Winding number for stationary Gaussian processes using real variables

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Abstract

We consider the winding number of planar stationary Gaussian processes defined on the line. Under mild conditions, we obtain the asymptotic variance and the Central Limit Theorem for the number of winding turns as the time horizon tends to infinity. In the asymptotic regime, our discrete approach is equivalent to the continuous one studied previously in the literature and our main result extends the existing ones. Our model allows for a general dependence of the coordinates of the process and non-differentiability of one of them. Furthermore, beyond our general framework, we consider as examples an approximation to the winding number of a process whose coordinates are both non-differentiable and the winding number of a process which is not exactly stationary.

Keywords Gaussian process, Stationary process, Winding number, Wiener chaos expansions, Fourth moment theorem
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1 Introduction

The notion of the number of winding turns for a planar Gaussian process has been studied for a long time. The first paper, dating from 1954, dealing with this matter in the case of planar Brownian motion is due to Spitzer [18]. Concerning processes with irregular paths,
the argument in the complex plane is defined and studied with the help of the stochastic calculus. The asymptotic behaviour of the argument as the time horizon tends to infinity was determined in some important cases starting with planar Brownian motion. Among a huge number of results, it is worth mentioning again the theorem of Spitzer [18], which states that, with a normalization by log \( t \), the argument of the Brownian curve tends in law to a Cauchy distribution. Later on, a similar result was obtained for the Ornstein–Uhlenbeck process, see [20]. Note that Messulam and Yor [14] proved that the occurrence of the Cauchy distribution as a limit (and consequently, the non-existence of the mean of the limit random variable) is due to the fact that the Brownian motion makes many turns when it is close to the origin. If the process is restricted to remain outside a neighbourhood of the origin, then the limiting distribution has moments of all orders.

In the 1980s, intense research was carried out on these matters, the reader may consult the recent paper [7], written in memory of Yor, for a modern view of the existing results and their relations with a deep problem from mathematical physics. Another key reference is [10], which provides a very interesting discussion of recent advances, applications to physics and also the paper gives a significant number of references on such themes. Physical applications involve the physics of polymers, flux lines in superconductors, and the quantum Hall effect.

Following these results for non-differentiable random processes, the study of the winding number was extended to stationary Gaussian processes whose paths are sufficiently smooth. This allows using the classical formula of complex analysis for the argument function to define the winding number. If \( A(t) \) denotes the argument at time \( t \), defining \( \Delta(T) = A(T) - A(0) \), we have the following formula:

\[
N_W = \left\lfloor \frac{A(T)}{2\pi} \right\rfloor - \left\lfloor \frac{A(0)}{2\pi} \right\rfloor,
\]

where \( N_W \) is the winding number. Using the complex analysis representation for \( A(t) \) and techniques for complex-valued processes, Buckley and Feldheim [4] studied the asymptotic behaviour of the variation of the argument \( \Delta(T) \), see [4, Eq. (1)] and Remark 2 below, of a complex random process \( X(t) \) following the circularly symmetric model over the interval \( [0, T] \) as \( T \to \infty \). For a definition of this and related classes of processes, see below, in subsection 2.2. They computed the moments and obtained a central limit theorem.

In the present paper we use a completely different approach to the same problem, counting the number of winding turns around the origin: they are counted by the up-crossings minus the down-crossings of the half line \( \{x_1 > 0, x_2 = 0\} \). We will make this definition precise and establish the link with the definition using complex variables in subsection 2.3.

Our approach may seem less intuitive and more involved since we replace a continuous functional by a discrete one, a priori more difficult to handle. Nevertheless, by doing so we can
profit from the extensive machinery developed for the study of crossings, such as Kac–Rice type formulas [2], chaos (also known as Wiener–Itô or Hermite) expansions of level functionals of Gaussian processes, diagram formula [19] Lem. 3.2 and the so called Fourth Moment Theorem and its generalizations [16, 15]. Actually, our work can be considered as a response to the sentence in [4]: ‘It may well be the case that the more sophisticated methods of Wiener–Itô expansions could be useful’. Nevertheless, the complex integral representation of the variation of the argument $\Delta(T)$ in [4, Eq. (1)] does not seem well adapted to obtain the chaos expansion, see Remark 2 below. On the other hand, the crossings of a half-line are well suited for this purpose.

Our results are true under wider conditions, while the proofs are simpler. Theorem 1 below extends and unifies the existing results, such as those of [10] and [4]. In particular, we consider less symmetric models, including general dependencies between the coordinates of the process and the case where one of them is not differentiable. This shows that real analysis is a very good alternative to complex analysis in this particular case. In addition, the real representation allows obtaining a more explicit description of the different sub-models, see subsections 2.1 and 2.2.

Though we restrict ourselves to the stationary case, we point out that both the Kac–Rice formulas and chaos expansions can be used in the non-stationary case. For instance, in [3, 6], these techniques are used in a non-stationary framework. The main advantage of assuming stationarity is that it implies many symmetries and independences, thus simplifying the computations.

This paper is organized as follows. Section 2 introduces the model as well as some particular cases. Our main result, Theorem 1 is presented in Section 3. The proof is presented in Sections 4, 5 and 6. Section 7 is dedicated to some examples, one of which is not exactly stationary. Section 8 contains some ancillary computations used in the proof of the main results.

## 2 Description of the model

### 2.1 Generalities

Consider a stationary mean-zero vectorial Gaussian process defined on $\mathbb{R}$,

$$X(\cdot) = (X_1(\cdot), X_2(\cdot)) \in \mathbb{R}^2.$$ 

Let $r_i(\cdot)$ denote the covariance function of $X_i$, $i = 1, 2$ and $r_{12}(\cdot)$ their cross-covariance function:

$$r_i(t) = \mathbb{E}(X_i(t)X_i(0)), i = 1, 2; \quad r_{12}(t) = \mathbb{E}(X_1(t)X_2(0)); \quad t \in \mathbb{R}.$$
Without loss of generality, after a spatial scaling, we can assume that for each $t$ the variance–covariance matrix of $X(t)$ is the identity matrix $I$. That is, $r_i(0) = 1$, $i = 1, 2$ and $r_{12}(0) = 0$. Indeed, in Remark 1 it will be justified that a spatial scaling plays no role in the asymptotic study of winding turns. In addition, note that
\[\mathbb{E}(X_1(0)X_2(t)) = r_{12}(-t); \quad \mathbb{E}(X_1(0)X'_2(t)) = -r'_{12}(-t).\]
Clearly, $r_1, r_2$ and $r_{12}$ determine the distribution of $X(\cdot)$.

To be able to compute winding turns we assume that one of the coordinates is differentiable in quadratic mean, say $X_2$. The fact that $X_2$ has a derivative is equivalent to
\[-r''_2(0) =: \lambda_{2,2} < \infty.\]
After a scaling in time, we can assume w.l.o.g. that $\lambda_{2,2} = 1$.

### 2.2 Some particular models

For the sake of ease of comparison with the literature, and to describe the submodels, consider now the complex counterpart of $X(\cdot)$, namely, the centred stationary complex Gaussian process $\mathbb{X}(\cdot)$ s.t.
\[\mathbb{X}(t) := X_1(t) + iX_2(t).\]
The distribution of $\mathbb{X}(\cdot)$ is determined by its covariance and pseudo-covariance functions, given by
\[R(t) := \mathbb{E}(\mathbb{X}(0)\mathbb{X}(t)) = r_1(t) + r_2(t) - i(r_{12}(t) - r_{12}(-t));\]
\[C(t) := \mathbb{E}(\mathbb{X}(0)\mathbb{X}(t)) = r_1(t) - r_2(t) + i(r_{12}(t) + r_{12}(-t)).\]

As said in the Introduction, in the literature some symmetries are usually imposed (see [4, 10]). We present now the most common submodels.

1. **The circularly symmetric model.** Assume that $C(t) = 0$, or, equivalently, that $r_1 = r_2$ and that $r_{12}$ is an odd function (see for instance [4] and [8, Sec. 8.1, pp. 163]).

2. **The reflexional symmetric model.** Assume that
\[(X_1(\cdot), X_2(\cdot)) \overset{d}{=} (X_2(\cdot), X_1(\cdot)),\]
where $\overset{d}{=} \text{means that both sides have the same distribution}$. A simple computation shows that $R(t) = 2r(t)$ is real and the pseudo-covariance is purely imaginary, $C(t) = 2ir_{12}(t)$. Here, $r$ is the common value of $r_1$ and $r_2$.  

4
3. The independent model. This is the case where $X_1(\cdot)$ and $X_2(\cdot)$ are independent, or, equivalently, where $r_{12} \equiv 0$.

4. The i.i.d. model. Here, $X_1(\cdot)$ and $X_2(\cdot)$ are independent and have the same distribution. This model is often considered by physicists [10].

The intersection of any two of the models (1), (2), and (3) yields model (4).

2.3 Real variable definition of winding

Define the number of winding turns around the origin by

$$N_W([0,T]) = \# \{ t \leq T : X_1(t) > 0, X_2(t) = 0, X_2'(t) > 0 \} - \# \{ t \leq T : X_1(t) > 0, X_2(t) = 0, X_2'(t) < 0 \}. \quad (2)$$

Thus, $N_W([0,T])$ is just the number of up-crossings minus the number of down-crossings of $X_2$ conditioned on the event $X_1(t) > 0$.

