Equivalence of Liouville measure and Gaussian free field

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Abstract

Given an instance $h$ of the Gaussian free field on a planar domain $D$ and a constant $\gamma \in (0, 2)$, one can use various regularization procedures to make sense of the Liouville quantum gravity area measure $\mu := e^{\gamma h(z)}dz$. It is known that the field $h$ a.s. determines the measure $\mu_h$. We show that the converse is true: namely, $h$ is measurably determined by $\mu_h$. More generally, given a random closed fractal subset $A$ endowed with a Frostman measure $\sigma$ whose support is $A$ (independent of $h$), a Gaussian multiplicative chaos measure $\mu_{\sigma,h}$ can be constructed. We give a mild condition on $(A, \sigma)$ under which $\mu_{\sigma,h}$ determines $h$ restricted to $A$, in the sense that it determines its harmonic extension off $A$. Our condition is satisfied by the occupation measures of planar Brownian motion and SLE curves under natural parametrizations. Along the way we obtain general positive moment bounds for Gaussian multiplicative chaos. Contrary to previous results, this does not require any assumption on the underlying measure $\sigma$ such as scale invariance, and hence may be of independent interest.

1 Introduction

1.1 Motivation and statement of the problem

Let $\gamma \in (0, 2)$. Liouville quantum gravity (LQG) is the random planar geometry whose volume measure, is given (formally) by

$$\mu_h = e^{\gamma h}dz,$$

where $h$ is a variant of the two-dimensional Gaussian free field (GFF), and $dz$ is the Lebesgue measure on $\mathbb{C}$. The GFF is not a pointwise defined function but is rather a random distribution, so that making sense of $\mu_h$ in the above definition requires some nontrivial work.

This measure, sometimes known as Liouville measure, can be constructed via a regularization and normalization procedure known as Gaussian multiplicative chaos; see (1). This fact goes back in some form to the pioneering work of Kahane [22] and (for a smaller range of values of the parameter $\gamma$) to the earlier work of Høegh-Krohn [20]. It was subsequently rediscovered in [14] (see also [32]) which formulated a framework for the construction of Liouville quantum gravity, as arising in the work of Polyakov [31] in string theory. By the so-called DDK ansatz this geometry is also predicted to describe the scaling limit of natural random planar maps models. See
for physics and mathematics background on LQG and in particular for the conjectured relation to random planar maps.

The construction of the LQG area measure was later complemented by that of further LQG observables, associated with Schramm–Loewner evolution [37, 12] and planar Brownian motion [16, 3]. In both cases we have a conformally invariant process whose trajectory is independent from the GFF but whose parametrization respects the LQG geometry. In the latter case, the reparametrized Brownian motion is called Liouville Brownian motion. More recently, a random metric associated with LQG was constructed in a remarkable series of works culminating in [9, 19]. (In the case where the parameter $\gamma = \sqrt{8/3}$, this metric coincides with the one constructed earlier in [29, 26, 27] via an indirect mechanism called quantum Loewner evolution). All these observables have natural interpretations in light of the conjectured link with random planar maps.

The LQG area measure is perhaps the simplest and most well studied LQG observable at this point. Moreover, much of the theory of LQG is based on the implicit assumption that this observable captures all of the geometry of LQG. For example, the coordinate change formula dictating whether two LQG surfaces should be viewed as conformally equivalent, is prescribed by the requirement that the LQG area measure must transform covariantly. Moreover, in the mating of trees theory (as introduced in [12]) a pair of Brownian motions $(L, R)$ is used to describe the LQG area measure decorated by an SLE curve. Based on this assumption, the LQG field and hence the entire underlying surface is captured by $(L, R)$; see Corollary 1.3.

This suggests that the Liouville measure is an observable that is sufficiently rich to capture all the relevant geometry. The main goal of this paper is precisely to put the assumption above on solid ground, by demonstrating that the underlying Gaussian free field $h$ can be measurably recovered from its Liouville measure (Theorem 1.1). While this is intuitive at some level, this is also far from obvious a priori, since it is well known that the Liouville measure concentrates on points that are in some sense exceptional for the GFF, indeed such points (the $\gamma$-thick points of the field) have Hausdorff dimension less than 2.

In fact, we consider the same question in a more general (and, as we will see, more challenging) yet also very natural setup. Let $A$ be a closed set and let $\sigma$ be a measure supported on $A$. We suppose that $h$ is a variant of the two-dimensional GFF and $\sigma$ (and hence also $A$) is either fixed deterministic or random but independent from $h$. Let $d$ be the dimension of $\sigma$ (the maximal value such that $\sigma$ has finite $d$-dimensional energy) and let $\gamma < \sqrt{2d}$. Then $h$ induces a Gaussian multiplicative chaos measure $\mu_{\sigma, h}$ supported on $A$ with parameter $\gamma$, that is, informally,

$$d\mu_{\sigma, h} = e^{\gamma h}d\sigma.$$

In the special case where $h$ is a GFF with Dirichlet boundary conditions on a domain $D$ and $\sigma$ is the Lebesgue measure (henceforth the Lebesgue case) we recover the above setup. In Theorem 1.4 we will show that under mild assumptions on $\sigma$, $\mu_{\sigma, h}$ entirely determines $h$ restricted to $A$: more precisely (since $h$ is not defined pointwise), $\mu_{\sigma, h}$ determines the harmonic extension of $h$ off $A$ (which is well defined under our assumptions). We emphasise that the nontrivial part of Theorem 1.4 concerns the case where $A$ has no interior point, which is why we consider the harmonic extension of the field off $A$ instead of the field itself; when $A$ has a nontrivial interior, the field restricted to $A$ can already be recovered from the measure by Theorem 1.1.

As an application of this general framework, we obtain the following corollaries:
(1) a single trajectory of Liouville Brownian motion, viewed as a parameterized curve up to any given time \( t \in [0, \infty] \), determines the Gaussian free field entirely on its range up to time \( t \). In particular, if \( t = \infty \) and the underlying domain is the whole complex plane, since the infinite range of Liouville Brownian motion is dense, that trajectory determines all of the GFF.

(2) the quantum length of an SLE curve (as defined e.g. in [14] or [12]) determines the GFF on its range.

Along the way, we will obtain some moments estimates (both positive and negative) for Gaussian multiplicative chaos on such fractals which we feel are of independent interest, see Theorems 3.1 and 3.6 respectively.

1.2 The equivalence theorem for the LQG area measure

Let \( D \subset \mathbb{C} \) be a bounded domain and \( h \) be a Gaussian free field with zero boundary condition (zero boundary GFF) on \( D \). See (7) for its definition. We can use a regularization procedure to define an area measure on \( D \):

\[
\mu_h := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz,
\]

where \( dz \) is Lebesgue measure on \( D \), \( h_\varepsilon(z) \) is the mean value of \( h \) on the circle \( \{ w \in \mathbb{C} : |w - z| = \varepsilon \} \), and \( \gamma \in (0, 2) \) is the GMC parameter. The limit holds in probability for the topology of weak convergence in the space of measures on \( D \), and is independent of the choice of regularization \([5],[35]\). In particular, \( h \) determines \( \mu_h \) almost surely. In fact the convergence is almost sure for the above circle averages if \( \varepsilon \) is restricted to powers of two \([14]\) (more recently, the restriction to powers of two was removed in \([38]\)).

Note that if \( h = h_0 + g \), where \( h_0 \) is a zero-boundary GFF on \( D \), and \( g \) is a possibly random continuous function on \( D \), the LQG area measure \( \mu_h \) associated to \( h \) can still be defined as the almost sure limit from (1), hence is determined by \( h \).

Our first main result shows that the converse measurability holds.

**Theorem 1.1.** Let \( h = h_0 + g \) where \( h_0 \) is a zero boundary GFF on a bounded domain \( D \subset \mathbb{C} \) and \( g \) is a random continuous function on \( D \). Denote by \( \mu_h \) its LQG area measure with parameter \( \gamma \in (0, 2) \). Then \( h \) is determined by \( \mu_h \) almost Surely. That is, for an arbitrary smooth test function \( f \) compactly supported in \( D \), \( (h, f) \) is measurable with respect to the \( \sigma \)-algebra generated by \( \{ \mu_h(A) : A \text{ open in } D \} \).

Once \((h, f)\) is measurable with respect to \( \mu_h \) for arbitrary smooth compactly supported test function \( f \) in \( D \), it is easy to see that we also recover \( h \) both as a random distribution in \( D \) and as a stochastic process indexed by \( H^{-1}(D) \), the Sobolev space of index \(-1\) on \( D \); see [4]. If \( g \in H_0^1(D) \) is deterministic, then we note that by the Cameron–Martin theorem \( h + g \) is absolutely continuous with respect to \( h \) so in this case the theorem follows trivially from the case \( g \equiv 0 \). But here we only assume that \( g \) is continuous in \( D \), so \( g \) can be much rougher (e.g., it does not need to be continuous on \( D \)). Moreover \( g \) may depend on \( h \).
Remark 1.2. Theorem 1.1 covers various types of GFFs (Dirichlet boundary conditions, Neumann boundary conditions, mixed boundary conditions, the whole plane GFF, etc.) via the domain Markov property. Likewise, by absolute continuity, Theorem 1.1 also covers the quantum surfaces defined in [12] including quantum cones, wedges, spheres and disks. By [21], for a large class of log-correlated fields, the field can be written in the form $h_0 + g$ when restricted to small enough neighborhood, as in Theorem 1.1. Therefore this extends the result to fields considered in [21].

Application to the mating of trees theorem. We now briefly explain an application of Theorem 1.1 to the LQG mating-of-trees framework developed in [12]. In the main result of that paper (Theorem 9.1), the authors consider a space-filling variant of SLE$_{\kappa'}$, $\kappa' = 16/\gamma^2$, on top of a $\gamma$-quantum cone, where the curve $\eta'$ is parametrized by its quantum area (i.e., $\mu_h(\eta'([s, t])) = t - s$ for all $s \leq t \in \mathbb{R}$). We refer to [12] and [37] for the notion of quantum cone, while the space-filling variant of SLE was introduced in [28]. The main theorem of [12] is that the left and right boundary quantum lengths of the curve $\eta'([0, t])$, relative to time 0, evolve as a certain two dimensional Brownian motion $(L_t, R_t)_{t \in \mathbb{R}}$ whose correlation coefficient is given by $\cos(\pi \gamma^2/4)$. (In fact, this formula was only proved for $\gamma \in [\sqrt{2}, 2)$ in [12], and the corresponding result for $\gamma \in [0, \sqrt{2})$ is addressed in [17].) In [12, Chapter 10], the authors proved that this pair of Brownian motions in fact determines the Liouville measure $\mu_h$ on the $\gamma$-quantum cone as well as the space-filling SLE almost surely, up to rotations. More precisely, we say that two functions or measures are equivalent up to rotations if one can be obtained from the other by a rotation of the complex plane about the origin. It was proved in [12, Chapter 10] that the Liouville measure $\mu_h$ modulo this equivalence relation is a.s. determined by the Brownian motions. Using Theorem 1.1 of this paper they concluded that this in turn determines the free field $h$ up to rotations.

Corollary 1.3. Modulo rotation, the field $h$ defining the $\gamma$-quantum cone is almost surely measurable with respect to the Brownian motion $(L_t, R_t)_{t \in \mathbb{R}}$.

Other applications of Theorem 1.1 (all subsequent to the time that a first version of this paper was made available on arxiv) can be found e.g. in [27, 1, 11].

1.3 Equivalence theorem in a more general setup

Recall that a Borel measure on a domain $D$ is locally finite if every point has a neighborhood of finite measure (or equivalently, if every compact set has finite measure). We will be interested in random pairs $(\sigma, A)$, where $\sigma$ is any (possibly random) locally finite measure on $D$ and $A$ is the support of $\sigma$. Here by support we mean the topological support, i.e., the unique closed set whose complement is the union of all open sets of measure 0. For example, $A$ could be one of the random fractal sets that arise in SLE theory, and $\sigma$ could be a ‘natural’ fractal measure associated to $A$. Let $h$ be an instance of the GFF on $D$ with some boundary conditions chosen independently from $(\sigma, A)$.

Fix $d \in (0, 2]$ and assume that $\sigma$ has finite $(d - \varepsilon)$-dimensional energy for all $\varepsilon \in (0, d)$, i.e.,

$$\mathcal{E}_{d-\varepsilon} := \int \int \frac{1}{|x - y|^{d-\varepsilon}} \sigma(dx)\sigma(dy) < \infty, \quad \text{for all } \varepsilon > 0. \quad (2)$$
The reader may recall that, by Frostman’s theorem, the Hausdorff dimension of a closed set $\mathcal{A}$ is the largest value of $d$ for which there exists a non-trivial measure $\sigma$ supported on $\mathcal{A}$ satisfying (2); in particular, $\dim(\mathcal{A}) \geq d - \varepsilon$. However, in the discussion below, we will not require that $d$ is the dimension of $\mathcal{A}$, or that $\sigma$ is in any sense an optimal Frostman measure on $\mathcal{A}$. Instead, we only assume (2).

