Adaptive invariant density estimation for continuous-time mixing Markov processes under sup-norm risk

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Abstract. Up to now, the nonparametric analysis of multidimensional continuous-time Markov processes has focussed strongly on specific model choices, mostly related to symmetry of the semigroup. While this approach allows to study the performance of estimators for the characteristics of the process in the minimax sense, it restricts the applicability of results to a rather constrained set of stochastic processes and in particular hardly allows incorporating jump structures. As a consequence, for many models of applied and theoretical interest, no statement can be made about the robustness of typical statistical procedures beyond the beautiful, but limited framework available in the literature. To contribute to the statistical understanding in more general situations, we demonstrate how combining $\beta$-mixing assumptions on the process and heat kernel bounds on the transition density representing controls on the long- and short-time transitional behaviour, allow to obtain sup-norm and $L^2$ kernel invariant density estimation rates that match the well-understood case of reversible multidimensional diffusion processes and are faster than in a sampled discrete data scenario. Moreover, we demonstrate how, up to log-terms, optimal sup-norm adaptive invariant density estimation can be achieved within our framework, based on tight uniform moment bounds and deviation inequalities for empirical processes associated to additive functionals of Markov processes. The underlying assumptions are verifiable with classical tools from stability theory of continuous-time Markov processes and PDE techniques, which opens the door to evaluate statistical performance for a vast amount of popular Markov models. We highlight this point by showing how multidimensional jump SDEs with Lévy-driven jump part under different coefficient assumptions can be seamlessly integrated into our framework, thus establishing novel adaptive sup-norm estimation rates for this class of processes.

MSC2020 subject classifications: Primary 62M05; secondary 62G05, 62G20, 60G10, 60J25

Keywords: nonparametric statistics, sup-norm risk, adaptive estimation, statistics for stochastic processes, jump SDEs

1. Introduction

There exist various probabilistic concepts that permit the investigation of quantitative ergodic properties of Markov processes, providing a number of approaches to analyzing the rate of convergence of the process to equilibrium. Such results actually present precious tools for an adequate statistical modelling of complex systems. Markov models, especially of (jump) diffusion-type, find numerous applications in biology, chemistry, natural resource management, computer vision, Bayesian inference in machine learning, cloud computing and many more [3, 17, 36, 37, 39, 43, 79, 82], and ergodicity can usually be seen as some kind of minimum requirement for the development of a fruitful statistical theory. While the probabilistic picture of quantitative ergodic properties is now quite clear, there are still open questions regarding the statistical implications. With this paper, we want to contribute to closing some gaps concerning adaptive nonparametric invariant density estimation for multivariate Markov processes with no specific structural assumptions on their dynamics.

In contrast to the highly-developed statistical theory for scalar diffusion processes, there are relatively few references for nonparametric or high-dimensional general Markov models. To not let sampling effects obscure the statistical implications, it is natural to base the statistical analysis in this context on a continuous observation scheme (i.e., one assumes that a complete trajectory of the process is available). A substantial point of reference for a thorough statistical analysis of ergodic multivariate diffusion processes is provided by the article [28] where the fundamental question of asymptotic statistical equivalence is investigated. Apart from its principal central statement, the work also nicely demonstrates the implications of probabilistic properties of processes on quantitative statistical results. Specifically, heat kernel bounds and the spectral gap inequality are used to prove tight variance bounds for integral functionals which in turn provide fast convergence rates for the specific problem of invariant density estimation. Similar techniques can be used for the in-depth
analysis of other statistical questions such as (adaptive) estimation of the drift vector of an ergodic diffusion (cf. [76], [77]). The results in [28, 76, 77] are developed for diffusion processes with drift of gradient-type and unit diffusion matrix. While in this specific case the reversibility assumption is directly verified, the condition of symmetry of the process presents a significant constraint, in particular for solutions of SDEs with jump noise.

More recently, a Bayesian approach to drift estimation of multivariate diffusion processes is undertaken in [63] and [42]. Whilst [42] work in a reversible setting since their approach relies on placing a Gaussian prior on the potential \( B \) of the drift \( b = \nabla B \) instead of tackling the drift directly, [63] approach drift estimation for non-reversible diffusions by employing PDE techniques to a penalized likelihood estimator. This opens up an excitingly different viewpoint on the statistical handling of multivariate diffusion processes and in case of [63] avoids the need for reversibility. However, both approaches restrict the setting to assumed periodicity of the drift coefficient. While this assumption (similar to reversibility) can certainly be justified for specific applications, the approach does not yet provide an answer to the question of how to conduct a statistical analysis of multidimensional Markov processes without strong structural constraints on the coefficients. From a different perspective, the very recent contribution [4] yields the remarkable observation that quantitatively similar statistical results as in the reversible diffusion case can also be proven for jump diffusions with Lévy-driven jump part, without the need to rely on a reversible or periodic setting, by focusing on assumptions on the characteristics of the process which guarantee exponential ergodicity as the driving force of the statistical approach.

Another branch of the literature that does not consider specific structural assumptions on the process is based on the so called Castellana–Leadbetter condition or variations thereof [13, 18, 50], which imposes finiteness of the integrated uniform distance between the density of the bivariate law of \((X_0, X_t)\) of a stationary Markov process \(X\) with stationary density \(\rho\) and the product density \(\rho \otimes \rho\). This assumption yields dimension independent parametric estimation rates of the invariant density and is thus not suitable for our goal to extend the dimension dependent minimax optimal estimation rates for continuous diffusion processes to more general classes of multidimensional Markov processes, introduced below.

Throughout, we suppose that \((X, (\mathbb{P}^x)_{x \in \mathbb{R}^d})\) is a non-explosive Borel right Markov process with state space \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\) and semigroup \((P_t)_{t \geq 0}\) defined by

\[
P_t(x, B) := \mathbb{P}^x(X_t \in B), \quad x \in \mathbb{R}^d, B \in \mathcal{B}(\mathbb{R}^d),
\]

see Definition 8.1 in [74] for an exact characterization. This general class of right-continuous Markov processes includes the more specific class of standard processes, which form the basis of the classical textbook [12], and even more specific Feller processes, i.e., càdlàg Markov processes with a strongly continuous semigroup mapping the more specific class of standard processes, which form the basis of the classical textbook [12], and even more specifically Definition 8.1 in [74] for an exact characterization. This general class of right-continuous Markov processes includes the more specific class of standard processes, which form the basis of the classical textbook [12], and even more specifically Feller processes, i.e., càdlàg Markov processes with a strongly continuous semigroup mapping the more specific class of standard processes, which form the basis of the classical textbook [12], and even more specifically Definition 8.1 in [74] for an exact characterization. This general class of right-continuous Markov processes includes the more specific class of standard processes, which form the basis of the classical textbook [12], and even more specifically Feller processes, i.e., càdlàg Markov processes with a strongly continuous semigroup mapping the more specific class of standard processes, which form the basis of the classical textbook [12], and even more specifically Definition 8.1 in [74] for an exact characterization. This general class of right-continuous Markov processes includes the more specific class of standard processes, which form the basis of the classical textbook [12], and even more specifically Feller processes, i.e., càdlàg Markov processes with a strongly continuous semigroup mapping the more specific class of standard processes, which form the basis of the classical textbook [12], and even more specifically Definition 8.1 in [74] for an exact characterization. This general class of right-continuous Markov processes includes the more specific class of standard processes, which form the basis of the classical textbook [12], and even more specifically Feller processes, i.e., càdlàg Markov processes with a strongly continuous semigroup mapping the more specific class of standard processes, which form the basis of the classical textbook [12], and even more specifically Definition 8.1 in [74] for an exact characterization.
With a view towards applicability of the statistical results we turn away from the functional inequality approach to stability and focus on mixing conditions verifiable through Lyapunov-type criteria that are applicable for a vast amount of structurally diverse Markov processes. Making the mixing behaviour of the process a cornerstone of the statistical analysis is completely natural when comparing to discrete time theory. For discrete observations it is well-established in the field of weak dependence that different sets of mixing assumptions (e.g., \(\alpha\)-mixing or \(\beta\)-mixing) and relaxations thereof can produce variance bounds and deviation inequalities that hold up to analogous results from i.i.d. observations to yield sharp nonparametric estimation results, see \([30, 70]\) for an overview. We provide an answer to the statistically fundamentally interesting question under which conditions on a multivariate continuous-time mixing Markov process drawing inference based on full observations can yield better estimation rates than under partial observations corresponding to a weakly dependent discrete observation sequence.

From a statistical perspective, this extended range of application comes at the price that mixing assumptions are in general not suited to conducting the nonparametric statistical analysis in the usual minimax sense. The reason for this is that the mixing constants are non-explicit in most cases of interest, rendering the uniform analysis of upper bounds on the statistical error over whole classes of processes impossible. Our focus is therefore on analyzing the \(\sup\)-norm risk of kernel invariant density estimators of a given single Markov process with the known minimax rates of multivariate reversible diffusion processes serving as benchmark results.

Our particular interest in \(\sup\)-norm adaptive invariant density estimation is not only rooted in the higher degree of interpretability of such statements compared to the pointwise \(L^2\) risk, but also comes from the observation that certain problems from applied probability can only be handled with statistical tools when \(\sup\)-norm estimation bounds of a quantity of interest are available. This point is highlighted in \([22]\), where results from this paper are implemented for the development of data-driven stochastic optimal control strategies for the probabilistically quite diverse problems of optimally reflecting underlying diffusions and Lévy processes. Moreover, the general issue of adaptive invariant density estimation is not only interesting for purely intrinsic mathematical reasons, but is also highly relevant for related nonparametric statistical questions in an ergodic setting such as drift estimation for stochastic differential equations via Nadaraya–Watson type path estimators, whose analysis requires sharp invariant density estimation rates. This has been demonstrated for continuous diffusion processes under different risk measures and coefficient assumptions \([27, 32, 76, 77]\), but can potentially also be extended to additional Lévy jump structures. The application of our general kernel invariant density estimation results to Lévy driven SDEs in this paper can therefore serve as basis for future drift estimation investigations of such processes. We also emphasize that the uniform moment bounds for path integrals of Markov processes with general mixing rates that we develop as the fundamental tool for invariant density estimation under exponential mixing can also be utilized for \(\sup\)-norm risk analysis of invariant density and drift estimators in subexponentially ergodic SDE models—which even in the continuous diffusion case is not well-understood in the literature.

In order to obtain a clear picture and benchmark results that are not distorted by discretization errors, we work under the assumption that a continuous observation of a trajectory \(X^T = (X_t)_{t \in [0,T]}\) of \(X\) is available. For the analysis of statistical methods (e.g., for estimating the characteristics of \(X\)), variance bounds and deviation inequalities are of central importance. Section 2 focuses on the analysis of the variance of additive functionals of the form \(\int_0^T f(X_s) \, ds\) for the ergodic process \(X\). We introduce sets of general assumptions on transition and invariant density which allow to prove tight variance bounds (cf. Propositions 2.1 and 2.5). Here, we consider an on-diagonal heat kernel bound to regulate the short-time transitional behaviour of the process and either local uniform transition density convergence to the invariant distribution at sufficient speed for any dimension \(d \in \mathbb{N}\) or exponential \(\beta\)-mixing in dimension \(d \geq 2\) to obtain tight controls on the long-time transitions of the process. The combination of heat kernel bound and local uniform transition density convergence can be interpreted as a localized version of the Castellana–Leadbetter condition that separates the short- and long-time effects and considerably weakens the inherent assumptions on the speed at which the law of \(X_t\) approaches a singular distribution as \(t \downarrow 0\) in higher dimensions. We give a detailed analysis of this condition. We demonstrate how total variation convergence at sufficient speed implies the local uniform transition density assumption and argue that in case of \(\mu\)-a.s. exponential ergodicity of the process, exponential \(\beta\)-mixing and local uniform transition density convergence are essentially equivalent, giving a homogeneous picture of our different sets of assumptions.

In Section 3 we proceed by showing how the \(\beta\)-mixing property of \(X\)—which is satisfied for a wide range of Markov processes appearing in applied and theoretical probability theory—is reflected in uniform moment bounds on empirical processes associated to integral functionals of \(X\). More precisely, for countable classes \(\mathcal{G}\) of bounded measurable functions \(g\), we establish an upper bound on

\[
\left( \mathbb{E} \left[ \sup_{g \in \mathcal{G}} \left| \frac{1}{T} \int_0^T g(X_s) \, ds - \int g \, d\mu \right|^p \right] \right)^{1/p}, \quad p \geq 1,
\]

(cf. Theorem 3.1) stated in terms of entropy integrals related to \(\mathcal{G}\) and the variance of the integral functionals. This result holds for \(\beta\)-mixing Borel right processes on general state spaces without any assumptions on the existence of transition
densities, i.e., Assumption (A20) is diminished to stationarity which further increases the applicability of our findings for future investigations. Such moment bounds and associated uniform deviation inequalities are generally the focal point for efficient implementation of adaptive estimation procedures, both for the \( \sup \)-norm as well as the pointwise and integrated \( L^2 \) risk. In our concrete estimation context, we use the uniform moment bounds together with the variance bounds from Section 2 to establish sharp deviation inequalities for the \( \sup \)-norm risk of a kernel invariant density estimator that is essential for the adaptive estimation scheme considered in Section 4 that we describe below.

In presence of additional information on the irregularity of paths provided by the heat-kernel estimate, we establish in Section 4 that the stationary density of exponentially \( \beta \)-mixing Markov processes can be estimated in any dimension at optimal rates both wrt. \( \sup \)-norm risk and pointwise \( L^2 \) risk—where optimality is understood relative to the benchmark minimax rates known for continuous reversible diffusion processes. We go even further by showing that in dimension \( d \geq 3 \)—where the optimal bandwidth choice depends on the typically unknown degree of Hölder smoothness \( \beta \)—fitting a Lepski type adaptive bandwidth selection scheme proposed in [41] for i.i.d. data to our needs provides optimal estimation rates up to iterated \( \log \)-factors (see also [52] for an adaptive scheme for anisotropic \( \sup \)-norm estimation for i.i.d. observations). More precisely, our main result Theorem 4.2 shows that, given a kernel estimator \( \hat{\rho}_{h,T} \) for the unknown invariant density \( \rho \) with bandwidth choice

\[
h \equiv h(T) \sim \begin{cases} \log^2 T / \sqrt{T}, & d = 1, \\ \log T / T^{1/4}, & d = 2, \\ (\log T / T)^{1/(2\beta + d - 2)}, & d \geq 3, \end{cases}
\]

we have, for any \( p \geq 1 \) and a bounded open domain \( D \),

\[
\mathbb{E} \left[ \sup_{x \in D} |\hat{\rho}_{h,T}(x) - \rho(x)|^p \right]^{1/p} \lesssim \begin{cases} O(\sqrt{\log T / T}), & d = 1, \\ O(\log T / \sqrt{T}), & d = 2, \\ O((\log T / T)^{\frac{\beta}{2\beta + d - 2}}), & d \geq 3. \end{cases}
\]

Although we consider a nonparametric framework, the question of data-driven estimation only arises in dimension \( d \geq 3 \). In this case, we suggest to replace the smoothness-dependent bandwidth choice \( h(T) \) by the adaptive selector \( \hat{h}_{T} \equiv \hat{h}_{T}^{(k)} \) introduced in (4.2). Then, if the order of the kernel is sufficiently large and for \( \log_{(k)} T \) denoting the \( k \)-th iterated logarithm,

\[
\mathbb{E} \left[ \sup_{x \in D} |\hat{\rho}_{\hat{h}_{T},T}(x) - \rho(x)| \right] \lesssim O\left( \left( \log_{(k)} T \log T \right)^{\frac{\beta}{2\beta + d - 2}} \right),
\]

where \( k \in \mathbb{N} \) can, in principle, be chosen arbitrarily large—which however decreases the size of the set of candidate bandwidths for the adaptive selection procedure given a finite observation horizon. We emphasize that the logarithmic gap could be avoided if constants appearing in the uniform deviation inequality from Section 3 were explicitly calculated. This, however, requires exact knowledge of the ergodic and short-time behaviour of the process, contradicting a truly adaptive nature of the approach.

Such \( \sup \)-norm adaptive multivariate estimation results are completely new and complement adaptive \( L^2 \) estimation procedures based on model selection considered in [24] for discrete time mixing chains and in [4] for Lévy driven jump-diffusions. We emphasize that [24] also consider estimation of continuous-time mixing processes in terms of their sampled skeletons. However, our faster adaptive estimation rates in presence of heat kernel bounds demonstrate that such approach can be considerably improved by not taking a Markov chain viewpoint under partial observations but by exploiting continuous-time probabilistic structures under full observations.

As a concrete example, we investigate multidimensional SDEs with Lévy-driven jump part, i.e., Markov processes associated to the solution of

\[
dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t + \gamma(X_{t-}) \, dZ_t, \quad X_0 = x \in \mathbb{R}^d, \tag{1.1}
\]

where \( W \) is \( d \)-dimensional Brownian motion and \( Z \) is a pure jump Lévy process independent of \( W \). In Section 4.2, we investigate Lévy driven Ornstein–Uhlenbeck processes as the basic class of Lévy driven jump diffusions with unbounded drift coefficient. In presence of non-trivial Gaussian part and very mild moment assumptions on the Lévy measure, we infer optimal \( \sup \)-norm and pointwise \( L^2 \) invariant density estimation results in any dimension. In this case, an adaptive estimation procedure is not necessary, since the invariant density is a smooth function. In Section 4.3, we allow for more flexible dispersion and jump coefficients \( \sigma, \gamma \) with the price to be paid being boundedness of the drift \( b \). By considering
Basic notation. A set $B \in \mathcal{B}(\mathbb{R}^d)$ is called $\mu$-full if $\mu(B) = 1$. We say that the Borel right Markov process $X$ is $\mu$-a.s. $V$-ergodic at speed $\xi$ if, for some $\mu$-full set $\Lambda$,

$$
\|P_t(x, \cdot) - \mu\|_{TV} \leq CV(x)\xi(t), \quad t \geq 0, x \in \Lambda,
$$

where $V : \mathbb{R}^d \to [0, \infty]$ with $V1_\Lambda(x) < \infty$ and, for a signed measure $\nu$, $\|\nu\|_{TV} := \sup_{|f| \leq 1} |\nu(f)|$ denotes its total variation norm. If (1.2) holds with $\xi(t) = (1 + t)^{-\alpha}$ for some $\alpha > 0$, we say that $X$ is $\mu$-a.s. $V$-polynomially ergodic of degree $\alpha$. If $\xi(t) = e^{-\kappa t}$ for some $\kappa > 0$, then $X$ is called $\mu$-a.s. $V$-exponentially ergodic. When $\Lambda = \mathbb{R}^d$ and $V(x) < \infty$ for any $x \in \mathbb{R}^d$, we just say that $X$ is $V$-ergodic at speed $\xi$ (resp., $V$-polynomially ergodic and $V$-exponentially ergodic).

For any multi-index $\alpha \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$, set $|\alpha| = \sum_{i=1}^d \alpha_i$ and $x^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$. For $\|\beta\|$ denoting the largest integer strictly smaller than $\beta$, introduce the Hölder class on an open domain $D \subset \mathbb{R}^d$.

$$
\mathcal{H}_D(\beta, L) = \left\{ f \in C^{\|\beta\|}(D, \mathbb{R}) : \max_{|\alpha| = \|\beta\|} \sup_{x,y \in D, x \neq y} \frac{|f^{(\alpha)}(x) - f^{(\alpha)}(y)|}{|x - y|^{\beta - \|\alpha\|}} \leq L, \sup_{x \in D} |f(x)| \leq L \right\},
$$

where $f^{(\alpha)} := \frac{\partial^{(|\alpha|)}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f$. Recall that a kernel function $K : \mathbb{R}^d \to \mathbb{R}$ is said to be of order $\ell \in \mathbb{N}$ if, for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq \ell$, $x \mapsto x^\alpha K(x)$ is integrable and, moreover, $\int_{\mathbb{R}^d} K(x) \, dx = 1$, $\int_{\mathbb{R}^d} K(x)\, x^\alpha \, dx = 0$, for $\alpha \in \mathbb{N}^d, |\alpha| \in \{1, \ldots, \ell\}$.

