Asymptotic Enumeration and Limit Laws for Multisets: the Subexponential Case

Konstantinos Panagiotou\(^a\) and Leon Ramzews\(^a,\ast\)

\(^a\)Department of Mathematics, Ludwig-Maximilians-Universität München, Munich, Germany E-mail: kpanagio@math.lmu.de; \(^\ast\)ramzews@math.lmu.de

Abstract. For a given combinatorial class \(C\) we study the class \(G = \text{MSET}(C)\) satisfying the multiset construction, that is, any object in \(G\) is uniquely determined by a set of \(C\)-objects paired with their multiplicities. For example, \(\text{MSET}(\mathbb{N})\) is (isomorphic to) the class of number partitions of positive integers, a prominent and well-studied case. The multiset construction appears naturally in the study of unlabelled objects, for example graphs or various structures related to number partitions. Our main result establishes the asymptotic size of the set \(G_{n,N}\) that contains all multisets in \(G\) having size \(n\) and being comprised of \(N\) objects from \(C\), as \(n\) and \(N\) tend to infinity and when the counting sequence of \(C\) is governed by subexponential growth. Moreover, we study the component distribution of random objects from \(G_{n,N}\) and we discover a phenomenon that we baptise extreme condensation: taking away the largest component as well as all the components of the smallest possible size, we are left with an object which converges in distribution as \(n,N \to \infty\). The distribution of the limiting object is also retrieved. Moreover and rather surprisingly, in stark contrast to analogous results for labelled objects, the results here hold uniformly in \(N\).

Résumé. In French.

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1. Introduction & Main Results

Let \(C\) be a combinatorial class, that is, a countable set endowed with a size function \(|\cdot|: C \to \mathbb{N}\) such that \(C_n := \{C \in C: |C| = n\}\) contains only finitely many objects for all \(n \in \mathbb{N}\). Then the class of \(C\)-multisets \(G = \text{MSET}(C)\) consists of all objects of the form

\[
\{(C_1, d_1), \ldots, (C_k, d_k)\}, \quad k \in \mathbb{N}, \quad C_i \in C, d_i \in \mathbb{N} \text{ for all } 1 \leq i \leq k,
\]

where \((C_i)_{1 \leq i \leq k}\) are pairwise distinct and \(d_i\) is the multiplicity of the object \(C_i\) in the multiset. In simple words, a \(C\)-multiset is a finite unordered collection of elements from \(C\) such that multiple occurrences of each element are admissible. For example, if \(C = \mathbb{N}\), then \(\text{MSET}(C)\) contains all partitions of natural numbers, a prominent object. The multiset construction is omnipresent in combinatorial settings, for example when \(C\) is some class of connected unlabelled graphs; this makes \(G\) the class of unlabelled graphs having connected components in \(C\). For many historical references and examples we refer the reader to the excellent books [7, 19]. An alternative and instructive way to describe multisets of size \(n \in \mathbb{N}\) is to make the connection to number partitions explicit as follows. First, choose a number partition of \(n\). Then, assign to each of the parts an element of that size from \(C\). Hence, multisets are also called weighted integer partitions, frequently
encountered in the context of statistical physics of ideal gas. There, \( c_k := |\{ C \in C : |C| = k \}| \) describes the different possible states of a particle at energy level \( k \in \mathbb{N} \), see [48] for a thorough overview.

Given \( G = \{(C_1, d_1), \ldots , (C_k, d_k)\} \in \mathcal{G} \) we denote by \( |G| := \sum_{1 \leq i \leq k} d_i |C_i| \) the size and by \( \kappa(G) := \sum_{1 \leq i \leq k} d_i \) the number of components of \( G \). We further set

\[
\mathcal{G}_n := \{ G \in \mathcal{G} : |G| = n \} \quad \text{and} \quad \mathcal{G}_{n,N} := \{ G \in \mathcal{G}_n : \kappa(G) = N \}, \quad n, N \in \mathbb{N}.
\]

Additionally, we define \( \mathcal{G}_n \) and \( \mathcal{G}_{n,N} \) to be multisets drawn uniformly at random from \( \mathcal{G}_n \) and \( \mathcal{G}_{n,N} \), respectively.

A vast amount of literature is dedicated to the enumerative problem of determining \( g_n := |\mathcal{G}_n| \), and sometimes also \( g_{n,N} := |\mathcal{G}_{n,N}| \), under various general assumptions or for specific examples such as integer partitions, plane partitions or unlabelled (un-)rooted forests, see e.g. [22–25, 31, 36] for \( g_n \) and [26, 29] for \( g_{n,N} \). Note that determining \( g_n \) and \( g_{n,N} \) is directly related to the limiting distribution and limit theorems for the number of components in \( \mathcal{G}_n \), for example investigated in [5, 17, 27, 33]. Another closely related topic that has received a lot of attention is devoted to finding explicit.

\[
\text{cf. [3, 4, 17, 32, 34, 47]. Section 1.1 highlights some of these results in more detail and makes the connection to this work explicit.}
\]

We associate to \( C \) and \( \mathcal{G} \) the (ordinary) generating series in two formal variables \( x \) and \( y \)

\[
C(x) := \sum_{k \in \mathbb{N}} |C_k| x^k \quad \text{and} \quad G(x, y) := \sum_{k, \ell \in \mathbb{N}} |G_{k,\ell}| x^k y^\ell,
\]

and we use the standard notation \( g_{n,N} = |\mathcal{G}_{n,N}| = |x^n y^N|G(x, y) \) for all \( n, N \in \mathbb{N} \). These two power series are known to fulfill the fundamental relation, see for example [7, 19],

\[
G(x, y) = \exp \left( \sum_{j \geq 1} \frac{C(x^j) y^j}{j} \right).
\]

In this paper we consider the prominent and broad case in which the counting sequence \( (c_n)_{n \in \mathbb{N}} \) is subexponential, and our aim is to study the class \( \mathcal{G}_{n,N} \) – what is \( g_{n,N} \), how do typical objects look like? – as \( n \to \infty \) and for all \( 1 \leq N \leq n \).

Subexponential sequences appear naturally in combinatorial contexts, the main reason being the presence of square-root singularities in the analysis of associated generating functions. Here is an example for a prototypical application.

**Example.** Let \( \mathcal{T} \) be the class of unlabelled trees, that is, isomorphism classes of connected and acyclic graphs. Then \( \mathcal{F} = \text{MSET}(\mathcal{T}) \) is the class of unlabelled forests. Moreover, see [35], the number of unlabelled trees satisfies

\[
|T_n| \sim c \cdot n^{-5/2} \cdot \rho^{-n}
\]

for some \( c > 0 \) and \( 0 < \rho < 1 \). What can we say about \( \mathcal{F}_{n,N} \)?

Similar counting sequences, in particular with a polynomial term \( n^{-\alpha} \) for some \( \alpha > 1 \), appear in a variety of contexts in graph enumeration; so-called subcritical graph classes [12] that include trees, outerplanar and series-parallel graphs are prominent examples. All these counting sequences – and many more – are subexponential. Let us proceed with a formal definition. In order to do so, we step back from our combinatorial setting and let \( (c_k)_{k \in \mathbb{N}} \) be a real-valued non-negative sequence. Then we say that \( C(x) = \sum_{k \geq 1} c_k x^k \), or \( (c_k)_{k \geq 1} \) respectively, is subexponential with radius of convergence \( \rho > 0 \), if

\[
\frac{c_n}{c_{n+1}} \sim \rho \quad \text{and} \quad \frac{c_n}{c_{n+1}} \sum_{1 \leq k \leq n-1} c_k c_{n-k} \sim 2C(\rho) < \infty, \quad n \to \infty.
\]

Important examples for subexponential sequences are of the form \( c_n \sim \lambda(n) \cdot n^{-\alpha} \cdot \rho^{-n} \) for \( \alpha > 1 \) and \( \lambda(n) \) any slowly varying function.

Let us now return to the question investigated in this paper. Given a subexponential \( C(x) \) we want to study for \( n \in \mathbb{N} \) and all \( 1 \leq N \leq n \) the number \( g_{n,N} = [x^n y^N]G(x, y) \), and moreover, if \( C(x) \) is the generating series of some combinatorial class (that is, \( (c_k)_{k \in \mathbb{N}} \) is an integer sequence), typical properties of the random multiset \( \mathcal{G}_{n,N} \). A directly related result in this context is [5], where the authors show that the (limiting) distribution of the number of components in a random C-multiset \( \mathcal{G}_n \) is given by a weighted sum of independent Poisson random variables. Equivalently, this means that \( g_{n,N} \) can be determined asymptotically for fixed \( N \) as \( n \to \infty \); thus the enumeration problem is well understood.
for a bounded number of components. On the other end of the spectrum, let $m = m(C) \in \mathbb{N}$ be such that $c_m > 0$ and $c_1 = \cdots = c_{m-1} = 0$. When $C(x)$ is the generating function of a combinatorial class, this means that the size of the smallest possible object in $C$ is $m$, and so any $C$-multiset in $G_n$ has at most $n/m$ components. In particular, $n - mN \geq 0$, and if $n - mN = O(1)$ then the structure of any $C$-multiset of size $n$ with $N$ components is rather simple: except for a bounded number of components of bounded size, all other components are of the smallest possible size $m$. Our first main result addresses the enumeration problem in all other remaining cases, namely when $N, n - mN \to \infty$.

**Theorem 1.1.** Suppose that $C(x)$ is subexponential and $0 < \rho < 1$. Let $m = \min\{k \in \mathbb{N} : c_k > 0\}$. Then,

$$[x^n y^N]G(x, y) \sim A \cdot N^{c_m - 1} \cdot c_{n-m(N-1)}, \quad \text{as} \quad n, N, n-mN \to \infty. \quad (2)$$

where

$$A = \frac{1}{\Gamma(c_m)} \exp \left( \sum_{j \geq 1} \frac{C(j \rho) - c_m j \rho^m}{j \rho^m} \right).$$

The proof is in Section 3.3. Some discussion and remarks are in place. First, by considering real-valued sequences $(c_k)_{k \in \mathbb{N}}$ the formula in Theorem 1.1 gives us the asymptotic behaviour of the coefficients of $G(x, y)$ with a priori no combinatorial interpretation. However, if the sequence is integer-valued and corresponds to the counting sequence of a combinatorial class $C$, then Theorem 1.1 is an enumeration result: it provides us with the number of $C$-multisets of size $n$ and $N$ components, where $N, n - mN \to \infty$. Second, in combinatorial applications, note that we always have that $\rho < 1$, as otherwise the subexponentiality of $C(x)$ would imply that $c_k \to 0$ as $k \to \infty$. That is, the assumption $0 < \rho < 1$ imposes no restriction in the combinatorial setting.

Let us make a third remark that will pave the way to the following results. From here on we solely consider the combinatorial setting. Note the right hand side of (2); this formula establishes an explicit connection between $g_{n, N}$ and $c_{n-m(N-1)}$, that is, we do not need the actual counting sequence of $C$ to make statements about $g_{n, N}$. Moreover, a closer look at this formula reveals an unexpected fact. The number of possible ways to choose a multiset of $C$-multisets of size $n$ and $N$ components, where $N, n - mN \to \infty$. Second, in combinatorial applications, note that we always have that $\rho < 1$, as otherwise the subexponentiality of $C(x)$ would imply that $c_k \to 0$ as $k \to \infty$. That is, the assumption $0 < \rho < 1$ imposes no restriction in the combinatorial setting.

Our next result formalizes this intuition. For $G = \{(C_1, d_1), \ldots, (C_k, d_k)\} \in G$ denote by $L(G) := \max_{1 \leq i \leq k} |C_i|$ the size of one of its largest components. We show that except for a term $O_p(1)$, that is, a quantity that is bounded in probability, the largest component in a uniformly drawn object $G_{n, N}$ from $G_{n, N}$ has indeed size very close to $n - mN$.

**Theorem 1.2.** Suppose that $C(x)$ is subexponential and $0 < \rho < 1$. Then, as $n, N, n-mN \to \infty$,

$$L(G_{n, N}) = n - mN + O_p(1).$$

The proof is in Section 3.4. We call the phenomenon established in Theorem 1.2 extreme condensation: we observe that typically our objects have a giant component that is essentially as large as possible; its size is close to the largest possible size $n - mN - 1$. In particular, the other $O_p(1)$ components are of smallest possible size $m$. We are not aware of any other object with a comparable behaviour, at least not in the analytical ($\rho > 0$) setting considered here. Moreover, this behaviour is surprising for one more reason: if we consider the labelled counterparts of our unlabelled objects, in our running example trees, then the typical structure is well known to undergo various phase transitions (from subcritical to condensation) depending on the number of components, but it never becomes as extreme as observed here. See [28, 37] and Section 1.2 for a more detailed discussion.

