathbb{E}[|Z_t| + |Z_t|]}{2 r_\gamma(t)} dt
\]

\[
+ \sqrt{8\gamma^{-1} \text{urc}(Z_t, W_t) \frac{|(1 - 2r)| Z_t + 2\gamma^{-1} W_t|}{2 r_\gamma(t)}} \cdot e_t e_t^T dB_t^\text{re},
\]

where \(c_1\) is given in Lemma 19. Note that \(\frac{d}{dt} f(D_K + \epsilon x) = f'(D_K + \epsilon x)\). Further, since \(f(D_K + \epsilon x)\) is a concave function, \(\frac{d^2}{dx^2} f(D_K + \epsilon x)\) is negative. By Itô’s formula, we obtain

\[
d \rho(t) = df(D_K + \epsilon r(t)) dt
\]

(61)

\[
\leq f'(D_K + \epsilon r(t)) \left(-c_1 r(t) + \frac{|(1 - 2r)| Z_t + 2\gamma^{-1} W_t| \tilde{L} u \mathbb{E}[|Z_t| + |Z_t|]}{2 \gamma r(t)} \right) dt + d\tilde{M}_t,
\]

where \(\tilde{M}_t\) is a martingale given by

\[
\tilde{M}_t = \int_0^t \frac{f'(D_K + \epsilon r(s))}{2 r_\gamma(s)} \sqrt{8\gamma^{-1} \text{urc}(Z_s, W_s)} \frac{|(1 - 2r)| Z_s + 2\gamma^{-1} W_s|}{2 r_\gamma(s)} \cdot e_s e_s^T dB_s^\text{re}.
\]

We split the first term of (61) and bound each part applying (39),

\[
-\frac{f'(D_K + \epsilon r(t))}{2} c_1 r(t) \leq -\left\{ \inf_{q \geq 0} \frac{f'(q)}{f(q)} \right\} \frac{c_1 \gamma r(t)}{2(D_K + \epsilon r(t))} \rho(t) \leq -f'(R_1) \frac{c_1 \gamma r(t)}{2(D_K + \epsilon r(t))} \rho(t)
\]

and

\[
-\frac{f'(D_K + \epsilon r(t))}{2} c_1 r(t) \leq -f'(R_1) \frac{c_1 \gamma r(t)}{2} r(t).
\]

Since \(\Delta(t) > D_K\), it holds

\[
\frac{r_\gamma(t)}{D_K + \epsilon r(t)} \geq \frac{r_\gamma(t)}{r_\gamma(t)} \geq \mathcal{E},
\]

where \(\mathcal{E}\) is given in (30). Hence, we obtain for the first term of (61), by (63), (64) and (65)

\[
-f'(D_K + \epsilon r(t)) c_1 r(t) \leq -f'(R_1) \frac{c_1 \gamma \mathcal{E}}{2} \rho(t) - f'(R_1) \frac{c_1 \gamma}{2} r(t).
\]

For the second term of (61), we obtain

\[
f'(D_K + \epsilon r(t)) \frac{|(1 - 2r)| Z_t + 2\gamma^{-1} W_t|}{2 \gamma r(t)} \leq \frac{\epsilon}{2 \gamma} \sqrt{\frac{(1 - 2r)^2 |Z_t|^2 + 4(1 - 2r) \gamma^{-1} Z_t \cdot W_t + 4\gamma^{-2} |W_t|^2}{(1/2)(1 - 2r)^2 |Z_t|^2 + (1 - 2r) \gamma^{-1} Z_t \cdot W_t + \gamma^{-2} |W_t|^2}} \leq \frac{\epsilon}{\gamma}.
\]

Combining (66) and (67) yields,

\[
d \rho(t) \leq -f'(R_1) \frac{c_1 \gamma \mathcal{E}}{2} \rho(t) dt - f'(R_1) \frac{c_1 \gamma}{2} r(t) dt + \gamma^{-1} \tilde{L} u \mathbb{E}[|Z_t| + |Z_t|] dt + d\tilde{M}_t
\]

(68)

\[
\leq -f'(R_1) \frac{c_1 \gamma \mathcal{E}}{2} \rho(t) dt - f'(R_1) \frac{c_1 \gamma}{2} \sqrt{\gamma \omega \gamma^{-2} |Z_t|^2 + \frac{1}{2} \gamma^{-1} \tilde{L} u \mathbb{E}[|Z_t| + |Z_t|] dt + d\tilde{M}_t,
\]
where \( r_j(t) \geq \sqrt{k}\gamma^{-2}|Z_t| \) and \( 2c \leq 1 \) are applied and where \( \langle M_t \rangle_{t \geq 0} \) is given in (62).

Combining (60) and (68), taking expectation and \( \xi \to 0 \), yields
\[
\frac{d}{dt} \mathbb{E}[\rho(t)] \leq -\min \left( c_2, f'(R_1) \frac{c_1 t \epsilon E}{2} \right) \mathbb{E}[\rho(t)] - \min \left( f'(R_1) \frac{c_1 t \epsilon E}{8}, f'(R_1) \frac{c_1 t \epsilon E}{2} \sqrt{k}\gamma^{-2}|Z_t| + \gamma^{-1}\hat{L}u\mathbb{E}[|Z_t|] \right)
\]
\[
\leq -\min \left( c_2, f'(R_1) \frac{c_1 t \epsilon E}{2} \right) \mathbb{E}[\rho(t)],
\]
where we used (25) and (28) in the second step. By applying Grönwall’s inequality, we obtain
\[
\mathcal{W}_{1,\rho}(\bar{\mu}_t, \bar{\nu}_t) \leq \mathbb{E}[\rho(t)] \leq e^{-c_3 t} \mathbb{E}[\rho(0)]
\]
with
\[
(69) \quad c_3 = \min \left( \frac{2}{\gamma u^{-1} \int_0^{R_1} \Phi(s) \phi(s) - 1} ds, \frac{2}{8} f'(R_1) \gamma \epsilon E \right).
\]
The term \( \epsilon E \) is bounded from below by \( E \) given in (22). For the first two arguments in the minimum we note that
\[
\int_0^{R_1} \int_0^s \exp \left( -\frac{\alpha \gamma^2 s^2}{2} \right) ds \leq \sqrt{\frac{2}{\pi}} \left( \alpha \gamma^2 \right)^{-1/2} \int_0^{R_1} \exp \left( -\frac{\alpha \gamma^2 s^2}{2} \right) ds
\]
\[
\leq \sqrt{\frac{2}{\pi}} \left( \alpha \gamma^2 \right)^{-1/2} \left( \frac{\alpha \gamma^2 R_1^2}{4} \right) \left( \frac{\alpha \gamma^2 R_1^2}{4} \right) \leq 4 \left( \frac{\alpha \gamma^2 R_1^2}{4} \right) \left( \frac{\alpha \gamma^2 R_1^2}{4} \right) \left( \frac{\alpha \gamma^2 R_1^2}{4} \right)
\]
since \( \int_0^s \exp(r^2/2) dr \leq 2x^{-1} \exp(x^2/2) \), and
\[
(70) \quad \frac{R_1 \Phi(R_1)}{\Phi(R_1)} \geq \frac{R_1 \exp(-\frac{\alpha \gamma^2 R_1^2}{4})}{\sqrt{\frac{2}{\pi} \left( \frac{\alpha \gamma^2 R_1^2}{4} \right)^{-1/2}}} = \frac{2}{\sqrt{\pi}} \left( \frac{\alpha \gamma^2 R_1^2}{4} \right)^{1/2} \exp \left( -\frac{\alpha \gamma^2 R_1^2}{4} \right) \geq \left( \frac{\alpha \gamma^2 R_1^2}{4} \right)^{1/2} \exp \left( -\frac{\alpha \gamma^2 R_1^2}{4} \right).
\]
Hence, \( \mathcal{W}_{1,\rho}(\bar{\mu}_t, \bar{\nu}_t) \leq \mathbb{E}[\rho(t)] \leq e^{-c_3 t} \mathbb{E}[\rho(0)] \) with \( \bar{c} \) given by
\[
(72) \quad \bar{c} = \gamma \exp(-\Lambda) \min \left( \frac{\left( L_K + L_\xi \right) u \gamma^{-2}}{4}, \frac{1}{8} \Lambda^{1/2}, \frac{1}{4} \Lambda^{1/2}, \frac{1}{4} \right)
\]
with \( \Lambda, \tau \) and \( E \) given in (20), (21) and (22). Taking the infimum over all couplings \( \omega \in \Pi(\bar{\mu}_0, \bar{\nu}_0) \) concludes the proof of the first result.

By (42), the second result holds with \( M_1 = C_2/C_1 \) given by (24).

**Proof of Theorem 5.** Theorem 5 forms a special case of Theorem 12. We obtain analogously to Lemma 19 for \( \Delta(t) \geq D_K \)
\[
dr(t) \leq -c_1 r(t) dt + \sqrt{8\gamma^{-1}urc(Z_t, W_t)(r(t) - 1/2)((1 - 2\tau)Z_t + 2\gamma^{-1}W_t) \cdot e_t e^T d\xi_t},
\]
where \( c_1 = \tau \gamma / 2 \) with \( \tau \) given in (21). Similarly as in Lemma 20, we get for \( \Delta(t) < D_K \)
\[
df(r_s(t)) \leq -c_2 f(r_s(t)) dt + (1 + \alpha) \xi \gamma dt + dM_t,
\]
where \( M_t \) is a martingale, \( \alpha \) is defined in (28), \( f \) is defined in (36) and \( c_2 \) is given in (56). Here \( \hat{L} = 0 \). Combining the two local contraction results as in the proof of Theorem 12 gives the desired result with contraction rate
\[
(73) \quad c = \min \left( 2 \int_0^{R_1} \Phi(s) \phi(s) - 1} ds, \frac{2}{8} f'(R_1) \gamma \epsilon E \right).
\]
Note that the last two terms in the minimum differ by a factor of 2 from the last two terms in (69), as the first terms in (61) and (59) are not split up to compensate for the interaction term as in the nonlinear term. \( \square \)
6.3. Proof of Section 5

Fix \( N \in \mathbb{N} \). To show propagation of chaos in Theorem 17 we construct in the same line as in Section 4.2 a coupling between a solution to (3) and \( N \) copies of solutions to (1). We fix a positive constant \( \xi \), which we take in the end to the limit \( \xi \to 0 \). Let \( \{B_i^{t,(\xi)}\}_{t \geq 0} : i = 1, \ldots, N \} \) and \( \{\overline{B}_i^{t,rc}\}_{t \geq 0} : i = 1, \ldots, N \} \) be \( 2N \) independent \( d \)-dimensional Brownian motions and let \( \mu_0 \) and \( \overline{\mu}_0 \) be two probability measures on \( \mathbb{R}^{2d} \). The coupling \( \{(\overline{X}_i^t, \overline{Y}_i^t), (X_i^t, Y_i^t)\}_{t \geq 0} \) is a solution to the SDE on \( \mathbb{R}^{2Nd} \times \mathbb{R}^{2Nd} \) given by

\[
\begin{align*}
\overline{X}_i^t &= Y_i^t \, dt \\
\overline{Y}_i^t &= (-\gamma \overline{Y}_i^t + u(b(\overline{X}_i^t)) + \int_{\mathbb{R}^d} \tilde{b}(\overline{X}_i^t, z) \overline{\mu}_i^T (dz) ) \, dt + \sqrt{2\gamma \text{usc}(\overline{Z}_i^t, W_i^t)} \, dB_i^{t,sc} + \sqrt{2\gamma \text{usc}(\overline{Z}_i^t, W_i^t)} \, dB_i^{t,rc} \\
X_i^t &= Y_i^t \, dt \\
Y_i^t &= (-\gamma Y_i^t + u(X_i^t)) + \int_{\mathbb{R}^d} \tilde{b}(X_i^t, X_i') (dz) + \sqrt{2\gamma \text{usc}(Z_i^t, W_i^t)} \, dB_i^{t,sc} \\
&\quad + \sqrt{2\gamma \text{usc}(Z_i^t, W_i^t)} (\text{Id} - 2e_i^T e_i) \, dB_i^{t,rc}
\end{align*}
\]