2. Basic framework and variance analysis of integral functionals of general Markov processes

This section focuses on the analysis of the variance of integral functionals of the form $\int_0^t f(X_s) \, ds$ for the ergodic process $X = (X_s)_{0 \leq s \leq t}$ under different sets of general assumptions on $X$ that will carry us through the rest of the paper. Such variance bounds are indispensable tools for statistical applications since (as we will see in Section 3) the variance of integral functionals naturally appears in associated deviation inequalities and related moment bounds and thus requires tight estimates. All proofs for this section can be found in Appendix A.2.

2.1. Variance analysis under assumptions on transition and invariant density

Recall the definition of Assumption (\mathcal{A}0) from the introduction. We start by working under the following set of additional assumptions:

(\mathcal{A}1) In case $d = 1$, there exists a non-negative, measurable function $\alpha : (0, 1] \to \mathbb{R}_+$ such that, for any $t \in (0, 1],$

$$
\sup_{x,y \in \mathbb{R}} p_t(x, y) \leq \alpha(t) \quad \text{and} \quad \int_0^1 \alpha(t) \, dt = c_1 < \infty,
$$

and, in case $d \geq 2$, there exists $c_2 > 0$ such that the following on-diagonal heat kernel estimate holds true:

$$
\forall t \in (0, 1] : \sup_{x,y \in \mathbb{R}^d} p_t(x, y) \leq c_2 t^{-d/2}.
$$

(\mathcal{A}2) There exists a $\mu$-full set $\Lambda$ such that, for any compact set $\mathcal{S} \subset \mathbb{R}^d$, there exists a non-negative, measurable function $r_\mathcal{S} : (0, \infty) \to \mathbb{R}_+$ such that

$$
\forall t > 1 : \sup_{x,y \in \mathbb{R}^d} |p_t(x, y) - \rho(y)| \leq r_\mathcal{S}(t) \quad \text{with} \quad \int_1^\infty r_\mathcal{S}(t) \, dt = c_\mathcal{S} < \infty.
$$

An essential aspect of the statistical analysis of stochastic processes is the influence of the dimension of the underlying process. It is known that certain phenomena (as compared, e.g., to estimation based on i.i.d. observations) occur in the one-dimensional case. However, these phenomena can usually only be detected by means of specific techniques that take
advantage of the unique probabilistic characteristics of scalar processes such as local time for one-dimensional diffusion processes. A “standardized” statistical framework which covers all dimensions with similar conditions cannot capture these phenomena. Our assumptions may therefore be understood as an attempt to find general conditions that make no reference to dimension or process specific phenomena, yet yield variance bounds which are tight enough to allow proving optimal convergence rates for nonparametric procedures.

In this regard, they should be compared to the Castellana–Leadbetter condition [18] requiring that

\[ \int_{(0, \infty)} \sup_{x, y \in \mathbb{R}^d} |\rho(x)p_t(x, y) - \rho(x)p(y)| \, dt < \infty, \tag{2.3} \]

which allows \( L^2 \) estimation of the invariant density via a kernel estimator at parametric (or superoptimal [14]) rate \( 1/T \) in any dimension \( d \geq 1 \). Since (A1) implies that \( \rho \) is bounded, (A2) can be understood as a localized, unweighted alternative to (2.3) away from 0, which captures the mixing behaviour of the process as we discuss below. Our assumption (A1) corresponds to the integral part of (2.3) close to 0 and guarantees that the distribution of \( X_t \) is not too close to a singular distribution. However, in dimension \( d \geq 2 \) this assumption is much milder than (2.3) since heat kernel bounds on the transition density are quite common for many multidimensional Markov processes such as strong solutions of (jump) SDEs. On the other hand, (2.3) is too strong for such Markov processes, since, e.g., the minimax optimal \( L^2 \) rate for multivariate diffusions processes is known to be worse than \( 1/T \) and hence the variance bound implied by (2.3) cannot be achieved.

Also note that the transition density bounds formulated in (A1) are weak compared to related literature dealing with statistical estimation of jump processes. E.g., [4] construct their assumptions on the coefficients and the jump measure of a \( d \)-dimensional Lévy-driven jump diffusion to guarantee a heat kernel-type estimate of the form

\[ p_t(x, y) \leq t^{-d/2} e^{-\lambda_2|x-y|^2} + \frac{t}{\sqrt{t + \|y - x\|^2d + \gamma}}, \quad x, y \in \mathbb{R}^d, t \in (0, T], \]

for the estimation horizon \( T > 0 \), where \( \alpha \in (0, 2) \) is the self-similarity index of a strictly \( \alpha \)-stable Lévy process whose Lévy measure is assumed to dominate the Lévy measure governing the jumps of the SDE. Clearly, this condition is stronger than what we require and is fitted to the concrete probabilistic setting. The reason for this specific choice becomes apparent from Corollary 4.7 in Section 4.3, but our approach reveals that (A1) is sufficient to obtain tight variance bounds in a general multivariate setting. Let us now give the variance bounds implied in our framework.

**Proposition 2.1.** Suppose that (A1) and (A2) are satisfied, and let \( f \) be a bounded function with compact support \( S \) fulfilling \( \lambda(S) < 1 \). Then, there exists a constant \( C > 0 \) independent of \( f \), such that, for any \( T > 0 \),

\[ \text{Var} \left( \int_0^T f(X_t) \, dt \right) \leq C(1 \lor c_S)T \|f\|_2^2 \lambda(S) \mu(S) \psi^2_d(\lambda(S)), \]

with \( \psi_d(x) := \begin{cases} 1, & d = 1, \\ \sqrt{1 + \log(1/x)}, & d = 2, \\ x^{d - \frac{1}{2}}, & d \geq 3, \end{cases} \]

where the variance is taken with respect to \( \mathbb{P} \).

To get an impression of the usefulness of the above result, let us discuss the relation of the local uniform transition density convergence assumption (A2) to more general and often conveniently verifiable stability conditions on \( X \). In [83], conditions on the characteristic function \( \varphi_{X_t}^\lambda(\lambda) := \mathbb{E} e^{i \langle X_t, \lambda \rangle} \) of \( X_t \) and the Fourier transform \( \{ \mathcal{F}_\mu \}(\lambda) = \int_{\mathbb{R}^d} \exp(i \langle x, \lambda \rangle) \mu(dx) \) were formulated in the scalar setting \( d = 1 \) that imply finiteness of the integral part away from 0 in the Castellana–Leadbetter condition (2.3). A straightforward adaption to our multivariate localized setting yields the following result, with the proof being omitted.

**Lemma 2.2.** Suppose that \( X \) is \( V \)-polynomially ergodic of degree \( \gamma_1 > q/(q - 1) \) for some locally bounded function \( V \) and \( q > 1 \). If there exists \( \gamma_2 > q d \) such that

\[
\begin{align*}
(\gamma^1) \quad & |\varphi_{X_t}^\lambda(\lambda) - \{ \mathcal{F}_\mu \}(\lambda)| \leq V(x)(1 + t)^{-\gamma_1}, \quad t \geq 1, x, \lambda \in \mathbb{R}^d \\
(\gamma^2) \quad & |\varphi_{X_t}^\lambda(\lambda)| \lor |\{ \mathcal{F}_\mu \}(\lambda)| \leq (1 + \|\lambda\|)^{-\gamma_2}, \quad x, \lambda \in \mathbb{R}^d, t \geq 1,
\end{align*}
\]

then (A2) is satisfied with \( \Lambda = \mathbb{R}^d, r_S(t) \sim \sup_{x \in S} V(x)(1 + t)^{-\gamma_1} \) for compacts \( S \).

Note that (\( \gamma^2 \)) implies that the Fourier transforms of \( P_t(x, \cdot) \) and \( \mu \) are integrable and hence the Fourier inversion theorem guarantees that continuous bounded transition and invariant densities exist. Moreover, as remarked in [83], (\( \gamma^1 \)) is fulfilled whenever \( X \) is \( V \)-polynomially ergodic with rate \( \gamma_1 > 1 \).
Condition (\(\mathcal{F}2\)) is quite natural in a statistical estimation context since it essentially encodes a certain amount of smoothness of the transition and stationary density. However, the following simple observation demonstrates that the additional growth conditions on the characteristic function are not needed in presence of sufficiently fast total variation convergence. Concerning the specific set of assumptions (\(\mathcal{A}0\))–(\(\mathcal{A}2\)), it is established with this result in Section 4.2 that they are satisfied, e.g., for a large class of multivariate Lévy-driven Ornstein–Uhlenbeck processes.

**Lemma 2.3.** Suppose that \(\|p_t\|_\infty < \infty\) and that \(X\) is \(\mu\)-a.s. \(V\)-ergodic at speed \(\xi\) such that \(V1_\Lambda\) is locally bounded and \(\int_0^\infty \xi(t)\,dt < \infty\). Then, (\(\mathcal{A}2\)) holds with

\[
 r_S(t) = 2C\|p_t\|_\infty \sup_{x \in S\setminus \Lambda} V(x)\xi(t) - 1, \quad t > 1.
\]

Recall that the stationary Markov process \(X\) is said to be \(\beta\)-mixing if

\[
 \beta(t) := \int_{\mathbb{R}^d} \|P_t(x, \cdot) - \mu(\cdot)\|_{TV} \,\mu(dx) \to 0, \quad t \to \infty.
\]

Hence, if \(X\) is \(\mu\)-a.s. \(V\)-ergodic at speed \(\xi\) for a function \(V\) such that \(\mu(V) < \infty\), then \(\beta(t) \lesssim \xi(t)\), i.e., \(X\) is \(\beta\)-mixing at speed \(\xi\). If there exist constants \(\kappa, c_\xi > 0\) such that \(\beta(t) \leq c_\xi e^{-\kappa t}\) for any \(t > 0\), then \(X\) is said to be exponentially \(\beta\)-mixing. This is always true for \(\mu\)-a.s. \(V\)-exponentially ergodic Markov processes since in this case we can always find a nonnegative function \(\bar{V} \in L^1(\mu)\) such that \(X\) is \(\bar{V}\)-exponentially ergodic as well, which follows from a straightforward extension of [64, Theorem 6.14.(iii)] to the continuous-time case. Conversely, it follows from combining [65, Theorem 1] and [64, Theorem 6.14.(iii)] that if \(X\) is exponentially \(\beta\)-mixing, then \(X\) is \(\mu\)-a.s. \(V\)-exponentially ergodic for some function \(V\) satisfying \(\mu(V) < \infty\). See also [20, Lemma 8.9] or [16, Theorem 3.7] for these statements. Exponential \(\beta\)-mixing is formulated as assumption (\(\mathcal{A}2\)) in the next section and will be one of the pillars of our statistical analysis for the sup-norm risk. It is therefore critical for us to understand the exact relationship between exponential \(\beta\)-mixing and (\(\mathcal{A}2\)). To this end, as a partial converse to Lemma 2.3, we explore in Appendix A.1 under which additional (quite natural) conditions, (\(\mathcal{A}2\)) implies the exponential \(\beta\)-mixing property of \(X\). Our main findings, taking account of Lemma 2.3, Appendix A.1 and the developments in Section 2.2, are summarized in Figure 1.

A clear picture is drawn, demonstrating that local uniform transition density convergence at exponential speed is intimately connected with exponential \(\beta\)-mixing of the process—both concepts having \(\mu\)-a.s. exponential ergodicity as the driving force behind them in most concrete applications. Both conditions (\(\mathcal{A}2\)) and (\(\mathcal{A}2\)) gain substantial additional statistical power via the smoothing assumption (\(\mathcal{A}1\)), which allows obtaining tight variance bounds that yield superior estimation properties under continuous observations compared to incomplete information via sampling procedures. This will be demonstrated in Section 4. Moreover, the slightly more specific localized Castellana–Leadbetter condition provides the advantage of optimal estimation also in the scalar case \(d = 1\) and wrt the \(L^2\) risk under less restrictive assumptions on the speed of convergence of the process (polynomial is sufficient) in any dimension, which justifies us studying this concept separately from exponential \(\beta\)-mixing.
2.2. Variance analysis under exponential $\beta$-mixing

In this subsection, we specify our study to multidimensional stochastic processes by restricting the analysis to dimension $d \geq 2$. While we further assume that the on-diagonal heat kernel bound on the transition density \((\rho_0)\) from \((\rho_1)\) still holds, we drop the transition density rate assumption \((\rho_2)\) and instead impose exponential $\beta$-mixing of $X$. Note that this is implied by \((\rho_2)\) under suitable technical conditions on $X$ (see Figure 1 and Propositions A.1 and A.2 in Appendix A.1).

\((\rho_\beta)\) The process $X$ started in the invariant measure $\mu$ is exponentially $\beta$-mixing, i.e., there exist constants $c_\kappa, \kappa > 0$ such that

$$
\int \| P_t(x, \cdot) - \mu \|_{TV} \mu(dx) \leq c_\kappa e^{-\kappa t}, \quad t \geq 0.
$$

Let us emphasize that in presence of the heat kernel bound \((\rho_1)\), Lemma 2.4 below shows that Assumption \((\rho_0)\) is strengthened to the existence of a bounded invariant density since the transition density of any skeleton chain is uniformly bounded for fixed $t > 0$. That is, the following assumption is in place.

\((\rho_0+)\) Assumption \((\rho_0)\) holds and the invariant density has a bounded version $\rho$, i.e., $\|\rho\|_{\infty} < \infty$.

**Lemma 2.4.** Assume that $X$ has an invariant distribution $\mu$ and that there is some $\Delta > 0$ such that the transition density $p_\Delta$ exists and $\sup_{x,y \in \mathbb{R}^d} p_\Delta(x, y) \leq c$ for some constant $c > 0$. Then, $\mu$ admits a bounded density.

The next result gives a tight variance bound on the integral $\int_0^T f(X_t) \, dt$ under $\beta$-mixing. Its effectiveness for sup-norm estimation of general Markov processes will be demonstrated in Section 4. Note in particular that, using boundedness of $\rho$ under \((\rho_0+)\) and \((\rho_1)\), the same rate can be obtained under \((\rho_2)\) from Proposition 2.1. Recall the definition of $\psi_d: (0, e) \rightarrow \mathbb{R}_+$ in (2.4).

**Proposition 2.5.** Grant assumptions \((\rho_1)\) and \((\rho_\beta)\), and let $f$ be a bounded function with compact support $S$ fulfilling $\lambda(S) < 1$. Then, for any $d \geq 2$, there exists a constant $C > 0$ not depending on $f$ such that, for any $T > 0$,

$$
\text{Var} \left( \int_0^T f(X_t) \, dt \right) \leq C T \| f \|^2_{\infty} \| \rho \|^2_{\infty} \lambda^2(S) \psi^2_d(\lambda(S)).
$$

(2.5)

**Notation.** Throughout the sequel, we denote by $\Sigma$ the class of non-explosive, exponentially $\beta$-mixing Borel right Markov processes $X$ such that assumptions \((\rho_0+)\) and \((\rho_1)\) hold (and hence \((\rho_0+)\) is in place, i.e., the invariant density $\rho$ is bounded). Moreover, in dimension $d = 1$ we assume that \((\rho_2)\) is in place with a rate function $r_S$ which is monotone wrt the compact sets $S$ in the sense that

$$
S_1 \subseteq S_2 \Rightarrow c_{S_1} = \int_1^{\infty} r_{S_1}(t) \, dt \leq \int_1^{\infty} r_{S_2}(t) \, dt = c_{S_2} < \infty.
$$

(2.6)

Alternatively, if we do not want to restrict to exponentially $\beta$-mixing processes, consider the class of processes $\Theta$ consisting of $d$-dimensional non-explosive Borel right processes such that \((\rho_0+)\)–\((\rho_2)\) hold, where again the constants $c_S$ appearing in \((\rho_2)\) satisfy (2.6). Note that if $\tilde{\Theta}$ is the restriction of $\Theta$ containing the class of processes $X$ satisfying the assumptions of Proposition A.1 or Proposition A.2, then $\tilde{\Theta} \subseteq \Sigma$.

3. Uniform moment bounds for path integrals

We now turn to deriving uniform moment bounds for integral functionals of the ergodic process $X$. These are intimately connected with Bernstein-type tail inequalities, which due to their crucial importance for many probabilistic and statistical applications—such as the derivation of limit theorems or upper bound statements for nonparametric estimation procedures—have been excessively studied in the literature (see Section 1.1 of [40] for an overview). Both a Lyapunov function method and a functional inequalities approach can be used for deriving results on the concentration behaviour of additive functionals of $X$. [19] establish non-asymptotic deviation bounds for

$$
\mathbb{P} \left( \frac{1}{t} \int_0^t f(X_s) \, ds - \int f \, d\mu \geq r \right), \quad f \in L^1(\mu),
$$

where $r \geq 0$. 

The next result establishes uniform moment bounds for $\mathbb{E} \left( \left| \int_0^T f(X_s) \, ds \right|^d \right)$ with $d$-th moment bounds for $f(X_s)$, and it is obtained by using the following inequality:

$$
\left| \int_0^T f(X_s) \, ds \right|^d \leq d_T f(X_s)^d,
$$

where $d_T f(X_s)$ is the derivative of $f(X_s)$ with respect to $X_s$. The proof of this inequality is based on the following lemma:

**Lemma 3.1.** Let $f$ be a bounded function with compact support $S$ fulfilling $\lambda(S) < 1$. Then, for any $d \geq 2$, there exists a constant $C > 0$ not depending on $f$ such that, for any $T > 0$

$$
\text{Var} \left( \int_0^T f(X_t) \, dt \right) \leq C T \| f \|^2_{\infty} \| \rho \|^2_{\infty} \lambda^2(S) \psi^2_d(\lambda(S)).
$$

(2.5)
using different moment assumptions for \( f \) and regularity conditions for \( \mu \), “regularity” referring to the condition that \( \mu \) may satisfy various functional inequalities (F-Sobolev, generalized Poincaré, etc.). In a symmetric Markovian setting and assuming a spectral gap, Lezaud [53] uses Kato’s perturbation theory for proving Bernstein-type concentration inequalities for empirical means of the form \( \int_0^t f(X_s) \, ds \), the upper bound depending on the asymptotic variance of \( f \). Amongst other methods, [40] exploit both a Lyapunov function method and a functional inequalities approach for extending Lezaud’s result to inequalities for possibly unbounded \( f \). Going beyond the symmetric case, Lyapunov-type conditions can also be used for verifying exponential mixing properties, paving the way to generalizing concentration results based on independent observations to the dependent case. For corresponding results for discrete random (Markov) sequences under different mixing or ergodicity assumptions, we refer to [1, 2, 10, 23, 31, 51, 58, 66, 71].

### 3.1. General framework

Our main focus in this subsection is on deriving corresponding uniform moment inequalities of empirical processes, using results based on independent observations to the dependent case. For corresponding results for discrete random (Markov) processes with arbitrary topological state space \( X \), not necessarily equal to \( \mathbb{R}^d \), and general mixing rate. That is, we suppose in this section that for some rate function \( \beta \) densities is needed, but that we only work within an ergodic \( \beta \) conditions on the process. We emphasize that for this section no assumption on the existence of transition or invariant densities is needed, but that we only work within an ergodic \( \beta \)-mixing framework. Moreover, the results are established for \( \beta \)-mixing Markov processes with arbitrary topological state space \( X \), not necessarily equal to \( \mathbb{R}^d \), and general mixing rate. That is, we suppose in this section that

\[
\beta(t) = \int_X \| P_t(x, \cdot) - \mu \|_{TV} \, \mu(dx) \leq \Xi(t),
\]

for some rate function \( \Xi(t) \) decreasing to 0 as \( t \to \infty \). We aim to prove moment bounds for suprema of the form

\[
\sup_{g \in G} |G_t(g)| := \|G_t\|_{G}, \quad \text{for } G_t(g) := \frac{1}{\sqrt{t}} \int_0^t g(X_s) \, ds,
\]

where the supremum is taken over entire (possibly infinite-dimensional) function classes \( G \subset B_0(\mathcal{X}) \) of \( \mu \)-centered measurable bounded functions on \( \mathcal{X} \). Similarly to [9] and [33], we apply the generic chaining device for the derivation of our result. The basic strategy of the proof is splitting the integral into blocks of length \( m_t \), constructing an independent Berbee coupling based on the \( \beta \)-mixing property as described in Viennet [84], and then using the classical Bernstein inequality for i.i.d. random variables for the coupled integral blocks to drive the chaining procedure from [33]. The use of Berbee’s coupling lemma is a well-established method for studying empirical processes of discrete \( \beta \)-mixing sequences, see [69, Chapter 8], and has recently been employed in [4] for establishing \( L^2 \) oracle bounds for an adaptive estimator of the invariant density of a class of exponentially \( \beta \)-mixing Lévy-driven jump diffusions.