Our final main result addresses the last remaining bit and describes the shape of a typical object from $G_{n, N}$ when we remove a component of largest size and all components of the smallest possible size $m$. This remainder is a multiset of stochastically bounded size and number of components, and we determine the limiting distribution. To formulate our statement we need some additional notation. Define the class $C_{>m} = \bigcup_{k > m} C_k$ equipped with the modified size function $|C|_{>m} := |C| - m$ for $C \in C_{>m}$. The associated generating function $C_{>m}(x)$ thus equals $(C(x) - c_m x^m)/x^m$; here subtracting $c_m x^m$ accounts for the fact that we remove objects of (the smallest) size $m$ and dividing through $x^m$ all

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1For example, it is known that a factorial weight sequence induces extreme condensation in the balls-in-boxes model, see [28, Example 19.36]. In such situations the respective generating series has radius of convergence 0.
objects in $\mathcal{C}_k$, $k > m$, are treated as objects with size $k - m$. Similar to the formula in (1) (setting $y = 1$) the class of all multisets $\mathcal{G}_{>m} := \text{MSET}(\mathcal{C}_{>m})$ therefore has generating series

$$G_{>m}(x) := \exp \left( \sum_{j \geq 1} \frac{C(x^j) - c_m x^j}{j x^m} \right).$$

Further, the size of an object $G$ in $\mathcal{G}_{>m}$ is given by $|G|_{>m} := |G| - m \kappa(G)$. As the coefficients of $C_{>m}(x)$ are given by $(c_{k+1})_{k \in \mathbb{N}}$ we deduce that $G_{>m}(x)$ is also subexponential with radius of convergence $\rho$ and $G_{>m}(\rho) < \infty$. Define a random variable $\Gamma_{G_{>m}}(\rho)$ on $\mathcal{G}_{>m}$ specified by

$$\Pr[\Gamma_{G_{>m}}(\rho) = G] = \frac{\rho |G|_{>m}}{G_{>m}(\rho)} = \exp \left( - \sum_{j \geq 1} \frac{C(\rho^j) - c_m \rho^j}{j \rho^m} \right) \rho^{|G| - m \kappa(G)}, \quad G \in \mathcal{G}_{>m}. \quad (3)$$

We remark in passing that this is the – well-known – Boltzmann distribution (on $\mathcal{G}_{>m}$) about which we will talk later extensively. For $G \in \mathcal{G}$ let the remainder $\mathcal{R}(G)$ be the multiset obtained after removing all tuples $(C, d) \in G$ with $C \in \mathcal{C}_m$ and one object of largest size from $G$ (this choice can be done in a canonical way by numbering all elements in $C$). That means, if the object of largest size has multiplicity $d > 1$ replace $d$ by $d - 1$, otherwise remove the object and its multiplicity 1 completely from the set. Then the distribution in (3) is the limit of the remainder $\mathcal{R}(G_{n,N})$, see Section 3.5 for the proof.

**Theorem 1.3.** Suppose that $C(x)$ is subexponential and $0 < \rho < 1$. Then, as $n, N, n - mN \to \infty$, in distribution $\mathcal{R}(G_{n,N}) \to \Gamma_{G_{>m}}(\rho)$.

Note that the convergence in distribution of discrete random variables with the same domain is equivalent to their total variation distance tending to zero, see for example [28, Lem. 19.5]. We close this introduction and the presentation of the main results by catching up with our previous example regarding the class $\mathcal{T}$ of unlabelled trees. We readily obtain that the Theorem 1.1 is directly applicable to the class of unlabelled trees. We readily obtain that the number of unlabelled forests of size $n$ with $N$ components satisfies

$$f_{n,N} \sim A \cdot \frac{|\mathcal{T}_{n-N+1}|}{|n - N|^{5/2}} \cdot \rho^{-n+N}, \quad \text{for } n, N, n - N \to \infty$$

and for some constants $A, A' > 0$. Moreover, for this range of $N$, we obtain that with high probability, a random unlabelled forest contains a huge tree with $n - N + O(1)$ vertices, and $n + O(1)$ “trivial” trees that consist of a single vertex. This is in stark contrast to the known behaviour of random labelled forests, see Section 1.2 for a detailed discussion, but also from unlabelled models such as random unrooted ordered forests, cf. [9].

We proceed with an application of our results to Benjamini-Schramm convergence of unlabelled graphs with many components. The Benjamini-Schramm limit of a sequence of graphs describes what a uniformly at random chosen vertex typically sees in its neighbourhood and is a special instance of local weak convergence, see also [2, 6]. Given a graph $G = (V, E)$ we form the rooted graph $(G, o)$ by distinguishing a vertex $o \in V$. Let $\mathcal{B}$ be the collection of all these rooted graphs. Then two graphs $(G, o)$ and $(G', o')$ in $\mathcal{B}$ are called isomorphic, $(G, o) \simeq (G', o')$, if there exists an edge-preserving bijection $\Phi$ on the vertex sets of $G$ and $G'$ such that $\Phi(o) = o'$. Hence, the collection $\mathcal{B}_o = \mathcal{B}/\approx$ of equivalence classes in $\mathcal{B}$ under the relation $\approx$ contains all unlabelled rooted graphs.

Set $B_k(G, o)$ to be the induced subgraph of $(G, o) \in \mathcal{B}_o$ containing all vertices within graph distance $k$ from the root $o$. Then we say that a sequence of (labelled or unlabelled) simple connected locally finite graphs $(G_n)_{n \geq 1}$ (possibly random) converges in the Benjamini-Schramm (BS) sense to a limiting object $(G, o) \in \mathcal{B}_o$ if for a vertex $o_n$ being selected uniformly at random from $G_n$

$$\lim_{n \to \infty} \Pr[B_k(G_n, o_n) \simeq (G, o)] = \Pr[B_k(G, o) \simeq (G, o)], \quad k \in \mathbb{N}, (G, o) \in \mathcal{B}_o. \quad (4)$$

Back to our setting, we consider $\mathcal{C}$ to be a class of unlabelled finite connected graphs (with subexponential counting sequence and $m \in \mathbb{N}$ denotes the size of the smallest possible graph in $\mathcal{C}$) such that $\hat{\mathcal{C}} = \text{MSET}(\mathcal{C})$ is the class of unlabelled graphs with connected components in $\mathcal{C}$. Let us before $G_{n,N}$ be drawn uniformly at random from $G_{n,N}$. In order to adapt to the setting above we let $(G_{n,N}, o_n)$ denote the connected component around a uniformly at random chosen root $o_n$ in $G_{n,N}$. Let $C_n$ be drawn uniformly at random from $C_n := \{ C \in \mathcal{C} : |C| = n \}$. With this at hand, the extension of BS convergence to non-connected graphs is evident and we obtain the following result.
Proposition 1.4. Suppose that $C(x)$ is subexponential and $0 < \rho < 1$. Assume that $mN/n \to \lambda \in [0, 1)$ as $n, N \to \infty$. If the sequence $(C_n)_{n \geq 1}$ converges to a limit object $(\mathcal{C}, \varnothing)$ in the BS sense, then $G_{n,N}$ converges as $n, N \to \infty$ to a limit object $(\mathcal{G}, \varnothing)$ in the BS sense given by the law

$$(1 - \lambda)\delta_{(\mathcal{C}, \varnothing)} + \lambda\delta_{(\mathcal{C}_n, o_m)},$$

where $o_m$ is a vertex chosen uniformly at random among the $m$ vertices in $\mathcal{C}_m$. In particular, if $N = o(n)$ we have that $(\mathcal{G}, \varnothing) = (\mathcal{C}, \varnothing)$.

The proof is found in Section 3.6. The authors of [21] show that any subcritical class $C$ of connected unlabelled graphs fulfils the conditions of Proposition 1.4. In the subcritical setting the BS limit of connected unlabelled rooted graphs is also the BS limit of the respective unrooted graphs as shown in [43]. In particular, prior to these works it was shown in [44, 46] that the BS limits of unlabelled unrooted trees and of unlabelled rooted trees, also called Pólya trees, both exist and coincide. Additionally, this limit, say $(\mathcal{T}, \varnothing)$, is made explicit in these publications.

Example (further continued) We obtain with Proposition 1.4 that the BS limit $(\mathcal{F}, \varnothing)$ of $F_{n,N}$ drawn uniformly at random from all unlabelled forests of size $n$ and being composed of $N$ trees has law, assuming that $N/n \to \lambda \in [0, 1)$ as $n, N \to \infty$,

$$(1 - \lambda)\delta_{(\mathcal{T}, \varnothing)} + \lambda\delta_{X},$$

where $X$ is a single rooted vertex. In other words, with probability $1 - \lambda$ the neighbourhood of a uniformly at random chosen vertex from $F_{n,N}$ looks like the infinite tree $\mathcal{T}$ and with probability $\lambda$ the neighbourhood is empty.

Proof Strategy The main idea in the proof is to consider a randomized algorithm/stochastic process that generates $C$-multisets. As it turns out, such an algorithm that outputs elements from $\mathcal{G}$ (with a priori no control on the size or the number of components!) can be designed by defining the so-called Boltzmann distribution on $\mathcal{G}$, see Section 3.1 for all details. The crucial property of this algorithm is that all choices it makes are independent. Our first contribution is to establish explicitly the connection between the choices of the algorithm and its output; hence the probability that the output is in $\mathcal{G}_{n,N}$ can be linked to an event regarding the actual choices of the algorithm. Our second and main contribution is to actually compute the probability of this event; as we will see, this is not at all an easy task, since the involved random variables are not identically distributed and interfere in a complex way with the parameters of the generated object.

In contrast to this work, most proofs in the literature about enumeration of multisets are either conducted from a purely analytical generating function perspective or involve somehow the conditioning relation representing the (heavily dependent) number of component frequencies in $G_n$ of a particular size by independent random variables with negative binomial distributions. This is related to the alternative (and equivalent) representation of the generating function for $C$-multisets (1) given by

$$G(x, y) = \prod_{k \geq 1} (1 - yx^k)^{-ek}.$$  \hspace{1cm} (5)

As opposed to analysing the component spectrum in [3, 4, 22, 33], our arguments are in the style of [38, 40, 44, 45, 47]; we use the Boltzmann model representing $G_n$ as a collection of random $C$-objects attached to cycles of a random permutation, which is helpful to get rid of cumbersome appearances of symmetries and which gives rise to Poisson distributions instead of negative binomials; this difference is reflected by the two different representations (1) and (5). Then we show that the size associated to fixpoints is dominant and the subexponentiality-feature often referred to as “single big jump” guarantees that in fact only one object receives the entire possible size.

Plan of the Paper Subsequently, we embed our results in the corpus of existing literature in Section 1.1. In Section 1.2 we compare the labelled and the unlabelled setting in light of our results. Then we collect and prove some results about subexponential power series tailored to our needs in Section 2. In Section 3 all proofs are presented, where each of the main results is treated in an extra subsection, such that Theorem 1.1 is proven in Section 3.3, Theorem 1.2 in Section 3.4 and Theorem 1.3 in Section 3.5.

Notation We shall use the following (standard) notation. Given two real-valued sequences $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ with $b_k \neq 0$ for all $k \geq k_0$ for some $k_0 \in \mathbb{N}$, we write, as $n \to \infty$,

i) $a_n \sim b_n$ (“$a_n$ is asymptotically equal to $b_n$”) if $\lim_{n \to \infty} a_n/b_n = 1$,
ii) \( a_n \propto b_n \) ("\( a_n \) is asymptotically proportional to \( b_n \)") if there exist \( 0 < A_1 \leq A_2 \) such that
\[
A_1 \leq \liminf_{n \to \infty} \left| \frac{a_n}{b_n} \right| \leq \limsup_{n \to \infty} \left| \frac{a_n}{b_n} \right| \leq A_2.
\]

iii) \( a_n = o(b_n) \) if \( \lim_{n \to \infty} a_n/b_n = 0 \).

For a sequence of real-valued random variables \( (X_k)_{k \in \mathbb{N}} \) and a non-negative sequence \( (a_k)_{k \in \mathbb{N}} \) we write \( X_n = O_p(a_n) \) ("\( X_n \) is stochastically bounded by \( a_n \)") if for all \( \varepsilon > 0 \) there exists \( K > 0 \) such that \( \limsup_{n \to \infty} \Pr \left[ |X_n| \geq Ka_n \right] \leq \varepsilon \). In the case \( a_k \equiv 1 \) we simply say "\( X_n \) is stochastically bounded".

We will use the following notation for formal power series. For a monomial \( x \) we write \( x^d \) for the monomial \( x_1^d_1 \cdots x_k^d_k \). A multivariate power series with real-valued coefficients is given by \( A(x) = \sum_{d \in \mathbb{N}^k_0} a_d x^d \), where the \( a_d \)'s are in \( \mathbb{R} \). For \( d \in \mathbb{N}^k_0 \) we write \( [x^d] A(x) = a_d \) for the coefficient of \( x^d \).

### 1.1. (More) Related Work

In this section we put our results in the broader context of (asymptotic) enumeration of multisets/weighted integer partitions. The most prominent assumption is that the counting sequence of \( C \) fulfils \( c_n \sim \lambda(n) \cdot n^{-\alpha} \cdot \rho^{-n} \) as \( n \to \infty \) for some slowly varying function \( \lambda(\cdot) \) and \( \alpha \in \mathbb{R}, 0 < \rho \leq 1 \). Then there are three cases depending on \( \alpha \), each giving rise to a fundamentally different picture.