Let $\gamma < \sqrt{2d}$. By Kahane’s theory of multiplicative chaos (as explained, e.g., in Theorem 1.1 in [5]) there is a way to define a measure $\mu_{\sigma, h}$ (which depends on $h$ and $\sigma$) that can be formally written as:

$$\mu_{\sigma, h}(dz) = \exp(\gamma h(z) - \frac{\gamma^2}{2} \mathbb{E}(h(z)^2)) \sigma(dz),$$

(3)

as a limit in probability (for the topology of weak convergence) of measures $\mu_{h, \sigma, \varepsilon}$ of the form (3) but with $h$ replaced by an $\varepsilon$-regularisation $h_\varepsilon$ of $h$; furthermore $\mu_{h, \sigma, \varepsilon}(B)$ converges in $L^1$ to $\mu_{h, \sigma}$ for fixed open sets $B$. In particular, $\mathbb{E}[\mu_{\sigma, h}(B)] = \sigma(B)$. Strictly speaking, Theorem 1.1 in [5] is under the assumption that $\sigma$ is a Radon measure. But every locally finite Borel measure on $\mathbb{R}^2$ is Radon. We will not explain the details of this construction here, but we emphasize that the measure $\mu_{\sigma, h}$ is non-trivial, in the sense that $\mu_{\sigma, h}(A) \in (0, \infty) \text{ a.s.}$ and that its support is also $\mathcal{A}$. To see this, we first note that for each open set $B$ such that $\sigma(B) > 0$, from a standard 0-1 law argument based on the orthonormal decomposition of GFF (see (8)), we must have that $\mathbb{P}[^{\mu_{\sigma, h}(B)} > 0] \in \{0, 1\}$. Since $\mathbb{E}[\mu_{\sigma, h}(B)] = \sigma(B) > 0$, we have $\mu_{\sigma, h}(B) > 0$ a.s. Let $QB$ be the set of balls with rational center, rational radius, and a positive $\mu_{\sigma, h}$-mass. Then almost surely $\mu_{\sigma, h}(B) > 0$ for all $B \in QB$. Therefore, almost surely, for each $x \in \mathcal{A}$ and open set $B \ni x$, we have $\mu_{\sigma, h}(B) > 0$ because we can find $B' \in QB$ such that $B' \subset B$. Write supp($\mu_{\sigma, h}$) as the support of $\mu_{\sigma, h}$. We have $\mathcal{A} \subset \text{supp}(\mu_{\sigma, h})$ a.s. On the other hand, for each open set $B$ with $\sigma(B) = 0$, we must have $\mu_{\sigma, h}(B) = 0$ a.s. Therefore supp($\mu_{\sigma, h}$) $\subset \mathcal{A}$ a.s.

We can now formulate the question we have in mind:

**Question:** To what extent does the measure $\mu_{\sigma, h}$ determine the field $h$?

Clearly $\mu_{\sigma, h}$ can only determine the field $h$ ‘restricted to $\mathcal{A}$’ in some sense. The issue of whether the restriction of $h$ to a fractal subset $\mathcal{A}$ makes sense is itself not obvious. But if $\mathcal{A}$ is deterministic (or more generally random but independent from $h$) then there is a natural way to define the harmonic extension (to the complement of $\mathcal{A}$) of the values of $h$ on $\mathcal{A}$. (In fact this is even possible for ‘local’ set coupled with $h$ that are not independent of it, although this will not be needed in the paper, see [34].) (If a Brownian motion hits $\mathcal{A}$ with probability 0, then this extension is just the a priori expectation of $h$.) In other words, we apply the **domain Markov property** of $h$ (Theorem 1.26 in [4]) to write

$$h = \tilde{h} + h^\text{har}$$

(4)

where $\tilde{h}, h^\text{har}$ are independent, $\tilde{h}$ is a zero-boundary GFF on $D \setminus \mathcal{A}$ and $h^\text{har}$ is harmonic in $D \setminus \mathcal{A}$.

In this paper we give the following condition on $(\mathcal{A}, \sigma)$ ensuring that the measure $\mu_{\sigma, h}$ a.s. determines the harmonic extension $h^\text{har}$ of $h$ off $\mathcal{A}$. For concreteness, we assume that $\sigma$ is a deterministic finite measure, $D$ is the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{A} \subset \mathbb{D}$. In our applications to random fractals where $(\sigma, \mathcal{A})$ and $h$ are independent, we will check that the condition holds almost surely, see Remark 1.5.
We say that \((A, \sigma)\) satisfies **Property (P)** if and only if there exist positive constants \(c, r\) depending on \((A, \sigma)\) such that
\[
\min_{x \in A} \sigma(B_\varepsilon(x)) \geq ce^r \quad \forall \varepsilon \in (0, 1),
\]
where \(B_\varepsilon(z)\) is the Euclidean ball of radius \(\varepsilon\) centered at 0.

We say that \(A\) satisfies **Property (Q)** if and only if there exist positive constants \(c, q\) depending on \(A\) such that
\[
\omega(z, B_\varepsilon(x)) \leq \frac{ce^q}{|z-x|^q} \quad \forall \varepsilon \in (0, 1) \text{ and } x \in A, \ z \in \mathbb{D} \setminus A
\]
where \(\omega(z, dx)\) is the harmonic measure on \(\partial A\) viewed from \(z\); we view this harmonic measure as a measure defined on all of \(\mathbb{D}\) which is supported on \(\partial A\), so that \(\omega(z, B_\varepsilon(x)) = \omega(z, B_\varepsilon(x) \cap \partial A)\). We will comment on these properties in Section 1.4; in particular see Lemma 1.8 and Lemma 1.9 for some simple conditions guaranteeing that these properties hold together with concrete examples of interest.

**Theorem 1.4.** Let \(\sigma\) be a (deterministic) finite measure with closed support \(A \subset \mathbb{D}\) such that Properties (P) and (Q) hold. Let \(h\) be a zero boundary GFF on \(\mathbb{D}\). Given \(d \in (0, 2]\) satisfying (2) and \(\gamma \in (0, \sqrt{2d})\), the measure \(\mu_{\sigma, h}\) from (3) almost surely determines the function \(h_{\text{har}}\) from (4).

**Remark 1.5.** By conditioning, the result also holds for random sets \(A \subset \mathbb{D}\) that are independent of \(h\) such that (P) and (Q) hold almost surely, with constants \(c > 0\) that are allowed to be random. By the same reduction in the proof of Theorem 1.1, we can extend Theorem 1.4 to the cases where the field is a GFF plus a continuous function and the domain is a general open set.

Along the way, we derive general positive moments bounds for the mass of Gaussian multiplicative \(\mu_{\sigma, h}(\mathbb{D})\). Earlier works on this question (see e.g. [33]) assume either that \(\sigma\) is the Lebesgue measure or, more generally, that it has nice scaling properties. The result we present below does not make any assumption on \(\sigma\) beyond having finite \(d - \varepsilon\) energy as in (2) (in particular, properties (P) or (Q) are not assumed).

**Theorem 1.6.** Fix \(0 < d \leq d\) such that \(\mathcal{E}_d = \iint |x-y|^{-d} \sigma(dx)\sigma(dy) < \infty\) and \(0 < \gamma < \sqrt{2d}\). For each \(\alpha \in (1, \frac{2d}{\gamma} \wedge 2)\), there exists a constant \(c = c(d, \gamma, \alpha)\) such that
\[
\mathbb{E}[\mu_{\sigma, h}(\mathbb{D})^\alpha] \leq c\sigma(\mathbb{D})^{3-\alpha} (\mathcal{E}_d)^{\alpha-1} < \infty.
\]

Theorem 1.6 will be proved as Theorem 3.1 in Section 3. Bounds on \(\mathbb{E}[\mu_{\sigma, h}(\mathbb{D})^\alpha]\) for negative \(\alpha\) are also given in [15]. However, in contrast to Theorem 1.6, the estimate in [15] is only made explicit in terms of \(\sigma(\mathbb{D})\) and \(\mathcal{E}_d\) when \(\gamma \in (0, \sqrt{d})\), namely, the \(L^2\) regime; see [15, Corollary 3.2]. In Section 3 we use Theorem 1.6 to give an estimate on negative moments with an explicit dependence on \(\sigma(\mathbb{D})\) for all \(\gamma \in (0, \sqrt{d})\), which will be crucial to our proof of Theorem 1.4.

**Theorem 1.7.** Suppose \(\sigma(\mathbb{D}) \leq 1\). There exists \(\ell = \ell(\gamma, d) > 0\) such that the following holds. For any \(s \in (0, \ell)\) and \(\mathcal{E}' \geq \mathcal{E}_d\), there exists a constant \(c = c(\mathcal{E}', s, d, \gamma)\) such that
\[
\mathbb{E}[\mu_h(\mathbb{D})^{-s}] \leq c(\mathcal{E}', s, d, \gamma)\sigma(\mathbb{D})^{-s/\ell} < \infty.
\]

Theorem 1.7 will be proved as Theorem 3.6 in Section 3.
1.4 Examples

Property (P) is a quantification of the statement that $\sigma$ is sufficiently spread out on $\mathcal{A}$. It is rather weak since $r$ is allowed to be arbitrarily large. For example, it is satisfied by occupation measures of Hölder continuous curves.

**Lemma 1.8.** Suppose $\eta : [0,1] \to \mathbb{D}$ is a Hölder continuous curve. Namely, there exist positive constants $C, \alpha$ such that $|\eta(s) - \eta(t)| \leq C |s - t|^\alpha$ for $s, t \in [0,1]$. Let $\mathcal{A} = \eta([0,1])$. Let $\sigma$ be the occupation measure of $\eta$, which means for each Borel set $B \subset \mathbb{D}$, $\sigma(B)$ measures the amount of time $\eta$ spends in $B$. Then $(\mathcal{A}, \sigma)$ satisfies Property (P) with $q = \frac{1}{\alpha}$.

*Proof.* For $x \in \mathcal{A}$, suppose $\eta(t) = x$ for some $t \in [0,1]$. Then $\eta(t') \in B_\varepsilon(x)$ if $C|t - t'|^\alpha < \varepsilon$. Therefore $\sigma(B_\varepsilon(x)) > (\varepsilon/C)^{1/\alpha}$. □

The following sufficient condition for Property (Q) is a direct consequence of Beurling’s estimate (see [23] Equation (3.17)).

**Lemma 1.9.** Property (Q) is satisfied for $\mathcal{A}$ with $q = \frac{1}{2}$ as long as $\mathcal{A}$ is connected.

**Liouville Brownian motion.** Let $\mathcal{B} = (\mathcal{B}_t, t \leq T_D)$ be a (standard) Brownian motion starting from 0, run until it leaves $D$, where $D$ is a domain such that $0 \in D$ and $D \cup \partial D \subset \mathbb{D}$. Let $\sigma$ denote the corresponding occupation measure: as above, for a Borel set $B \subset D$, $\sigma(B) = \int_0^{T_D} 1_{\{\mathcal{B}_s \in B\}} ds$. Its closed support $\mathcal{A}$ is the range $\{\mathcal{B}_t, t \leq T_D\}$ of $\mathcal{B}$. It is well known that the dimension of planar Brownian motion is almost surely 2 and so condition (2) is a.s. satisfied by $\sigma$ with $d = 2$. Furthermore, by Lemmas 1.8 and 1.9, both Properties (P) and (Q) are satisfied by $(\sigma, \mathcal{A})$ almost surely, so that Theorem 1.4 applies.

For $\gamma \in (0, 2)$, let $h$ be a zero-boundary GFF on $\mathbb{D}$, independent of $\mathcal{B}$. Following [3, 16], the following limit makes sense in probability and defines an increasing process $(\phi(t), 0 \leq t \leq T_D)$, the quantum clock:

$$\phi(t) := \lim_{\varepsilon \to 0} e^{\gamma \varepsilon^2/2} \int_0^t e^{\gamma h_\varepsilon(B_s)} ds.$$  

By definition, Liouville Brownian motion (LBM) is the reparametrization of $\mathcal{B}$ by its quantum clock: namely,

$$Z_t := \mathcal{B}(\phi^{-1}(t)); \quad \text{for } t \leq \tau_D := \phi(T_D).$$

The quantum clock can be rewritten in terms of the Gaussian multiplicative chaos $\mu_{\sigma,h}$ associated to $h$ and $\sigma$ in the sense of (3) as follows. For any $t \leq T_D$,

$$\phi(t) = \int_D 1_{\{x \in \mathcal{B}[0,t]\}} d\mu_{\sigma,h}(x),$$

namely, $\phi(t)$ is the $\mu_{\sigma,h}$-mass of $\mathcal{B}[0,t]$. Note that $(\mathcal{B}[0,t], t \leq T_D)$ is measurable with respect to $(Z_t, t \leq \tau_D)$: indeed it suffices to reparametrize $Z$ by its quadratic variation. Note that $T_D$ is the Euclidean exit time of $D$ but $\tau_D$ is the exit time of $D$ measured in the quantum clock of Liouville Brownian motion. Note also that in particular, $(\phi(t), 0 \leq t \leq T_D)$ is a measurable function of $(Z_t, t \leq \tau_D)$. As a consequence, we see that $\mu_{\sigma,h}$ is a measurable function of $(Z_t, t \leq \tau_D)$. To see
this, for \( \varepsilon > 0 \), we set \( \phi_\varepsilon(t) := \varepsilon^{\gamma/2} \int_0^t e^{\gamma h_\varepsilon(B_s)} ds \). Then \( \phi_\varepsilon \) converge to \( \phi \) a.s. in the uniform topology ([3, Theorem 1.2]). Set \( \mu^\varepsilon_{\sigma,h}(dx) = \varepsilon^{\gamma/2} e^{\gamma h_\varepsilon(x)} \sigma(dx) \). Then for a Borel set \( B \subset D \) we have

\[
\mu^\varepsilon_{\sigma,h}(B) = \varepsilon^{\gamma/2} \int_0^{\tau_D} e^{\gamma h_\varepsilon(B_s)} 1_{B_\varepsilon \in B} ds = \int_0^{\tau_D} 1_{B_\varepsilon \in B} d\phi_\varepsilon(s).
\]

Sending \( \varepsilon \to 0 \), we have \( \mu_{\sigma,h}(B) = \int_0^{\tau_D} 1_{B_\varepsilon \in B} d\phi(s) \), which gives the desired measurability for \( \mu_{\sigma,h} \).