(74)

\((\overline{X}_0^t, \overline{Y}_0^t) \sim \mu_0, \ (X_0^t, Y_0^t) \sim \mu_0\)

for \( i = 1, \ldots, N \), where \( \overline{\mu}_i^T = \text{Law}(\overline{X}_i^t) \) for all \( i \). Further, \( \overline{Z}_i^t = \overline{X}_i^t - X_i^t \), \( \overline{W}_i^t = \overline{Y}_i^t - Y_i^t \), \( Q_i^t = Z_i^t + \gamma^{-1} W_i^t \), and \( e_i^T = Q_i^t/|Q_i^t| \) if \( Q_i^t \neq 0 \) and \( e_i^T = 0 \) if \( Q_i^t = 0 \). As in Section 4.2, the functions \( \text{rc, sc} : \mathbb{R}^{2d} \rightarrow [0, 1) \) are Lipschitz continuous and satisfy \( \text{rc}^2 + \text{sc}^2 \equiv 1 \) and \( (44) \). We note that by Levy’s characterization, for any solution of (74) the processes

\[
\begin{align*}
B_i^t &:= \int_0^t \text{rc}(Z_i^s, W_i^s) \, dB_i^s + \int_0^t \text{sc}(Z_i^s, W_i^s) \, dB_i^s \\
\tilde{B}_i^t &:= \int_0^t \text{sc}(Z_i^s, W_i^s) \, dB_i^s + \int_0^t \text{rc}(Z_i^s, W_i^s) (\text{Id} - e_i^T e_i) \, dB_i^s
\end{align*}
\]

are \( d \)-dimensional Brownian motions. Therefore, (74) defines a coupling between \( N \) copies of solutions to (1) and a solution to (3). The processes \( \{(\overline{X}_i^t)_{i=1}^N, (\overline{Y}_i^t)_{i=1}^N\}_{t \geq 0}, \{(W_i^t)_{i=1}^N\}_{t \geq 0} \) and \( \{(Q_i^t)_{i=1}^N\}_{t \geq 0} \) satisfy the stochastic differential equations given by

\[
\begin{align*}
dZ_i^t &= W_i^t \, dt = (Q_i^t - \gamma Z_i^t) \, dt \\
dW_i^t &= (-\gamma W_i^t + u(b(\overline{X}_i^t)) - b(X_i^t)) + \int_{\mathbb{R}^d} \tilde{b}(\overline{X}_i^t, z) \overline{\mu}_i^T (dz) - N^{-1} \sum_{j=1}^N \tilde{b}(X_i^t, X_j^t)) \) \, dt \\
&\quad + \sqrt{8\gamma \text{usc}(\overline{Z}_i^t, W_i^t)} e_i^T \, dB_i^{t,rc} \\
\end{align*}
\]

\[
\begin{align*}
dQ_i^t &= \gamma^{-1} u(b(\overline{X}_i^t)) - b(X_i^t)) + \int_{\mathbb{R}^d} \tilde{b}(\overline{X}_i^t, z) \overline{\mu}_i^T (dz) - N^{-1} \sum_{j=1}^N \tilde{b}(X_i^t, X_j^t)) \) \, dt \\
&\quad + \sqrt{8\gamma^{-1} \text{usc}(Z_i^t, W_i^t)} e_i^T \, dB_i^{t,rc},
\end{align*}
\]

for all \( i = 1, \ldots, N \).

The proof of Theorem 17 relies on three auxiliary lemmata. We abbreviate \( r_i(t) = r_t((\overline{X}_i^t, \overline{Y}_i^t), (X_i^t, Y_i^t)) \), \( r_s(t) = r_s((\overline{X}_i^t, \overline{Y}_i^t), (X_i^t, Y_i^t)) \) and \( \Delta^t(t) = \Delta((\overline{X}_i^t, \overline{Y}_i^t), (X_i^t, Y_i^t)) \).

Lemma 21. Suppose Assumption 2 and Assumption 3 hold. Suppose that (18) holds. Let \( \tau > 0 \) be given by (21). Let \( \{(\overline{X}_i^t, \overline{Y}_i^t), (X_i^t, Y_i^t)\}_{i=1}^N \) be a solution to (74). Then for \( i \in \{1, \ldots, N\} \) with \( \Delta^t(t) \geq D_K \), it holds

\[
\begin{align*}
dr_i^t(t) &\leq -c_1 r_i^t(t) \, dt + \frac{|(1 - 2\tau) Z_i^t + 2\gamma^{-1} W_i^t|}{2\gamma r_i^t(t)} u(L N^{-1} \sum_{j=1}^N (|Z_j^t| + |Z_i^t|) + A_i^t) \, dt \\
&\quad + \sqrt{2\gamma^{-1} \text{usc}(Z_i^t, W_i^t)} \frac{(1 - 2\tau) Z_i^t + 2\gamma^{-1} W_i^t}{r_i^t(t)} \cdot e_i^T \, dB_i^{t,rc},
\end{align*}
\]

(76)
where $c_1 = \tau \gamma / 2$ and $\{ A_i^j \}_{i=1}^N$ is given by

\begin{equation}
A_i^j := \left| \int_{\mathbb{R}^d} \tilde{b}(X_i^j, z) \tilde{\mu}_i^j(dz) - N^{-1} \sum_{j=1}^N \tilde{b}(X_i^j, X_i^j) \right| \quad \text{with } \tilde{\mu}_i^j = \text{Law}(X_i^j).
\end{equation}

**Proof.** By Itô’s formula, it holds for $((Z_i^j, W_i^j)_{j=1}^N)_{t \geq 0} = ((\bar{X}_i^j - X_i^j, \bar{Y}_i^j - Y_i^j)_{j=1}^N)_{t \geq 0}$,

\[
\left\{ \begin{aligned}
\mathrm{d}Z_i^j &= W_i^j \mathrm{d}t \\
\mathrm{d}W_i^j &= \left( -\gamma W_i^j + u(\tilde{b}(X_i^j) - \tilde{b}(X_i^j)) + N^{-1} \sum_{j=1}^N (\tilde{b}(X_i^j, \bar{X}_i^j) - \tilde{b}(X_i^j, X_i^j)) + \tilde{A}_i^j \right) \mathrm{d}t \\
&\quad + \sqrt{8\gamma \text{urc}(Z_i^j, W_i^j)} \epsilon_i^T \mathrm{d}B_i^T,
\end{aligned} \right.
\]

where

\[
\tilde{A}_i^j := \left( \int_{\mathbb{R}^d} \tilde{b}(X_i^j, z) \tilde{\mu}_i^j(dz) - N^{-1} \sum_{j=1}^N \tilde{b}(X_i^j, X_i^j) \right) \quad \text{with } \tilde{\mu}_i^j = \text{Law}(X_i^j)
\]

for all $i = 1, \ldots, N$. Hence, by Itô’s formula it holds for the positive matrices $A, B, C$ given in (54),

\[
d(Z_i^j \cdot (AZ_i^j) + Z_i^j \cdot (BW_i^j) + W_i^j \cdot (CW_i^j)) \leq 2(AZ_i^j) \cdot W_i^j \mathrm{d}t + \left( (BW_i^j) \cdot (-BZ_i^j) + (\gamma W_i^j + uKZ_i^j) + L_g u(1 - 2\tau)\gamma^{-1} |Z_i^j|^2 \cdot 1_{\{|Z_i^j| < R\}} \right) \mathrm{d}t \\
+ \left( -2\gamma W_i^j \cdot (CW_i^j) - 2u(CW_i^j) \cdot (KZ_i^j) + 2L_g u |C|^{1/2} W_i^j |C|^{1/2} Z_i^j \right) \mathrm{d}t \\
+ |BZ_i^j + 2CW_i^j| u \left( \tilde{L}N^{-1} \sum_{j=1}^N (|Z_i^j| + |Z_i^j|) + A_i^j \right) \\
+ \gamma^{-2} 8\gamma \text{urc}(Z_i^j, W_i^j)^2 \mathrm{d}t + \sqrt{8\gamma \text{urc}(Z_i^j, W_i^j)} (BZ_i^j + CW_i^j) \cdot \epsilon_i^T \mathrm{d}B_i^T,
\]

with $\{ A_i^j \}_{i=1}^N$ given by (77). By (21) and (55),

\[
\mathrm{d}r_i^j(t)^2 = d(Z_i^j \cdot (AZ_i^j) + Z_i^j \cdot (BW_i^j) + W_i^j \cdot (CW_i^j)) \\
\leq -2\tau r_i^j(t)^2 \mathrm{d}t + |(1 - 2\tau)Z_i^j + 2\gamma^{-1} W_i^j| u \left( \tilde{L}N^{-1} \sum_{j=1}^N (|Z_i^j| + |Z_i^j|) + A_i^j \right) \mathrm{d}t + \gamma^{-1} \text{urc}(Z_i^j, W_i^j)^2 \mathrm{d}t \\
+ \gamma^{-1} (1 - 2\tau)L_g u |Z_i^j|^2 \cdot 1_{\{|Z_i^j| < R\}} \mathrm{d}t + \sqrt{8\gamma^{-1} \text{urc}(Z_i^j, W_i^j)} (1 - 2\tau)Z_i^j + 2\gamma^{-1} W_i^j \cdot \epsilon_i^T \mathrm{d}B_i^T.
\]

Since $\Delta_i^j(t) \geq D_{Kc}$, it holds $r_i^j(t)^2 > R$ by (32) and (33). By (34) and (44),

\[
\tau r_i^j(t)^2 + \gamma^{-1} (1 - 2\tau)L_g u |Z_i^j|^2 \cdot 1_{\{|Z_i^j| < R\}} + 8\gamma^{-1} \text{urc}(Z_i^j, W_i^j)^2 \leq \tau R + L_g u R^2 \gamma^{-1} + 8\gamma^{-1} u \mathbb{1}_{\{R > 0\}} \leq 0.
\]

By Itô’s formula and since the second derivative of the square root is negative,

\[
\mathrm{d}r_i^j(t)^2 \leq (2r_i^j(t))^{-1} \mathrm{d}(2r_i^j(t))^2 \leq -c_1 r_i^j(t) \mathrm{d}t + \frac{|(1 - 2\tau)Z_i^j + 2\gamma^{-1} W_i^j|}{2\gamma r_i^j(t)} u \left( \tilde{L}N^{-1} \sum_{j=1}^N (|Z_i^j| + |Z_i^j|) + A_i^j \right) \mathrm{d}t \\
+ \sqrt{2\gamma^{-1} \text{urc}(Z_i^j, W_i^j)r_i^j(t)^{-1} (1 - 2\tau)Z_i^j + 2\gamma^{-1} W_i^j} \cdot \epsilon_i^T \mathrm{d}B_i^T,
\]

which concludes the proof. \qed
Lemma 22. Suppose Assumption 2 and Assumption 3 hold. Let \( (\{X^i_t, Y^i_t, X^i_t, Y^i_t\}_{i=1}^N)_{t \geq 0} \) be a solution to (74). Let \( r_s \) be given in (27) with \( \alpha \) defined in (28). If \( \Delta^i(t) < D_K \) with \( D_K \) given in (32), it holds

\[
d f(r^i_s(t)) \leq -c_2 f(r^i_s(t))dt + \gamma^{-1} \hat{L}uN^{-1} \sum_{j=1}^N (|Z^j_t| + |Z^j_t|)dt - \frac{\alpha |Z^j_t|}{4} f'(R_1)|Z^j_t|dt
\]

\[
+ \gamma^{-1} u \left( \int_{R^d} \tilde{b}(X^i_t, z) \tilde{u}_t(dz) - N^{-1} \sum_{j=1}^N \tilde{b}(X^i_t, X^j_t) \right) dt + (1 + \alpha) \gamma \xi dt + dM^i_t,
\]

where \( f \) is given in (36), \( (M^i_t)_{t \geq 0} \) is a martingale and \( c_2 \) is given in (56).

