Yaglom limit for unimodal Lévy processes

Gavin Armstrong\textsuperscript{a}, Krzysztof Bogdan\textsuperscript{b,*} Tomasz Grzywny\textsuperscript{b,†} Łukasz Leżaj\textsuperscript{b,‡} and Longmin Wang\textsuperscript{c}

\textsuperscript{a}Mathematics Department, Central Washington University, 400E University Way, Ellensburg, WA 98926, USA, E-mail: gavin.armstrong@cwu.edu
\textsuperscript{b}Department of Mathematics, Wrocław University of Science and Technology, wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland, E-mail: *krzysztof.bogdan@pwr.edu.pl; †tomasz.grzywny@pwr.edu.pl; ‡lukasz.lezaj@pwr.edu.pl
\textsuperscript{c}School of Statistics and Data Science, Nankai University, Tianjin 300071, P. R. China, E-mail: wanglm@nankai.edu.cn

Abstract. We prove universality of the Yaglom limit of Lipschitz cones among all unimodal Lévy processes sufficiently close to the isotropic $\alpha$-stable Lévy process.

Résumé. On prouve l’universalité de la limite de Yaglom dans les cônes de Lipschitz parmi tous les processus de Lévy unimodaux suffisamment proches du processus de Lévy isotrope $\alpha$-stable.

MSC2020 subject classifications: Primary 60G51, 60J50; secondary 60G18, 60J35

Keywords: Yaglom limit, Lévy process, Lipschitz cone, boundary limit

1. Introduction

Let $d \geq 1$ and $X = (X_t : t \geq 0)$ be a pure-jump isotropic unimodal Lévy process in $\mathbb{R}^d$ with non-integrable Lévy density $\nu$ and the characteristic exponent $\psi$. Thus,

\begin{equation}
\mathbb{E}e^{i\xi \cdot X_t} = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d, \quad t \geq 0,
\end{equation}

where

\begin{equation}
\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos \xi \cdot z) \nu(z) \, dz, \quad \xi \in \mathbb{R}^d,
\end{equation}

\begin{equation}
\int_{\mathbb{R}^d} \nu(z) \, dz = \infty \quad \text{and} \quad \int_{\mathbb{R}^d} (|z|^2 \wedge 1) \nu(z) \, dz < \infty,
\end{equation}

and $\nu(z)$ is a radial function, non-increasing in $|z|$. For normalization we also assume that $\psi(1) = 1$. We let $\Gamma$ be an arbitrary Lipschitz cone on $\mathbb{R}^d$ with the vertex at the origin and we denote by $\tau_\Gamma$ the first exit time from the cone,

$$
\tau_\Gamma = \inf \{ t > 0 : X_t \notin \Gamma \}.
$$

Our main objective is to determine under which conditions there exist a function $g : [0, \infty) \mapsto (0, \infty)$ and a probability measure $\mu$ on $\Gamma$ such that

\begin{equation}
\lim_{t \to \infty} \mathbb{P}_x \left( \frac{X_t}{g(f)} \in A \bigg| \tau_\Gamma > t \right) = \mu(A), \quad x \in \Gamma, A \subset \Gamma.
\end{equation}

The above limit, if it exists, is called the Yaglom limit of $\Gamma$ for $X$, after Akiva Moiseevich Yaglom, who first identified quasi-stationary distributions for the subcritical Bienaymé-Galton-Watson branching processes in [60]. Generally speaking, Yaglom limits describe the large-time limiting behaviour of processes conditioned not to become extinct or absorbed.
Accordingly, the random time which defines the conditioning in (1.4) and in similar expressions is sometimes referred to as absorption time. For example, in the one-dimensional case, the absorption time is usually the first hitting time of the origin, see, e.g., Haas and Rivero [34], or the first hitting time of the negative half-line \((-\infty, 0)\), see, e.g., A. Kyprianou and Palmowski [43]. Over the last years the interest in the existence of Yaglom limits and quasi-stationary distributions was steadily growing. This is due to the fact that the topic is mathematically challenging and in various applications it is natural to ask about the limiting behaviour conditional on non-extinction. However, so far the existence of the Yaglom limit for Lévy processes was studied mainly in the one-dimensional setting or in the finite volume setting, with the notable exception of the result Bogdan, Palmowski and Wang [15] explained below, which we extend in this work.

We write \(\nu(r) := \nu(x)\) if \(|x| = r\) for some \(x \in \mathbb{R}^d\) (the radial profile of the Lévy density). The following additional assumptions on \(X\) will be explicitly made when needed:

**A1** There is \(\beta \in (0, 2)\) and \(M > 1\) such that

\[
\frac{\nu(r_1)}{\nu(r_2)} \leq M \left(\frac{r_1}{r_2}\right)^{d-\beta}, \quad 0 < r_1 < r_2 < \infty.
\]  

**A2** \(\nu(r)\) is regularly varying at infinity with index \(-d - \alpha\) for some \(\alpha \in (0, 2)\).

**A3** There exists \(t_0 > 0\) such that \(e^{-t_0 \psi} \in L^1\).

If A1, A2 and A3 are satisfied, then we say in short that A holds. We remark that the condition A1 is important in the probabilistic potential theory, in particular for estimating the Dirichlet heat kernel for \(X\), see Bogdan, Grzywny and Ryznar [11]. Assumption A2 is crucial for asymptotic analysis, in particular it puts the one-dimensional distributions of \(X\) in the domain of attraction of the isotropic \(\alpha\)-stable law. By Cygan, Grzywny and Trojan [19, Theorem 7 and Proposition 2], A2 is equivalent to \(\psi\) being regularly varying at 0 with index \(\alpha\). The condition A3 allows to formulate our results in terms of convergence of probability densities. Noteworthy, by Knopova and Schilling [39, the proof of Lemma 2.6] and (1.1), A3 is equivalent to the boundedness of the transition density of \(X\) for some (large) time \(t\).

For example, the geometric \(\alpha\)-stable Lévy process with \(\alpha \in (0, 2)\) satisfies A, since A2 and A3 follow from the formula \(\psi(x) = \log(1 + |x|^\alpha)\) and A1 follows from Grzywny and Ryznar [30, p. 10]; see Grzywny, Ryznar and Trojan [31] and Šikić, Song and Vondraček [59] for further properties of this important process.

The focal point of our discussion, however, is the isotropic \(\alpha\)-stable process in \(\mathbb{R}^d\) with \(\alpha \in (0, 2)\), which we denote by \(X^\alpha = (X^\alpha_t : t \geq 0)\) [10], see also below. Of course, the Lévy-Khinchine exponent of \(X^\alpha\) is \(\psi(\xi) = |\xi|^\alpha\) and the process satisfies A. For the sake of clarity and consistency, the objects pertaining to \(X^\alpha\) will be marked by a superscript \(\alpha\), which is then not an exponent. For instance, we write \(\tau^\alpha_t = \tau^\alpha(X^\alpha)\). Let \(\mu^\alpha\) be the Yaglom limit of the isotropic \(\alpha\)-stable process for the Lipschitz cone \(\Gamma\), given in [15, Theorem 1.1]:

\[
\lim_{t \to \infty} \mathbb{P}_x \left( t^{-1/\alpha} X^\alpha_t \in A \mid \tau^\alpha_t > t \right) = \mu^\alpha(A), \quad A \subset \Gamma, \quad x \in \Gamma.
\]  

By [15, Theorem 3.3], the probability measure \(\mu^\alpha\) has a density \(n^\alpha\) with respect to the Lebesgue measure and the following convergence of densities holds,

\[
n^\alpha(y) = \lim_{\Gamma \ni x \to 0} \frac{p^\Gamma,\alpha_x(y, y)}{\mathbb{P}_x(\tau^\alpha_t > 1)}, \quad y \in \Gamma,
\]  

\(p^\Gamma,\alpha\) being the Dirichlet heat kernel of the cone \(\Gamma\) for \(X^\alpha\), see (2.3).

Here is the main result of the paper. As usual, we let

\[
\psi^{-1}(u) = \sup\{r \geq 0 : \psi(r) = u\}, \quad u \geq 0.
\]  

**Theorem 1.1.** Let \(X\) be a pure-jump isotropic unimodal Lévy process with characteristic exponent \(\psi\). Assume A. Let \(\Gamma\) be a Lipschitz cone with vertex at the origin. Then,

\[
\lim_{s \to \infty} \mathbb{P}_x \left( \psi^{-1}(1/s) X^\alpha_s \in A \mid \tau^\alpha_t > s \right) = \mu^\alpha(A), \quad x \in \Gamma, \quad A \subset \mathbb{R}^d.
\]  

For example, the rescaling factor is \(\psi^{-1}(1/s) = (e^{1/s} - 1)^{1/\alpha}\) for the geometric \(\alpha\)-stable Lévy process. A qualitatively different rescaling can be found in Example 4.14, so the universality of the Yaglom limit for the considered class of processes refers to the limit but not the rescaling. The proof of Theorem 1.1 is given at the end of Section 4.4. In fact, the whole paper is devoted to this proof, but the setting and some auxiliary results may be of independent interest, e.g.,
the fact that in general $X$ is not self-similar but we have pointwise convergence of density functions and uniformity of potential-theoretic estimates and asymptotics for the rescaled processes $(\psi^{-1}(1/s)X_{st}: t > 0)$ parametrized by $s \geq 1$. We note in passing that (1.8) also holds for $\Gamma = \mathbb{R}^d$, as a consequence of Lemma 4.2 and Scheffé’s lemma, which may also be of interest.

Let us discuss in more detail some classical results on quasi-stationary distributions. The seminal work of Yaglom appeared in 1947. Since then a huge number of papers have dealt with this problem; a comprehensive survey is given by van Doorn and Pollet [58]. In particular, the discrete-time Markov chains were analysed by Seneta and Vere-Jones [54] in 1967 and by Tweedie [56] in 1974. In 1995 Jacka and Roberts [37] studied the continuous-time case. In 1971 E. Kyprianou [44] considered a conditional limit distribution for the virtual waiting time in queues with Poisson arrival. Quasi-stationary distributions for Markov chains on non-negative integers with absorption at the origin were considered by Ferrari et al. [22], Flaspohler and Holmes [24], and San Martín [47] in 1994. The result is an analogue of the random walk case studied by Iglehart [36] in 1974. The case of Bienaymé-Galton-Watson process. Finally, Bean et al. [6] gave Yaglom limits for stochastic fluid models.

In general, studies on quasi-stationary distributions are model specific — each class of stochastic processes is treated in a different way. Let us further mention the quasi-birth-and-death processes (see, e.g., Bean et al. [3–5]), Fleming-Viot processes (Asselah et al. [1] or Ferrari and Marič [23]) and branching processes (Lambert [45], see also Ren et al. [50, 51] and Harris et al. [35] for the so-called critical neutron transport), which are the continuous-space-time analogues of the Bienaymé-Galton-Watson process. Finally, Bean et al. [6] gave Yaglom limits for stochastic fluid models.

Let us turn our attention back to Lévy processes. The case of the Brownian motion with drift was resolved by Martínez and San Martín [47] in 1994. The result is an analogue of the random walk case studied by Iglehart [36] in 1974. The case of the so-called spectrally one-sided Lévy processes was investigated in [43] in 2006, using the Wiener-Hopf factorisation. The absorption time in [43] is taken as the first ruin time:

$$\tau^{-}_0 = \inf \{ t \geq 0 : X_t < 0 \}. $$

A similar problem was investigated by Czarna and Palowski [20] for the Parisian ruin,

$$\tau^0 = \inf \{ t > 0 : t - g_t > e^{\theta} \}. $$

Here $g_t = \sup \{ s \leq t : X_s \geq 0 \}$ and $e^{\theta}$ is an independent exponential random variable with intensity $\theta > 0$, so the ruin occurs if negative values of $X$ persist longer than $\exp(\theta)$.

The spectrally one-sided Lévy processes were also considered by Mandles et al. in [46] and Palmowski and Vlasiou [48] provided the speed of convergence to the quasi-stationary distribution, which turned out surprisingly slow.

Limits similar to (1.4) for self-similar Markov processes in the one-dimensional case were studied by Haas and Rivero [34]. Note that the normalization by $g$ in [34] strongly depends on the tails of the Lévy measure.

To the best of our knowledge, the only study of Yaglom limits for unbounded sets in the multidimensional case is [15], except for results similar to Zhang et al. [61] on Markov processes killed upon leaving a set of bounded volume. The volume boundedness allows to employ the first eigenfunction, however it is a prohibitive restriction in our setting.

Finally, let us note that the conditioning defined by (1.4) is related to but different than the conditioning of the process to stay forever in a set; we refer to Bertoin and Doney [8] for the case of random walks and to Bertoin [7], Chaumont [16] and Chaumont and Doney [17] for the case of continuous-time processes.

The proof of Theorem 1.1 is inspired by [15], the limit resulting from boundary asymptotics of Green potentials, but there are fundamental difficulties to overcome, mostly the lack of self-similarity (exact scaling) of $X$, low regularity of the heat kernel of the process, highly non-trivial uniform in $s \geq 1$ estimates for the rescaled processes $(\psi^{-1}(1/s)X_{st}: t > 0)$, and delicate considerations related to continuity of their random functionals, mainly the first exit time of $\Gamma$, in the Skorokhod topology as $s \to \infty$. A crucial step in our development is Theorem 4.7, which establishes stability, or uniformity, of limits of ratios of harmonic functions at the boundary of the cone. Namely, we refine the already general statement of Kwaśnicki and Juszczyzyn [42], by proving that for the rescaled processes the ratios converge uniformly. Furthermore, we avoid superficial technical assumptions on $\nu$ – this makes the arguments on convergence of integrals tricky, but statements simple. Of course, a number of further questions is now open: the case of non-Lipschitz cones $\Gamma$, the Yaglom limit for anisotropic stable Lévy processes (even for processes with independent components) and universality classes for Yaglom limits in each such case.

The structure of the paper is as follows. In Section 2 we discuss definitions and basic results. In Section 3 we give estimates of the (Dirichlet) heat kernel of $\Gamma$ for the process $X$. In Section 4 we analyse the effects of rescaling of $X$, including the convergence of densities of the rescaled processes. In particular, Section 4.2 is devoted to estimates of harmonic functions and oscillation-reduction, uniform for the rescaled processes, and in Section 4.3 we verify the convergence of the survival probabilities. In Section 4.4 we obtain normalized limits of Green potentials, expressed in terms of the Martin kernel, which we then bootstrap to the level of the heat kernel and the heat kernel conditioned by the survival probability, to finally prove Theorem 1.1 using the uniformity of relative estimates and asymptotics with respect to the scaling parameter $s$, as $s \to \infty$. 

Yaglom limit for unimodal Lévy processes

$3$
2. Notation and preliminaries

By $\mathcal{R}_\alpha^0$ we denote the class of (positive) functions $f: (0, \infty) \mapsto (0, \infty)$ which are regularly varying at the origin with index $\alpha$. Thus, $f \in \mathcal{R}_\alpha^0$ if for every $\lambda > 0$,
\[
\lim_{x \to 0^+} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha.
\]
Furthermore, we say that $f$ is regularly varying at infinity with index $\alpha$ and we write $f \in \mathcal{R}_\alpha^\infty$ if for every $\lambda > 0$,
\[
\lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha.
\]

Notation $c = c(a, b, \ldots)$ means that number $c \in (0, \infty)$, called constant, may be so chosen to depend only on $a, b, \ldots$. For non-negative functions $f, g$ we use the notation $f \lesssim g$ if there is a constant $c > 0$ such that $f \leq cg$ and we write $f \approx g$ if $c^{-1}g \leq f \leq cg$. If the constant $c$ is not important, we may write $f \approx g$. As usual, $B(x, r) = \{y \in \mathbb{R}^d: |y - x| < r\}$ is the open ball in $\mathbb{R}^d$ with radius $r > 0$ and center $x \in \mathbb{R}^d$. We let $B_r := B(0, r)$. For $x \in D \subset \mathbb{R}^d$ we let $\delta_D(x) = \inf\{|y - x|: y \in D^c\}$. All the considered sets, measures and functions are tacitly assumed Borel.

As in Introduction, $X = (X_t: t \geq 0)$ is a non-trivial pure-jump isotropic unimodal Lévy process in $\mathbb{R}^d$ with $d \geq 1$. The Lévy density $\nu$ is non-integrable but satisfies the Lévy measure condition in (1.3) and the Lévy-Khinchine exponent $\psi$ is given by (1.2). The Lévy-Khinchine formula (1.1) relates the distribution of $X_t$ to $\psi$ and $\nu$. Since $X$ is isotropic unimodal and has infinite Lévy measure, the distribution of $X_0$ is absolutely continuous if $t > 0$. We denote the density by $p_t$. In fact, due to Kulczycki and Ryznar [40, Lemma 2.5], $p_t$ is continuous on $\mathbb{R}^d \setminus \{0\}$. We define
\[
(2.1) \quad p_t(x, y) = p_t(y - x) \quad \text{and} \quad \nu(x, y) = \nu(y - x), \quad t > 0, x, y \in \mathbb{R}^d,
\]
the heat kernel (the transition probability density) and the jumping kernel of $X$, correspondingly. Of course, $p_t(x, y) = p_t(y, x)$ and $\nu(x, y) = \nu(y, x)$. Let
\[
P_t(x, A) = \int_A p_t(x, y) \, dy, \quad t > 0, x \in \mathbb{R}^d, A \subset \mathbb{R}^d.
\]
It is the transition probability of $X$. We also define the operator semigroup
\[
P_tf(x) = \int_{\mathbb{R}^d} f(y)p_t(x, y) \, dy, \quad f \in C_0(\mathbb{R}^d), \ x \in \mathbb{R}^d, \ t > 0.
\]
Its infinitesimal generator is
\[
\mathcal{L}\varphi(x) = \lim_{\varepsilon \to 0^+} \int_{|y| > \varepsilon} \left(\varphi(x + y) - \varphi(x)\right)\nu(y) \, dy,
\]
for $\varphi \in C_c^\infty(\mathbb{R}^d)$, see, e.g., Sato [52, Theorem 31.5]. In particular, we let
\[
\nu^\alpha(y) = A(d, \alpha)|y|^{d - \alpha}, \quad y \in \mathbb{R}^d,
\]
where $\alpha \in (0, 2)$ and
\[
A(d, \alpha) = \frac{\alpha 2^{\alpha - 1} \Gamma((d + \alpha)/2)}{\pi^{d/2} \Gamma(1 - \alpha/2)}.
\]
Then,
\[
(2.2) \quad \psi(\xi) = \psi^\alpha(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) \, \nu^\alpha(y) \, dy = |\xi|^\alpha, \quad \xi \in \mathbb{R}^d,
\]
and $\mathcal{L} = \Delta^{\alpha/2} := -(\Delta)^{\alpha/2}$, the fractional Laplacian; see, e.g., Kwaśnicki [41].