Applying Theorem 1.4, we deduce that the harmonic extension of \( h \) off the range of \( B_{[0,T_D]} \) is a.s. determined by \( (Z_t, 0 \leq t \leq \tau_D) \).

**SLE.** Let \( \kappa < 8 \) and let \( \eta \) denote a chordal SLE\(_\kappa\) from 0 to \( \infty \) in the upper half plane \( \mathbb{H} \). It is shown in [24] that an \( \eta \) a.s. has nontrivial \( d \)-dimensional Minkowski content with \( d = (1 + \frac{\kappa}{8}) \wedge 2 \), which defines the so-called natural measure on the curve. It is proved in [24] that SLE is Hölder continuous under its natural parametrization, for any \( \kappa \in (0, 8) \). (The optimal Hölder exponent is achieved in [39].) By Lemmas 1.8 and 1.9, Property (P) holds almost surely for the occupation measure \( \sigma \) of SLE under this natural parametrization, and Property (Q) holds for an SLE curve segment.

For \( \kappa \in (0, 4) \), it was proved by [2] that if \( h \) is an independent Neumann GFF (with an additional logarithmic singularity at the origin), then the corresponding GMC measure with parameter \( \gamma/2 \) with respect to the reference measure \( \sigma \), \( e^{\frac{2}{\gamma} \sigma} \) is identical to the quantum length measure of SLE\(_\kappa\), as defined in [37], where \( \gamma = \sqrt{\kappa} \). (The intricacy of [2] lies in that it does not assume the existence of Minkowski content from [24]. Instead, it shows that if one takes the conditional expectation of the quantum length given the SLE curve, this defines a measure over the range of \( \eta \) which can be shown to satisfy the set of axioms in [25], which characterize the natural parametrization uniquely: hence this conditional expectation is \( \sigma \). In any case, as a consequence of this result, by Theorem 1.4, the quantum length of \( \eta \) determines the restriction of \( h \) to \( \eta \) (or, more precisely, the harmonic extension \( h^{\text{har}} \) off of \( \eta \)).

In fact, for \( \kappa \in (4, 8) \), the counterpart of [2] should still hold, with the quantum length replaced by the so-called natural time considered in [12], and the factor \( \frac{1}{2} \) in the GMC adjusted according to the KPZ relation.

**Remark 1.10.** We expect that Property (Q) holds for Cantor-like random fractals such as cut points of Brownian motion, cut points and double points of SLE\(_\kappa\) with \( \kappa \in (4, 8) \). Moreover, Property (P) holds for natural measures on them defined via the Minkowski content.

**Notations**

The following notations are used throughout this paper. For \( r > 0 \), \( z \in \mathbb{C} \), we let \( B_r(z) \subset \mathbb{C} \) be the ball of radius \( r \) centered at \( z \) and \( \mathbb{D} = B_1(0) \), \( r\mathbb{D} = B_r(0) \). The symbol \( \varepsilon \) always represents a small enough positive number (e.g. \( \varepsilon \in (0, 1/100) \)). Suppose \( h \) is a variant of GFF, let \( h_\varepsilon(z) \) be \( h \) averaged along \( \partial B_\varepsilon(z) \). We let \( C \) be constants arising in the argument that can vary line by line. If a function \( \psi \) is such that \( \psi(\varepsilon) = o(\varepsilon^{-p}) \) for all \( p > 0 \), we call \( \psi \) a sub-polynomial function. The main example of sub-polynomial functions are powers of \( |\log \varepsilon| \).
Outline of the paper

In Section 2.1, we provide some background on the Gaussian free field and prove a useful preliminary estimate. In Section 2 we give the proof of Theorem 1.1. Although it will be a consequence of Theorem 1.4, the proof is fairly elementary and contains the high level ideas of the more general case. In Section 4; we prove Theorem 1.4.

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2 Proof of Theorem 1.1: the Lebesgue case

2.1 Preliminaries

We start by briefly recalling some background on the Gaussian free field (GFF) with zero (sometimes also known as Dirichlet) boundary conditions, before stating a useful estimate which will be used repeatedly in this paper.

Let $D$ be a domain in $\mathbb{C}$ with harmonically nontrivial boundary (i.e. the harmonic measure of $\partial D$ is positive as seen from any point in $D$). We denote by $H_0(D)$ the Hilbert-space closure of $C_0^\infty(D)$ (the space of compactly supported smooth functions in $D$), equipped with the Dirichlet inner product

$$ (f, g)_D = \frac{1}{2\pi} \int_D \nabla f(z) \cdot \nabla g(z) \, dz. \quad (7) $$

A zero boundary Gaussian free field on $D$ is given by the formal sum

$$ h = \sum_{n=1}^\infty \alpha_n f_n, \quad (\alpha_n) \text{ i.i.d } N(0, 1) \quad (8) $$

where $\{f_n\}$ is an orthonormal basis for $H_0(D)$. Although this expansion of $h$ does not converge in $H_0(D)$, it can be shown that the convergence holds almost surely in the space of distributions. See [36, 4] for more details. Throughout this section recall that we denote by $\mu_h$ the Liouville measure defined in (1), this also corresponds to a choice of $\sigma$ being absolutely continuous with respect to Lebesgue measure.
The **domain Markov property** of the GFF (see Theorem 1.27 in [4]) states that given an open subdomain $U \subset D$ and a zero boundary GFF $h$ on $D$, we can write

$$h = h^{\text{har}} + h^{\text{supp}}$$

(9)

where $h^{\text{har}}, h^{\text{supp}}$ are independent; $h^{\text{har}}$ restricted to $U$ is almost surely a harmonic function in $U$; and $h^{\text{supp}}$ is a zero boundary GFF on $U$ extended to be zero on $D \setminus U$. Recall also that $h$ is **conformally invariant** (i.e., its law is invariant under conformal transformations of the complex plane – see Theorem 1.30 in [4]). In particular, $h$ is scale-invariant.

We record a lemma which will be used frequently; this gives a quantitative control on the fluctuations of the harmonic part in a ball of radius $\varepsilon$ arising in the Markov decomposition (9) when $U \subset D$ is taken to be a ball of radius $r > 4\varepsilon$.

**Lemma 2.1.** Let $D$ be a simply connected domain, and let $h$ be a zero boundary GFF on $D$. Suppose that $z \in D$ and $r > 0$ is such that $U = B_r(z) \subset D$. Let $h^{\text{har}}$ denote the harmonic part in the Markov decomposition (9) associated to $U$. For $\varepsilon < r/4$, let

$$\Delta_\varepsilon(z) = \max_{x \in B_{\varepsilon}(z)} h^{\text{har}}(x) - \min_{x \in B_{\varepsilon}(z)} h^{\text{har}}(x).$$

Then $E[\Delta_\varepsilon^2(z)] \leq C(\varepsilon/r)^2$ where $C$ is a universal constant independent of $\varepsilon, r, z, D$.

**Proof.** By translation and scaling invariance of the GFF, it suffices to prove the lemma in the case $z = 0, r = 1$. We first control the gradient of $h^{\text{har}}$ in $B_{1/4}(0)$. In the proof of [14, Lemma 4.5], the authors show that the minimum of $h^{\text{har}}$ in $B_{1/2}(0)$ has a super exponential tail which is independent of the domain $D$ containing the unit disk. (In fact, we point out that a simpler proof of that lemma can be obtained using the Borell–Tsirelson inequality for Gaussian processes). The same is true for the maximum of $h^{\text{har}}$. In particular, the second moment of $\|h^{\text{har}}\|_{\infty, B_{1/2}(0)}$ is bounded by a universal constant $C$. By a standard gradient estimate for harmonic functions (see e.g. Lemma 2.8 in [6]), $\|\nabla h^{\text{har}}\|_{\infty, B_{1/4}(0)} \leq C\|h^{\text{har}}\|_{\infty, B_{1/2}(0)}$. So for $\varepsilon < 1/4$, we have

$$E[\Delta_\varepsilon^2(z)] \leq C\varepsilon^2E[\|\nabla h^{\text{har}}\|_{\infty, B_{1/4}(0)}^2] \leq C\varepsilon^2,$$

as desired. \qed

We will prove Theorem 1.1 by taking the logarithm of the volume of a ball of radius $\varepsilon$ (rescaled by $1/\gamma$) and showing that this does not deviate much from the circle average $h_\varepsilon$ of the field at scale $\varepsilon$ (up to a deterministic term). In other words, let us define $h^\varepsilon$ by setting

$$e^{\gamma h^\varepsilon(z)} = \mu_h(B_z(\varepsilon)), \quad \text{i.e.,} \quad h^\varepsilon(z) = \gamma^{-1} \log \mu_h(B_z(\varepsilon)).$$

(10)

Roughly speaking, we will show that $h^\varepsilon - E[h^\varepsilon]$ converges to $h$ in probability as $\varepsilon \to 0$ when we integrate against a test function, see Proposition 2.4 for a precise statement. Since $h^\varepsilon$ is determined by $\mu_h$, so is $h$. We will achieve this via the following two lemmas which will be proved in Sections 2.2 and 2.3.
Lemma 2.2 (Variance estimate). Suppose $D$ is a simply connected domain and $D' = \{ x \in D | \text{dist}(x, \partial D) > \varepsilon_0 \}$, where $\varepsilon_0$ is a fixed constant. Suppose $h$ is the zero boundary GFF on $D$ and $h_\varepsilon$ is as in (10). For all $z \in D'$, $0 < \varepsilon < \frac{\varepsilon_0}{4}$, let $k_\varepsilon(z) = h_\varepsilon(z) - h(z)$. Then we have

$$\text{Var}[k_\varepsilon(z)] \leq C \log(\varepsilon_0/\varepsilon),$$

where $C$ is a universal constant independent of $D, \varepsilon_0$ and $z$.

Lemma 2.3 (Covariance estimate). Let $h$ be the zero boundary GFF on $D$ and $k_\varepsilon$ be defined as in Lemma 2.2 for $D = \mathbb{D}$. Then for $x_1, x_2 \in \mathbb{R}D$ and $|x_1 - x_2| > \varepsilon^{1/2}$,

$$\text{Cov}[k_\varepsilon(x_1), k_\varepsilon(x_2)] \leq C \varepsilon \log^{1/2} \left( \frac{|x_1 - x_2|}{\varepsilon} \right)$$

where $C_r$ only depends on $r \in (0, 1)$.

Given Lemma 2.2 and Lemma 2.3, we can get Theorem 1.1 in the case where $h$ is the zero boundary GFF on $\mathbb{D}$.

Proposition 2.4. If $h$ is a zero boundary GFF on $\mathbb{D}$ and $0 < \gamma < 2$, then $\mu_h$ determines $h$ almost surely.

Proof. Suppose $\rho$ is a smooth function supported on $r\mathbb{D}$ for some $r < 1$. It is sufficient to show that $(h, \rho)$ is measurable with respect to $\mu_h$. We start by noting that

$$\text{Var}[(k_\varepsilon, \rho)] = \int_{\mathbb{D} \times \mathbb{D}} dx dy \text{Cov}[k_\varepsilon(x), k_\varepsilon(y)] \rho(x) \rho(y)$$

$$= \int_{\{|x-y| > \varepsilon^{1/2}, x, y \in r\mathbb{D}\}} dx dy \text{Cov}[k_\varepsilon(x), k_\varepsilon(y)] \rho(x) \rho(y) + \int_{\{|x-y| < \varepsilon^{1/2}\}} dx dy \text{Cov}[k_\varepsilon(x), k_\varepsilon(y)] \rho(x) \rho(y).$$

By Lemma 2.3

$$1_{\{|x-y| > \varepsilon^{1/2}, x, y \in r\mathbb{D}\}} \text{Cov}[k_\varepsilon(x), k_\varepsilon(y)] \leq C_r \varepsilon^{1/2} \log^{1/2}(\varepsilon^{-1}).$$

Therefore

$$\lim_{\varepsilon \to 0} \int_{\mathbb{D} \times \mathbb{D}} dx dy 1_{\{|x-y| > \varepsilon^{1/2}, x, y \in r\mathbb{D}\}} \text{Cov}[k_\varepsilon(x), k_\varepsilon(y)] \rho(x) \rho(y) = 0.$$ 

On the other hand, by the Cauchy–Schwarz inequality and Lemma 2.2 (applied with $\varepsilon_0 = r$),

$$\int_{\mathbb{D} \times \mathbb{D}} dx dy 1_{\{|x-y| < \varepsilon^{1/2}\}} \text{Cov}[k_\varepsilon(x), k_\varepsilon(y)] \rho(x) \rho(y) \leq C \varepsilon \log \frac{r}{\varepsilon}.$$ 

Therefore $\lim_{\varepsilon \to 0} \text{Var}[(k_\varepsilon, \rho)] = 0$. Note that $\mathbb{E}(h_\varepsilon(\rho)) = 0$. Hence

$$\lim_{\varepsilon \to 0} (h_\varepsilon, \rho) - \mathbb{E}[(h_\varepsilon, \rho)] - (h_\varepsilon, \rho) = 0 \text{ in } L^2.$$  \hspace{1cm} (11)

To conclude, it remains to recall the standard fact that $h_\varepsilon$ approximates $h$ in $L^2$ (when tested against $\rho$).
Lemma 2.5. As $\varepsilon \to 0$, $(h_\varepsilon - h, \rho)$ converges to 0 in $L^2(\mathbb{P})$.