We now formulate a crucial tool for deriving upper bounds on the \( \sup \)-norm risk of estimators of the invariant density of processes \( X \in \Sigma \). Our final moment bound on the supremum of the process \( G_t \) is stated in terms of entropy integrals of the indexing function class \( G \). In many applications, the corresponding assumption is straightforward to verify. For any given \( \varepsilon > 0 \), denote by \( N(\varepsilon, G, d) \) the covering number of \( G \), i.e., the smallest number of balls of \( d \)-radius \( \varepsilon \) needed to cover \( G \). Furthermore, given \( f, g \in G \), let \( d_{\varepsilon, t}(f, g) := \|f - g\|_{\infty} \) and

\[
d_{\varepsilon, t}(f, g) = \sigma_t^2(f - g), \quad \text{where } \sigma_t^2(f) := \text{Var} \left( \frac{1}{\sqrt{t}} \int_0^t f(X_s) \, ds \right).
\]

**Theorem 3.1.** Suppose that \( X \) is \( \beta \)-mixing with rate function \( \Xi(t) \). Let \( G \) be a countable class of bounded real-valued functions with \( \mu(g) = 0 \) and let \( m_t \in (0, t/4] \). Then, there exist \( \tau \in [m_t, 2m_t] \) and constants \( \bar{C}_1, \bar{C}_2 > 0 \) such that, for any \( 1 \leq p < \infty \),

\[
(\mathbb{E} [\|G_t\|_G^p])^{1/p} \leq \bar{C}_1 \int_0^\infty \log N(u, G, \frac{2m_t}{\sqrt{t}} \|d_\infty\|) \, du + \bar{C}_2 \int_0^\infty \sqrt{\log N(u, G, d_{G, \tau})} \, du + 4 \sup_{g \in G} \left( \frac{2m_t}{\sqrt{t}} \|g\|_{\infty} \bar{c}_1 \bar{p} + \|g\|_{G, \tau} \bar{c}_2 \sqrt{\bar{p}} + \frac{1}{2} \|g\|_{\infty} \sqrt{\bar{t}} \Xi(m_t)^{1/p} \right),
\]

for positive constants \( \bar{c}_1, \bar{c}_2 \) defined in (B.2).

**Remark 3.2.** Consider \( p = 1 \) and the specific choice of \( m_t = \kappa^{-1} \log t \) in case of exponential \( \beta \)-mixing rate \( \Xi(t) = c_\kappa \exp(-\kappa t) \). Then, the above result implies that

\[
\mathbb{E} [\|G_t\|_G] \lesssim \int_0^\infty \log N(u, G, \frac{\log t}{\sqrt{t}} \|d_\infty\|) \, du + \int_0^\infty \sqrt{\log N(u, G, d_{G, \tau})} \, du + \sup_{g \in G} \left( \frac{\log t}{\sqrt{t}} \|g\|_{\infty} + \|g\|_{G, \tau} \right).
\]
If we considered the related discrete time problem of finding uniform moment bounds for additive functionals
\[ \frac{1}{\sqrt{n}} \sum_{k=0}^{n} g(X_k) \] of a Markov chain \((X_n)_{n \in \mathbb{N}_0} \) and assumed exponential ergodicity of the chain, using the state of the art
Bernstein inequality given in [1, Theorem 6] (see also [51]) for the generic chaining procedure would yield an analogous
result with an asymptotic version of the variance norm. In particular, the \( \log \)-scaling of the \( \sup \)-norm is also present in the
discrete time case as a consequence of exponential ergodicity, whereas in the i.i.d. case this factor would disappear. Our
direct coupling approach therefore yields tight uniform moment bounds and makes the contribution of the mixing term
transparent, which paves the way for studying nonparametric implications of sub-exponential mixing rates for \( \sup \)-norm
estimation problems in continuous time.

To get a first taste of the consequences of Theorem 3.1, consider the trivial situation where \( G \) is a singleton set. This allows the study of rates for the \( L^p \)-version of von Neumann’s ergodic theorem\(^1\) for continuous-time ergodic Markov processes which states that, for \( g \in L^p(\mu) \),
\[
\frac{1}{T} \int_0^T g(X_t) \, dt \rightarrow \mu(g), \quad \text{in } L^p(\mathbb{P}).
\]
Indeed, \( \beta \)-mixing implies strong mixing such that the \( \sigma \)-algebra of shift invariant sets is \( \mathbb{P} \)-trivial and hence the ergodic theorem is satisfied.

**Corollary 3.3.** Suppose that \( X \) is exponentially \( \beta \)-mixing. Then, there exists a constant \( C > 0 \) such that, for any \( T > 0 \),
\[ 1 \leq p < \infty \text{ and any bounded, measurable function } g, \]
\[
\left\| \frac{1}{T} \int_0^T g(X_t) \, dt - \mu(g) \right\|_{L^p(\mathbb{P})} \leq C_p \|g\|_{\infty} \frac{1}{\sqrt{T}}.
\]
If \( X \) is polynomially mixing of degree \( \alpha > 1 \), i.e., \( \Xi(t) \lesssim t^{-\alpha} \), then for any \( p \geq 1 \) and \( T \geq 4(\alpha + p)/\alpha \) we have
\[
\left\| \frac{1}{T} \int_0^T g(X_t) \, dt - \mu(g) \right\|_{L^p(\mathbb{P})} \lesssim \|g\|_{\infty} T^{-\left(\frac{1}{2} \wedge \frac{\alpha}{\alpha + p}\right)}.
\]

### 3.2. Deviation inequalities for suprema of empirical Markov processes

Theorem 3.1 provides a foundation for the derivation of deviation inequalities, as they are needed, for example, for bounding the \( \sup \)-norm risk of estimators and for the convergence analysis of adaptive estimation procedures. We will focus on the question of invariant density estimation for Borel right Markov processes, introduced and discussed in Section 2. Recall the definition of \( \Sigma \) and \( \Theta \) at the end of that section. Given the observation \((X_s)_{0 \leq s \leq T}\), a natural kernel estimator for the invariant density \( \rho \) on a domain \( D \) of a Markov process \( X \in \Sigma \cup \Theta \) is given by
\[
\hat{\rho}_{h,T}(x) = \frac{1}{T} \int_0^T K(h(x - X_s)) \, ds, \quad x \in \mathbb{R}^d, \quad \text{where } K_h(\cdot) := h^{-d} K((\cdot)/h), \quad h > 0,
\]
for some smooth, Lipschitz continuous kernel function \( K : \mathbb{R}^d \rightarrow \mathbb{R} \) with compact support \([-1/2, 1/2]^d\). The knowledge of the invariant density is not only a question of its own interest, but is also needed, among other things, for the implementation of drift estimation procedures or data-driven methods of stochastic control. Furthermore, this specific estimation problem can be regarded as an acid test for the quality of the statistical analysis: It is known that the invariant density of (possibly multidimensional) diffusion processes can be estimated with a faster convergence rate than is feasible in the classical discrete i.i.d. or weak dependency context. However, these superior convergence rates can only be verified with sufficiently tight estimates in the proof of the upper bound, more precisely, for the stochastic error part appearing in the decomposition
\[
\hat{\rho}_{h,T}(x) - \rho(x) = \mathbb{H}_{h,T}(x) + (\rho * K_h - \rho)(x), \quad \text{for } \mathbb{H}_{h,T}(x) := \hat{\rho}_{h,T}(x) - \mathbb{E}[\hat{\rho}_{h,T}(x)].
\]
While the bias part is bounded using standard arguments, tight upper bounds on (the supremum of) the stochastic error
require specific probabilistic tools.

The following uniform deviation inequality is central for our statistical analysis. Its proof requires bounding
\( \mathbb{E}[\sup_{x \in D} |\mathbb{H}_{h,T}(x)|^p] \) which is done by applying Theorem 3.1 to the function class
\[
\mathcal{G} := \left\{ \mathcal{K}((x - \cdot)/h) : x \in D \cap \mathbb{Q}^d \right\}, \quad \text{where } \mathcal{K}((x - \cdot)/h) = K((x - \cdot)/h) - \mu(K((x - \cdot)/h)),
\]
\(^1\)Not referring to the \( L^p \)-statement as Birkhoff’s ergodic theorem is not without reason, see [89].
for some kernel function $K$ as in (3.2) with Lipschitz constant $L$ wrt to the sup-norm $\|\cdot\|_\infty$, and the bandwidth $h$ chosen in $(0, 1)$.

Recall that any $X \in \Sigma$, by definition, is exponentially $\beta$-mixing, i.e., $\beta$-mixing with rate function $\Xi(t) = c_\kappa e^{-\kappa t}$ for some constants $c_\kappa, \kappa > 0$.

**Lemma 3.4.** Suppose that $X \in \Theta \cup \Sigma$ and additionally assume in case $X \in \Theta$ that $X$ is $\beta$-mixing with strictly decreasing rate function $\Xi(t)$. Then, for any $u_T \geq 1$ such that $\Xi^{-1}(T^{-u_T}) \in o(T)$ and $T^{-2} \leq h = h_T \in o(1)$, there exists a constant $c^* > 0$ such that for large enough $T$

$$P \left( \|\hat{\rho}_{h,T} - E\hat{\rho}_{h,T}\|_{L_\infty(D)} \geq c^* \left( \frac{u_T + \log(T)}{Th^d} \Xi^{-1}(T^{-u_T}) + T^{-\frac{d}{2}} \psi_d(h^d) \sqrt{u_T \log(h^{-1})} \right) \right) \leq e^{-u_T}.$$

In particular, when $X \in \Sigma$, for any $\gamma > 0$ and $u_T \in [1, \gamma \log T]$ there exists a constant $c_\gamma > 0$ such that for large enough $T$

$$P \left( \|\hat{\rho}_{h,T} - E\hat{\rho}_{h,T}\|_{L_\infty(D)} \geq c_\gamma \Upsilon_{h,T}(u_T) \right) \leq e^{-u_T},$$

where

$$\Upsilon_{h,T}(u) := \frac{u(\log(T))^2}{Th^d} + T^{-\frac{d}{2}} \psi_d(h^d) \sqrt{u \log(h^{-1})}, \quad u \geq 1. \tag{3.5}$$

### 4. sup-norm adaptive estimation of the stationary density

In this section, we demonstrate the effectiveness of our previous results and probabilistic tools in a concrete statistical application. We already introduced the general form of the kernel invariant density estimator in (3.2). In order to quantify the speed of convergence, we will now analyse its convergence behaviour under standard Hölder smoothness assumptions, i.e., we focus on the problem of estimating the invariant density $\rho$ on a domain $D$ of a Markov process $X \in \Sigma \cup \Theta$ with $|\rho|_D \in H_D(\beta, L)$ (as introduced in (1.3)). For stating our statistical results, we define

$$\Phi_{d,\beta}(T) := \begin{cases} 1/\sqrt{T}, & d = 1, \\ \sqrt{\frac{\log T}{T}}, & d = 2, \\ T^{-\frac{d}{2d+\beta-2}}, & d \geq 3, \end{cases} \quad \Psi_{d,\beta}(T) := \begin{cases} \sqrt{\frac{\log T}{T}}, & d = 1, \\ \frac{\log T}{\sqrt{T}}, & d = 2, \\ \left( \frac{\log T}{T} \right)^{\frac{\beta}{d(\beta-2)}}, & d \geq 3. \tag{4.1} \end{cases}$$

Note that these convergence rates have already been identified in the literature as being optimal in the minimax sense, where $\Phi_{d,\beta}(\cdot)$ is associated with the pointwise or $L^2$ risk, while $\Psi_{d,\beta}(\cdot)$ contains an additional logarithmic factor as it inevitably arises when passing to the examination of the sup-norm risk.

For the case $d = 1$, we refer to [49, Section 4.2]. The multivariate case is less classical. For a class of multivariate reversible diffusion processes satisfying spectral gap and Nash-type inequalities, $\Psi_{d,\beta}(\cdot)$, $d \geq 3$, has been identified in [78] as the minimax optimal convergence rate (cf. Theorem 3.4 and Theorem 3.6). For $d = 2$, a uniform upper bound of order $\Psi_{2,\beta}(\cdot)$ is stated in (1.7) in [78], which can be complemented with a lower bound obtained using similar arguments to the proof of Theorem 5 from the current reference [5]. Adapting the strategy from [78] to the simpler case of pointwise $L^2$ risk one can verify minimax optimality of the rates $\Phi_{d,\beta}$ for reversible diffusions, which has also been carried out explicitly in [5] for a class of exponentially ergodic diffusions under explicit boundedness and smoothness constraints on the coefficients.

Given the benchmark results mentioned above, the rates introduced in (4.1) will be referred to as “optimal” in what follows. However, we actually do not target the verification of optimality in the minimax sense, since this would require in particular to verify upper bound statements holding uniformly over entire classes of processes. Controlling the constants involved in mixing inequalities is known to be extremely challenging and can only be achieved by adding further assumptions to the processes under consideration.

All proofs of this section are given in Appendix C. Throughout, $K$ denotes a $\|\cdot\|_\infty$-Lipschitz kernel of order $\ell$ and with Lipschitz constant $L$ that is supported on $[-1/2, 1/2]^d$.

#### 4.1. General framework

Depending on the concrete application, one might be interested in quantifying the accuracy of estimators in terms of different risk measures. Our findings from Section 2 immediately imply an upper bound on the classical mean squared error at some fixed point $x \in \mathbb{R}^d$. 
Corollary 4.1. Suppose that $X \in \Sigma \cup \Theta$. For $x \in \mathbb{R}^d$ such that there exists an open neighbourhood $D \subset \mathbb{R}^d$ of $x$ such that $\rho|_D \in \mathcal{H}_D(\beta, L)$, $\beta \in (0, \ell + 1]$, it holds for the kernel estimator

$$
\mathbb{E} \left[ (\hat{\rho}_{h,T}(x) - \rho(x))^2 \right] \in \mathcal{O} \left( \Phi_{d,\beta}(T) \right), \quad \text{if } h = h(T) \sim \begin{cases} T^{-1/\gamma}, & d \leq 2, \gamma \in (0, \beta], \\ T^{-1/(2\beta + d - 2)}, & d \geq 3. \end{cases}
$$

We now turn our focus to the technically significantly more involved problem of sup-norm adaptive invariant density estimation for processes from the class $\Sigma$ having Hölder continuous invariant densities. We demonstrate that optimal estimation rates in any dimension are achieved by kernel estimators $\hat{\rho}_{T,h}$ as introduced in (3.2) for a suitable choice of the bandwidth $h$. While in dimension $d = 1, 2$ the optimal bandwidth has the remarkable property of being independent of the (typically unknown) order $\beta$ of Hölder smoothness, this is not the case in higher dimensions $d \geq 3$. In order to remove $\beta$ from the bandwidth choice, we need to find a data-driven substitute for the upper bound on the bias in the balancing process. Heuristically, this is the idea behind the Lepski-type selection procedure suggested now:

1. Specify the discrete set of candidate bandwidths

$$
\mathcal{H}_T \equiv \mathcal{H}_T^{(k)} := \left\{ h_1 = \eta^{-l}, \ l \in \mathbb{N}_0, \ \eta^{-l} > \left( \frac{\log(k) T (\log T)^5}{T} \right)^{\frac{1}{\pi^2}}, \ \eta > 1 \text{ arbitrary}, \right\}
$$

for arbitrarily chosen $k \in \mathbb{N}$, and denote by $h_{\min}$ the smallest element in the grid $\mathcal{H}_T$. Here, $\log(k) T$ denotes the $k$-th iterated logarithm, iteratively specified by $\log(k) T := \log \log(k-1) T$ and $\log(0) T = T$, which is well-defined for $T$ large enough.

2. Define $\hat{h}_T \equiv \hat{h}_T^{(k)}$ by letting

$$
\hat{h}_T := \max \left\{ h \in \mathcal{H}_T : \| \hat{\rho}_{h,T} - \hat{\rho}_{g,T} \|_{L^\infty(D)} \leq \sqrt{\| \hat{\rho}_{h_{\min},T} \|_{L^\infty(D)} \sigma(g,T)} \ \forall g \leq h, \ g \in \mathcal{H}_T \right\}, \quad (4.2)
$$

where, for $\psi_d(\cdot)$ introduced in (2.4),

$$
\sigma(h, T) := \frac{\log(k) T (\log T)^2}{T h^d} \log(h^{-1}) + \psi_d(h^d) \sqrt{\log(k) T \log(h^{-1})}, \quad h \in \mathcal{H}_T. \quad (4.3)
$$

Letting $\| \cdot \|_{L^\infty(D)}$ denote the restriction of the sup-norm to a domain $D \subset \mathbb{R}^d$, we obtain the following result.

Theorem 4.2. Suppose that $X \in \Sigma$. Let $D \subset \mathbb{R}^d$ be open and bounded. Suppose that $\rho|_D \in \mathcal{H}_D(\beta, L)$ with $\beta \in (1, \ell + 1]$ for $d = 1$ and $\beta \in (2, \ell + 1]$ for $d \geq 2$. Then, for any $p \geq 1$,

$$
\left( \mathbb{E} \left[ \| \hat{\rho}_{h,T} - \rho \|^p_{L^\infty(D)} \right] \right)^{1/p} \in \mathcal{O} \left( \Psi_{d,\beta}(T) \right), \quad \text{if } h = h(T) \sim \begin{cases} \log^2 T / \sqrt{T}, & d = 1, \\ \log T / T^{1/4}, & d = 2, \\ (\log T / T)^{(1/2\beta + d - 2)}, & d \geq 3. \end{cases}
$$

For the adaptive bandwidth scheme, let $\hat{h}_T = \hat{h}_T^{(k)}$ be selected according to (4.2) for some $k \in \mathbb{N}$. Then, if $\rho|_D \in \mathcal{H}_D(\beta, L)$ with $\beta \in (2, \ell + 1]$, we have in any dimension $d \geq 3$,

$$
\mathbb{E} \left[ \| \hat{\rho}_{h,T} - \rho \|_{L^\infty(D)} \right] \in \mathcal{O} \left( \left( \Phi_{d,\beta}(T) \log T \right)^{\frac{d}{\pi^2 + d - 2}} \right). \quad (4.4)
$$

While the scheme of the proof of (4.4) is close to the proof of the result on nonparametric density estimation based on i.i.d. observations in [41], the established convergence rate (recall the definition (4.1)) clearly reflects the fact that the invariant density of stochastic processes can be estimated faster than in the classical i.i.d. context. This is well-known for ergodic continuous diffusion processes (see [28, 78]), but, as we will show in the sequel, the result is fulfilled for a much larger class of stochastic processes. The additional log-factor occurring in the definition of $\Psi_{d,\beta}(\cdot)$ represents the common price to be paid when switching from the pointwise error control (described by $\Phi_{d,\beta}(\cdot)$) to bounding the sup-norm risk.

Remark 4.3. (a) The conditions on the Hölder index $\beta$ stated in Theorem 4.2 are due to two different reasons: On the one hand, in dimension $d \leq 2$, we chose a bandwidth not depending on $\beta$ which still achieves the optimal balance
between bias and stochastic error. By choosing a bandwidth dependent on $\beta$ (as in Corollary 4.1), restrictions on $\beta$ could be avoided. However, for the implementation of estimators it is advantageous to be able to choose a bandwidth independent of the typically unknown smoothness $\beta$. On the other hand, in dimension $d \geq 3$, the assumption on $\beta$ is an unavoidable effect. The coupling error leaves us no other choice but to select the interval block length $n_{\text{IT}}$ in the decomposition of (3.2) of order $\log T$, which forces $\beta > 2$ to balance out bias and stochastic sensitivity of the estimator. We emphasize that this is not an artifact of our proof strategy since the additional log-factor also appears in the optimal Bernstein inequalities for geometrically ergodic Markov chains in [1, 51]. The restriction on $\beta$ can therefore be considered as a price that must be paid for the generality of our exponential $\beta$-mixing assumption.