For an open non-empty set $D \subset \mathbb{R}^d$ we let
\[
\tau_D = \inf\{t > 0: X_t \notin D\},
\]
the first exit time from $D$. The transition density of the process killed on exiting $D$ is defined by Hunt’s formula:

\begin{equation}
\label{eq:2.3}
p_t^D(x, y) := p_t(x, y) - \mathbb{E}_x[\tau_D < t; p_{t-\tau_D}(X_{\tau_D}, y)], \quad t > 0, \ x, y \in \mathbb{R}^d.
\end{equation}

We have $0 \leq p_t^D(x, y) \leq p_t(x, y)$ for all $t > 0$ and $x, y \in \mathbb{R}^d$. It is well known that $p_t^D$ is symmetric: $p_t^D(x, y) = p_t^D(y, x)$ for all $x, y \in \mathbb{R}^d$ and $t > 0$. Moreover, the Chapman-Kolmogorov equations (semigroup property) hold for $p_t^D$:

\[ p_{t+s}^D(x, y) = \int_{\mathbb{R}^d} p_t^D(x, z)p_s^D(z, y) \, dz, \quad t, s > 0, \ x, y \in \mathbb{R}^d. \]

For every non-negative or bounded function $f$ we let

\[ P_t^D f(x) = \mathbb{E}_x[\tau_D > t; f(X_t)] = \int_{\mathbb{R}^d} p_t^D(x, y)f(y) \, dy, \quad x \in \mathbb{R}^d, \ t > 0. \]

A function $u : \mathbb{R}^d \to \mathbb{R}$ is called harmonic with respect to $X$ on an open set $D \subset \mathbb{R}^d$ if

\[ u(x) = \mathbb{E}_x u(X_{\tau_D}), \quad x \in B, \]

for all open bounded sets $B$ such that $\overline{B} \subset D$. Here we assume that the integral on the right-hand side is absolutely convergent. The function $u$ is called regular harmonic in $D$ if the identity above is satisfied with $B = D$. The concept of regular harmonicity is related to the notion of harmonic measure $P_D(x, \cdot)$, i.e., the distribution of $X_{\tau_D}$:

\[ P_D(x, A) = \mathbb{P}_x(X_{\tau_D} \in A), \quad A \subset \mathbb{R}^d, \ x \in \mathbb{R}^d. \]

Namely, a function $u$ regular harmonic on $D$ satisfies

\[ u(x) = \int_{D^c} u(y) P_D(x, dy), \quad x \in D. \]

The density of the harmonic measure on $\overline{D^c}$ is called the Poisson kernel and is denoted by $P_D(x, z)$ for $x \in D$, $z \in \overline{D^c}$ (see also \eqref{eq:2.7} below). For simplicity we write $P_{B^c} := P_r$.

The Green function of $D$ is given by

\[ G_D(x, y) = \int_0^\infty p_t^D(x, y) \, dt, \quad x, y \in \mathbb{R}^d. \]

In the case of $D = \mathbb{R}^d$ we write $U := G_{\mathbb{R}^d}$ and $U$ is then called the potential kernel of $X$. Note that in view of \eqref{eq:2.1} we may write $U(x, y) = U(y - x)$ for $x, y \in \mathbb{R}^d$. If $\psi \in C_0^\infty$, for $\alpha \in (0, 2)$, and $d \geq 2$, then by the Chung-Fuchs criterion (see, e.g., \cite[Corollary 37.6]{52}), $X$ is transient and the potential kernel is finite a.e. The Green function gives rise to the Green operator, defined as

\[ G_D f(x) = \int_D G_D(x, y)f(y) \, dy, \quad x \in \mathbb{R}^d, \]

for non-negative or integrable functions $f$. Then, by the Fubini’s theorem,

\[ \mathbb{E}_x \int_0^{\tau_D} f(X_t) \, dt = \int_D G_D(x, y)f(y) \, dy. \]

In particular, by letting $f = 1$ we obtain $G_D 1(x) = \mathbb{E}_x \tau_D$.

For $x \in D$, the distribution of $(\tau_D, X_{\tau_D}, X_{\tau_D})$ restricted to the event $\{X_{\tau_D} \neq X_{\tau_D^c}\}$ has the following density function:

\[ (0, \infty) \times D \times D^c \ni (u, y, z) \mapsto \nu(y, z)p_u^D(x, y), \]

that is for $I \subset (0, \infty)$, $A \subset D$ and $B \subset D^c$ we have

\begin{equation}
\label{eq:2.4}
\mathbb{P}_x(X_{\tau_D} \neq X_{\tau_D^c}, \tau_D \in I, X_{\tau_D^c} \in A, X_{\tau_D^c} \in B) = \int_I \int_A \int_B p_u^D(x, y)\nu(y, z) \, dz \, dy \, du.
\end{equation}
This is called the Ikeda-Watanabe formula. Denote
\[
\kappa_D(y) = \int_{D^c} \nu(y, z) \, dz, \quad y \in D.
\]

We call \(\kappa_D\) the killing intensity.

Recall that the set \(D \subset \mathbb{R}^d\) is called Lipschitz if there are numbers \(R_0 > 0\) and \(\lambda > 0\) such that for every \(Q \in \partial D\) there exist an orthonormal coordinate system \(CS_Q\) and a Lipschitz function \(f_Q : \mathbb{R}^{d-1} \to \mathbb{R}\) with Lipschitz constant \(\lambda\) such that if \(y = (y_1, \ldots, y_n)\) in \(CS_Q\) coordinates, then
\[
D \cap B(Q, R_0) = \{ y : y_n > f_Q(y_1, \ldots, y_{n-1}) \} \cap B(Q, R_0).
\]

Every Lipschitz set \(D\) is \(\kappa\)-fat, i.e., there exists \(\kappa \in (0, 1)\) and \(R > 0\) such that for every \(r \in (0, R)\) and \(Q \in \overline{D}\) there is a point \(A = A_r(Q) \in D \cap B(Q, r)\) such that \(B(A, \kappa r) \subset D \cap B(Q, r)\). The pair \((\kappa, R)\) is sometimes called the characteristics of a \(\kappa\)-fat set \(D\). Note that usually \(A_r(Q)\) is not uniquely determined.

If the set \(D\) is Lipschitz, then, by Szymonik [55, Theorem 1], for all \(x \in D\) we have \(P_x(X_{\tau_D} = \partial D) = 0\) and
\[
\mathbb{P}_x(X_{\tau_D -} = X_{\tau_D}) = 0.
\]

Letting \(I = (t, \infty)\), \(A = D\) and \(B = D^c\) in (2.4), by Chapman-Kolmogorov equations,
\[
\mathbb{P}_x(\tau_D > t) = \int_t^\infty \int_D p^D_t(x, y) \kappa_D(y) \, dy \, du
= \int_0^\infty \int_D \int_D p^D_t(x, w) p^D_t(w, y) \, dw \kappa_D(y) \, dy \, du
= G_D P^D_t \kappa_D(x), \quad x \in D.
\]

Furthermore, \(\mathbb{P}_x\text{-a.s.} \) we have \(\tau_D = 0\) for every \(x \in D^c\), so
\[
\mathbb{P}_x(\tau_D > t) = G_D P^D_t \kappa_D(x), \quad x \in \mathbb{R}^d.
\]

The following identities, also known as Ikeda-Watanabe formulae, are vital for our development. Namely, by setting \(I = (0, \infty)\) and \(A = D\) in (2.4) we obtain
\[
P_D(x, z) = \int_D G_D(x, y) \nu(y - z) \, dy, \quad x \in D, z \in \overline{D}^c.
\]

Consequently, for every function \(u\) regular harmonic in \(D\) with respect to \(X\) we have
\[
u(x) = \int_{D^c} P_D(x, z) u(z) \, dz, \quad x \in D.
\]

As in Introduction, \(\Gamma \subset \mathbb{R}^d\) is a Lipschitz cone with vertex at the origin, i.e., (non-empty) open Lipschitz set such that \(0 \in \partial \Gamma \) and \(r \gamma \in \Gamma\) whenever \(y \in \Gamma\) and \(r > 0\). Note that in this way we exclude \(\Gamma = \mathbb{R}^d\) and \(\Gamma = \{ 0 \}\) from our discussion (but see a remark following Theorem 1.1). Without essential loss of generality we may and do assume that \(\Gamma = (0, 1)\). We note that if \(d = 1\), then necessarily \(\Gamma = (0, \infty)\) and our results do not cover \(\Gamma = \mathbb{R} \setminus \{ 0 \}\). We also observe that \(\Gamma\) is \(\kappa\)-fat with characteristics \((\kappa, R)\) for every \(R > 0\), with \(\kappa\) independent of \(R\).

We further note that \(\mathbb{P}_x(\tau_\Gamma = 0) = 1\) for \(x \in \Gamma^c\) by the exterior cone condition and the Blumenthal 0-1 law, which implies \(p^\Gamma_t(x, y) = 0\) whenever \(x \in \Gamma^c\) or \(y \in \Gamma^c\), and consequently,
\[
\mathbb{P}_x(\tau_\Gamma > t) = \int_\Gamma p^\Gamma_t(x, y) \, dy, \quad x \in \mathbb{R}^d, \ t > 0.
\]

A particular role in this paper will be played by the Green function of \(\Gamma\). Let \(r > 0\) and consider the truncated cone:
\[
\Gamma_r = \Gamma \cap B_r.
\]

The strong Markov property implies that for all \(t > 0\) and \(x, y \in \mathbb{R}^d\),
\[
p^\Gamma_t(x, y) = p^{\Gamma_r}_t(x, y) + \mathbb{E}_x \left[ \tau_{\Gamma_r} < t; p^{\Gamma_{\Gamma_r}}_{t - \tau_{\Gamma_r}}(X_{\tau_{\Gamma_r}}), y \right].
\]
Integrating the identity with respect to $dt$ we obtain

$$(2.10) \quad G_{\Gamma}(x, y) = G_{\Gamma_r}(x, y) + \mathbb{E}_x G_{\Gamma}(X_{\tau_{\Gamma_r}}, y), \quad x, y \in \mathbb{R}^d. $$

It follows that $x \mapsto G_{\Gamma}(x, y)$ is regular harmonic on $\Gamma_r$ if $|y| \geq r$.

After Pruitt [49], we define the concentration functions for the Lévy density $\nu$,

$$K(r) = r^{-2} \int_{B_r} |z|^2 \nu(z) dz \quad \text{and} \quad h(r) = \int_{\mathbb{R}^d} \left( \frac{|z|^2}{r^2} \wedge 1 \right) \nu(z) dz, \quad r > 0.$$  

We note that $h$ is strictly decreasing and for all $r > 0$ and $\lambda \leq 1$,

$$(2.11) \quad \lambda^2 h(\lambda r) \leq h(r) \leq h(\lambda r).$$

In particular, $h$ is doubling on $(0, \infty)$. Furthermore, by Grzywny [27, Lemma 4],

$$(2.12) \quad \frac{1}{8(1 + 2d)} h(1/r) \leq \psi^*(r) \leq 2h(1/r), \quad r > 0,$$

where $\psi^*$ is the radially non-decreasing majorant of $\psi$, i.e.,

$$\psi^*(r) = \sup_{|z| \leq r} \psi(z), \quad r \geq 0.$$  

By Bogdan, Grzywny and Ryznar [12, Proposition 2],

$$(2.13) \quad \psi(r) \leq \psi^*(r) \leq \pi^2 \psi(r), \quad r \geq 0,$$

where $\psi(r) := \psi(|x|) \text{ for } r = |x|$ and $x \in \mathbb{R}^d$.

We conclude this section by collecting some consequences of $A1$, $A2$ or $A3$. First we observe that $A1$ implies that for every $r_0 > 0$ there is $c = c(r_0)$ such that

$$(2.14) \quad c \nu(r) \leq \nu(r + 1) \leq \nu(r), \quad r > r_0.$$  

Next, note that the monotonicity of the Lévy density entails that

$$(2.15) \quad \nu(r) \leq c(d) K(r) r^{-d}, \quad r > 0.$$  

**Proposition 2.1. Assume $A1$. For every $\lambda \leq 1$ and $r > 0$, $K(\lambda r) \leq M \lambda^{-\beta} K(r)$.**

Furthermore,

$$\nu(r) \approx r^{-d} K(r), \quad r > 0,$$

with the comparability constant depending only on $d$, $M$ and $\beta$. If we additionally assume $A2$, then for every $R > 0$ there exists constant $c > 0$ such that

$$h(r) \leq c K(r), \quad r \geq R.$$  

**Proof.** The assumption $A1$ together with Grzywny and Szczypkowski [33, Lemma A.3] immediately imply the first claim. For large $R > 0$ the last claim is a consequence of $A2$, Potter bounds for $\nu$ and Grzywny and Szczypkowski [32, Lemma 2.5]. Using positivity and monotonicity of $\nu$ it is easy to make the threshold $R > 0$ arbitrary. \hfill \Box

**Lemma 2.2. Assume $A1$. If $d \geq 2$, then there is a constant $c = c(d, M, \beta)$ such that

$$(2.16) \quad U(x) \leq \frac{c}{h(|x|)|x|^d}, \quad x \neq 0.$$  

For $d = 1$ we have

$$(2.17) \quad G_{(0, \infty)}(x, y) \leq \frac{c}{y \sqrt{h(x)h(y)}}, \quad 0 < x < y.$$
**Proof.** First note that due to the Chung-Fuchs criterion the process is transient if \( d \geq 2 \). By [32, Lemma 2.2] and (2.12) with (2.13) we have, for \( s > 1 \) and \( x \in \mathbb{R}^d \),
\[
\psi(sx) \leq c Ms^\beta \psi(s),
\]
where \( c \) depends only on \( d \). Then the claim is a consequence of Bogdan, Grzywny and Ryznar [13, Lemma 5.6] and [27, Theorem 3]. If \( d = 1 \), then the assumption A1 implies global scale invariant Harnack inequality for the process due to Grzywny and Kwaśnicki [28, Theorem 1.9 and Remark 1.10 e)]. With this in hand one can repeat the proof of Corollary 5.6 in Grzywny, Leżaj and Miśta [29] to get the claim. \( \square \)

**Proposition 2.3.** Assume A2. Then \( h^{-1} \) has doubling property on \( (0,r) \) for every \( r > 0 \). Furthermore, for every \( r > 0 \) there is a constant \( c > 0 \) such that
\[
\frac{c^{-1}}{h^{-1}(u)} \leq \psi^{-1}(u) \leq \frac{c}{h^{-1}(u)}, \quad u < r.
\]

**Proof.** Observe that the regular variation of \( \psi \) together with (2.13) entails B3 in [32, Lemma 2.5]. Thus, the first claim follows by a standard extension argument and monotonicity of \( h \). Next observe that by (2.12),
\[
\frac{1}{h^{-1}(r/2)} \leq \psi^{-1}(r) \leq \frac{1}{h^{-1}(8(1+2d)r)}, \quad r > 0.
\]

Now we may apply the doubling property of \( h^{-1} \) to conclude the proof. \( \square \)

Now let \( u \) be a harmonic function with respect to \( X \) in an open set \( D \). By the Poisson formula (2.8),
\[
u(x) = \int_{|z| > r} P_r(0,z) u(x+z) \, dz,
\]
if only \( \overline{B(x,r)} \subset D \). Recall that \( P_r(x, \cdot) \) is the Poisson kernel for the ball \( B_r \). Then [28, Lemma 2.2] entails
\[
u(x) \geq c \frac{1}{h(r)} \int_{|z| > r} \nu(z) u(x+z) \, dz
\]
for some constant \( c \in (0,1] \). Therefore, using (2.14) we conclude that

(2.16) \[
\int_{\mathbb{R}^d} u(x) (1 \wedge \nu(x)) \, dx < \infty.
\]

Let \( D \) be an arbitrary open set.

**Proposition 2.4.** Assume A1 and let \( x_0 \in \mathbb{R}^d \) and \( r > 0 \). Suppose that non-negative functions \( f, g \) are regular harmonic in \( D \cap B(x_0,2r) \) and vanish on \( D^c \cap B(x_0,2r) \). Then
\[
f(x) \overset{C_{\text{BH}}}{\approx} \mathbb{E}_{x,T \subset B(x_0,4r/3)} \int_{\mathbb{R}^d \setminus B(x_0,5r/3)} f(y) \nu(|y-x_0|) \, dy
\]
for \( x \in D \cap B(x_0,r) \), where \( C_{\text{BH}} = C_{\text{BH}}(d,M,\beta) \), and
\[
f(x)g(y) \leq C_{\text{BH}} f(y)g(x), \quad x, y \in D \cap B(x_0,r),
\]
with \( C_{\text{BH}} = C_{\text{BH}}^d \).