Proof. The proof works similarly as the proof of Lemma 20. First, note that for all \( i, (Z^i_t)_{t \geq 0} \) is almost surely continuously differentiable with derivative \( dZ^i_t/\gamma = -\gamma Z^i_t + \gamma Q^i_t \) and hence \( t \to |Z^i_t| \) is almost surely absolutely continuous with

\[
\frac{d}{dt} |Z^i_t| = \frac{Z^i_t}{|Z^i_t|} \cdot (-\gamma |Z^i_t| + \gamma Q^i_t)
\]

for a.e. \( t \) such that \( Z^i_t \neq 0 \) and

\[
\frac{d}{dt} |Z^i_t| \leq \gamma |Q^i_t|
\]

for a.e. \( t \) such that \( Z^i_t = 0 \).

and therefore

\[
\frac{d}{dt} |Z^i_t| \leq -\gamma |Z^i_t| + \gamma |Q^i_t|
\]

for a.e. \( t \geq 0 \).

By Itô’s formula and by Assumption 2 and Assumption 3, we obtain for \( |Q^i_t| \),

\[
d |Q^i_t| = \gamma^{-1} u e^i_t \cdot \left( b(X^i_t) - b(X^i_t) + \int_{R^d} \tilde{b}(X^i_t, z) \tilde{u}_t(dz) - N^{-1} \sum_{j=1}^N \tilde{b}(X^i_t, X^j_t) \right) dt + \sqrt{8 \gamma^{-1}} u \tilde{c}_i^T dE^i_t
\]

\[
\leq \gamma^{-1} u (L_K + L_g)|Z^i_t|dt + \gamma^{-1} u (A^i_t + N^{-1} \sum_{j=1}^N \tilde{L}(|Z^j_t| + |Z^j_t|))dt + \sqrt{8 \gamma^{-1}} u \tilde{c}_i^T dE^i_t
\]

where \( A^i_t \) is given by (77). Note that there is no Itô correction term, since \( \partial^2_{q^2/|q|} = 0 \) for \( q \neq 0 \) and \( rc = 0 \) for \( Q_t = 0 \). Combining this bound and (78) yields for \( f(r^i_s(t)) \) by Itô’s formula,

\[
d f(r^i_s(t)) = f'(r^i_s(t)) \left( ((L_K + L_g)u \gamma^{-2} - \alpha) |Z^i_t| + \alpha |Q^i_t| + \gamma^{-1} u \left( A^i_t + N^{-1} \sum_{j=1}^N \tilde{L}(|Z^j_t| + |Z^j_t|) \right) \right) dt
\]

\[
+ f'(r^i_s(t)) \sqrt{8 \gamma^{-1}} u \tilde{c}_i^T dE^i_t + f''(r^i_s(t)) \gamma^{-1} u \tilde{c}_i^T dE^i_t + f''(r^i_s(t)) \gamma^{-1} u \tilde{c}_i^T dE^i_t
\]

\[
\cdot \sqrt{8 \gamma^{-1}} u \tilde{c}_i^T dE^i_t.
\]

Case 1: Consider \( \Delta^i(t) < D_K \) and \( |Q^i_t| > \xi \), then \( rc(Z^i_t, W^i_t) = 1 \) and \( r^i_s(t) < R_1 \). Hence, by (58) we obtain

\[
d f(r^i_s(t)) \leq f'(r^i_s(t)) \alpha \gamma r^i_s(t)dt + f''(r^i_s(t)) \gamma^{-1} u dt + \gamma^{-1} u \left( A^i_t + N^{-1} \sum_{j=1}^N \tilde{L}(|Z^j_t| + |Z^j_t|) \right) dt
\]

\[
- f'(R_1) \frac{1}{2} \gamma |Z^i_t|dt + dM^i_t
\]

\[
\leq -2 \gamma f(r^i_s(t))dt + \gamma^{-1} u \left( A^i_t + N^{-1} \sum_{j=1}^N \tilde{L}(|Z^j_t| + |Z^j_t|) \right) dt - f'(R_1) \frac{\gamma \alpha}{2} |Z^i_t|dt + dM^i_t
\]

\[
\leq -c_2 f(r^i_s(t))dt + \gamma^{-1} u \left( A^i_t + N^{-1} \sum_{j=1}^N \tilde{L}(|Z^j_t| + |Z^j_t|) \right) dt - f'(R_1) \frac{\gamma \alpha}{2} |Z^i_t|dt + dM^i_t.
\]
Case 2: Consider $\Delta^i(t) < D_K$ and $|Q_i| \leq \xi$, then $\alpha|Z_i| = r^i_s(t) - |Q_i| \geq r^i_s(t) - \xi$. We note that
\[(L_K + L_g)u\gamma^{-2} - \alpha)|Z_i| + \alpha|Q_i| \leq -\frac{1}{2}r^i_s(t) + (1 + \alpha)\xi.
\]

Since the second derivative of $f$ is negative and $\psi(s) \in [1/2, 1]$, it holds
\[
df(r^i_s(t)) \leq -\frac{\gamma}{2}r^i_s(t)f'(r^i_s(t))dt + (1 + \alpha)\gamma\xi dt + \gamma^{-1}u\left(A^i + N^{-1}\sum_{j=1}^N \tilde{L}(|Z^j| + |Z^j|)\right) dt + dM^i_t
\]
\[
\leq -\frac{\gamma}{8}\inf_{r \leq R_1} \frac{r\phi(r)}{\Phi(r)}(r^i_s(t))dt - \frac{\gamma\alpha}{4}|Z^i|f'(R_1)dt + (1 + \alpha)\gamma\xi dt
\]
\[
+ \gamma^{-1}u\left(A^i + N^{-1}\sum_{j=1}^N \tilde{L}(|Z^j| + |Z^j|)\right) dt + dM^i_t
\]
\[
\leq -\frac{\gamma R_1\phi(R_1)}{8}\frac{f(r^i_s(t))dt - \gamma\alpha}{4}|Z^i|f'(R_1)dt + (1 + \alpha)\gamma\xi dt
\]
\[
+ \gamma^{-1}u\left(A^i + N^{-1}\sum_{j=1}^N \tilde{L}(|Z^j| + |Z^j|)\right) dt + dM^i_t.
\]

Combining the two cases, we obtain the result by using the definition of $c_2$ given in (56).

\[\square\]

Lemma 23. (Moment control for Langevin dynamics) Suppose that Assumption 2 and Assumption 3 hold. Suppose that \(18\) and \(25\) hold. Let $(X_t, Y_t)_{t \geq 0}$ be a solution to (1) with $E[|X_0|^2 + |Y_0|^2] < \infty$. Then there exists a finite constant $C_2 > 0$ such that
\[
\sup_{t \geq 0} E[|X_t|^2] \leq C_2.
\]

The constant $C_2$ depends on $\gamma$, $E[|X_0|^2 + |Y_0|^2]$, $d$, $R$, $\kappa$, $L_g$, $u$ and $\tilde{L}$.

Proof. We adapt the proof idea from [22, Lemma 8]. By Itô’s formula, by Assumption 2 and by Assumption 3, it holds
\[
d(\gamma^{-2}u\tilde{X}_t \cdot (K\tilde{X}_t) + \frac{1}{2}|(1 - 2\tau)\tilde{X}_t + \gamma^{-1}\tilde{Y}_t|^2 + \frac{1}{2}\gamma^{-2}|\tilde{Y}_t|^2)
\]
\[
\leq \left(2\gamma^{-2}u\tilde{X}_t \cdot (K\tilde{Y}_t) + (1 - 2\tau)^2\tilde{X}_t \cdot \tilde{Y}_t + \gamma^{-1}(1 - 2\tau)|\tilde{Y}_t|^2\right) dt + \gamma^{-1}(1 - 2\tau)\left(-u\tilde{X}_t \cdot (K\tilde{X}_t) - \gamma\tilde{X}_t \cdot \tilde{Y}_t\right) dt
\]
\[
+ 2\gamma^{-2}\left(-u(K\tilde{Y}_t) \cdot \tilde{X}_t + L_g|\tilde{Y}_t||\tilde{X}_t| - \gamma|\tilde{Y}_t|^2\right) dt + \frac{u}{\gamma}(1 - 2\tau)|\tilde{X}_t| + 2\gamma^{-1}\tilde{Y}_t\left(L(E[|X_t|] + |X_t|) + \tilde{b}(0, 0, 0)\right) dt
\]
\[
+ (1 - 2\tau)\gamma^{-1}u(L_g|X_t|^2 + |g(0)||\tilde{X}_t|)1_{\{|X_t| < R\}} dt + 2\gamma^{-2}u|\tilde{Y}_t|g(0) dt + 2\gamma^{-1}ud\tau
\]
\[
+ \sqrt{2\gamma^{-1}}u((1 - 2\tau)\tilde{X}_t + 2\gamma^{-1}\tilde{Y}_t)dB_t
\]
\[
\leq -\gamma^{-1}u(1 - 2\tau)\tilde{X}_t \cdot (K\tilde{X}_t) - 2\gamma\gamma^{-2}|\tilde{Y}_t|^2 + (1 - 2\tau)\gamma^{-1}\tilde{X}_t \cdot \tilde{Y}_t + \gamma^{-3}u^2L_g^2|\tilde{X}_t|^2
\]
\[
+ \frac{u}{\gamma}(1 - 2\tau)|\tilde{X}_t| + 2\gamma^{-1}\tilde{Y}_t\left(L(E[|X_t|] + |X_t|) + \tilde{b}(0, 0, 0)\right) dt + (1 - 2\tau)\gamma^{-1}u(L_g|X_t|^2 + |g(0)||\tilde{X}_t|)1_{\{|X_t| < R\}} dt
\]
\[
+ 2\gamma^{-2}u|\tilde{Y}_t|g(0) dt + 2\gamma^{-1}ud\tau + \sqrt{2\gamma^{-1}}u((1 - 2\tau)\tilde{X}_t + 2\gamma^{-1}\tilde{Y}_t)dB_t
\]