Proof. Since $(h_\varepsilon - h, \rho)$ is Gaussian, the result follows at once from the fact that $\text{Cov}(h_\varepsilon(x), h_\varepsilon(y))$ converges to $G(x, y)$ uniformly over compact subsets of $\mathbb{D} \times \mathbb{D}$ away from the diagonal, and from the bound
\[
\text{Cov}(h_\varepsilon(x), h_\varepsilon(y)) \leq -\log(|x - y| \vee \varepsilon) + O(1)
\] (see, e.g., Lemma 3.5 of [5]).

Combining (11) and Lemma 2.5 we get that $(h_\varepsilon, \rho) - \mathbb{E}[(h_\varepsilon, \rho)]$ tends to $(h, \rho)$ in $L^2$. This implies that the random variable $(h, \rho)$ is measurable with respect to $\mu_h$.

So far, we have proved that for any smooth function $\rho$ supported on $\mathbb{D}$, $(h, \rho)$ is measurable with respect to $\mu_h$. This yields that $h$ is determined by $\mu_h$ almost surely. \(\square\)

With this in hand it is not hard to get a proof of Theorem 1.1.

Proof of Theorem 1.1. We first assume $D = \mathbb{D}$ and write $h$ as $h_0 + g$ where $h_0$ is an instance of a zero boundary GFF on $\mathbb{D}$ and $g$ is a random continuous function as in Theorem 1.1. Let $\mu_{h_0}$ be the Liouville quantum measure of $h_0$. Define $h_\varepsilon$ and $h_0^\varepsilon$ by
\[
e^{-\varepsilon h(x)} = \mu_h(B_\varepsilon(x)) \quad \text{and} \quad e^{-\varepsilon h_0^\varepsilon(x)} = \mu_{h_0}(B_\varepsilon(x)).
\]
By the intermediate value theorem, for all $x$, there is a $\xi_x$ (which also depends on $\varepsilon$) such that
\[
\mu_h(B_\varepsilon(x)) = \exp\{\gamma g(\xi_x)\}\mu_{h_0}(B_\varepsilon(x)), \quad |\xi_x - x| \leq \varepsilon.
\] (13)
Let $g^\varepsilon(x) = g(\xi_x)$. By taking the logarithm on either side of (13), we have $h_\varepsilon = h_0^\varepsilon + g^\varepsilon$. Let $h_\varepsilon, h_0^\varepsilon, g^\varepsilon$ be the circle average process of $h, h_0, g$ respectively. Then $\forall \rho \in C^\infty_c(D),
\[
(h_\varepsilon, \rho) - (h_0^\varepsilon, \rho) - (h_0, \rho) + (g^\varepsilon - g^\varepsilon, \rho).
\] (14)
By the argument in Proposition 2.4, $(h_0^\varepsilon, \rho) - (h_0, \rho) - \mathbb{E}[(h_0^\varepsilon, \rho)]$ tends to 0 in $L^2$ as $\varepsilon$ tends to 0. Let $\omega_g$ be the modulus of continuity of $g$:
\[
\omega_g(x, \varepsilon) = \max\{|g(x) - g(y)| : |y - x| \leq \varepsilon\}, \quad \forall x \in D, \varepsilon < \text{dist}(x, \partial D).
\] (15)
Since $g$ is a continuous function,
\[
\lim_{\varepsilon \to 0} \omega_g(x, \varepsilon, \rho(x)) = 0 \text{ a.s.} \quad \forall \rho \in C^\infty_c(D).
\] (16)
By (16), $(g^\varepsilon - g^\varepsilon, \rho)$ tends to 0 a.s. as $\varepsilon$ tends to 0. Hence
\[
\lim_{\varepsilon \to 0} (h_\varepsilon, \rho) - \mathbb{E}[(h_0^\varepsilon, \rho)] = (h, \rho) \text{ in probability.}
\]
As the same argument at the end of the proof of Proposition 2.4, we conclude the proof of Theorem 1.1 for $D = \mathbb{D}$. By translation and scaling we get Theorem 1.1 if $D$ is a ball in $\mathbb{C}$.

For a general domain $D$, by the domain Markov property the field $h$ restricted to any ball $B \subset D$ can be written as a zero boundary GFF on $B$ plus a continuous function. Therefore $h_{|B}$ is determined by $\mu_h$ almost surely. Varying $B$ we obtain Theorem 1.1. \(\square\)
2.2 The variance estimate

Proof of Lemma 2.2. Consider first the disk $U = B_{\xi_0/2}(z)$, and apply the Markov decomposition (9) to write $h = h^{\text{supp}}_0 + h^{\text{har}}_0$ as in that statement, where $h^{\text{supp}}_0$ is a Dirichlet GFF on $U$ and $h^{\text{har}}_0$, restricted to $U$, is harmonic in $U$ (and coincides with $h$ on $D \setminus U$).

As in (13), by the intermediate value theorem let $\xi_0$ be a point in $B_\varepsilon(z)$ such that

$$
\mu_h(B_\varepsilon(z)) = e^{\gamma h^{\text{har}}_0(\xi_0)} \mu_{h^{\text{supp}}_0}(B_\varepsilon(z)).
$$

Since $h^{\text{har}}_0$ has the mean value property, the circle average of this function is equal to its value at the center of the circle, so we have

$$
k_\varepsilon(z) = [h^{\text{har}}_0(\xi_0) - h^{\text{har}}_0(z)] + [\gamma^{-1} \log \mu_{h^{\text{supp}}_0}(B_\varepsilon(z)) - h^{\text{supp}}_0(z)].
$$

(17)

By Lemma 2.1,

$$
\text{Var}[h^{\text{har}}_0(\xi_0) - h^{\text{har}}_0(z)] \leq \mathbb{E}[(h^{\text{har}}_0(\xi_0) - h^{\text{har}}_0(z))^2] \\
\leq \mathbb{E}(\Delta_\varepsilon(z)^2) \\
\leq C(\varepsilon/\varepsilon_0)^2
$$

Applying the scaling and translation $x \mapsto 2 \cdot \frac{x - z}{\varepsilon_0}$ to $h^{\text{supp}}_0$ in (17) (recall the conformal invariance of the Dirichlet GFF), in order to prove Lemma 2.2 it suffices (by Cauchy–Schwarz) to show

$$
\text{Var}(I) \leq C|\log \varepsilon|.
$$

(18)

where

$$
I = \frac{1}{\gamma} \log \mu_h(B_\varepsilon(0)) - h_\varepsilon(0)
$$

and $h$ is the zero boundary GFF on $\mathbb{D}$.

We now start proving (18). We assume without loss of generality $\varepsilon = 2^{-n}$ for some integer $n$, and recursively decompose $I$ as follows. Let $h^0 = h$. For $0 \leq k < n - 1$, let $U_k = B_{2^{-k}}(0)$, and apply inductively the domain Markov property onto each $U_k$. That is, having defined $h^1, \ldots, h^k$ as Dirichlet GFFs on $U_1, \ldots, U_k$ respectively, and $h^{\text{har}}_1, \ldots, h^{\text{har}}_k$ whose respective restrictions to $U_1, \ldots, U_k$ are harmonic functions in the relevant domains, we apply the domain Markov property to $h^k$ and the subdomain $U_{k+1}$ to get a Dirichlet GFF $h^{k+1}$ on $U_{k+1}$ and an independent field $h^{\text{har}}_{k+1}$ which is harmonic in $U_{k+1}$ such that

$$
h^k = h^{k+1} + h^{\text{har}}_{k+1}.
$$

In this way, for each $k \geq 1$, $h^{k+1}$ and $h^{\text{har}}_{k+1}$ are independent and more generally, $h^{k+1}$ is independent from the collection of random variables $h^{\text{har}}_1, \ldots, h^{\text{har}}_k$. Furthermore,

$$
h = \sum_{i=1}^{n-1} h^{\text{har}}_i + h^{n-1}
$$

and so taking the circle average at 0, by the mean value property of harmonic functions, we have

$$
h_\varepsilon(0) = \sum_{i=1}^{n-1} h^{\text{har}}_i(0) + h^{n-1}_\varepsilon(0).
$$

(19)
Let $\xi_{k+1}$ be a point in $B_\varepsilon(z)$ (which may be chosen in a measurable way with respect to $h^k$) such that
\[ \mu_{h^k}(B_\varepsilon(0)) = e^\gamma h_{k+1}(\xi_{k+1}) \mu_{h^{k+1}}(B_\varepsilon(0)), \]
whence
\[ \frac{1}{\gamma} \log \mu_h(B_\varepsilon(0)) = \sum_{i=1}^{n-1} h_{\text{har}}(\xi_i) + \frac{1}{\gamma} \log \mu_{h^{n-1}}(B_\varepsilon(0)). \tag{20} \]
Combining (19) and (20), we can write
\[ I = \sum_{k=1}^{n-1} \Delta_k + R, \tag{21} \]
where
\[ \Delta_k = h_{k+1}(\xi_k) - h_{k+1}(0) \quad \text{and} \quad R = \frac{1}{\gamma} \log \mu_{h^{n-1}}(B_\varepsilon(0)) - h_{n-1}(0). \]
The random variables $R$ and $\sum_{k=1}^{n-1} \Delta_k$ are not independent in general (because $\xi_k$ depends a priori on all of $h_k$). Nevertheless, by the Cauchy–Schwarz inequality,
\[ \text{Var}(I) \leq 2 \text{Var}\left(\sum_{k=1}^{n-1} \Delta_k\right) + 2 \text{Var}(R). \tag{22} \]
Mapping $B_{2^{-n+1}}(0) = B_{2\varepsilon}(0)$ to $\mathbb{D}$ by scaling and applying the LQG change of coordinates (see [14, Proposition 2.1]), we have
\[ R \overset{d}{=} \frac{1}{\gamma} \log \mu_h(B_{1/2}(0)) - h_{1/2}(0) - \frac{1}{\gamma} Q \log 2\varepsilon, \tag{23} \]
where $h$ is the zero boundary GFF on $\mathbb{D}$, and $h_{1/2}(0)$ refers to the circle average of $\partial B_{1/2}(0)$. Since $\mu_h(B_{1/2}(0))$ has moments of positive and negative orders (see Theorems 2.11 and 2.12 in [33] or Theorems 3.27 and 3.37 in [4]), we deduce that its logarithmic moments are also finite. Therefore $\text{Var}[R]$ is a constant independent of $\varepsilon$.