(b) The logarithmic gap (of arbitrary iterative order $k$) between the adaptive rate (see (4.4)) and the optimal rate $\Psi_{h,\beta}$ in dimension $d \geq 3$ (see (4.1)) is not a consequence of suboptimality of arguments used in the proof. Rather, it is a deliberate choice motivated by our desire to introduce a truly adaptive selection procedure that does not rely on the specification of obscure constants. To be more precise, a key step in the proof of the upper bound for the adaptive approach requires quantifying the concentration of the estimator $\hat{\psi}_{h,T}$ around the variance proxy $\sigma(h, T)$ from (4.3), which is handled with the deviation inequality from Lemma 3.4 involving the term $T_{h,T}(\gamma \log T)$ (see (3.5)). If we remove the factor $\log(k)$ in the variance proxy $\sigma(h, T)$, we obtain

$$\frac{(\log T)^2}{Th_d^d} \log(h^{-1}) + \psi_d(h^d) \sqrt{\frac{\log(1)}{T}} \sim \Psi_{h,T}(\gamma \log T).$$

In this case, an exact quantification of the constant $c_d$ from Lemma 3.4 is mandatory, which would then be included as an additional factor in the specification of $h_T$ in (4.2). Together with an adjustment of the candidate bandwidths $\mathcal{A}_T$, this would allow us to close the logarithmic gap and hence obtain optimal rates for the adaptive procedure. However, $c_d$ is of the form $\gamma \times C(D, L, \kappa, c_1, c_2)$—where we recall that $c_1, c_2$ determine the mixing coefficient and $c_1$ is a constant appearing in the heat kernel bound from Assumption $(\mathcal{A}_1)$—and therefore can only be bounded with explicit knowledge/assumptions on the process. We avoid this fundamental problem in our procedure to not shift the problem from unknown exact smoothness to unknown exact ergodic and small time behaviour, with the price to be paid being a logarithmic loss. In this regard, our approach differs from the bandwidth selection procedure for the $L^2$ risk in [4], which relies on the choice of a “sufficiently large” constant $k$ that cannot be exactly specified or efficiently chosen in a data-driven way.

Our previous results rely on the very general conditions $(\mathcal{A}_0)$ and $(\mathcal{A}_1)$ as well as assumptions related to the speed of convergence to the invariant distribution, $(\mathcal{A}_2)$ and $(\mathcal{A}_\beta)$. For statistical purposes, however, it is essential to derive results under conditions on the coefficients of the underlying process as easily verifiable as possible. For this reason, the next two subsections are devoted to investigating specific classes of jump diffusion processes and explicit conditions on their underlying characteristics such that the above assumptions are satisfied and hence statistical conclusions can be drawn from our general theory.

### 4.2. Example: Lévy-driven Ornstein–Uhlenbeck processes

As a first example, we discuss estimation rates of $d$-dimensional Lévy-driven Ornstein–Uhlenbeck processes as representatives of Lévy-driven jump diffusions with unbounded drift coefficient by establishing assumptions on the characteristics of the Lévy process that guarantee $X \in \Sigma \cup \Theta$.

Let $Z$ be a $d$-dimensional Lévy process with generating triplet $(a, Q, \nu)$, where $a \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d \times d}$ is a symmetric positive semidefinite matrix and $\nu$ is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\nu(dx) < \infty$ such that $\mathbb{E}_0[\exp(i\langle Z_1, \theta \rangle)] = \exp(\psi(\theta))$ with

$$\psi(\theta) = i\langle a, \theta \rangle - \frac{1}{2}\langle Q\theta, \theta \rangle + \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{i\langle x, \theta \rangle} - 1 - i\langle x, \theta \rangle 1_{B(0, 1)}(x)\right) \nu(dx), \quad \theta \in \mathbb{R}^d,$$

where $B(0, 1) = \{x \in \mathbb{R}^d : |x| < 1\}$. Then, given some matrix $B \in \mathbb{R}^{d \times d}$, a Lévy-driven Ornstein–Uhlenbeck process $X$ is a solution to the SDE

$$dX_t = -BX_t dt + dZ_t,$$

given by

$$X_t = e^{-tB}X_0 + \int_0^t e^{-(t-s)B} dZ_s, \quad t \geq 0.$$
We suppose that the real parts of all eigenvalues of $B$ are positive, implying that $e^{-tB} \to 0_{d \times d}$ as $t \to \infty$, and assume the following moment condition
\[
\int_{\|z\| > 2} \log\|z\| \nu(dz) < \infty. \tag{4.5}
\]

Then, $X$ is a Markov process on $\mathbb{R}^d$ with invariant distribution $\mu$ such that
\[
\{\mathcal{F}_t \mu\}(u) = \exp \left( \int_0^\infty \psi(e^{-sB^T} u) \, ds \right), \quad u \in \mathbb{R}^d,
\]
and $\varphi_{X_t} (u) = \exp \left( i(x, e^{-tB^T} u) + \int_0^t \psi(e^{-sB^T} u) \, ds \right)$, $u, x \in \mathbb{R}^d, t > 0$,

see [72, Theorem 3.1, Theorem 4.1]. Let us now introduce the following conditions.

(\theta 1) $\text{rank}(Q) = d$;
(\theta 2) $\int_{\|x\| > 1} \|x\|^p \nu(dx) < \infty$ for some $p > 0$;
(\theta 3) $\int_{\|x\| > 1} (\log\|x\|)^\alpha \nu(dx) < \infty$ for some $\alpha > 2$.

These assumptions are borrowed from [55], [56] and [48], where (sub)-exponential ergodicity and exponential $\beta$-mixing of OU-processes are investigated. (\theta 1) guarantees the strong Feller property of $X$ and the existence of a $C^\infty$-density for $P_t(x, \cdot), x \in \mathbb{R}^d$ ([55, Theorem 3.1]). Similar arguments to the ones in [55, Theorem 3.2] also show that under (\theta 1), $\mu$ admits a $C^\infty$-density $p$. (\theta 2) and (\theta 3) are moment assumptions on $Z$, where (\theta 3) in absence of (\theta 2) corresponds to an extremely heavy-tailed distribution and represents a minor strengthening of the necessary and sufficient criterion (4.5) for stationarity of $X$.

Based on the results from [48, 55, 56] together with our investigations in Sections 2 and 4.1, we can obtain the following result which is proved in Appendix C.

**Theorem 4.4.** Suppose that (\theta 1) holds. Then, in any dimension $d \in \mathbb{N}$, (\theta 1) holds with
\[
\sup_{x, y \in \mathbb{R}^d} p_t(x, y) \lesssim t^{-d/2}, \quad t \in (0, 1]. \tag{4.6}
\]

If, additionally,

(i) (\theta 2) holds for some $p > 0$, then, for any $d \geq 1$, $X \in \Sigma \cap \Theta$;

(ii) (\theta 3) holds, then, for $d = 1$, $X \in \Theta$.

Let $d \geq 1$ in scenario (i) and $d = 1$ in scenario (ii). Then, for arbitrary $\beta \in (0, \ell + 1]$, we obtain for any $x \in \mathbb{R}^d$ that

\[
\mathbb{E} \left[ (\hat{\rho}_{h,T}(x) - \rho(x))^2 \right] \in O(\Psi^2_{d, \beta}(T)), \quad \text{if } h = h(T) \sim \begin{cases} T^{-1}, & d \leq 2, \\ T^{-1/(2\beta + d - 2)}, & d \geq 3, \end{cases}
\]

and for any bounded, open domain $D \subset \mathbb{R}^d$ and $p \geq 1$ that in scenario (i)

\[
\mathbb{E} \left[ \|\hat{\rho}_{h,T} - \rho\|^p_{L^\infty(D)} \right]^{1/p} \in O(\Psi_{d, \beta}(T)), \quad \text{if } h = h(T) \sim \begin{cases} \log^2 T/\sqrt{T}, & d = 1, \\ \log T/T^{1/4}, & d = 2, \\ (\log T/T)^{1/(2\beta + d - 2)}, & d \geq 3. \end{cases}
\]

**Remark 4.5.**

(a) Since we can choose $\beta > 0$ arbitrarily large, we make the remarkable observation that, in the scenarios described above, for any $\varepsilon > 0$ we can obtain the almost superoptimal rates $T^{-(1+\varepsilon)}$ and $(\log T/T)^{1/(2(1+\varepsilon))}$ in any dimension $d \geq 3$ for the pointwise $L^2$ and $\text{sup}$-norm risk, respectively. Moreover, in any dimension, an adaptive choice of the bandwidth is not necessary.

(b) The result demonstrates that even under much less stringent assumptions (logarithmic moments and unbounded drift) compared to the class of processes studied in the next section, there are examples of jump diffusions with Lévy-driven jump part for which optimal estimation results are feasible. It is therefore an interesting question for future research to determine more general coefficient assumptions based on a linear growth condition on the drift that yield optimal estimation properties.
4.3. Example: Non-reversible Lévy-driven jump diffusion processes

The goal of this section is to show that solutions of the d-dimensional SDE, \( d \in \mathbb{N} \),

\[
X_t = X_0 + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s + \int_0^t \gamma(X_{s-}) \, \tilde{N}(ds, dz) \tag{4.7}
\]
satisfy assumptions (A',0), (A'1) and (A' \beta) which then allows using Theorem 4.2 to bound the sup-norm risk of the kernel invariant density estimator. Here, \( \sigma: \mathbb{R}^d \to \mathbb{R}^{d \times d}, \gamma: \mathbb{R}^d \to \mathbb{R}^{d \times d}, b: \mathbb{R}^d \to \mathbb{R}^d \). \( W \) denotes an \( \mathbb{R}^d \)-valued Brownian motion, \( N \) is a Poisson random measure on \([0, \infty) \times \mathbb{R}^d \{0\}\) with intensity measure \( \mu(ds, dz) = ds \otimes \nu(dz) \), and \( \tilde{N} \) denotes the compensated Poisson random measure. Moreover, \( \nu \) is a Lévy measure and we assume that \( N, W \) and \( X_0 \) are independent. Note that, if \( z \mapsto \gamma(x)z \) is in \( L^1(\mathbb{R}^d \setminus B_1, \nu) \) for all \( x \in \mathbb{R}^d \), (4.7) is equivalent to

\[
X_t = X_0 + \int_0^t b^*(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s + \int_0^t \gamma(X_{s-}) \, \tilde{N}(ds, dz) \tag{4.8}
\]

with \( b^*(x) := b(x) - \int_{\|z\| > 1} \gamma(x)z \nu(dz) \) and \( B_1 := \{ z \in \mathbb{R}^d : \|z\| \leq 1 \} \). We assume the following.

(\( J1 \)) The functions \( b, \gamma, \sigma \) are globally Lipschitz continuous, \( b \) and \( \gamma \) are bounded, and, for \( 1_{d \times d} \) denoting the \( d \times d \)-identity matrix, there exists a constant \( c \geq 1 \) such that

\[
e^{-1}1_{d \times d} \leq \sigma \sigma^\top \leq c1_{d \times d},
\]

where the ordering is in the sense of Loewner for positive semi-definite matrices.

(\( J2 \)) \( \nu \) is absolutely continuous wrt the Lebesgue measure and, for an \( \alpha \in (0, 2) \),

\[
(x, z) \mapsto \|\gamma(x)z\|^{d+\alpha} \nu(z)
\]

is bounded and measurable, where, by abuse of notation, we denoted the density of \( \nu \) also by \( \nu \). Furthermore, if \( \alpha = 1 \),

\[
\int_{r < \|\gamma(x)z\| \leq R} \gamma(x)z \nu(dz) = 0, \quad \text{for any } 0 < r < R < \infty, \; x \in \mathbb{R}^d.
\]

(\( J3 \)) There exist \( c_1, c_2 > 0 \) and \( \eta_0 > 0 \) such that

\[
\{ x, b(x) \} \leq -c_1 \|x\|, \quad \forall x: \|x\| \geq c_2, \quad \text{and} \quad \int_{\mathbb{R}^d} \|z\|^{2\eta_0} \|z\| \nu(dz) < \infty.
\]

In [4], the authors also investigate \( L^2 \) invariant density estimation for jump diffusions and use a similar approach for formulating requirements on the diffusion coefficients which imply their respective heat kernel bound and mixture assumptions. The conditions however are more restrictive and, in particular, the case of continuous diffusions cannot be handled within their framework since it requires \( \text{supp} (\nu) = \mathbb{R}^d \) and \( \det(\gamma(x)) > c \) for some constant \( c > 0 \) and all \( x \in \mathbb{R}^d \). In [6], the authors improve the \( L^2 \) rate for dimension \( d = 1 \) from [4] to the parametric rate \( 1/T \) by imposing an additional smoothness restriction on the jump measure. Our main contribution in this section is to show that under the less stringent assumptions above, optimal convergence rates can be achieved not only wrt the \( L^2 \) risk but even wrt sup-norm risk in any dimension. In particular, reversible diffusion processes satisfying the drift and dispersion matrix assumptions fall into the above process class, such that the minimax lower bounds from the literature (see section 4.1) suggest sharpness of our estimation rates.

Note that (\( J1 \)) and (\( J3 \)) directly imply \( \gamma(x)z \in L^1(\mathbb{R}^d \setminus B_1, \nu) \), so (4.7) and (4.8) are equivalent. The subsequent lemma shows that, under the given assumptions, there exists a pathwise unique strong solution for (4.7) and that the conditions of Corollary 1.5 of [21] hold, implying the heat kernel bound (4.9). All proofs can be found in Appendix C.

**Lemma 4.6.** Let (\( J1 \))–(\( J3 \)) hold. Then, (4.7) admits a càdlàg, non-explosive, pathwise unique, strong solution possessing the strong Markov property, and the assumptions (H\( ^{\circ} \)) and (H\( ^{\bullet} \)) of [21] hold.

Let \( X \) be the unique solution of (4.7) described in Lemma 4.6.
Corollary 4.7. Let (\(\mathcal{F}_l\))–(\(\mathcal{F}_3\)) hold. Then, transition densities \((p_t)_{t \geq 0}\) exist and there are constants \(C, \lambda > 1\) such that the solution \(X\) of (4.7) satisfies the following heat kernel estimate for all \(x, y \in \mathbb{R}^d, 0 < t \leq 1,\)

\[
C^{-1} \left( t^{-d/2} \exp(-\lambda \|x - y\|^2/t) + (\inf_{x \in \mathbb{R}^d} \text{ess inf}_{z \in \mathbb{R}^d} \kappa_\alpha(x, z)) t(\|x - y\| + t^{1/2})^{-d-\alpha} \right) 
\leq p_t(x, y) \leq C(t^{-d/2} \exp(-\|x - y\|^2/(\lambda t)) + \|\kappa_\alpha\|_\infty t(\|x - y\| + t^{1/2})^{-d-\alpha}),
\]

(4.9)

where \(\kappa_\alpha(x, z) = \|\gamma(x)z\|^{d+\alpha} \nu(z)\). In particular, assumption (\(\mathcal{A}1\)) is satisfied.

Now our goal is to show that the solution \(X\) of (4.7) fulfills the fundamental assumption (\(\mathcal{A}0^+\)) and exponential ergodicity along with the mixing property (\(\mathcal{A}\beta\)). First, observe that (\(\mathcal{A}1\)) implies that \(b \in C_0(\mathbb{R}^d; \mathbb{R}^d)\) and \(\sigma, \gamma \in C_0(\mathbb{R}^d; \mathbb{R}^{d \times d})\) and hence Theorem 6.7.4 in [7] guarantees that the unique càdlàg Markov process \(X\) solving (4.7) is Feller and therefore Borel right. Further, Corollary 4.7 in particular implies the existence of bounded transition densities and thus, by Lemma 2.4, it suffices to show the existence of an invariant distribution. This will be done as a byproduct while proving exponential ergodicity and the exponential mixing property (\(\mathcal{A}\beta\)). For this, we will employ results of Masuda [56] which are again based on the theory of stability of continuous-time Markov processes of Meyn and Tweedie [62]. These lead us to the following proposition.

Proposition 4.8. Grant assumptions (\(\mathcal{A}1\))–(\(\mathcal{A}3\)). Then, an invariant distribution exists, \(X\) is \(V\)-exponentially ergodic with locally bounded \(V\) and the process \(X\) started in an arbitrary state is strictly positive. If for \(\mathcal{A}0\)–(\(\mathcal{A}2\)) and (\(\mathcal{A}\beta\)) are fulfilled for the solution \(X\) of (4.7), i.e., \(X \in \Sigma \cap \Theta\). In particular, the results from Section 4.1 can be applied.

Theorem 4.9. Let \(D \subset \mathbb{R}^d\) be open and bounded and assume (\(\mathcal{A}1\))–(\(\mathcal{A}3\)). If \(\rho|_D \in \mathcal{H}_D(\beta, \mathcal{L})\) with \(\beta \in (1, \ell + 1)\) for \(d = 1\) and \(\beta \in (2, \ell + 1)\) for \(d \geq 2\), then, the sup-norm risk of the kernel estimator defined in (3.2) is of order

\[
\mathbb{E}\left[\left\|\hat{\rho}_{h,T} - \rho\right\|_{L^\infty(D)}\right]^{1/p} \in \begin{cases} O\left(\frac{\log T}{T}\right), & d = 1, \\ O\left(\frac{\log T}{\sqrt{T}}\right), & d = 2, \quad \text{if } h \sim \left\{ \frac{\log^2 T}{\sqrt{T}}, \frac{\log T}{T^{1/4}}, \left(\log T/T\right)^{-1/(2\beta+d-2)} \right\}, & d = 3. \end{cases}
\]

for any \(p \geq 1\). If \(\hat{h}_T = \hat{h}_T^{(k)}\) is chosen adaptively according to (4.2) for some \(k \in \mathbb{N}\), then for any \(d \geq 3,\)

\[
\mathbb{E}\left[\left\|\hat{\rho}_{h,T} - \rho\right\|_{L^\infty(D)}\right] \in O\left(\frac{\log(k) \log T \log T}{T} \right)^{\beta/(2\beta+d-2)).}
\]

Moreover, for any \(x \in \mathbb{R}^d\) such that \(\rho|_D \in \mathcal{H}_D(\beta, \mathcal{L})\) for some \(\beta \in (0, \ell + 1]\) and a neighborhood \(D\) of \(x\), we have the pointwise \(L^2\) risk estimate

\[
\mathbb{E}\left[ (\hat{\rho}_{h,T}(x) - \rho(x))^2 \right] \in \begin{cases} O\left(1/T\right), & d = 1, \\ O\left(\log T/T\right), & d = 2, \quad \text{if } h \sim \left\{ T^{-1/\gamma}, T^{-1/(2\beta+d-2)} \right\}, & d = 2, \gamma \leq \beta, \\ O\left(T^{-2\beta/(2\beta+d-2)}\right), & d \geq 3. \end{cases}
\]

Appendix A: Supplements of Section 2

A.1. Assumption (\(\mathcal{A}2\)) and the exponential \(\beta\)-mixing property

As in the rest of the paper, we will assume in this section that \(X\) is a Borel right Markov process with unique invariant distribution \(\mu\) possessing a Lebesgue density \(\rho\). Let us start by collecting some important definitions in the realm of stability theory of Markov processes. We say that \(X\) is \(\psi\)-irreducible for some \(\sigma\)-finite measure \(\psi\) on its state space if \(\psi(B) > 0\) for some Borel set \(B\) implies

\[
U(x, B) := \int_0^\infty P_t(x, B) dt = \mathbb{E}^x[\eta_B] > 0
\]

for any \(x \in \mathbb{R}^d\), i.e., the expected sojourn time \(\eta_B\) of \(X\) in \(B\) (or, equivalently, the potential of \(B\), where \(\eta_B = \int_0^\infty 1_{\{X_t \in B\}} dt\), when \(X\) is started in an arbitrary state is strictly positive. If for \(B \in \mathcal{B}(\mathbb{R}^d), \psi(B) > 0\) even implies
$\mathbb{P}^x(\eta_B = \infty) = 1$ for any $x \in \mathbb{R}^d$, we say that $X$ is *Harris recurrent* and that $\psi$ is a Harris measure. Harris recurrent Markov processes having an invariant distribution (which is unique in this case) are called positive Harris recurrent. A Borel set $C$ is called *small* if there exists $T > 0$ and a non-trivial measure $\nu$ on the state space such that

$$P_T(x, \cdot) \geq \nu(\cdot), \quad x \in C.$$  

*Petite* sets generalize the notion of small sets. We call a Borel set $C$ petite if there exists a sampling distribution $a$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and a non-trivial measure $\nu_a$ on the state space $s.t.$

$$K_a(x, \cdot) := \int_0^\infty P_t(x, \cdot) a(dt) \geq \nu_a(\cdot), \quad x \in C,$$

i.e., small sets are petite sets with sampling distribution $a = \delta_T$ for some $T > 0$. All three concepts have obvious counterparts for discrete-time chains. If moreover the $\psi$-irreducible process $X$ possesses a small set $C$ such that $\psi(C) > 0$ and there is $T > 0$ such that $P_t(x, C) > 0$, $\forall x \in C$, $t \geq T$, we say that $X$ is *aperiodic*.