**Proof.** By [28, Remark 1.10d] we get that \( C_{\text{BH}} \) depends only on the characteristics of the process \( X \). With the notation from [28] we have \( R_\infty = \infty \), \( \alpha = \beta \), and \( M \) in A1 is the same as in [28]. Therefore, \( C_{\text{BH}} = C_{\text{BH}}(d,M,\beta) \) by [28, Theorem 1.9]. \( \square \)

**Proposition 2.5.** Assume A1 and let \( \phi \in C_c^\infty(\Gamma) \). Then \( G_T \mathcal{L} \phi \) is well defined and
\[
G_T \mathcal{L} \phi(x) = -\phi(x), \quad x \in \mathbb{R}^d.
\]
Proof. Since \( \nu \) is symmetric, for every \( y \in \mathbb{R}^d \) we have
\[
\|\mathcal{L}\phi(y)\| \leq \|\phi\|_{C^2(\mathbb{R}^d)} \int_{\mathbb{R}^d} (1 \wedge |z|^2) \nu(z) \, dz < \infty.
\]

Fix \( x \in \mathbb{R}^d \). Choose \( R_1 \) so that supp \( \phi \subset B_{R_1} \) and set \( R = 2R_1 + |x| \). By Dynkin’s formula [21, (5.8)], for \( r > 2R \) we have
\[
-\phi(x) = \mathbb{E}_x \int_0^{\tau_R} \mathcal{L}\phi(X_t) \, dt = \int_0^{\infty} \mathbb{E}_x [\tau_r > t; \mathcal{L}\phi(X_t)] \, dt = \int_{\tau_r} \mathcal{G}_\tau(x,y) \mathcal{L}\phi(y) \, dy.
\]

The application of the Fubini theorem is justified by the facts that \( \mathcal{L}\phi \) is bounded on \( \mathbb{R}^d \) and \( \mathbb{E}_x \tau_R < \infty \) (see, e.g., [13]). We split the integral as follows:
\[
-\phi(x) = \int_{\Gamma_R} \mathcal{G}_\tau(x,y) \mathcal{L}\phi(y) \, dy + \int_{\Gamma_R \setminus \Gamma_r} \mathcal{G}_\tau(x,y) \mathcal{L}\phi(y) \, dy =: I_1(r) + I_2(r).
\]

By Proposition 2.4, for \( y \in \Gamma_R \) and \( v \in \Gamma \setminus \Gamma_{2R} \) and some fixed \( y_1 \in \Gamma_R \) and \( y_2 \in \Gamma \setminus \Gamma_{2R} \),
\[
\frac{\mathcal{G}_\Gamma(v,y)}{\mathcal{G}_\Gamma(y_1,y_2)} \leq C_{BH1} \frac{\mathcal{G}_\Gamma(v,y_1)}{\mathcal{G}_\Gamma(y_1,y_2)}.
\]

This and (2.10) imply that
\[
\mathcal{G}_\Gamma(x,y) = \mathcal{G}_{\Gamma_{2R}}(x,y) + \mathbb{E}_x \mathcal{G}_\Gamma(X_{\tau_{2R}},y)
\leq \mathcal{G}_{\Gamma_{2R}}(x,y) + c \mathbb{E}_x \mathcal{G}_\Gamma(X_{\tau_{2R}},y_1) \cdot \frac{\mathcal{G}_\Gamma(y_2,y_2)}{\mathcal{G}_\Gamma(y_1,y_2)}
\leq \mathcal{G}_{\Gamma_{2R}}(x,y) + c \mathcal{G}_\Gamma(x,y_1) \cdot \frac{\mathcal{G}_\Gamma(y_2,y_2)}{\mathcal{G}_\Gamma(y_1,y_2)}.
\]

Since \( \mathcal{G}_\Gamma(y_2,y_2) \) is regular harmonic on \( \Gamma_{2R} \) and vanishes on \( \Gamma^c \), it is bounded on \( \Gamma_R \) by Proposition 2.4. Therefore, by the boundedness of \( \mathcal{L}\phi \), (2.17) and the dominated convergence theorem,
\[
\lim_{r \to \infty} I_1(r) = \int_{\Gamma_R} \mathcal{G}_\Gamma(x,y) \mathcal{L}\phi(y) \, dy.
\]

Note that by Proposition 2.1, for \( y \in \Gamma^c_R \),
\[
|\mathcal{L}\phi(y)| \leq \int_{B_{R_1}} |\phi(z)| \nu(y-z) \, dz \leq ||\phi||_\infty |B_{R_1}| \nu(|y| - R_1) \lesssim \nu(2^d) \lesssim K(|y|) \frac{2^d}{|y|^d}.
\]

This and Lemma 2.2 yield
\[
\mathcal{G}_\Gamma(x,y) |\mathcal{L}\phi(y)| \lesssim |y|^{-2d}.
\]

Now, the dominated convergence theorem implies that
\[
\lim_{r \to \infty} I_2(r) = \int_{\Gamma \setminus \Gamma_R} \mathcal{G}_\Gamma(x,y) \mathcal{L}\phi(y) \, dy.
\]

Combining (2.18) and (2.20) we complete the proof.

3. Heat kernel estimates

We apply to \( \Gamma \) standard geometric considerations on \( \kappa \)-sets, see Bogdan, Grzywny and Ryznar [10, Definition 2] and Chen, Kim and Song [18, Figure 1]. Namely, for \( x \in \Gamma \) and \( r > 0 \) we let
\[
U^{x,r} = B(x, |x - A_r(x)| + \kappa r/3) \cap \Gamma, \quad B^x_r = B(A_r(x), \kappa r/3),
\]
so that $B_{2}^{x,r} \subset U_{x,r}$. There is also $A'(x)$ and $B_{2}^{x,r} = B(A'(x), \kappa r/6)$ so that $B(A'(x), \kappa r/3) \subset B(A_r(x), \kappa r) \setminus U_{x,r}$ and consequently $\text{dist}(U_{x,r}, B_{2}^{x,r}) \geq \kappa r/6$, see Figure 1.

We next focus on sharp estimates of the heat kernels $p_t$ and $p_{\Gamma}^t$.

**Proposition 3.1.** Assume A. There exists $T_1 > 0$ such that

\[ p_t(x) \approx p_t(0) \wedge t \nu(x) \approx \left(\psi^{-1}(1/t)\right)^d \wedge t \nu(x), \quad x \in \mathbb{R}^d, \ t \geq T_1. \]

**Proof.** Since $X_t$ is symmetric, by [31, Theorem 5.4] and Proposition 2.1 we get

\[ p_t(x) \leq p_t(0) \wedge c(d) t K(|x|)|x|^{-d} \lesssim p_t(0) \wedge t \nu(x), \quad t > 0, \ x \in \mathbb{R}^d. \]

Next, note that (2.13) and the Potter bounds for $\psi$ imply the condition D3 in [32, Theorem 3.12]. This, in view of Proposition 2.3, entails the existence of $T_1 > 0$ such that the claimed upper bound of $p_t(x)$ holds for $t \geq T_1$.

Furthermore, [31, Proposition 5.3] gives us

\[ p_t(x) \geq c_1 t \nu(x) e^{-c_2 t \psi(1/|x|)}, \quad t > 0, \ x \in \mathbb{R}^d. \]

By Proposition 2.1, (2.12) and (2.13) we have $t \nu(x) \approx \left(\psi^{-1}(1/t)\right)^d$ if $t \psi(1/|x|) = 1$ and $t \geq T_1$. This and the radial monotonicity of $p_t$ imply the lower bound. \hfill \Box

**Proposition 3.2.** Assume A1 and A2. For every $T > 0$ there is $c > 0$ such that

\[ \mathbb{E}_z \tau_{B(z,r) \cap \Gamma} \leq c t \mathbb{P}_z(\tau_{\Gamma} > t), \quad z \in \Gamma, \ t \geq T, \]

where $r = 1/\psi^{-1}(1/t)$. Furthermore,

\[ \mathbb{P}_z(\tau_{\Gamma} > t/2) \approx \mathbb{P}_z(\tau_{\Gamma} > t), \quad z \in \Gamma, \ t \geq T. \]
Proof. Let \( r = 1/\psi^{-1}(1/t) \). Set \( D = B(z, r) \cap \Gamma \) and \( A = A_{3r/\kappa}(z) \). First let us consider the case \( |z - A| \leq r \). Then \( D = B(z, r) \), and Pruitt’s estimates [49] and (2.12) yield,

\[
\mathbb{E}_z \tau_D \leq \frac{c}{b(r)} \leq 2ct, \quad t > 0,
\]

where \( c \) depends only on \( d \). Furthermore, by [31, Proposition 5.2],

\[
\mathbb{P}_z(\tau_D > t) \geq c_1 e^{-c_2 t \psi'(1/r)} = c(d), \quad t > 0.
\]

Since \( \mathbb{P}_z(\tau_D > t) \leq \mathbb{P}_z(\tau_T > t) \), we get (3.2) in this case.

Now suppose that \( |z - A| > r \). Let \( V \subset \overline{D}^c \). By the Ikeda-Watanabe formula,

\[
\mathbb{P}_z(X_{\tau_D} \in V) = \int_D G_D(x,w)\nu(V-w)dw \geq \inf_{w \in D} \nu(V-w)\mathbb{E}_z \tau_D.
\]

The condition \( |z - A| > r \) allows for \( A' \in B(A, 3r) \) such that \( B(A', 2r) \subset B(A, 3r) \setminus D \). Hence, for \( V = B(A', r) \) we get, with the aid of Proposition 2.1 and \( A_1 \),

\[
\inf_{w \in D} \nu(V-w) \geq \nu(r(2 + 3/\kappa))|V| \approx K(r) \approx t^{-1},
\]

the last comparability resulting from Proposition 2.1 and (2.12). Thus, by (3.6) and (3.5),

\[
\mathbb{E}_z \tau_D \lesssim t\mathbb{P}_z(X_{\tau_D} \in V) \lesssim t\mathbb{E}_z \left[ X_{\tau_D} \in V; \mathbb{P}_{X_{\tau_D}}(\tau_B(X_{\tau_D}, r) > t) > 0 \right] \leq t\mathbb{P}_z(\tau_T > t),
\]

so (3.2) is proved.

We now turn our attention to the proof of (3.3). In the first case \( |z - A| \leq r \), by (3.5),

\[
1 \geq \mathbb{P}_z(\tau_T > t/2) \geq \mathbb{P}_z(\tau_T > t) \geq \frac{1}{2}.
\]

Next, note that for every \( z \in \Gamma \) we have from [13, Lemma 2.1] and (2.12),

\[
\mathbb{P}_z(X_{\tau_D} \in \Gamma) \leq \mathbb{P}_z(|X_{\tau_D} - z| \geq r) \leq 24h(r)\mathbb{E}_z \tau_D \leq C(d)t^{-1}\mathbb{E}_z \tau_D, \quad t > 0.
\]

Therefore, in the case \( |z - A| > r \), by Markov inequality and (3.8),

\[
\mathbb{P}_z(\tau_T > t/2) \leq \mathbb{P}_z(\tau_D > t/2) + \mathbb{P}_z(X_{\tau_D} \in \Gamma) \leq ct^{-1}\mathbb{E}_z \tau_D.
\]

The application of (3.2) yields (3.3). \( \square \)

The next proposition provides lower estimates on the Dirichlet heat kernel. Recall that \( \delta_T(x) = \text{dist}(x, \Gamma^c) \).

Proposition 3.3. Assume \( A_1 \) and \( A_2 \). For every \( T > 0 \) there is \( b \geq 1 \) and \( c > 0 \) such that for all \( x, y \in \Gamma \) and \( t \geq T \) satisfying \( \delta_T(x) \wedge \delta_T(y) \geq b/\psi^{-1}(1/t) \),

\[
p_T^\Gamma(x, y) \geq c(\psi^{-1}(1/t))^d \quad \text{if} \quad |x - y| \leq b/\psi^{-1}(1/t),
\]

and

\[
p_T^\Gamma(x, y) \geq ct\nu(x - y) \quad \text{if} \quad |x - y| \geq b/\psi^{-1}(1/t).
\]

Proof. Set \( r = 1/\psi^{-1}(1/t) \). First we consider \( |x - y| \leq br \). The Hunt formula (2.3) is

\[
p_T^\Gamma(x, y) = p_t(y - x) - \mathbb{E}_x[\tau_T < t; p_{t - \tau_T}(y - X_{\tau_T})], \quad x, y \in \mathbb{R}^d.
\]

By the radial monotonicity of \( p_t \) and (3.1), for every \( t > 0 \),

\[
p_t(y - x) \geq p_t(br) \geq c_1 t\nu(br)e^{-c_2 t \psi^{-1}(1/t)/b} \geq t\nu(br),
\]

Yaglom limit for unimodal Lévy processes 11
with the implied constant depending only on $d$. Furthermore, by [31, Theorem 5.4], Proposition 2.1 and monotonicity of $\nu$,

$$\mathbb{E}_x \left[ \tau_T < t; p_{t-\tau_T}(y - X_{\tau_T}) \right] \leq \mathbb{E}_x \left[ \tau_T < t; (t - \tau_T) \nu(y - X_{\tau_T}) \right]$$

$$\leq t \nu (\delta_T(y)) \mathbb{P}_x (\tau_T < t)$$

$$\leq t \nu (br) \mathbb{P}_x (\tau_B(x, \delta_T(x)) < t)$$

$$\leq t \nu (br) \mathbb{P}_x (\tau_B(x, br) < t).$$

By Potter’s bounds for $\psi$, there exists $w > 0$ such that

$$\psi(u) \leq 2 \left( \frac{u}{s} \right)^{3a/4} \psi(s), \quad u \leq s \leq w.$$  

Set $\lambda = \min \{ 1, w/\psi^{-1}(1/T) \}$. The above inequality, Pruitt’s estimates [49] and (2.11) together with (2.12) imply, for $t \geq T$,

$$\mathbb{P}_0 (\tau_B(x, r) \leq t) \leq c_1 t\lambda^{-2} \psi^* (\lambda \psi^{-1}(1/t)/b)$$

$$\leq 2c_2 \lambda^{-2} t^3/4 \psi^* (\lambda \psi^{-1}(1/t)) \leq 2c_2 \lambda^{-2} t^{-3a/4},$$

where $c_2$ depends only on $d$. Thus, by fixing $b$ large enough and putting together (3.10) with (3.11) we conclude that for $t \geq T$,

$$p^*_1(x, y) \approx t \nu (br) \approx M^{-1} t b^{-d - \beta} \nu (r).$$

By Proposition 2.1 and (2.12) we get the first claim.

Now assume that $|x - y| \geq br$. By [11, Lemma 1.10] with $D_1 = B(x, br/2)$ and $D_3 = B(y, br/2)$ we have

$$p^*_1(x, y) \geq t\mathbb{P}_x (\tau_D > t) \mathbb{P}_y (\tau_D > t) \inf_{u \in D_1, z \in D_3} \nu (u - z).$$

Observe that by (2.11) and (2.12),

$$th(b/2) \approx t \psi^* \approx t \psi^* (\psi^{-1}(1/t) \lambda) \approx 1,$$

with comparability constants depending only on $d$ and $T$. Thus, in view of [31, Proposition 5.2], with constant depending only on the dimension we have the comparison:

$$\mathbb{P}_x (\tau_D > t) \approx 1.$$

Similarly, $\mathbb{P}_y (\tau_D > t) \approx 1$. Moreover, for $u \in D_1$ and $z \in D_3$ we clearly have $|u - z| \leq |x - y| + br \leq 2|x - y|$. From the monotonicity of the Lévy density and A1,

$$\inf_{u \in D_1, z \in D_2} \nu (u - z) \geq \nu (2|x - y|) \geq \nu (|y - x|),$$

with comparability constant depending only on $d$, $M$ and $\beta$. The proof is complete. 

\ \ 

**Proposition 3.4.** Assume A1 and A2. For each $T > 0$ there is $c > 0$ such that

$$\mathbb{P}_x (\tau_T \cap B(x, r) > t) \geq c \mathbb{P}_x (\tau_T > t), \quad x \in \Gamma, \ t \geq T,$$

where $r = 1/\psi^{-1}(1/t)$.