Remark 1. The choice of the semi-axis $\{ X_1 > 0, X_2 = 0 \}$ is arbitrary: we can replace it by any other half-line starting from zero. This fact explains why, without loss of generality, we can perform the spatial scaling of Section 2.1.

Remark 2. In [4], for a complex stationary Gaussian process $\mathbb{X}(t) = X_1(t) + iX_2(t)$, the increment of the argument is defined by

$$\Delta(T) = \int_0^T \frac{X_2'(t)X_1(t) - X_1'(t)X_2(t)}{X_1^2(t) + X_2^2(t)} dt. \quad (3)$$

Our approach is linked to that of the paper above because the increment of the argument $\Delta(T)$ and the number of turns $N_W([0,T])$ by [7] verify

$$\left| \frac{\Delta(T)}{2\pi} - N_W([0,T]) \right| < 1. \quad (4)$$

Thus, the difference between $\frac{1}{2\pi} \Delta(T)$ and $N_W([0,T])$ plays no role in our asymptotic study, because our normalisation $\sqrt{T}$ tends to infinity.

In Section 5, we will obtain a chaos expansion for $N_W([0,T])$ that allows obtaining a CLT for this random variable and consequently also for $\Delta(T)$ because of (4). But one should be aware that we do not provide any chaos expansion for $\Delta(T)$. We point out that such a representation seems very difficult to obtain due to the lack of integrability, see Remark 7 below, while the Kac-type integral representation of $N_W([0,T])$ is well adapted to obtain the expansion.
3 Main results

Consider the following conditions.

(G) Assume that $X_2(\cdot)$ satisfies
\[
\int \frac{\lambda_{22} + r_{22}''(t)}{t} \, dt \text{ converges at zero.}
\]

It is well known that this condition is necessary and sufficient for having a finite second factorial moment for the number of zeros of $X_2$. Geman proved this equivalence in [9].

(A) Set
\[
m(t) = \max \{|r_2(t)|, |r_2''(t)|, |r_1(t)|, |r_{12}(t)|, |r_{12}'(t)|\},
\]
and assume that $m \in L^2([0, \infty))$ and $m(t) \to 0$ as $t \to \infty$.

(A') Assume that $r_1(t), r_2(t) \to 0$ as $t \to +\infty$ and
\[
\int_{\mathbb{R}} r_2^2 + (r_{12}')^2 + (r_2')^2 + |r_{12}| < +\infty. \tag{5}
\]

These mixing conditions differ slightly from the one introduced by Arcones [1, Lem. 1].

Set
\[
N^*_W([0, T]) := \frac{N_W([0, T]) - \mathbb{E}N_W([0, T])}{\sqrt{T}}. \tag{6}
\]

**Theorem 1.** Consider $X(\cdot)$ as in Section 2. Let $N_W([0, T])$ and $N^*_W([0, T])$ be defined as in (2) and (6) respectively. Then,

1. For each $T > 0$, $\mathbb{E}(N_W([0, T])) = -\frac{T}{2\pi} r_{12}'(0)$.

2. Assume that conditions (G) and (A') hold. Then, there exists $V_\infty < \infty$ s.t.
\[
\lim_{T \to \infty} \frac{\text{Var}(N_W([0, T]))}{T} = V_\infty. \tag{7}
\]

3. Under conditions (G) and (A), as $T \to \infty$, the distribution of $N^*_W([0, T])$ converges towards the centred normal distribution with variance $V_\infty$.

Statements 1, 2 and 3 are proven in Sections 4.1, 4.2 and 6 respectively. Some remarks are in order.
Remark 3. 1. Finiteness of the expectation and of the variance. For each $T > 0$, $\mathbb{E}N_W([0,T])$ is finite and under $(G)$, $\text{Var}(N_W([0,T]))$ is finite, see Section 4.

2. Note that Condition $(A')$ is weaker than Condition $(A)$.

3. Extension of Theorems 1 and 3 of [4]. If $r'_1$ exists, $\int |r_1r''_2| < \infty$ can be replaced, in (5), by $\int |r'_1r'_2| < \infty$. From this fact we can see that this theorem extends Theorem 3 in [4]. Furthermore, in (13) of Section 4.2 we give an explicit integral formula for the variance of $N_W([0,T])$ for each $T > 0$. This formula is analogous to the one exhibited in Theorem 1 of [4] but recall that the functional $N_W([0,T])$ does not coincide with $\Delta(T)$ in [4].

Remark 4. Chaos expansions may allow obtaining a quantitative version of the Central Limit Theorem (CLT), e.g. obtaining bounds for the distance, in a suitable sense, of the law of the normalized winding turns from the standard Gaussian law. We do not consider this problem in this paper.

Under more restrictive hypotheses we can give more precise results. The next proposition, whose proof is deferred to Section 8, concerns the positivity of the asymptotic variance. It is convenient to consider the condition

(S) Assume that $X_1$ and $X_2$ have spectral densities; they will be denoted by $f_1$ and $f_2$.

Note that Condition (S) is weaker than (A) but not weaker than (A').

Proposition 1. Under condition (S) and assuming that $r'_{12}(0) = 0$, we have that $V_\infty > 0$.

Remark 5. The general case $r'_{12}(0) \neq 0$ could be dealt with by the same techniques, but the computations involved become burdensome.

The next theorem, whose proof is deferred to Section 4.3, deals with the simpler case of independent coordinates.

Theorem 2. Assume that $X_1$ and $X_2$ are independent and that the covariance $r_1(\cdot)$ is differentiable except at the origin. Then, a sufficient condition to have a finite asymptotic variance (for the r.v. defined in (2)) is

$$I := \int_0^\infty \frac{r'_1(t)}{\sqrt{1 - r^2_1(t)}} \frac{r'_2(t)}{\sqrt{1 - r^2_2(t)}} \, dt$$

is convergent in the sense of Riemann.

The asymptotic variance takes the value

$$\lim_{T \to \infty} \frac{\text{Var}(N_W([0,T]))}{T} = \frac{1}{\pi} \left( \pi \left( \frac{1}{2} + I \right) \right).$$
In the particular case where the distributions of \(X_1\) and \(X_2\) are equal (the i.i.d. model), the asymptotic variance is equal to

\[
\frac{1}{\pi} \left( \frac{\pi}{2} + \int_0^\infty \frac{r'^2(t)}{1 - r^2(t)} \, dt \right),
\]

where \(r(\cdot)\) denotes the common value of \(r_i(\cdot), i = 1, 2\). This expression coincides with the results of [4] and [10].

**Remark 6.** Note that the integrability condition in Theorem 2 is weaker than condition (G).

**Remark 7.** By defining

\[H(x_1, x_2, x_3, x_4) = \frac{(x_4x_1 - x_3x_2)}{x_1^2 + x_2^2},\]

we can write

\[\Delta(T) = \int_0^T H(X_1(t), X_2(t), X_1'(t), X_2'(t)) \, dt.\]

Nevertheless, the CLT of Theorem 7 is not a direct application of a continuous time vectorial Breuer–Major Theorem and condition (A), because in the present case \(H\) does not belong to the space of square-integrable functions with respect to the four-dimensional standard Gaussian measure.

### 4 Moments

In this section we compute the first two moments of \(N_W([0,T])\).

We start by expressing \(N_W([0,T])\) by a Kac-type counting formula, as

\[
N_W([0,T]) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^T 1_{[-\delta,\delta]}(X_2(t))X'_2(t)1_{[0,\infty)}(X_1(t)) \, dt,
\]

where the limit is in the a.s. sense, see [2] Lem. 3.1, pp 70-71] where similar level functionals are treated.

#### 4.1 The expectation of the winding number

The expectation can be computed by the Kac–Rice formula, as we do below, but it can also be deduced from the Hermite expansion, something that will be presented in Section 5.

From [8] and a proof similar to that of the Kac–Rice formula [2] Rk 8, pp. 85, we have
\[ \mathbb{E}(N_W([0, T])) = \frac{T}{\sqrt{2\pi}} \mathbb{E}\left( \left[ (X_2')^+ - (X_2')^- \right] 1_{X_1 > 0} X_2 = 0 \right) \]
\[ = \frac{T}{\sqrt{2\pi}} \mathbb{E}\left( X_2' 1_{X_1 > 0} X_2 = 0 \right) = -\frac{T r'_{12}(0)}{2\pi}. \]  
(9)

Here, \( X^+(\cdot) \) (resp. \( X^-(\cdot) \)) denotes the positive part (resp. negative part) of \( X(\cdot) \). Note that \( r_{12}(t) = \mathbb{E}[X_1(t)X_2(0)] = \mathbb{E}[X_1(0)X_2(-t)] \). We need to consider \( \mathbb{E}[X_2'(0)X_1(0)] \), but stationarity implies \( r'_{12}(t) = -\mathbb{E}[X_1(0)X_2'(t)] = -\mathbb{E}[X_1(t)X_2'(0)] \). Thus \( -r'_{12}(0) = \mathbb{E}[X_1(0)X_2'(0)] \). Then \( -r_{12}(0) \) exists and by the Cauchy–Schwarz inequality is finite.