These notions are of central importance in the theory of stability of Markovian processes on general state spaces in both discrete as well as continuous time. In discrete time, the existence of small sets allows the construction of a related Markov chain via the technique of Nummelin splitting, which shares the same stability properties with the original chain but possesses an atom. This in turn allows to transfer well-known reasoning in Markov chain theory on countable state spaces to the general state space situation with renewal arguments. With the Meyn and Tweedie approach to stability of continuous-time Markov processes, which heavily involves the aforementioned concept of aperiodicity, we can then infer stability properties through sampled chains, generalizing discrete-time results to continuous time. For a complete picture in discrete time, we refer to the monograph [59]. Continuous-time theory was developed in the 1990s in a series of papers [35, 60–62] and many other subsequent contributions.

We see that these concepts are quite natural when we aim to infer stability of general Markov processes, and we need no more than irreducibility as well as the property that compact sets are small together with exponential decay in (2.2) to infer exponential $\beta$-mixing of the process.

**Proposition A.1.** Suppose that $X$ is $\psi$-irreducible and that every compact set $S \subset \mathbb{R}^d$ is small. Moreover, let (A.2) be satisfied with $\Lambda = \mathbb{R}^d$ and

$$r_S(t) := C_S e^{-\kappa_S t}, \quad t > 0,$$  

(A.1)

with constants $C_S, \kappa_S > 0$. Then, $X$ is exponentially $\beta$-mixing.

**Proof.** Let $S \subset \mathbb{R}^d$ be compact such that $\lambda(S) > 0$. Since $\mathbb{R}^d$ can be covered by countably many compact sets and the irreducibility measure $\psi$ is $\sigma$-finite, we can also assume that $\psi(S) > 0$ and $\mu(S) > 0$. Letting $(P_t)_{t \geq 0}$ denote the semigroup associated to $X$, we obtain from (2.2) and (A.1) that, for any $x \in S$ and $t > 0$,

$$|P_t(x, S) - \mu(S)| \leq \int_S |p_t(x, y) - \rho(y)| \, dy \leq C_se^{-\kappa_S t} \lambda(S) = \tilde{C}_S e^{-\kappa_S t},$$

with $\tilde{C}_S = C_S \lambda(S)$. Since $\mu(S) > 0$, this implies in particular that there exists $T(S) > 0$ such that $P_t(x, S) > 0$ for all $t \geq T(S)$ and $x \in S$. Since $S$ is small by assumption, it follows that $X$ is aperiodic. Hence, by Theorem 5.3 in [35] and the remarks thereafter, there exists (a) an extended real-valued measurable function $V \geq 1$ such that, for some $T > 0$, we have

$$P_T V(x) \leq \lambda V(x) + b1_\Theta$$

(A.2)

for some $0 < \lambda < 1$, $b \geq 0$ and a small set $\Theta \in \mathcal{B}(\mathbb{R}^d)$ and (b) a set $S_V \subset \{ V < \infty \}$, which is full and absorbing—that is, $\mu(S_V) = 1$ and $P_T(x, S_V) = 1$ for any $x \in S_V$—such that $X$ restricted to $S_V$ is exponentially ergodic in the sense

$$\|P_t(\cdot, \cdot) - \mu\|_{TV} \leq C V(x) e^{-\kappa t}, \quad x \in S_V,$$

(A.3)

for some constants $C, \kappa > 0$. Noting that (A.2) implies

$$\Delta \tilde{V} \leq -V + \frac{b}{1 - \lambda}1_\Theta$$
with \( \tilde{V} = V/(1 - \lambda) \geq 0 \) and \( \Delta := P_T - I \), it follows from Theorem 14.0.1 in [59] that \( \mu(V) < \infty \). The claim on exponential \( \beta \)-mixing of the process now follows from (A.3) since

\[
\int_{\mathbb{R}^d} \|P_t(x, \cdot) - \mu\|_{TV} \mu(dx) = \int_{S_V} \|P_t(x, \cdot) - \mu\|_{TV} \mu(dx) \leq C e^{-\kappa t} \int_{S_V} V(x) \mu(dx) = \tilde{C} e^{-\kappa t},
\]

for any \( t \geq 0 \), where finiteness of \( \tilde{C} = C \mu(V) \) was discussed above and for the first equality we used that \( S_V \) is full. \( \square \)

Compactness of small sets can be inferred for a quite general class of Markov processes. We say that \( X \) is a \( T \)-process if there exists a non-trivial continuous component for some sampled chain, i.e., there exists a sampling distribution \( a \) on \( (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \) and a non-trivial, lower semi-continuous kernel \( T \) on the state space s.t.

\[
K_a(x, \cdot) \geq T(x, \cdot), \quad x \in \mathbb{R}^d.
\]

Many processes in applied probability can be shown to be \( T \)-processes such as price processes driven by Lévy risk and return processes [67], certain piecewise deterministic Markov processes used for MCMC [11] or queueing networks [34]. Moreover, any open set irreducible weak \( C_0 \)-Feller process is a \( T \)-process (cf. [81, Theorem 7.1]). Markov processes having the strong Feller property—that is, the semigroup satisfies \( P_t B_0(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d) \) for all \( t \geq 0 \)—are trivially \( T \)-processes, since any operator \( P_t \) is a continuous component for itself. Here, we denoted by \( C_0(\mathbb{R}^d) \) the family of bounded, continuous functions on \( \mathbb{R}^d \) and by \( B_0(\mathbb{R}^d) \) the family of bounded Borel functions on \( \mathbb{R}^d \). The strength of Markov processes with the strong Feller property—and \( T \)-processes as a generalization of such processes—comes from making possible to connect distributional properties of the Markov process induced by the semigroup and topological properties of the state space, thus allowing to use knowledge of the topology to infer strong stability results of the Markov process. Classical examples of Markov processes with the strong Feller property are Lévy processes with absolutely continuous semigroup with respect to the Lebesgue measure [45, Theorem 2.2], diffusion processes with hypoelliptic Fisk–Stratonovich-type generator [46, Lemma 5.1], diffusion processes on Hilbert spaces under appropriate assumptions on the coefficients [68, Theorem 1.2], or solutions of different classes of parabolic SPDEs [25, 26, 38, 54]. More recently, the strong Feller property was discussed for switching (jump-)diffusions [86, 88], for jump-diffusions with non-Lipschitz on the coefficients [68, Theorem 1.2], or solutions of different classes of parabolic SPDEs [25, 26, 38, 54].

Let us now infer the exponential \( \beta \)-mixing property for \( T \)-processes given exponential decay in (2.2) and, as a natural mixing requirement, ergodicity in the sense of total variation convergence to the invariant distribution, i.e., \( \|P_t(x, \cdot) - \mu\|_{TV} \xrightarrow{t \to \infty} 0, \forall x \in \mathbb{R}^d \). Note that indeed, dominated convergence shows that any stationary, ergodic Markov process is \( \beta \)-mixing.

**Proposition A.2.** Let \( X \) be an ergodic \( T \)-process such that (as/2) is satisfied for \( r \) given as in (A.1) and \( \Lambda = \mathbb{R}^d \). Then, \( X \) is positive Harris recurrent, every compact set is small and \( X \) is exponentially \( \beta \)-mixing.

**Proof.** For the exponential \( \beta \)-mixing property, it suffices to check that every compact set is small by Proposition A.1, since ergodicity clearly implies \( \mu \)-irreducibility of \( X \). We prove this property together with positive Harris recurrence at once. To this end, for a given \( \varepsilon > 0 \), choose a compact set \( C \subseteq \mathbb{R}^d \) such that \( \mu(C) \geq 1 - \varepsilon \). Then, for fixed \( x \in \mathbb{R}^d \), ergodicity guarantees that \( \lim_{t \to \infty} P^x(t) \in C \geq 1 - \varepsilon \), and hence \( X \) is bounded in probability on average as defined on p. 495 of [60]. Since \( X \) is an irreducible \( T \)-process, Theorem 3.2 and Theorem 4.1 of the same reference yield Harris recurrence and petiteness of compact sets. It remains to show that small and peti setso coincide for the given process. The reverse implication of Theorem 6.1 in [60] guarantees that there exists an irreducible skeleton \( X_\Delta = (X_{n\Delta})_{n \in \mathbb{N}_0} \) for some \( \Delta > 0 \) thanks to ergodicity and positive Harris recurrence of \( X \). Proposition 6.1 in [60] therefore implies equivalence of small and peti setso, which finishes the proof. \( \square \)

**A.2. Proofs for Section 2**

**Proof of Proposition 2.1.** Without loss of generality, let \( T \geq 1 \) be fixed. Then, using the Markov property and the invariance of \( \mu \), for any \( \delta \in [0, 1] \),

\[
\text{Var} \left( \int_0^T f(X_s) \, ds \right) = \mathbb{E} \left[ \left( \int_0^T (f(X_s) - \mathbb{E} f(X_0)) \, ds \right)^2 \right]
\]
with analogous arguments. Note first that it remains to consider the first parts of the integral. We now restrict to dimension $d$ and (substituting $x,y$)

It follows from the assumption on the convergence of the transition density in (2.2) that

**Proof of Lemma 2.3.**

By the semigroup property of $(P_t)_{t \geq 0}$ and invariance of $\mu$ we have for any $t > 1$ and $y \in \mathbb{R}^d$ and $\mu$-a.e. $x \in \mathbb{R}^d$,

\[
|p_t(x,y) - \rho(y)| \leq \int_{\mathbb{R}^d} p_1(z,y)|p_{t-1}(x,z) - \rho(z)|\,dz
\]

\[
\leq \|p_1\|_{\infty} \int_{\mathbb{R}^d} |p_{t-1}(x,z) - \rho(z)|\,dz
\]

\[
= 2\|p_1\|_{\infty} \|P_{t-1}(x,\cdot) - \mu\|_{TV} \leq 2\|p_1\|_{\infty} CV(x)\xi(t-1),
\]

where the equality follows from Scheffé’s theorem, see [80, Lemma 2.1]. Thus, for any compact set $S$ and $r_S(t) = 2C\|p_1\|_{\infty} \sup_{x \in S \cap \Lambda} V(x)\xi(t-1)$, it follows that

\[
\int_1^\infty \sup_{x \in S \cap \Lambda, y \in S} |p_t(x,y) - \rho(y)|\,dt \leq \int_1^\infty r_S(t)\,dt \lesssim \sup_{x \in S \cap \Lambda} V(x) \int_0^\infty \xi(t) < \infty,
\]
by local boundedness of $V \mathbf{1}_A$ and the convergence assumption on $\xi$, which yields $(\mathcal{A}/2)$.

**Proof of Lemma 2.4.** Let $B \in \mathcal{B}(\mathbb{R}^d)$ such that $\lambda(B) = 0$. Then, it holds that

$$\mu(B) = \int_{\mathbb{R}^d} \int_B p_\Delta(x, y) \, dy \, \mu(dx) = 0,$$

which yields the existence of a Lebesgue density $\rho$ of $\mu$ by the Radon–Nikodym theorem. Now, let $B \in \mathcal{B}(\mathbb{R}^d)$ such that $\lambda(B) > 0$. Arguing as above and using boundedness of $p_\Delta$, we get

$$\frac{\int_B \rho(x) \lambda(dx)}{\lambda(B)} \leq c.$$

Now the Lebesgue differentiation theorem yields $\text{ess sup} \rho \leq c$, and defining

$$\rho_b(x) = \rho(x) \mathbf{1}_{[0,c]}(\rho(x)), \quad x \in \mathbb{R}^d,$$

we have $\rho = \rho_b$ almost everywhere and $\rho_b \leq c$ which completes the proof.

**Proof of Proposition 2.5.** Let $0 < \delta < 1 \leq D$. Analogously to the proof of Proposition 2.1, one can compute that

$$\text{Var} \left( \int_0^T f(X_t) \, dt \right) = 2 \int_0^T (T-v) \int_{\mathbb{R}^{d \times d}} f(x) f(y) (p_v(x,y) - \rho(y)) \mu(dx) \, dy \, dv$$

$$\leq 2T \|f\|_\infty^2 \left( \int_0^T \int_{\mathbb{R}^d} p_v(x,y) \mu(dx) \, dy \, dv + \int_0^T \int_{\mathbb{R}^d} (p_v(x,y) - \rho(y)) \mu(dx) \, dy \, dv \right)$$

$$= 2T \|f\|_\infty^2 (I_\delta + I_D + I_T),$$

where $I_\delta := \int_0^\delta \int_{\mathbb{R}^2} p_v(x,y) \mu(dx) \, dy \, dv$, $I_D := \int_0^D \int_{\mathbb{R}^2} p_v(x,y) \mu(dx) \, dy \, dv$ and

$$I_T := \int_0^T \int_{\mathbb{R}^d} (P_v(x,S) - \mu(S)) \mu(dx) \, dv.$$

As before (see (A.4)) and under our additional assumption that $\rho$ is bounded, it holds

$$I_\delta \leq \mu(S) \delta \leq \|\rho\|_\infty \lambda(S) \delta. \tag{A.5}$$

Furthermore, exploiting the mixing property of $X$,

$$I_T \leq \int_0^T \int_D \|P_v(x,\cdot) - \mu(\cdot)\|_{TV} \mu(dx) \, dv \leq c_\kappa \int_0^T e^{-\kappa v} \, dv \leq \frac{c_\kappa}{\kappa} e^{-\kappa D} \mathbf{1}_{(D,\infty)}(T). \tag{A.6}$$

By assumption $(\mathcal{A}/1)$, $p_v(x,y) \leq c_2 v^{-d/2}$, for $0 < t \leq 1$. Hence, we have $p_{t/2}(x,y) \leq c_2 2^{d/2} =: c_p$ which implies

$$p_v(x,y) = \int p_{t-1/2}(x,z) p_{t/2}(z,y) \, dz \leq c_p,$$

for all $t > 1/2$. Since $\delta < 1 \leq D$, it follows

$$\int_\delta^D p_v(x,y) \, dv \leq c_2 \int_\delta^1 v^{-d/2} \, dv + c_p D \mathbf{1}_{(1,\infty)}(D) \leq c_{\delta,D} \left( \int_\delta^1 v^{-d/2} \, dv + D \mathbf{1}_{(1,\infty)}(D) \right)$$

for $c_{\delta,D} := c_2 + c_p$. For $d \geq 3$, this implies

$$\int_\delta^D p_v(x,y) \, dv \leq c_{\delta,D} \left( \int_\delta^1 v^{-d/2} \, dv + D \mathbf{1}_{(1,\infty)}(D) \right)$$

$$\leq c_{\delta,D} \left( \frac{(d/2 - 1)^{-1} \delta^{1-d/2}}{d} + D \mathbf{1}_{(1,\infty)}(D) \right)$$

$$\leq c'_{\delta,D} \left( \delta^{1-d/2} + D \mathbf{1}_{(1,\infty)}(D) \right), \tag{A.7}$$
where \( c_{\delta,D}' := 2c_{\delta,D} \). Letting \( \delta = \lambda(S)^{2/d} \), \( D = (1 - \frac{2}{\kappa} \log(\lambda(S))) \land T \), (A.7) and \( \lambda(S) < 1 \) imply

\[
\int_\delta^D p_v(x,y) \, dv \leq c_{\delta,D}' \left( \lambda(S)^{2/d - 1} + \frac{2}{\kappa} \log(\lambda(S)^{-1}) \right) \leq c_{\delta,D}' \left( \lambda(S)^{2/d - 1} + \frac{2}{\kappa(1-2/d)} \lambda(S)^{2/d} \right)
\]

\[
\leq c_{\delta,D}' \lambda(S)^{2/d - 1}, \quad \text{for } c_{\delta,D}'' := c_{\delta,D}' \left( 1 + \frac{2}{\kappa(1-2/d)} \right),
\]

where we have used the well-known inequality \( \log x \leq nx^{1/n}, \); \( x, n > 0 \). Using Fubini’s theorem, this directly implies

\[
\mathcal{I}_D = \int_\delta^D \int_{S_\delta} p_v(x,y) \mu(dx) \, dy \, dv \leq c_{\delta,D}'' \mu(\lambda(S)^{2/d}) \leq c_{\delta,D}'' \|\mu\|_{\infty} \lambda(S)^{2/d+1}
\]

(A.8)

for \( d \geq 3 \). Noting that our choice of \( \delta \) and \( D \) implies by (A.5) and (A.6) that

\[
\mathcal{I}_d \leq \|\mu\|_{\infty} \lambda(S)^{2/d+1} \quad \text{and} \quad \mathcal{I}_T \leq \frac{c_k}{\kappa} \lambda(S)^{2/d+1},
\]

(2.5) follows for any \( d \geq 3 \) by combining these estimates with (A.8). The case \( d = 2 \) is treated by similar arguments. \( \square \)

Appendix B: Proofs for Section 3

Proof of Theorem 3.1. We start by splitting the process \( (X_s)_{0 \leq s \leq t} \) with Borel state space \( \mathcal{X} \) into \( 2n_t \) parts of length \( m_t \), where \( t = 2m_t n_t, n_t \in \mathbb{N}, m_t \in \mathbb{R}_+ \). More precisely, for \( j \in \{1, \ldots, n_t\} \), define the processes

\[
X^{j,1} := (X_s)_{s \in [2(j-1)m_t, (2j-1)m_t]}, \quad X^{j,2} := (X_s)_{s \in [(2j-1)m_t, 2jm_t]}.
\]

Since \( X \) is a stationary Markov process, the \( \beta \)-mixing assumption is equivalent to

\[
\Xi(s) \geq \int_{\mathbb{R}^d} \|P_s(x, \cdot) - \mu\|_{TV} \, \mu(dx) = \mathbb{E} \left[ \|P(\cdot|\mathcal{F}_0) - \mathbb{P}\|_{TV|\mathcal{F}_s} \right] = \mathbb{E} \left[ \|P(\cdot|\mathcal{F}_t) - \mathbb{P}\|_{TV|\mathcal{F}_{t+s}} \right],
\]

for any \( s, t > 0 \), see Proposition 1 in [29]. Here, \( (\mathcal{F}_t = \sigma(X_s, s \leq t))_{t \geq 0} \) denotes the natural filtration of \( X \), \( (\mathcal{F}_t = \sigma(X_s, s \geq t))_{t \geq 0} \) the filtration of the future of \( X \) and, for a signed measure \( \mu \) and a sub-\( \sigma \)-algebra \( \mathcal{A} \) on a measure space \((\Omega, \mathcal{F})\), \( \|\mu\|_{TV|\mathcal{A}} \) denotes the total variation norm of \( \mu \) restricted to \( \mathcal{A} \). As demonstrated in [85, Lemma 1.4],

\[
\mathbb{E} \left[ \|P(\cdot|\mathcal{F}_t) - \mathbb{P}\|_{TV|\mathcal{F}_{t+s}} \right] = \beta(\mathcal{F}_t, \mathcal{F}_{t+s})
\]

where for two sub-\( \sigma \)-algebras \( \mathcal{A}, \mathcal{B} \subset \mathcal{G} \) and a probability measure \( \mathbb{P} \) on \((\Omega, \mathcal{G})\), the classical \( \beta \)-mixing coefficient \( \beta(\mathcal{A}, \mathcal{B}) \) is given by

\[
\beta(\mathcal{A}, \mathcal{B}) = \sup_{C \in \mathcal{A} \otimes \mathcal{B}} \left| \mathbb{P}(C \cap A \otimes B) - \mathbb{P}(A) \otimes \mathbb{P}(B) \right|.
\]