**Proof.** We follow the proof of [10, Lemma 1]. Let $A = A_r(x), A' = A'_r(x)$ and denote $D = \Gamma \cap B(x, r)$. If $|x - A| \leq \kappa r/2$ then $B(x, \kappa r/2) \subset D$. Since $th(\kappa r/2) \approx 1$ (see the proof of Proposition 3.3), by [31, Proposition 5.2] we get that $\mathbb{P}_x (\tau_B(x, \kappa r/2) > t) \approx 1$. Hence

$$\mathbb{P}_x (\tau_T > t) \approx \mathbb{P}_x (\tau_D > t), \quad t > 0.$$
We thus assume that $|x - A| > \kappa r/2$. For simplicity we write $U = U^{x,r}$, $B_1 = B_1^{x,r}$ and $B_2 = B_2^{x,r}$. By the Ikeda-Watanabe formula (2.7) we see that, for $t \geq T$ and $y \in U$,

$$
\mathbb{P}_y (X_{\tau_U} \in B_2) = \int_U G_U (y, w) \nu (B_2 - w) \, dw 
$$

(3.12)

$$
\approx r^d \nu (r) \mathbb{E}_y \tau_U 
$$

$$
\approx t^{-1} \mathbb{E}_y \tau_U,
$$

where the last two comparisons follow from Proposition 2.1. Next, by Proposition 2.4,

$$
\mathbb{P}_x (X_{\tau_U} \in \Gamma) \leq C_{\text{BHI}} \mathbb{P}_A (X_{\tau_U} \in \Gamma), 
$$

$$
\mathbb{P}_x (X_{\tau_U} \in B_2) \leq C_{\text{BHI}} \mathbb{P}_A (X_{\tau_U} \in B_2).
$$

By Pruitt’s estimates we obtain $\mathbb{E}_A \tau_U \approx t$. It follows that

$$
\mathbb{P}_x (X_{\tau_U} \in \Gamma) \lesssim \mathbb{P}_x (X_{\tau_U} \in B_2).
$$

(3.13)

Thus, by the Markov inequality, (3.13) and (3.12),

$$
\mathbb{P}_x (\tau_U > t) \leq \mathbb{P}_x (\tau_U > t) + \mathbb{P}_x (X_{\tau_U} \in \Gamma) \lesssim t^{-1} \mathbb{E}_x \tau_U.
$$

(3.14)

Moreover, by the verbatim repetition of the argument from [18, (4.8)–(4.9)] we get

$$
\mathbb{P} (\tau_{B(x,|x - A| + \kappa r)} > t) \mathbb{P}_x (X_{\tau_U} \in B_2) \leq \mathbb{P}_x (\tau_V > t)
$$

with $V = B(x,|x - A| + \kappa r) \cap \Gamma$. Therefore, combining [31, Proposition 5.2], (3.12) and (3.14) we obtain the claim. \( \square \)

**Lemma 3.5.** Assume A. Then there exists $T_2 > t_0$ such that

$$
p_t^\Gamma (x, y) \approx \mathbb{P}_x (\tau_T > t) \mathbb{P}_y (\tau_T > t) p_t (y - x), \quad x, y \in \mathbb{R}^d, \quad t \geq T_2,
$$

Proof. First we focus on the upper bound. Observe that by the semigroup property,

$$
p_t^\Gamma (x, y) \leq p_{t/2} (0) \mathbb{P}_x (\tau_T > t/2), \quad x, y \in \Gamma, \quad t > 0.
$$

Thus, by Potter bounds for $\psi^{-1}$ together with Propositions 3.1 and 3.2, for $t \geq 2T_1$,

$$
p_t^\Gamma (x, y) \leq p_t (0) \mathbb{P}_x (\tau_T > t), \quad x, y \in \Gamma.
$$

Set $r = 1/(4 \psi^{-1}(1/t))$. Let us consider the case $\psi^* (1/|x - y|) \leq 1$. Note that this implies that $|x - y| \geq 4r$. Let $D_1 = B(x, r) \cap \Gamma$, $D_2 = \Gamma \setminus B(x, |x - y|/2)$ and $D_3 = \Gamma \setminus (D_1 \cup D_2)$. Then by radial monotonicity, [31, Theorem 5.4], Proposition 2.1 and A1,

$$
\sup_{s < t, x \in D_2} p_s (z, y) \leq \sup_{s \leq t} \nu (|x - y|/2) \leq C(d) t \nu (|x - y|).
$$

(3.15)

Moreover, by A1 we get

$$
\sup_{z \in D_1, w \in D_3} \nu (z - w) \leq \nu (|x - y|/4) \leq C(d, M, \beta) \nu (|x - y|).
$$

(3.16)

Thus, using [11, Lemma 1.10], (3.15) and (3.16) we obtain

$$
p_t^\Gamma (x, y) \leq c(d, M, \beta) \left( t \mathbb{P}_x (X_{\tau_{D_1}} \in D_2) + \mathbb{E}_x \tau_{D_1} \right) \nu (|x - y|).
$$

Therefore, by (3.8) and Proposition 3.2 we conclude that for $t \psi^* (1/|x - y|) \leq 1$,

$$
p_t^\Gamma (x, y) \leq c(d, M, \beta) \mathbb{P}_x (\tau_T > t) \nu (|x - y|).
$$
By Proposition 2.1, (2.12) and [19, Remark 2], $\nu(|x - y|) \approx p_t(0)$ for $t\psi^*(1/|x - y|) \approx 1$, so Proposition 3.1 entails

$$p_{t/3}^\Gamma(x,y) \leq c(d,M,\beta)\mathbb{P}_x(\tau_T > t)p_t(x-y), \quad x,y \in \Gamma, \quad t \geq 2T_1. $$

Applying symmetry of $p_t^\Gamma$, the semigroup property and Proposition 3.2, we arrive at the desired upper bound.

Now we turn to the lower bound. This part of proof is inspired by the proofs of [10, Lemma 5] and [18, Theorem 1.3].

Let $b$ be taken from Proposition 3.3 and set $r_1 = 6br/\kappa$ with $r = 1/\psi^{-1}(1/t)$. By the semigroup property,

$$p_t^\Gamma(x,y) = \int_{\Gamma \times \Gamma} p_{t/3}^\Gamma(x,u)p_{t/3}^\Gamma(u,v)p_{t/3}^\Gamma(v,y) \, du \, dv \geq \int_{B_{2r_1}^* \times B_{2r_1}^*} p_{t/3}^\Gamma(x,u)p_{t/3}^\Gamma(u,v)p_{t/3}^\Gamma(v,y) \, du \, dv \geq \inf_{u \in B_{2r_1}^*, v \in B_{2r_1}^*} \int_{B_{2r_1}^*} p_{t/3}^\Gamma(x,u) \int_{B_{2r_1}^*} p_{t/3}^\Gamma(u,v) \, du \, dv.$$

Note that by the definition of $r_1$ we have $\delta_1(u) \land \delta_1(v) \geq b/\psi^{-1}(1/t)$. Thus, by Proposition 3.3 and 3.1 there is $c > 0$ such that for $t \geq T_1$,

$$\inf_{u \in B_{2r_1}^*, v \in B_{2r_1}^*} p_{t/3}^\Gamma(u,v) \geq c \left( (\psi^{-1}(1/t))^d \land \nu(y-x) \right) \geq cp_t(y-x), \quad x,y \in \Gamma. $$

By [11, Lemma 1.10], for $x \in \Gamma, u \in B_{2r_1}^*$, $D_1 = U_{x,r_1}$ and $D_3 = B(A_{1}^*(x), \kappa r_1/4)$ we get

$$p_{t/3}^\Gamma(x,u) \geq ct\mathbb{P}_x(\tau_{D_1} > t/3)\mathbb{P}_u(\tau_{D_3} > t/3) \inf_{z \in D_1, w \in D_3} \nu(w-z).$$

By the radial monotonicity of $\nu$, A1, Proposition 2.1 and (2.12),

$$t \inf_{z \in D_1, w \in D_3} \nu(w-z) \geq t\nu(br/(2\kappa)) \geq c(d,M,\beta)\nu(1/\psi^{-1}(1/t)) \geq c(\psi^{-1}(1/t))^d, \quad t > T_1.$$

Next, we observe that by the same argument as in the proof of (3.5),

$$\mathbb{P}_u(\tau_{D_3} > t) \geq \mathbb{P}_u(\tau_{B(u,\kappa r_1/12)} > t) \gtrsim 1, \quad t > 0, \quad u \in B_{2r_1}^*.$$

Furthermore, since for $r_2 = 1/\psi^{-1}(3/t)$ we have $r_1 \geq r \geq r_2$, by Proposition 3.4,

$$\mathbb{P}_x(\tau_{D_1} > t/3) = \mathbb{P}_x(\tau_{D \cap B(x,r_1)} > t/3) \geq \mathbb{P}_x(\tau_{D \cap B(x,r_2)} > t/3) \geq c\mathbb{P}_x(\tau_{D > t/3}) \geq c\mathbb{P}_x(\tau_{D > t})$$

for $t \geq T_1$, with $c > 0$. Thus, combining (3.18), (3.19) and (3.20) we conclude that

$$\int_{B_{2r_1}^*} p_{t/3}^\Gamma(x,u) \, du \geq c|B_{2r_1}^*|\left(\psi^{-1}(1/t))^d\mathbb{P}_x(\tau_{D > t}) = c\mathbb{P}_x(\tau_{D > t})$$

for $t \geq T_1$. In the same spirit we prove that

$$\int_{B_{2r_1}^*} p_{t/3}^\Gamma(y,v) \, dv \geq c\mathbb{P}_x(\tau_{D > t}), \quad t \geq T_1.$$

Putting this together with (3.17) completes the proof. \[\square\]
4. Rescaled process

Let $s \geq 1$. As aforementioned in Introduction, we define the rescaled process $X^s = \{X^s_t: t \geq 0\}$ by setting

\[ X^s_t = \psi^{-1}(1/s)X_{st}. \]

For the sake of clarity and consistency of notation, every object corresponding to the rescaled process $X$ will be marked by a superscript $s$. For instance, we write $\psi^s$ for the characteristic exponent of $X^s$. Observe that for every $\xi \in \mathbb{R}^d$,

\[ E e^{i\xi \cdot X^s_t} = E e^{i\xi \cdot \psi^{-1}(1/s)X_{st}} = e^{-s\psi(\psi^{-1}(1/s)\xi)}. \]

Thus,

\[ \psi^s(\xi) = s\psi(\psi^{-1}(1/s)\xi), \quad \xi \in \mathbb{R}^d. \tag{4.1} \]

The reader may also easily verify that

\[ \nu^s(x) = \frac{s}{(\psi^{-1}(1/s))^d} \nu \left( \frac{x}{\psi^{-1}(1/s)} \right), \quad x \in \mathbb{R}^d. \tag{4.2} \]

Let us collect some basic properties of $X^s$. First, observe that $A_1$ holds for $\nu^s$ with the same constant $M$ (this simple observation will have profound implications). Next, since $A_2$ implies that $\psi \in \mathcal{R}_0^d$ (see the discussion after introducing assumptions $A_1 - A_3$), it follows that for every $\xi \in \mathbb{R}^d$,

\[ \lim_{s \to \infty} \psi^s(\xi) = \psi(\psi^{-1}(1/s)\xi) = |\xi|^\alpha. \tag{4.3} \]

Furthermore, under $A_2$ for every $x \in \mathbb{R}^d \setminus \{0\}$ we have

\[ \lim_{s \to \infty} \nu^s(x) = c_{d,\alpha}|x|^{-d-\alpha}. \tag{4.4} \]

Indeed,

\[ \nu^s(x) = \frac{s}{(\psi^{-1}(1/s))^d} \nu \left( \frac{x}{\psi^{-1}(1/s)} \right) = |x|^{-d} \nu \left( \frac{x}{\psi^{-1}(1/s)} \right) \frac{\psi(\psi^{-1}(1/s))}{\psi(\psi^{-1}(1/s))}. \]

Therefore, the claim follows by [19, Theorem 7(iii)] and the fact that $\psi \in \mathcal{R}_0^d$.

**Proposition 4.1.** Assume $A_2$. Then for every $x \in \Gamma$,

\[ \lim_{s \to \infty} \kappa^s_t(x) = \kappa^s_t(x). \]

Furthermore, if we additionally assume $A_1$, then for every $r > 0$,

\[ \lim_{s \to \infty} h^s(r) = h^\nu(r). \]

**Proof.** Fix $x \in \Gamma$ and $r > 0$. Recall that

\[ \kappa^s_t(x) = \int_{\nu^s} \nu^s(y - x) \, dy. \]

Since $\nu \in \mathcal{R}_{-d-\alpha}$, by Potter’s bounds there is $r > 0$ such that for all $|z_1| \geq |z_2| \geq r$,

\[ \frac{\nu(z_1)}{\nu(z_2)} \leq 2 \left( \frac{|z_1|}{|z_2|} \right)^{-d-\alpha/2}. \]

Therefore, due to definition of $\nu^s$ (4.2) there is $s_0 \geq 1$ such that for all $s \geq s_0$,\n
\[ \frac{\nu^s(z_1)}{\nu^s(z_2)} \leq 2 \left( \frac{|z_1|}{|z_2|} \right)^{-d-\alpha/2} \]

Finally, we state a result on the Yaglom limit for unimodal Lévy processes.

**Proposition 4.2.** Assume $A_1$. Then for every $\gamma > 0$,

\[ \lim_{s \to \infty} \kappa^s_\gamma(x) = \kappa^\nu_\gamma(x). \]

**Proof.** Fix $x \in \Gamma$ and $\gamma > 0$. Recall that

\[ \kappa^s_\gamma(x) = \int_{\nu^s} \nu^s(y - x) \, dy. \]

Since $\nu \in \mathcal{R}_{-d-\alpha}$, by Potter’s bounds there is $\gamma > 0$ such that for all $|z_1| \geq |z_2| \geq \gamma$,

\[ \frac{\nu(z_1)}{\nu(z_2)} \leq 2 \left( \frac{|z_1|}{|z_2|} \right)^{-d-\alpha/2}. \]

Therefore, due to definition of $\nu^s$ (4.2) there is $s_0 \geq 1$ such that for all $s \geq s_0$,\n
\[ \frac{\nu^s(z_1)}{\nu^s(z_2)} \leq 2 \left( \frac{|z_1|}{|z_2|} \right)^{-d-\alpha/2} \]
for all $|z_1| \geq |z_2| \geq \delta r(x)$. Therefore, for $s \geq s_0$,

$$\nu^s(y-x) \leq 2\delta r(x)^{d+\alpha/2}\nu^s(\delta r(x))|y-x|^{-d-\alpha/2}, \quad y \in \Gamma^c.$$  

Moreover, by (4.4) we get that $\nu^s(\delta r(x)) \leq 2\nu^s(\delta r(x))$ for $s$ large enough, so an application of the dominated convergence theorem finishes the proof of the first part.

Next, observe that for every $s \geq 1$,

$$h^s(r) = K^s(r) + \int_{|x|>r} \nu^s(z) \, dz, \quad r > 0.$$  

The convergence of the second component is established in the same way as $\kappa^s$. Furthermore, by A1,

$$|y|^2 \nu^s(y) \leq M s^d |y|^{-d-\beta+2}, \quad |y| \leq r.$$  

Since $\nu^s(r) \leq 2\nu^s(r)$ due to (4.4) for $s$ large enough, another application of the dominated convergence theorem finishes the proof.

We define

$$\psi^{-1,s}(u) = \sup\{r > 0: \psi^s(r) = u\},$$

where $\psi^s(r) = \sup_{|z| \leq r} \psi^s(z)$. Then,

$$\lim_{s \to \infty} \psi^{-1,s}(u) = u^{1/\alpha}.$$  

Indeed,

$$\psi^s(r) = s \sup_{|z| \leq r} \psi(\psi^{-1}(1/s)z) = s \psi^s(\psi^{-1}(1/s)r),$$

therefore,

$$\psi^{-1,s}(u) = \sup\{r > 0: s \psi^s(\psi^{-1}(1/s)r) = u\} = \frac{\psi^{-1}(u/s)}{\psi^{-1}(1/s)}.$$  

Since $\psi^{-1} \in \mathcal{R}^{0}_{1/\alpha}$ by [9, the proof of Theorem 1.5.12], we obtain (4.5).

Let us derive a formula for the heat kernel $p^s_t$. We have

$$\int_{\mathbb{R}^d} e^{ix\xi} p^s_t(x) \, dx = e^{-t\psi^s(\xi)} = e^{-t\psi(\psi^{-1}(1/s)\xi)} = \int_{\mathbb{R}^d} e^{i\psi^{-1}(1/s)\xi} p^s_t(x) \, dx.$$  

By a change of variables we get

$$\int_{\mathbb{R}^d} e^{ix\xi} p^s_t(x) \, dx = \frac{1}{(\psi^{-1}(1/s))^d} \int_{\mathbb{R}^d} e^{i\xi u} p^s_t \left( \frac{u}{\psi^{-1}(1/s)} \right) \, du,$$

thus,

$$p^s_t(x) = \frac{1}{(\psi^{-1}(1/s))^d} p^s_t \left( \frac{x}{\psi^{-1}(1/s)} \right).$$  

Our next goal is to prove that the heat kernel of the rescaled process converges to the heat kernel of the limiting $\alpha$-stable Lévy process.

### 4.1. Convergence of the heat and potential kernels

With the tools from the previous subsection at hand, we are now able to prove the convergence of heat kernels. In fact, we will prove that the same holds for the Dirichlet heat kernel of $\Gamma$ and the Green function of $\Gamma$.

**Lemma 4.2.** Assume A2 and A3. Then, for every $t > 0$ and $x \in \mathbb{R}^d$,

$$\lim_{s \to \infty} p^s_t(x) = p^0_t(x).$$
\textbf{Proof.} Fix } t > 0. \textit{By a change of variables, we get that } e^{-t\psi} \in L^1(\mathbb{R}^d) \textit{ for } s > t_0/t. \textit{Thus, by the Fourier inversion formula,}

\[ p_t^s(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} e^{-t\psi^s(\xi)} d\xi, \quad x \in \mathbb{R}^d, \]

\textit{if } s \textit{ is large enough. \textbf{For any given } } \delta, s > 0 \textit{ we split the above integral according to:}

\[ \mathbb{R}^d = \{ \xi : \psi^{-1}(1/s) |\xi| < \delta \} \cup \{ \xi : \psi^{-1}(1/s) |\xi| \geq \delta \} = D_1(\delta, s) \cup D_2(\delta, s). \]

\textit{Let us first consider } D_1(\delta, s). \textit{Since } \psi \in \mathcal{R}_0^0, \textit{by Potter’s bounds there exists } \delta_1 > 0 \textit{ such that, for all } 0 < |x|, |y| \leq \delta_1,

\begin{equation}
\psi(y) \geq \frac{1}{2} \psi(x) \left( \frac{|y|}{|x|} \right)^\alpha \min \left\{ \frac{|y|}{|x|}, \frac{|y|}{|y|} \right\} \alpha/2.
\end{equation}

\textit{If } s \geq 1/\psi(\delta_1), \textit{then we can let } x = \psi^{-1}(1/s) 1 \textit{ and } y = \psi^{-1}(1/s) \xi \textit{ with } \xi \in D_1(\delta_1, s), \textit{so}

\[ ts\psi(\psi^{-1}(1/s) \xi) \geq \frac{1}{2} t|\xi|^\alpha \min \{ |\xi|, |\xi|^{-1} \} \alpha/2. \]

\textit{Thus,}

\begin{equation}
\begin{aligned}
& e^{-ts\psi(\psi^{-1}(1/s) \xi)} \leq e^{-\frac{1}{2} t|\xi|^\alpha \min \{ |\xi|, |\xi|^{-1} \} \alpha/2}, \\
& \text{on } D_1(\delta_1, s), \text{ provided that } s \geq 1/\psi(\delta_1). \\
& \text{Next, we turn our attention to the second integral for } \delta = \delta_1. \textit{Using the substitution } u = \psi^{-1}(1/s) \xi \textit{ and applying (2.13) we obtain}
\end{aligned}
\end{equation}

\[ \left| \int_{|\xi| \geq \delta} e^{-i\xi \cdot x} e^{-ts\psi(\psi^{-1}(1/s) \xi)} d\xi \right| \leq \int_{|u| \geq \delta} e^{-ts\psi(u)} (\psi^{-1}(1/s))^d du \\
\leq (\psi^{-1}(1/s))^{-d} \int_{|u| \geq \delta} e^{-ts\psi^\ast(u)/2} du \\
\leq (\psi^{-1}(1/s))^{-d} e^{-(st-t_0)\psi^\ast(\delta)/2} \int_{|u| \geq \delta} e^{-tu\psi^\ast(u)/2} du.
\]

\textit{Recall that the assumption } \psi \in \mathcal{R}_0^0 \textit{ implies that } \psi^{-1} \in \mathcal{R}_1^{0,0}. \textit{It follows by A3 that the above integral tends to 0 when } s \textit{ goes to infinity. This, the dominated convergence theorem, (4.3) and the Fourier inversion formula yield}

\[ \lim_{s \to \infty} p_t^s(x) = p_t^0(x), \]

\textit{as claimed.} \hfill \Box

\textbf{Now we turn our attention to uniform estimates for the heat kernel of } X^s.