In the **expansive** case \( \alpha < 1 \), which in particular includes the classical and prominent setting \( c_k \equiv 1 \) of integer partitions, the number of \( \mathcal{C} \)-multisets \( g_n = |G_n| \) is well-understood [22]. However, general results about the uniformly drawn element \( G_n \) in particular the distribution of the number of components that is of interest here – are not known without any extra conditions. For example, under the assumption of Meinardus scheme of conditions, a set of analytic assumptions that in particular imply that \( \rho = 1 \), the number of components of \( G_n \) fulfils various limit theorems, see [33]. The authors of [24] state that expansive multisets with counting sequence \( c_k = Ck^{-\alpha} \) for some \( C > 0 \) fulfil Meinardus conditions; hence, they call the Meinardus case quasi-expansive. For quasi-expansive sequences it is established in [34] that the size of the largest component of \( G_n \) is with high probability of size \( \Theta(1) \cdot n^{1/(2 - \alpha)} \log n \). The broader picture here is that the number of components in \( G_n \) is typically unbounded and the size of the largest component is sublinear in \( n \). On the other hand, much less is known for the number of \( \mathcal{C} \)-multisets with \( N \) components \( g_{n,N} = |G_{n,N}| \) and the typical shape of \( G_{n,N} \) for arbitrary \( 1 \leq N \leq n \). Nevertheless, there is one important exception, namely the case of integer partitions.

There, depending on \( N \) is \( \mathcal{O}(n^{1/2}) \) or \( \omega(n^{1/2}) \) the asymptotic behaviour of the number of partitions of \( n \) into \( N \) parts is given by different formulas, see [29]. For \( N \geq (1 + \varepsilon)(\sqrt{2/3\pi})\sqrt{n \log n} \) for any \( 0 < \varepsilon < 1 \) it is even true that \( g_{n,N} \sim g_{n,N-1} \) [26]. In general, it is reasonable to conjecture that the shape of \( G_{n,N} \) depends on the asymptotic regime of \( N \); this, however, is a topic for a completely different paper.

The **logarithmic** case, where \( \alpha = 1 \) and \( \lambda \equiv \lambda(\cdot) \) constant, is concomitant with similar effects. The number of components in \( G_n \) is typically of order \( \log n \) [3, Thm. 8.21] and, denoting by \((L_1, \ldots, L_N)\) the \( N \) largest component sizes of \( G_n \), then \( n^{-1}(L_1, \ldots, L_N) \) has a limiting distribution that is Poisson-Dirichlet [3, Thm. 6.8] implying that \( G_n \) is composed of several \( \log \) "large" objects. The same is true for \( G_{n,N} \) with \( N \in \mathbb{N} \) fixed, where the \( N - 1 \) smallest components have with high probability sizes \( \left(n^{U_i}\right)_{1 \leq i \leq N - 1} \) for iid uniformly distributed random variables \((U_i)_{1 \leq i \leq N - 1} \) [3, Thm. 6.9]. To our knowledge, there are no results (regarding asymptotic enumeration or structural properties) for all other \( N \).

In contrast to all previous cases, the phenomenon of condensation is observed in the **convergent** case, where \( \alpha > 1 \): the single largest component of \( G_n \) is of size \( n - \mathcal{O}_p(1) \) and its number of components converges in distribution, see [4]. In accordance with the results observed for the convergent case the number of components in \( G_n \) in the subexponential setting has a limiting distribution given by a weighted sum of independent Poisson random variables [5]. Equivalently, this means that \( g_{n,N} \) can be determined asymptotically for fixed values of \( N \in \mathbb{N} \). As for the global shape of \( G_n \), the results in [47] also establish the distribution of the remainder obtained after removing the largest component.

### 1.2. Discussion – The Unlabelled vs. the Labelled Setting

In what follows we will have a closer look at the resemblances and surprising disparities between multisets, which are typically associated to unlabelled structures, and sets of labelled combinatorial structures. We refer the reader to the books [7, 19] for an excellent exposition to combinatorial classes. Another vast source of references and examples is the tour-de-force paper [28] entailing many results about the balls-in-boxes model (Section 11), which by choosing the weight sequence \((c_k/k!)_{k \in \mathbb{N}} \) implies the labelled set-construction.

Given a labelled class \( \mathcal{C} \) we may form the analogon to the multiset construction discussed in this work. Initially we pick \( C_1, \ldots, C_k \) from \( \mathcal{C} \) and let \( n \) be the total size. Then we partition \( \{1, \ldots, n\} \) into sets \( L_1, \ldots, L_k \) such that \( L_i = |C_i| \).
for all $i$. Subsequently, we canonically assign the labels in $L_i$ to $C_i$ for all $i$. The outcome of this procedure is a labelled set, where each of the labels in $\{1, \ldots, n\}$ appears exactly once. The notion of size and number of components carries over from the multiset construction. Let us call the collection of all such labelled sets $G^i = \text{SET}(C^i)$ and introduce the sets $G^i_n$ and $G^i_{n,N}$ of objects in $G^i$ of size $n$ and of size $n$ having $N$ components, respectively. Further, let $c^i_k := |\{C \in C^i : |C| = k\}|$ for $k \in \mathbb{N}$. Similarly to (1) the bivariate (exponential) generating series in this case is, see [7, 19].

$$G^i(x, y) = \exp \left( y C^i(x) \right), \quad \text{where } C^i(x) = \sum_{k \geq 1} c^i_k \frac{x^k}{k!}.$$  

In complete analogy to the unlabelled case we are here interested in the number of unlabelled forests of size $n$ to $c$.

First of all, the case $N = \mathcal{O}(1)$ is treated (together with the unlabelled case) in [5]. There it is shown that in the subexponential setting both $\kappa(G_n)$ and $\kappa(G^i_n)$ converge in distribution; hence, for a fixed number of components the labelled and unlabelled cases behave qualitatively the same. Also the global structure of the associated random variables $G^i_n$ and $G_n$ is in both cases governed by the same condensation effect, see [45, 47]. However, the situation changes as $N \to \infty$. The works [28, 37] treat this topic extensively: under the condition that $c^i_n \sim bn^{-(1+\alpha)}\rho^{-n}n!$ for $b > 0$ and $\alpha > 1$ as $n \to \infty$ there emerges a “trichotomy” ($1 < \alpha \leq 2$) and in some cases a “dichotomy” ($\alpha > 2$) depending on the asymptotic regime of $N$. To illustrate the nature of these results, let us consider the class of labelled trees $T^*$ such that $\mathcal{F}^* = \text{SET}(T^*)$ is the class of labelled forests. The well-known formula by Cayley states that $t^*_n = n^{n-2} \sim (2n)^{-1/2}n^{-5/2}e^{n/2}n!$, so that $\alpha = 3/2$. Abbreviating by $f^i_{n,N}$ the number of forests on $n$ nodes and $N$ trees, the following detailed result exposing a phase transition is known. Let $N := \lfloor \lambda n \rfloor$, then

$$\frac{N!}{n!} \cdot f^i_{n,N} \sim \begin{cases} c_-(\lambda)n^{-3/2}e^{\rho n 2^{-N}}, & \lambda \in (0, 1/2) \\ cn^{-3/2}e^{\rho n 2^{-N}}, & \lambda = 1/2, \ n \to \infty, \\ c_+(\lambda)n^{-1/2}f(\lambda)^n, & \lambda \in (1/2, 1) \end{cases}$$

for positive real-valued continuous functions $c_{\pm}(\lambda), f(\lambda)$ and a $c > 0$; note that the critical exponent jumps from $3/2$ to $2/3$ and then to $1/2$. All in all, the main results of the present paper reveal substantial differences between the labelled and the unlabelled case already at the level of the counting sequences: as we stated before in our example, the number of unlabelled forests of size $n$ with $N = \lfloor \lambda n \rfloor$ components is asymptotically equal to $A \cdot (1 - \lambda)^{-5/2}n^{-5/2}e^{n/2}n!$; in particular, the critical exponent does not vary.

The aforementioned variation in the critical exponent has also important consequences for the global structure of a labelled forest $F_{n,N}$ drawn uniformly at random from the set of labelled forests of size $n$ composed of $N$ trees. Three different cases emerge as $n$ approaches infinity:

1. In the case where there are “few” components ($0 < \lambda < 1/2$), most of the mass is concentrated in one large tree containing a linear fraction (that is $1 - 2\lambda$) of all nodes and the remaining $N - 1$ trees all have size $O_p(n^{2/3})$.
2. If $\lambda = 1/2$ all trees have size $O_p(n^{2/3})$.
3. Whenever there are “many” components with respect to the total number of nodes ($1/2 < \lambda < 1$), all trees are small in the sense that their size is stochastically bounded by $\log n$.

For a detailed discussion for what happens near the critical point $\lambda = 1/2$ see also [30]. This is again substantially different to the unlabelled case, where we showed that extreme condensation dominates the picture for all values of $N$.

2. Subexponential Power Series

In this section we collect (and prove) some properties of subexponential power series that will be quite handy. Many of the definitions and statements are taken from [16, 20] and adapted to the discrete case, see also [47].

**Definition 2.1.** A power series $C(x) = \sum_{k \geq 0} c_k x^k$ with non-negative coefficients and radius of convergence $0 < \rho < \infty$ is called subexponential if

$$\frac{1}{cn} \sum_{0 \leq k \leq n} c_{n-k}c_k \sim 2C(\rho) < \infty \quad \text{and} \quad (S_1)$$

$$\frac{c_{n-1}}{cn} \sim \rho, \quad n \to \infty. \quad (S_2)$$

\[2\text{In particular we want to highlight [28, Theorems 18.12, 18.14, 19.34, 19.49].}\]
Note that the radius of convergence of a power series $C(x)$ satisfying $(\mathcal{S}_2)$ (in particular of any subexponential power series) is $\rho$ and that eventually $[x^n]C(x) > 0$, where as usual, $[x^n]C(x) = c_n$ denotes the coefficient of $x^n$ in $C(x)$. Any arbitrary subexponential power series $C(x)$ with radius of convergence $\rho$ induces the probability generating series of a $\mathbb{N}_0$-valued random variable by setting

$$d_k := \frac{c_k \rho^k}{C(\rho)}, \quad n \in \mathbb{N}_0.$$ 

Then $D(x) = \sum_{k \geq 0} d_k x^k$ is subexponential with $\rho = 1$ and $D(\rho) = 1$. There are several results about the asymptotic behaviour of sums of random variables with such a subexponential generating series. Here we will need Lemma 2.2 (i) below, which corresponds to determining the probability that a randomly stopped sum of random variables with such a subexponential generating series. Here we will need Lemma 2.2 (iii) which corresponds to determining the probability that a randomly stopped sum of random variables with subexponential power series has radius of convergence $\rho$.

**Lemma 2.2.** Let $(D_i)_{i \in \mathbb{N}}$ be iid $\mathbb{N}_0$-valued random variables with probability generating function $D(x)$. Assume that $D(x)$ is subexponential with radius of convergence 1. For $p \in \mathbb{N}$ let $S_p := \sum_{1 \leq i \leq p} D_i$ and $M_p := \max\{D_1, \ldots, D_p\}$. Then the following statements are true.

(i) [20, Theorem 4.30] Let $\tau$ be a $\mathbb{N}_0$-valued random variable independent of $(D_i)_{i \in \mathbb{N}}$. Further, assume that the probability generating function of $\tau$ is analytic at 1. Then

$$\Pr[\tau = n] \sim \mathbb{E}[\tau] \Pr[D_1 = n], \quad n \to \infty.$$ 

(ii) [20, Theorem 4.11] For every $\delta > 0$ there exists $n_0 \in \mathbb{N}$ and a $C > 0$ such that

$$\Pr[S_p = n] \leq C(1 + \delta)^p \Pr[D_1 = n], \quad \text{for all } n \geq n_0, \; p \in \mathbb{N}.$$ 

(iii) For any $p \geq 2$

$$(M_p \mid S_p = n) = n + O(1), \quad n \to \infty.$$ 

**Proof of Lemma 2.2 (iii).** Let $\varepsilon > 0$ be arbitrary. To prove the claim we will establish the existence of $K \in \mathbb{N}$ such that

$$\lim_{n \to \infty} \Pr[|M_p - n| \geq K \mid S_p = n] < \varepsilon.$$ 

Clearly under the condition $S_p = n$ we have that $n/p \leq M_p \leq n$. Thus, for any $K \in \mathbb{N}$

$$\Pr[|M_p - n| \geq K \mid S_p = n] = \sum_{K \leq k \leq (1-\varepsilon^{-1})n} \Pr[M_p = n - k \mid S_p = n].$$

(6)

Since $D_1, \ldots, D_p$ are iid we obtain for any $k \geq K$

$$\Pr[M_p = n - k \mid S_p = n] \leq \Pr \left[ \bigcup_{1 \leq i \leq p} \{D_i = n - k\} \mid S_p = n \right] \leq p \frac{\Pr[D_1 = n - k] \Pr[S_{p-1} = k]}{\Pr[S_p = n]}.$$ 

Together with Lemma 2.2 (ii) we find some constant $C > 0$ such that for $k \geq K$ sufficiently large

$$\Pr[S_{p-1} = k] \leq C(1 + \varepsilon)^{p-1} \Pr[D_1 = k].$$

Part (i) asserts for $n$ sufficiently large that $\Pr[S_p = n] \geq (1 - \varepsilon)p \Pr[D_1 = n]$. All in all, for a suitably chosen constant $C' = C'(p)$ the expression in (6) can be estimated by

$$\Pr[|M_p - n| \geq K \mid S_p = n] \leq C' \sum_{K \leq k \leq (1-\varepsilon^{-1})n} \frac{\Pr[D_1 = n - k] \Pr[D_1 = k]}{\Pr[D_1 = n]}.$$ 

Property $(\mathcal{S}_1)$ and $p \geq 2$ then imply that this is smaller than $\varepsilon$ by choosing $K$ large enough. This finishes the proof. □
The following lemma establishes asymptotics for the coefficients of the product of two power series.