Taking expectation, we obtain
\[
\frac{d}{dt}E[\gamma^{-2}u\tilde{X}_t \cdot (K\tilde{X}_t) + \frac{1}{2}|(1 - 2\tau)\tilde{X}_t + \gamma^{-1}\tilde{Y}_t|^2 + \frac{1}{2}\gamma^{-2}|\tilde{Y}_t|^2]
\]
\[
\leq -\gamma^{-1}u(1 - 2\tau)E[\tilde{X}_t \cdot (K\tilde{X}_t)] + \gamma^{-3}u^2L_g^2E[|\tilde{X}_t|^2] - 2\gamma\gamma^{-2}E[|\tilde{Y}_t|^2] + (1 - 2\tau)\gamma^{-1}E[|X_t| \cdot \tilde{Y}_t]
\]
\[
+ (1 - 2\tau)\gamma^{-1}u(L_gR^2 + R|g(0)|) + 2\gamma^{-1}ud + u\gamma^{-1}E\left[((1 - 2\tau)\tilde{X}_t + 2\gamma^{-1}\tilde{Y}_t)\left(L(E[|X_t|] + |X_t|) + \tilde{b}(0, 0, 0)\right)\right]
\]
We note that by (25) and by Young’s inequality,
\[
\gamma^{-1}E[(1 - 2\tau)\tilde{X}_t + 2\gamma^{-1}\tilde{Y}_t|u(\tilde{L}(\tilde{E}[|\tilde{X}_t|] + |\tilde{X}_t|) + \tilde{b}(0,0))]
\leq T\frac{\kappa u}{8}E[(1 - 2\tau)\tilde{X}_t + 2\gamma^{-1}\tilde{Y}_t|[(\tilde{E}[|\tilde{X}_t|] + |\tilde{X}_t|)] + \gamma^{-1}uE[(1 - 2\tau)\tilde{X}_t + 2\gamma^{-1}\tilde{Y}_t)|\tilde{b}(0,0))]
\leq T\frac{\kappa u}{4}E[(1 - 2\tau)\tilde{X}_t + 2\gamma^{-1}\tilde{Y}_t]^2 + \frac{4u^2}{\tau\gamma^3}|\tilde{b}(0,0)|^2
\]
and
\[
2\gamma^{-2}uE[|\tilde{Y}_t|]|g(0)| \leq \frac{\tau\gamma}{2}E[|\gamma^{-1}\tilde{Y}_t|^2] + \frac{4u^2}{\tau\gamma^3}|g(0)|^2.
\]
Then by (55),
\[
\frac{d}{dt}E\left[\gamma^{-2}u\tilde{X}_t \cdot (K\tilde{X}_t) + \frac{1}{2}|(1 - 2\tau)\tilde{X}_t + \gamma^{-1}\tilde{Y}_t|^2 + \frac{1}{2}\gamma^{-2}|\tilde{Y}_t|^2\right]
\leq -2\tau\gamma E\left[\gamma^{-2}u\tilde{X}_t \cdot (K\tilde{X}_t) + \frac{1}{2}|(1 - 2\tau)\tilde{X}_t + \gamma^{-1}\tilde{Y}_t|^2 + \frac{1}{2}\gamma^{-2}|\tilde{Y}_t|^2\right] + (1 - 2\tau)\gamma^{-1}u(L_\varphi R^2 + R|g(0)|)
\]
\[
+ \tau\gamma\left(\gamma^{-2}E[|\tilde{X}_t|^2] + \frac{1}{2}E[|(1 - 2\tau)\tilde{X}_t + \gamma^{-1}\tilde{Y}_t|^2 + |\gamma^{-1}\tilde{Y}_t|^2]\right) + u^2(\tau\gamma^3)^{-1}(|\tilde{b}(0,0)|^2 + |g(0)|^2) + 2\gamma^{-1}ud
\]
\[
\leq -\tau\gamma E\left[\gamma^{-2}u\tilde{X}_t \cdot (K\tilde{X}_t) + \frac{1}{2}|(1 - 2\tau)\tilde{X}_t + \gamma^{-1}\tilde{Y}_t|^2 + \frac{1}{2}\gamma^{-2}|\tilde{Y}_t|^2\right] + (1 - 2\tau)\gamma^{-1}u(L_\varphi R^2 + R|g(0)|)
\]
\[
+ u^2(\tau\gamma^3)^{-1}(|\tilde{b}(0,0)|^2 + |g(0)|^2) + 2\gamma^{-1}ud.
\]
By Grönwall’s inequality, there exists a constant C such that
\[
\sup_{t \geq 0}E\left[\gamma^{-2}u\tilde{X}_t \cdot (K\tilde{X}_t) + \frac{1}{2}|(1 - 2\tau)\tilde{X}_t + \gamma^{-1}\tilde{Y}_t|^2 + \frac{1}{2}\gamma^{-2}|\tilde{Y}_t|^2\right] \leq C < \infty.
\]
Thus, we obtain the result for \( C_2 = C/(\kappa u\gamma^{-2}) \).

Proof of Theorem 17. To prove uniform in time propagation of chaos, we consider the coupling \((\{\tilde{X}_t^i, \tilde{Y}_t^i\}_{i=1}^{N})_{t \geq 0}\) given in (74) and combine the results of Lemma 21 and Lemma 22. The second moment control given in Lemma 22 will be essential to bound the terms involving the non-linearity. We write here \( r_s(t) = r_s((\tilde{X}_t^i, \tilde{Y}_t^i), (X_t^i, Y_t^i)), r_t^i(t) = r_t^i((\tilde{X}_t^i, \tilde{Y}_t^i), (X_t^i, Y_t^i)), \Delta^i(t) = r_s^i(t) - cr_t^i(t) \) and \( \rho^i(t) = f((\Delta^i(t) \land D\kappa) + cr_t^i(t)) \).

We distinguish two cases for all particles \( i = 1, \ldots, N \):

Case 1: Consider \( \Delta^i(t) < D\kappa \). Then \( \rho^i(t) = f(r_s^i(t)) \), and by Lemma 22 it holds for \( \xi > 0 \)
\[
d\rho^i(t) = df(r_s^i(t))dt \leq -c_2f(r_s^i(t))dt + \gamma^{-1}u\left(A_t^i + N^{-1}\sum_{j=1}^{N}|\tilde{L}(\tilde{Z}_t^i)| + |Z_t^i|\right)dt - \frac{\alpha\gamma}{8}f'(R_1)|Z_t^i|dt
\]
\[
+ (1 + \alpha)\gamma \xi dt + dM_t^i
\]
(79) \[
\leq -c_2f(r_s^i(t))dt + \gamma^{-1}u\left(A_t^i + N^{-1}\sum_{j=1}^{N}|\tilde{L}(\tilde{Z}_t^i)|\right)dt - \frac{\alpha\gamma}{8}f'(R_1)|Z_t^i|dt + (1 + \alpha)\gamma \xi dt + dM_t^i,
\]
where \( A_t^i \) is given in (77) and \( c_2 \) is given by (56). Note the last step holds by (25).

Case 2: Consider \( \Delta^i(t) \geq D\kappa \). We obtain by Lemma 21,
\[
dr_t^i(t) \leq -c_1r_t^i(t)dt + \frac{(1 - 2\tau)|\tilde{Z}_t^i| + 2\gamma^{-2}W_t^i|}{2r_t^i(t)}u\left(A_t^i + N^{-1}\sum_{j=1}^{N}|\tilde{L}(\tilde{Z}_t^i)| + |Z_t^i|\right)dt
\]
Hence, 
\[
\begin{align*}
\frac{d}{dt}f(D_K + \epsilon x) &= \epsilon f'(D_K + \epsilon x). \quad \text{Further, since } f(D_K + \epsilon x) \text{ is a concave function, } \\
\frac{d^2}{dt^2} f(D_K + \epsilon x) &= -\frac{d}{dt} f(D_K + \epsilon x) < 0. \quad \text{By Itô’s formula, we obtain} \\
\frac{d\rho(t)}{dt} &= d f(D_K + \epsilon r_i^2(t)) \\
&\leq f'(D_K + \epsilon r_i^2(t)) \left( -c_1 r_i^2(t) + \frac{1}{2} (1 - 2\tau) Z_i^2(t) + 2\gamma^{-1} W_i^2(t) \cdot e_i^T \epsilon_t^T d B_t^i \right) \\
&\quad + \frac{\epsilon f'(D_K + \epsilon r_i^2(t))}{r_i^2(t)} \sqrt{2\gamma^{-1} \text{var}(Z_i^2(t) W_i^2(t))(1 - 2\tau) Z_i^2(t) + 2\gamma^{-1} W_i^2(t) \cdot e_i^T \epsilon_t^T d B_t^i}. \\
\end{align*}
\]

By (66) and (67), which holds in the same line as in the proof of Theorem 12, it holds
\[
\begin{align}
\frac{d\rho(t)}{dt} &\leq -f'(R_1) \frac{c_1 \epsilon}{2} \min \left( \frac{\sqrt{\kappa u \gamma^{-1}}}{\sqrt{8 \alpha}}, \frac{1}{2} \right) \rho(t) dt - f'(R_1) \frac{c_1 \epsilon}{2} \sqrt{\kappa u \gamma^{-2}} Z_i^2(t) dt \\
&\quad + 2\epsilon \gamma^{-1} u \left( A_i^1 + N^{-1} \sum_{j=1}^{N} \bar{L}(|Z_i^1(t)| + |Z_i^1|) \right) dt + dM_i^t, \\
\end{align}
\]
where \((\{M_i^t\})_{i=1}^{N})_{t\geq 0} \text{ is some martingale.}

Combining (79) and (80), taking expectations and summing over \(i = 1, \ldots, N\) yields
\[
\begin{align}
\frac{d}{dt} \mathbb{E} \left[ N^{-1} \sum_{i=1}^{N} \rho(t) \right] &\leq -\min \left( c_2, f'(R_1) \frac{c_1 \epsilon}{2} \min \left( \frac{\sqrt{\kappa u \gamma^{-1}}}{\sqrt{8 \alpha}}, \frac{1}{2} \right) \right) \mathbb{E} \left[ N^{-1} \sum_{i=1}^{N} \rho^i(t) \right] + \gamma^{-1} u \mathbb{E} \left[ N^{-1} \sum_{i=1}^{N} A_i^1 \right] \\
&\quad - \min \left( f'(R_1) \frac{\gamma \alpha}{8}, f'(R_1) \frac{c_1 \epsilon}{2} \sqrt{\kappa u \gamma^{-2}} \right) \mathbb{E} \left[ N^{-1} \sum_{i=1}^{N} |Z_i| \right] + \bar{L} \gamma^{-1} \mathbb{E} \left[ N^{-1} \sum_{i=1}^{N} |Z_i| \right] \\
&\leq -\min \left( c_2, f'(R_1) \frac{c_1 \epsilon}{2} \min \left( \frac{\sqrt{\kappa u \gamma^{-1}}}{\sqrt{8 \alpha}}, \frac{1}{2} \right) \right) \mathbb{E} \left[ N^{-1} \sum_{i=1}^{N} \rho^i(t) \right] + \gamma^{-1} u \mathbb{E} \left[ N^{-1} \sum_{i=1}^{N} A_i^1 \right], \\
\end{align}
\]
where we used \(2\epsilon \leq 1\) for the last term and (25).

To bound \(\mathbb{E}[A_i^1]\), we note that given \(X_i^1, \bar{X}_i, j \neq i\) are identically and independently distributed with law \(\bar{\mu}_i^2\) and
\[
\mathbb{E} \left[ \bar{b}(\bar{X}_i^1, \bar{X}_j^1) \right] = \int_{\mathbb{R}^d} \bar{b}(\bar{X}_i^1, z) \bar{\mu}_i^2 (dz). \\
\]
Hence,
\[
\begin{align*}
\mathbb{E} \left[ \left| \int_{\mathbb{R}^d} \bar{b}(\bar{X}_i^1, z) \bar{\mu}_i^2 (dz) - \frac{1}{N} \sum_{j=1}^{N} \bar{b}(\bar{X}_i^1, \bar{X}_j^1) \right|^2 \right] &\leq \frac{4L^2}{N} \int_{\mathbb{R}^d} |x|^2 \bar{\mu}_i^2 (dx) + \frac{4L^2}{N^2} \int_{\mathbb{R}^d} |x|^2 \bar{\mu}_i^2 (dx) \\
&\quad + \frac{8L^2}{N} \int_{\mathbb{R}^d} |x|^2 \bar{\mu}_i^2 (dx). \\
\end{align*}
\]
Then, by Jensen’s inequality
\[
E[A_i^t] \leq \frac{4L}{N^{1/2}} \left( \int_{R^d} |x|^2 \tilde{\mu}_t^x (dx) \right)^{1/2}.
\]

By Lemma 23, there exists a finite constant \( C_1 \) such that for \( N \geq 2 \) and all \( i = 1, \ldots, N \),
\begin{equation}
\sup_{t \geq 0} E[A_i^t] \leq \gamma u^{-1} C_1 N^{-1/2}.
\end{equation}

Note that \( C_1 \) depends on \( \gamma, E[|\tilde{X}_0|^2 + |\tilde{Y}_0|^2], d, u, R, \kappa, L_g \) and \( \tilde{L} \). Inserting the bound for \( E[A_i^t] \) in (81) yields
\[
\frac{d}{dt} \left[ \frac{1}{N} \sum_{i=1}^N \rho^i(t) \right] \leq - \min \left( c_2, f'(R_1) \frac{C_1 L}{2} \left( \frac{\sqrt{\kappa u} \gamma^{-1}}{\sqrt{8\alpha}} \right) \right) \frac{1}{N} \sum_{i=1}^N \rho^i(t) + \frac{C_1}{N^{1/2}}.
\]

Applying Gronwall’s inequality and (70) and (71) yields
\[
W_{1, \rho_N}(\tilde{\mu}_t^x, \tilde{\mu}_t^y) \leq \left[ \frac{1}{N} \sum_{i=1}^N \rho^i(t) \right] \leq e^{-\epsilon t} \left[ \frac{1}{N} \sum_{i=1}^N \rho^i(0) \right] + C_1 N^{-1/2} \epsilon^{-1}.
\]

with \( \epsilon \) given in (72). Taking the infimum over all couplings \( \omega \in \Pi(\tilde{\mu}_0^x, \tilde{\mu}_0^y) \) concludes the proof of the first result.