Furthermore, by Lemma 2.1 and the definition of $\Delta_k$, $\mathbb{E}[\Delta_k^2] \leq C 4^{-k-n}$ where $C$ is independent of $\varepsilon$. Thus, again by the Cauchy–Schwarz inequality,
\[ \text{Var}(\sum_{k=1}^{n-1} \Delta_k) \leq \mathbb{E}[\left(\sum_{k=1}^{n-1} \Delta_k\right)^2] \leq n \mathbb{E}\left[\sum_{k=1}^{n-1} \Delta_k^2\right] \leq C \log^{-1} \varepsilon. \]
Together with (22), this proves (18) and hence Lemma 2.2. \hfill \square

### 2.3 The covariance estimate

**Proof of Lemma 2.3.** For fixed $x_1$ and $x_2$ in $r\mathbb{D}$, let $L$ be the line segment orthogonally bisecting the line segment $[x_1, x_2]$. Let $U_i(i=1, 2)$ be the connected component of $\mathbb{D} \setminus L$ containing $x_i$. Apply the domain Markov property in $U_1$ and $U_2$ to write
\[ h = h_{\text{har}} + h^1 + h^2 \]
where $h_{\text{har}}$ is harmonic on $U_1$ and $U_2$, $h^1$ and $h^2$ are GFF (with Dirichlet boundary conditions) in $U_1$ and $U_2$ respectively, and all three terms are mutually independent. Then for $|x_1 - x_2| \geq \varepsilon^{1/2} > 2\varepsilon$, the mean value property of $h_{\text{har}}$ gives

$$k_\varepsilon(x_i) = \gamma^{-1} \log \int_{B_\varepsilon(x_i)} e^{\gamma h_{\text{har}}(z)} d\mu_{h^1}(z) - h_{\text{har}}^1(x_i) - h_{\varepsilon}^1(x_i).$$

As in the proof of Lemma 2.2, let $\xi_i$ be a point in $B_\varepsilon(x_i)$ (chosen in a measurable way with respect to $h^1$ and $h_{\text{har}}$) such that

$$
\int_{B_\varepsilon(x_i)} e^{\gamma h_{\text{har}}(z)} d\mu_{h^1} = e^{\gamma h_{\text{har}}(\xi_i)} \mu_{h^1}(B_\varepsilon(x_i)).
$$

Let

$$\Gamma_\varepsilon(x_i) = \gamma^{-1} \log \mu_{h^1}(B_\varepsilon(x_i)) - h_{\varepsilon}^1(x_i) \quad \text{and} \quad \Delta_i = h_{\text{har}}(\xi_i) - h_{\text{har}}^1(x_i).$$

Then $k_\varepsilon(x_i) = \Gamma_\varepsilon(x_i) + \Delta_i$. Therefore

$$\text{Cov}[k_\varepsilon(x_1), k_\varepsilon(x_2)] = \text{Cov}[\Gamma_\varepsilon(x_1), \Gamma_\varepsilon(x_2)] + \text{Cov}[\Gamma_\varepsilon(x_1), \Delta_2] + \text{Cov}[\Gamma_\varepsilon(x_2), \Delta_1] + \text{Cov}[\Delta_1, \Delta_2].$$

By Lemma 2.2, $\text{Var}[\Gamma_\varepsilon(x_1)] \leq C_\tau \log \frac{|x_1 - x_2|}{\varepsilon}$. By Lemma 2.1, $\text{Var}[\Delta_i] \leq C \frac{\varepsilon^2}{|x_2 - x_1|^2}$. $\Gamma_\varepsilon(x_1)$ and $\Gamma_\varepsilon(x_2)$ are independent, which means $\text{Cov}[\Gamma_\varepsilon(x_1), \Gamma_\varepsilon(x_2)] = 0$. Therefore, by Cauchy–Schwarz inequality we have,

$$\text{Cov}[k_\varepsilon(x_1), k_\varepsilon(x_2)] \leq (\text{Var}[\Gamma_{\varepsilon}(x_1)] \text{Var}[\Delta_2])^{1/2} + (\text{Var}[\Gamma_{\varepsilon}(x_2)] \text{Var}[\Delta_1])^{1/2} + (\text{Var}[\Delta_1] \text{Var}[\Delta_2])^{1/2}$$

$$\leq C_\tau \frac{\varepsilon}{|x_2 - x_1|} \log^{1/2} \frac{|x_1 - x_2|}{\varepsilon}.$$

This concludes the proof of Lemma 2.3, and with it the proof of Theorem 1.1. \qed

3 Moments on GMC over fractals

As should be clear from the previous section, getting bounds (both positive and negative moments) for Gaussian multiplicative chaos with respect to the reference measure $\sigma$ supported on the (typically fractal) set $A$ is a key part of the argument (e.g. in the proof of Lemma 2.2). In the Lebesgue case, such moments are usually derived from scale invariance considerations and Kahane’s inequality. Clearly, such arguments are not directly applicable when the measure $\sigma$ is not itself assumed to be scale-invariant in any reasonable sense. We hence need to develop arguments to control these moments. We believe these results are of independent interest.

In this section, we let $h$ be the zero boundary GFF on $D$. We suppose that $D$ is bounded; after Lemma 3.4 we will take $D = \mathbb{D}$. Let $\sigma$ satisfy (2) and $\sigma(D) < \infty$. Let $0 \leq \gamma < \sqrt{2d}$ (so we only consider the subcritical case). Fix a $d \in (\gamma^2/2, d)$ such that $\gamma < \sqrt{2d}$ and

$$E_d := \int_{D^2} |x - y|^{-d} \sigma(dx)\sigma(dy) < \infty. \quad (24)$$

Let $\mu_h = \mu_{\sigma,h}$ be defined as in (3) for this parameter $\gamma$. 

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**Theorem 3.1.** For each $\alpha \in (1, \frac{2d}{\gamma} \wedge 2)$, there exists a constant $c = c(d, \gamma, \alpha)$ such that
\[
\mathbb{E}[\mu_h(D)^\alpha] \leq \sigma(D)^{3-\alpha}(\mathcal{E}_d)^{\alpha-1} < \infty.
\]

**Proof.** Let $\bar{x}, \bar{y}$ be two i.i.d. samples from $\sigma(\cdot \cap D)/\sigma(D)$. Note that $\mathcal{E}_d = \sigma(D)^2 \mathbb{E}(|\bar{x} - \bar{y}|^{-d})$.

Set $\delta = \alpha - 1 \in (0, 1)$, and write
\[
\mathbb{E}[\mu_h(D)^\alpha] = \mathbb{E}[\mu_h(D)\mu_h(D)^\delta] = \mathbb{E}[\mu_h(D)] \mathbb{E}^*[\mu_h(D)^\delta] = \sigma(D) \mathbb{E}^*[\mu_h(D)^\delta]
\]
where $\mathbb{P}^*$ denotes the law of the field biased by $\mu_h(D)$. Using Girsanov’s theorem (see Theorem 17 in [35] and Theorem 3.15 in [4]), we can rewrite this as
\[
\mathbb{E}^*[\mu_h(D)^\delta] = \int_D \sigma(dx) \mathbb{E}(\left(\int_D e^{\gamma G(x,y)} \mu_h(dy)\right)^\delta)
\]
where $G(x,y) = G_D(x,y)$ is the Green function in $D$ with Dirichlet (zero) boundary conditions, normalized so that
\[
G(x,y) = -\log |x - y| + O(1) \text{ as } y \to x.
\]

For any integer $n \geq 0$, let $A_n(x)$ denote the annulus at distance $2^{-n}$ from $x$, i.e., $A_n(x) = \{y : |y - x| \in [2^{-n-1}, 2^{-n})\}$. Then using (25) and the fact (coming from concavity of the function $t > 0 \mapsto t^\delta$) that $(a_1 + \ldots + a_n)\delta \leq a_1^\delta + \ldots + a_n^\delta$ for any $0 < \delta < 1$ (i.e., $\alpha < 2$ as per our assumptions) and $a_i > 0$, we see that for some constant $C$ depending on $\delta$ and $\gamma$,
\[
\mathbb{E}^*[\mu_h(D)^\delta] \leq C \sum_n \int_D \sigma(dx) \mathbb{E}(\left(\int_{A_n(x)} e^{-\gamma^2 \log |x - y|} \mu_h(dy)\right)^\delta)
\]
and
\[
\leq C \sum_n 2^{n\gamma^2\delta} \int_D \sigma(dx) \mathbb{E}(\mu_h(A_n(x))^{\delta}).
\]

For each fixed $x$, consider instead of $h$ an exactly scale invariant field $X$ centered around $x$, i.e., so that for any $\lambda < 1$,
\[
\{X(x + \lambda z)\}_{z \in B_1(0)} = \{\tilde{X}(z) + \Omega_{-\log \lambda}\}_{z \in B_1(0)},
\]
where $\Omega_{r}$ is a Gaussian with variance $r$ independent of $\tilde{X}$ (see Theorem 3.21 in [4] for a construction).

Set $\lambda = 2^{-n} \leq 1$. Note that for $\delta < 1$, and fixed $n$, making the change of variables $y = x + \lambda z$, and denoting $\sigma_{\lambda,x}(dz)$ the corresponding image measure (so that $y \in A_n(x)$ means $z \in A_1(0)$ and the total mass is $\sigma_{\lambda,x}(A_1(0)) = \sigma(A_n(x)))$.

\[
\mathbb{E}\left[\left(\int_{A_n(x)} e^{\gamma X_{\lambda,x}(y)}(\lambda \varepsilon)^{\gamma^2/2} \sigma(dy)\right)^\delta\right] \leq \lambda^{\gamma^2/2} \mathbb{E}\left[\left(\int_{A_1(0)} e^{\gamma X_{\lambda,x}(x+\lambda z)}(\lambda \varepsilon)^{\gamma^2/2} \sigma_{\lambda,x}(dz)\right)^\delta\right]
\]

\[
= \lambda^{\gamma^2/2} \mathbb{E}\left[\left(\int_{A_1(0)} e^{\gamma \tilde{X}_{\varepsilon}(z)}(\varepsilon)^{\gamma^2/2} \sigma_{\lambda,x}(dz)\right)^\delta\right]
\]

\[
= \lambda^{\gamma^2/2} \mathbb{E}\left[\left(\int_{A_1(0)} e^{\gamma \tilde{X}_{\varepsilon}(z)}(\varepsilon)^{\gamma^2/2} \sigma_{\lambda,x}(dz)\right)^\delta\right]
\]

\[
\leq \lambda^{\gamma^2/2 - \delta^2\gamma^2/2} \mathbb{E}\left[\left(\int_{A_1(0)} e^{\gamma \tilde{X}_{\varepsilon}(z)}(\varepsilon)^{\gamma^2/2} \sigma_{\lambda,x}(dz)\right)^\delta\right],
\]

16
where the last inequality is by Jensen’s inequality since $\delta \leq 1$. As a consequence, by Kahane’s inequality (see Theorem 3.12 in [4] for a statement of Kahane’s inequality, and see in particular (3.39) and (3.45) in [4] for its applicability), there exists a constant $C > 0$ depending only on $\gamma$ and $\delta$ but not $n$ or $\varepsilon$, such that

$$
\mathbb{E}[\mu_{h,\varepsilon}2^{-n}(A_n(x))^{\delta}] \leq C\gamma^{2\delta/2-\delta^2\gamma^2/2}\sigma(A_n(x))^{\delta},
$$

where as usual $\mu_{h,\varepsilon}(dx) = \varepsilon^{\gamma^2/2}e^{\gamma h\varepsilon(x)}dx$. Letting $\varepsilon \to 0$, we get that for any $n \geq 0$,

$$
\mathbb{E}[\mu_h(A_n(x))^{\delta}] \leq C2^{-n(\gamma^2/2-\delta^2\gamma^2/2)}\sigma(A_n(x))^{\delta}.
$$

We deduce

$$
\mathbb{E}^*[\mu_h(D)^\delta] \leq C\sum_n 2^n\gamma^2n\int \sigma(dx)\sigma(A_n(x))^{\delta}.
$$

Now observe that

$$
\sigma(A_n(x)) \leq \sigma(D)\mathbb{P}[|\bar{y} - x| \leq 2^{-n}]
$$

so that by Jensen’s inequality again (since $\delta \leq 1$)

$$
\int_D \sigma(dx)\sigma(A_n(x))^{\delta} \leq \sigma(D)^{1+\delta}\mathbb{P}(|\bar{x} - \bar{y}| < 2^{-n}|\bar{x})^{\delta})
\leq \sigma(D)^{1+\delta}\mathbb{P}(|\bar{x} - \bar{y}| \leq 2^{-n})^{\delta}
\leq \sigma(D)^{1+\delta}\mathbb{E}(|\bar{x} - \bar{y}|^{-d})^{\delta}/2^{nd}\delta,
$$

where in the last inequality we have used Markov’s inequality. By choice of $d$, the numerator in right hand side is finite. Now $\delta < 2d/\gamma^2 - 1$ implies that $\delta(\gamma^2/2 - d) + \gamma^2\delta^2/2 < 0$. Putting everything together, we can find $c = c(d, \gamma, \delta)$ such that

$$
\mathbb{E}^*[\mu_h(D)^\delta] \leq c\sigma(D)^{1+\delta}\mathbb{E}[|\bar{x} - \bar{y}|^{-d}]^{\delta}.
$$

Recall that $\mathbb{E}[\mu_h(D)^{\alpha}] = \sigma(D)^{\alpha}\mathbb{E}^*[\mu_h(D)^{\delta}]$. Since $\mathcal{E}_d = \sigma(D)^2\mathbb{E}(|\bar{x} - \bar{y}|^{-d})$ and $\delta = \alpha - 1$, we conclude the proof.

**Remark 3.2.** It would be natural to try to extend our Theorem 3.1 to all $\alpha \in (1, 2d/\gamma^2)$ which is the correct range of finite moments of GMC in the Lebesgue case. However, using our current approach, we would need to assume that integrals of the form $\int_{Dx \times \cdots \times D} \prod_{1 \leq i \neq j \leq n}|x_i - x_j|^{-\alpha} \prod_{1 \leq i \leq n} \sigma(dx_i)$ can be bounded in terms of $\mathcal{E}_d$ and $\sigma(D)$. We do not know if this holds without further assumptions on $A$ and $\sigma$, and will not pursue this direction in the current paper.

We now turn to negative moments, and take $D = \mathbb{D}$. For $\gamma \in (0, \sqrt{d})$, the so-called $L^2$ regime, the following estimate about the tail of the Laplace transform of $\mu_h(\mathbb{D})$ was established in [15].