**Lemma 2.3.** [10, Thm. 3.42] or [42, Ex. 178] Let \( A(x), B(x) \) be power series such that \( A \) satisfies (\( S_2 \)). Assume that the radii of convergence \( \rho_A \) and \( \rho_B \) of \( A \) and \( B \), respectively, satisfy \( 0 < \rho_A < \rho_B \) and \( B(\rho_A) \neq 0 \). Then

\[
[x^n]A(x)B(x) \sim B(\rho_A) \cdot [x^n]A(x), \quad n \to \infty.
\]

Note that Lemma 2.3 does not require \( A \) to be subexponential, neither does it require that \( B \) has non-negative coefficients only. We will later apply the lemma with (powers of)

\( [10, \text{Thm. 3.42}] \) or \( [42, \text{Ex. 178}] \) Let

\( \rho \)

the radii of convergence

\( \rho \)

Section 3.6.

Subsequently, we present the proofs of our three main theorems in Sections 3.3-3.5. At last we prove Proposition 1.4 in Section 3.6.
3.1. (Combinatorial) Setup and Notation

In this section we will introduce the Boltzmann model from the pioneering paper [15], which has found various applications in the study of the typical shape of combinatorial structures, see for example [1, 8, 11, 13, 14, 39, 41, 45]. Assume that \( z \in \mathbb{R}_+ \) is chosen such that \( C(z) > 0 \) is finite. The Boltzmann model defines a random variable \( \Gamma C(z) \) taking values in the entire space \( C \) through

\[
\Pr[\Gamma C(z) = C] = \frac{z^{|C|}}{C(z)}, \quad C \in C.
\]

In complete analogy the random variable \( \Gamma G(z) \) is defined on \( G = \text{MSET}(C) \), where in this case the parameter \( z > 0 \) is such that \( G(z) := G(z, 1) < \infty \) in (1). In the rest of this section we fix \( z = \rho \) recalling that \( 0 < \rho < 1 \) is the radius of convergence of \( C \). Then, in virtue of Lemma 3.1, \( G \) has radius of convergence \( \rho \) and \( G(\rho) < \infty \), so that both \( \Gamma C(\rho), \Gamma G(\rho) \) are well-defined, and we just write \( \Gamma C, \Gamma G \).

Let \( g_n \) be the number of objects of size \( n \) in \( G \) and \( g_{n,N} \) those of size \( n \) comprised of \( N \) components. By using Bayes’ Theorem and that the Boltzmann model induces a uniform distribution on objects of the same size, we immediately obtain

\[
\frac{g_{n,N}}{g_n} = \Pr[\kappa(\Gamma G) = N \mid |\Gamma G| = n] = \Pr[|\Gamma G| = n \mid \kappa(\Gamma G) = N] \frac{\Pr[\kappa(\Gamma G) = N]}{\Pr[|\Gamma G| = n]}, \quad n, N \in \mathbb{N}.
\]

To get a handle on this expression we exploit a powerful description of the distribution of \( \Gamma G(z) \) in terms of \( \Gamma C(\gamma) \), derived in [18]. In the next steps, the notation \( \bigcup_{j \in J} A_j \) is used to denote a multiset of elements \( A_j \) from a set \( A, j \in J \) being indices in some countable set \( J \). That is, multiple occurrences of identical elements are allowed and \( \bigcup_{j \in J} A_j \) is completely determined by the different elements it contains and their multiplicities.

1. Let \( P_j \) be independent random variables, where \( P_j \sim P_0 \left( C(\rho^j)/j \right) \).
2. Let \( \gamma_{j,i} \) be independent random variables with \( \gamma_{j,i} \sim \Gamma C(\rho^j) \) for \( j, i \geq 1 \).
3. For \( j, i \geq 1 \) and \( k \leq j \) set \( \gamma_{j,i} = \gamma_{j,i} \), that is, make \( j \) copies of \( \gamma_{j,i} \). Let \( \Lambda G := \bigcup_{j \geq 1} \bigcup_{1 \leq i \leq P_j} \bigcup_{1 \leq k \leq j} \gamma_{j,i} \).

Intuitively, we interpret \( P_j \) as the number of \( j \)-cycles in some not further specified permutation, and to each cycle of length \( j \) we attach \( j \) times an identical copy of a \( \Gamma C(\rho^j) \)-distributed \( C \)-object. Afterwards we discard the permutation and the cycles and keep the multiset of the generated \( C \)-objects.

**Lemma 3.3. [18, Prop. 2.1]** The distributions of \( \Gamma G \) and \( \Lambda G \) are identical.

This statement paves the way to study \( G \). In particular, if we write \( C_{j,i} = |\gamma_{j,i}| \), note that the definition of \( \Lambda G \) guarantees that in distribution

\[
\kappa(\Gamma G) = \sum_{j \geq 1} j P_j \quad \text{and} \quad |\Gamma G| = \sum_{j \geq 1} j \sum_{1 \leq i \leq P_j} C_{j,i}.
\]

So, let us for \( n, N \in \mathbb{N} \) define the events

\[
\mathcal{P}_N := \left\{ \sum_{j \geq 1} j P_j = N \right\} \quad \text{and} \quad \mathcal{E}_n := \left\{ \sum_{j \geq 1} j \sum_{1 \leq i \leq P_j} C_{j,i} = n \right\}.
\]

With \( \Pr[\mathcal{E}_n] = \Pr[|\Lambda G| = n] = g_n \rho^n / G(\rho) \) at hand, Lemma 3.3 and (9) then guarantee that

\[
g_{n,N} = G(\rho) \rho^{-n} \Pr[\mathcal{E}_n \mid \mathcal{P}_N] \Pr[\mathcal{P}_N].
\]

Note that for all \( 1 \leq i \leq P_j \) and \( j \in \mathbb{N} \), we have

\[
\Pr[C_{j,i} = k] = \frac{c_k \rho^j k}{C(\rho^j)}, \quad k \in \mathbb{N}.
\]

Equation (11) enables us to reduce the problem of determining \( g_{n,N} = [x^n y^N] G(x, y) \) to the problem of determining the probability of the events \( \mathcal{P}_N \) and \( \mathcal{E}_n \) conditioned on \( \mathcal{P}_N \).
3.2. Real-valued Sequences in Theorem 1.1

In Theorem 1.1 we consider \((c_k)_{k \in \mathbb{N}}\) to be a real-valued non-negative sequence and assume \(0 < \rho < 1\). In complete analogy to the discussion prior to this subsection let \(P_j \sim \text{Po}\left(C(\rho^j)/j\right)\) for \(j \in \mathbb{N}\), \((C_{j,1}, \ldots, C_{j,p_j})_{j \in \mathbb{N}}\) be as in (12), and assume that all these variables are independent. As a matter of fact, also in this (more general) case we obtain exactly the same representation of \([x^n y^N]G(x, y)\) in terms of \(E_n\) and \(P_N\) defined in (10) without using the combinatorial Boltzmann model.

**Lemma 3.4.** Let \(C(x)\) be a power series with non-negative real-valued coefficients and radius of convergence \(0 < \rho < 1\) at which \(C(\rho) < \infty\). Then

\[[x^n y^N]G(x, y) = G(\rho)\rho^{-n}\Pr[E_n | P_N]\Pr[P_N], \quad n, N \in \mathbb{N}.

**Proof.** We begin with the simple observation

\[\Pr[P_N, E_n] = [x^n y^N] \sum_{k \geq 0} \sum_{\ell \geq 0} \Pr[P_k, E_\ell] x^\ell y^k\]

\[= [x^n y^N] \sum_{k \geq 0} y^k \sum_{j_{\ell \geq 1}, j_{p_j = k} \geq 1} \prod_{\ell \geq 0} \Pr[P_j = p_j] \sum_{j \geq 1} \sum_{1 \leq i \leq p_j} C_{j,i} = \ell x^\ell. \tag{13}\]

We will study this expression by first simplifying the sum over \(\ell\), then the sum over all \(p_j\)'s, and eventually the sum over \(k\). We begin with the sum over \(\ell\). For a \(\mathbb{N}_0\)-valued random variable \(A\) let \(A(x) := \sum_{\ell \geq 0} \Pr[A = \ell] x^\ell\) denote its probability generating series. Then, if \((A_j)_{j \in \mathbb{N}}\) is a sequence of independent \(\mathbb{N}_0\)-valued random variables,

\[(A_1 + \cdots + A_m)(x) = \prod_{1 \leq j \leq m} A_j(x), \quad m \in \mathbb{N}. \tag{14}\]

Let us write \(C_{j,i}(x)\) for the probability generating series of \(jC_{j,i}\); note that the actual value of \(i\) is not important, since the \((C_{j,i})_{i \in \mathbb{N}}\) are iid. Then, whenever \(\sum_{j \geq 1} p_j\) is finite, (14) implies

\[\sum_{\ell \geq 0} \left[ \sum_{j \geq 1} \sum_{1 \leq i \leq p_j} C_{j,i} = \ell \right] x^\ell = \prod_{j \geq 1} C_{j}(x)^{p_j}. \]

Noting that \(jC_{j,1}\) takes only values in the lattice \(j\mathbb{N}_0\), we obtain

\[C_j(x) = \sum_{\ell \geq 0} \Pr[jC_{j,1} = \ell] x^\ell = \sum_{\ell \geq 0} \Pr[C_{j,1} = \ell] x^{j\ell} = \frac{1}{C(\rho^j)} \sum_{\ell \geq 0} c_{\ell j} \rho^{\ell j} x^{j\ell} = \frac{C((\rho x)^j)}{C(\rho^j)}. \]

We deduce

\[\sum_{\ell \geq 0} \left[ \sum_{j \geq 1} \sum_{1 \leq i \leq p_j} C_{j,i} = \ell \right] x^\ell = \prod_{j \geq 1} \left( \frac{C((\rho x)^j)}{C(\rho^j)} \right)^{p_j}. \]

This puts the sum over \(\ell\) in (13) in compact form. To simplify the sum over the \(p_j\)'s in (13) define independent random variables \((H_j)_{j \geq 1}\) with \(H_j \sim \text{Po}\left(C((\rho x)^j)/j\right)\). Then

\[\sum_{\ell \geq 0} \prod_{j_{\ell \geq 1}, j_{p_j = k} \geq 1} \Pr[P_j = p_j] \left( \frac{C((\rho x)^j)}{C(\rho^j)} \right)^{p_j} = \frac{G(\rho x, 1)}{G(\rho, 1)} \Pr \left[ \sum_{j \geq 1} jH_j = k \right]. \]

By similar reasoning as before the probability generating function of \(jH_j\) is given by

\[\sum_{\ell \geq 0} \Pr[H_j = \ell] y^{j\ell} = \exp \left( -C((\rho x)^j)/j \right) \sum_{\ell \geq 0} \frac{(C((\rho x)^j)y^{j\ell})}{\ell!} = \exp \left( C((\rho x)^j)y^{j}/j \right). \]
Applying (14), where we set \( A_j := jH_j \), in combination with this identity and plugging everything into (13) yields
\[
\sum_{k \geq 0} \Pr \left[ \sum_{j \geq 1} jH_j = k \right] y^k = \frac{G(\rho x, y)}{G(\rho x, 1)}.
\]

All in all, we have shown that \( \Pr [\mathcal{P}_N, \mathcal{E}_n] = G(\rho)^{-1} [x^n y^N]G(\rho x, y) \). With \( [x^n] F(ax) = a^n [x^n] F(x) \) for any power series \( F \) and \( a \in \mathbb{R} \) we finish the proof.

3.3. Proof of Theorem 1.1

Let \( \mathcal{P}_N, \mathcal{E}_n \) be as in the previous section, see (10), where \( P_j \sim \text{Po} \left( \frac{C(\rho^j)}{j} \right) \) and \( C_{j,1}, \ldots, C_{j,P_j} \) for \( j \in \mathbb{N} \) have the distribution specified in (12). Moreover, we assume that all these random variables are independent. Equipped with Lemma 3.4 from the previous section, the proof of Theorem 1.1 boils down to estimating \( \Pr [\mathcal{E}_n | \mathcal{P}_N] \) and \( \Pr [\mathcal{P}_N] \).

Before we actually do so, let us introduce some more auxiliary quantities. Set
\[
P := \sum_{j \geq 1} jP_j \quad \text{and} \quad P(\ell) := \sum_{j \geq \ell} jP_j, \quad \ell \in \mathbb{N}_0.
\]

With this notation, \( \mathcal{P}_N \) is the same as \( \{P = N\} \) and \( \{P(0) = N\} \). Moreover, recall (8) and set
\[
L := \sum_{1 \leq i \leq P_1} (C_{1,i} - m) \quad \text{and} \quad R := \sum_{j \geq 2} j \sum_{1 \leq i \leq P_j} (C_{j,i} - m).
\]

With this notation
\[
\Pr [\mathcal{E}_n | \mathcal{P}_N] = \Pr [L + R = n - mN | \mathcal{P}_N].
\]

The driving idea behind these definitions is that the random variables \( C_{j,i} - m \), for \( j \geq 2 \), have exponential tails, and these tails get thinner as we increase \( j \); in particular, the probability that \( C_{j,i} - m = 0 \) approaches one exponentially fast as we increase \( j \). However, things are not so easy, since we always condition on \( \mathcal{P}_N \), and in this space some of the \( P_j \)'s might be large. This brings us to our general proof strategy. First of all, we will study our probability space conditioned on \( \mathcal{P}_N \); in particular, in Corollary 3.7 and Lemma 3.8 below we describe the joint distribution of \( P_1, \ldots, P_N \) given \( \mathcal{P}_N \). More specifically, these results show that the \( P_j \)'s are (more or less) distributed like Poisson random variables with bounded expectations. This will allow us then in Lemma 3.9 to show that \( L \) dominates the sum \( L + R \) in the sense that \( \Pr [L + R = n - mN | \mathcal{P}_N] \sim \Pr [L = n - mN | \mathcal{P}_N] \) as \( n, N, n - N \to \infty \). Subsequently, in Lemma 3.10 we exploit the subexponentiality and establish that this last probability is essentially a multiple of \( \Pr [C_{1,1} = n - mN] \). Just as a side remark and so as to make the notation more accessible: it is instructive to think of the random variable \( L \) as something (that will turn out to be) large, and \( R \) as some remainder (that will turn out to be small with exponential tails).