The second bound holds by (47) with \( M_1 \) given in (24) and \( M_2 = \sqrt{2/C_1} \) given in (48).

\[ \square \]

Appendix A: Unconfined nonlinear Langevin dynamics

A.1. Contraction for unconfined nonlinear Langevin dynamics

Consider the unconfined nonlinear Langevin dynamics given by
\begin{equation}
\begin{cases}
    d\tilde{X}_t = \tilde{Y}_t dt \\
    d\tilde{Y}_t = (-\gamma \tilde{Y}_t + u \int_{R^d} \tilde{b}(\tilde{X}_t, z) \tilde{\mu}_t^x (dz)) dt + \sqrt{2 \gamma u} dB_t, \\
    (\tilde{X}_0, \tilde{Y}_0) \sim \tilde{\mu}_0,
\end{cases}
\end{equation}

where \( \gamma, u > 0, \tilde{\mu}_0 \) is a probability measure on \( R^{2d} \), \( \tilde{\mu}_t^x = \text{Law}(\tilde{X}_t) \) and \( (B_t)_{t \geq 0} \) is a \( d \)-dimensional standard Brownian motion. We impose for the function \( \tilde{b} \) and for the initial distribution:

Assumption 4. The function \( \tilde{b} : R^{2d} \to R^d \) is Lipschitz continuous, and there exist a function \( \tilde{g} : R^d \to R^d \) and a positive definite matrix \( \tilde{K} \in R^{d \times d} \) with smallest eigenvalue \( \tilde{\kappa} \in (0, \infty) \) and largest eigenvalue \( L_{\tilde{K}} \in (0, \infty) \) such that
\[
\tilde{b}(x, y) = -\tilde{K}(x - y) + \tilde{g}(x - y) \quad \text{for all } x, y \in R^d,
\]
and \( \tilde{g} \) is Lipschitz continuous with Lipschitz constant \( L_{\tilde{g}} \in (0, \infty) \) and anti-symmetric, i.e., \( \tilde{g}(-z) = -\tilde{g}(z) \) for all \( z \in R^d \).

Assumption 5. Let \( \tilde{\mu}_0 \in P(R^{2d}) \) satisfy \( \int_{R^{2d}} |(x, y)|^2 \tilde{\mu}_0 (dx dy) < \infty \) and \( \int_{R^{2d}} (x, y) \tilde{\mu}_0 (dx dy) = 0 \).

By Assumption 4, it holds \( \frac{d}{dt} \mathcal{E}[(X_t, Y_t)] = \mathcal{E}[(Y_t, -\gamma Y_t)] \) and hence by Assumption 5 \( \mathcal{E}[(X_t, Y_t)] = 0 \) for all \( t \geq 0 \). Note that this observation is crucial in our analysis, since in general convergence to equilibrium cannot be guaranteed for the unconfined dynamics unless the solution is centered or a recentering of the center of mass is considered.

We establish contraction in Wasserstein distance with respect to the distance function \( \tilde{r} : R^{2d} \times R^{2d} \to [0, \infty) \) given by
\begin{equation}
\tilde{r}((x, y), (\tilde{x}, \tilde{y}))^2 = \gamma^2 u (x - \tilde{x}) \cdot (\tilde{K}(x - \tilde{x})) + \frac{1}{2} (1 - 2\sigma) (x - \tilde{x}) + \gamma^{-1} (y - \tilde{y})^2 + \frac{1}{2} \gamma^{-2} (y - \tilde{y})^2,
\end{equation}
for \((x, y), (\tilde{x}, \tilde{y}) \in R^{2d}\) where \( \sigma \) is given by
\begin{equation}
\sigma = \min(1/8, \tilde{\kappa} u \gamma^{-2}/2).
\end{equation}
\textbf{Theorem 24} (Contraction for nonlinear unconfined Langevin dynamics in }L^2\text{ and }L^1\text{ Wasserstein distance). Suppose Assumption 4 holds. Let }\tilde{\mu}_t\text{ and }\tilde{\nu}_t\text{ be two probability distributions on }\mathbb{R}^{2d}\text{ satisfying Assumption 5. For }t \geq 0\text{, let }\tilde{\mu}_t\text{ and }\tilde{\nu}_t\text{ be the law of the processes }\{\tilde{X}_t, \tilde{Y}_t\}\text{ and }\{\tilde{X}_t', \tilde{Y}_t'\}\text{, respectively, where }\{\tilde{X}_t, \tilde{Y}_t\}_{s \geq 0}\text{ and }\{\tilde{X}_t', \tilde{Y}_t'\}_{s \geq 0}\text{ are solutions to (85) with initial distribution }\tilde{\mu}_0\text{ and }\tilde{\nu}_0\text{, respectively. If}

\begin{equation}
L_{\tilde{g}} \leq \sqrt{\gamma/\gamma}(1/2) \min(1/8, \tilde{\kappa}u\gamma^{-2}/2),
\end{equation}

then

\begin{equation}
\mathcal{W}_2,\hat{r}(\tilde{\mu}_t, \tilde{\nu}_t) \leq e^{-\hat{r}t}\mathcal{W}_2,\hat{r}(\tilde{\mu}_0, \tilde{\nu}_0) \quad \text{and} \quad \mathcal{W}_2(\tilde{\mu}_t, \tilde{\nu}_t) \leq M_3e^{-\hat{r}t}\mathcal{W}_2(\tilde{\mu}_0, \tilde{\nu}_0),
\end{equation}

where \(\hat{r}\) is defined in (86) and where the contraction rate \(\hat{c}\) is given by

\begin{equation}
\hat{c} = \min(\gamma/16, \tilde{\kappa}^{-1}/4).
\end{equation}

The constant \(M_3\) is given by

\begin{equation}
M_3 = \max(\sqrt{L_{\tilde{g}} + \gamma^2}, \sqrt{3/2}) \max(\sqrt{\tilde{\kappa}u^{-1}}, \sqrt{2}).
\end{equation}

Moreover, there exists a unique invariant probability measure \(\tilde{\mu}_\infty\) for (85) and convergence in \(L^2\) Wasserstein distance to \(\tilde{\mu}_\infty\) holds.

\textbf{Proof.} The proof uses a synchronous coupling and is postponed to Appendix A.3. \(\square\)

\textbf{Remark 25.} Note that (89) implies directly a bound in \(L^p\) Wasserstein distance for \(1 \leq p < 2\), i.e., by Jensen’s inequality it holds

\(\mathcal{W}_p(\tilde{\mu}_t, \tilde{\nu}_t) \leq \mathcal{W}_2(\tilde{\mu}_t, \tilde{\nu}_t) \leq M_0M_3e^{-\hat{r}t}\mathcal{W}_p(\tilde{\mu}_0, \tilde{\nu}_0),\)

where \(M_0 = \mathcal{W}_2(\tilde{\mu}_0, \tilde{\nu}_0)/\mathcal{W}_p(\tilde{\mu}_0, \tilde{\nu}_0).\) The additional constant \(M_3\) is finite by Assumption 5, but it might be very large. Here, contraction in \(L^1\) Wasserstein distance is stated separately and (93) is proven directly.

\textbf{Remark 26.} By (88) and (92), it holds \(L_{\tilde{g}} \leq \tilde{\kappa}/8\) and \(L_{\tilde{g}} \leq \tilde{\kappa}/16\), respectively. Hence, contraction is proven for \(b\) being a small perturbation of a linear function. Further, the contraction rate is maximized for \(\gamma = 2\sqrt{\tilde{\kappa}u}.

\textbf{Remark 27.} Note that the underlying distance \(\hat{r}\) is defined similarly as \(r_1\) in (26) and coincides with \(r\) defined in (35) if \(\tilde{K} = K, \sigma = \tau\) and \(K = \{(0, 0)\}.\) Moreover, \(\tilde{r}\) is equivalent to the Euclidean distance on \(\mathbb{R}^{2d},\) i.e.,

\begin{equation}
\min(\tilde{\kappa}u/2, 1/4)\gamma^{-2}(|x - \bar{x}| + |y - \bar{y}|)^2 \leq \min(\kappa u, 1/2)\gamma^{-2}(|x - \bar{x}, \bar{y}|)^2 \leq \hat{r}((x, y), (\bar{x}, \bar{y}))^2
\end{equation}

\begin{equation}
\leq \max(L_gu\gamma^{-2} + 1, 3/2)\gamma^{-2}(|x - \bar{x}| + |y - \bar{y}|)^2.
\end{equation}

\subsection{A.2. Uniform in time propagation of chaos in the unconfined case}

Next, we establish uniform in time propagation of chaos bounds for the unconfined Langevin dynamics. Fix \(N \in \mathbb{N}.\) We consider the functions \(\tilde{\rho}_N, \tilde{\rho}_N : \mathbb{R}^{2Nd} \times \mathbb{R}^{2Nd} \to [0, \infty)\) given by

\begin{equation}
\tilde{\rho}_N((x, y), (\bar{x}, \bar{y})) := N^{-1} \sum_{i=1}^{N} \hat{r}(\pi(x, y), \pi(\bar{x}, \bar{y}))^2,
\end{equation}

and

\begin{equation}
\tilde{\rho}_N((x, y), (\bar{x}, \bar{y})) := N^{-1} \sum_{i=1}^{N} \hat{r}(\pi(x, y), \pi(\bar{x}, \bar{y}))\quad \text{for all } x, y, \bar{x}, \bar{y} \in \mathbb{R}^{Nd},
\end{equation}

where \(\hat{r}\) is given by (86) and (90).
where $\hat{r}$ is given in (86) and $\pi : \mathbb{R}^{2Nd} \to \mathbb{R}^{2Nd}$ is given by

$$
\pi(x, y) = \left( x^i - \bar{N}^{-1} \sum_{j=1}^{N} x^j, y^j - \bar{N}^{-1} \sum_{j=1}^{N} y^j \right)_{i=1}^{N} \quad \text{for } (x, y) \in \mathbb{R}^{2Nd}.
$$

The function $\pi$ defines a projection from $\mathbb{R}^{2Nd}$ to the hyperplane $H^N = \{(x, y) \in \mathbb{R}^{2Nd} : (\sum_i x^i, \sum_i y^j) = 0\}$. We note that distances $\hat{\rho}_N$ and $\bar{\rho}_N$ are equivalent to $\bar{\rho}_N$ given by

$$
\bar{\rho}_N((x, y), (\tilde{x}, \tilde{y})) = \bar{\rho}_N(\pi(x, y), \pi(\tilde{x}, \tilde{y})), \quad \text{for all } x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^{Nd},
$$

with $\rho = 1$ and $p = 2$, respectively.

**Theorem 28** (Propagation of chaos for unconfined Langevin dynamics in $L^2$ and $L^1$ Wasserstein distance). Suppose Assumption 4 holds. Let $\bar{\mu}_0$ and $\mu_0$ be two probability distributions on $\mathbb{R}^{2d}$ satisfying Assumption 5. For $t \geq 0$, let $\mu_t$ be the law of the process $(X_t, Y_t)$, where $(X_s, Y_s)_{s \geq 0}$ is a solution to (85) with initial distribution $\bar{\mu}_0$. Let $\mu_t^N$ be the law of $\{X_t^{i,N}, Y_t^{i,N}\}_{i=1}^{N}$, where $\{(X_s^{i,N}, Y_s^{i,N})\}_{i=1}^{N}$, $s \geq 0$ is a solution to (3) with $b = 0$ and with initial distribution $\mu_0^N = \mu_0^N$. If $L_\bar{\gamma}$ satisfies (88), then

$$
W_{2,\bar{\rho}_N}(\bar{\mu}_0^N, \mu_t^N) \leq e^{-\ell/2}W_{2,\bar{\rho}_N}(\bar{\mu}_0^N, \mu_0^N) + \hat{\epsilon}^{-1/2}C_3N^{-1/2} \quad \text{and}
$$

$$
W_{2,\bar{\rho}_N}(\bar{\mu}_0^N, \mu_t^N) \leq \sqrt{2M_3}e^{-\ell/2}W_{2,\bar{\rho}_N}(\bar{\mu}_0^N, \mu_0^N) + M_4\hat{\epsilon}^{-1/2}C_4N^{-1/2},
$$

where $\hat{\epsilon}, \bar{\rho}_N$ and $M_3$ are given in (90), (98) and (91), respectively. The constant $M_4$ is given by

$$
M_4 = \gamma \max(\sqrt{2/\bar{\kappa}}, 2),
$$

and $C_3$ is a positive constant depending on $\gamma, d, \bar{\kappa}, L_{\bar{\kappa}}, L_{\bar{\gamma}}, u$ and on the second moment of $\bar{\mu}_0$. If $L_{\bar{\gamma}}$ satisfies (92), then

$$
W_{1,\bar{\rho}_N}(\bar{\mu}_0^N, \mu_t^N) \leq e^{-\ell t}W_{1,\bar{\rho}_N}(\bar{\mu}_0^N, \mu_0^N) + \hat{\epsilon}^{-1/2}C_4N^{-1/2} \quad \text{and}
$$

$$
W_{1,\bar{\rho}_N}(\bar{\mu}_0^N, \mu_t^N) \leq \sqrt{2M_3}e^{-\ell t}W_{1,\bar{\rho}_N}(\bar{\mu}_0^N, \mu_0^N) + M_4\hat{\epsilon}^{-1/2}C_4N^{-1/2},
$$

where $C_4$ is a positive constant depending on $\gamma, d, \bar{\kappa}, L_{\bar{\kappa}}, L_{\bar{\gamma}}, u$ and on the second moment of $\bar{\mu}_0$.