\textbf{Lemma 4.3.} \textbf{Assume A. For all } s \geq 1 \textbf{ we have}

\[ p_t^s(x) \approx p_t^0(0) \wedge tu^s(x) \approx (\psi^{-1,s}(1/t))^d \wedge tu^s(x), \quad x \in \mathbb{R}^d, t \geq T_1/s, \]

\textit{with the same comparability constant as in Proposition 3.1.}

\textbf{Proof.} First, observe that the case } s = 1 \textbf{ is the contents of Proposition 3.1. We will extend the result to arbitrary } s \geq 1. \textbf{Observe that by (4.2), (4.6), (4.7) and Proposition 3.1 we have}

\begin{align*}
p_t^s(x) &= \frac{1}{(\psi^{-1}(1/s))^d} P_{st} \left( \frac{x}{\psi^{-1}(1/s)} \right) \\
&\approx \frac{1}{(\psi^{-1}(1/s))^d} \left( (\psi^{-1}(1/(st)))^d \wedge ts\nu \left( \frac{x}{\psi^{-1}(1/s)} \right) \right) \\
&= \psi^{-1,s}(1/t) \wedge tu^s(x), \quad t \geq T_1/s,
\end{align*}
with the implied constant independent of $s$. The proof is completed. 

We next deal with the Dirichlet heat kernel. Since $\Gamma$ is a cone,

\begin{equation}
\tau^s_\Gamma = \inf \{ t > 0 : \psi^{-1}(1/s)X_{ts} \notin \Gamma \} = \frac{\tau^s_p}{s},
\end{equation}

so

\begin{equation}
P_x(\tau^s_\Gamma > t) = P_{x/\psi^{-1}(1/s)}(\tau^s_p > st).
\end{equation}

Here and below we write $\tau^s_\Gamma = \tau^s_p(X^s_p)$ to point out that the functional $\tau^s_\Gamma$ is applied to the rescaled process. By (4.7), (4.11) and the Hunt formula,

\begin{equation}
p^s_{t,\Gamma}(x, y) = p^s_t(x, y) - \mathbb{E}_x \left[ \tau^s_\Gamma < t; p^s_{t-\tau^s_\Gamma}(X_{\tau^s_\Gamma}, y) \right] \\
= \frac{1}{(\psi^{-1}(1/s))^d}p_{ts} \left( \frac{y-x}{\psi^{-1}(1/s)} \right) \\
- \mathbb{E}_{x/\psi^{-1}(1/s)} \left[ \tau^s_\Gamma < st; \frac{1}{(\psi^{-1}(1/s))^d}p_{st-\tau^s_\Gamma} \left( \frac{\psi^{-1}(1/s)X_{\tau^s_\Gamma} - y}{\psi^{-1}(1/s)} \right) \right] \\
= \frac{1}{(\psi^{-1}(1/s))^d}p_{ts} \left( \frac{y-x}{\psi^{-1}(1/s)} \right) - \mathbb{E}_{x/\psi^{-1}(1/s)} \left[ \tau^s_\Gamma < st; p_{st-\tau^s_\Gamma} \left( X_{\tau^s_\Gamma} - \frac{y}{\psi^{-1}(1/s)} \right) \right],
\end{equation}

therefore,

\begin{equation}
p^s_{t,\Gamma}(x, y) = \frac{1}{(\psi^{-1}(1/s))^d}p^s_{st} \left( \frac{x}{\psi^{-1}(1/s)}, \frac{y}{\psi^{-1}(1/s)} \right).
\end{equation}

**Lemma 4.4.** Assume A2 and A3. Then for every $t > 0$ and $x, y \in \Gamma$, we have

\[
\lim_{s \to \infty} p^s_{t,\Gamma}(x, y) = p^{\Gamma,\alpha}_t(x, y).
\]

**Proof.** Fix $t > 0$ and $x, y \in \Gamma$. By [12, Corollary 7], (2.13) and (4.1), we have that

\[
p^s_t(u) \leq C \frac{s t \psi(\psi^{-1}(1/s)/|u|)}{|u|^d}.
\]

Since $\psi \in \mathcal{R}_0^\beta$, from Potter’s theorem there is $s_0$ such that

\[
s \psi \left( \psi^{-1}(1/s)/|u| \right) \leq 2|u|^{-\alpha/2}, \quad |u| \geq \delta_\Gamma(y),
\]

if only $s > s_0$. Therefore for $s$ large enough,

\begin{equation}
p^s_t(u) \leq \frac{C t}{|u|^{d+\alpha/2}}, \quad |u| \geq \delta_\Gamma(y).
\end{equation}

Next, note that $|X^s_{\tau^s_\Gamma} - y| \geq \delta_\Gamma(y)$. It follows that

\begin{equation}
\mathbb{E}_x \left[ t - \tau^s_\Gamma \leq \varepsilon; p^s_{t-\tau^s_\Gamma}(X^s_{\tau^s_\Gamma}, y) \right]
\end{equation}

is uniformly small provided $\varepsilon$ is small enough. We have also from (4.14) that

\begin{equation}
\mathbb{E}_x \left[ |X^s_{\tau^s_\Gamma}| > R; p^s_{t-\tau^s_\Gamma}(X^s_{\tau^s_\Gamma}, y) \right] \to 0 \quad \text{as} \quad R \to \infty
\end{equation}

uniformly in $s$ large enough. Recall that by the proof of Lemma 4.2,
\[ p_t^s(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} e^{-t\psi^s(\xi)} d\xi \]

for \( s \) large enough. Then,

\[ \left| \frac{\partial}{\partial t} p_t^s(x) \right| + \left| \nabla p_t^s(x) \right| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t\psi^s(\xi)} (\psi^s(\xi) + |\xi||\xi|) d\xi, \]

is uniformly bounded for \( \varepsilon \leq t \leq T \) and \( |x| \leq R \) by the same arguments as in the proof of Lemma 4.2. Therefore, the functions \( \{p_t^s(\cdot): s \geq s_0\} \) are equicontinuous for \( \varepsilon \leq t \leq T \), \( |x| \leq R \) and some \( s_0 \geq 1 \), hence

\[ \lim_{s \to \infty} p_t^s(x) = p_t^\alpha(x) \]

uniformly in \( \varepsilon \leq t \leq T \) and \( |x| \leq R \). This and (4.16) imply that

\[ \text{(4.17)} \quad \mathbb{E}_x \left[ \tau_t^s < t - \varepsilon; p_{t - \tau_t^s}^s \left( X_{t - \tau_t^s}^s, y \right) \right] - \mathbb{E}_x \left[ \tau_t^\alpha < t - \varepsilon; p_{t - \tau_t^\alpha}^\alpha \left( X_{t - \tau_t^\alpha}^\alpha, y \right) \right] \to 0 \]

as \( s \to \infty \). Now, in view of (4.3) and Jacod and Shiryaev [38, Corollary VII.3.6], the distribution of \( X^s \) converges weakly in \( D_{[0,T]} \) to the distribution of \( X^\alpha \). Thus, by Proposition A.1 and Corollary A.2 we have for every \( \varepsilon > 0 \) that

\[ \text{(4.18)} \quad \mathbb{E}_x \left[ \tau_t^s < t; p_{t - \tau_t^s}^s \left( X_{t - \tau_t^s}^s, y \right) \right] \to \mathbb{E}_x \left[ \tau_t^\alpha < t; p_{t - \tau_t^\alpha}^\alpha \left( X_{t - \tau_t^\alpha}^\alpha, y \right) \right] \]

as \( s \to \infty \). Therefore, combining (4.15), (4.16), (4.17) and (4.18) we arrive at

\[ \mathbb{E}_x \left[ \tau_t^s < t; p_{t - \tau_t^s}^s \left( X_{t - \tau_t^s}^s, y \right) \right] \to \mathbb{E}_x \left[ \tau_t^\alpha < t; p_{t - \tau_t^\alpha}^\alpha \left( X_{t - \tau_t^\alpha}^\alpha, y \right) \right], \]

and by the Hunt formula (2.3) with Lemma 4.2,

\[ \lim_{s \to \infty} p_t^\Gamma^s(x, y) = p_t^\Gamma^\alpha(x, y). \]

\[ \square \]

**Lemma 4.5. Assume A. For all \( s \geq 1 \) we have**

\[ p_t^\Gamma^s(x, y) \approx \mathbb{P}_x(\tau_t^s > t) \mathbb{P}_y(\tau_t^s > t)p_t^s(y - x), \quad x, y \in \Gamma, t \geq T_2/s, \]

*where the comparability constant is taken from Lemma 3.5.*

**Proof.** The case \( s = 1 \) follows from Lemma 3.5. Now we use the same extension technique as in the proof of Lemma 4.3. By (4.7), (4.12), (4.13) and Lemma 3.5, for all \( t \geq T_2/s \),

\[ p_t^\Gamma^s(x, y) = \frac{1}{(\psi^{-1}(1/s))} \int_{\mathbb{R}^d} p_{t-s}^s \left( \frac{x}{\psi^{-1}(1/s)}, \frac{y}{\psi^{-1}(1/s)} \right) \]

\[ \approx \frac{1}{(\psi^{-1}(1/s))} \int_{\mathbb{R}^d} \mathbb{P}_x(\tau_t^s > t) \mathbb{P}_y(\tau_t^s > t)p_{t-s}^s(x, y), \]

with the implied constant independent of \( s \).

\[ \square \]

**Proposition 4.6. Assume A. If \( d \geq 2 \) then for every \( x \in \mathbb{R}^d \),**

\[ \text{(4.19)} \quad \lim_{s \to \infty} U^s(x) = U^\alpha(x). \]

*Furthermore, for \( d \geq 1 \),

\[ \lim_{s \to \infty} G_t^s(x, y) = G_t^\alpha(x, y), \]

*for all \( x, y \in \Gamma \).*
Proof. The first claim is an easy consequence of [19, Corollary 3] for $d \geq 3$. Since $\alpha < 2$, one can obtain the similar result also for $d = 2$. The proof of the second claim is similar to that of Lemma 4.4. Recall that we adopt the notation $U^s(x, y) = \frac{1}{d} \int_0^\infty p^s_t(x - y) dt$. Then by the Hunt formula we have

\[(4.20) \quad G^\alpha_T(x, y) = U^s(x, y) - \mathbb{E}_x \left[ U^s \left( X^{\alpha, \tau}_t, y \right) \right].\]

We first refine (4.19) and prove that $U^s(u)$ in fact converges uniformly on compact subsets of $\mathbb{R}^d \setminus \{0\}$ to $U^\alpha(u)$ as $s \to \infty$. By (4.14), we have that

\[(4.21) \quad \lim_{s \to \infty} \int_0^s p^s_t(u) dt = 0\]

uniformly in $|u| \geq \delta_T(y)$. Next, note that Lemma 4.3 implies, for $T \geq T_1/s$,

\[
\int_T^\infty p^s_t(u) dt \lesssim \int_T^\infty (\psi^{-1}(s/t))^{d} dt.
\]

Since $\psi^{-1} \in L^d_1$, using (4.6) and Potter’s bounds for every $\eta > 0$, we get the existence of $\delta > 0$ with

\[(4.22) \quad \psi^{-1}(s/t) \leq 2 \left( \frac{u}{T} \right)^{1/\alpha - \eta}, \quad t \geq u \geq 1/(s\delta).
\]

Hence,

\[
\int_T^\infty p^s_t(u) dt \lesssim T (\psi^{-1}(s/T))^{d} \leq 2T (\psi^{-1}(1))^{d} T^{-d/\alpha + \eta} \leq 4T^{-d/\alpha + 1 + d\eta},
\]

which tends to 0 as $T \to \infty$ uniformly in $u$, provided $d \geq 2$ and $\eta < 1/\alpha - 1/d$.

For $\varepsilon \leq t \leq T$, we have from the proof of Lemma 4.4 that $\lim_{s \to \infty} p^s_t(u) = p^\alpha_t(u)$ uniformly in $\varepsilon \leq t \leq T$ and $|u| \leq R$. It follows that uniformly in $|u| \leq R$,

\[(4.23) \quad \lim_{s \to \infty} \int_{\varepsilon}^T p^s_t(u) dt = \int_{\varepsilon}^T p^\alpha_t(u) dt.
\]

Combining (4.21) – (4.23), we get the claimed uniform convergence.

Now, proceeding as in the proof of Lemma 4.4 we conclude that for every $R > 2|y|$,

\[
\lim_{s \to \infty} \mathbb{E}_x \left[ |X_s^{\alpha, \tau}| \leq R; U^s(X^{\alpha, \tau}_t, y) \right] = \mathbb{E}_x \left[ |X^{\alpha, \tau}_0| \leq R; U^\alpha(X^{\alpha, \tau}_0, y) \right].
\]

The unimodality of the potential kernel and the fact that $\lim_{|u| \to \infty} U^\alpha(u) = 0$ ends the proof in the case $d \geq 2$.

If $d = 1$, then $\Gamma = (0, \infty)$; therefore, by Lemma 4.5 and 4.3 together with [13, Proposition 2.6] and (2.12),

\[
p^\Gamma_t (x, y) \lesssim \frac{1}{\sqrt{h^s(x)h^s(y)}} \frac{1}{t} \psi^{-1}(1/t), \quad x, y > 0, \quad t \geq T_2/s.
\]

Hence, for sufficiently large $s$, we have, by Proposition 4.1 and (4.22),

\[
p^\Gamma_t (x, y) \lesssim (xy)^{\alpha/2} t^{-1/\alpha - 1 + \eta}, \quad x, y > 0, \quad t \geq 1.
\]

This together with (4.14) allow us to use the dominated convergence theorem and the claim in this case follow by Lemma 4.4.

\[\square\]

4.2. Uniform BHP and estimates of harmonic functions

Let us turn to the boundary Harnack inequality and its consequences for harmonic functions. A very general version of BHP was proved by Bogdan, Kumagai and Kwasniewski [14]. It was later simplified in [28] in the case of unimodal Lévy processes. Therefore, we will rather use the results from [28], as they better serve our purpose. Next, using ideas from [42] we will show that the boundary limits of ratios of harmonic functions exist and are uniform in the sense of [42].
Remark 3]. This observation will be crucial in the proof of Lemma 4.10, which in turn is essential for Lemma 4.11 to hold.

First we prove the uniform boundary Harnack inequality and generalize [42, Remark 3] to the whole family of Lévy processes \( \{X^s : s \geq 1\} \). To this end we closely examine assumptions imposed in [14], [28] and [42] and verify that constants appearing therein are in fact independent of \( s \). First, let us show that the boundary Harnack inequality holds with the same constant for every \( X^s \). Recall that \( A1 \) holds true for all \( X^s \) for all \( s \geq 1 \) with the same parameters \( M \) and \( \beta \). Therefore, by Proposition 2.4, \( C_{BHI} \) is also independent on \( s \). In particular, the uniform boundary Harnack inequality holds: for all \( s \geq 1 \), \( x_0 \in \partial D \) and two functions \( f^s, g^s \geq 0 \) which are regular harmonic with respect to \( X^s \) in \( D \cap B(x_0,2r) \) and vanish on \( D^c \cap B(x_0,2r) \),

\[
\frac{f^s(x)}{g^s(x)} \leq C_{BHI} \frac{f^s(y)}{g^s(y)}, \quad x, y \in D \cap B(x_0,r),
\]

(BHP)

with \( C_{BHI} = \tilde{C}_{BHI}^s \). For the existence of the boundary limits we verify the assumptions of [42, Theorem 2]. Assumption (ii) therein is satisfied for every unimodal Lévy process. Our goal is to prove that (ii) and (iii) hold uniformly in \( s \). First, we check that for every \( R > 0 \), the constant \( C_{\text{Lévy}}^s(r,R) \) satisfying

\[
\nu^s(t-r) \leq C_{\text{Lévy}}^s(r,R) \nu^s(t+r),
\]

for all \( t > R \), may be chosen independent of \( s \) (which justifies writing \( C_{\text{Lévy}}^s = C_{\text{Lévy}} \)). Indeed, assume \( A1 \) and \( A2 \). We clearly have \( C_{\text{Lévy}}^s(r,R) \geq 1 \) for all \( 0 < r < R \) and \( s \geq 1 \). Furthermore,

\[
C_{\text{Lévy}}^s(r,R) = \sup_{t \geq R} \nu^s(t-r) = M \sup_{t \geq R} \left( \frac{t+r}{t-r} \right)^{d+\beta} = \left( \frac{R+r}{R-r} \right)^{d+\beta} M,
\]

which proves the independence of the constant from \( s \). In particular, by setting \( R = 2r \) we see that the condition (iii) holds uniformly in \( s \). Finally, we note that condition (iv) actually holds independently of \( s \) only the ingredient in the proof of [28, Theorem 1.11] is the scaling condition \( A1 \), which is independent of \( s \).

