Reflection of Stochastic Evolution Equations in Infinite Dimensional Domains

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Abstract: In this paper, we establish the existence and uniqueness of solutions of stochastic evolution equations (SEEs) with reflection in an infinite dimensional ball. Our framework is sufficiently general to include e.g. stochastic Navier-Stokes equations.

Key Words: Stochastic evolution equations, stochastic evolution equations with reflection, random measures, Sobolev embedding.

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1 Introduction

Let $D := B(0,1)$ be the open unit ball in a separable Hilbert space $H$ endowed with an inner product denoted by $(\cdot, \cdot)$. Let $A$ be a self-adjoint, positive definite operator on the Hilbert space $H$ and let $B$ be a certain unbounded bilinear map from $H \times H$ to $H$.

In this article we consider the following stochastic evolution equations (SEEs) with reflection. To be more specific, given a filtered probability space $\mathcal{F} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, satisfying the so-called usual condition, a real-valued Brownian Motion $W = (W(t) : t \geq 0)$ defined on $\mathbb{F}$, and an element $u_0 \in D$ we will be looking for a pair $(u, L)$ which solves, in the sense that will be made precise below, the following initial value problem

\begin{align}
    du(t) + Au(t) \, dt &= f(u(t)) \, dt + B(u(t), u(t)) \, dt + \sigma(u(t)) \, dW(t) + dL(t), \quad t \geq 0, \quad (1.1) \\
    u(0) &= u_0, \quad (1.2)
\end{align}

where $u$ and $L$ are respectively $D$ and $H$-valued adapted stochastic processes, with continuous, and respectively, of locally bounded variation, see [D, §17 and Theorem III.2.1, p. 358], trajectories. We assume that the coefficients $f$ and $\sigma$ are measurable maps from $H$ to $H$ and that $B : V \times V \to V^*$ is a bilinear measurable function whose properties will be specified later. Here $V = D(A^{1/2})$, by $V^*$ we denote the dual of $V$, we identify $H^*$, the dual of $H$, with $H$ so that we have a Gelfand triple

\begin{equation}
    V \hookrightarrow H = H^* \hookrightarrow V^*. \quad (1.3)
\end{equation}

The following is the definition of a solution to Problem (1.1)-(1.2).

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Definition 1.1. A pair \((u, L)\) is said to be a solution of Problem (1.1)-(1.2) iff the following conditions are satisfied

(i) \(u\) is a \(\bar{D}\)-valued continuous and \(\mathbb{F}\)-progressively measurable stochastic process with \(u \in L^2([0,T], V)\), for any \(T > 0, \mathbb{P}\)-a.s.

(ii) the corresponding \(V\)-valued process is strongly \(\mathbb{F}\)-progressively measurable;

(iii) \(L\) is a \(H\)-valued, \(\mathbb{F}\)-progressively measurable stochastic process of paths of locally bounded variation such that \(L(0) = 0\) and

\[
\mathbb{E}[|\text{Var}_H(L)([0,T])|^2] < +\infty, \quad T \geq 0, \tag{1.4}
\]

where, for a function \(v : [0, \infty) \to H\), \(\text{Var}_H(v)([0,T])\) is the total variation of \(v\) on \([0,T]\) defined by

\[
\text{Var}_H(v)([0,T]) := \sup \sum_{i=1}^{n} |v(t_i) - v(t_{i-1})|, \tag{1.5}
\]

where the supremum is taken over all partitions \(0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T, \ n \in \mathbb{N}\), of the interval \([0,T]\);

(iii) \((u, L)\) satisfies the following integral identity in \(V^*\), for every \(t \geq 0, \mathbb{P}\)-almost surely,

\[
u(t) + \int_0^t A u(s) ds - \int_0^t f(u(s)) ds - \int_0^t B(u(s), u(s)) ds = u(0) + \int_0^t \sigma(u(s)) dW(s) + L(t); \tag{1.6}
\]

(iv) for every \(T > 0\), and \(\phi \in C([0,T], \bar{D})\), \(\mathbb{P}\)-almost surely,

\[
\int_0^T (\phi(t) - u(t), L(dt)) \geq 0. \tag{1.7}
\]

where the integral on the LHS is the Riemann-Stieltjes integral of the \(H\)-valued function \(\phi - u\) with respect to an \(H\)-valued bounded-variation function \(L\), see [BP, p. 47 in section 1.3.3].