**Proof.** The proof is postponed to Appendix A.3. □

**Remark 29.** For $t \geq 0$, let $\mu_t^N$ and $\nu_t^N$ denote the law of $\{X_t^{i,N}, Y_t^{i,N}\}_{i=1}^{N}$ and $\{X_t^{i,N}, Y_t^{i',N}\}_{i=1}^{N}$, where the processes $\{(X_s^{i,N}, Y_s^{i,N})\}_{i=1}^{N}$, $s \geq 0$ and $\{(X_s^{i,N}, Y_s^{i,N})\}_{i=1}^{N}$, $s \geq 0$ are solutions to (3) with initial distributions $\mu_0^N$ and $\nu_0^N$, respectively, and for which Assumption 4 is supposed. An easy adaptation of the proof of Theorem 17 shows that if (88) holds, then

$$
W_{2,\bar{\rho}_N}(\mu_0^N, \nu_t^N) \leq e^{-\ell t}W_{2,\bar{\rho}_N}(\mu_0^N, \nu_0^N) \quad \text{and} \quad W_{2,\bar{\rho}_N}(\mu_t^N, \nu_0^N) \leq \sqrt{2M_3}e^{-\ell t}W_{2,\bar{\rho}_N}(\mu_0^N, \nu_0^N),
$$

and if (92) holds, then

$$
W_{1,\bar{\rho}_N}(\mu_0^N, \nu_t^N) \leq e^{-\ell t}W_{1,\bar{\rho}_N}(\mu_0^N, \nu_0^N) \quad \text{and} \quad W_{1,\bar{\rho}_N}(\mu_t^N, \nu_0^N) \leq \sqrt{2M_3}e^{-\ell t}W_{1,\bar{\rho}_N}(\mu_0^N, \nu_0^N),
$$

where $\hat{\epsilon}$ and $M_3$ are given in (90) and (91), respectively. For the proof, a coupling of two copies of $N$ particle systems is constructed in the same lines as (108). As it will clarify by an inspection of the proof of Theorem 17, we can obtain a slightly better contraction rate in $L^2$ Wasserstein distance for the particle system compared to the rate in the propagation of chaos result.
A.3. Proof of Section A.1 and Section A.2

**Proof of Theorem 24.** Given two probability measures $\mu_0, \nu_0$ on $\mathbb{R}^{2d}$ and a $d$-dimensional Brownian motion $(B_t)_{t \geq 0}$, we consider the synchronous coupling $((\tilde{X}_t, \tilde{Y}_t), (\tilde{X}_t', \tilde{Y}_t'))_{t \geq 0}$ of two copies of solutions to (85) on $\mathbb{R}^{2d} \times \mathbb{R}^{2d}$ given by

\[
\begin{align*}
\frac{d\tilde{X}_t}{dt} &= \tilde{Y}_tdt \\
\frac{d\tilde{Y}_t}{dt} &= -\gamma\tilde{Y}_t + u\int_{\mathbb{R}^d} \tilde{b}(\tilde{X}_t, x) \nu_1^x(dx) + \sqrt{2\gamma}d\tilde{B}_t, \\
\frac{d\tilde{X}'_t}{dt} &= \tilde{Y}'_tdt \\
\frac{d\tilde{Y}'_t}{dt} &= -\gamma\tilde{Y}'_t + u\int_{\mathbb{R}^d} \tilde{b}(\tilde{X}'_t, x) \nu_1^x(dx) + \sqrt{2\gamma}d\tilde{B}_t,
\end{align*}
\]  

(100)

where $\mu_t^x = \text{Law}(\tilde{X}_t), \nu_t^x = \text{Law}(\tilde{X}_t')$. We set $\tilde{Z}_t = \tilde{X}_t - \tilde{X}_t'$ and $\tilde{W}_t = \tilde{Y}_t - \tilde{Y}_t'$. By Assumption 4 the process $(\tilde{Z}_t, \tilde{W}_t)_{t \geq 0}$ satisfies

\[
\begin{align*}
\frac{d\tilde{Z}_t}{dt} &= \tilde{W}_tdt \\
\frac{d\tilde{W}_t}{dt} &= (\gamma\tilde{W}_t - u\int_{\mathbb{R}^d} \tilde{b}(\tilde{X}_t, x) \mu_t^x(dx) - u\int_{\mathbb{R}^d} \tilde{b}(\tilde{X}_t', x) \nu_t^x(dx))dt \\
&= (-\gamma\tilde{W}_t - u\tilde{K}\tilde{Z}_t + u\int_{\mathbb{R}^d} \tilde{g}(\tilde{X}_t - z) \mu_t^x(dx) - u\int_{\mathbb{R}^d} \tilde{g}(\tilde{X}_t' - z) \nu_t^x(dx))dt,
\end{align*}
\]  

(101)

where we used that $E[\tilde{Z}_t] = 0$, which holds by Assumption 4 and Assumption 5. Let $\tilde{A}, \tilde{B}, \tilde{C} \in \mathbb{R}^{d \times d}$ be positive definite matrices given by

\[
\tilde{A} = \gamma^{-2}u\tilde{K} + (1/2)(1 - 2\sigma)^2\text{Id}, \quad \tilde{B} = (1 - 2\sigma)\gamma^{-1}\text{Id}, \quad \text{and} \quad \tilde{C} = \gamma^{-2}\text{Id},
\]

where $\sigma$ is given by (87). Then, by Itô’s formula,

\[
\begin{align*}
\frac{d}{dt}(\tilde{Z}_t \cdot (\tilde{A}\tilde{Z}_t) + \tilde{Z}_t \cdot (\tilde{B}\tilde{W}_t) + \tilde{W}_t \cdot (\tilde{C}\tilde{W}_t)) \\
&\leq 2(\tilde{A}\tilde{Z}_t) \cdot \tilde{W}_tdt + (\tilde{W}_t \cdot (\tilde{B}\tilde{W}_t) - (\tilde{B}\tilde{Z}_t) \cdot (\gamma\tilde{W}_t + u\tilde{K}\tilde{Z}_t))dt - 2(\tilde{C}\tilde{W}_t) \cdot (\gamma\tilde{W}_t + u\tilde{K}\tilde{Z}_t)dt \\
&\quad + L_\tilde{g}|\tilde{B}|\tilde{Z}_t + 2\tilde{C}\tilde{W}_t(|\tilde{Z}_t| + E[|\tilde{Z}_t|])dt \\
&\leq \tilde{Z}_t \cdot ((-\gamma\tilde{B}) + \tilde{Z}_t \cdot (2\tilde{A} - \gamma\tilde{B} - 2u\tilde{K}\tilde{C})\tilde{W}_t + \tilde{W}_t \cdot ((\tilde{B} - \gamma\tilde{C})\tilde{W}_t) \\
&\quad + L_\tilde{g}|\tilde{B}|\tilde{Z}_t + 2\tilde{C}\tilde{W}_t(|\tilde{Z}_t| + E[|\tilde{Z}_t|]) \\ 
&\leq -2\sigma\gamma(\tilde{Z}_t \cdot (\tilde{A}\tilde{Z}_t) + \tilde{Z}_t \cdot (\tilde{B}\tilde{W}_t) + \tilde{W}_t \cdot (\tilde{C}\tilde{W}_t)) + L_\tilde{g}|\tilde{B}|\tilde{Z}_t + 2\tilde{C}\tilde{W}_t(|\tilde{Z}_t| + E[|\tilde{Z}_t|]),
\end{align*}
\]

where we applied (87) in the last step. More precisely, it holds for all $z \in \mathbb{R}^d$

\[
\begin{align*}
z \cdot ((-\gamma(1 - 4\sigma)\gamma^{-1})z) \leq (\gamma/2)z^2 \leq -\gamma\sigma|z|^2 \leq -\gamma\sigma(1 - 2\sigma)^2|z|^2
\end{align*}
\]  

(103)

and therefore

\[
\begin{align*}
z \cdot ((-\gamma(1 - 2\sigma)\gamma^{-1})z) \leq -2\gamma\sigma\gamma^{-2}(\gamma/2)(1 - 2\sigma)^2|z|^2.
\end{align*}
\]

Then for $\tilde{r}(t) = \tilde{r}((\tilde{X}_t, \tilde{Y}_t), (\tilde{X}'_t, \tilde{Y}'_t)) = (\tilde{Z}_t \cdot (\tilde{A}\tilde{Z}_t) + \tilde{Z}_t \cdot (\tilde{B}\tilde{W}_t) + \tilde{W}_t \cdot (\tilde{C}\tilde{W}_t))^{1/2}$ given in (86),

\[
\frac{d\tilde{r}(t)^2}{dt} \leq -2\sigma\gamma\tilde{r}(t)^2dt + L_\tilde{g}|\tilde{B}|\tilde{Z}_t + 2\gamma\tilde{W}_t(|\tilde{Z}_t| + E[|\tilde{Z}_t|])dt.
\]

By taking expectation, it holds

\[
\frac{d}{dt}E[\tilde{r}(t)^2] \leq -2\sigma\gamma E[\tilde{r}(t)^2] + L_\tilde{g}|\tilde{B}|\tilde{Z}_t + 2\gamma\tilde{W}_t(|\tilde{Z}_t| + E[|\tilde{Z}_t|]).
\]

By (88), (87) and Young’s inequality, we obtain for the last term

\[
L_\tilde{g}|\tilde{B}|\tilde{Z}_t + 2\gamma\tilde{W}_t(|\tilde{Z}_t| + E[|\tilde{Z}_t|]) \leq \frac{\sqrt{k\gamma}}{2}E[(|\tilde{Z}_t| + 2\gamma\tilde{W}_t)|\tilde{Z}_t| + E[|\tilde{Z}_t|])]
\]