Therefore, by [42, Theorem 2 and Remark 1] we have the following.

**Theorem 4.7.** Assume A. Let \( D \) be an open Lipschitz set, \( x_0 \in \partial D \) and \( R > 0 \). Then

\[
\lim_{r \to 0^+} \sup_{s \geq 1} \sup_{f^s, g^s} \frac{\sup_{x \in D \cap B(x_0,r)} f^s(x)/g^s(x)}{\inf_{x \in D \cap B(x_0,r)} f^s(x)/g^s(x)} = 1,
\]

where \( \sup_{f^s, g^s} \) is taken over all non-negative functions \( f^s, g^s \) that are regular harmonic in \( D \cap B(x_0,R) \) with respect to \( X^s \) and are equal to zero in \( B(x_0,R) \setminus D \).

**Proof.** Without loss of generality we may and do assume that \( x_0 = 0 \). First of all, we show that \( x_0 \in \partial D \) is an accessible boundary point (see [42, Remark 1]) and then the proof will follow by inspection of the proof of Theorem 2 therein in the case of accessible boundary points in [42, Section 4.3]. We remark that in this case the assumption (ii) of [42, Theorem 2] is redundant. Namely, since \( D \) is a Lipschitz set, there exists \( R > 0 \) and an open right circular cone \( \Gamma \) with apex at \( x_0 \) such that \( \Gamma \subset D \). By isotropy of \( X \) we may and do assume that the axis of \( \Gamma \) is the line \( \ell = \{t \mathbf{1} : t \in \mathbb{R}\} \). By \( \kappa \)-fatness of \( \Gamma \) and due to the Pruitt bounds we have, for all \( y \in \Gamma_R \) with \( \text{dist}(y,\ell) \leq \kappa |y|/2 \) that

\[
\mathbb{E}_y \tau^s_{D \cap B_R} \geq \mathbb{E}_y \tau^s_{\Gamma_R} \geq \mathbb{E}_y \tau^s_{B(y,\kappa |y|/2)} \geq \frac{c(d)}{h^s(\kappa |y|/2)} \geq \frac{c(d)\kappa^2}{4 h^s(|y|)}.
\]

Hence, by Proposition 2.1 and the isotropy of \( X \), for large \( s \) and small \( \delta \),

\[
\int_{D \cap B_R \setminus B_{2\delta}} \mathbb{E}_y \tau^s_{D \cap B_R} \nu^s(y) dy \geq c(d, M, \beta, \kappa) \int_{B_{R/2} \setminus B_{2\delta}} \frac{1}{h^s(|y|)} \frac{K^s(|y|)}{|y|^{d}} dy \geq c \int_{2\delta}^{R/2} \frac{K^s(u)}{h^s(u)} \frac{1}{u} du = c(\ln h^s(2\delta) - \ln h^s(R/2)),
\]

(4.24)
Observe that, since \( h^s \) blows up as \( \delta \to 0^+ \), \( x_0 \) is an accessible boundary point for \( X^s \). We will now prove that the blow-up is uniform in \( s \). Indeed, suppose that

\[
\sup_{\delta < R^s \geq 1} \inf_{\delta < R^s \geq 1} \frac{h^s(2\delta)}{h^s(R/2)} < \infty.
\]

Note that

\[
\frac{h^s(2\delta)}{h^s(R/2)} = \frac{h(2\delta/\psi^{-1}(1/s))}{h(R(2\psi^{-1}(1/s)))}.
\]

By Proposition 4.1 and continuity of \( h \), for all small enough \( \delta > 0 \) there is \( s = s(\delta) \) with

\[
\limsup_{\delta \to 0^+} \frac{h^s(2\delta)}{h^s(R/2)} < \infty.
\]

Moreover, we observe that the expression \( \delta/\psi^{-1}(1/s(\delta)) \) is bounded in \( \delta \). Indeed, should the converse be true, (4.26), (2.12) and (2.13) would imply that

\[
\sup_{\delta < R^s \geq 1} \frac{h^s(2\delta)}{h^s(R/2)} = \infty,
\]

so \( x_0 \) is an accessible boundary point for \( X^s \) and the lower bounds above is independent of \( s \). This implies that every constant that appears in Section 4.3 in [42] is in fact independent of \( s \) and therefore the limit proven there is uniform with respect to \( s \) and functions \( f^s \) and \( g^s \).

\[\square\]

**Lemma 4.8.** Let \( D \) be a \( \kappa \)-fat set. Assume \( A1 \). There is a constant \( C = C(d, M, \beta, \kappa) \) such that for all \( s \geq 1, Q \in \partial D, r > 0 \) and non-negative functions \( u^s \) regular harmonic in \( D \cap B(Q, 2r) \) with respect to \( X^s \) and vanishing on \( D^c \cap B(Q, 2r) \),

\[ u^s(x) \leq Cu^s(A_r(Q)), \quad x \in D \cap B(Q, r). \]

**Proof.** By Proposition 2.4 and discussion before (BHP) we get

\[
u^s(x)\frac{u^s(A_r(Q))}{u^s(A_r(Q))} \leq C_{BHI} \frac{\mathbb{E}_x \tau_{D \cap B(Q, 4r/3)}}{\mathbb{E}_{A_r(Q)} \tau_{D \cap B(Q, 4r/3)}}.
\]

By Pruitt’s estimates,

\[
\mathbb{E}_x \tau_{D \cap B(Q, 4r/3)} \leq \frac{c(d)}{h(4r/3)}
\]

and

\[
\mathbb{E}_{A_r(Q)} \tau_{D \cap B(Q, 4r/3)} \geq \frac{c(d)}{h(3r)}.
\]

An application of (2.11) yields the claim. \[\square\]
4.3. Uniform integrability

Since a dilation of a Lipschitz set is Lipschitz, by (2.5) we get $\mathbb{P}_x(X_{t_D}^s = X_{t_D}^s) = 0$. Thus, by (2.4), and analogous results for $X^\alpha$ we have

$$1 = \int_0^\infty \int_\Gamma \int_{t_D} p_{u}^{\Gamma_{\alpha}}(x,y)\nu^\alpha(y,z) \, dz \, dy \, du$$

$$\xrightarrow{s \to \infty} 1 = \int_0^\infty \int_\Gamma \int_{t_D} p_{u}^{\Gamma_{\alpha}}(x,y)\nu^\alpha(y,z) \, dz \, dy \, du$$

Here and below $x \in \Gamma$ is fixed but arbitrary, and we consider the integrands as functions parametrized by $s \to \infty$. Due to Lemma 4.4 and (4.4), the integrands converge, too, so by Vitali’s convergence theorem, the integrand in (4.27) is uniformly integrable, see, e.g., [53, Chapter 22]. Conversely, by Vitali’s theorem and uniform integrability, for each bounded function $f$,

$$\xrightarrow{s \to \infty} \int_0^\infty \int_\Gamma \int_{t_D} p_{u}^{\Gamma_{\alpha}}(x,y) f(u,y,z)\nu^\alpha(y,z) \, dz \, dy \, du$$

For instance, taking arbitrary $t \geq 0$ and letting $f = 1_{u \geq t}$, we get

$$\mathbb{P}_x(\tau_t^\Gamma > t) \to \mathbb{P}_x(\tau_t^\alpha > t) \quad s \to \infty.$$

Lemma 4.9. For all $x \in \Gamma$ and $t > 0$,

$$\lim_{s \to \infty} P_{t}^{F_{\alpha}} \kappa_{t}^\alpha(x) = P_{t}^{\Gamma_{\alpha}} \kappa_{t}^\alpha(x).$$

Proof. By the semigroup property,

$$P_{t}^{\Gamma_{\alpha}} \kappa_{t}^\alpha(x) = \int_\Gamma \int_\Gamma \int_\Gamma p_{u}^{\Gamma_{\alpha}}(x,y)\nu^\alpha(y,z) \, dz \, dy$$

$$= \frac{2}{t} \int_0^{t/2} \int_\Gamma \int_\Gamma \int_\Gamma p_{u}^{\Gamma_{\alpha}}(x,v)p_{u}^{\Gamma_{\alpha}}(v,y)\nu^\alpha(y,z) \, dz \, dy \, dv \, du.$$

We claim that there is an integrable function $g \geq 0$ on $\mathbb{R}^d$ such that

$$p_{r}^{\Gamma_{\alpha}}(x,v) \leq g(v), \quad r \in (t/2, t), v \in \mathbb{R}^d,$$

if $s$ is large enough. Indeed, by Lemma 4.3 and (2.14) there is $c = c(d, M, \beta, x)$ such that

$$p_{r}^{\Gamma_{\alpha}}(x,v) \leq p_{r}^{\alpha}(x,v) \leq cp_{r}^{\alpha}(v).$$

Next, again by Lemma 4.3, monotonicity of $\psi^{-1.\alpha}$ and (4.5),

$$p_{r}^{\alpha}(v) \leq c\left(\psi^{-1.\alpha}(1/r)\right)^d \leq c\left(\psi^{-1.\alpha}(2/t)\right)^d \leq 2c\left(\psi^{-1.\alpha}(2/t)\right)^d$$

for $s$ large enough. Thus, with the aid of (4.14) we conclude (4.31). Note that

$$\int_\Gamma \int_0^\infty \int_\Gamma g(v)p_{u}^{\Gamma_{\alpha}}(v,y)\nu^\alpha(y,z) \, dz \, dy \, dv$$

$$\xrightarrow{r \to \infty} \int_\Gamma g(v) \, dv \xrightarrow{r \to \infty} \int_\Gamma g(v) \, dv$$

$$\xrightarrow{r \to \infty} \int_\Gamma \int_0^\infty \int_\Gamma g(v)p_{u}^{\Gamma_{\alpha}}(v,y)\nu^\alpha(y,z) \, dz \, dy \, dv.$$
Lemma 4.10. Assume we get that $M_{0}^{\alpha}$ is homogeneous of degree $\alpha - d - \beta$, i.e.

\[ M_{0}^{\alpha}(x) = |x|^\alpha d - \beta M_{0}^{\alpha}(x/|x|), \]

where $0 < \beta < \alpha$ is the homogeneity degree of the Martin kernel at infinity for $\Gamma$ (see [15, (2.17)]). Furthermore, since the Martin kernel at infinity for $\Gamma$ is locally bounded in $\mathbb{R}^{d}$ (see Bañuelos and Bogdan [2, Theorem 3.2]), by [15, (2.18)] we get that $M_{0}^{\alpha}$ is locally bounded on $\mathbb{R}^{d} \setminus \{0\}$.

Lemma 4.10. Assume A. For every $y \in \Gamma$,

\[ \lim_{s \to \infty} \frac{G_{1}^{\Gamma}(x\psi^{-1}(1/s), y)}{G_{1}^{\Gamma}(x\psi^{-1}(1/s), 1)} = M_{0}^{\alpha}(y). \]

Proof. Fix $y \in \Gamma$. We will prove the lemma by verifying that

\[ \lim_{(\Gamma, r, +, (x, |x|)/s) \to (0, 0)} \frac{G_{1}^{\Gamma}(x, y)}{G_{1}^{\Gamma}(x, 1)} = \lim_{r \to 0^+} \sup_{x \in \Gamma} \frac{G_{1}^{\Gamma}(x, y)}{G_{1}^{\Gamma}(x, 1)} = \lim_{r \to 0^+} \inf_{x \in \Gamma} \frac{G_{1}^{\Gamma}(x, y)}{G_{1}^{\Gamma}(x, 1)} = M_{0}^{\alpha}(y). \]

To this end we will justify the application of the Moore-Osgood theorem. First we observe that in view of Proposition 4.6, for every $x \in \Gamma$,

\[ \lim_{s \to \infty} G_{1}^{\Gamma}(x, y) = \frac{G_{1}^{\Gamma}(x, y)}{G_{1}^{\Gamma}(x, 1)}. \]

Next, we note that Theorem 4.7 yields

\[ \lim_{r \to 0^+} \sup_{x \in \Gamma} \frac{G_{1}^{\Gamma}(x, y)}{G_{1}^{\Gamma}(x, 1)} = 1. \]

That is, for every $\eta > 0$ there exists $r = r(\eta)$ such that

\[ \sup_{x \in \Gamma} \frac{G_{1}^{\Gamma}(x, y)}{G_{1}^{\Gamma}(x, 1)} - \inf_{x \in \Gamma \setminus r} \frac{G_{1}^{\Gamma}(x, y)}{G_{1}^{\Gamma}(x, 1)} \leq \eta \inf_{x \in \Gamma} \frac{G_{1}^{\Gamma}(x, y)}{G_{1}^{\Gamma}(x, 1)}, \quad r < r(\eta), s \geq 1. \]

We claim that $G_{1}^{\Gamma}(x, y)/G_{1}^{\Gamma}(x, 1)$ converges as $\Gamma \ni x \to 0$, uniformly in $s \geq s_{0}$ for some $s_{0} \geq 1$. Fix $x_{0} \in \Gamma_{|y|/2}$. By (BHP),

\[ \sup_{x \in \Gamma} \frac{G_{1}^{\Gamma}(x, y)}{G_{1}^{\Gamma}(x, 1)} \leq c_{1} \frac{G_{1}^{\Gamma}(x_{0}, y)}{G_{1}^{\Gamma}(x_{0}, 1)}, \]

if $r$ is small enough. Since both $y$ and $x_{0} = x_{0}(y)$ are fixed, Proposition 4.6 entails that

\[ \sup_{x \in \Gamma} \frac{G_{1}^{\Gamma}(x, y)}{G_{1}^{\Gamma}(x, 1)} \leq 2c_{1} \frac{G_{1}^{\Gamma}(x_{0}, y)}{G_{1}^{\Gamma}(x_{0}, 1)} = c_{2}, \]

if $r > 0$ is small enough and $s \geq s_{0}$, for $s_{0}$ large enough. We note that both $s_{0}$ and $c_{2}$ depend only on $y$ and in particular do not depend on $r$. Now, if we fix $\varepsilon > 0$ and set $\eta = \varepsilon/c_{2}$ then, using (4.35) and (4.36) for $|x_{1}|, |x_{2}| < r(\eta(\varepsilon)) = r(\varepsilon)$, we obtain for $s \geq s_{0}$,

\[ \frac{G_{1}^{\Gamma}(x_{1}, y)}{G_{1}^{\Gamma}(x_{1}, 1)} - \frac{G_{1}^{\Gamma}(x_{2}, y)}{G_{1}^{\Gamma}(x_{2}, 1)} \leq \sup_{x \in \Gamma_{r(\varepsilon)}} \frac{G_{1}^{\Gamma}(x, y)}{G_{1}^{\Gamma}(x, 1)} - \inf_{x \in \Gamma_{r(\varepsilon)}} \frac{G_{1}^{\Gamma}(x, y)}{G_{1}^{\Gamma}(x, 1)} \]
uniformly bounded on 
for each 
boundedness of 
immediately.

and the claim is proved. Thus, by the Moore-Osgood theorem [26, Chapter VII] we obtain (4.34) and the lemma follows immediately.

Lemma 4.11. Assume A. Let \( f^s \) be a family of measurable non-negative functions which are uniformly bounded on \( \Gamma_r \) for each \( r \geq 1 \) and \( s \geq s_0 \) with some \( s_0 \geq 1 \). Suppose that \( f^s \rightarrow f^\alpha \) a.e. \( \Gamma \) as \( s \rightarrow \infty \). We also assume that \( G^s f^s \) are uniformly bounded on \( \Gamma \) and there is \( x_0 \in \Gamma \) such that \( \lim_{s \rightarrow \infty} G^s f^s(x_0) = G^\alpha f(x_0) \). Then \( \int_{\Gamma} M_0^\alpha(y) f^\alpha(y) dy < \infty \) and for every \( x \in \Gamma \),

\[
\lim_{s \rightarrow \infty} \frac{G^s f^s(x) - (1/s) x}{G^s(x, 1)} = \int_{\Gamma} M_0^\alpha(y) f^\alpha(y) dy.
\]

\[\text{Proof.}\] We closely follow the proof of [15, Lemma 3.5] and adapt it to our setting. Fix \( x \in \Gamma \), let \( 0 < \delta < R \) and denote \( x_s = x\psi^{-1}(1/s) \). By \( \text{(BHP)} \) we have, for large enough \( s > s_0 = s_0(\delta) \) and some \( x_1 \in \Gamma_{\delta/2} \),

\[G^s(x_s, y) \leq C_{\text{BHI}} G^s(x_1, y), \quad |y| > \delta.
\]

Next, by Proposition 4.6 we have \( G^s(x_1, 1) \geq \frac{1}{s^2} G^s(x_1, 1) \) for sufficiently large \( s \geq s_0 \). Furthermore, by (4.33) and local boundedness on \( \mathbb{R}^d \setminus \{0\} \) we get that \( M_0^\alpha \) is integrable at the origin; thus, by (4.38) with \( \delta = 1 \), the assumption of uniform boundedness of \( f^s \) on \( \Gamma \) and Fatou’s lemma,

\[
\int_{\Gamma} M_0^\alpha(y) f^\alpha(y) dy \leq c \left( \int_{\Gamma_1} M_0^\alpha(y) dy + \frac{1}{G^s(x_1, y)} \liminf_{s \rightarrow \infty} \int_{\Gamma} G^s(x, y) f^s(y) dy \right) < \infty,
\]

which proves that the right-hand side of (4.37) is finite.