**Lemma 3.3 (Corollary 3.2 in [15]).** For $\gamma \in (0, \sqrt{d})$, let $\ell := \frac{d - \gamma^2}{d + \gamma^2}$. Then

$$
\mathbb{E}[\exp(-t\mu_h(\mathbb{D}))] \leq \frac{32}{\sigma(\mathbb{D})^{t\ell}} \quad \text{if} \quad t\ell \geq 2^{4t+5}\mathcal{E}_d\sigma(\mathbb{D})^{-1}.
$$
Lemma 3.3 allows us to control the small ball probability of $\mu_h(D)$ (i.e., the probability that this positive random variable is small) and hence its negative moments. In [15], $\mu_h(D)$ was shown to have negative moments of all order and for all $\gamma \in (0, \sqrt{2d})$. However, the estimate is only made explicit in terms of $\sigma(D)$ and $\mathcal{E}_d$ for $\gamma \in (0, \sqrt{d})$, thanks to Lemma 3.3. We now give a variant of Lemma 3.3 for $\gamma \in [\sqrt{d}, 2\sqrt{d}]$; namely Lemma 3.5 below. Surprisingly it relies on the positive moments bound we just proved in Theorem 3.1. The fact that positive moments can be used to control negative moments is implicit in Lemma 3.4 which comes from [15], and is related to the use of Girsanov’s theorem.

Another key input is the following lemma taken from [15, Lemma 3.1].

**Lemma 3.4.** Set $\bar{\beta} = \max\{d, \sqrt{2d}\gamma\}$. Fix any choice of $\delta > 0$, let

$$\beta = \frac{1 + \delta}{1 + 2\delta} \beta + \frac{\delta}{1 + 2\delta} \gamma^2 \in (\gamma^2, \bar{\beta}) \quad \text{and} \quad \ell = \frac{\beta - \gamma^2}{\beta + \gamma^2\delta} = \frac{\bar{\beta} - \gamma^2}{\beta + 2\gamma^2\delta} \in (0, 1).$$

Let $\phi_{\beta}(x) = \int_D |x - y|^{-\beta} \mu_h(dy)$ denote the (quantum) energy function with power exponent $\beta$ and let $\bar{x}$ be a point sampled proportionally to the measure $\sigma(dx)$. Then $P[\phi_{\beta}(\bar{x}) < \infty] = 1$ and

$$\mathbb{E}[\exp(-t\mu_h(D))] \leq \frac{32}{\sigma(D)t^{\ell}} \quad \text{if} \quad P\left(\phi_{\beta}(\bar{x}) \leq 2^{-4(1+\delta)t}t^{\delta}\right) \geq 1/2. \quad (26)$$

**Lemma 3.5.** For $\gamma \in [\sqrt{d}, 2\sqrt{d}]$ and $\delta > 0$, let $\ell$ and $\beta$ be defined as in Lemma 3.4 in terms of $(d, \gamma, \delta)$. Let $\alpha = \frac{1 + (2d\gamma^2 - 2)\wedge - \gamma^2}{2}$. Then there exists $c_* = c_*(d, \gamma, \delta)$ such that

$$\mathbb{E}[\exp(-t\mu_h(D))] \leq \frac{32}{\sigma(D)t^{\ell}} \quad \text{if} \quad t^{\delta\ell} \geq c_* \mathcal{E}_d \sigma(D)^{-1+\frac{1}{\alpha-1}}.$$

**Proof.** Note that $\beta/\gamma < \sqrt{2d}$ by our assumption on $\gamma$. Thus let $\mu_h^{\gamma^{-1}\beta}$ be the Gaussian multiplicative chaos with respect to the reference measure $\sigma$ on the set $\mathcal{A}$, with parameter $\gamma^{-1}\beta$, which is well defined as a result of the assumption on $\beta$ and $\gamma$. For $r > 0$, using the fact that $G(x, y) = -\log|x - y| + O(1)$, and Girsanov’s theorem (see Theorem 17 in [35] and Theorem 3.15 in [4]), we get

$$\mathbb{E}[\phi_{\beta}(\bar{x})^r] \leq c\mathbb{E}\left[\int_D \left(\int_D e^{\beta G(x, y)} \mu_h(dy)\right)^r \frac{\sigma(dx)}{\sigma(D)}\right] \leq c\sigma(D)^{-1} \mathbb{E}\left[\left(\mu_h(D)^\gamma\right)^r \mu_h^{\gamma^{-1}\beta}(D)\right].$$

Now suppose also that $0 < r < 1$ and set $r = 1 - \alpha^{-1}$. By Hölder’s inequality,

$$\mathbb{E}[\phi_{\beta}(\bar{x})^r] \leq \sigma(D)^{-1} \mathbb{E}\left[\mu_h(D)^r\right] \mathbb{E}\left[\left(\mu_h^{\gamma^{-1}\beta}(D)\right)^{\alpha^{-1}}\right].$$

Since $\alpha \in (1, 2d\gamma^2 - 2 \wedge 2)$, we can use Theorem 3.1 to bound $\mathbb{E}\left[\left(\mu_h^{\gamma^{-1}\beta}(D)\right)^{\alpha}\right]$ and obtain

$$\mathbb{E}\left[\left(\mu_h^{\gamma^{-1}\beta}(D)\right)^{\alpha}\right] \leq c\sigma(D)^{3-\alpha}(\mathcal{E}_d)^{-\alpha^{-1}}.$$
On the other hand, \( \mathbb{E}[\mu_h(\mathbb{D})]^r \) is bounded by \( \sigma(\mathbb{D})^r \). Therefore, plugging in \( r = 1 - \alpha^{-1} \) we see that there exists \( c_0 = c_0(d, \gamma, \delta, r) \) such that

\[
\mathbb{E}[\phi^r_\beta(\bar{x})] \leq c_0 \sigma(\mathbb{D})^{-1 + r + {3 - \alpha \over \alpha}} (\mathcal{E}_d)^r = c_0 \sigma(\mathbb{D})^{1 - r} (\mathcal{E}_d)^r.
\]

Now let \( t > 0 \) be such that \( t^{\frac{1}{1+r}} \geq 2^{1+4(1+\delta)} \times c_0 \sigma(\mathbb{D})^{1 - r} (\mathcal{E}_d)^r \). Then by Markov’s inequality

\[
\mathbb{P}\left[ \phi^r_\beta(\bar{x}) \geq 2^{-4(1+\delta)} t^{\frac{1}{1+r}} \right] \leq 2^{4(1+\delta)} \mathbb{E}[\phi^r_\beta(\bar{x})] t^{-\frac{1}{1+r}} \leq 1/2.
\]

Since \( \alpha r = \alpha - 1 \), Lemma 3.5 follows from Lemma 3.4.

As a consequence, we obtain negative moments for \( \mu_h(\mathbb{D}) \). The novel part in comparison to [15] is when \( \gamma \in [\sqrt{d}/\sqrt{2d}] \). But we give a uniform statement for all \( \gamma \in (0, \sqrt{2d}) \) for completeness.

**Theorem 3.6.** Suppose \( \sigma(\mathbb{D}) \leq 1 \). There exists \( \ell = \ell(\gamma, d) > 0 \) such that the following holds. For any \( s \in (0, \ell) \) and \( \mathcal{E}' \geq \mathcal{E}_d \), there exists a constant \( c = c(\mathcal{E}', s, d, \gamma) \) such that

\[
\mathbb{E}[\mu_h(\mathbb{D})^{-s}] \leq c(\mathcal{E}', s, d, \gamma) \sigma(\mathbb{D})^{-s/\ell} < \infty.
\]

The assumption that \( \sigma(\mathbb{D}) \leq 1 \) is to ensure the appearance of \( \sigma(\mathbb{D})^{-s/\ell} \) in the upper bound. By considering the restriction of \( \sigma \) to a small enough ball, the statement \( \mathbb{E}[\mu_h(\mathbb{D})^{-s}] < \infty \) holds without this assumption. The appearance of an arbitrary \( \mathcal{E}' \geq \mathcal{E}_d \) in the statement might look initially puzzling but it is crucial for later use of this theorem in Lemma 4.5 to have this flexibility.

**Proof of Theorem 3.6.** We first assume \( \gamma \in [\sqrt{d}, \sqrt{2d}] \) so that Lemma 3.5 applies. Note that

\[
\int_0^\infty t^{s-1} e^{-t \mu_h(\mathbb{D})} dt = \Gamma(s) \mu_h(\mathbb{D})^{-s},
\]

where \( \Gamma(\cdot) \) is the Gamma function. Set \( \delta = 1 \), and take \( \ell = \frac{\beta - \gamma^2}{\beta + 2\delta \gamma} \) as in Lemma 3.4, and let \( \mathcal{E}' \geq \mathcal{E}_d \). Since \( \sigma(\mathbb{D}) \leq 1 \) and \( \alpha > 1 \), so that \( 1/((1/\alpha - 1) > 0 \), we see that \( \sigma(\mathbb{D})^{-1+1/(\alpha - 1)} \leq \sigma(\mathbb{D})^{-1} \). Hence the estimate of Lemma 3.5 is valid as soon as \( t^\ell \geq c_\ast \mathcal{E}'/\sigma(\mathbb{D}) \).

We therefore obtain the following bound:

\[
\mathbb{E}[\mu_h(\mathbb{D})^{-s}] \lesssim \int_0^{(c \mathcal{E}'/\sigma(\mathbb{D}))^{1/\ell}} t^{s-1} dt + \int_{(c \mathcal{E}'/\sigma(\mathbb{D}))^{1/\ell}}^{\infty} 32 \sigma(\mathbb{D})^{-s-1-\ell} dt \lesssim \frac{1}{s} \left( \frac{\mathcal{E}'}{\sigma(\mathbb{D})} \right)^{\frac{s}{\ell}} + \frac{1}{\sigma(\mathbb{D})} \left( \frac{\mathcal{E}'}{\sigma(\mathbb{D})} \right)^{\frac{s-\ell}{\ell}} \lesssim \sigma(\mathbb{D})^{-s/\ell},
\]

where the implicit constants only depend on \( \mathcal{E}', s, d, \gamma \). For \( \gamma \in (0, \sqrt{d}) \), set \( \ell = \frac{d - \gamma^2}{d + \gamma^2} \). Then the same argument with Lemma 3.3 in place of Lemma 3.5 gives the desired result.

**4 Proof of Theorem 1.4**

Now suppose that \( \mathcal{A} \subset \mathbb{D} \) is a compact closed set. Since \( \sigma \) is usually clear from the context, we simply write \( \mu_{\sigma, h} \) as \( \mu_h \).
For \( z \in \mathbb{D} \), recall that \( \omega(z, dx) \) is the harmonic measure of \( \mathbb{D} \setminus \mathcal{A} \) viewed from \( z \). For \( \varepsilon \in (0, 1/2) \), let \( h_\varepsilon \) be the circle average process of \( h \) at \( z \). Let

\[
\begin{align*}
\tilde{h}_\varepsilon^{\text{har}}(z) &= \int \omega(z, dx) \left[ \gamma^{-1} \log \mu_h(B_\varepsilon(x)) \right], \\
h_\varepsilon^{\text{har}}(z) &= \int \omega(z, dx) h_\varepsilon(x), \\
g_\varepsilon(z) &= \int k_\varepsilon(x) \omega(z, dx) = \tilde{h}_\varepsilon^{\text{har}}(z) - h_\varepsilon^{\text{har}}(z), \quad (27)
\end{align*}
\]

where

\[
k_\varepsilon(x) = \gamma^{-1} \log \mu_h(B_\varepsilon(x)) - h_\varepsilon(x), \quad \forall x \in \mathcal{A}. \quad (28)
\]

Let \( h^{\text{har}} \) be the harmonic extension of \( h \) off \( \mathcal{A} \). Following the strategy in Section 2, we will show that \( h_\varepsilon^{\text{har}} \) is a good estimator of \( h^{\text{har}} \) (Lemma 4.1) and \( g_\varepsilon \) tends to 0 (Lemma 4.2 and 4.3). Before going into the proof, we first record a useful Brownian motion fact.

In Theorem 1.4 we assume that \( \mathcal{A} \subset \mathbb{D} \) satisfies Property (Q). This assumption is used to justify the intuitively obvious claim that \( h_\varepsilon^{\text{har}} \) is a good estimator for \( h^{\text{har}} \). We will see from the following proof that a much weaker condition than Property (Q) would suffice.

**Lemma 4.1.** \( \forall \rho \in C^\infty_0(D) \), \( E[(h_\varepsilon^{\text{har}}, \rho)] = 0 \) and \( (h_\varepsilon^{\text{har}}, \rho) \) tends to \((h^{\text{har}}, \rho)\) in probability.