Here, \( \mathbb{P}|_{\mathcal{A} \otimes \mathcal{B}} \) is the restriction to \((\Omega \times \Omega, \mathcal{A} \otimes \mathcal{B})\) of the image measure of \( \mathbb{P} \) under the canonical injection \( \iota(\omega) = (\omega, \omega) \). Clearly, if \( A_1 \subset A_2 \), we have \( \beta(A_1, B) \leq \beta(A_2, B) \). Observe that \( X^{j,1} \), as a mapping from \( \Omega \) to \( \mathcal{X}^{2(2j-1)m_t, (2j-1)m_t} \), is both \( \mathcal{F}_{(2j-1)m_t} \)-measurable and \( \mathcal{T}_{2(2j-1)m_t} \)-measurable. It now follows from the above discussion for \( j, k \in \{1, \ldots, n_t\}, j < k \), that

\[
\beta(X^{j,1}, X^{k,1}) := \beta(\sigma(X^{j,1}), \sigma(X^{k,1})) \leq \beta(\mathcal{F}_{(2j-1)m_t}, \mathcal{F}_{(2k-1)m_t}) \leq \Xi((2(k - j) - 1)m_t) \leq \Xi((k - j)m_t).
\]

In the same way, we obtain \( \beta(X^{j,2}, X^{k,2}) \leq \Xi((k - j)m_t) \). Arguing as in the proof of Proposition 5.1 of [84], we can then construct a process \((\tilde{X}_s)_{0 \leq s \leq t}\) by Berbee’s coupling method, such that for \( k = 1, 2 \),

1. \( X^{j,k} \overset{(d)}{=} \tilde{X}^{j,k} \), for all \( j \in \{1, \ldots, n_t\} \),
2. \( \mathbb{P}(X^{j,k} \neq \tilde{X}^{j,k}) \leq \Xi(m_t) \) for all \( j \in \{1, \ldots, n_t\} \),
3. \( \tilde{X}^{1,k}, \ldots, \tilde{X}^{n_t,k} \) are independent,
where $\tilde{X}^{j,k}$ is defined analogously to $X^{j,k}$ for $j \in \{1, \ldots, n_t\}$ and $k = 1, 2$. In order to ease the notation, define for $j \in \{1, \ldots, n_t\}$

$$I_g(X^{j,1}) := \int_{2(j-1)m_t}^{(2j-1)m_t} g(X_s) \, ds, \quad I_g(X^{j,2}) := \int_{(2j-1)m_t}^{2jm_t} g(X_s) \, ds,$$

and, analogously, define $I_g(\tilde{X}^{j,k})$ for $k = 1, 2, j \in \{1, \ldots, n_t\}$. Fix $p \geq 1$. Then,

$$(\mathbb{E} [||G_t||_G^p])^{1/p} \leq \left( \mathbb{E} \left[ \sup_{g \in G} \frac{1}{\sqrt{t}} \int_0^t g(\tilde{X}_s) \, ds \right]^p \right)^{1/p} + \left( \mathbb{E} \left[ \sup_{g \in G} \frac{1}{\sqrt{t}} \int_0^t (g(X_s) - g(\tilde{X}_s)) \, ds \right]^p \right)^{1/p}$$

$$= \left( \mathbb{E} \left[ \sup_{g \in G} \frac{1}{\sqrt{t}} \sum_{k=1}^{2^n_t} \sum_{j=1}^{n_t} I_g(\tilde{X}^{j,k}) \right]^p \right)^{1/p} + \left( \mathbb{E} \left[ \sup_{g \in G} \frac{1}{\sqrt{t}} \sum_{k=1}^{2^n_t} \sum_{j=1}^{n_t} (I_g(X^{j,k}) - I_g(\tilde{X}^{j,k})) \right]^p \right)^{1/p}. \tag{B.1}$$

The classical Bernstein inequality (cf. Theorem 2.10 in [15]) implies that for $u > 0$

$$\mathbb{P} \left( \frac{1}{\sqrt{t}} \sum_{j=1}^{n_t} I_g(\tilde{X}^{j,k}) \geq \sqrt{\text{Var} \left( \frac{1}{m_t} \int_0^{m_t} g(X_s) \, ds \right) u} + \frac{m_t \|g\|_{\infty}}{\sqrt{t}} \right) \leq 2e^{-u},$$

where we used that $2n_t/t = 1/m_t$. Consequently, denoting

$$\tilde{c}_1 := 2e^{1/(2e)}\sqrt{2\pi}e^{-11/12}, \quad \tilde{c}_2 := 2(2e)^{-1/2}e^{1/(2e)}\sqrt{\pi}e^{1/6}, \tag{B.2}$$

Lemma A.2 in [33] gives, for $k \in \{1, 2\}$,

$$\left( \mathbb{E} \left[ \frac{1}{\sqrt{t}} \sum_{j=1}^{n_t} I_g(\tilde{X}^{j,k}) \right]^p \right)^{1/p} \leq \|g\|_{\infty} \tilde{c}_1 p + \sqrt{\text{Var} \left( \frac{1}{\sqrt{m_t}} \int_0^{m_t} g(X_s) \, ds \right) \tilde{c}_2 \sqrt{p}}. \tag{B.3}$$

In addition, Theorem 3.5 in [33] implies that there exist positive constants $\tilde{C}_1, \tilde{C}_2$ such that

$$\left( \mathbb{E} \left[ \sup_{g \in G} \frac{1}{\sqrt{t}} \sum_{j=1}^{n_t} I_g(\tilde{X}^{j,k}) \right]^p \right)^{1/p} \leq \frac{\tilde{C}_1}{2} \int_0^\infty \log \mathcal{N}(\bar{u}, G, \frac{m_t}{\sqrt{t}}d\bar{u}) \, d\bar{u} + \frac{\tilde{C}_2}{2} \int_0^\infty \sqrt{\log \mathcal{N}(\bar{u}, G, d\bar{u}, m_t)} \, d\bar{u}$$

$$+ 2 \sup_{g \in G} \left( \mathbb{E} \left[ \frac{1}{\sqrt{t}} \sum_{j=1}^{n_t} I_g(\tilde{X}^{j,k}) \right]^p \right)^{1/p}. \tag{B.4}$$

Here, we bounded the $\gamma_{\alpha}$-functionals appearing in the original statement of the theorem by the corresponding entropy integrals. Note further that the last term on the rhs of (B.1) is upper bounded by

$$\left( \mathbb{E} \left[ \sup_{g \in G} \frac{1}{\sqrt{t}} \sum_{k=1}^{2^n_t} \sum_{j=1}^{n_t} (I_g(X^{j,k}) - I_g(\tilde{X}^{j,k})) \cdot 1_{X^{j,k} \neq \tilde{X}^{j,k}} \right]^p \right)^{1/p} \leq \frac{4n_t m_t}{\sqrt{t}} \sup_{g \in G} \|g\|_{\infty} (P(X^{j,k} \neq \tilde{X}^{j,k}))^{1/p}$$

$$\leq 2 \sup_{g \in G} \|g\|_{\infty} \sqrt{t}(m_t)^{1/p}. \tag{B.5}$$
Plugging the upper bounds (B.3), (B.4) and (B.5) into (B.1) yields

\[
\left( E \left[ \sup_{g \in G} |G_t(g)|^p \right] \right)^{1/p} \leq \frac{1}{m_0^2} \frac{n}{\sqrt{t}} \log N \left( \frac{2}{m_0} \frac{1}{\sqrt{t}} d_{\infty} \right) du + \tilde{C}_2 \int_0^\infty \sqrt{\log N \left( \frac{u}{G, d_{\infty}} \right)} du \\
+ 4 \sup_{g \in G} \left( \frac{m_t}{\sqrt{t}} \|g\|_{\infty} c_1 + \frac{1}{2} \frac{\log \left( m_t \right)}{\sqrt{t}} \sqrt{\log \left( m_t \right)} \right) \eta \left( \frac{m_t}{\sqrt{t}} \right)^{1/p}. \tag{B.6}
\]

For general \( m_t \in (0, 1] \), let \( \tilde{m}_t = \left\lfloor \frac{1}{2m_t} \right\rfloor \), where \( \left\lfloor x \right\rfloor \) denotes the largest integer smaller or equal to \( x \geq 1 \). Then, for \( \tilde{m}_t := \frac{t}{2m_t} \), we have \( m_t \leq \tilde{m}_t \), and from \( \tilde{m}_t \geq \frac{1}{2m_t} - 1 = \frac{t - 2m_t}{2m_t} \) and \( m_t \leq \frac{1}{4} \), we get

\[
\tilde{m}_t = \frac{t}{2m_t} \leq \frac{t m_t}{t - 2m_t} \leq 2m_t.
\]

Since \( \tilde{m}_t \in \mathbb{N} \), (B.6) holds with \( \tilde{m}_t \in [m_t, 2m_t] \) and \( m_t \) being replaced by \( \tilde{m}_t \), and combining this with the computations above yields

\[
\left( E \left[ \sup_{g \in G} |G_t(g)|^p \right] \right)^{1/p} \leq \frac{1}{m_0^2} \frac{n}{\sqrt{t}} \log N \left( \frac{2}{m_0} \frac{1}{\sqrt{t}} d_{\infty} \right) du + \tilde{C}_2 \int_0^\infty \sqrt{\log N \left( \frac{u}{G, d_{\infty}} \right)} du \\
+ 4 \sup_{g \in G} \left( \frac{2m_t}{\sqrt{t}} \|g\|_{\infty} c_1 + \frac{1}{2} \frac{\log \left( m_t \right)}{\sqrt{t}} \sqrt{\log \left( m_t \right)} \right) \eta \left( \frac{m_t}{\sqrt{t}} \right)^{1/p},
\]

which completes the proof.

**Proof of Corollary 3.3.** In case of exponential \( \beta \)-mixing we obtain, similarly to the proof of Proposition 2.5, for any \( t > 0 \),

\[
\|g\|^2_{G_{\beta,t}} = \frac{1}{t} \text{Var} \left( \int_0^t g(X_s) ds \right) \leq \frac{1}{2} \int_0^t \int \|P_s(x, \cdot) - \mu\|_{TV} \mu(dx) ds \leq \frac{1}{2} \|g\|^2_{\infty} \frac{c_0}{\kappa}.
\]

Choosing \( m_T = \sqrt{T} \) and plugging this into (3.1) therefore yields the assertion for the exponential mixing case. For the \( \alpha \)-polynomial case we obtain the assertion similarly by the minimizing choice \( m_T = T^{p/(\alpha + p)} \), where \( T \geq 4(\alpha + p)/\alpha \) guarantees that \( m_T \leq T/4 \) and the assumption \( \alpha > 1 \) is needed to guarantee uniform boundedness of \( \|g\|^2_{G_{\beta,t}} \) in \( t \).

For the proof of bounds on the stochastic error \( h_{\beta,T} \) defined in (3.3), we start with the following preparatory lemma that provides bounds of the covering numbers of the function class \( G \) introduced in (3.4) with respect to the norms appearing in Theorem 3.1. By a slight abuse of notation, we do not distinguish notationally between the \( \sup \)-norm on \( \mathbb{R}^d \) and the function space \( B_{0}^{d}(\mathbb{R}^d) \).

**Lemma B.1.** Let \( D \subset \mathbb{R}^d \) be a bounded set and, given some Lipschitz continuous kernel \( K \) with Lipschitz constant \( L \) and compact support \([-1/2, 1/2]^d\), define the function class \( G \) according to (3.4). Then, for any \( \varepsilon > 0 \),

\[
\mathcal{N}(\varepsilon, G, \|f\|d_{\infty}) \leq \left( \frac{4L \text{diam}(D)}{\varepsilon h} \right)^d,
\]

and if moreover \( X \in \Sigma \cup \Theta \), then there exists a constant \( \kappa > 0 \) such that, for any \( \varepsilon > 0 \) and \( t > 0 \),

\[
\mathcal{N}(\varepsilon, G, \|f\|G_{\beta,t}) \leq \left( \frac{2L \text{diam}(D) \sqrt{\kappa} \|f\|d_{\infty} h^{d-1} \psi_d(h^d)}{\varepsilon} \right)^d.
\]

**Proof.** For \( x \in \mathbb{R}^d \), we obtain by Lipschitz continuity of \( K \) that

\[
B_{d_{\infty}}(K((x - \cdot)/h), \varepsilon) = \{ K((y - \cdot)/h) : y \in \mathbb{R}^d, \|K((x - \cdot)/h) - K((y - \cdot)/h)\|_{\infty} \leq \varepsilon \} \\
\subseteq \{ K((y - \cdot)/h) : y \in \mathbb{R}^d, \|x - y\|_{\infty} \leq \varepsilon h/(2L) \}.
\]

(B.7)
Let $Q \supset D$ be a cube of side length $\text{diam}(D) < \infty$ and choose for 
\[
\pi := \left(\frac{2L\text{diam}(D)}{\varepsilon h}\right)^d
\]
points $x_1, \ldots, x_\pi \in Q$ such that \{ $B_{d,h}(x_i, \varepsilon h/(2L)) : i = 1, \ldots, \pi$ \} covers $Q$ and therefore $D$. From (B.7), it follows that 
\{ $B_{d,h}(\mathcal{K}(x_i - \cdot)/h, \varepsilon) : i = 1, \ldots, \pi$ \} is an external covering of $\mathcal{G}$. The external covering number $\mathcal{N}_\text{ext}(\varepsilon, \mathcal{G}, d_{\infty})$ is thus bounded by $(2L\text{diam}(D)/(\varepsilon h))^d$. Hence,
\[
\mathcal{N}(\varepsilon, \mathcal{G}, d_{\infty}) \leq \mathcal{N}_\text{ext}(\varepsilon/2, \mathcal{G}, d_{\infty}) \leq \left(\frac{4L\text{diam}(D)}{\varepsilon h}\right)^d.
\]
Similarly, for 
\[
\tilde{\mathcal{G}} = \{ K(x - \cdot)/h : x \in D \cap \mathbb{Q}^d \},
\]
we obtain
\[
\mathcal{N}(\varepsilon, \tilde{\mathcal{G}}, d_{\infty}) \leq \left(\frac{2L\text{diam}(D)}{\varepsilon h}\right)^d.
\]
The variance term is bounded by means of Propositions 2.1 and 2.5, respectively. In case $d = 1$ for $X \in \Theta$ or any dimension for $X \in \Sigma$, boundedness of $\rho$, Proposition 2.1 and (2.6) yield that, for $h \in (0, 1)$ and some constant $C$ independent of $\lambda(\text{supp}(K((x - \cdot)/h))) = h^d$, 
\[
\text{Var} \left( \int_0^T K \left( \frac{x - X_t}{h} \right) \, dt \right) \leq C(1 + c_{\tilde{D}})T \| K \|_2^2 \| \rho \|_\infty h^{2d} \psi_2^2(h^d),
\]
where $\tilde{D}$ is a compact set containing $D + [-1/2, 1/2]^d$. Hence, for any dimension $d$ and $X \in \Sigma \cup \Theta$, we obtain together with Proposition 2.5 that there exists some global constant $\Lambda$ independent of $h$ such that for any $h \in (0, 1)$, $t > 0$ and $g \in \tilde{\mathcal{G}}$
\[
\text{Var} \left( \frac{1}{\sqrt{t}} \int_0^t g(X_s) \, ds \right) \leq \Lambda \| g \|_\infty^2 \| \rho \|_\infty h^{2d} \psi_2^2(h^d),
\]
and hence
\[
\| g \|_{\mathcal{G}, \tau} \leq \sqrt{\Lambda} \| \rho \|_\infty h^d \psi_d(h^d) \| g \|_\infty.
\]
Consequently, with the first part of the proof we obtain
\[
\mathcal{N}(\varepsilon, \mathcal{G}, \| \cdot \|_{\mathcal{G}, \tau}) = \mathcal{N}(\varepsilon, \tilde{\mathcal{G}}, \| \cdot \|_{\mathcal{G}, \tau}) \leq \mathcal{N}(\varepsilon(\sqrt{\Lambda} \| \rho \|_\infty h^d \psi_d(h^d))^{-1}, \tilde{\mathcal{G}}, \| \cdot \|_\infty) \leq \left(\frac{2L\text{diam}(D)\sqrt{\Lambda} \| \rho \|_\infty h^{d-1} \psi_d(h^d)}{\varepsilon}\right)^d.
\]
\[\Box\]

**Proof of Lemma 3.4.** Let $X \in \Theta \cup \Sigma$. We start with bounding $\mathbb{E}[\sup_{x \in D} \| x_{h,T}(x) \|]$. Let $m_T \in (0, T/4]$ and $\tau \in [m_T, 2m_T]$ as in Theorem 3.1. Using (B.10) and $\sup_{f,g \in \tilde{\mathcal{G}}} \| f - g \|_\infty \leq 2 \| K \|_\infty$ for $\tilde{\mathcal{G}}$ defined in (B.8), we obtain 
\[
\sup_{f,g \in \tilde{\mathcal{G}}} \| f - g \|_{\mathcal{G}, \tau} \leq \sqrt{\Lambda} \| \rho \|_\infty \sup_{f,g \in \tilde{\mathcal{G}}} \| f - g \|_\infty h^d \psi_d(h^d) \leq 2\sqrt{\Lambda} \| \rho \|_\infty \| K \|_\infty h^d \psi_d(h^d) =: \mathcal{V}(h),
\]
such that $\mathcal{N}(u, \tilde{\mathcal{G}}, \| \cdot \|_{\mathcal{G}, \tau}) = 1$ for $u \geq \mathcal{V}(h)$. Consequently, using $\int_0^C \sqrt{\log(M/u)} \, du \leq 4C \sqrt{\log(M/C)}$ provided $\log(M/C) \geq 2$, e.g., p. 592 of Giné and Nickl [41], and the covering number bound from Lemma B.1, it follows for $h \leq e^{-2L\text{diam}(D)}/\| K \|_\infty$ that
\[
\int_0^\infty \sqrt{\log \mathcal{N}(u, \mathcal{G}, d_{\mathcal{G}, \tau})} \, du = \int_0^\infty \sqrt{\log \mathcal{N}(u, \tilde{\mathcal{G}}, d_{\tilde{\mathcal{G}}, \tau})} \, du \leq \int_0^{\mathcal{V}(h)} \sqrt{d \log \left( \frac{L\text{diam}(D)\mathcal{V}(h)}{uh \| K \|_\infty} \right)} \, du 
\leq 2\mathcal{V}(h) \sqrt{d \log \left( \frac{L\text{diam}(D)}{\| K \|_\infty h} \right)}.\]
Moreover, since $\sup_{f \in \mathcal{G}} \| f - g \|_{\infty} \leq 4 \| K \|_{\infty}$, it follows that $\mathcal{N}(u, \mathcal{G}, d_{\infty}) = 1$ for all $u \geq 4 \| K \|_{\infty}$ and hence we obtain by the covering number bound with respect to the sup-norm from Lemma B.1 and elementary calculations

$$
\int_{0}^{\infty} \log \mathcal{N}(u, \mathcal{G}, 2mu/sqrt{T} d_{\infty}) \, du = 2mu/sqrt{T} \int_{0}^{4K \| K \|_{\infty}} \log \mathcal{N}(u, \mathcal{G}, d_{\infty}) \, du \leq 8 mu/sqrt{T} K \| K \|_{\infty} \left(1 + \log \left(\frac{\text{Ldiam}(D)}{\| K \|_{\infty}}\right)\right).
$$

Denseness of $Q^d$ in $\mathbb{R}^d$, continuity of $x \mapsto \mathbb{H}_{h,T}(x)$ and Theorem 3.1 thus imply for $h \leq e^{-2(\text{Ldiam}(D)/\| K \|_{\infty}}$.