Note that under our hypotheses, the expectation is always finite. Note also that in submodels (2) – (4) of Section 2.2, it vanishes. In particular, we have obtained the result of [4].

4.2 The variance of \( N_W([0, T]) \)

In this section we assume that \( X_2(\cdot) \) satisfies condition (G). As we said before, see [9], this is a necessary and sufficient condition to ensure that the number of zeros of \( X_2(\cdot) \) has a finite second moment. This implies that \( N_W([0, T]) \) has finite variance.

To compute the variance of the random variable \( N_W([0, T]) \), we use the Kac–Rice formula [2 Rk 8, pp. 85] and the equality \( \text{Var}(N) = \mathbb{E}(N(N - 1)) - \mathbb{E}^2(N) + \mathbb{E}(N) \). We use the short hand notation \( \mathbb{E}_c(\cdot) = \mathbb{E}(\cdot \mid X_2(0) = X_2(t) = 0) \). Hence,

\[ \mathbb{E}\left( N_W([0, T])(N_W([0, T]) - 1) \right) = 2 \int_0^T (T-t) \mathbb{E}_c\left[ 1_{[0,\infty)}(X_1(0)) 1_{[0,\infty)}(X_1(t)) X_2'(0)X_2'(t) \right] \frac{dt}{2\pi \sqrt{1 - r^2_{12}(t))}}. \]  
(10)

The following lemma, whose proof is postponed to Section 8 helps us to compute the conditional expectation. It is a particular case of the celebrated Diagram formula, cf. [19 Lem. 3.2].

**Lemma 1.** Let \((Z_1, Z_2, Z_3, Z_4)\) be a centred Gaussian vector with variance 1 and covariances \( \rho_{ij}, 1 \leq i < j \leq 4 \). Then,

\[ \mathbb{E}[Z_1Z_2 1_{[0,\infty)}(Z_3) 1_{[0,\infty)}(Z_4)] = \frac{\rho_{12}}{4} + \frac{\rho_{12}}{2\pi} \arcsin(\rho_{34}) + \frac{\rho_{13}\rho_{24} + \rho_{14}\rho_{23}}{2\pi} \frac{1}{\sqrt{1 - \rho_{34}^2}}. \]

As a consequence, when \( \rho_{34} \to 0 \), we get the expansion

\[ \mathbb{E}[Z_1Z_2 1_{[0,\infty)}(Z_3) 1_{[0,\infty)}(Z_4)] = \frac{1}{2\pi} (\rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23}) + \frac{1}{4} \rho_{12} + O(\rho_{34}^2). \]  
(11)
The next step is to compute the (conditional) covariances involved in the factorial moment $E(N_W([0, T])(N_W([0, T]) - 1))$. This is done in the following lemma, which is a direct consequence of Gaussian Regression, see [2, Proof of Prop. 4.1, p. 96] for a similar computation.

**Lemma 2 (The variance–covariance matrix).**

Set $r_1, r_2, r'_2, r''_2$ for $r_1(t), r_2(t), r'_2(t), r''_2(t)$ for short and because of the asymmetry, we keep the notation $r_{12}(t)$ and $r_{12}(-t)$. The conditional variance–covariance matrix of $(X_2(0), X'_2(t), X_1(0), X_1(t))$ given $X_2(0) = X_2(t) = 0$ has the expression

$$
\begin{pmatrix}
1 - \frac{(r'_2)^2}{1-r^2_2} & -r''_2 - \frac{r_2(r'_2)^2}{1-r^2_2} & -r'_2(0) + \frac{r'_2 r_{12}(-t)}{1-r^2_2} & -r'_2(t) - \frac{r_2 r'_2 r_{12}(t)}{1-r^2_2} \\
1 - \frac{(r'_2)^2}{1-r^2_2} & -r'_2(t) + \frac{r_2 r'_2 r_{12}(-t)}{1-r^2_2} & -r'_2(0) + \frac{r'_2 r_{12}(-t)}{1-r^2_2} & r_2 + \frac{r_2 r_{12}(t)r_{12}(-t)}{1-r^2_2} \\
1 - \frac{r'_2(-t)}{1-r^2_2} & 1 - \frac{r_2 r_{12}(-t)}{1-r^2_2} & 1 - \frac{r'_2(-t)}{1-r^2_2} & 1 - \frac{r_{12}(0)}{1-r^2_2}
\end{pmatrix}.
$$ (12)

In conclusion, we get

$$
\text{Var}(N_W([0, T]) = \mathbb{E}(N_W([0, T])(N_W([0, T]) - 1)) + \mathbb{E}(N_W([0, T]) - \mathbb{E}^2(N_W([0, T]))
$$

$$
= 2 \int_0^T (T-t) \left( \frac{\rho_{12}}{4} + \frac{\rho_{12} \arcsin(\rho_{34})}{2\pi} + \frac{\rho_{13} \rho_{24} + \rho_{14} \rho_{23}}{2\pi} \right) dt
$$

$$
\frac{1}{\sqrt{1-r^2_3}} \frac{2\pi}{\sqrt{(1-r^2_2(t))}} \frac{2\pi}{1-r^2_2(t) - \left( \frac{Tr'_{12}(0)}{2\pi} \right)^2},
$$ (13)

where the $\rho_{ij}$ are given by the entries of (12).

This gives an expression which is analogous to that given in [4, Th. 1]. We recall that since the studied quantities are not exactly equal, their variances also differ.

### 4.3 Asymptotic study of the variance

In this subsection we study the asymptotic behaviour of the variance of the random sequence $N_W([0, T])$, obtaining point 2 of Theorem [1] and the result of Theorem 2.

For the sake of readability we study first the independent model.

#### 4.3.1 Independent case

In this case, the complexity of the computations is drastically simplified. Note that this condition is assumed in [10].
**Proof of Theorem** Since (9) implies that the expectation of $N_W$ vanishes, the variance equals the second factorial moment and is given by the Kac–Rice formula (10).

$$V_T := \frac{1}{T} \text{Var}(N_W([0,T]))$$

$$= \frac{1}{\pi} \int_0^T \frac{(T-t)}{T \sqrt{1 - r_2^2(t)}} \mathbb{E}_c(X'_2(0)X'_2(t)) \mathbb{P}\{X_1(0) > 0; X_1(t) > 0\} dt.$$

By Lemma 4.3 of [2],

$$\mathbb{P}\{X_1(0) > 0; X_1(t) > 0\} = \arctan \frac{1 + r_1(t)}{1 - r_1(t)} = \arccos \frac{1 - r_1(t)}{2}.$$

Using the covariances given in Lemma 2, we get

$$V_T = -\frac{1}{\pi} \int_0^T \frac{(T-t)}{T} \frac{r''_2(t)(1 - r_2^2(t)) + r_2(t)(r'_2(t))^2}{(1 - r_2^2(t))^{3/2}} \cdot \arccos \frac{1 - r_1(t)}{2} dt.$$

We have the following identities for $t > 0$:

$$\left( \arccos \frac{1 - r_1(t)}{2} \right)' = \frac{r'_1(t)}{\sqrt{1 - r_1^2(t)}},$$

$$\frac{r''_2(t)(1 - r_2^2(t)) + r_2(t)(r'_2(t))^2}{(1 - r_2^2(t))^{3/2}} = \left( \frac{r'_2(t)}{\sqrt{1 - r_2^2(t)}} \right)'.$$

Now, we set

$$W_T := -\int_0^T \frac{r''_2(t)(1 - r_2^2(t)) + r_2(t)(r'_2(t))^2}{(1 - r_2^2(t))^{3/2}} \arccos \frac{1 - r_1(t)}{2} dt;$$

$$w_T := \int_0^T t \frac{r''_2(t)(1 - r_2^2(t)) + r_2(t)(r'_2(t))^2}{(1 - r_2^2(t))^{3/2}} \arccos \frac{1 - r_1(t)}{2} dt.$$

Thus,

$$V_T = \frac{1}{\pi} W_T + \frac{1}{2\pi T} w_T.$$

By integration by parts, we get that

$$W_T = \frac{\pi}{2} - \frac{r'_2(T)}{\sqrt{1 - r_2^2(T)}} \arccos \frac{1 - r_1(T)}{2} + \int_0^T \frac{r'_2(t)}{\sqrt{1 - r_2^2(t)}} \frac{r'_1(t)}{\sqrt{1 - r_1^2(t)}} dt.$$
where we have used that, since the second spectral moment of $X_2$ is finite, $r'_2$ exists and the Taylor expansion for $r_2$ implies that \( \frac{r'_2(t)}{\sqrt{1 - r'_2^2(t)}} \to -1 \) as $t \to 0^+$.

Let us look at the second term. Again by integration by parts, we get that

\[
   w_T = [ -tW_t ]_0^T + \int_0^T W_t dt.
\]

We are now in a position to prove that for all $T$, the variance is finite. Consider first $T$ sufficiently small that $1 - r'_2(t)$ and $1 - r'_2(T)$ are bounded away from zero. The calculation above proves that the variance $\text{Var}(N_W([0, T])) = T \cdot V_T$ is finite. For the second step, we apply the Minkowsky inequality to get that the variance $\text{Var}(N_W([0, T]))$ is finite for all $T$.