**Proof.** Without loss of generality, we can assume \( \rho \) is a continuous probability density function. A general \( \rho \) can be write as \( c_1 \rho_1 - c_2 \rho_2 \) where \( \rho_1, \rho_2 \) are such functions. Now we sample \( z \) according to \( \rho(z) dz \) and then sample a Brownian motion from \( z \). Suppose \( X \) is the exit location of the domain \( \mathbb{D} \setminus \mathcal{A} \) for this Brownian motion and \( U \) is a uniform unit 2D vector independent of \( X \). Let \( \hat{\rho}(x) dx \) and \( \hat{\rho}_e(x) dx \) be the distribution of \( X \) and \( X + \varepsilon U \) respectively. Then \((h_\varepsilon^{\text{har}}, \rho) = (h, \hat{\rho}) \) and \((h_\varepsilon^{\text{har}}, \rho) = (h, \hat{\rho}_e) \). Since \( h \overset{d}{=} -h \), we have \( E[(h_\varepsilon^{\text{har}}, \rho)] = 0 \). It suffices to prove that \( \text{Var}[(h, \hat{\rho}_e - \hat{\rho})] \to 0 \).

Suppose \( X \) and \( Y \) are two independent copies sampled from \( \hat{\rho}(z) dz \), and \( U_1, U_2 \) are two independent uniform unit vectors. Then it suffices to show that

\[
\lim_{\varepsilon \to 0} E[G_\mathbb{D}(\bullet, \bullet)] = E[G_\mathbb{D}(X, Y)] \quad (29)
\]

where \( \bullet \) represents either \( X \) or \( X + \varepsilon U_1 \), \( * \) represents either \( Y \) or \( Y + \varepsilon U_2 \), and \( G_\mathbb{D} \) is the Green function in \( \mathbb{D} \). Here we extend \( G_\mathbb{D} \) to \( \mathbb{C}^2 \) by setting \( G_\mathbb{D} = 0 \) on \( \mathbb{C}^2 \setminus \mathbb{D} \times \mathbb{D} \).

(29) has to be shown in all four possible combinations of the terms. We only explain the case \( G_\mathbb{D}(X + \varepsilon U_1, Y + \varepsilon U_2) \), since other cases are similar and simpler.

By the mean value property of \( G_\mathbb{D} \) off the diagonal, we have

\[
E[G_\mathbb{D}(X + \varepsilon U_1, Y + \varepsilon U_2)1_{\{|X-Y|>2\varepsilon, X,Y \in \mathcal{A}\}}] = E[G_\mathbb{D}(X,Y)1_{\{|X-Y|>2\varepsilon, X,Y \in \mathcal{A}\}}].
\]

On the other hand, by (12), we have

\[
E[G_\mathbb{D}(X + \varepsilon U_1, Y + \varepsilon U_2)1_{\{|X-Y| \leq 2\varepsilon\}}] \leq C|\log \varepsilon| \mathbb{P}[|X - Y| < 2\varepsilon].
\]
It suffices to show that $\lim_{\varepsilon \to 0} |\log \varepsilon| \mathbb{P}[|X - Y| < 2\varepsilon] = 0$. Since $\mathcal{A}$ satisfies Property (Q), we have

$$\mathbb{P}[|X - Y| < 2\varepsilon] \leq \max_{x \in \mathcal{A}} \mathbb{P}[|Y - x| < 2\varepsilon] \leq \max_{x \in \mathcal{A}} \int_{\mathbb{D}} \rho(z) \omega(z, B_{2\varepsilon}(x)) dz \leq \max_{x \in \mathcal{A}} \int_{2\mathbb{D}} \left( \frac{\varepsilon}{|z - x|} \wedge 1 \right)^q dz.$$ 

Here $f \lesssim g$ means that $f \leq Cg$ for some constant $C > 0$ that does not depend on the parameters in $f$ and $g$. As the last expression is bounded by $C\varepsilon^{-q/2}$ and $\lim_{\varepsilon \to 0} |\log \varepsilon| \varepsilon^{-q/2} = 0$, we obtain Lemma 4.1.

The following two crucial lemmas (which are the analogues of Lemmas 2.2 and 2.3 respectively) will be proved in Sections 4.2 and 4.3.

**Lemma 4.2.** There exists a constant $C$ and a sub-polynomial function $\psi$ such that

$$\text{Var}[g_\varepsilon(z)] \leq \psi(\varepsilon) \quad \forall z \in \mathbb{D}.$$ (30)

**Lemma 4.3.** Fix a sub-polynomial function $\psi$. We have

$$\text{Cov}[g_\varepsilon(z_1), g_\varepsilon(z_2)] = o_\varepsilon(1)$$ (31)

uniformly in $z_1, z_2 \in \mathbb{D}$ and $|z_1 - z_2| > 1/\psi(\varepsilon)$.

Given these two lemmas, we quickly verify the proof of Theorem 1.4.

**Proof of Theorem 1.4 given Lemmas 4.2 and 4.3.** For $\rho \in C_0^\infty(\mathbb{D})$, we have

$$\text{Var}[(g_\varepsilon, \rho)] = \int \int \text{Cov}[g_\varepsilon(z_1), g_\varepsilon(z_2)] \rho(z_1) \rho(z_2) dz_1 dz_2.$$ 

Let $\psi_1(\varepsilon)$ be the sub-polynomial function in Lemma 4.2 and $\psi_2(\varepsilon) = \psi_1(\varepsilon) \log(1/\varepsilon)$, which is also sub-polynomial. Combining Lemmas 4.2 and 4.3 and the Cauchy–Schwarz inequality, we have

$$\text{Cov}[g_\varepsilon(z_1), g_\varepsilon(z_2)] \leq 1_{\{|z_1 - z_2| \leq 1/\psi_2(\varepsilon)\}} \psi_1(\varepsilon) + o_\varepsilon(1) 1_{\{|z_1 - z_2| \geq 1/\psi_2(\varepsilon)\}}.$$ 

Since the $o_\varepsilon(1)$ is uniform in $z_1, z_2$ provided that $|z_1 - z_2| > 1/\psi_2(\varepsilon)$, we can integrate the above and obtain $\lim_{\varepsilon \to 0} \text{Var}[(g_\varepsilon, \rho)] = 0$. By Lemma 4.1, $(h_{\text{har}}, \rho)$ is measurable w.r.t. $\mu_h$. 

### 4.1 Uniform estimate for logarithmic moments for small balls

Suppose we are in the setting of Theorem 1.4, As in Section 3, we fix a $d \in (\gamma^2/2, d)$ so that $\gamma < \sqrt{2d}$ and $E_d < \infty$ as defined in (24) is finite. We may and will assume without loss of generality that $\sigma(\mathbb{D}) \leq 1$.

Both Lemmas 4.2 and 4.3 will ultimately follow from the proposition below, which is where Property (P) is used.

**Proposition 4.4.** For $x \in \mathcal{A}$ and $\varepsilon \in (0, \frac{1}{2})$, let $h_x$ be a zero boundary GFF on $B_{2\varepsilon}(x)$. Let $\mu_x := \mu_{\sigma, h_x}$ be the GMC of $h_x$ with parameter $\gamma$ over the reference measure $\sigma$; see (3). Then there exists a sub-polynomial function $\psi$ depending on $(\gamma, d, \sigma)$ such that

$$\mathbb{E}\left[|\log \mu_x(B_\varepsilon(x))|^4\right] \leq \psi(\varepsilon) \quad \text{for each } \varepsilon \in (0, \frac{1}{2}) \text{ and } x \in \mathcal{A}.$$ (32)
Note that the law of $\mu_x(B_\varepsilon(x))$ depends on $x$ as the measure $\sigma$ is not in general translation invariant. Our proof of Proposition 4.4 relies on the following corollary of Theorem 3.6.

**Lemma 4.5.** Let $\ell = \ell(\gamma, d) > 0$ be as in Theorem 3.6. In the setting of Proposition 4.4, for each $p \in (0, 1)$, there exists a constant $C = C(\mathcal{E}_d, p, d, \gamma)$ such that

$$\mathbb{E}[\mu_x(B_\varepsilon(x))^{-p\ell}] \leq C\varepsilon^{-\gamma^2/2} \sigma(B_\varepsilon(x))^{-p}, \quad \text{for each } \varepsilon \in (0, 1/2) \text{ and } x \in A.$$

**Proof.** Consider the map $y \mapsto (2\varepsilon)^{-1}(y-x)$ from $B_{2\varepsilon}(x)$ to $\mathbb{D}$. Let $\sigma_{x,\varepsilon}$ be the pushforward measure of $\sigma_{|B_{\varepsilon}(x)}$ under this map; thus $\sigma_{x,\varepsilon}((\mathbb{D}) = \sigma(B_{2\varepsilon}(x)) \leq \sigma(\mathbb{D}) \leq 1$. Then $\mu_x(B_\varepsilon(x))$ can be written as $(2\varepsilon)^{\gamma^2/2} \int_{\mathbb{D}} e^{\gamma h(z)} \sigma_{x,\varepsilon}(dz)$, where $h$ is a zero boundary GFF on $\mathbb{D}$.

Let $X := \int_{\mathbb{D}} e^{\gamma h(z)} \sigma_{x,\varepsilon}(dz)$. To apply Theorem 3.6, let

$$X_{\varepsilon, \ell} := \int_{\mathbb{D} \times \mathbb{D}} |z-w|^{-d} \sigma_{x,\varepsilon}(dw) \sigma_{x,\varepsilon}(dz) = (2\varepsilon)^d \int_{B_{\varepsilon}(x) \times B_{\varepsilon}(x)} |z-w|^{-d} \sigma(dw) \sigma(dz).$$

Since $X_{\varepsilon, \ell} \leq (2\varepsilon)^d \varepsilon_{\ell} \leq \mathcal{E}_d$ for $\varepsilon \in (0, 1/2)$, we can apply Theorem 3.6, with $(\mathcal{E}', \sigma, s)$ in that theorem taken here to be equal to $(\mathcal{E}_d, \sigma_{x,\varepsilon}, p\ell)$, and $\mathcal{E}_d$ in that theorem being taken here to be equal to $X_{\varepsilon, \ell}$. We therefore are able to bound $\mathbb{E}[X^{-p\ell}]$. Since $\sigma_{x,\varepsilon}(\mathbb{D}) = \sigma(B_\varepsilon(x))$ by the definition of $\sigma_{x,\varepsilon}$, we obtain Lemma 4.5.

**Proof of Proposition 4.4.** For each $a > 0$, we have $|\log x|^4 \leq c(x + x^{-a})$ for all $x > 0$ for some constant $c = c(a)$. Note that $\mathbb{E}[\mu_x(B_\varepsilon(x))] \lesssim \sigma(B_\varepsilon(x))$. Fix $a \in (0, \ell)$. Lemma 4.5 shows that

$$\mathbb{E}[\mu_x(B_\varepsilon(x))^{-a}] \lesssim \varepsilon^{-\gamma^2/2} \sigma(B_\varepsilon(x))^{-a/\ell}.$$ 

Using Property (P), we get $\mathbb{E}[|\log \mu_x(B_\varepsilon(x))|^4] \lesssim \varepsilon^{-\gamma^2/2 - qa/\ell}$ where the implicit constant depends on $a, \sigma$ and $\gamma$. Since $a$ can be arbitrarily small, this shows that the left hand side is sub-polynomial, as desired.

\[\square\]

### 4.2 Proof of Lemma 4.2: the variance estimate

By the Cauchy-Schwarz inequality and (27), Lemma 4.2 follows from

**Lemma 4.6.** There exist a sub-polynomial function $\psi$ such that

$$\int \text{Var}[k_\varepsilon(x)|\omega(z,dx)] \leq \psi(\varepsilon), \quad \forall z \in \mathbb{D}. \quad (33)$$

**Proof.** We will recursive decomposition and idea as in the proof of Lemma 2.2 (see Section 2.2). Without loss of generality we can assume $\varepsilon = 2^{-n-2}$. Fix $x \in \frac{1}{2}\mathbb{D}$. We then apply the Markov property of $h$ inductively to decompose $h$ into

$$h = \sum_{i=0}^{n-1} h_i^{har} + h^{n-1}$$

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where \( h_i^{\text{har}} (0 \leq i \leq n - 1) \) is harmonic on \( B_{2^{-i-2}}(x) \), and supported on \( B_{2^{-i-1}}(x) \) (except for \( i = 0 \)); while \( h^{n-1} \) is a Dirichlet GFF in \( B_{2^{-n-1}}(x) = B_{2^x}(x) \). Furthermore, \( h^{n-1} \) is independent from \( (h_i^{\text{har}})_{0 \leq i \leq n-1} \). As in the proof of Lemma 2.2 (see again Section 2.2), applying the intermediate value theorem in each successive ball and the mean value property of \( h_i^{\text{har}} \) for each \( 0 \leq i \leq n-1 \), we can write

\[
 k_{\varepsilon}(x) = \sum_{i=0}^{n-1} (h_i^{\text{har}}(\xi_i) - h_i^{\text{har}}(x)) + \gamma^{-1} \log \mu_{h_{n-1}}(B_{\varepsilon}(x)) - h_{\varepsilon}^{n-1}(x),
\]

where \( \xi_i \in B_{\varepsilon_i}(x) \) comes from the intermediate value theorem, \( \mu_{h_{n-1}} = \mu_{\sigma,h_{n-1}} \) and \( h_{\varepsilon}^{n-1} \) is the \( \varepsilon \)-circle average of \( h^{n-1} \).