(106)

\[
\leq \sigma\gamma\left(\frac{1}{4}E[|\tilde{Z}_t|^2] + \frac{1}{4}E[(1 - 2\sigma)|\tilde{Z}_t| + 2\gamma\tilde{W}_t]^2]\right) \leq \sigma\gamma E[\tilde{r}(t)^2].
\]
By inserting this bound in (105), we obtain by Grönwall’s inequality,
\[
\mathcal{W}_{2,\mathcal{F}}(\bar{\mu}_t, \bar{\nu}_t)^2 \leq E[\hat{r}(t)^2] \leq e^{-2\hat{c}t} E[\hat{r}(0)^2]
\]
with \(\hat{c}\) given in (90). By taking the square root and the infimum over all couplings \(\omega \in \Pi(\bar{\mu}_0, \bar{\nu}_0)\), we obtain the first result in \(L^2\) Wasserstein distance. The second bound holds by (94) with \(M_2\) given by (91). To obtain contraction in \(L^1\) Wasserstein distance, we take the square root in (104).
\[
d\hat{r}(t) \leq -\sigma \gamma \hat{r}(t) dt + L\bar{g} u \gamma^{-1} \frac{|(1 - 2\sigma)\hat{Z}_t + 2\gamma^{-1}\hat{W}_t|}{2\hat{r}(t)} (||\hat{Z}_t|| + E[||\hat{Z}_t||]) dt
\]
where the last step holds by (107)
\[
\frac{|(1 - 2\sigma)\hat{Z}_t + 2\gamma^{-1}\hat{W}_t|}{2\hat{r}(t)} \leq \frac{1}{2} \left( \frac{(1 - 2\sigma)^2|\hat{Z}_t|^2 + 4(1 - 2\sigma)\gamma^{-1}\hat{Z}_t \cdot \hat{W}_t + 4\gamma^{-2}|\hat{W}_t|^2}{(\bar{\kappa} w \gamma^{-2} + (1/2)(1 - 2\sigma)^2)||\hat{Z}_t||^2 + (1 - 2\sigma)\gamma^{-1}|\hat{Z}_t \cdot \hat{W}_t + \gamma^{-2}|\hat{W}_t||} \right)^{1/2} \leq 1.
\]
Taking expectation and applying (92) we obtain
\[
\frac{d}{dt} E[\hat{r}(t)] \leq -\sigma E[\hat{r}(t)] + 2L\bar{g} u \gamma^{-1} E[||\hat{Z}_t||] \leq -\sigma E[\hat{r}(t)] + \frac{\sigma^2}{2} E[\sqrt{\bar{\kappa} w \gamma^{-2} ||\hat{Z}_t||}] \leq -\frac{\sigma^2}{2} E[\hat{r}(t)].
\]
Hence by Grönwall’s inequality,
\[
\mathcal{W}_{1,\mathcal{F}}(\bar{\mu}_t, \bar{\nu}_t) \leq e^{-\hat{c}t} E[\hat{r}(0)],
\]
where \(\hat{c}\) is given in (90). Taking the infimum over all couplings \(\omega \in \Pi(\bar{\mu}_0, \bar{\nu}_0)\), we obtain the first bound in \(L^1\) Wasserstein distance. The second bound follows by (94) with \(M_3\) given in (91).

To prove Theorem 28, we establish a second moment bound of the solution to the nonlinear unconfined Langevin equation.

Lemma 30 (Moment control for unconfined Langevin dynamics). Suppose that Assumption 4 and (88) hold. Let \((X_t, Y_t)_{t \geq 0}\) be a solution to (85) with initial distribution satisfying Assumption 5. Then there exists a finite constant \(C_5 > 0\) such that
\[
\sup_{t \geq 0} E[||\bar{X}_t||^2] \leq C_5.
\]
The constant \(C_5\) depends on \(\gamma, d, \bar{\kappa}, L\bar{g}, u\) and the second moment of the initial distribution.

Proof. As in the proof of Lemma 23, we apply the idea of the proof from [22, Lemma 8]. First, we note that by Assumption 4 and Assumption 5, \(E[\bar{X}_t] = E[\bar{Y}_t] = 0\) for all \(t \geq 0\), since by anti-symmetry of \(\bar{g}\)
\[
\frac{d}{dt} E[\bar{X}_t] = E[\bar{Y}_t], \quad \frac{d}{dt} E[\bar{Y}_t] = -\gamma E[\bar{Y}_t],
\]
and \(E[\bar{X}_0] = E[\bar{Y}_0] = 0\). Hence, \(\bar{X}_t \cdot E[\bar{X}_t] = \bar{Y}_t \cdot E[\bar{X}_t] = 0\). Further, we bound \(E_x \cdot \bar{g}(\bar{X}_t, x)| \leq L\bar{g}(||\bar{X}_t|| + E[||\bar{X}_t||])\).

By Itô’s formula and Assumption 4, it holds for \(\sigma \in (0, 1/2)\),
\[
d(\gamma^{-2}u\bar{X}_t \cdot (\bar{K}\bar{X}_t) + (1/2)(1 - 2\sigma)\bar{X}_t + \gamma^{-1}\bar{Y}_t|^2 + (1/2)\gamma^{-2}||\bar{Y}_t||^2)
\]
\[
\leq (2\gamma^{-2}u\bar{X}_t \cdot (\bar{K}\bar{Y}_t) + (1 - 2\sigma)\bar{X}_t \cdot \bar{Y}_t)dt + (1 - 2\sigma)\gamma^{-1}||\bar{Y}_t||^2 - \bar{X}_t \cdot (u\bar{K}\bar{X}_t) - \gamma \bar{X}_t \cdot \bar{Y}_t)dt
\]
\[
+ \gamma^{-2}(-2||\bar{Y}_t||^2 - 2(u\bar{K}\bar{Y}_t) \cdot \bar{X}_t)dt + L\bar{g} u((1 - 2\sigma)\gamma^{-1}\bar{X}_t + 2\gamma^{-2}\bar{Y}_t(||\bar{X}_t|| + E[||\bar{X}_t||])dt
\]
\[
+ 2\gamma^{-1}u dd + \sqrt{2\gamma^{-1}u((1 - 2\sigma)\bar{X}_t + 2\gamma^{-1}\bar{Y}_t)dB_t
\]
\[
\leq -((1 - 2\sigma)\gamma^{-1}u\bar{X}_t \cdot (\bar{K}\bar{X}_t) - 2\gamma((1 - 2\sigma)\gamma^{-1}\bar{X}_t \cdot \bar{Y}_t + \gamma^{-2}||\bar{Y}_t||^2)dt + 2\gamma^{-1}u dd
\]
\[
+ L\bar{g} u((1 - 2\sigma)\gamma^{-1}\bar{X}_t + 2\gamma^{-2}\bar{Y}_t(||\bar{X}_t|| + E[||\bar{X}_t||])dt + \sqrt{2\gamma^{-1}u((1 - 2\sigma)\bar{X}_t + 2\gamma^{-1}\bar{Y}_t)dB_t
\]
Then by (103) we obtain after taking expectation
\[
\frac{d}{dt} \mathbb{E}[\gamma^{-2} u \tilde{X}_t \cdot (\tilde{K} \tilde{X}_t) + \frac{1}{2}((1-2\sigma) \tilde{X}_t + \gamma^{-1} \tilde{Y}_t)^2 + \frac{1}{2} \gamma^{-2} |\tilde{Y}_t|^2]
\]
\[
\leq -2\sigma \gamma \mathbb{E}[\gamma^{-2} u \tilde{X}_t \cdot (\tilde{K} \tilde{X}_t) + \frac{1}{2}((1-2\sigma) \tilde{X}_t + \gamma^{-1} \tilde{Y}_t)^2 + \frac{1}{2} \gamma^{-2} |\tilde{Y}_t|^2 + 2\gamma^{-1} u d] + L_3 u \gamma^{-1} \mathbb{E}[|| (1-2\sigma) \tilde{X}_t + \gamma^{-1} \tilde{Y}_t || (|\tilde{X}_t| + \mathbb{E}[|\tilde{X}_t|])].
\]
By (88) and Young’s inequality, we bound the last term similarly as (106) by
\[
L_3 u \gamma^{-1} \mathbb{E}[|| (1-2\sigma) \tilde{X}_t + \gamma^{-1} \tilde{Y}_t || (|\tilde{X}_t| + \mathbb{E}[|\tilde{X}_t|])] \leq \sigma \gamma \left( \hat{\kappa} u \gamma^{-2} \mathbb{E}[|\tilde{X}_t|^2] + \frac{1}{2} \mathbb{E}[|| (1-2\sigma) \tilde{X}_t + \gamma^{-1} \tilde{Y}_t ||^2] + \frac{1}{2} \mathbb{E}[|\tilde{Y}_t|^2] \right).
\]
Hence,
\[
\frac{d}{dt} \mathbb{E}[\gamma^{-2} u \tilde{X}_t \cdot (\tilde{K} \tilde{X}_t) + \frac{1}{2}((1-2\sigma) \tilde{X}_t + \gamma^{-1} \tilde{Y}_t)^2 + \frac{1}{2} \gamma^{-2} |\tilde{Y}_t|^2]
\]
\[
\leq -\sigma \gamma \mathbb{E}[\gamma^{-2} u \tilde{X}_t \cdot (\tilde{K} \tilde{X}_t) + \frac{1}{2}((1-2\sigma) \tilde{X}_t + \gamma^{-1} \tilde{Y}_t)^2 + \frac{1}{2} \gamma^{-2} |\tilde{Y}_t|^2 + 2\gamma^{-1} u d].
\]
Then by Grönwall’s inequality, there exists a constant \( C \) such that
\[
\sup_{t \geq 0} \mathbb{E}[\gamma^{-2} u \tilde{X}_t \cdot (\tilde{K} \tilde{X}_t) \frac{1}{2}((1-2\sigma) \tilde{X}_t + \gamma^{-1} \tilde{Y}_t)^2 + \frac{1}{2} \gamma^{-2} |\tilde{Y}_t|^2] \leq C < \infty
\]
and we obtain the result for \( C_5 = C / (\hat{\kappa} \gamma^{-2} u) \).

\[\square\]

**Proof of Theorem 28.** We consider a synchronous coupling approach of solutions to (85) and (3) with \( b = 0 \). Fix \( N \in \mathbb{N} \). Let \( \{ (B^i_t)_{t \geq 0} \}_{i=1}^{N} \) be \( N \) independent \( d \)-dimensional Brownian motions and let \( \mu_0 \) and \( \tilde{\mu}_0 \) be two probability measures on \( \mathbb{R}^{2d} \). The coupling \( \{ (X^i_t, Y^i_t), (X^i_t, Y^i_t) \}_{i=1}^{N} \) \( (t \geq 0) \) of \( N \) copies of a solution to (85) and a solution to (3) with \( b = 0 \) is given on \( \mathbb{R}^{2Nd} \times \mathbb{R}^{2Nd} \) by
\[
\begin{align*}
\frac{dX^i_t}{dt} &= Y^i_t dt \\
\frac{dY^i_t}{dt} &= (-\gamma Y^i_t + u \int_{\mathbb{R}^d} \tilde{b}(X^i_t, z) \tilde{\mu}^i_t(dz))dt + \sqrt{2\gamma u} dB^i_t, \quad (X^i_0, Y^i_0) \sim \tilde{\mu}_0, \\
\frac{dX^i_t}{dt} &= Y^i_t dt \\
\frac{dY^i_t}{dt} &= (-\gamma Y^i_t + u N^{-1} \sum_{j=1}^{N} b(\tilde{X}^j_t, \tilde{X}^i_t))dt + \sqrt{2\gamma u} dB^i_t, \quad (X^i_0, Y^i_0) \sim \mu_0
\end{align*}
\]
for \( i = 1, \ldots, N \), where \( \tilde{\mu}^i_t = \text{Law}(\tilde{X}^i_t) \) for all \( i \). For simplicity, we omitted the parameter \( N \) in the index of \( (X^i_t, Y^i_t) \) in the particle model. We set \( \tilde{Z}^i_t = \tilde{X}^i_t - X^i_t - N^{-1} \sum_{j=1}^{N} (\tilde{X}^j_t - X^i_t) \) and \( \tilde{W}^i_t = \tilde{Y}^i_t - Y^i_t - N^{-1} \sum_{j=1}^{N} (\tilde{Y}^j_t - Y^i_t) \). By Assumption 4, the process \( \{ (\tilde{Z}^i_t, \tilde{W}^i_t)_{i=1}^{N} \}_{t \geq 0} \) satisfies
\[
\begin{cases}
\frac{d\tilde{Z}^i_t}{dt} = \tilde{W}^i_t dt \\
\frac{d\tilde{W}^i_t}{dt} = -\tilde{W}^i_t dt + u \left( \int_{\mathbb{R}^d} \tilde{b}(X^i_t, z) \tilde{\mu}^i_t(dz) - N^{-1} \sum_{j=1}^{N} \int_{\mathbb{R}^d} \tilde{b}(\tilde{X}^j_t, \tilde{X}^i_t, \tilde{Z}^i_t) \tilde{\mu}^j_t(d\tilde{Z}^j_t) \right)dt \\
- N^{-1} \sum_{j=1}^{N} b(\tilde{X}^j_t, \tilde{X}^i_t) + N^{-2} \sum_{j,k=1}^{N} \tilde{b}(\tilde{X}^j_t, \tilde{X}^k_t) \right)dt \\