We now study the asymptotic behaviour of $V_T$ as $T \to \infty$. Under (S) and applying the Riemann–Lebesgue lemma we get that $W_T - I$ converges to $\pi/2$. Hence,

\[
   \lim_{T \to \infty} \frac{w_T}{T} = -W_\infty + W_\infty = 0,
\]

proving that

\[
   \lim_{T \to \infty} V_T = \int_0^\infty \frac{r'_2(t)}{\sqrt{1 - r'_2^2(t)}} \frac{r'_1(t)}{\sqrt{1 - r'_1^2(t)}} dt.
\]

The result follows.

4.3.2 General case

This section deals mainly with the proof of point 2 of Theorem [1]. In the proof we will use systematically the results of Lemma [2].

Writing $N_W$ for $N_W([0, T])$, we have

\[
   \frac{\text{Var}(N_W)}{T} = \frac{\mathbb{E}(N_W(N_W - 1)) + \mathbb{E}(N_W) - (\mathbb{E}(N_W))^2}{T}.
\]

Since $\mathbb{E}(N_W) = -\frac{r'_{12}(0)}{2\pi} - T$,

\[
   \frac{\text{Var}(N_W)}{T} = -\frac{r'_{12}(0)}{2\pi} + \frac{2}{(2\pi)^2} \int_0^T \frac{T - t}{T} \left( \frac{2\pi E_c}{\sqrt{1 - r'_2^2}} - (r'_{12}(0))^2 \right) dt,
\]

with $E_c := \mathbb{E}_c \left[ 1_{[0, \infty)}(X_1(0))1_{[0, \infty)}(X_1(t))X'_2(0)X'_2(t) \right]$. 

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Our next goal is to apply Lemma 1 with the law of the Gaussian vector \((Z_1, Z_2, Z_3, Z_4)\) being equal to the conditional law of \((X_2'(0), X_2'(t), X_1(0), X_1(t))\) given \(X_2(0) = X_2(t) = 0\). As we have assumed that \(r_1(t) \rightarrow 0\) as \(t \rightarrow +\infty\) and Condition (A'), we get
\[
\rho_{34} = r_1 + \frac{r_2r_{12}(t)r_{12}(-t)}{1 - r_2^2} \rightarrow 0.
\]
Hence, we can apply (11) to get the following terms:

- **\(\rho_{12}\)**. We have
\[
\rho_{12} = -r_2'' - \frac{r_2r_2}{1 - r_2^2}.
\]
A sufficient condition to ensure the finiteness of the contribution of the second term is
\[
\int |r_2|^2(r_2')^2 < +\infty.
\]
As for the first one, its contribution is (up to a multiplicative constant that plays no role)
\[
\frac{1}{T} \int_0^T ds \int_0^T - \frac{r_2''(t - s)}{\sqrt{1 - r_2^2(t - s)}} dt.
\]
By a first integration by parts we get that this quantity is equal to
\[
I_1 := \frac{1}{T} \int_0^T r_2'(-s) \frac{1}{\sqrt{1 - r_2^2(-s)}} + \frac{r_2''(T - s)}{\sqrt{1 - r_2^2(T - s)}} ds
\]
plus a term which is convergent as long as \(\int |r_2|(r_2')^2 < +\infty\). We perform now a second integration by parts to get that
\[
I_1 = \frac{1}{T} \left( \frac{-r_2'(T - s)}{\sqrt{1 - r_2^2(T - s)}} \right)_0^T + \left( \frac{-r_2'(-s)}{\sqrt{1 - r_2^2(-s)}} \right)_0^T + I_2.
\]
The first term clearly tends to zero. The integral \(I_2\) is convergent as long as \(\int r_2^2|r_2'| < +\infty\). A sufficient (simpler) condition for the convergence of the integral of \(\rho_{12}\) is
\[
\int r_2^2 + (r_2')^2 < \infty.
\]

- **\(\rho_{12}\rho_{34}\)**. We have
\[
\rho_{12}\rho_{34} = \left(-r_2'' - \frac{r_2r_2^2}{1 - r_2^2}\right) \left(r_1 + \frac{r_2r_{12}(t)r_{12}(-t)}{1 - r_2^2}\right).
\]
Integrating by parts the term involving \(r_2r_2''\) we get that it is integrable as long as
\[
\int (r_2')^2 < \infty, \quad \int |r_2''| r_1 < \infty.
\]
\( \rho_{13} \rho_{24} \). We have

\[
\rho_{13} \rho_{24} = (r'_{12}(0))^2 - 2 \frac{r'_{12}(0) r_2 r_{12}(-t)}{1 - r_2^2} + \frac{(r_2')^2 (r_{12}(-t))^2}{(1 - r_2^2)^2}.
\]

The first term is compensated for by the term \(-(r'_{12}(0))^2\) appearing in (14). Since

\[
\frac{1}{\sqrt{1 - r_2^2}} \approx 1 + \frac{r_2^2}{2},
\]

a term appears that is equivalent to

\[
\text{Const } r_2^2.
\]

A sufficient condition for the convergence of the remaining terms is

\[
\int (r_2')^2 < +\infty, \quad \int |r_2' r_{12}(-t)| < +\infty.
\]

\( \rho_{14} \rho_{23} \). We have

\[
\rho_{14} \rho_{23} = (r'_{12}(t))^2 - \frac{r_2^2 r_{12}^2(r_{12}(t))^2}{(1 - r_2^2)^2} \leq (r'_{12}(t))^2;
\]

a sufficient condition for this is

\[
\int (r_{12}'(t))^2 dt < +\infty.
\]

Gathering all these conditions together, the result (7) follows.

5 Chaos expansion of the number of winding turns

In order to prove the asymptotic normality of the standardized winding number, we need to work with Hermite polynomials: they are defined by

\[
H_n(x) = (-1)^n \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2})e^{\frac{1}{2}x^2}, \quad x \in \mathbb{R}, n \geq 0.
\]

They form a complete orthogonal system in the space \( L^2(\mathbb{R}, \phi(x)dx) \) of square integrable functions with respect to the standard Gaussian measure \( \phi(x)dx \). One of the key properties of
Hermite polynomials, known as Mehler’s formula, establishes that for a vector \((X, Y)\) of standard Gaussians with correlation \(\rho\),

\[
\mathbb{E}[H_n(X)H_m(Y)] = \delta_{n,m} n! \rho^n.
\]

We now give the Hermite expansion for \(N_W([0, T])\). The proof is similar to the analogous expansions in [11] and [17]. Recall that in [8], \(N_W([0, T])\) is written a.s. as

\[
N_W([0, T]) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^T \mathbf{1}_{[-\delta, \delta]}(X_2(t))X_2'(t) \mathbf{1}_{[0, \infty)}(X_1(t)) dt.
\]

In order to take advantage of the independence, we perform a regression of \(X_1(t)\) on \(X_2'(t)\) (note that \(X_1(t)\) and \(X_2(t)\) are independent since \(r_{12}(0) = 0\)). Thus, we write for each \(t \in [0, \infty)\)

\[
X_1(t) = \rho_1 X_2'(t) + \rho_2 Z(t),
\]

with \(\rho_1 = r_{12}'(0), \rho_2 = \sqrt{1 - \rho_1^2}\) (\(\rho_2 \neq 0\) to avoid trivialities) and \(Z(t)\) a standard Gaussian r.v. independent from \(X_2(t), X_2'(t)\). Note that if \(\rho_2 = 0\), the number of winding turns is simply equal to the number of up-crossings of \(X_2(\cdot)\). Note also that

\[
r_Z(t) = \frac{1}{\rho_2} r_1(t) - \frac{\rho_1}{\rho_2} (r_{12}'(t) - r_{12}'(\cdot)) - \frac{\rho_1^2}{\rho_2} r_{12}''(t).
\]

Set

\[
g(x', z) = x' \mathbf{1}_{[0, \infty)}(\rho_1 x' + \rho_2 z) \in L^2(\mathbb{R}^2, \phi_2(dx)),
\]

with \(\phi_2\) the standard Gaussian density in \(\mathbb{R}^2\).