Our first aim is to study the distribution – in particular the tails – of \( P \) and \( P(\ell) \), that is, we want to estimate the probability of \( \mathcal{P}_N \). To this end, consider the probability generating series \( F(x) \) and \( F(\ell)(x) \) of \( P \) and \( P(\ell) \), respectively, that is
\[
F(\ell)(x) = \frac{1}{G(\ell)(\rho)} \cdot \exp \left( \sum_{j > \ell} \frac{C(\rho^j)}{j} x^j \right), \quad \text{where} \quad G(\ell)(\rho) := \exp \left( \sum_{j > \ell} \frac{C(\rho^j)}{j} \right)
\]
and \( F(x) = F(0)(x), \) \( G(0)(\rho) = G(\rho) \). Hence, the distribution of \( P(\ell) \) (and \( P \)) is given by \( \{ \Pr [P(\ell) = N] \}_{N \geq 0} = ([x^N] F(\ell)(x))_{N \geq 0} \). In Lemma 3.6 we determine the precise asymptotic behaviour of these probabilities. But first, we need a simple auxiliary statement.

**Proposition 3.5.** There exists \( A > 0 \) such that, for all \( 0 < z \leq \rho \) and \( j \in \mathbb{N} \)
\[
1 \leq \frac{C(z^j)}{c_m z^m} \leq 1 + Az^j.
\]

**Proof.** The first inequality follows directly from the definition of \( C \) and \( m \). Note that
\[
\frac{C(z^j)}{c_m z^m} \leq 1 + z^j \frac{1}{c_m} \sum_{k > m} c_k \rho^{j-k(m+1)} = 1 + z^j \frac{\rho^{-2m}}{c_m} \sum_{k > m} c_k \rho^{j-k(m+1)+2m}.
\]
Similarly, for \( \ell \) large of \( L(\rho) < \infty \) and the radius of convergence of Lemma 3.6.

Moreover, for any \( \ell \) dependent, but the corollary says that this effect vanishes for large \( N \). Moreover, we study the moments of \( P_1 \) given \( \mathcal{P}_N \).

**Corollary 3.7.** Let \( \ell \) and \( (p_1, \ldots, p_\ell) \in \mathbb{N}_0^\ell \). Then

\[
\operatorname{Pr} \left[ \bigcap_{1 \leq j \leq \ell} \{ P_j = p_j \} \mid \mathcal{P}_N \right] \to \prod_{1 \leq j \leq \ell} \operatorname{Pr} \left[ \text{Po} \left( \frac{C(\rho)}{j \rho^m} \right) = p_j \right], \quad N \to \infty.
\]

Moreover, for any \( z \in \mathbb{R} \), as \( N \to \infty \)

\[
\mathbb{E} \left[ z^{P_1} \mid \mathcal{P}_N \right] \to \mathbb{E} \left[ z^{\text{Po} \left( \frac{C(\rho)}{\rho^m} \right)} \right] = e^{\frac{C(\rho)}{\rho^m} (z - 1)}, \quad \mathbb{E} \left[ P_1 \mid \mathcal{P}_N \right] \to \mathbb{E} \left[ \text{Po} \left( \frac{C(\rho)}{\rho^m} \right) \right] = C(\rho)\rho^{-m}.
\]
Proof. Let \( s = \sum_{1 \leq j \leq t} j p_j \). Using the definition of conditional probability we obtain readily

\[
\Pr \left[ \bigcap_{1 \leq j \leq t} \{ P_j = p_j \} \mid \mathcal{P}_N \right] = \frac{\Pr \left[ \bigcap_{1 \leq j \leq t} \{ P_j = p_j \} \cap \{ P^{(t)} = N - s \} \right]}{\Pr [ P = N ]}.
\]

Since \( P_1, \ldots, P_t, P^{(t)} \) are independent, the right-hand size equals

\[
\prod_{1 \leq j \leq t} \Pr [ P_j = p_j ] \cdot \frac{1}{[x^{N-s}] F^{(t)}(x) / [x^N] F(x)},
\]

and (19) follows by applying Lemma 3.6. We will next show \( P_t \) given \( \mathcal{P}_N \) has exponential moments. Abbreviate \( B := C(\rho) \rho^{-m} \). Note that (19) (where we use \( \ell = 1 \)) yields for any fixed \( K \in \mathbb{N} \)

\[
\sum_{0 \leq k \leq K} z^k \Pr [ P = k \mid \mathcal{P}_N ] \leq \sum_{0 \leq k \leq K} z^k \Pr [ \text{Po}(B) = k ] , \quad N \to \infty.
\]

Let \( \varepsilon > 0 \). Note that we can choose \( K \) large enough such that the right hand side differs at most \( \varepsilon / \text{corollary 3.7} \). In order finish the proof we will argue that if \( K \) and \( N \) are large enough, then \( \sum_{K \leq k \leq N} z^k \Pr [ P_1 = k \mid \mathcal{P}_N ] \leq \varepsilon \) as well. First, by Lemma 3.6

\[
z^N \Pr [ P_1 = N \mid \mathcal{P}_N ] \leq z^N \Pr [ P_1 = N ] = \frac{e^{-C(\rho)} B^{(0)} N^{-c_m+1} z^B N}{N!} \to 0, \quad N \to \infty.
\]

Moreover, according to Lemma 3.6 and Equation (7) there exists a constant \( A_1 > 0 \) such that \( [x^{N-k}] F^{(1)}(x) / [x^N] F(x) \leq A_1 \cdot (1 - k/N)^{c_m-1} \rho^{-mk} \) for all \( 0 \leq k \leq N - 1 \). Then with (20) we obtain

\[
\sum_{K \leq k \leq N-1} z^k \Pr [ P_1 = k \mid \mathcal{P}_N ] \leq A_1 \sum_{K \leq k \leq N-1} t_k, \quad \text{where} \quad t_k := (1 - k/N)^{c_m-1} \frac{(zB)^k}{k!}.
\]

Note that we can choose \( K \) large enough such that, say, \( t_{k+1} \leq t_k / 2 \) for all \( K \leq k < N - 1 \). Then the sum is bounded by \( 2t_K \), and choosing \( K \) once more large enough gives \( 2t_K < \varepsilon \).

Note that Corollary 3.7 (only) holds for a fixed \( \ell \in \mathbb{N} \); it does not tell us anything about \( (P_1, \ldots, P_t) \) in the case where \( \ell \) is not fixed, or, more importantly, when \( \ell = N \) (note that \( P_N = 0 \) for all \( N' > N \) if we condition on \( \mathcal{P}_N \)). Regarding this general case, the following statement gives an upper bound for the probability of the event \( \bigcap_{1 \leq j \leq N} \{ P_j = p_j \} \) that is not too far from the right-hand side in Corollary 3.7. For the remainder of this section it is convenient to define

\[
\Omega_N := \left\{ (p_1, \ldots, p_N) \in \mathbb{N}_0^N : \sum_{1 \leq j \leq N} j p_j = N \right\}, \quad N \geq 2.
\]

In what follows we derive a stochastic upper bound for the distribution of \( (P_1, \ldots, P_N) \) conditioned on \( \mathcal{P}_N \).

**Lemma 3.8.** There exists an \( A > 0 \) such that for all \( N \) and all \( (p_1, \ldots, p_N) \in \Omega_N \)

\[
\Pr \left[ \bigcap_{1 \leq j \leq N} \{ P_j = p_j \} \mid \mathcal{P}_N \right] \leq A \cdot N \cdot \prod_{1 \leq j \leq N} \Pr [ \text{Po} \left( \frac{C(\rho^j)}{j \rho^m} \right) = p_j ] .
\]

**Proof.** Using the definition of conditional probability and recalling that the \( P_j \)'s are independent and \( P_j \sim \text{Po} \left( \frac{C(\rho^j)}{j \rho^m} \right) \)

\[
\Pr \left[ \bigcap_{1 \leq j \leq N} \{ P_j = p_j \} \mid \mathcal{P}_N \right] \leq \frac{1}{\Pr [ \mathcal{P}_N ]} \prod_{1 \leq j \leq N} \left( \frac{C(\rho^j)}{j \rho^m} \right)^{p_j} \frac{1}{p_j!}.
\]

\[
= \frac{1}{\Pr [ \mathcal{P}_N ]} \cdot \exp \left( \sum_{1 \leq j \leq N} \frac{C(\rho^j)}{j \rho^m} \right) \rho^m N \cdot \prod_{1 \leq j \leq N} \Pr [ \text{Po} \left( \frac{C(\rho^j)}{j \rho^m} \right) = p_j ] .
\]
With Lemma 3.6 we obtain the existence of $B_1 > 0$ such that for $N$ large enough
\[
\Pr[P_N]^{-1} \leq B_1 \rho^{-mN} N^{1-c_m}.
\]
By Proposition 3.5 there exists a constant $B_2 > 0$ such that $C(\rho^j)/\rho^m \leq c_m + B_2 c_m \rho^j$. Consequently, since $\rho \in (0, 1)$ there exists $B_3 > 0$ such that
\[
\exp \left( \sum_{1 \leq j \leq N} \frac{C(\rho^j)}{j \rho^m} \right) \leq B_3 N^{c_m},
\]
which concludes the proof.

With this result at hand we are ready to study the distribution of $R$, cf. (15). As it will be necessary later, we show uniform tails bounds that hold for the joint distribution of $P_1$ and $R$ conditioned on $P_N$.

**Lemma 3.9.** There exist $A > 0$ and $0 < a < 1$ such that
\[
\Pr[P_1 = p, R = r \mid P_N] \leq A \cdot a^{p+r}, \quad p, r, N \in \mathbb{N}.
\]

**Proof.** We will prove the claimed bound by showing appropriate bounds for the moment generating function $\mathbb{E}[e^{\lambda R} \mid P_N]$. Let us fix any $0 < \lambda < -\log(\rho)/2$ such that $\rho e^{\lambda} < 1$. Then $\rho^j e^{\lambda j} < \rho$ for all $j \geq 2$. Recall that $\Pr[C_{j,i} = k] = c_k \rho^j / C(\rho^j)$, $k \in \mathbb{N}$, $j \geq 2$, $i \geq 1$, see (12). We obtain that
\[
\mathbb{E}[e^{\lambda (j(C_{j,i} - m))}] = \sum_{s \geq 0} \Pr[C_{j,i} = s + m] e^{\lambda j s} = e^{-\lambda j m} \frac{C(\rho^j e^{\lambda j})}{C(\rho^j)}, \quad i \geq 1, j \geq 2.
\]
Let $\Omega_{N,p}$ be the set of all $(p_2, \ldots, p_N) \in \mathbb{N}_0^{N-1}$ such that $(p, p_2, \ldots, p_N) \in \Omega_N$, i.e. $p = N - \sum_{2 \leq j \leq N} j p_j$, and let $\mathcal{E}_p$ be the event
\[
\mathcal{E}_p := \{P_1 = p\} \cap \bigcap_{2 \leq j \leq N} \{P_j = p_j\}.
\]
Then by Markov’s inequality and the independence of the $C_{j,i}$’s and the $P_j$’s, for any $p \in \Omega_{N,p}$
\[
\Pr[P_1 = p, R \geq r \mid \mathcal{E}_p] = \Pr[e^{\lambda R} \geq e^{\lambda r} \mid \mathcal{E}_p] \leq e^{-\lambda r} \mathbb{E}[e^{\lambda R} \mid \mathcal{E}_p] = e^{-\lambda r} \prod_{j=2}^{N} \left( \frac{C(\rho^j e^{\lambda j})}{C(\rho^j) e^{\lambda j m}} \right)^{p_j}.
\]
Abbreviate $\tau_j := C((\rho e^{\lambda})^j)/(\rho e^{\lambda})^{jm}$ for $j \in \mathbb{N}$. By Lemma 3.8 there exists $A_1 > 0$ such that
\[
\Pr[P_1 = p, R \geq r \mid P_N] = \sum_{p \in \Omega_{N,p}} \Pr[R \geq r \mid \mathcal{E}_p] \Pr[\mathcal{E}_p \mid P_N] 
\leq A_1 e^{-\lambda r} N \exp \left( - \sum_{2 \leq j \leq N} \frac{C(\rho^j)}{j \rho^m} \right) \frac{(C(\rho)/\rho^m)^p}{p!} \sum_{p \in \Omega_{N,p}} \prod_{2 \leq j \leq N} \frac{(\tau_j/j)^{p_j}}{p_j!}.
\]
(22)