Let

\[
 \Delta_i(x) = h_i^{\text{har}}(\xi_i) - h_i^{\text{har}}(x) \quad \text{and} \quad R_{\varepsilon}(x) = \log \mu_{h_{n-1}}(B_{\varepsilon}(x)).
\]

Then

\[
 k_{\varepsilon}(x) = \sum_{i=1}^{n} \Delta_i(x) - h_{\varepsilon}^{n-1}(x) + \gamma^{-1} R_{\varepsilon}(x).
\]

Let \( \overline{\Delta}_i(x) = \max_{y_1,y_2 \in B_{\varepsilon}(x)} \{ h_i^{\text{har}}(y_1) - h_i^{\text{har}}(y_2) \} \). By Lemma 2.1,

\[
 E[\Delta_i^2(x)] \leq E[\overline{\Delta}_i(x)^2] \leq C 4^{i-n}.
\]

Using (36) and the Cauchy–Schwarz inequality, we see that

\[
 \text{Var} \left[ \sum_{i=1}^{n} \Delta_i(x) \right] \leq n \sum_{i=1}^{n} E[\Delta_i^2(x)] \leq C |\log \varepsilon|.
\]

Observe also that \( \text{Var}[h_{\varepsilon}^{n-1}(x)] = O(1) \). Hence by Cauchy–Schwarz, to finish the proof of Lemma 4.2, it only remains to find a sub-polynomial function \( \psi \) such that

\[
 \int \text{Var} [R_{\varepsilon}(x)] \omega(z,dx) \leq \psi(\varepsilon).
\]

From Proposition 4.4, we can actually find a sub-polynomial function \( \psi \) such that \( E[R_{\varepsilon}(x)^4] \leq \psi(\varepsilon) \) uniform in \( x \in \mathcal{A} \). This concludes the proof of Lemma 4.2.

\[ \square \]

### 4.3 Proof of Lemma 4.3: the covariance estimate

\[ \Omega_{x,r} = \{(x_1,x_2) : |x_1 - x_2| \geq \varepsilon^{\frac{1}{2}}, x_1, x_2 \in \frac{1}{2} \mathbb{D}\} \]. Recalling (28), write

\[
 E[\text{Cov}[g_{\varepsilon}(z_1), g_{\varepsilon}(z_2)]] = M(z_1,z_2) + R(z_1,z_2),
\]

where

\[
 M(z_1,z_2) = E \left[ \int_{\Omega_{x,r}} \text{Cov} \left[ k_{\varepsilon}(x_1), k_{\varepsilon}(x_2) \right] \omega(z_1, dx_1) \omega(z_2, dx_2) \right],
\]

\[
 R(z_1,z_2) = E \left[ \int_{|x_1 - x_2| < \varepsilon^{\frac{1}{2}}} \text{Cov} \left[ k_{\varepsilon}(x_1), k_{\varepsilon}(x_2) \right] \omega(z_1, dx_1) \omega(z_2, dx_2) \right].
\]

We claim that:
Lemma 4.7. We have \( M(z_1, z_2) = o_\varepsilon(1) \) uniformly in \( z_1, z_2 \in D \).

Lemma 4.8. Fix a sub-polynomial function \( \psi \). We have

\[
R(z_1, z_2) = o_\varepsilon(1)
\]

uniformly in \( z_1, z_2 \in D \) with \( |z_1 - z_2| \geq 1/\psi(\varepsilon) \).

Note that Lemma 4.7 and Lemma 4.8 together will imply Lemma 4.3.

Proof of Lemma 4.7. For \( x_1, x_2 \in \Omega_{\varepsilon,r} \), let \( L \) be the straight line orthogonally bisecting the line segment \([x_1x_2]\) as in the proof of Lemma 2.3. Let \( U_i \) be the connected component of \( D \setminus L \) containing \( x_i \), \( i = 1, 2 \). By the domain Markov property, write

\[
h = h_{\text{har}} + h_1 + h_2,
\]

where \( h_i \) are independent Gaussian free fields with Dirichlet boundary conditions in \( U_i \) \( (i = 1, 2) \) and \( h_{\text{har}} \) is independent, and harmonic in \( U_1 \cup U_2 \). Let \( \mu_{\sigma, h_i} \) \( (i = 1, 2) \) be the GMC measure \( \mu_{\sigma, h_i} \) in \( U_i \)

\[
\Gamma_\varepsilon(x_i) = \gamma^{-1} \log \mu_{h_i}(B_\varepsilon(x_i)) - h_i^\varepsilon(x_i),
\]

\[
\Delta_i = \gamma^{-1} \log \int_{B_\varepsilon(x_i)} e^{\gamma h_{\text{har}}(x)} d\mu_{h_i}(x) - \gamma^{-1} \log \mu_{h_i}(B_\varepsilon(x_i)) - h_{\text{har}}(x_i).
\]  (41)

As in the proof of Lemma 2.3 (see Section 2.3),

\[
k_\varepsilon(x_i) = \Gamma_\varepsilon(x_i) + \Delta_i.
\]  (42)

By the independence of \( h_1, h_2 \), for \( |x_1 - x_2| > \varepsilon^2 \),

\[
\text{Cov} [\Gamma_\varepsilon(x_1), \Gamma_\varepsilon(x_2)] = 0.
\]

Therefore by (42)

\[
M(z_1, z_2) = I(z_1, z_2) + II(z_1, z_2) + III(z_1, z_2)
\]

where

\[
I(z_1, z_2) = \int_{\Omega_{\varepsilon,r}} \text{Cov} [\Gamma_\varepsilon(x_1), \Delta_2] \omega(z_1, dx_1) \omega(z_2, dx_2),
\]

\[
II(z_1, z_2) = \int_{\Omega_{\varepsilon,r}} \text{Cov} [\Delta_1, \Gamma_\varepsilon(x_2)] \omega(z_1, dx_1) \omega(z_2, dx_2),
\]

\[
III(z_1, z_2) = \int_{\Omega_{\varepsilon,r}} \text{Cov} [\Delta_1, \Delta_2] \omega(z_1, dx_1) \omega(z_2, dx_2).
\]

Recall (41). The intermediate value theorem (applied to \( h_{\text{har}} \)) shows that \( \Delta_i \) can be rewritten as \( h_{\text{har}}^\varepsilon(\xi_i) - h_{\text{har}}^\varepsilon(x_i) \) for some point \( \xi_i \in B_\varepsilon(x_i) \). Hence using Lemma 2.1 we obtain for \( x_1, x_2 \in \Omega_{\varepsilon,r} \),

\[
\text{Var} [\Delta_i] \leq \mathbb{E} (\Delta_i^2) \leq C \varepsilon^2 / |x_2 - x_1|^2 \leq C \varepsilon,
\]  (43)

where \( C \) does not depend on \( z_1, z_2 \). By the Cauchy–Schwarz inequality, we have \( III(z_1, z_2) \leq C \varepsilon \).
We now switch out attention to $I(z_1, z_2)$ and $II(z_1, z_2)$. By the Cauchy–Schwarz inequality (twice),

\[
I(z_1, z_2) \leq \iint_{\Omega_{\varepsilon, r}} \text{Var}^{1/2}[\Gamma_\varepsilon(x_1)] \text{Var}^{1/2}[\Delta_2] \omega(z_1, dx_1) \omega(z_2, dx_2)
\leq C \varepsilon^{1/2} \iint_{\Omega_{\varepsilon, r}} \text{Var}[\Gamma_\varepsilon(x_1)] \omega(z_1, dx_1) \omega(z_2, dx_2)
\leq C \varepsilon^{1/2} \left( \iint_{\Omega_{\varepsilon, r}} \text{Var}[\Gamma_\varepsilon(x_1)] \omega(z_1, dx_1) \omega(z_2, dx_2) \right)^{1/2}.
\]

Furthermore, by (42) and (43),

\[
\iint_{\Omega_{\varepsilon, r}} \text{Var}[\Gamma_\varepsilon(x_1)] \omega(z_1, dx_1) \omega(z_2, dx_2)
\leq 2 \int \text{Var}[k_\varepsilon(x_1)] \omega(z_1, dx_1) + 2 \iint_{\Omega_{\varepsilon, r}} \text{Var}[\Delta_i] \omega(z_1, dx_1) \omega(z_2, dx_2)
\leq 2 \int \text{Var}[k_\varepsilon(x_1)] \omega(z_1, dx_1) + 2 C \varepsilon^{1/2},
\]

where $C$ is the constant in (43). A similar estimate holds for $II(z_1, z_2)$. Now Lemma 4.7 follows from Lemma 4.6. 

\[\square\]

**Proof of Lemma 4.8.** Let

\[
I'(z_1, z_2) = \iint 1_{|x_1 - x_2| < \varepsilon^{1/2}} \omega(z_1, dx_1) \omega(z_2, dx_2),
\]

\[
II'(z_1, z_2) = \iint_{|x_1 - x_2| < \varepsilon^{1/2}} \text{Var}[k_\varepsilon(x_1)] \omega(z_1, dx_1) \omega(z_2, dx_2),
\]

\[
III'(z_1, z_2) = \iint_{|x_1 - x_2| < \varepsilon^{1/2}} \text{Var}[k_\varepsilon(x_2)] \omega(z_1, dx_1) \omega(z_2, dx_2).
\]

By the Cauchy-Schwarz inequality,

\[
R^2(z_1, z_2) \leq \left( \iint_{|x_1 - x_2| < \varepsilon^{1/2}} \text{Var}^{1/2}[k_\varepsilon(x_1)] \text{Var}^{1/2}[k_\varepsilon(x_2)] \omega(z_1, dx_1) \omega(z_2, dx_2) \right)^2
\leq II'(z_1, z_2) III'(z_1, z_2).
\]

(44)

We first estimate $I'(z_1, z_2)$. Assume $\varepsilon$ is small enough such that $\psi(\varepsilon) > 3\varepsilon^{1/2}$. Then $|z_1 - z_2| > \psi(\varepsilon)$ and $|x_1 - x_2| < \varepsilon^{1/2}$ imply that either $|z_1 - x_2|$ or $|z_2 - x_1|$ is bigger than $\psi(\varepsilon)/3$. Note that

\[
\iint 1_{|x_1 - x_2| < \varepsilon^{1/2}, |z_1 - x_2| > \psi(\varepsilon)/3} \omega(z_1, dx_1) \omega(z_2, dx_2) \leq \iint 1_{|z_1 - x_2| > \psi(\varepsilon)/3} \omega(z, B_\varepsilon^{1/2}(x_2)) \omega(z_2, dx_2)
\]

Since $\mathcal{A}$ satisfy property (Q) with exponent $q$ as in (6),

\[
\iint 1_{|z_1 - x_2| > \psi(\varepsilon)/3} \omega(z, B_\varepsilon^{1/2}(x_2)) \omega(z_2, dx_2) \leq \varepsilon^{q/2} \psi(\varepsilon)^{-q} = o(\varepsilon^{q/3}).
\]

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We get a similar estimate if $(z_1, x_2)$ is switched with $(z_2, x_1)$. Therefore $I'(z_1, z_2) = o(\epsilon^{2/3})$.

We now estimate $II'(z_1, z_2)$ and $III'(z_1, z_2)$. Recall $R_\epsilon(x)$ from the recursive decomposition in (34). By (35) and (37), we have

$$\text{Var}[k_\epsilon(x) - \gamma^{-1}R_\epsilon(x)] \leq C|\log \epsilon|.$$  

Then

$$II'(z_1, z_2) \leq C|\log \epsilon|I'(z_1, z_2) + C \int_{|x_1 - x_2| < \epsilon^{1/2}} \text{Var}[R_\epsilon(x_1)] \omega(z_1, dx_1)\omega(z_2, dx_2).$$

(45)

For the second term on the right hand side of (45), by Proposition 4.4 and Jensen’s inequality $\text{Var}[R_\epsilon(x_1)]^2 \leq E[R_\epsilon(x_1)^4]$. Hence this second term is bounded by $\sqrt{E[R_\epsilon(x)^4]}I'(z_1, z_2)$. Therefore

$$II'(z_1, z_2) \leq C \left(|\log \epsilon| + \sqrt{E[R_\epsilon(x)^4]}\right) I'(z_1, z_2).$$

As argued at the end of the proof of Lemma 4.6, $E[R_\epsilon(x)^4]$ is sub-polynomial. Combining with our earlier bound on $I'(z_1, z_2)$, we deduce that $II'(z_1, z_2) = o_\epsilon(1)$.

Similarly, $III'(z_1, z_2) = o_\epsilon(1)$. Now Lemma 4.8 follows from (44).

References


[27] Jason Miller and Scott Sheffield. Liouville quantum gravity and the Brownian map III: the

[28] Jason Miller and Scott Sheffield. Imaginary geometry IV: interior rays, whole-plane reversibil-

[29] Jason Miller and Scott Sheffield. Liouville quantum gravity and the Brownian map I: the


[32] Rémi Rhodes and Vincent Vargas. KPZ formula for log-infinitely divisible multifractal random


[34] Oded Schramm and Scott Sheffield. A contour line of the continuum Gaussian free field.


[37] Scott Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper.

[38] Scott Sheffield and Menglu Wang. Field-measure correspondence in Liouville quantum gravity

[39] Dapeng Zhan. Optimal Hölder continuity and dimension properties for SLE with Minkowski