Now, we split the integral as follows:

\[
\frac{G^s f^s(x_s)}{G^s(x_s, 1)} = \int_{\Gamma} \frac{G^s(x_s, y)}{G^s(x_s, 1)} f^s(y) dy = \left( \int_{\Gamma_1} + \int_{\Gamma_1 \setminus \Gamma_1} + \int_{\Gamma_1 \setminus \Gamma_1} \right) \frac{G^s(x_s, y)}{G^s(x_s, 1)} f^s(y) dy
\]

\[=: I_1(s) + I_2(s) + I_3(s).
\]

For \( d \geq 2 \), by (4.38) and Proposition 4.6 we have, for \( s \) large enough and \( |y| > \delta \),

\[
\frac{G^s(x_s, y)}{G^s(x_s, 1)} \leq \frac{U^s(x_1 - y)}{G^s(x_1, 1)} \leq \frac{U^s(\delta/2)}{G^s(x_1, 1)} \leq c \frac{U^\alpha(\delta/2)}{G^\alpha(x_1, 1)}
\]

for some \( c = c(\delta) \). A similar bound, for \( d = 1 \), is a consequence of Lemma 2.2, Proposition 4.1 and the fact that \( u \mapsto u^2 h(u) \) is non-decreasing. Therefore, since \( f_s \) are uniformly bounded on \( \Gamma_R \), by the dominated convergence theorem and Lemma 4.10,

\[
\lim_{s \rightarrow \infty} I_2(s) = \lim_{s \rightarrow \infty} \int_{\Gamma_1 \setminus \Gamma_1} \frac{G^s(x_s, y)}{G^s(x_s, 1)} f^s(y) dy = \int_{\Gamma_1 \setminus \Gamma_1} M_0^\alpha(y) f^\alpha(y) dy.
\]

Next, let \( x_0 \) be such that

\[
\lim_{s \rightarrow \infty} G^s f^s(x_0) = G^\alpha f(x_0).
\]

We may and do assume that \( R > 2|x_0| \). Again by \( \text{(BHP)} \), for \( s \) large enough,

\[
\frac{G^s(x_s, y)}{G^s(x_s, 1)} \leq C_{\text{BHI}} \frac{G^s(x_0, y)}{G^s(x_0, 1)}, \quad |y| \geq R.
\]
Therefore, by Proposition 4.6,

\[ I_3(s) = \int_{\Gamma \setminus \Gamma_R} G_\Gamma^s(x, y) f^s(y) \, dy \leq C_{\text{BHI}} \int_{\Gamma \setminus \Gamma_R} \frac{G_\Gamma^s(x, y)}{G_\Gamma^s(x, 1)} f^s(y) \, dy \]

\[ \leq c \int_{\Gamma \setminus \Gamma_R} G_\Gamma^s(x, y) f^s(y) \, dy, \]

for large enough \( s \geq s_0 \). Denote \( c_1 = c/G_\alpha^s(x_0, 1) \). By the Fatou lemma,

\[ \lim_{s \to \infty} I_3(s) \leq c_1 \left( \limsup_{s \to \infty} G_\Gamma^s f^s(x_0) - \liminf_{s \to \infty} \int_{\Gamma_R} G_\Gamma^s(x, y) f^s(y) \, dy \right) \]

\[ \leq c_1 \left( G_\Gamma^\alpha f^\alpha(x_0) - \int_{\Gamma_R} G_\Gamma^\alpha(x, y) f^\alpha(y) \, dy \right) \]

\[ = c_1 \int_{\Gamma \setminus \Gamma_R} G_\Gamma^\alpha(x, y) f^\alpha(y) \, dy. \]

(4.40)

It remains to estimate \( I_1(s) \). Note that, by symmetry, \( y \mapsto G_\Gamma^s(v, y) \) is regular harmonic on \( \Gamma_\delta \) when \( v \in \Gamma \setminus \Gamma_2 \). Using (BHP) we obtain, for \( y \in \Gamma_\delta \) and \( v \in \Gamma \setminus \Gamma_2 \),

\[ \frac{G_\Gamma^s(v, y)}{G_\Gamma^s(y, 2\delta \cdot 1)} \leq C_{\text{BHI}} \frac{G_\Gamma^s(v, \delta/2 \cdot 1)}{G_\Gamma^s(\delta/2 \cdot 1, 2\delta \cdot 1)}. \]

(4.41)

Therefore, for \( s \) large enough and \( y \in \Gamma_\delta \), by (4.41) and (2.10),

\[ G_\Gamma^s(x, y) = G_{\Gamma_2}^s(x, y) + \mathbb{E}_{x, z} G_\Gamma^s(x, z) \cdot G_\Gamma^s(z, y) \]

\[ \leq G_{\Gamma_2}^s(x, y) + c \mathbb{E}_{x, z} \left[ G_\Gamma^s(x, z) \cdot G_\Gamma^s(z, y) \right] \]

\[ \leq G_{\Gamma_2}^s(x, y) + c G_\Gamma^s(x, \delta/2 \cdot 1) \cdot \frac{G_\Gamma^s(y, 2\delta \cdot 1)}{G_\Gamma^s(\delta/2 \cdot 1, 2\delta \cdot 1)}. \]

(4.42)

Thus, by uniform boundedness of \( f_s \) on \( \Gamma_\delta \) and (4.42),

\[ \int_{\Gamma_\delta} G_\Gamma^s(x, y) f^s(y) \, dy \leq c \left( \int_{\Gamma_\delta} G_{\Gamma_2}^s(x, y) \, dy + \frac{G_\Gamma^s(x, \delta/2 \cdot 1)}{G_\Gamma^s(\delta/2 \cdot 1, 2\delta \cdot 1)} \int_{\Gamma_\delta} G_\Gamma^s(y, 2\delta \cdot 1) \, dy \right). \]

(4.43)

Lemma 4.8 implies

\[ \int_{\Gamma_\delta} G_\Gamma^s(y, 2\delta \cdot 1) \, dy \leq c_5 \delta^d G_\Gamma^s(\delta/2 \cdot 1, 2\delta \cdot 1). \]

(4.44)

Indeed, observe that there is \( \tilde{\kappa} \leq \kappa \) such that \( B(1/2 \cdot 1, \tilde{\kappa}) \subset \Gamma_1 \), thus after a possible change of \( \kappa \) we may and do assume that \( A_1(0) = 1/2 \cdot 1 \). Since \( \Gamma_1 \) is a cone, it follows immediately that \( A_\delta(0) = \delta/2 \cdot 1 \). Moreover, observe that \( \Gamma_\delta \) is created from \( \Gamma_1 \) by scaling and the Lipschitz constant as well as \( \kappa \) are not affected by the operation; therefore, \( c_5 \) is independent of \( \delta \). Furthermore, we have by Proposition 2.4,

\[ \int_{\Gamma_\delta} G_{\Gamma_2}^s(x, y) \, dy \leq C_{\text{BHI}} \mathbb{E}_{x, z} \tau_{\Gamma_2}^s \int_{\Gamma_\delta} G_\Gamma^s(y, 1) \nu^s(y) \, dy \]

(4.45)

We claim that

\[ G_\Gamma^s(y, 1) \approx \mathbb{E}_y \tau_{\Gamma_2}^s \]

(4.46)

for large \( s \) and \( y \in \Gamma_{1/2} \), with the implied constant independent of \( s \). Indeed, by (BHP) we have,

\[ G_\Gamma^s(y, 1) \approx \mathbb{E}_y \tau_{\Gamma_2/3}^s \int_{\Gamma_{1/2}} G_\Gamma^s(w, 1) \nu^s(w) \, dw, \]

(4.47)
thus it is enough to show that the integral in (4.47) is uniformly bounded. For the lower bound we use the monotonicity of the Lévy density, (4.4), Pruitt estimates [49] and Proposition 4.1, and we get
\[
\int_{\Gamma_{5/6}} G_t^\alpha(y,1)\nu^\alpha(y) dy \geq c \int_{\Gamma_{5/6}\setminus\Gamma_{3/2}} \mathbb{E}_y \tau_{\Gamma_{3/2}}^\alpha \nu^\alpha(y) dy \geq c \ln \frac{h^\alpha(3d)}{h^\alpha(1/2)},
\]
where the last inequality is a consequence of (4.24). Thus, using (4.43), (4.44) and (4.45) together with (4.48) we infer that
\[
I_1(s) \leq c_6 \left( \frac{1}{\ln \frac{h^\gamma(3\delta)}{h^\gamma(1/2)}} + \delta^\beta \frac{G_t^\alpha(x_s, \delta/2 \cdot 1)}{G_t^\alpha(x_s, 1)} \right).
\]
Lemma 4.10 and the fact that \( h^\alpha(\rho) = c \rho^{-\alpha} \) now yield
\[
\limsup_{s \to \infty} I_1(s) \leq c_7 \left( -\ln \delta + \delta^\beta M_0^\alpha(\delta/2 \cdot 1) \right).
\]
Finally, using the homogeneity of \( M_0^\alpha \) (4.33) we get
\[
\limsup_{s \to \infty} I_1(s) \leq c_7 \left( -\ln \delta + \delta^{\alpha-\bar{\beta}} M_0^\alpha(1) \right).
\]
Now, by (4.39), (4.40), (4.49) and the Fatou lemma,
\[
\int_{\Gamma_{6}\setminus\Gamma_s} M_0^\alpha(y) f^\alpha(y) dy \leq \liminf_{s \to \infty} \frac{G_t^\alpha f^\alpha(x_s)}{G_t^\alpha(x_s, 1)} \leq \limsup_{s \to \infty} \frac{G_t^\alpha f^\alpha(x_s)}{G_t^\alpha(x_s, 1)} \leq c_8 \left( -\ln \delta + \delta^{\alpha-\bar{\beta}} \right) + \int_{\Gamma_R \setminus \Gamma_s} M_0^\alpha(y) f^\alpha(y) dy + c_1 \int_{\Gamma_{6}\setminus\Gamma_R} G_t^\alpha(x_0, y) f^\alpha(y) dy.
\]
Recalling that \( \alpha > \bar{\beta} \) and letting \( \delta \to 0 \) and \( R \to \infty \) we end the proof.

**Theorem 4.12.** Assume A. Let \( x_s = x \psi^{-1}(1/s) \). For every \( t > 0 \) we have
\[
\lim_{s \to \infty} \mathbb{P}_{x_s}(\tau_t^\alpha > t) G_t^\alpha(x_s, 1) = C_t,
\]
where \( C_t \in (0, \infty) \) is given by
\[
C_t = \int_{\Gamma} \int M_0^\alpha(y) \rho_t^\alpha(y, z) \kappa_t^\alpha(z) dz dy = \int_{\Gamma} M_0^\alpha(y) \int P_t^\alpha(y, z) \kappa_t^\alpha(z) dy.
\]
Recall that \( t > 0 \). We verify that \( f^s(x) = P^{\Gamma,s}_t \kappa^s_t(x) \) satisfies the assumptions of Lemma 4.11; then the theorem will follow immediately. Indeed, first observe that pointwise convergence is a claim of Lemma 4.9. Fix \( r > 0 \) and let us verify the uniform boundedness of \( f^s \) on \( \Gamma_r \). By Lemma 4.5 and A1 we have

\[
P^{\Gamma,s}_t \kappa^s_t(x) \approx \mathbb{P}_x(\tau^{\Gamma,s}_t > t) \int_{\Gamma} \mathbb{P}_y(\tau^{\Gamma,s}_t > t) p^s_t(y) \kappa^s(y) dy
\]

for \( s \) large enough, with the comparability constant dependent only on \( d, M, \beta \) and \( r \). By (2.6), we have \( G^s_t P^{\Gamma,s}_t \kappa^s(x) \leq 1 \) for all \( x \in \mathbb{R}^d \). Since the survival probability is bounded from above by 1, it remains to find the upper bound for the integral. If \( \delta = \delta_\Gamma(1) \), then by Pruitt’s estimates, Proposition 4.1 and the argument from (3.5),

\[
1 \geq c \int_{B(1,r)} G^s_t(1, z) \int_{\Gamma} \mathbb{P}_y(\tau^{\Gamma,s}_t > t) p^s_t(y) \kappa^s(y) dy \, dz
\]

\[
\geq c \mathbb{E} \tau_{B_1} \int_{\Gamma} \mathbb{P}_y(\tau^{\Gamma,s}_t > t) p^s_t(y) \kappa^s(y) dy
\]

\[
\geq \frac{c}{h^s(\delta)} \int_{\Gamma} \mathbb{P}_y(\tau^{\Gamma,s}_t > t) p^s_t(y) \kappa^s(y) dy
\]

\[
\geq \frac{c}{h^s(\delta)} \int_{\Gamma} \mathbb{P}_y(\tau^{\Gamma,s}_t > t) p^s_t(y) \kappa^s(y) dy.
\]

Therefore the functions \( P^{\Gamma,s}_t \kappa^s \) are uniformly bounded on \( \Gamma_r \) for every \( r > 0 \).

Finally, by Corollary A.3 we have \( \lim_{s \to \infty} G^s_t f^s(x) = G^s_t f^0(x) \) for every \( x \in \Gamma \), hence an application of Lemma 4.11 ends the proof.

The following theorem refines [15, Theorem 3.3], which is a special case for \( X \) being the isotropic \( \alpha \)-stable Lévy process in \( \mathbb{R}^d \), but we note that part of the proof relies on the consequences of [15, Theorem 1.1].

**Theorem 4.13.** Let \( X \) be a pure-jump isotropic unimodal Lévy process. Assume A. Let \( \Gamma \) be a Lipschitz cone. Then the following limit exists

\[
\lim_{s \to \infty} \frac{p^{\Gamma,s}_t(x, y)}{\mathbb{P}_{x_s}(\tau^{\Gamma,s}_t > 1)} = n^\alpha(y),
\]

where \( n^\alpha(y) \) is the function from (1.7).

**Proof.** The proof follows directly the proof of [15, Theorem 3.3]. Fix \( x \in \Gamma \) and \( t > 0 \). Consider the family of measures defined as follows,

\[
(4.50) \quad \mu^s_t(A) = \int_A f^s_t(y) \, dy = \frac{\int_A p^{\Gamma,s}_t(x, y) \, dy}{\mathbb{P}_{x_s}(\tau^{\Gamma,s}_t > 1)}, \quad A \subset \mathbb{R}^d,
\]

with \( x_s = \psi_t^{-1}(1/s) x \). We claim that the family \( \{\mu^s_t : s \geq s_0\} \), where \( s_0 \) will be specified later in the proof, is tight. Indeed, proceeding as in the proof of Proposition 4.1 and applying (4.5) we have that

\[
(4.51) \quad \left( \psi_t^{-1}(1/t)^d \right)^r \wedge t \nu^s(y) \lesssim 1 \wedge \nu^s(y) \lesssim 1 \wedge |y|^{d-\alpha/2}, \quad s \geq s_0, y \in \mathbb{R}^d,
\]

for some \( s_0 > 1 \), with the implied constant independent of \( s \). It follows from Lemma 4.3 and 4.5, and (2.14) that we may bound the densities \( f^s_t \) by a fixed integrable function, i.e.,

\[
(4.52) \quad \frac{p^{\Gamma,s}_t(x, y)}{\mathbb{P}_{x_s}(\tau^{\Gamma,s}_t > 1)} \approx \mathbb{P}_y(\tau^{\Gamma,s}_t > t) p^s_t(y - x_s) \lesssim p^s_t(y) \lesssim 1 \wedge |y|^{d-\alpha/2}, \quad s \geq s_0, y \in \mathbb{R}^d.
\]

Recall that \( t \) is fixed and the implied constant depends only on \( d, M \) and \( \beta \). Consider an arbitrary sequence \( \{s_n\} \) with \( \lim_{n \to \infty} s_n = \infty \). By the Prokhorov theorem, there is a subsequence \( \{s_{n_k}\} \) such that \( \mu^s_{n_k} \) converges weakly to a probability measure \( \mu_t \) as \( k \to \infty \).
Let \( \phi \in C_c^\infty(\Gamma) \) and set \( u_\phi^s = -L^s\phi \). For every \( s \geq 1 \), \( u_\phi^s \) is bounded, continuous, and, in view of Proposition 2.5, \( G^s_t u_\phi^s(y) = \phi(y) \) for \( y \in \mathbb{R}^d \). By (2.19),

\[
(4.53) \quad |L^s\phi(y)| \lesssim 1 \land \nu^s(y), \quad y \in \mathbb{R}^d.
\]

In view of Lemma 4.3 and (4.5), \( |u_\phi^s(y)| \leq cp_1^s(y) \). Note that \( c \) here may depend on \( s \) but it is irrelevant for the proof of (4.54). It follows that

\[
P_t^{\Gamma,s}u_\phi^s(y) \leq cp_t(y),
\]

and consequently, for every \( y \in \Gamma \),

\[
G^s_t P_t^{\Gamma,s}u_\phi^s(y) \leq c \int_{\mathbb{R}^d} G^s_t(y,z)p_{t+1}(z) dz < \infty,
\]

where the last inequality follows from Lemma 4.3 and (2.16). By the Fubini-Tonelli theorem, for every \( s \geq 1 \),

\[
(4.54) \quad G^s_t P_t^{\Gamma,s}u_\phi^s(y) = P_t^{\Gamma,s}G^s_t u_\phi^s(y) = P_t^{\Gamma,s}\phi(y).
\]

Next, observe that by \( A_1, (4.4) \) and (4.51), for \( s \) large enough,

\[
|\phi(y + z) - \phi(y)| |\nu^s(z) | \leq ||\phi||_{C^2(\mathbb{R}^d)} (|z|^2 \land 1) |\nu^s(z) | \lesssim |z|^{2-d} |B_{1}\nu^s(z) | + |z|^{-d-\alpha/2} 1_{B_1}\nu^s(z), \quad y, z \in \mathbb{R}^d,
\]

with the implied constant independent of \( s \). Thus, the dominated convergence theorem entails that for every \( y \in \Gamma \),

\[
\lim_{s \to \infty} u_\phi^s(y) = u_\phi^0(y) = -\Delta_{\alpha/2}\phi(y).
\]

Moreover, in view of (4.51) we may refine (4.53) so that

\[
|L^s\phi(y)| \leq c \left( 1 \land |y|^{-d-\alpha/2} \right), \quad y \in \mathbb{R}^d,
\]

for \( s \) large enough with \( c = c(d, M, \beta) \). Thus, by Lemma 4.3, (4.5) and the dominated convergence theorem, for every \( y \in \Gamma \),

\[
\lim_{s \to \infty} P_t^{\Gamma,s}u_\phi^s(y) = P_t^{\Gamma,0}u_\phi^0(y).
\]

The same argument yields that \( P_t^{\Gamma,s}u_\phi^s \) are uniformly bounded and that \( G_t^{\Gamma,s}P_t^{\Gamma,s}u_\phi^s = P_t^{\Gamma,s}\phi \) are uniformly bounded. We also get that

\[
\lim_{s \to \infty} G_t^{\Gamma,s}P_t^{\Gamma,s}u_\phi^s(y) = \lim_{s \to \infty} P_t^{\Gamma,s}\phi(y) = P_t^{\Gamma,0}\phi(y) = G_t^{\Gamma,0}P_t^{\Gamma,0}u_\phi^0(y), \quad y \in \Gamma.
\]

Thus, by Lemma 4.11,

\[
\lim_{s \to \infty} \frac{P_t^{\Gamma,s}\phi(x_s)}{G_t^{\Gamma}(x_s, 1)} = \lim_{s \to \infty} \frac{G_t^{\Gamma,s}P_t^{\Gamma,s}u_\phi^s(x_s)}{G_t^{\Gamma}(x_s, 1)} = \int_{\Gamma} M_0^{\alpha}(y)P_t^{\Gamma,0}u_\phi^0(y) dy,
\]

If we denote \( \mu_t^s(\phi) = \int_{\Gamma} \phi(y) \mu_t^s(dy) \), then by Theorem 4.12 and the identity above we conclude that there is a finite limit

\[
\lim_{s \to \infty} \mu_t^s(\phi) = \lim_{s \to \infty} \frac{P_t^{\Gamma,s}\phi(x_s)}{P_{x_s}(T^{\Gamma}_t > t)} = \int_{\Gamma} M_0^{\alpha}(y)P_t^{\Gamma,0}u_\phi^0(y) dy/
\]

In particular, \( \mu_t(\phi) = \lim_{s \to \infty} \mu_t^{sn_k}(\phi) \) does not depend on the choice of subsequence \( s_{nk} \). Therefore, \( \mu_t^s \) converges weakly to \( \mu_t \) as \( s \to \infty \).