With Proposition 3.5 we find $A_2 > 0$ with
\[
\exp \left( - \sum_{2 \leq j \leq N} \frac{C(\rho^j)}{j \rho^m} \right) \leq \exp \left( -c_m \sum_{2 \leq j \leq N} \frac{1}{j} \right) \leq A_2 N^{-c_m}.
\]
Let $H_j \sim \text{Po}(\tau_j/j)$ be independent for $j = 2, \ldots, N$ and set $\tau := \exp(\sum_{2 \leq j \leq N} \tau_j/j)$. Moreover, abbreviate $B := C(\rho)/\rho^m$. From (22) we obtain that there is an $A_3 > 0$ such that
\[
\Pr[P_1 = p, R \geq r \mid P_N] \leq A_3 e^{-\lambda r} N^{1-c_m} \cdot \tau \cdot \frac{B^p}{p!} \sum_{p \in \Omega_{N,p}} \prod_{j=2}^{N} \Pr[H_j = p_j].
\]
(23)
Note that
\[
\sum_{p \in \Omega_{n,p}} \prod_{j=2}^{N} \Pr[H_j = p_j] = \Pr \left[ \sum_{j=2}^{N} j H_j = N - p \right] = \tau^{-1} \cdot [x^{N-p}] \exp \left( \sum_{j \geq 2} \frac{\tau_j x^j}{j} \right).
\]

Observe that in the last expression we actually have to restrict the summation to the interval $2 \leq j \leq N$; however, $[x^M] \exp(\sum_{j \geq 2} \tau_j x^j / j) = [x^M] \exp(\sum_{2 \leq j \leq M} \tau_j x^j / j)$ for all $M \in \mathbb{N}$. Then
\[
\exp \left( \sum_{j \geq 2} \frac{\tau_j x^j}{j} \right) = \exp \left( c_m \sum_{j \geq 1} x^j / j \right) \cdot \exp \left( -c_m x + \sum_{j \geq 2} \frac{x^j}{j} (\tau_j - c_m) \right) =: G(x) \cdot H(x).
\]

By Proposition 3.5 there exists a constant $A_4 > 0$ such that $\tau_j \leq c_m (1 + A_4 (pe^\lambda)^j)$. With this at hand we deduce that $H(x)$ has radius of convergence (at least) $(pe^\lambda)^{-1}$, which by our choice of $\lambda$ is $> 1$. Note that $G(x) = (1 - x)^{-c_m}$, which shows together with Lemma 3.2 that $G$ has property $(S_2)$ with radius of convergence 1. As $G(x)$ only has positive coefficients, by Lemma 2.3 and the remark in (7) there is an $A_5 > 0$ such that
\[
[x^{N-p}] G(x) H(x) \leq A_5 (N - p)^{c_m - 1}, \quad p = 0, \ldots, N - 1.
\]

All in all,
\[
\Pr \left[ \sum_{2 \leq j \leq N} j H_j = N - p \right] \leq A_5 \tau^{-1} (N - p)^{c_m - 1}, \quad p = 0, \ldots, N - 1.
\] (24)

For the case $p = N$ note that the probability that $\sum_{2 \leq j \leq N} j H_j = 0$ equals $\tau^{-1}$. Putting the pieces together, we get from (23) that there is an $A_6 > 0$ such that
\[
\Pr[P_1 = p, R \geq r \mid \mathcal{P}_N] \leq A_6 e^{-\lambda r} N^{1 - c_m} \left( \frac{B^N}{N!} + \frac{B^p}{p!} (N - p)^{c_m - 1} \cdot 1[p \neq N] \right).
\] (25)

Observe that $N^{1 - c_m} B^N / N! \leq e^{-\lambda N}$ for $N$ large enough. Additionally, if $N/2 \leq p < N$, then $(1 - p/N)^{c_m - 1} \leq \max \{2^{1 - c_m}, N \}$ so that for $N$ large enough
\[
N^{1 - c_m} \left( \frac{(e^\lambda B)^p}{p!} (N - p)^{c_m - 1} = (1 - p/N)^{c_m - 1} \frac{(e^\lambda B)^p}{p!} \right) \leq 2p \cdot \frac{(e^\lambda B)^p}{p!} \leq 1
\]
and for $0 \leq p \leq N/2$
\[
N^{1 - c_m} \left( \frac{(e^\lambda B)^p}{p!} (N - p)^{c_m - 1} \leq \max \{2^{1 - c_m}, 1 \} \cdot e^{\lambda B} \Pr[P_0 = e^\lambda B] \right) \leq \max \{2^{1 - c_m}, 1 \} \cdot e^{\lambda B}
\]
is also bounded. Plugging these bounds into (25) completes the proof. \hfill \Box

We have just proven that $P_1, R$ have (joint) exponential tails when conditioned on $\mathcal{P}_N$. The next lemma is the last essential step towards the proof of Theorem 1.1, where we estimate $\Pr[\mathcal{E}_n \mid \mathcal{P}_N]$. Recall from (16) that
\[
\Pr[\mathcal{E}_n \mid \mathcal{P}_N] = \Pr[L + R = n - m N \mid \mathcal{P}_N], \quad \text{where} \quad L = \sum_{1 \leq i \leq P_1} (C_{1,i} - m).
\]

**Lemma 3.10.** Let $C(x)$ be subexponential. Then
\[
\Pr[\mathcal{E}_n \mid \mathcal{P}_N] \sim c_{n-m(N-1)} p^{n-m N}, \quad n, N, n - m N \rightarrow \infty.
\]

**Proof.** For the entire proof we abbreviate $\bar{N} := n - m N$. Then
\[
\Pr[\mathcal{E}_n \mid \mathcal{P}_N] = \sum_{p \geq 0} \sum_{r \geq 0} \Pr[L = \bar{N} - r \mid \mathcal{P}_N, P_1 = p, R = r] \Pr[P_1 = p, R = r \mid \mathcal{P}_N].
\] (26)
For brevity, let us write in the remainder
\[ D_{N,p,r} = \mathcal{P}_N \cap \{ P_1 = p \} \cap \{ R = r \} \quad \text{and} \quad Q_N := \Pr[C_{1,1} = \bar{N} + m] = \frac{c_{n-m(N-1)p^{n-m(N-1)}}}{C(p)}. \]

We will show that
\[ \Pr[L = \bar{N} - r \mid D_{N,p,r}] \sim p \cdot Q_N \quad \text{for} \quad p, r \in \mathbb{N}_0, \quad \text{as} \quad \bar{N} \to \infty. \] (27)

Let \( a \in (0, 1) \) be the constant guaranteed to exist from Lemma 3.9, and choose \( \delta > 0 \) such that \((1 + \delta)a < 1\). We will also show that there are \( C' > 0, N_0 \in \mathbb{N} \) such that
\[ \Pr[L = \bar{N} - r \mid D_{N,p,r}] \leq C(1 + \delta)^{p+r} \cdot Q_N \quad \text{for all} \quad p, r \in \mathbb{N}_0, \quad \bar{N} \geq N_0. \] (28)

From the two facts (27) and (28) the statement in the lemma can be obtained as follows. We will assume throughout that \( \delta \) is fixed as described above, say for concreteness \( \delta = (a^{-1} - 1)/2 \), and choose an \( 0 < \varepsilon < 1 \) arbitrarily. Moreover, we will fix \( K \in \mathbb{N} \) in dependence of \( \varepsilon \) only, and we will split the double sum in (26) in three (overlapping) parts with \( (p, r) \) in the sets
\[ B_{\leq} = \{ (p, r) : 0 \leq p, r \leq K \}, \quad B_{<} = \{ (p, r) : p > K, r \in \mathbb{N}_0 \}, \quad B_{>} = \{ (p, r) : p \in \mathbb{N}_0, r > K \}. \]

We will show that the main contribution to \( \Pr[\mathcal{E}_n \mid \mathcal{P}_N] \) stems from \( B_{\leq} \), while the other two parts contribute rather insignificantly. Let us begin with treating the latter parts. Observe that using Lemma 3.9 and (28) we obtain that there is a constant \( C'' > 0 \) such that for all \( r \in \mathbb{N}_0 \) and \( K \geq K_0(\varepsilon) \)
\[ \sum_{p \geq K} \Pr\left[L = \bar{N} - r \mid D_{N,p,r}\right] \Pr[P_1 = p, R = r \mid \mathcal{P}_N] \leq C'' \sum_{p \geq K} (1 + \delta)^{p+r} \cdot a^{p+r} \cdot Q_N \leq \varepsilon \cdot ((1 + \delta)a)^r \cdot Q_N. \]

Since \((1 + \delta)a < 1\), summing this over all \( r \) readily yields for \( c = (1 - (1 + \delta)a)^{-1} \)
\[ \sum_{(p, r) \in B_{<}} \Pr\left[L = \bar{N} - r \mid D_{N,p,r}\right] \Pr[P_1 = p, R = r \mid \mathcal{P}_N] \leq c\varepsilon \cdot Q_N. \] (29)

Completely analogously with the roles of \( p, r \) interchanged we obtain that also
\[ \sum_{(p, r) \in B_{>} \mid \mathcal{P}_N} \Pr\left[L = \bar{N} - r \mid D_{N,p,r}\right] \Pr[P_1 = p, R = r \mid \mathcal{P}_N] \leq c\varepsilon \cdot Q_N. \] (30)

It remains to handle the part of the sum in (26) with \( p, r \in B_{\leq} \). Using (27) we infer that
\[ \sum_{(p, r) \in B_{\leq}} \Pr\left[L = \bar{N} - r \mid D_{N,p,r}\right] \Pr[P_1 = p, R = r \mid \mathcal{P}_N] \sim \sum_{(p, r) \in B_{\leq}} p \Pr[P_1 = p, R = r \mid \mathcal{P}_N] \cdot Q_N. \]

Using Lemma 3.9 once again note that we can choose \( K \) large enough such that
\[ \sum_{0 \leq p \leq K} \sum_{r \geq K} p \Pr[P_1 = p, R = r \mid \mathcal{P}_N] \leq A \sum_{0 \leq p \leq K} \sum_{r \geq K} p a^{p+r} \leq \varepsilon \]
and that
\[ \left| \sum_{p \geq 0} p \Pr[P_1 = p \mid \mathcal{P}_N] - \sum_{0 \leq p \leq K} p \Pr[P_1 = p \mid \mathcal{P}_N] \right| = \left| \sum_{|p > K} p \Pr[P_1 = p \mid \mathcal{P}_N] \right| \leq \varepsilon. \]

Altogether this establishes that
\[ \left| \sum_{(p, r) \in B_{\leq}} \Pr\left[L = \bar{N} - r \mid D_{N,p,r}\right] \Pr[P_1 = p, R = r \mid \mathcal{P}_N] - \mathbb{E}[P_1 \mid \mathcal{P}_N] Q_N \right| \leq 2\varepsilon Q_N. \]
Corollary 3.7 asserts that $E[P_3 \mid P_N] \to C(\rho)\rho^{-m}$. Since $\varepsilon > 0$ was arbitrary, combining this with (29) and (30) we obtain from (26) that $Pr[\mathcal{E}_n \mid P_N] \sim C(\rho)\rho^{-m} \cdot Q_N$, which is the claim of the lemma.

In order to complete the proof it remains to show the two claims (27) and (28). We begin with (27). Note that for $p, r \in \mathbb{N}_0$

$$\Pr[L = \bar{N} - r \mid \mathcal{P}_N; P_1 = p, R = r] = \Pr\left[\sum_{1 \leq i \leq p} C_{1,i} = \bar{N} - r + pm\right].$$

Recall that $\Pr[C_{1,1} = k] = e_k \rho^k / C(\rho)$, where $\rho$ is the radius of convergence of $C$. Since $C$ is subexponential, $c_{k-1} \sim \rho c_k$ and thus the distribution of the $C_{1,i}$’s is also subexponential with $Pr[C_{1,1} = k - 1] \sim Pr[C_{1,1} = k]$. We obtain with Lemma 2.2 (i) that the probability (31) is $\sim pPr[C_{1,1} = \bar{N} - r + pm]$, as $\bar{N} \to \infty$. Moreover, as $\bar{N} \to \infty$, $\Pr[C_{1,1} = \bar{N} - r + pm] \sim Q_N$, and (27) is established.

We finally show (28). Our starting point is again (31). Note that with Lemma 2.2 (ii) there are $\omega \geq \rho$ large enough such that in addition $Pr[C_{1,1} = \bar{N} - r + pm] \leq (1 + \delta)^{-m} Q_N$. This establishes (28) if $\bar{N} - r + pm \geq \omega$. To treat the remaining cases, note that in this situation we have $r > \bar{N} - \omega$. Since $(c_n)_{n \in \mathbb{N}}$ is subexponential we obtain that for any $\varepsilon > 0$ and $\bar{N}$ sufficiently large

$$Q_N = e_{\bar{N} + m} \rho^{\bar{N} + m} / C(\rho) \geq \frac{(1 - \varepsilon)^m}{C(\rho)} (1 - \varepsilon)^{\bar{N}}.$$

Choosing $\varepsilon$ such that $(1 + \delta)(1 - \varepsilon) > 1$ we obtain that $(1 + \delta)^{-m} Q_N > 1$ for sufficiently large $\bar{N}$; thus (28) is trivially true in this case.