Put \(k := (k_1, k_2, k_3)\) and \(|k| = \sum_i k_i\). We have

\[
N^*_W([0, T]) = \sum_{q=1}^{\infty} I_q(T);
\]

\[
I_q(T) = \sum_{|k|=q} \frac{a_{k_1} d_{k_2} d_{k_3}}{\sqrt{T}} \int_0^T H_{k_1}(X_2(t))H_{k_2}(X_2'(t))H_{k_3}(Z(t)) dt,
\]

where the coefficients \(a_{k_1}\) are the Hermite coefficients of the Dirac delta distribution, [17],

\[
a_{2k_1} = \frac{H_{2k_1}(0)}{(2k_1)! \sqrt{2\pi}} = \frac{(-1)^{k_1} (2k_1 - 1)!!}{\sqrt{2\pi} (2k_1)!} = \frac{(-1)^{k_1}}{\sqrt{2\pi} 2^{k_1} k_1!}, k_1 \geq 0;
\]
and \( a_{k_1} = 0 \) if \( k_1 \) is odd. We remark that
\[
a_{2k_1}(2k_1)! \leq \text{Const.} \tag{19}
\]
In addition, the \( d_{k_2,k_3} \) are the Hermite coefficients of \( g \), i.e.
\[
d_{k_2,k_3} = \frac{1}{k_2!k_3!} \int_{\mathbb{R}^2} g(x', z) H_{k_2}(x') H_{k_3}(z) \phi_2(x', z) dx' dz. \tag{20}
\]
Moreover, since \( \left\{ \frac{1}{\sqrt{n!}} H_n(\cdot) : n \geq 0 \right\} \) form an orthonormal basis of \( L^2(\phi(x)dx) \),
\[
||g(\cdot, \cdot)||^2 = \sum_{q=0}^{\infty} \sum_{k_2+k_3=q} d_{k_2,k_3}^2 k_2!k_3! < \infty. \tag{21}
\]

Now, we write this Hermite expansion as a Wiener Chaos expansion, that is, we write \( I_q(T) \) as a multiple stochastic integral w.r.t. a standard Brownian motion \( B = \{ B(\lambda) : \lambda \in [0, \infty) \} \).

Since condition (S) holds, for \( i = 1, 2 \), \( X_i \) has spectral density \( f_i \). Thus, we have the spectral representation
\[
X_2(t) = \int_0^\infty \cos(t \lambda) \sqrt{f_2(\lambda)} dB(\lambda); \quad X'_2(t) = -\int_0^\infty \sin(t \lambda) \lambda \sqrt{f_2(\lambda)} dB(\lambda).
\]
It is easy to get a similar representation for \( Z \) using the same Brownian motion \( B \).

For \( t \in [0, \infty) \), let
\[
Y(t) = (Y_1(t), Y_2(t), Y_3(t)) := (X_2(t), X'_2(t), Z(t)),
\]
and, for \( i = 1, 2, 3 \), let \( \varphi_{i,t} \in L^2([0, \infty)) \) be such that
\[
Y_i(t) = \int_0^\infty \varphi_{i,t}(\lambda) dB(\lambda) =: I^B_1(\varphi_{i,t}),
\]
and
\[
\mathbb{E}(I^B_1(\varphi_{i,t}) I^B_1(\varphi_{j,t}')) = \langle \varphi_{i,t}, \varphi_{j,t}' \rangle_{L^2([0,\infty))}.
\]
By the properties of Hermite polynomials and stochastic integrals,
\[
H_{k_1}(X_2(t)) H_{k_2}(X'_2(t)) H_{k_3}(Z(t)) = I^B_q \left( \varphi_{1,t}^{\otimes k_1} \otimes \varphi_{2,t}^{\otimes k_2} \otimes \varphi_{3,t}^{\otimes k_3} \right).
\]
Hence,
\[
N^*_W([0, T]) = \sum_{q=1}^{\infty} I^B_q \left( \sum_{|k|=q} \frac{a_{k_1} d_{k_2,k_3}}{\sqrt{T}} \int_0^T \varphi_{1,t}^{\otimes k_1} \otimes \varphi_{2,t}^{\otimes k_2} \otimes \varphi_{3,t}^{\otimes k_3} dt \right). \tag{22}
\]
6 Central Limit Theorem

We now prove part 3 of Theorem 1, the CLT for $N^*_W([0,T])$. We use [15, Th. 6.3.1, pp. 125–126] and [16, Th. 1]. These theorems are extensions of the so called Fourth Moment Theorem [15, Th. 5.2.7, pp. 99–100]. They provide a simple and powerful characterization of the CLT based on the chaos decomposition. We will proceed in two steps: the first one proves that the variance of $\pi^Q(N^*_W([0,T])) := \sum_{q \geq Q} I_q(T)$ is negligible when $T \to \infty$ for $Q$ large enough in this precise order; the second step establishes the asymptotic normality of $\pi^Q(N^*_W([0,T])) := \sum_{1 \leq q \leq Q} I_q(T)$.

These results allow obtaining a CLT as follows. Let $d$ be any distance that metricizes the weak convergence of probability measures. By the triangular inequality we have

$$d(N^*_W([0,T]), N(0, V_\infty)) \leq d(N^*_W([0,T]), \pi^Q(N^*_W([0,T]))) + d(\pi^Q(N^*_W([0,T])), N(0, \sigma^2_Q)) + d(N(0, \sigma^2_Q), N(0, V_\infty)),$$

(23)

where $\sigma^2_Q$ is the limit variance of $\pi^Q(N^*_W([0,T]))$ and $N(0, \gamma^2)$ denotes the law of a centred Gaussian random variable with variance $\gamma^2$. Similarly to the proof of Lemma 2 of [12] we also obtain that $\lim_{Q \to \infty} d(N(0, \sigma^2_Q), N(0, V_\infty)) = 0$. Summing up we get

$$\lim_{Q \to \infty} \sup_{T \to \infty} d(N^*_W([0,T]), N(0, V_\infty)) = 0.$$

One of the main tools of the proof is Arcones’ inequality (see [1]), which is used to show the asymptotic negligibility of the tail of the expansion. For completeness, we give here a statement of this inequality adapted to our framework. We restrict the inequality to two three-dimensional standard Gaussian random vectors $Z = (Z_1, Z_2, Z_3)$ and $W = (W_1, W_2, W_3)$. For $1 \leq j, k \leq 3$, set $\gamma_{ij} = \mathbb{E}[Z_i W_j]$ and set

$$\psi := \sup_{1 \leq i \leq 3} \sum_{j=1}^3 |\gamma_{ij}| \vee \sup_{1 \leq j \leq 3} \sum_{i=1}^3 |\gamma_{ij}|.$$

Let $F$ be a function s.t. $\mathbb{E}(F(Z)) = \mathbb{E}(F(W)) = 0$ and $\|F\|^2 := \mathbb{E}(F(Z)^2) < \infty$. We consider its expansion in the Hermite basis

$$F(x_1, x_2, x_3) = \sum_{(\sum_{i=1}^3 k_i) \geq \tau} F_{k_1, k_2, k_3} H_{k_1}(x_1) H_{k_2}(x_2) H_{k_3}(x_3).$$

The number $\tau$, the first time when the coefficients do not vanish, is the Hermite rank of function $F$. Hence, if $\psi \leq 1$, Arcones’ inequality can be written as

$$|\mathbb{E}(F(Z)F(W))| \leq \psi^\tau \mathbb{E}(F^2(Z)) = \psi^\tau \|F\|^2.$$  

(24)
Let’s recall that \(|F|^2 = \sum (\sum_{i=1}^{i=k_i} k_i)! k_1! k_2! k_3! |F_{k_1k_2k_3}^2|\).

**Step 1:** Here, it is convenient to use the expansion (17). Consider first w.l.o.g. \(T = n \in \mathbb{N}\). In fact, if \(T > 0\), write \(N_W([0,T]) = N_W([0,[T]]) + N_W((T,T-[T]))\). Clearly, for \(s \in [0,1]\), \(\text{Var}(N_W([0,s]))\) is finite and continuous w.r.t \(s\), thus \(\text{Var}(N_W([T,T-[T]])) \leq \text{Const}\). Hence, the second term does not contribute. Indeed, we have

\[
N^*([0,T]) = N^*_W([0,[T]]) \sqrt{[T]} + \frac{N_W([T,T-[T]])}{\sqrt{T}}.
\]

Then, by the above computation, it holds that \(N^*_W([0,[T]])\) → 0 in probability, consequently it is enough to prove that \(N^*_W([0,[T]])\) satisfies a Central Limit Theorem.