Moreover, we observe that for \( t = 1 \) the limit measure \( \mu_1 \) is exactly the same as the one in the proof of [15, Theorem 3.3]. By a repetition of the arguments in the proofs of [15, (3.16), Theorem 3.1 and Theorem 3.3] one can conclude that the same holds true for every \( t > 0 \), i.e., the measures

\[
\bar{\mu}_{x,t}(A) = \int_{A} P_t^{\Gamma,0}(x, y) dy/
\]
in [15] converge weakly as $\Gamma \ni x \to 0$ to the same measure $\mu_t$. It is therefore appropriate to use the notation $\mu_t^s$ instead of $\mu_t$. Moreover, also from the proof of [15, Theorem 3.3] we may conclude that the limit

$$f_t^s(y) = \lim_{\Gamma \ni x \to 0} \frac{p_{t,x}^s(x,y)}{\mathbb{P}_x(\tau_t^s > t)}$$

exists and is the density function of the measure $\mu_t^s$. Note here that $f_t^s = n^\alpha$. We now prove that

$$\lim_{s \to \infty} f_t^s(y) = \lim_{s \to \infty} \frac{p_{1/2}^{s}(x,y)}{\mathbb{P}_x(\tau_t^s > 1)} = n^\alpha(y), \ y \in \Gamma.$$  

Indeed, fix $y \in \Gamma$ and we denote $\phi_y^\alpha(\cdot) = p_{1/2}^{s}(\cdot,y)$. By the Chapman-Kolmogorov equation,

$$p_{1/2}^{s}(u,y) = \int_{\Gamma} p_{1/2}^{s}(u,z) p_{1/2}^{s}(z,y) \, dz = P_{1/2}^{s} \phi_y^\alpha(u), \ u \in \mathbb{R}^d.$$  

Thus,

$$\frac{p_{1/2}^{s}(x,y)}{\mathbb{P}_x(\tau_t^s > 1)} = \frac{P_{1/2}^{s} \phi_y^\alpha(x)}{\mathbb{P}_x(\tau_t^s > 1)} = \mu_t^s(\phi_y^\alpha).$$  

We claim that

$$\lim_{s \to \infty} \mu_t^s(\phi_y^\alpha - \phi_y^\alpha) = 0.$$  

Thus, in order to get (4.56) we need to show that

$$\lim_{s \to \infty} \mu_t^s(\phi_y^\alpha - \phi_y^\alpha) = 0.$$  

Recall that $\mu_t^s(\phi_y^\alpha - \phi_y^\alpha) = 0$.

By (4.52) and [15, (4.16)] we see that $f_t^s$ and $f_t^s$ are uniformly bounded by a fixed bounded integrable function $g(z) = 1 \land |z|^{-d-\alpha/2}$, if only $s$ is large enough. Thus, for every $R > 0$ we have, by Lemma 4.4, (4.5) and the dominated convergence theorem,

$$\lim_{s \to \infty} \int_{\Gamma} |\phi_y^s(z) - \phi_y^\alpha(z)| \, f_t^s(\phi_y^s - \phi_y^\alpha) \, dz = 0.$$
By letting \( R \) to infinity, we obtain (4.57), so (4.56) follows.

We thus have proved that the left-hand side limit in (4.55) exists for all \( y \in \Gamma \). Let us denote \( \lim_{y \to \infty} f^\ast_y(y) = \bar{n}(y) \) for \( y \in \Gamma \). By weak convergence, (4.52) and the dominated convergence theorem, for every bounded continuous function \( \phi \) we have

\[
\int_{\Gamma} n^\alpha(y)\phi(y) \, dy = \mu^\alpha(\phi) = \lim_{s \to \infty} \int_{\Gamma} P^s_\Gamma(x, y) \phi(y) \, dy = \int_{\Gamma} \bar{n}(y)\phi(y) \, dy.
\]

Therefore, \( n^\alpha = \bar{n} \) and (4.55) follows. \( \square \)

We are ready to prove the main result of the paper.

**Proof of Theorem 1.1.** Using (4.13), (2.9) and changing variables we get, for \( x \in \Gamma \) and \( s \geq 1 \),

\[
\mathbb{P}_x\left( \psi^{-1}(1/s)X_s \in A \mid \tau_\Gamma > s \right) = \frac{\mathbb{P}_x\left( \psi^{-1}(1/s)X_s \in A, \tau_\Gamma > s \right)}{\mathbb{P}_x(\tau_\Gamma > s)} = \frac{\int_{A/\psi^{-1}(1/s)} P^s(x, y) \, dy}{\int_{A} P^s(x, y) \, dy} = \frac{\int_{A} P^s_\Gamma(x, y) \, dy}{\int_{A} P^s_\Gamma(x, y) \, dy} = n^\alpha(A).
\]

By (4.55) and (4.52) and an application of [15, Lemma 4.1] we conclude that

\[
\lim_{s \to \infty} \mathbb{P}_x\left( \psi^{-1}(1/s)X_s \in A \mid \tau_\Gamma > s \right) = \int_{A} n^\alpha(y) \, dy = \mu^\alpha(A).
\]

The proof is complete. \( \square \)

**Example 4.14.** Consider the Lévy density \( \nu(x) = A(d, \alpha) \ln^\beta(e + |x|)/|x|^{d+\alpha} \) on \( \mathbb{R}^d \), with \( \beta \in \mathbb{R} \) and, of course, \( \alpha \in (0, 2) \). Then \( A \) is satisfied, for instance \( \psi(r) \sim r^\alpha \ln r^\alpha(1/r) \) for small \( r > 0 \), see [19, Proposition 2]. In this case, the rescaling in Theorem 1.1 is by \( \psi^{-1}(1/s) \sim s^{-1/\alpha} \ln^{-\beta/\alpha}(s^{1/\alpha}) \), which is qualitatively different than in (1.6).

**Appendix A: Weak convergence and continuity in the Skorokhod topology**

This appendix is devoted to weak convergence and continuity in the Skorokhod topology.

Let us recall the Skorokhod topology and its basic properties. The main references here are the books [38] and Gihman and Skorokhod [25]. In view of [38, Theorem VI.1.14], the space \( \mathcal{D}(\mathbb{R}^d) = \mathcal{D}_{[0,\infty)}(\mathbb{R}^d) \) of all càdlàg functions \( \omega : [0, \infty) \to \mathbb{R}^d \), may be endowed with a topology for which it is a complete separable metric space. The notion of convergence is described as follows: let \( \Lambda \) be a set of strictly increasing continuous functions \( \lambda : [0, \infty) \to [0, \infty) \) such that \( \lambda(0) = 0 \) and \( \lambda(t) \to \infty \) as \( t \to \infty \). Then \( \omega_n \) converges to \( \omega \) in \( \mathcal{D}(\mathbb{R}^d) \) as \( n \to \infty \) if and only if there is a sequence \( \{\lambda_n : n \in \mathbb{N}\} \subset \Lambda \) such that

\[
\lim_{n \to \infty} \sup_{t \in [0, \infty)} |\lambda_n(t) - t| = 0,
\]

and

\[
\lim_{n \to \infty} \sup_{t \leq T} |\omega_n(\lambda_n(t)) - \omega(t)| = 0 \quad \text{for every } T > 0.
\]

Let \( Y \) be the canonical projection or a coordinate process in \( \mathcal{D}(\mathbb{R}^d) \), i.e.

\[
Y_t = Y_t(\omega) = \omega(t)
\]
for $t \geq 0$ and $\omega \in D(\mathbb{R}^d)$. As before, we define for open set $D \subset \mathbb{R}^d$,

$$\tau_D(\omega) = \inf\{ t > 0 : \omega(t) \notin D \}.$$ 

Let $\mathbb{P}_x$ be a probability measure on $D(\mathbb{R}^d)$ such that $\mathbb{P}_x(Y_0 = x) = 1$. We assume throughout this appendix that the process $Y$ is quasi-left-continuous under $\mathbb{P}_x$ and for every $x \in D$, $\mathbb{P}_x(Y_{\tau_D} \in \partial D) = 0$.

**Proposition A.1.** Let $\mathbb{D}$ be the set of discontinuities of the functions $\tau_D$, $Y_{\tau_D -}$ and $Y_{\tau_D}$ on $D(\mathbb{R}^d)$. Then $\mathbb{P}_x(\mathbb{D}) = 0$ for all $x \in D$.

**Proof.** For $m \geq 1$ we define $D_m = \{ x \in D : \text{dist}(x, D^c) > 1/m \}$ and $\tau_m = \inf\{ t : Y_t \notin D_m \}$. Of course, $\tau_m \leq \tau_D$. If $\tau_\infty := \lim_{m \to \infty} \tau_m = \infty$, then $\tau_D = \tau_\infty$. On $\{ \tau_\infty < \infty \}$ we have from the quasi-left-continuity of $Y$ that $\lim_{m \to \infty} \tau_m = \tau_\infty$ a.s. It follows that $\tau_\infty \in D^c$, hence $\tau_\infty \geq \tau_D$ and so $\tau_\infty = \tau_D$. Thus, $\lim_{m \to \infty} \tau_m \uparrow \tau_D$.

By our assumption, $\mathbb{P}_x \left( \tau_{\tau_D} \in D, Y_{\tau_D} \in \overline{D} \right) = 1$. Fix $\omega \in D(\mathbb{R}^d)$ such that $Y_{\tau_D -}(\omega) \in D$, $Y_{\tau_D} \in \overline{D}$ and $\lim_{m \to \infty} \tau_m(\omega) = \tau_D(\omega)$. We will prove that the functionals $\tau_D$, $Y_{\tau_D -}$ and $Y_{\tau_D}$ are continuous at $\omega$.

Let $\{ \omega_n : n \in \mathbb{N} \}$ be a sequence in $D(\mathbb{R}^d)$ such that $\omega_n \to \omega$ as $n \to \infty$. By the definition of the Skorokhod topology, there exist $\{ \lambda_n : n \in \mathbb{N} \} \subset \Lambda$ such that $\sup_{s > 0} |\lambda_n(s) - s| \to 0$ and $\sup_{s \leq T} |Y_{\lambda_n(s)}(\omega_n) - Y_s(\omega)| \to 0$ as $n \to \infty$ for every $T > 0$. Let $t = \tau_{\tau_D}(\omega)$ and $t_n = \lambda_n(t)$. Then $t_n \to t$, $Y_{t_n}(\omega_n) \to Y_t(\omega)$ and $Y_{t_n -}(\omega_n) \to Y_{t -}(\omega)$ as $n \to \infty$.

Since $Y_t(\omega) \in D$, we have $Y_{t_n}(\omega_n) \in \overline{D}$ so $\tau_{\tau_D}(\omega_n) \leq t_n$ for $n$ large enough. Therefore,

$$\limsup_{n \to \infty} \tau_{\tau_D}(\omega_n) \leq \tau_{\tau_D}(\omega). \quad (A.3)$$

For every $\varepsilon > 0$, there exists $m_0$ such that $t - \varepsilon < \tau_{m_0}(\omega_n) \leq t$. Note that for $n$ sufficiently large, $\sup_{u \leq t - \varepsilon} |\lambda_n(u) - u| < \varepsilon$ and $\sup_{u \leq t - \varepsilon} |Y_{\lambda_n(u)}(\omega_n) - Y_u(\omega)| < 1/m_0$. Since $Y_u(\omega) \in D_{m_0}$ we have $Y_{\lambda_n(u)}(\omega_n) \in D$ for every $u \leq t - \varepsilon$. Therefore $\tau_{\tau_D}(\omega_n) \geq \lambda_n(t - \varepsilon) \geq t - 2\varepsilon$. As $\varepsilon$ is arbitrary, $\liminf_{n \to \infty} \tau_{\tau_D}(\omega_n) \geq t$. This and (A.3) prove the continuity of $\tau_{\tau_D}$ at $\omega$.

Note that we have shown that $\lim_{n \to \infty} \tau_{\tau_D}(\omega_n) = t$ and for $n$ large enough, $\tau_{\tau_D}(\omega_n) \leq t$. Applying [38, Proposition VI.2.1 (b.2)] we get that $\lim_{n \to \infty} Y_{\tau_{\tau_D}-}(\omega_n) = Y_{\tau_{\tau_D}-}(\omega)$.

If $\tau_{\tau_D}(\omega_n) < t_n$ for a subsequence $\{ n_k : k \in \mathbb{N} \}$, then, by [38, Proposition VI.2.1 (b.1)], we have $\lim_{n \to \infty} Y_{\tau_{\tau_D}}(\omega_{n_k}) = Y_{\tau_{\tau_D}}(\omega)$. This contradicts the facts that $Y_{\tau_{\tau_D}}(\omega_n) \in D^c$ and $Y_{\tau_{\tau_D}-}(\omega) \in D$. So we must have that $\tau_{\tau_D}(\omega_n) = t_n$ for $n$ large enough. Therefore, $\lim_{n \to \infty} Y_{\tau_{\tau_D}}(\omega_n) = Y_{\tau_{\tau_D}}(\omega)$.

**Corollary A.2.** For every $t > 0$, $\mathbb{P}_x(\partial \{ \tau_{\tau_D} > t \}) = 0$.

**Proof.** The boundary of $\{ \tau_{\tau_D} > t \}$ in the Skorokhod topology is contained in $\mathbb{D} \cup \{ \tau_{\tau_D} = t \}$, where $\mathbb{D}$ is the set of discontinuities of $\tau_D$. By Proposition A.1 we have that $\mathbb{P}_x(\mathbb{D}) = 0$. Since $Y_{\tau_D} = Y_t$ a.s., we have that $\mathbb{P}_x(\tau_{\tau_D} = t) \leq \mathbb{P}_x(\tau_{\tau_D} \in \partial D) = 0$.

Now assume that, under $\mathbb{P}_x$, $Y$ is a pure-jump Lévy process in $\mathbb{R}^d$ with Lévy exponent $\psi$. For $s \geq 1$ let $\{ Y^s : t \geq 0 \}$ be a Lévy processes with Lévy exponent $\psi^s$. Set

$$\tau_D^s = \inf\{ t > 0 : Y^s_t \notin D \}.$$ 

Suppose that for each $\xi \in \mathbb{R}^d$, $\psi^s(\xi) \to \psi(\xi)$ as $s \to \infty$. Then $Y^s$ converges in distribution to $Y_1$. It follows from [38, Corollary VII.3.6] that $Y^s$ converges in distribution to $Y$ in the Skorokhod space $D(\mathbb{R}^d)$ as $s \to \infty$. By Corollary A.2 we have the following result.

**Corollary A.3.** Let $D$ be an open subset of $\mathbb{R}^d$ such that $\mathbb{P}_x(Y_{\tau_D} \in \partial D) = 0$. Then for every $t > 0$ and $x \in D$,

$$\lim_{s \to \infty} \mathbb{P}_x(\tau_{\tau_D} > t) = \mathbb{P}_x(\tau_D > t).$$

**Funding**

Gavin Armstrong, Tomasz Grzywny and Łukasz Leźaj were partially supported by the National Science Centre (Poland): grant 2016/23/B/ST1/01665. Krzysztof Bogdan was partially supported by the National Science Centre (Poland): grant 2017/27/B/ST1/01339. Longmin Wang was partially supported by the National Natural Science Foundation of China: Grant 11801283.
References