With all these facts at hand the proof of Theorem 1.1 is straightforward. With Lemma 3.4 and 3.6 we obtain as $n, N, n - mN \to \infty$,

$$[x^ny^N]G(x, y) = G(\rho)\rho^{-m} \Pr[\mathcal{E}_n \mid \mathcal{P}_N] \Pr[\mathcal{P}_N] \sim \frac{1}{\Gamma(c_m)} \exp\left(\sum_{j \geq 1} \frac{C(\rho^j) - c_m \rho^{jm} \rho^{-jm}}{j} \right) N^{c_m - 1} c_m^{-m(N - 1)}.$$

### 3.4. Proof of Theorem 1.2

Let us begin with (re-)collecting all basic definitions that will be needed in the proof. Suppose that $C(x)$ is subexponential with radius of convergence $0 < \rho < 1$ and set $m := \min\{k \in \mathbb{N} : c_k > 0\}$, see also (8). Moreover, let $P_j \sim \text{Po}(C(\rho^j)/j), j \in \mathbb{N}$ and $C_{1,1}, \ldots, C_{j,P_j}, j \in \mathbb{N}$ have the distribution specified in (12), that is, $Pr[C_{j,i} = k] = e_k \rho^k / C(\rho^j), k, i, j \in \mathbb{N}$. We assume that all these random variables are independent. Let $\mathcal{P}_N, \mathcal{E}_n$ be as in (10), that is, with

$$P = \sum_{j \geq 1} j P_j, \quad L = \sum_{1 \leq i \leq P_1} (C_{1,i} - m), \quad R = \sum_{j \geq 1} \sum_{1 \leq i \leq P_j} (C_{j,i} - m)$$

we have that $\mathcal{P}_N = \{P = N\}$ and $\mathcal{E}_n = \{L + R = n - mP\}$.

With this notation at hand, let $G_{n,N}$ be a uniformly drawn random object from $G_{n,N}$, meaning that the number of atoms is $n$ and the number of components $N$. According to Lemma 3.3 and using that the Boltzmann model induces the uniform distribution on objects of the same size, we infer that

$$\Pr[G_{n,N} = G] = \frac{1}{|G_{n,N}|} = \frac{\rho^n / C(\rho)}{|G_{n,N}| \rho^n / C(\rho)} = \frac{\Pr[A_G = G]}{\Pr[\mathcal{P}_N, \mathcal{E}_n]} = \Pr[A_G = G \mid \mathcal{P}_N, \mathcal{E}_n], \quad G \in G_{n,N},$$

that is, studying the distribution of $G_{n,N}$ boils down to considering the distribution of $A_G$ conditional on both $\mathcal{P}_N, \mathcal{E}_n$. This is the starting point of our investigations. In particular, $G_{n,N}$ has $N$ components with sizes given by the vector $(C_{j,i} : 1 \leq j \leq N, 1 \leq i \leq P_j)$. Our aim is here to study the properties of that vector in the conditional space given by $\mathcal{P}_N, \mathcal{E}_n$. To this end, set

$$M^* := \max_{j \geq 1, 1 \leq i \leq P_j} C_{j,i} \quad \text{and} \quad C_p^* := \max\{C_{1,1}, \ldots, C_{1,p}\} \quad \text{for } p \in \mathbb{N}.$$
Then the statement of the theorem is that, conditional on $\mathcal{P}_N, \mathcal{E}_n$, we have that $M^* = n - mN + \mathcal{O}_p(1)$; since the total number of atoms is $n$, the number of components is $N$, and the smallest component contains $m$ atoms, this immediately implies that there are $N + \mathcal{O}_p(1)$ components with exactly $m$ atoms, and all remaining components have a total size of $\mathcal{O}_p(1)$ as well.

The general proof strategy in the remaining section is as follows. We first show in Lemma 3.11 that both $P_1, R$ are “small” in the conditioned space; this makes sure that only a bounded number of entries in the vector $(C_{j,i})_{j\geq 2, 1\leq i \leq P_1}$ are larger than $m$, and that this total excess is bounded. Hence, the remaining number of $n - (N - P_1)m + \mathcal{O}_p(1)$ atoms is to be found in the components with sizes in $(C_{1,i})_{1\leq i \leq P_1}$. In Lemma 3.11 we exclude that $P_1$ grows too large conditioned on $\mathcal{E}_n, \mathcal{P}_N$; indeed, we show that it is stochastically bounded. Then the property of subexponentiality guarantees that only the maximum of the $C_{1,i}$’s dominates the entire sum, cf. Lemma 2.2 (iii), and Theorem 1.2 follows.

Let us now fill this overview with details. Recall Lemma 3.9, which says that $P_1, R$ have (joint) exponential tails given $\mathcal{P}_N$. We show that conditioning in addition to $\mathcal{E}_n$ does not change the behaviour qualitatively. The proof can be found at the end of the section.

**Lemma 3.11.** There exist $A > 0$ and $0 < a < 1$ such that for all sufficiently large $n - mN$

$$\Pr[P_1 = p, R = r \mid \mathcal{E}_n, \mathcal{P}_N] \leq A \cdot a^{p+r}, \quad p, r \in \mathbb{N}.$$ 

With this lemma the proof of the theorem can be completed as follows. Let $\varepsilon > 0$ be arbitrary. Abbreviate $\tilde{N} = n - m\tilde{N}$. With $M^*$ as in (32) we will show that there is $K \in \mathbb{N}$ such that

$$\Pr\left[|M^* - \tilde{N}| \geq K \mid \mathcal{E}_n, \mathcal{P}_N\right] < \varepsilon$$

for $n, N, \tilde{N}$ sufficiently large, which is the statement of the theorem. According to Lemma 3.11 there exist constants $C_R, C_P \in \mathbb{N}$ such that

$$\Pr\left[\{R \geq C_R\} \cup \{P_1 \geq C_P\} \mid \mathcal{E}_n, \mathcal{P}_N\right] < \varepsilon/2, \quad \tilde{N} \text{ sufficiently large.}$$

We deduce

$$\Pr\left[|M^* - \tilde{N}| \geq K \mid \mathcal{E}_n, \mathcal{P}_N, R = r, P_1 = p\right] \leq \frac{\varepsilon}{2} + \sum_{0 \leq r \leq C_R} \sum_{1 \leq p \leq C_P} \Pr\left[|M^* - \tilde{N}| \geq K \mid \mathcal{E}_n, \mathcal{P}_N, R = r, P_1 = p\right]. \tag{33}$$

This allows us to view $p, r$ as fixed. Further note that we only need to consider values of $p$ which are larger than 1 as $p = 0$ excludes $R = r \leq C_R < \tilde{N}$. The event “$\mathcal{E}_n, \mathcal{P}_N, R = r, P_1 = p^*$” implies that $|C_{j,i}| \leq m + r$ for all $j \geq 2, 1 \leq i \leq P_j$, and $S_p := \sum_{1 \leq i \leq S_P} C_{1,i} = \tilde{N} - r + p(m + r)$. Recall the definition of $C^*$ from (32). Assume that $C^*_p \leq m + r$, then we get the contradiction $\tilde{N} - r + pm = S_p \leq p(m + r) < \tilde{N} - r + pm$ for $\tilde{N}$ large enough. It follows that $C^*_p > m + r$ and hence $C^*_p = M^*$ in this conditioned space. That yields

$$\Pr\left[|M^* - \tilde{N}| \geq K \mid \mathcal{E}_n, \mathcal{P}_N, R = r, P_1 = p\right] = \Pr\left[C^*_p - \tilde{N} \geq K \mid S_p = \tilde{N} - r + pm\right],$$

for $1 \leq p \leq C_P, 0 \leq r \leq C_R$. As $C^*_p$ is at most $\tilde{N} - r + pm$ under this condition, we particularly obtain that $\{C^*_p \geq \tilde{N} + K\} = \emptyset$ for $K \geq mC_P$ as long as $0 \leq p \leq C_P$ and $r \geq 0$. Consequently, for $1 \leq p \leq C_P, 0 \leq r \leq C_R$,

$$\Pr\left[C^*_p - \tilde{N} \geq K \mid S_p = \tilde{N} - r + pm\right] = \Pr\left[C^*_p \leq \tilde{N} - K \mid S_p = \tilde{N} - r + pm\right].$$

Now Lemma 2.2 (iii) is applicable as $C_{1,i}$ has subexponential distribution for $1 \leq i \leq p$ and hence for $1 \leq p \leq C_P, 0 \leq r \leq C_R$ we have $(C^*_p \mid S_p = \tilde{N} - r + pm) = \tilde{N} - r + pm + \mathcal{O}_p(1)$ as $\tilde{N} \to \infty$. Consequently, choosing $K$ large enough,

$$\Pr\left[C^*_p \leq \tilde{N} - K \mid S_p = \tilde{N} - r + pm\right] < \frac{\varepsilon}{2C_R C_P}, \quad 1 \leq p \leq C_P, 0 \leq r \leq C_R.$$ 

We conclude from (33)

$$\Pr\left[|M^* - \tilde{N}| \geq K \mid \mathcal{E}_n, \mathcal{P}_N\right] \leq \frac{\varepsilon}{2} + \sum_{0 \leq r \leq C_R} \sum_{1 \leq p \leq C_P} \Pr\left[C^*_p \leq \tilde{N} - K \mid S_p = \tilde{N} - r + pm\right] < \varepsilon.$$ 

Since $\varepsilon > 0$ was arbitrary we have just proven that the largest component satisfies $(M^* \mid \mathcal{E}_n, \mathcal{P}_N) = \tilde{N} + \mathcal{O}_p(1)$, and the proof is completed.
**Proof of Lemma 3.11.** First note that due to Lemma 3.4 we have that $\Pr[\mathcal{E}_n, \mathcal{P}_N] = \Pr[\mathcal{E}_n | \mathcal{P}_N]$ equals $[x^ny^N]G(x,y)\rho^NG^{-1}(\rho)$. For $n$ and $n-mN$ sufficiently large this expression is strictly greater than zero as $[x^ny^N]G(x,y) \geq c_N^{-1}e_m^{-m}$. Hence $\Pr[\mathcal{E}_n | \mathcal{P}_N] > 0$. We start with the observation

$$\Pr[P_1 = p, R = r | \mathcal{E}_n, \mathcal{P}_N] = \Pr[\mathcal{E}_n | P_1 = p, R = r, \mathcal{P}_N] \Pr[P_1 = p, R = r | \mathcal{P}_N] \Pr[\mathcal{E}_n | \mathcal{P}_N]^{-1}. \quad (34)$$

Set $\bar{N} := n - mN$ and $L_p := \sum_{1 \leq i \leq p}(C_{1,i} - m)$ for $p \in \mathbb{N}_0$ as well as $Q_{\bar{N}} = \Pr[C_{1,1} - m = \bar{N}]$. Let $0 < a < 1$ be the constant from Lemma 3.9 and let $\delta > 0$ be such that $(1 + \delta)a < 1$. From (28) we obtain that there exists $A_1 > 0$ such that for sufficiently large $\bar{N}$

$$\Pr[\mathcal{E}_n | P_1 = p, R = r, \mathcal{P}_N] = \Pr[L = \bar{N} - r | \mathcal{D}_{N,p,r}] \leq A_1(1 + \delta)^{p+r}Q_{\bar{N}}, \quad p, r, n, N \in \mathbb{N}.$$

Lemma 3.9 tells us that we find $A_2 > 0$ with

$$\Pr[P_1 = p, R = r | \mathcal{P}_N] \leq A_2a^{p+r}, \quad p, r, N \in \mathbb{N}.$$

Finally, according to Lemma 3.10 there is a $A_3 > 0$ such that for all sufficiently large $\bar{N}$

$$\Pr[\mathcal{E}_n | \mathcal{P}_N] \geq A_3Q_{\bar{N}}, \quad n, N \in \mathbb{N},$$

and the claim follows with $a$ replaced by $(1 + \delta)a < 1$ by plugging everything into (34).