Let us denote by

\[
Y^Q_i = \sum_{q=Q+1}^{\infty} \sum_{\|k\|=q} a_{k_1} d_{k_2k_3} \int_i^{i+1} H_{k_1}(X_{2}(t)) H_{k_2}(X'_{2}(t)) H_{k_3}(Z(t)) dt.
\]

Then,

\[
\pi^Q(N^*_W([0,n])) = \sum_{q=Q+1}^{\infty} \sum_{\|k\|=q} a_{k_1} d_{k_2k_3} \sqrt{\frac{n}{T}} \int_0^n H_{k_1}(X_{2}(t)) H_{k_2}(X'_{2}(t)) H_{k_3}(Z(t)) dt
\]

\[
= \sum_{i=0}^{n-1} \sum_{q=Q+1}^{\infty} \sum_{\|k\|=q} a_{k_1} d_{k_2k_3} \sqrt{\frac{n}{T}} \int_i^{i+1} H_{k_1}(X_{2}(t)) H_{k_2}(X'_{2}(t)) H_{k_3}(Z(t)) dt = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} Y^Q_i.
\]

The variance of this random variable is equal to

\[
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}[Y^Q_i Y^Q_j] = \frac{1}{n} \sum_{|i-j| \leq a} \mathbb{E}[Y^Q_i Y^Q_j] + \frac{1}{n} \sum_{|i-j| > a} \mathbb{E}[Y^Q_i Y^Q_j], \tag{25}
\]

where \(a > 1\) is a constant that will be chosen later on. Using the Cauchy–Schwarz inequality we easily get

\[
\left| \frac{1}{n} \sum_{|i-j| \leq a} \mathbb{E}[Y^Q_i Y^Q_j] \right| \leq \frac{\#\{(i,j) : |i-j| \leq a\}}{n} \mathbb{E}[Y^Q_i]^2 \leq \frac{\#\{(i,j) : |i-j| \leq a\}}{n} \mathbb{E}[N_W[0,1]^2] \rightarrow 0 \text{ when } n \rightarrow \infty.
\]
The above uses the stationarity of the process and the fact that $\mathbb{E}[(Y_1^Q)^2] \leq \mathbb{E}[(Y_1^0)^2] \leq \mathbb{E}[(N_W[0,1])^2]$. Moreover

\[ \mathcal{I}_{n,Q} := \frac{1}{n} \sum_{i,j}|i-j|>a \mathbb{E}[Y_i^Q Y_j^Q] = \frac{1}{n} \sum_{q=Q+1}^{\infty} \sum_{|i-j|>a} \int_i^{i+1} \int_j^{j+1} \mathbb{E}[F_q(t)F_q(s)]dtds. \]

Here,

\[ F_q(t) = \sum_{|k|=q} a_k d_{k_2,k_3} H_{k_1}(X_2(t))H_{k_2}(X_2'(t))H_{k_3}(Z(t)). \]

Assume now that $j > i$. Then $|s-t| > a - 1$. By Arcones’ inequality (24),

\[ |\mathbb{E}[F_q(t)F_q(s)]| \leq \psi(|t-s|)||F_q||^2. \]

Here,

\[ \psi(t) = \sup_{1 \leq i \leq 3} \left\{ \sum_{j=1}^3 |\mathbb{E}(\bar{X}_i(t)\bar{X}_j(0))| \right\} \leq \text{Const } m(t), \]

where we have set $\bar{X}_1(\cdot) = X_2(\cdot)$, $\bar{X}_2(\cdot) = X_2'(\cdot)$ and $\bar{X}_3(\cdot) = Z(\cdot)$.

Finally, by using (19) and (21),

\[ ||F_q||^2 \leq \text{Const } ||g(\cdot,\cdot)||^2, \]

where $g(\cdot,\cdot)$ was defined in (16). Choose $\rho > 1$ and $a$ such that $\psi(a - 1) < \rho < 1$. Then

\[ |\mathcal{I}_{n,Q}| \leq \sum_{q=Q+1}^{\infty} \rho^{q-1}||F_q||^2 \frac{1}{n} \sum_{|i-j|>a} \int_i^{i+1} \int_j^{j+1} \psi(t-s)dtds \\
\leq 2 \sup_{q} ||F_q||^2 \frac{\rho^Q}{1-\rho} \int_0^{\infty} \psi(s)ds. \rightarrow 0 \text{ as } Q \rightarrow \infty. \]

Thus taking first limit when $n \rightarrow \infty$ and then when $Q \rightarrow \infty$ the tail of the expansion is negligible. This result implies that the weak convergence of $\pi_Q(N_{W^*}([0,T]))$ implies that of the $N_W^*([0,T])$.

**Step 2:** Theorem 1 in [16] says us that it suffices to state the convergence towards a Gaussian r.v. of each term $I_q(T)$.

We consider separately the term $q = 1$. We know from (18) that $a_1 = 0$. Now, routine computations show that $d_{0,1} = 0$, and so

\[ I_1(T) = \frac{a_0 d_{1,0}}{\sqrt{T}} \int_0^T X_2'(s)ds. \]
Thus, \( I_1(T) \) is centred Gaussian with

\[
\text{Var}(I_1(T)) = \frac{a_0^2 d_1^2}{T} \int_0^T \int_0^T \mathbb{E}(X'_2(s)X'_2(t))dsdt = \frac{a_0^2 d_1^2}{T} \int_0^T \int_0^T r_2^2(t - s)dsdt \to 0.
\]

Hence, the term \( q = 1 \) converges weakly to Dirac’s distribution \( \delta_0 \), which is a Gaussian with variance equal to zero.

Now, fix \( q \geq 2 \) and \( k \in \mathbb{N}^3 \) s.t. \(|k| = q\). Here, it is convenient to use Expansion \( 22 \). Now consider

\[
J_{q,k}(T) = I^B_q(g_{q,k,T}),
\]

\[
g_{q,k,T} = \frac{1}{\sqrt{T}} \int_0^T \varphi_{1,t}^\otimes k_1 \otimes \varphi_{2,t}^\otimes k_2 \otimes \varphi_{3,t}^\otimes k_3 dt.
\]

We also define the symmetrized kernels

\[
\tilde{g}_{q,k,T}(\lambda_1, \ldots, \lambda_q) = \frac{1}{q!} \sum_{\sigma \in S_q} g_{q,k,T}(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(q)}),
\]

where \( S_q \) is the set of permutations of \( q \) elements.

Using the Fourth Moment Theorem (see [15] and [16] again) in order to prove the asymptotic normality of \( J_{q,k}(T) \) as \( T \to \infty \), it suffices to prove that for \( n = 1, \ldots, q - 1 \), the \( L^2 \)-norm of the so-called contractions

\[
\tilde{g}_{q,k,T} \otimes_n \tilde{g}_{q,k,T}(\lambda_1, \ldots, \lambda_{2q-2n})
\]

\[
= \int_{[0,\infty)^n} \tilde{g}_{q,k,T}(\lambda_1, \ldots, \lambda_{q-n}; z_1, \ldots, z_n)\tilde{g}_{q,k,T}(\lambda_{q-n+1}, \ldots, \lambda_{2q-2n}; z_1, \ldots, z_n)dz_1 \ldots dz_n
\]

tend to 0 as \( T \to \infty \). We show this fact in the rest of this step.

For ease of notation, we rename the kernels and their arguments in the following way:

\[
g_{q,k,T} = \frac{1}{\sqrt{T}} \int_0^T \otimes_{i=1}^q \psi_{i,t} dt,
\]

where we set \( \psi_{i,t} = \varphi_{1,t} \) for \( i = 1, \ldots, k_1 \); \( \psi_{i,t} = \varphi_{2,t} \) for \( i = k_1 + 1, \ldots, k_1 + k_2 \) and \( \psi_{i,t} = \varphi_{3,t} \) for \( i = k_1 + k_2 + 1, \ldots, q \). Write also

\[
(x_1, \ldots, x_q) = (\lambda_1, \ldots, \lambda_{q-n}; z_1, \ldots, z_n); \text{ and } (y_1, \ldots, y_q) = (\lambda_{q-n+1}, \lambda_{2q-2n}; z_1, \ldots, z_n).
\]
Hence,

\[
\tilde{g}_{q,k,T} \otimes_n \tilde{g}_{q,k,T}(x_1, \ldots, x_{q-n}; y_1, \ldots, y_{q-n})
\]

\[
= \frac{1}{T(q)!^2} \sum_{\sigma, \sigma' \in S_q} \int_0^T \int_0^T \prod_{j=1}^n \psi_{\sigma^{-1}(q-n+j), t}(z_j) \psi_{\sigma'^{-1}(q-n+j), t'}(z_j)dz_j
\]

\[
= \frac{1}{T(q)!^2} \sum_{\sigma, \sigma' \in S_q} \int_0^T \int_0^T \prod_{j=1}^n \langle \psi_{\sigma^{-1}(q-n+j), t}, \psi_{\sigma'^{-1}(q-n+j), t'} \rangle_{L^2([0,\infty))}
\]

\[
= \frac{1}{T(q)!^2} \sum_{\sigma, \sigma' \in S_q} \int_0^T \int_0^T \prod_{j=1}^n \langle \psi_{\sigma^{-1}(q-n+j), t}, \psi_{\sigma'^{-1}(q-n+j), t'} \rangle_{L^2([0,\infty))}
\]

In the second equality, we used the fact that \(z_j = x_{q-n+j} = y_{q-n+j}\). By the isometric property of the stochastic integrals, each inner product in the above integral equals the covariance of the r.v.’s associated to the corresponding kernels, namely, the covariance between some of \(X_2, X_2', Z\) at \(t\) and \(t'\).

Analogously, one sees that when taking the \(L^2\)-norm of \(\tilde{g}_{q,k,T} \otimes_n \tilde{g}_{q,k,T}\) one gets the integral of the product of 2q covariances of the same r.v.’s. Hence,

\[
\|\tilde{g}_{1,k,T} \otimes_n \tilde{g}_{1,k,T}\|^2_{L^2([0,\infty))^2} \leq \frac{1}{T^2} \int_{[0,T]^4} m(t - t')^m(s - s')^n(m(t - s)^q - m(t' - s')^q)dsds'dt'dt'.
\]

Here, we bounded the absolute value of each covariance by \(m\), the function defined in condition (A).