Remark 1.2. Here we choose to have a one-dimensional Brownian motion for the simplicity of the exposition. The method in this paper works for infinite dimensional Brownian motion as well. Let us emphasize that we do not allow the so-called gradient noise because the map \(\sigma\) maps \(H\) to \(H\) and not \(V\) to \(H\). We still believe that our results are true in the latter case under some additional coercivity assumptions, see e.g. [BM] and [BD]. However, we have not verified the details.

The existence and uniqueness of solutions of real-valued stochastic partial differential equations (SPDEs) with reflection at zero driven by space-time white noise were obtained by Nualart and Pardoux in [NP] for additive noise, by Donati-Martin and Pardoux in [DP1] for general diffusion coefficient \(\sigma\) without proving the uniqueness and by T. Xu and T. Zhang in [XZh] for general \(\sigma\) with also the proof of the uniqueness. Various properties of the solution of the real-valued SPDEs with reflection were studied in [DMZ],[DP2],[HP], [Zam] and [Zh]. SPDEs with reflection can be used to model the evolution of random interfaces near a hard wall. It was proved by T. Funaki and S. Olla in [FO] that the fluctuations of a
\( \nabla \phi \) interface model near a hard wall converge in law to the stationary solution of a SPDE with reflection. For stochastic Cahn-Hilliard equations with reflection, please see [DZ].

The purpose of this paper is to establish the existence and uniqueness of the reflection problem of stochastic evolution equation on an infinite dimensional ball in a separable Hilbert space. Under this setting, we like to mention two related papers. In [BDaPT-1] the authors considered a reflection problem for 2D stochastic Navier-Stokes equations with periodical boundary conditions in an infinite dimensional ball. The problem is formulated as a stochastic variational inequality, Galerkin approximations and Kolmogorov equations are used. A stochastic reflection problem was considered in [BDaPT] for stochastic evolution equations driven by additive noise on a closed convex subset in a Hilbert space. The solution is defined as a solution to a control problem. The approach is very much analytical.

In this paper, we consider the reflection problem for general stochastic evolution equation with mutiplicative noise on an infinite dimensional ball in a Hilbert space. The approach is stochastic and direct. We will solve the reflection problem using approximations of penalized stochastic evolution equations. To prove the convergence of the solutions of the approximating equations, due to the lack of the comparison theorems, we need to obtain a number of good estimates for the solutions of the penalized equations. Our approach is inspired by the work in [HP]. Our framework is quite general, which includes also stochastic Navier-Stokes equations.

The rest of the paper is organised as follows. In Section 2, we describe the setup and state the precise assumptions on the coefficients. In Section 3, we consider approximating penalized stochastic evolution equations and obtain a number of estimates for the solution which are used later. Section 4 is devoted to the statement and proof of our main result, i.e. Theorem 4.1, about the existence and uniqueness of the stochastic reflection problem. In Section 5 we discuss a concrete example of damped Navier-Stokes Equations in the whole Euclidean domain \( \mathbb{R}^2 \).

## 2 Framework

Now we introduce the setup of the paper and the assumptions. Let \( H \) be a separable Hilbert space with the norm \( \| \cdot \|_H \) or simply \( \| \cdot \| \) and the inner product \( \langle \cdot , \cdot \rangle \). Let \( A \) be a self-adjoint, positive definite operator on the Hilbert space \( H \) such that there exists \( \lambda_1 > 0 \) such that

\[
(Au, u) \geq \lambda_1 |u|^2, \quad u \in D(A). \tag{2.1}
\]

Set \( V := D(A^{1/2}) \), the domain of the operator \( A^{1/2} \). Then \( V \) is a Hilbert space with the inner product

\[
\langle (u, v) \rangle = (A^{1/2}u, A^{1/2}v), \tag{2.2}
\]

and the norm \( \| \cdot \| \). \( V^* \) denotes the dual space of \( V \). We also use \( \langle \cdot , \cdot \rangle_{V^*} \) to denote the duality between \( V \) and \( V^* \).