$$
\left(\mathbb{E} \left[\sup_{x \in D} \| \mathbb{H}_{h,T}(x) \| \right]^{1/p}\right)^{1/p}
\leq \frac{1}{\sqrt{T}h^d} \left(8\tilde{C}_1 \frac{m_T}{\sqrt{T}} d \| K \|_{\infty} \left(1 + \log \left(\frac{\text{Ldiam}(D)}{\| K \|_{\infty}}\right)\right) + 2\tilde{C}_2 \sqrt{p} + 4 \| K \|_{\infty} \sqrt{T} \mathcal{E}(m_T)^{1/p}\right).
$$

for all $h \geq T^{-2}$ and $h \in o(1)$ imply for the choice $m_T = \mathcal{E}^{-1}(T^{-u_T})$ that

$$
\mathbb{E} \left[\| \hat{\rho}_{h,T} - \mathbb{E} \hat{\rho}_{h,T} \|_{L^\infty(D)}\right] \leq c_{u_T} \left(\frac{\log T}{Th^d} \mathcal{E}^{-1}(T^{-u_T}) + T^{-\frac{1}{2}} \psi_d(h^d) \sqrt{\log(h^{-1})} + \frac{u_T}{Th^d} \mathcal{E}^{-1}(T^{-u_T}) + T^{-\frac{1}{2}} \psi_d(h^d) \sqrt{u_T + h^{-d}T^{-1}}\right)^{u_T},
$$

where the value of the constant $c$ changes from line to line. Hence Markov’s inequality implies that there exists some constant $c' > 0$ such that

$$
\mathbb{P} \left(\| \hat{\rho}_{h,T} - \mathbb{E} \hat{\rho}_{h,T} \|_{L^\infty(D)} \geq c' \left(\frac{u_T + \log T}{Th^d} \mathcal{E}^{-1}(T^{-u_T}) + T^{-\frac{1}{2}} \psi_d(h^d) \sqrt{u_T \log(h^{-1})}\right)\right) \leq e^{-u_T}.
$$

(B.13)

Suppose now that $X \in \Sigma$. Then, $X$ is exponentially $\beta$-mixing, i.e., $\Xi(t) = c_\kappa e^{-\kappa t}$, where without loss of generality we may assume that $c_\kappa \geq 1$. Then, for any $\gamma > 0$ and $1 \leq u_T \leq \gamma \log T$, it follows from $\mathcal{E}^{-1}(T^{-u_T}) \leq u_T \log T/\kappa$ and (B.13) that there exists some constant $c_G > 0$ such that

$$
\mathbb{P} \left(\| \hat{\rho}_{h,T} - \mathbb{E} \hat{\rho}_{h,T} \|_{L^\infty(D)} \geq c_G \left(\frac{u_T(\log T)^2}{Th^d} + T^{-\frac{1}{2}} \psi_d(h^d) \sqrt{u_T \log(h^{-1})}\right)\right) \leq e^{-u_T}.
$$

Appendix C: Proofs for Section 4

Proof of Corollary 4.1. Fix $x$ such that there exists an open neighbourhood $D$ of $x$ such that $\rho | D \in \mathcal{H}_D(\beta, L)$. The usual bias-variance decomposition gives

$$
\mathbb{E} \left[\left(\hat{\rho}_{h,T}(x) - \rho(x)\right)^2\right] = (\rho * K_h(x) - \rho(x))^2 + \text{Var}(\hat{\rho}_{h,T}(x)).
$$

(C.1)

For the bias term, since $\| \beta \| \leq \ell$, there exists a universal constant $M > 0$ such that

$$
|\rho * K_h(x)| = \left|h^{-d} \int K \left(\frac{x - y}{h}\right) (\rho(y) - \rho(x)) \, dy\right| \leq M h^d,
$$

(C.2)

see Proposition 1.2 in [80] for the case $d = 1$ and the analogous estimator for discrete observations, which can be extended to the general multivariate case under continuous observations without much effort. Moreover, for any dimension $d$ and
\(X \in \Sigma \cup \Theta\), it follows from (B.9) that for any \(h \in (0,1)\)

\[
\operatorname{Var}\left(\frac{1}{T} \int_0^T K_h(x - X_t) \, dt\right) \leq T^{-1} \|K\|_\infty^2 \|\rho\|_\infty \psi_d^2(h^d).
\]

The claim follows by plugging the specific choice of \(h\) into (C.2) and (B.9) and using (C.1).

\(\square\)

**Proof of Theorem 4.2.** Fix \(p \geq 1\), and recall the decomposition (3.3). By the assumption on the order of the kernel \(K\), the bias term \(\rho \cdot K_h - \rho\) is bounded by \(B(h) := MH^d\) for some universal constant \(M > 0\) as in the pointwise case (see (C.2)), while the upper bound on the stochastic error \(\mathbb{H}_{h,T}\) relies on a suitable specification on the upper bound in (B.12).

For \(d \geq 3\), set \(h = h(T) = (\log T/T)^{1/(2\beta + d - 2)}\) and \(m_T = p \log T/\kappa\) such that

\[
\frac{1}{\sqrt{T}} \psi_d(h^d) \in O(\log T)^{\beta/(2\beta + d - 2)} \quad \text{and} \quad \frac{m_T}{T h^d} = \left(\frac{\log T}{T}\right)^{2(\beta - 1)/(\beta + d - 2)}.
\]

Upon noting that \(\beta > 2\) implies \(2(\beta - 1) > \beta\), it follows from (B.12) that

\[
\left(\mathbb{E}\left[\sup_{x \in D} |\mathbb{H}_{h,T}(x)|^p\right]\right)^{1/p} \in O\left(\left(\frac{\log T}{T}\right)^{\beta/(2\beta + d - 2)}\right), \quad (C.3)
\]

Since \(h^\beta = (\log T/T)^{\beta/(2\beta + d - 2)}\), (3.3), (C.2) and (C.3) finally give \(\mathbb{E}[|\hat{\rho}_h - \rho|^p]^{1/p} \in O(\Psi_{d,\beta}(T))\) for \(d \geq 3\). For \(d = 1\) and \(d = 2\), the assertion follows by analogous arguments.

We now proceed with the proof of the convergence rate of the adaptive scheme for \(d \geq 3\). For the variance, we obtain from (B.12) that, for \(m_T := 2 \log(k) T (\log T)^2 / \kappa\) and whenever \(h \leq e^{-2} \text{diam}(D)/\|K\|_\infty\), there exists some constant \(C > 0\) such that

\[
\mathbb{E}\left[|\hat{\rho}_h - \mathbb{E}\hat{\rho}_h|^2\right] \leq C^2 \sigma^2(h, T),
\]

where \(\sigma^2(\cdot, \cdot)\) is defined according to (4.3). Define \(h_\rho\) by the balance equation

\[
h_\rho := \max\left\{h \in \mathcal{H}_T : B(h) \leq \frac{1}{4} \sqrt{0.8M}\sigma(h, T)\right\}, \quad \text{where} \quad M := \|\rho\|_{L^\infty(D)}.
\]

This definition implies that \(B(h_\rho) \simeq \sqrt{0.8M}\sigma(h_\rho, T)/4\) and, since \(\mathcal{H}_T \ni h_\rho \geq \left(\frac{\log(k) T \log T}{T}\right)^{\frac{\beta}{2(\beta + d - 2)}}
\]

\[
h_\rho^{2(\beta + d - 2)} \simeq \frac{\log(k) T \log T}{T} \quad \text{and} \quad \sigma(h_\rho, T) \simeq \left(\frac{\log(k) T \log T}{T}\right)^{\frac{\beta}{2(\beta + d - 2)}}.
\]

To justify this, define \(h_0 := (\log(k) T \log T)^{1/(2\beta + d - 2)}\). For large enough \(T\), we have \(\log(k) T \log T \leq (\log T)/2\) and hence

\[
\sigma(h_0, T) = \frac{\log(k) T (\log T)^2}{T h_0^d} \log(h_0^{-1}) + \psi_d(h_0^d) \sqrt{\frac{\log(k) T \log(h_0^{-1})}{T}}
\]

\[
\geq \sqrt{\frac{\log(k) T \log T}{2(2\beta + d - 2)T}} \psi_d(h_0^d) = \sqrt{\frac{1}{2(2\beta + d - 2)}} \left(\frac{\log(k) T \log T}{T}\right)^{\frac{\beta}{2(\beta + d - 2)}} = \mathcal{L}^{-1} B(h_0),
\]

for \(\mathcal{L} = \sqrt{2(2\beta + d - 2)M^2}\). Additionally, we get, since \(\beta > 2\),

\[
\sigma(h_0, T) = \frac{\log(k) T (\log T)^2}{T h_0^d} \log(h_0^{-1}) + \psi_d(h_0^d) \sqrt{\frac{\log(k) T \log(h_0^{-1})}{T}} \simeq \left(\frac{\log(k) T \log T}{T}\right)^{\frac{\beta}{2(\beta + d - 2)}}.
\]

In particular, it holds that \(h_0 \lesssim h_\rho\), which is clear if \(\mathcal{L} \leq \frac{1}{4} \sqrt{0.8M}\), and else follows by the fact that, for any \(0 < \lambda < 1\),

\[
B(\lambda h_0) = \lambda^\beta B(h_0) \leq \lambda^\beta \mathcal{L} \sigma(h_0, T) \leq \lambda^\beta \mathcal{L} \sigma(\lambda h_0, T).
\]
Lastly, we show \( h_\rho \lesssim h_0 \) by proving \( h_\rho^{2\beta + d - 2} h_0^{-(2\beta + d - 2)} \in O(1) \). Indeed, by the definition of \( h_\rho \),
\[
h_\rho^{2\beta + d - 2} \lesssim h_\rho^{d-2} \sigma^2(h_\rho, T)
\]
\[
\lesssim h_\rho^{d-2} \left( \log_k(T) \frac{1}{T} h_\rho^{d-2} + \psi_d(h_\rho) \right) 2
\]
\[
\lesssim \frac{(\log_k(T))^2 (\log(T))^6}{T^2} h_\rho^{-(2+d)} + h_\rho^{d-2} \psi_d^2(h_\rho) \frac{\log_k(T) T \log T}{T},
\]
and thus it holds that
\[
h_\rho^{2\beta + d - 2} h_0^{-(2\beta + d - 2)} \lesssim \frac{(\log_k(T))^5 (\log(T))^5}{T} h_\rho^{-(2+d)} + h_\rho^{d-2} \psi_d^2(h_\rho) \in O(1),
\]
thanks to \( h_\rho > (\log_k(T) (\log(T))^5/T)^{1/(d+2)} \).

**Case 1:** We first consider the case where \( \hat{h}_T \geq h_\rho \). To shorten notation, denote \( \hat{M} := \|\hat{\rho}_{h_{\min},T}\|_{L^\infty(D)} \). Then, exploiting
the definition of \( \hat{h}_T \) according to (4.2) and the bias and variance bounds,
\[
\mathbb{E} \left[ \|\hat{\rho}_{h_T,T} - \rho\|_{L^\infty(D)} \cdot 1_{\{\hat{h}_T \geq h_\rho\}} \cap (\hat{M} \leq 1.2M) \right]
\]
\[
\leq \mathbb{E} \left[ \left( \|\hat{\rho}_{h_T,T} - \hat{\rho}_{h,T}\|_{L^\infty(D)} + \|\hat{\rho}_{h,T} - \mathbb{E} \hat{\rho}_{h,T}\|_{L^\infty(D)} + B(h_\rho) \right) 1_{\{\hat{h}_T \geq h_\rho\}} \cap (\hat{M} \leq 1.2M) \right]
\]
\[
\leq \sqrt{1.2M} \sigma(h_\rho, T) + C \sigma(h_\rho, T) + \frac{1}{4} \sqrt{0.8M} \sigma(h_\rho, T) \in O(\sigma(h_\rho, T)).
\]
Similarly,
\[
\mathbb{E} \left[ \|\hat{\rho}_{h_T,T} - \rho\|_{L^\infty(D)} \cdot 1_{\{\hat{h}_T \geq h_\rho\}} \cap (\hat{M} > 1.2M) \right]
\]
\[
\leq \sum_{h \in \mathcal{J} : h \geq h_\rho} \mathbb{E} \left[ \|\hat{\rho}_h - \mathbb{E} \hat{\rho}_h\|_{L^\infty(D)} + B(h) \right] \cdot 1_{\{\hat{h}_T = h\}} \cap (\hat{M} > 1.2M)
\]
\[
\lesssim \log T (C \sigma(h_\rho, T) + B(1)) \sqrt{P(M > 1.2M)}.
\]
Now, for any \( T \) large enough,
\[
\mathbb{P} \left( |\hat{M} - M| > 0.2 \|\rho\|_{L^\infty(D)} \right) = \mathbb{P} \left( \|\hat{\rho}_{h_{\min},T}\|_{L^\infty(D)} - \|\rho\|_{L^\infty(D)} > 0.2M \right)
\]
\[
\leq \mathbb{P} \left( \|\hat{\rho}_{h_{\min},T} - \rho\|_{L^\infty(D)} - 0.2\|\rho\|_{L^\infty(D)} \right)
\]
\[
\leq \mathbb{P} \left( \|\hat{\rho}_{h_{\min},T} - \mathbb{E} \hat{\rho}_{h_{\min},T}\|_{L^\infty(D)} > 0.2\|\rho\|_{L^\infty(D)} - B(h_{\min}) \right)
\]
\[
\leq \mathbb{P} \left( \|\hat{\rho}_{h_{\min},T} - \mathbb{E} \hat{\rho}_{h_{\min},T}\|_{L^\infty(D)} > 0.1\|\rho\|_{L^\infty(D)} \right)
\]
\[
\leq \mathbb{P} \left( \|\hat{\rho}_{h_{\min},T} - \mathbb{E} \hat{\rho}_{h_{\min},T}\|_{L^\infty(D)} > \Upsilon_{h_{\min},T}(T \log T) \right)
\]
\[
\leq T^{-1},
\]
where, for the function \( \Upsilon_{h_{\min},T}(\cdot) \) defined according to (3.5), the last inequality follows from Lemma 3.4 and the last but
one inequality holds since there exists some constant \( C \) such that
\[
\Upsilon_{h_{\min},T}(T \log T) \leq C T^{-\frac{\beta+2d}{2\beta+2d}} (\log(T)^{\frac{\beta+2d}{2\beta+2d}} (\log_k(T) T)^{-\frac{d}{2\beta+2d}} + (\log(T)^{\frac{\beta+2d}{2\beta+2d}} (\log_k(T) T)^{\frac{d-2}{2\beta+2d}})
\]
\[
\leq 0.2\|\rho\|_{L^\infty(D)},
\]
for \( T \) sufficiently large. Thus, we can conclude that \( \mathbb{E} \left[ \|\hat{\rho}_{h_T,T} - \rho\|_{L^\infty(D)} \cdot 1_{\{\hat{h}_T \geq h_\rho\}} \right] \in O(\sigma(h_\rho, T)) \).
For the case $\hat{h}_T < h_\rho$, note first that the previous bias and variance bounds together with (C.4) imply that
\[
\begin{align*}
\mathbb{E} \left[ \| \hat{\rho}_{h,T} - \rho \|_{L^\infty(D)} \cdot 1_{\{ \hat{h}_T < h_\rho \}} \cap (\hat{M} < 0.8M) \right] \\
\leq \sum_{h \in \mathcal{H}_T : h < h_\rho} \mathbb{E} \left[ \left( \| \hat{\rho}_{h,T} - \mathbb{E} \hat{\rho}_{h,T} \|_{L^\infty(D)} + B(h) \right) \cdot 1_{\{ \hat{h}_T = h \}} \cap (\hat{M} < 0.8M) \right] \\
\leq \log T \left( C \sigma(h_{\min}, T) + B(h_\rho) \right) \mathbb{P}(\hat{M} < 0.8M) = O(\sigma(h_\rho, T)).
\end{align*}
\]
On the other hand,
\[
\begin{align*}
\mathbb{E} \left[ \| \hat{\rho}_{h,T} - \rho \|_{L^\infty(D)} \cdot 1_{\{ \hat{h}_T < h_\rho \}} \cap (0.8M \leq \hat{M}) \right] \\
\leq \sum_{h \in \mathcal{H}_T : h < h_\rho} \mathbb{E} \left[ \left( \| \hat{\rho}_{h,T} - \mathbb{E} \hat{\rho}_{h,T} \|_{L^\infty(D)} + B(h) \right) \cdot 1_{\{ \hat{h}_T = h \}} \cap (0.8M \leq \hat{M}) \right] \\
\leq \sum_{h \in \mathcal{H}_T : h < h_\rho} \sqrt{\mathbb{E} \left[ \| \hat{\rho}_{h,T} - \mathbb{E} \hat{\rho}_{h,T} \|_{L^\infty(D)}^2 \right]} \sqrt{\mathbb{E} \left[ 1_{\{ \hat{h}_T \geq h_\rho \}} \cap (0.8M \leq \hat{M}) \right]} + B(h_\rho) \\
\leq \sum_{h \in \mathcal{H}_T : h < h_\rho} C \sigma(h, T) \sqrt{\mathbb{P} \left( \{ \hat{h}_T \geq h_\rho \} \cap \{ 0.8M \leq \hat{M} \} \right) + O(\sigma(h_\rho, T))}.
\end{align*}
\]
Pick any $h \in \mathcal{H}_T$ such that $h < h_\rho$ and denote $h^+ := \min \{ g \in \mathcal{H}_T : g > h \} = \eta h$. It is then shown as in the proof of Theorem 2 in [41] that the verification of the fact that the first sum on the rhs of the last display is of order $O(\sigma(h_\rho, T))$ boils down to proving that
\[
\sum_{h \in \mathcal{H}_T : h < h_\rho} \sigma(h, T) \left( \sum_{g \in \mathcal{H}_T : g \leq h} \mathbb{P} \left[ \| \hat{\rho}_{h,T} - \hat{\rho}_{g,T} \|_{L^\infty(D)} > \sqrt{0.8M} \sigma(g, T) \right] \right)^{1/2} \in O(\sigma(h_\rho, T)).
\]
Following again the lines of [41], we obtain
\[
\begin{align*}
\mathbb{P} \left[ \| \hat{\rho}_{h,T} - \hat{\rho}_{g,T} \|_{L^\infty(D)} > \sqrt{0.8M} \sigma(g, T) \right] &\leq \mathbb{P} \left[ \| \hat{\rho}_{h,T} - \mathbb{E} \hat{\rho}_{h,T} \|_{L^\infty(D)} > \frac{1}{4} \sqrt{0.8M} \sigma(h^+, T) \right] \\
&+ \mathbb{P} \left[ \| \hat{\rho}_{g,T} - \mathbb{E} \hat{\rho}_{g,T} \|_{L^\infty(D)} > \frac{1}{4} \sqrt{0.8M} \sigma(g, T) \right].
\end{align*}
\]
Let $\gamma \geq 1$. Clearly, by definition of $\sigma(g, T)$, there exists $T(\gamma) > 0$ such that, for any $T \geq T(\gamma)$ and any $g \leq h_\rho$, $g \in \mathcal{H}_T$, we have
\[
\frac{1}{4} \sqrt{0.8M} \sigma(g, T) \geq c_\gamma T g (\gamma \log(g^{-1})) = c_\gamma \frac{\gamma (\log(g^{-1})) (\log T)^2}{T g^d} + \psi_d(g^d) \frac{\gamma \log(g^{-1})}{T},
\]
where $c_\gamma$ is the constant appearing in Lemma 3.4. Thus, using Lemma 3.4, we obtain for $T \geq T(\gamma)$ that
\[
\mathbb{P} \left[ \| \hat{\rho}_{g,T} - \mathbb{E} \hat{\rho}_{g,T} \|_{L^\infty(D)} > \frac{1}{4} \sqrt{0.8M} \sigma(g, T) \right] \leq e^{-\gamma \log(g^{-1})} = g^{-\gamma} =: \iota_\gamma(g)
\]
and hence
\[
\sum_{g \in \mathcal{H}_T : g \leq h} \mathbb{P} \left[ \| \hat{\rho}_{h,T} - \hat{\rho}_{g,T} \|_{L^\infty(D)} > \sqrt{0.8M} \sigma(g, T) \right] \leq \sum_{g \in \mathcal{H}_T : g \leq h} (\iota_\gamma(g) + \iota_\gamma(h^+)) \leq 2 \iota_\gamma(h) \log T.
\]
Thus, choosing $\gamma$ large enough demonstrates that
\[
\sum_{h \in \mathcal{H}_T : h < h_\rho} \sigma(h, T) \left( \sum_{g \in \mathcal{H}_T : g \leq h} \mathbb{P} \left[ \| \hat{\rho}_{h,T} - \hat{\rho}_{g,T} \|_{L^\infty(D)} > \sqrt{0.8M} \sigma(g, T) \right] \right)^{1/2}
\]
Let us first verify that under $(O1)$ the heat kernel bound $(A1)$ holds. Arguing as in the proof of Theorem 3.2 of Masuda [55], we see that $\mathcal{F}\mu$ and $\varphi_{Xt}$ are integrable for any $x \in \mathbb{R}$ and $t > 0$ and hence we can obtain the invariant density $\rho$ and the transition density $p_t$ of $X$ via inverse Fourier transformation through

$$
\rho(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i(y,\lambda)} \{\mathcal{F}\mu\}(\lambda) \, d\lambda, \quad y \in \mathbb{R}^d,
$$

and

$$
p_t(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i(y,\lambda)} \varphi_{Xt}(\lambda) \, d\lambda, \quad x, y \in \mathbb{R}^d, \ t > 0.
$$