---

3.5. **Proof of Theorem 1.3**

We begin with a simple definition. We define the family of multiplicity counting functions $(d_C(\cdot))_{C \subseteq C}$, where $d_C(G)$ is the multiplicity of $C \subseteq G$ in $G \subseteq G$. Note that for any $G$ we have that $d_C(G) = 0$ for all but finitely many $C \subseteq C$. Assume that $N(n) \equiv N$ is such that $N(n), n-mN(n) \rightarrow \infty$ as $n \rightarrow \infty$. Let us write $R_{n,N}$ for the object obtained after removing all objects of size $m$ and a largest component from $G_{n,N}$. The statement of the theorem is equivalent to showing that for any fixed $G \subseteq G_{n,N}$

$$\Pr[R_{n,N} = G] \rightarrow G_{>m}(\rho)^{-1}\rho^{|G| - \max(G)}, \quad \text{as } n \rightarrow \infty,$$

see also (3). We immediately obtain that

$$\Pr[R_{n,N} = G] = \Pr[\forall C \in C_{>m} : d_C(R_{n,N}) = d_C(G)].$$

In the remainder we write $d_C = d_C(G)$ for short. Let $S > \max\{m, |G|\}$ be some arbitrary integer to be specified later. We infer that

$$\Pr[R_{n,N} = G] \leq \Pr[\forall C \in C_{m+1,S} : d_C(R_{n,N}) = d_C].$$

To obtain a lower bound, since $S > |G|$, we observe that $\{\forall C \in C_{>m} : d_C(R_{n,N}) = d_C\}$ is the same as $\{\forall C \in C_{m+1,S} : d_C(R_{n,N}) = d_C\} \cap \{\forall C \in C_{>S} : d_C(R_{n,N}) = 0\}$. Moreover, note that $|R_{n,N}| \leq S$ implies $d_C(R_{n,N}) = 0$ for all $C \subseteq C_{>S}$. Thus

$$\Pr[R_{n,N} = G] \geq \Pr[\forall C \in C_{m+1,S} : d_C(R_{n,N}) = d_C, |R_{n,N}| \leq S]$$

$$\geq \Pr[\forall C \in C_{m+1,S} : d_C(R_{n,N}) = d_C] - \Pr[|R_{n,N}| > S].$$

Let $\varepsilon > 0$. According to Theorem 1.2 there is $S_1 > \max\{m, |G|\}$ so that $\Pr[|R_{n,N}| > S_1] < \varepsilon$. Hence $\Pr[R_{n,N} = G]$ differs by at most $\varepsilon$ from $\Pr[\forall C \in C_{m+1,S} : d_C(R_{n,N}) = d_C]$ for all $S > S_1$. Let us write $L_{n,N}$ for the size of a largest component in $G_{n,N}$. Theorem 1.2 guarantees that $L_{n,N}$ is unbounded whp, that is $\Pr[L_{n,N} = O(1)] = o(1)$, and so we obtain for any $S \in \mathbb{N}$

$$\Pr[\forall C \in C_{m+1,S} : d_C(R_{n,N}) = d_C] = \Pr[\forall C \in C_{m+1,S} : d_C(R_{n,N}) = d_C, |L_{n,N}| > S] + o(1).$$

However, the event $\{\forall C \in C_{m+1,S} : d_C(R_{n,N}) = d_C, |L_{n,N}| > S\}$ is equivalent to the event $\{\forall C \in C_{m+1,S} : d_C(G_{n,N}) = d_C, |L_{n,N}| > S\}$, since we obtain $R_{n,N}$ by removing all components with size $m$ and a largest component (of size $S$) from $G_{n,N}$. Now we add and subtract $\Pr[\forall C \in C_{m+1,S} : d_C(G_{n,N}) = d_C, |L_{n,N}| \leq S] = o(1)$ in order to get rid of the event $|L_{n,N}| > S$ and arrive at the fact

$$\Pr[\forall C \in C_{m+1,S} : d_C(R_{n,N}) = d_C] = \Pr[\forall C \in C_{m+1,S} : d_C(G_{n,N}) = d_C] + o(1).$$
Combining all previous facts yields that for $n$ sufficiently large
\[
\left| \Pr \left[ R_{n,N} = G \right] - \Pr \left[ \forall C \in C_{m+1,S} : d_C(G_{n,N}) = d_C \right] \right| \leq 2\varepsilon 
\] (35)
and thus we are left with estimating $\Pr \left[ \forall C \in C_{m+1,S} : d_C(G_{n,N}) = d_C \right]$. For $v_S := (v_C)_{C \in C_{m+1,S}}$ denote by $G(x,y,v_S)$ the generating series of $G$ such that $x$ marks the size, $y$ the number of components and $v_S$ the multiplicities of $(C)_{C \in C_{m+1,S}}$. In other words, for $\ell,k \in \mathbb{N}_0$, $t_s := (t_C)_{C \in C_{m+1,S}} \in \mathbb{N}_0^{C_{m+1,S}}$
\[
g_{\ell,k,t_s} = [x^\ell y^k v_S^t] G(x,y,v_S) = | \{ G \in \mathcal{G} : |G| = \ell, \kappa(G) = k, \forall C \in C_{m+1,S} : d_C(G) = t_C \} |.
\]
Setting $v_C = 1$ for all $C \in C_{m+1,S}$ we obtain the generating series $G(x,y)$ counting only size and number of components by $x$ and $y$ respectively. As $G_{n,N}$ is drawn uniformly at random from $\mathcal{G}_{n,N}$ the proof reduces to determining
\[
\Pr \left[ \forall C \in C_{m+1,S} : d_C(G_{n,N}) = d_C \right] = \frac{\left[ x^n y^n v_S^t \right] G(x,y,v_S)}{[x^n y^n] G(x,y)}.
\]
The following lemma, whose proof is shifted to the end of this section, accomplishes this task.

**Lemma 3.12.** Let $\mathbf{d} = (d_C)_{C \in C_{m+1,S}}$ with $D := \sum_{C \in C_{m+1,S}} |C| d_C$ and $D' := \sum_{C \in C_{m+1,S}} d_C$. Then
\[
\frac{[x^n y^n v_S^t] G(x,y,v_S)}{[x^n y^n] G(x,y)} \rightarrow \rho^{D-mD'} \prod_{C \in C_{m+1,S}} (1 - \rho^{|C|-m}), \quad n \rightarrow \infty.
\]

**Lemma 3.12** yields directly for sufficiently large $n$
\[
\left| \Pr \left[ \forall C \in C_{m+1,S} : d_C(G_{n,N}) = d_C \right] - \rho^{G(|\cdot|-m)} \prod_{C \in C_{m+1,S}} (1 - \rho^{|C|-m}) \right| < \varepsilon.
\]
Now observe that with defining $C_{m+1,S}(x) := \sum_{m<\ell \leq S} |C_\ell| x^\ell$ we obtain
\[
\lim_{S \rightarrow \infty} \prod_{C \in C_{m+1,S}} (1 - \rho^{|C|-m}) = \lim_{S \rightarrow \infty} \prod_{m<\ell \leq S} \exp \left( |C_\ell| \log (1 - \rho^{\ell-m}) \right) = \lim_{S \rightarrow \infty} \exp \left( - \sum_{j \geq 1} \frac{C_m+1,S(j)}{j \rho^{m}} \right).
\]
By the continuity of $\exp(\cdot)$ and monotone convergence this equals $G_{\geq m}(\rho)^{-1}$. Choose $S_2 > \max \{m,|G|\}$ large enough such that $\prod_{C \in C_{m+1,S}} (1 - \rho^{|C|-m})$ differs at most by $\varepsilon$ from $G_{\geq m}(\rho)^{-1}$ for all $S > S_2$. Summarizing, fixing $S \geq \max \{S_1, S_2\}$ we obtain for sufficiently large $n$
\[
\left| \Pr \left[ \forall C \in C_{m+1,S} : d_C(G_{n,N}) = d_C \right] - \rho^{G(|\cdot|-m)} G_{\geq m}(\rho)^{-1} \right| \leq 2\varepsilon.
\]
Since $\varepsilon > 0$ was arbitrary the proof of the theorem is finished with (35).

**Proof of Lemma 3.12.** First we determine $G(x,y,v_S)$ explicitly. Define the multivariate generating series
\[
C(x,y,v_S) = y \left( C(x) + \sum_{C \in C_{m+1,S}} (v_C - 1)x^{|C|} \right),
\]
where as usual $x$ marks the size, $y$ the number of components (which by convention is always 1 for $C \in \mathcal{C}$) and $v_S$ objects in $C_{m+1,S}$. Note that these parameters are clearly additive when forming multisets. Hence, according to [19, Theorem III.1] the formula (1) extends to the multivariate version
\[
G(x,y,v_S) = \exp \left( \sum_{j \geq 1} \frac{C(x^j, y^j, v_S^j)}{j} \right),
\] (36)
where $v^j_C = (v_C^j)_{C \in C_{m+1,S}}$. Setting $v_C = 1$ for all $C \in C_{m+1,S}$ we see that $G(x, y, 1) \equiv G(x, y)$ such that $[x^n y^N]G(x, y) = |G_n|^N$. By elementary algebraic manipulations we reformulate (36) to

$$G(x, y, v_S) = G(x, y) \exp \left( \sum_{C \in C_{m+1,S}} \left( \sum_{j \geq 1} \frac{(x^{\ell C})^j v_C^j}{j} - \sum_{j \geq 1} \frac{(x^{\ell C})^j}{j} \right) \right)$$

$$= G(x, y) \prod_{C \in C_{m+1,S}} \frac{1 - x^{\ell C} y^{\rho/m}}{1 - x^{\ell C} y^C}.$$  \hfill (37)

Let us now turn to the initial claim in Lemma 3.12. We obtain that

$$[x^n y^N v^d_S]G(x, y, v_S) = [x^n y^N]G(x, y) \prod_{C \in C_{m+1,S}} [v^d_C] \frac{1 - x^{\ell C} y}{1 - x^{\ell C} y^C}$$

$$= [x^{n-D^y} y^{N-D^y}]G(x, y) \prod_{C \in C_{m+1,S}} (1 - x^{\ell C} y).$$

Since $C_{m+1,S}$ does only have finitely many elements, there exist $L, K \in \mathbb{N}$ such that $[x^{\ell} y^k] \prod_{C \in C_{m+1,S}} (1 - x^{\ell C} y) = 0$ for all $\ell \geq L, k \geq K$. Recall that, using Theorem 1.1,

$$[x^n y^N]G(x, y) \sim \exp \left( \sum_{j \geq 1} \frac{C(\rho^j) - c_m \rho^{jm}}{j \rho^{jm}} \right) \frac{N^{-m-1}}{\Gamma(c_m)} |C_{n-m(N-1)}|, \quad n \to \infty,$$

and so $[x^{n-a} y^{N-b}]G(x, y) \sim [x^n y^N]G(x, y) \rho^{a-mb}$ for fixed $a, b \in \mathbb{N}$ as $C$ is subexponential. Hence, as $n \to \infty$,

$$[x^n y^N v^d_S]G(x, y, v_S) = \sum_{\ell \in [L], k \in [K]} [x^{n-D^y} y^{N-D^y-k}]G(x, y) [x^\ell y^k] \prod_{C \in C_{m+1,S}} (1 - x^{\ell C} y)$$

$$\sim [x^n y^N]G(x, y) \cdot \rho^{D^y-mD^y} \sum_{\ell \in [L], k \in [K]} \rho^{\ell - mk} [x^\ell y^k] \prod_{C \in C_{m+1,S}} (1 - x^{\ell C} y)$$

$$= [x^n y^N]G(x, y) \cdot \rho^{D^y-mD^y} \prod_{C \in C_{m+1,S}} (1 - \rho^{\ell C} - m),$$

which finishes the proof. \hfill \Box

### 3.6. Proof of Proposition 1.4

**Proof of Proposition 1.4.** It is a well-known fact that the weak convergence of $(G_n, o_n)$ to $(\mathcal{G}, \phi)$ in (4) is equivalent to showing that for any bounded and continuous function $f : B \to \mathbb{R}$

$$\lim_{n \to \infty} \mathbb{E} [f(G_n, o_n)] = \mathbb{E} [f(C, \phi)].$$

For any finite graph $G$ denote by $o_G$ a vertex chosen uniformly at random from its vertex set. Let $\mathcal{M}(G_n,N)$ denote a (canonically chosen) largest component of $G_n,N$ and $\mathcal{R}(G_n,N)$ the remainder after removing all objects of size $m$ and $\mathcal{M}(G_n,N)$. Let $f : B \to \mathbb{R}$ be an arbitrary bounded and continuous function. Then

$$\mathbb{E} [f(G_n, o_n)] = \mathbb{E} [f(\mathcal{M}(G_n,N), o_{\mathcal{M}(G_n,N)})] \Pr [o_n \in \mathcal{M}(G_n,N)]$$

$$+ \mathbb{E} [f(\mathcal{R}(G_n,N), o_{\mathcal{R}(G_n,N)})] \Pr [o_n \in \mathcal{R}(G_n,N)]$$

$$+ \mathbb{E} [f(C_m, o_m)] \Pr [o_n \notin \mathcal{M}(G_n,N) \cup \mathcal{M}(G_n,N)].$$

According to Theorem 1.2 we have that $|\mathcal{M}(G_n,N)| = n - mN + O_p(1)$ implying $\Pr [o_n \in \mathcal{M}(G_n,N)] \sim (n - mN)/n \to 1 - \lambda$. As the size of $\mathcal{M}(G_n,N) \in \mathcal{C}$ tends to infinity and $(C_n)_{n \geq 1}$ converges in the BS sense to $(\mathcal{C}, \phi)$ we have that

$$\mathbb{E} [f(\mathcal{M}(G_n,N), o_{\mathcal{M}(G_n,N)})] \Pr [o_n \in \mathcal{M}(G_n,N)] \to (1 - \lambda) \mathbb{E} [f(\mathcal{C}, \phi)], \quad n, N \to \infty.
Theorem 1.3 entails that $R(G_{n,N})$ has a limiting distribution and hence $\Pr[o_n \in R(G_{n,N})] \to 0$. As $f$ is bounded
\[
\mathbb{E} \left[ f(R(G_{n,N}), o_{R(G_{n,N})}) \right] \Pr[o_n \in R(G_{n,N})] \to 0, \quad n \to \infty.
\]
Finally, we obtain by combining Theorems 1.2 and 1.3 that $n - |R(G_{n,N}) \cup M(G_{n,N})| = mN + O_p(1)$ and hence $\Pr[o_n \notin R(G_{n,N}) \cup M(G_{n,N})] \sim mN/n \to \lambda$. Thus,
\[
\lim_{n,N \to \infty} \mathbb{E} \left[ f(G_{n,N}, o_n) \right] = (1 - \lambda)\mathbb{E} \left[ f(C, o) \right] + \lambda\mathbb{E} \left[ f(C_{m}, o_{m}) \right].
\]

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