= -\tilde{W}^i_t dt + u \left( -\bar{A} \tilde{Z}^i_t + N^{-1} \sum_{j=1}^{N} (\tilde{g}(\tilde{X}^j_t - \tilde{X}^i_t) - \tilde{g}(\tilde{X}^j_t - \tilde{X}^i_t)) + \tilde{A}^i_t + N^{-1} \sum_{j=1}^{N} \tilde{A}^i_t \right)dt,
\end{cases}
\]
where \( \bar{A} = \int_{\mathbb{R}^d} \tilde{b}(\tilde{X}^i_t, \tilde{Z}^i_t) \tilde{\mu}^i_t(d\tilde{Z}^i_t) - N^{-1} \sum_{j=1}^{N} \tilde{b}(\tilde{X}^j_t, \tilde{X}^i_t) \) for all \( k = 1, \ldots, N \). Hence, for the positive definite matrices \( \bar{A}, \tilde{B}, \tilde{C} \) given in (102), we obtain for \( i = 1, \ldots, N \),
\[
\frac{d(\tilde{Z}^i_t \cdot (\bar{A} \tilde{Z}^i_t) + \tilde{Z}^i_t \cdot (\tilde{B} \tilde{W}^i_t) + \tilde{W}^i_t \cdot (\tilde{C} \tilde{W}^i_t))}{dt} \\
\leq 2 \tilde{Z}^i_t \cdot (\bar{A} \tilde{W}^i_t) dt + (\tilde{W}^i_t \cdot (\tilde{B} \tilde{W}^i_t) - \gamma \tilde{Z}^i_t \cdot (\tilde{B} \tilde{W}^i_t) - (\gamma \tilde{Z}^i_t) \cdot (\bar{A} \tilde{W}^i_t)) dt + (\tilde{C} \tilde{W}^i_t) \cdot (2\gamma \tilde{W}^i_t - 2u \bar{K} \tilde{Z}^i_t) dt
\]
Hence, similarly as in (106) and
\[
L_i + 2 \tilde{C} \tilde{W}_i \bigg\{ L_{\tilde{\gamma}} N^{-1} \sum_{j=1}^{N} (|\tilde{Z}_i^j| + |\tilde{Z}_i^j|) + A^j_i + N^{-1} \sum_{i=j}^{N} A^j_i \bigg\} dt
\]
\[
\leq \left( -(u K \tilde{Z}_i) \cdot (\tilde{B} \tilde{Z}_i^j) + \tilde{Z}_i^j \cdot ((2 \tilde{A} - \gamma B - 2 u \tilde{K} \tilde{C}) \tilde{W}_i) + \tilde{W}_i \cdot ((\tilde{B} - 2 \gamma \tilde{C}) \tilde{W}_i^j) \right) dt
\]
\[
+ \int \frac{du}{\sigma} \bar{b}(X_i^k, \bar{z}) \mu_i^k (dz) - N^{-1} \sum_{j=1}^{N} \bar{b}(X_i^k, \bar{X}_i^j) \bigg\}
\]
\[
= \left( -(u K \tilde{Z}_i) \cdot (\tilde{B} \tilde{Z}_i^j) + \tilde{Z}_i^j \cdot ((2 \tilde{A} - \gamma B - 2 u \tilde{K} \tilde{C}) \tilde{W}_i) + \tilde{W}_i \cdot ((\tilde{B} - 2 \gamma \tilde{C}) \tilde{W}_i^j) \right) dt
\]
\[
+ \int \frac{du}{\sigma} \bar{b}(X_i^k, \bar{z}) \mu_i^k (dz) - N^{-1} \sum_{j=1}^{N} \bar{b}(X_i^k, \bar{X}_i^j)
\]
Then by (104) for \( \tilde{r}^i(t) = \tilde{r}((X_i^k, \tilde{Y}_i^k), (X_i^l, \tilde{Y}_i^l)) \)
\[
d\tilde{r}^i(t) = d(\tilde{Z}_i \cdot (\tilde{A} \tilde{Z}_i^l) + \tilde{Z}_i \cdot (\tilde{B} \tilde{W}_i^j) + \tilde{W}_i \cdot (\tilde{C} \tilde{W}_i^l))
\]
\[
\leq -2 \sigma^2 \gamma \tilde{r}^i(t)^2 dt + \gamma^{-1} |(1 - 2 \sigma) \tilde{Z}_i + 2 \gamma^{-1} \tilde{W}_i^l| \left( L_{\tilde{\gamma}} N^{-1} \sum_{j=1}^{N} (|\tilde{Z}_i^j| + |\tilde{Z}_i^j|) + A^j_i + N^{-1} \sum_{i=j}^{N} A^j_i \right) dt
\]
and hence, for \( \tilde{\rho}_t := \tilde{\rho}_N ((X_t, Y_t), (\tilde{X}_t, \tilde{Y}_t)) \) given in (95),
\[
d\tilde{\rho}_t \leq -2 \sigma^2 \gamma \tilde{\rho}_t dt + \frac{u}{\gamma} N^{-1} \sum_{j=1}^{N} \left( (1 - 2 \sigma) \tilde{Z}_i + 2 \gamma^{-1} \tilde{W}_i^l \right) \left( L_{\tilde{\gamma}} N^{-1} \sum_{j=1}^{N} (|\tilde{Z}_i^j| + |\tilde{Z}_i^j|) + A^j_i + N^{-1} \sum_{i=j}^{N} A^j_i \right) dt.
\]
For the last term, we obtain by (88) and Young’s inequality
\[
L_{\tilde{\gamma}} \tilde{\rho}_t \leq \frac{1}{2} N \sum_{i=1}^{N} \left( 1 - 2 \sigma \right) \tilde{Z}_i^j + 2 \gamma^{-1} \tilde{W}_i^l \left( \tilde{Z}_i^j + \tilde{Z}_i^j \right) \leq \sigma \gamma \tilde{\rho}_t
\]
similarly as in (106) and
\[
\frac{u}{\gamma} \frac{1}{N} \sum_{i=1}^{N} \left( 1 - 2 \sigma \right) \tilde{Z}_i^j + 2 \gamma^{-1} \tilde{W}_i^l \left( A_i^j + A_i^j \right) \leq \frac{\sigma \gamma}{N} \sum_{i=1}^{N} \left( 1 - 2 \sigma \right) \tilde{Z}_i^j + 2 \gamma^{-1} \tilde{W}_i^l \left( A_i^j + A_i^j \right) \leq \frac{\sigma \gamma}{2} \tilde{\rho}_t + \frac{8 u^2}{\gamma^3 \sigma} \sum_{i=1}^{N} (A_i^j)^2.
\]
Inserting these estimates in (111) and taking expectation yields
\[
\frac{d}{dt} \mathbb{E}[\tilde{\rho}_t] \leq -\frac{\sigma \gamma}{2} \mathbb{E}[\tilde{\rho}_t] + \frac{8 u^2}{\gamma^3 \sigma} \sum_{i=1}^{N} \mathbb{E}[(A_i^j)^2].
\]
We bound \( \mathbb{E}[A_i^j] \) similar as in the proof of Theorem 17. Note that by Assumption 4, \( \tilde{b} \) is Lipschitz continuous with a Lipschitz constant which is bounded from above by \( L_{\tilde{K}} + L_{\tilde{\gamma}} \). Hence, (82) and (83) hold here with \( L_{\tilde{K}} + L_{\tilde{\gamma}} \) instead of \( \tilde{L} \). Then,
\[
\mathbb{E}[A_i^j] \leq \mathbb{E} \left[ \int_{\mathbb{R}^d} \tilde{b}(X_i^k, z) \mu_i^k (dz) - \frac{1}{N} \sum_{j=1}^{N} \tilde{b}(X_i^k, \bar{X}_i^j)^2 \right] \leq \frac{16 (L_{\tilde{K}} + L_{\tilde{\gamma}})^2}{N} \int_{\mathbb{R}^d} |z|^2 \tilde{\mu}_t (dx).
\]
By Lemma 30, there exists a constant \( C_0 \) depending on \( \gamma, \mathbb{E}[|X_0|^2] + |Y_0|^2, d, \tilde{\kappa}, L_{\tilde{K}}, L_{\tilde{\gamma}}, u \) such that for \( N \geq 2 \) and \( i = 1, ..., N \),
\[
\sup_{t \geq 0} \mathbb{E}[A_i^j] \leq C_0 N^{-1}.
\]
Hence,
\[
\frac{d}{dt} \mathbb{E}[\tilde{\rho}_t] \leq -2 \sigma \gamma \mathbb{E}[\tilde{\rho}_t] + C_0^2 N^{-1} / 2,
\]
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where \( C_3^2 = \frac{8u^2}{\gamma^2} C_0 \). By Grönwall’s inequality,

\[
W_{2,\bar{\rho}_N}(\Law(X^1_t, \ldots, X^N_t), (\bar{\mu}_t)^{\otimes N})^2 \leq \mathbb{E}[\hat{\rho}_t^2] \leq e^{-\epsilon t} \mathbb{E}[\hat{\rho}_0^2] + \epsilon^{-1} C_4^2 N^{-1}
\]

with \( \epsilon \) given in (90). By taking the infimum over all couplings \( \omega \in \Pi(\bar{\mu}_0^N, \bar{\mu}_0^N) \), we obtain the first result in \( L^2 \) Wasserstein distance. The second bound holds by (94) with \( M_3 \) and \( M_4 \) given by (91) and (99), respectively. To obtain the bound in \( L^1 \) Wasserstein distance, we note that by (110)

\[
d\hat{\rho}_t^i(t) = \frac{1}{2\rho^i(t)} d\hat{\rho}_t^i(t)^2 \leq -\sigma \gamma \hat{\rho}_t^i(t) dt + \frac{|1 - 2\sigma| \tilde{Z}^i_t + 2\gamma^{-1} \tilde{W}_t|}{2\gamma^{-1} \tilde{W}_t} u \left( \frac{L^2}{N} \sum_j (|\tilde{Z}^j_t| + |\tilde{Z}^i_t|) + A^i_t + \frac{1}{N} \sum_{j=1}^N A^j_t \right) dt
\]

where the last step holds by (107). By summing over \( i \) and taking expectation, we obtain by (92) for \( \hat{\rho}_t := \bar{\rho}_N((X_t, Y_t), (\bar{X}_t, \bar{Y}_t)) \) given in (96),

\[
\frac{d}{dt} \mathbb{E}[\hat{\rho}_t] \leq -\sigma \gamma / 2 \mathbb{E}[\hat{\rho}_t] + \gamma^{-1} u N^{-1} \sum_{i=1}^N \mathbb{E}[A^i_t].
\]

By Assumption 4 and Lemma 30, there exists a constant \( C_4 \) depending on \( \gamma, \mathbb{E}[|\bar{X}_0|^2 + |\bar{Y}_0|^2], d, \tilde{k}, L_{\tilde{K}}, u \) and \( L_{\tilde{g}} \) such that

\[
\sup_{t \geq 0} \mathbb{E}[A^i_t] \leq C_4 \gamma N^{-1/2}
\]

similarly as in (84). Hence,

\[
\frac{d}{dt} \mathbb{E}[\hat{\rho}_t] \leq -\frac{\sigma \gamma}{2} \mathbb{E}[\hat{\rho}_t] + C_4 N^{-1/2}.
\]

By Grönwall’s inequality,

\[
W_{1,\bar{\rho}_N}(\bar{\mu}_t^{\otimes N}, \bar{\mu}_t^N) \leq \mathbb{E}[\hat{\rho}_t] \leq e^{-\epsilon t} \mathbb{E}[\hat{\rho}_0] + \epsilon^{-1} C_4 N^{-1/2}
\]

for \( \epsilon \) given in (90). Taking the infimum over all couplings \( \omega \in \Pi(\bar{\rho}_0^{\otimes N}, \bar{\mu}_0^N) \), we obtain the first result in \( L^1 \) Wasserstein distance. The second bound holds by (94) with \( M_3 \) and \( M_4 \) given in (91) and (99).

\[ \square \]

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