Again, as in the proof of Theorem 3.2 in [55], it follows that under $(O1)$,

$$
|\varphi_{Xt}(\lambda)| \leq \exp\left( -\frac{1}{2} \lambda^\top \left( \int_0^t e^{-sB}Qe^{-sB\top} \, ds \right) \lambda \right), \quad x, \lambda \in \mathbb{R}^d, \ t > 0.
$$

Thus, using the characterization of the multivariate normal distribution, we obtain

$$
p_t(x, y) \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left( -\frac{1}{2} \lambda^\top \left( \int_0^t e^{-sB}Qe^{-sB\top} \, ds \right) \lambda \right) \, d\lambda
= \frac{1}{(2\pi)^{d/2}} \left( \det \left( \int_0^t e^{-sB}Qe^{-sB\top} \, ds \right) \right)^{-1/2}.
$$

Observing that

$$
\lim_{t \downarrow 0} t^{d/2} \left( \det \left( \int_0^t e^{-sB}Qe^{-sB\top} \, ds \right) \right)^{-1/2} = \left( \det \left( \lim_{t \downarrow 0} \frac{1}{t} \int_0^t e^{-sB}Qe^{-sB\top} \, ds \right) \right)^{-1/2}
= \det(Q)^{-1/2} < \infty,
$$

where finiteness is a consequence of invertibility of $Q$ by $(O1)$, it follows that for any $d \geq 1$, there exists a constant $c > 0$ such that

$$
\sup_{x, y \in \mathbb{R}^d} p_t(x, y) \leq ct^{-d/2}, \quad t \in (0, 1].
$$

Thus indeed, for any dimension $d \in \mathbb{N}$, $(A1)$ holds. Next, in scenario (i), [55, Theorem 4.3] gives the exponential $\beta$-mixing property and the proof of Theorem 2.6 in [56] along with [56, Proposition 3.8] yields $V$-exponential ergodicity with $V(x) \sim (1 + \|x\|^p)$. This together with (4.6) entails that in scenario (i), we have $X \in \Sigma \cap \Theta$. Finally, $X \in \Theta$ in scenario (ii) follows from the considerations above and Lemma 2.3 due to the fact that the combination of $(O1)$ and the logarithmic moment condition imply that every compact set is small and hence petite since $X$ is strong Feller and by [47, Theorem 3.1] ergodic (see Proposition A.2) and hence $(O3)$ implies $V$-polynomial ergodicity of degree $\alpha - 1 > 1$ with $V(x) = C(\log|x|)^{\alpha}$ in dimension $d = 1$ by [48, Corollary 1]. The statements on the estimation rates are now an immediate consequence of Corollary 4.1 and Theorem 4.2 and the fact that $\rho \in C^\infty$ has arbitrary Hölder smoothness.

**Proof of Lemma 4.6.** We will employ Theorem 6.2.9 and Exercise 6.4.7 of [7] to show the first assertion. So first we must verify that condition $(C1)$ on page 365 of [7] holds. Since $(\mathcal{F}1)$ holds, we only have to show that there exists a constant $K_1 > 0$ such that, for all $x, y \in \mathbb{R}^d$,

$$
\sum_{i,j=1}^d (\sigma_{i,j}(x) - \sigma_{i,j}(y))^2 + \int_{\mathbb{R}^d} \|\gamma(x)z - \gamma(y)z\|^2 \nu(\,dz) \leq K_1 \|x - y\|^2,
$$

as desired. □
where \( \sigma_{i,j}(x) \) denotes the components of \( \sigma(x) \in \mathbb{R}^{d \times d} \) for any \( x \in \mathbb{R}^d \), (\( \mathcal{F}1 \)) implies that there exists a finite constant \( L_{i,j} > 0 \) for any \( i, j \in \{1, \ldots, d\} \), such that \( \sigma_{i,j} : \mathbb{R}^d \to \mathbb{R} \) is Lipschitz continuous with Lipschitz constant \( L_{i,j} > 0 \) and hence we have for \( x, y \in \mathbb{R}^d \)

\[
\sum_{i,j=1}^{d} (\sigma_{i,j}(x) - \sigma_{i,j}(y))^2 \leq 2d \max_{i,j \in \{1, \ldots, d\}} L_{i,j}^2 \|x - y\|^2.
\]

Furthermore, we have for \( x, y \in \mathbb{R}^d \) by the Lipschitz continuity of \( \gamma \)

\[
\int_{\mathbb{R}^d} \|\gamma(x)z - \gamma(y)z\|^2 \nu(dz) \leq L_{\gamma}^2 \|x - y\|^2 \int_{\mathbb{R}^d} \|z\|^2 \nu(dz),
\]

where we denote the Lipschitz constant of \( \gamma \) by \( L_{\gamma} \). By (\( \mathcal{F}3 \)), \( \int_{\mathbb{R}^d} \|z\|^2 \nu(dz) \) is finite and hence (\( C1 \)) holds. To verify the growth condition (\( C2 \)) on page 366 of [7], we have to show that there exists a constant \( K_2 \) such that, for all \( x \in \mathbb{R}^d \),

\[
\int_{\mathbb{R}^d} \|\gamma(x)z\|^2 \nu(dz) \leq K_2 (1 + \|x\|^2).
\]

Since \( \gamma \) is Lipschitz continuous by (\( \mathcal{F}1 \)), there exists a constant \( K > 0 \) such that the linear growth condition \( \|\gamma(x)\| \leq K(1 + \|x\|) \) holds for all \( x \in \mathbb{R}^d \), and thus we have, for \( x \in \mathbb{R}^d \),

\[
\int_{\mathbb{R}^d} \|\gamma(x)z\|^2 \nu(dz) \leq 2K^2 (1 + \|x\|^2) \int_{\mathbb{R}^d} \|z\|^2 \nu(dz).
\]

Again by (\( \mathcal{F}3 \)), \( \int_{\mathbb{R}^d} \|z\|^2 \nu(dz) \) is finite and hence (\( C2 \)) holds for \( K_2 = 2K^2 \int_{\mathbb{R}^d} \|z\|^2 \nu(dz) \). Since Assumption 6.2.8 in [7] is trivially fulfilled, the first assertion follows by Theorem 6.2.9 and Exercise 6.4.7 of [7].

We proceed by showing the second assertion. Equation (1.21) of [21] is in the setting of (4.7) equivalent to \( \kappa_\alpha(x, z) = \|\gamma(x)z\|^{d + \alpha} \nu(z) \geq 0 \) for all \( x \in \mathbb{R}^d \) and almost every \( z \in \mathbb{R}^d \). Since \( \nu \) is a density, this assumption is fulfilled.

For assumption (\( H^p \)) of [21] to hold, we only need to show that there exists a \( \beta \in (0, 1) \) such that the function \( \alpha(x) := \sigma(x)\sigma^T(x) \) is \( \beta \)-Hölder continuous. However this follows directly from the Lipschitz continuity and the boundedness of \( \sigma \) imposed in (\( \mathcal{F}1 \)), as can be seen in the proof of Lemma 1 of [4]. Now we note that assumption (\( H^p \)) of [21] follows by (\( \mathcal{F}2 \)).

**Proof of Corollary 4.7.** Since (\( \mathcal{F}1 \)) and (\( \mathcal{F}3 \)) imply that \( b^* \) is bounded, arguing as in the proof of Lemma 1 of [4] and using Lemma 4.6 entails that \( b^* \) belongs to the Kato class \( \mathcal{K}_2 \) for \( d \geq 2 \). For the definition of \( \mathcal{K}_2 \), see (2.28) in [21]. Existence of transition densities and the heat kernel estimate now follow directly from Corollary 1.5 of [21] and Lemma 4.6 for \( d \geq 2 \) and as described in Lemma 1 of [4], the same conclusions may be drawn for dimension \( d = 1 \) by adapting the arguments in [21]. Now note that (4.9), \( t \leq 1 \) and \( \alpha \in (0, 2) \) imply

\[
pt(x, y) \leq C(t^{-d/2} \exp(-t\|x - y\|^2/(\lambda t)) + \|\kappa_\alpha\|_\infty t\|x - y\| + t^{1/2})^{-d-\alpha})
\]

\[
\leq C(t^{-d/2} + t^{1-(d+\alpha)/2}) \leq Ct^{-d/2},
\]

where the value of \( C \) changes from line to line. This completes the proof.

**Proof of Proposition 4.8.** To verify the assertion, we show that the solution of (4.7) \( X \) satisfies the assumptions of Theorem 2.2 (ii) of [56] which are Assumption 1, 2(\( a' \)) and 3(\( a \)) of [56] and [57], respectively. Assumption 1 follows directly from (\( \mathcal{F}1 \)). Now, define \( b_\nu^*(x) := b^*(x) - \int_{u < \|z\| \leq 1} \gamma(x)z \nu(dz) = b(x) - \int_{\|z\| > u} \gamma(x)z \nu(dz) \), and let the diffusion process \( Y^u = (Y^u_t)_{t \geq 0} \) be given by

\[
Y^u_t = x + \int_0^t b_\nu^*(Y^u_s) \, ds + \int_0^t \sigma(Y^u_s) \, dW_s.
\]

For Assumption 2(\( a' \)) to be fulfilled, we first have to show that, for any \( u \in (0, 1) \), there exists \( \Delta > 0 \) such that \( P_x(Y^u_A \in B) > 0 \) for any \( x \in \mathbb{R}^d \) and any nonempty open set \( B \subseteq \mathbb{R}^d \). Since \( Y^u \) is a continuous diffusion process with bounded and Lipschitz coefficients \( b_\nu^* \), \( \sigma \) and \( \alpha = \sigma \sigma^T \) is uniformly elliptic, it follows from classical results, see e.g. [75, Theorem A], that for any \( x \in \mathbb{R}^d \) and \( t > 0 \), the transition function \( P_t(x, \cdot) \) of \( Y^u \) has a transition density with full support and
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\[ Q^* := \{ f : \mathbb{R}^d \to \mathbb{R}_+ : f \in C^2, f(x) \to \infty \text{ as } \|x\| \to \infty, \text{ and there exists a function } V \in Q^*, \text{ where} \]

\[ f(x + \gamma(x)z) \nu(dz) \leq f(x), \forall x \in \mathbb{R}^d \} \]

such that there are constants \( c_1, c_2 > 0 \), for which the Lyapunov drift criterion

\[ AV \leq -c_1 V + c_2 \]  

(C.5)

holds, where \( A \) denotes the extended generator of \( X \) acting on \( Q^* \) by

\[ Af(x) = (\nabla f(x), b^*(x)) + \frac{1}{2} \text{tr}(\nabla^2 f(x)\sigma(x)\sigma^T(x)) + \int_{\mathbb{R}^d} f(x + \gamma(x)z) - f(x) - 1_{\|z\|\leq 1}(\nabla f(x), \gamma(x)z)\nu(dz), \quad x \in \mathbb{R}^d, f \in Q^*. \]

Now, for \( \eta \in (0, \eta_0 c_1^{-1} \land 1) \), where \( c_1 := \|\gamma\|_{\infty} \), let \( V^\eta \) be a positive and increasing function in \( C^2(\mathbb{R}^d, \mathbb{R}) \) such that \( V^\eta = e^{\eta \|x\|} \) for all \( \|x\| > c_V \), where \( c_V > 0 \). Then, it holds for \( i \neq j \in \{1, \ldots, d\} \) and \( \|x\| > c_V \),

\[ \partial_i V^\eta(x) = \eta e^{\eta \|x\|} \frac{x_i}{\|x\|}, \]

\[ \partial_{ij}^2 V^\eta(x) = \eta^2 e^{\eta \|x\|} \frac{x_i x_j}{\|x\|^2} - \eta e^{\eta \|x\|} \frac{\|x\|^3}{\|x\|^2} \delta_{ij} \]  

(C.6)

where \( \delta_{ij} \) denotes the Kronecker delta. Furthermore, since \( V^\eta \in C^2(\mathbb{R}^d; \mathbb{R}) \), for \( i, j \in \{1, \ldots, d\} \) the functions \( V^\eta, \partial_i V^\eta, \partial_{ij}^2 V^\eta \) are bounded by a constant \( c_D > 0 \) for \( \|x\| \leq c_V \) and hence

\[ \int_{\|z\| > 1} V^\eta(x + \gamma(x)z) \nu(dz) \leq \int_{\|z\| > 1} (e^{\eta \|x\| + \gamma(x)\|z\|} + c_D) \nu(dz) \]

\[ \leq e^{\eta \|x\|} \int_{\|z\| > 1} e^{\gamma \|z\|} \nu(dz) + c_D \nu(\mathbb{R}^d \setminus B_1), \]

implying that \( V^\eta_2 \in Q^* \) for all \( \eta \leq \frac{c_D}{c_V} \). This last condition is satisfied by our choice of \( \eta \). To conclude the proof, the only thing left to show is that there exists \( 0 < \eta \leq \frac{c_D}{c_V} \) such that (C.5) holds for \( V^\eta \). Note that, by the mean value theorem, the definition of \( b^* \) and the Cauchy–Schwarz inequality, we have for any \( f \in Q^* \)

\[ Af(x) = (\nabla f(x), b(x)) + \frac{1}{2} \text{tr}(\nabla^2 f(x)\sigma(x)\sigma^T(x)) + \int_{\mathbb{R}^d} f(x + \gamma(x)z) - f(x) - (\nabla f(x), \gamma(x)z)\nu(dz) \]

\[ \leq (\nabla f(x), b(x)) + \frac{1}{2} \text{tr}(\nabla^2 f(x)\sigma(x)\sigma^T(x)) + \int_{\mathbb{R}^d \times (0,1)} \|\nabla f(x + t \gamma(x)z) - \nabla f(x)\| \|\gamma(x)z\| \nu(dz) \]

\[ \leq A_c f(x) + A_d f(x), \]

where, for \( \nabla^2 f(x) \) denoting the Hessian of \( f \) evaluated at \( x \),

\[ A_c f(x) := (\nabla f(x), b(x)) + \frac{1}{2} \text{tr}(\nabla^2 f(x)\sigma(x)\sigma^T(x)), \]

\[ A_d f(x) := c_d^2 \int_{\mathbb{R}^d \times (0,1)} \sup_{t \in (0,1)} \|\nabla^2 f(x + t \gamma(x)z)\| \|z\|^2 \nu(dz). \]

We start by investigating the jump part. By (C.6) and the fact that the operator norm can be bounded by the Frobenius norm \( \| \cdot \|_F \), we get for \( \|x\| > c_V \)

\[ \|\nabla^2 V^\eta(x)\|_F \leq \|\nabla^2 e^{\eta \|x\|}\|_F = \left( \sum_{i,j=1}^d \left( \eta^2 e^{\eta \|x\|} \frac{x_i x_j}{\|x\|^2} - \eta e^{\eta \|x\|} \frac{x_i x_j}{\|x\|^3} + \eta e^{\eta \|x\|} \|x\|^{-1} \delta_{ij} \right)^2 \right)^{1/2} \]
\[ \leq 2\eta^2e^{|x|/2} \left( \sum_{i,j=1}^d \left( \eta^2x_i^2x_j^2 + x_i^2x_j^2 \|x\|^2 + \|x\|^{-2}\delta_{ij} \right) \right)^{\frac{1}{2}} \leq 2^{3/2}\sqrt{d}\eta^2e^{|x|} (\eta^2 + 2\|x\|^{-2})^{\frac{1}{2}}. \]

Since we can choose \( c_V \) to be large, we can without loss of generality assume \( c_V \geq \sqrt{2}\eta^{-1} \) and, additionally, \( V^\eta \in C^2 \) implies that there exists a real-valued function \( cH(\eta) > 0 \) on \((0,\infty)\) such that \( \|H^2V^\eta(x)\| < cH(\eta) \) for all \( \|x\| \leq cV \).

Thus, we have \( \|H^2V^\eta(x)\| \leq 4\sqrt{d}\eta^2e^{|x|} + cH(\eta) \), \( x \in \mathbb{R}^d \), and we can conclude

\[
A_0 V^\eta(x) \leq 4c_0^2\sqrt{d}\eta^2 \int_{\mathbb{R}^d} \sup_{t \in [0,1]} e^{0_0|x+t\gamma(x)|\|x\|^2} \|z\|^2\nu(dz) + c_0^2 cH(\eta) \int_{\mathbb{R}^d} \|z\|^2\nu(dz)
\]

\[
\leq \eta^2e^{|x|} \left( \sum_{k=1}^d \left( \frac{c_0^2}{2} \eta \|x\| \frac{x_i^2}{\|x\|} + \eta \|x\|^{-1} \eta e^\eta \|x\| \frac{x_i^2}{\|x\|^2} \right) \right)
\]

and since \( V^\eta \in C^2(\mathbb{R}^d;\mathbb{R}) \), there exists a real-valued function \( c_\gamma(\eta) \) on \((0,\infty)\) such that \( A_\gamma V^\eta(x) \leq c_\gamma(\eta) \) for all \( \|x\| \leq cV \). Hence, we have

\[
A_\gamma V^\eta(x) \leq \eta e^{|x|} \left( -c_\gamma + \frac{3c_\gamma d}{2}\eta \right) + c_\gamma(\eta) + c_1 e^{cV} =: \eta e^{|x|} \left( -c_\gamma + \frac{3c_\gamma d}{2}\eta \right) + c_{\gamma,1}(\eta),
\]

where \( c_{\gamma,1} \) are positive and finite because of (J.3) and \( \eta < \eta_0\gamma^{-1} \). Now we turn our attention to the continuous part. From now on, without loss of generality, we assume that \( c_V \geq c_1 \) in (J.3). Then, for \( \|x\| > c_V \geq \eta^{-1} \), we have by (J.1), (J.3) and (C.6)

\[
A_\gamma V^\eta(x) \leq \eta e^{|x|} \left( -c_\gamma + \frac{3c_\gamma d}{2}\eta \right) + c_\gamma(\eta) + c_1 e^{cV} =: \eta e^{|x|} \left( -c_\gamma + \frac{3c_\gamma d}{2}\eta \right) + c_{\gamma,1}(\eta)
\]

Choosing \( \eta^* = 1 \wedge \eta_0\gamma^{-1} \wedge \frac{c_1}{3c_\gamma d + c_{\gamma,1}} \) implies

\[
A_\gamma V^\eta(x) \leq \frac{c_1\eta^*}{2} \eta e^{|x|} + c_{\gamma,2}(\eta^*) + c_{\gamma,1}(\eta^*),
\]

and thus (C.5) holds for \( V^\eta \in Q^* \). Now, Theorem 2.2 (ii) and Proposition 3.8 of [56] show the required assertion. \( \square \)

References

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