We consider the most difficult case: \(n = 1, q - n = 1\) (\(q = 2\)), which involves the lowest powers of \(m\). The remaining cases are easier or analogous to this one. Hence,

\[
\|\tilde{g}_{1,k,T} \otimes_1 \tilde{g}_{1,k,T}\|^2_{L^2([0,\infty))^2} \leq \frac{1}{T^2} \int_{[0,T]^4} m(t - t')m(s - s')m(t - s)m(t' - s')dsds'dt'dt'.
\]

Consider the isometric change of variables \((u_1, u_2, u_3, u_4) \mapsto (t - t', s - s', t - s, t')\), thus

\[
\|\tilde{g}_{1,k,T} \otimes_1 \tilde{g}_{1,k,T}\|^2_{L^2([0,\infty))^2}
\]

\[
= \frac{\text{Const}}{T^2} \int_{[0,T]^4} m(u_1)m(u_2)m(u_3)m(u_2 - u_1 - u_3)1_{\{u_2 - u_1 - u_3 \geq 0\}}du_1du_2du_3du_4
\]

\[
\leq \frac{\text{Const}}{T} \int_{[0,T]^3} m(u_1)m(u_2)m(u_3)m(u_2 - u_1 - u_3)1_{\{u_2 - u_1 - u_3 \geq 0\}}du_1du_2du_3.
\]

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Now, since $m \in L^2$, 
\[
\int_0^\infty m(u_3)m(u_2 - u_1 - u_3)1_{\{u_2 - u_1 - u_3 \geq 0\}}du_3 \leq \|m\|_{L^2([0,\infty))}^2.
\]

Besides, we claim 
\[
\frac{1}{\sqrt{T}} \int_0^T m(u)du \to_{T \to \infty} 0.
\]

Indeed, consider $\varepsilon > 0$ and $a$ such that $\int_a^\infty m^2(u)du \leq \varepsilon^2$. Then,
\[
\frac{1}{\sqrt{T}} \int_0^T m(u)du \leq \|m\|_{\infty} \frac{a}{\sqrt{T}} + \frac{1}{\sqrt{T}} \int_a^T m(u)du \\
\leq \|m\|_{\infty} \frac{a}{\sqrt{T}} + \frac{(T-a)^{1/2}}{\sqrt{T}} \left( \int_a^\infty m^2(u)du \right)^{\frac{1}{2}} \leq \|m\|_{\infty} \frac{a}{\sqrt{T}} + \varepsilon.
\]

Letting $T \to \infty$ we get $\limsup_{t \to \infty} \frac{1}{\sqrt{T}} \int_0^T m(u)du < \varepsilon$, and the claim follows.

These bounds prove that 
\[
\|\tilde{g}_{1,k,T} \otimes_1 \tilde{g}_{1,k,T}\|_{L^2([0,\infty)^2)} \to_{T \to \infty} 0,
\]
as we claimed. This completes the proof of the CLT.

7 Examples

We present four examples. The first and the third examples can not be obtained by other techniques, because they concern non-differentiable processes. In the second example, the conditions for the CLT are very simple. In our last example, we consider a process which slightly escapes from stationarity.

7.1 Bargmann–Fock and irregular processes

Assume that $X_2$ is a Bargmann–Fock process and that $X_1$ is an Ornstein–Uhlenbeck process independent from $X_2$, namely, for $t \geq 0$:
\[
r_1(t) = \exp(-t), \quad r_{12}(t) = 0, \\
r_2(t) = \exp(-t^2/2).
\]
In this case we know from \([9]\) that \(E(N_W([0, T])) = 0\) and that the asymptotic variance is given by
\[
\lim_{T \to \infty} \frac{\text{Var}(N_W([0, T]))}{T} = \frac{1}{2} \left( \frac{\pi}{2} \right) + \int_0^\infty \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \frac{te^{-t^2/2}}{\sqrt{1 - e^{-2t}}} \, dt.
\]
The convergence of the integral at \(+\infty\) is direct. As for the convergence at zero, the equivalent of the integrand is \(t^{-1/2}\) that ensures convergence.

As a consequence, the CLT holds. Note that this example is out of reach of other methods.

To generalize this example we need a definition.

**Definition 1.** Let \(0 < \alpha < 2\). We define an \(\alpha\)-process as a stationary Gaussian process with a covariance \(\rho(t)\) that satisfies

- \(\rho(t) = 1 - Ct^\alpha + o(t^\alpha), \ t \to 0, \ t > 0;\)
- \(\rho(\cdot)\) is differentiable except at the origin and
  \[\rho'(t) = -C\alpha t^{\alpha-1} + o(t^{\alpha-1}), \ t \to 0, \ t > 0;\]
- \(\rho(t) \to 0, \ t \to +\infty.\)

An example is given by \(\rho(t) = \exp\{-t^\alpha\}\).

Note that we can replace \(X_1(\cdot)\) in the example above by any \(\alpha\)-process and \(X_2(\cdot)\) by any differentiable process that satisfies condition (G) and s.t.
\[
\int_0^{+\infty} |r_1'(t)r_2'(t)| \, dt
\]
converges.

### 7.2 Correlated processes

Let \(X_2(\cdot)\) be a process that satisfies the Geman condition (G). This implies that it is differentiable in quadratic mean. Let \(Z(t)\) be an independent stationary Gaussian process. We set
\[
X_1(t) = \rho_1X_2'(t) + \rho_2Z(t), \quad \rho_1^2 + \rho_2^2 = 1.
\]
This model is a little more restrictive than model \([15]\). Indeed, in \([15]\), not the whole process \(Z(\cdot)\) but only its point values are independent. Let \(r_Z(\cdot)\) be the correlation function of \(Z(\cdot)\). Then,
\[
\begin{align*}
r_1(t) &= -\rho_1^2 r_2''(t) + \rho_2^2 r_Z(t), \\
r_{12}(t) &= \rho_2 r_2'(t).
\end{align*}
\]
To avoid particular situations, we assume that $\rho_i \neq 0$, $i = 1, 2$. Then we see that conditions for the CLT are

$$r_2, r'_2, r''_2, r_Z \in L^2.$$  

### 7.3 Two $\alpha$-processes

In this section we consider two independent processes. The first one, $X_1(\cdot)$, is an $\alpha_1$ process (in the sense of Definition 1). The second one, $X_2(\cdot)$, is an $\alpha_2$ process. We assume that

$$\alpha_1 + \alpha_2 > 2.$$  

Our goal is to prove that the number of winding turns of $X(\cdot)$ has a finite second moment. Note that none of the two coordinates is differentiable.

Let $\psi(\cdot)$ be a compactly supported smooth enough function; let $\psi_\epsilon(t) = \frac{1}{\epsilon} \psi(t/\epsilon)$ and let $X_{2,\epsilon}(\cdot)$ be the regularization of $X_2(\cdot)$ by pathwise convolution with $\psi_\epsilon(\cdot)$. We denote by $N_{W,\epsilon}([0, T])$ the number of winding turns of $(X_1(\cdot), X_2,\epsilon(\cdot))$. We only sketch the proof.

The number of turns $N_W([0, T])$ is well defined and a.s. finite. By homotopy arguments,

$$N_W([0, T]) \leq \liminf_{\epsilon \to 0} N_{W,\epsilon}([0, T]).$$  

So we can apply Fatou’s lemma to obtain

$$\mathbb{E}(N_W^2([0, T])) \leq \liminf_{\epsilon \to 0} \frac{1}{\pi} \int_0^T \frac{T - t}{\sqrt{1 - r_2^2(\epsilon)(t)}} \mathbb{E}(X_{2,\epsilon}(0)X_{2,\epsilon}'(t)) \mathbb{P}\{X_1(0) > 0; X_1(t) > 0\} dt$$

$$=: T \cdot V_{T,\epsilon}.$$  

At this stage, we perform the integration by parts of Section 4.3.1 to obtain, with the obvious notation,

$$T \cdot V_{T,\epsilon} = \frac{T}{\pi} W_{T,\epsilon} + \frac{1}{2\pi} w_{T,\epsilon} = \frac{T}{\pi} W_{T,\epsilon} + \frac{1}{2\pi} \left( \int_0^T W_{t,\epsilon} dt - W_{T,\epsilon} \right).$$  

with

$$W_{T,\epsilon} = \left[ \frac{\pi}{2} - \frac{r_2^2(\epsilon)(T)}{\sqrt{1 - r_2^2(\epsilon)(T)}} \arccos \sqrt{\frac{1 - r_1(T)}{2}} \right] + \int_0^T \frac{r_2^2(\epsilon)(t)}{\sqrt{1 - r_2^2(\epsilon)(t)}} \frac{r_1'(t)}{\sqrt{1 - r_1^2(t)}} dt.$$  

Now, it is easy to check the convergence as $\epsilon \to 0$. Eventually, we get that
\[ \mathbb{E}(N_W^2([0, T])) \text{ is finite} \]

\[ \limsup_{T \to +\infty} \frac{1}{T} \mathbb{E}(N_W^2([0, T])) \leq \int_0^\infty \frac{r'_1(t)}{\sqrt{1 - r_1^2(t)}} \frac{r'_2(t)}{\sqrt{1 - r_2^2(t)}} dt. \]

A direct calculation shows that the integral converges as long as

\[ \int_0^\infty r'_1(t) r'_2(t) dt \text{ converges.} \]

### 7.4 Non-exactly stationary processes.

In this last example, mainly inspired by Section 4.2 of [10], we consider an extension of our ideas to a class of non-stationary Gaussian processes. Assume that \( Y(t) = (Y_1(t), Y_2(t)) \) is a Gaussian planar process with i.i.d. coordinates. In addition, assume that \( r_{Y_1}(t, s) = f(s/t) \) for \( s \leq t \) and \( f \) a real function. In certain physics models, \( f(x) = e^{-|\log x|^\alpha} \) is chosen, where \( \alpha > 1 \).

Define \( X_1(t) = Y_1(e^t) \). Thus \( X_1 \) is a stationary and centred Gaussian process with covariance function \( e^{-|t|/\alpha} \). Put \( X(t) = (X_1(t), X_2(t)) \) where \( X_2 \) is an independent copy of \( X_1 \). We only consider the case \( \alpha = 2 \) because in this case the two coordinates are differentiable. Now, using a change of scale, we have the equality in law

\[ N_W^Y([0, T]) = N_W^X([0, \log T]). \]

Then, our results imply that

\[ \lim_{T \to \infty} \frac{\text{Var}(N_W^Y([0, T]))}{\log T} = \lim_{T \to \infty} \frac{\text{Var}(N_W^X([0, \log T]))}{\log T} = \frac{1}{\pi} \left( \frac{\pi}{2} + \int_0^\infty \frac{u^2 e^{-u^2}}{1 - e^{-u^2}} du \right). \]

A CLT can also be obtained with the above expression as the limit variance. This can be expressed as in Section 4.2 of [10] as

\[ \text{Var}(N_W^Y([0, s/t])) \approx \frac{1}{\pi} \left( \frac{\pi}{2} + \int_0^\infty \frac{u^2 e^{-u^2}}{1 - e^{-u^2}} du \right) \log \left( \frac{s}{t} \right), \quad s \leq t \text{ and } s \to \infty. \]

It is possible to consider also the cases \( 1 < \alpha < 2 \), as in example 7.3, but the non-differentiability of the coordinates makes the procedure more involved.
8 Ancillary computations

8.1 Proof of Proposition 1

We use the Hermite expansion (17). Since the r.v.’s $I_q(T)$ are orthogonal for different values of $q$,

$$\text{Var}(N^*_{W, ([0, T])}) = \sum_{q=1}^{\infty} \text{Var}(I_q(T)) \geq \text{Var}(I_2(T)) + \text{Var}(I_4(T)).$$

We consider now $I_2$. From (18) and (20), we know that the only non-vanishing coefficient in $I_2(T)$ is $a_0d_{1,1} = (2\pi)^{-1}$. Hence,

$$I_2(T) = \frac{1}{2\pi \sqrt{T}} \int_0^T X'_2(t)X_1(t)dt.$$

Thus,

$$V_\infty \geq \lim_{T \to \infty} \text{Var}(I_2(T))$$

$$= \lim_{T \to \infty} \frac{1}{4\pi^2T} \int_0^T 2(T-t) \left[-r''_2(t)r_1(t) + r'_{12}(-t)r'_{12}(t)\right]dt$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left[-r''_2(t)r_1(t) + r'_{12}(-t)r'_{12}(t)\right]dt$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left[r'_2(t)r'_1(t) + r'_{12}(-t)r'_{12}(t)\right]dt.$$

Now, we use the Plancherel equality:

$$\int_{-\infty}^{\infty} \left[r'_2(t)r'_1(t) + r'_{12}(-t)r'_{12}(t)\right]dt = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \lambda^2 \left[f_1(\lambda)f_2(\lambda) + |f_{12}(\lambda)|^2\right]d\lambda.$$

Bochner’s matricial theorem [5] implies that $|f_{12}|^2 \leq f_1 f_2$. Hence, $V_\infty > 0$ as long as $f_1(\lambda)f_2(\lambda) > 0$ with positive Lebesgue measure.

Otherwise, note that if $f_1(\lambda)f_2(\lambda) = 0$ a.e., then $f_{12}(\lambda) = 0$ a.e., and thus $r_{12}(t) = 0$ for every $t \in \mathbb{R}$. We consider $I_4$. Equations (17), (18) and (20), together with some routine computations, show that $I_4(T)$ is asymptotically equivalent to

$$\frac{1}{12\pi \sqrt{T}} \int_0^T \left[H_3(X'_2(t))X'_1(t) - X'_2(t)H_3(X_1(t))\right]dt.$$
Hence, since \( r_{12}(t) = 0 \) for every \( t \),

\[
\lim_{T \to \infty} \text{Var}(I_4(T)) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[ r_1^3(t)(-r_2''(t)) + r_2^3(t)(-r_1''(t)) \right] \, dt
\]

\[
= \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[ f_1^{(*)} * (\lambda^2 f_2) + f_2^{(*)} * (\lambda^2 f_1) \right] \, d\lambda > 0.
\]

where we used the usual properties of the Fourier transform and Parseval’s identity in the last equality. This finishes the proof.

8.2 Proof of Lemma \[1\]

In the first place, we claim that

\[
\mathbb{E}[Z_1 Z_2 H_{k_3}(Z_3) H_{k_4}(Z_4)] = q! \rho_{12}\rho_{34}^q + qq! \rho_{13}\rho_{24}\rho_{34}^{q-1} + qq! \rho_{14}\rho_{23}\rho_{34}^{q-1},
\]

for \( k_3 + k_4 = 2q \), with the convention that \( \rho_{34}^{-1} = 0 \).

We use the Diagram formula, for definitions and a proof see \[19\, \text{Lem. 3.2}\]. The graphs have one vertex associated with \( Z_1 \), another vertex associated with \( Z_2 \), \( k_3 \) vertices associated with \( Z_3 \), \( k_4 \) vertices associated with \( Z_4 \), and they have \( 2 + k_3 + k_4 = 2q + 2 \) edges joining the vertices associated to different r.v.’s. For computing the expectation we (only) need to consider the following graphs. (For ease of notation we write 1, 2, 3, 4 to represent any of the vertices associated respectively with \( Z_1, Z_2, Z_3 \) and \( Z_4 \)).

- The first one consists in joining the vertex 1 \( \rightarrow 2 \) and the vertex 3 \( \rightarrow 4 \). The computation gives \( \rho_{12}\rho_{34}^q \) but there are \( q! \) ways to join 3 \( \rightarrow 4 \). Thus this graph gives as contribution \( \rho_{12}\rho_{34}^q q! \).

- The second possible type of graph consists of one line 1 \( \rightarrow 3 \), another line 2 \( \rightarrow 4 \), and the remaining lines 3 \( \rightarrow 4 \). Thus, the contribution of each array of lines is \( \rho_{13}\rho_{24}\rho_{34}^{q-1} \) and there are \( q^2(q-1)! \) of these configurations. Hence, the contribution in this case is \( \rho_{13}\rho_{24}\rho_{34}^{q-1} q^2(q-1)! = \rho_{13}\rho_{24}\rho_{34}^{q-1} qq! \). The same can be done for the third graph, given \( \rho_{14}\rho_{23}\rho_{34}^{q-1} qq! \).

Summing up these contributions and taking into account that there are no other suitable diagrams, the claim follows.

Now, if \( G \in L^2(\mathbb{R}, \phi(dx)) \), we can expand it in terms of Hermite polynomials as \( G = \sum_{k=0}^{\infty} g_k H_k(x) \). We get

\[
\mathbb{E}[Z_1 Z_2 G(Z_3) G(Z_4)] = \rho_{12} \sum_{q=0}^{\infty} g_q^2 \rho_{34}^q q! + \rho_{13}\rho_{24} \sum_{q=1}^{\infty} g_q^2 \rho_{34}^{q-1} qq! + \rho_{14}\rho_{23} \sum_{q=1}^{\infty} g_q^2 \rho_{34}^{q-1} qq!.
\]

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In our case, \( G = 1_{[0,\infty)} \) and from Slud [17] we know that
\[
\hat{g}_0 = \frac{1}{2}; \quad \hat{g}_{2k+1} = \frac{1}{\sqrt{2\pi}} \frac{H_{2k+1}(0)}{(2k+1)!} = \frac{1}{\sqrt{2\pi}} \frac{(-1)^k}{2k!k^2(2k+1)}.
\]

Thus,
\[
\mathbb{E}[Z_1 Z_2 1_{[0,\infty)}(Z_3) 1_{[0,\infty)}(Z_4)] = \frac{\rho_{12}}{4} + \frac{\rho_{12}}{2\pi} \sum_{j=0}^{\infty} \frac{(2j)!}{2^{2j}(j!)^2(2j+1)!} \frac{\rho_{34}^{2j+1}}{2\pi} + \frac{\rho_{13}\rho_{24} + \rho_{14}\rho_{23}}{2\pi} \sum_{j=0}^{\infty} \frac{(2j)!}{2^{2j}(j!)^2} \frac{\rho_{34}^{2j}}{2\pi}.
\]

This completes the proof of the lemma.

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