We consider two measurable maps

\[
f : H \to V^* \tag{2.3}
\]
\[
\sigma : H \to H \tag{2.4}
\]
a bilinear map

\[ B : V \times V \to V^* \]  

and the corresponding trilinear form \( \bar{b} : \times V \times V \to R \) defined by

\[ \bar{b}(u, v, w) = _V \langle B(u, v), w \rangle_V , \quad u, v, w \in V. \]  

We introduce the following assumptions\(^1\) that we will be using in the paper.

(A.1) There exists a constant \( C \) such that

\[ |f(u) - f(v)|_{V^*} + |\sigma(u) - \sigma(v)|_H \leq C |u - v|_H , \quad \text{for all } u, v \in H. \]  

(A.2) The form \( \bar{b} \) satisfies the following conditions.

a) For all \( u, w, v \in V \),

\[ _V \langle B(u, v), w \rangle_V = \bar{b}(u, v, w) = -\bar{b}(u, w, v) = -_V \langle B(u, w), v \rangle_V. \]

b) For all \( u, w, v \in V \),

\[ |_V \langle B(u, v), w \rangle_V | \leq 2\|u\|_H^{\frac{1}{2}}|u|_H^{\frac{1}{2}}\|w\|_H^{\frac{1}{2}}|w|_H^{\frac{1}{2}}\|v\|. \]

Let us observe that the constant 2 could be replaced by any positive constant.

(A.3) Given is a \( \bar{D} \)-valued \( \mathcal{F}_0 \)-measurable random variable \( u_0 \); here \( \bar{D} \) is the closed unit ball in the Hilbert space \( H \).

Let us note that Assumption (A.2) particularly implies that

\[ \bar{b}(u, v, v) = 0 \quad \text{i.e.} \quad _V \langle B(u, v), v \rangle_V = 0 , \quad u, v \in V \]  

and

\[ \|B(u, u)\|_{V^*} \leq 2\|u\|_H , \quad u \in V. \]

Throughout the paper we assume that \( \mathfrak{P} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \) is a filtered probability space with filtration \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \), satisfying the usual hypothesis. We also assume that \( W = (W(t))_{t \geq 0} \) is an \( \mathbb{R} \)-valued Wiener process on \( \mathfrak{P} \). It should not be difficult to extend the results to a cylindrical Wiener process on some separable Hilbert space \( K \) provided that \( \sigma \) is a map \( H \to \gamma(K, H) \) such that the corresponding part of assumption (2.7) is replaced by

\[ \|\sigma(u) - \sigma(v)\|_{\gamma(K, H)} \leq C |u - v|_H , \quad \text{for all } u, v \in H \]  

and for some \( C > 0. \) \hspace{1cm} (2.12)

Here, by \( \gamma(K, H) \) we denote the space of all Hilbert-Schmidt operators from \( K \) to \( H \) and by \( \| \cdot \|_{\gamma(K, H)} \) we denote the corresponding Hilbert-Schmidt norm.

For a given \( T > 0 \) by \( X_T \) we will denote the Banach space

\[ X_T = C([0, T]; H) \cap L^2([0, T]; V) \]

\(^1\)One can easily weaken the assumptions of the vector field \( f \) as below. We will not dwell upon this in the current paper. \textit{There exists } \alpha \in [0, 1) \text{ such that } f : D(A^{\alpha}) \to V^* \text{ is globally Lipschitz.}
endowed with the natural norm:

$$\|u\|_{X_T}^2 := \sup_{t \in [0, T]} |u(t)|_H^2 + \int_0^T \|u(s)\|^2 ds.$$  

By $\mathcal{M}^2(0, T)$ we will denote the space of all $H$-valued continuous $\mathbb{F}$-progressively measurable processes $u$ such that $u$ has a $V$-valued $\mathcal{L}[0, T] \otimes \mathbb{F}$-measurable version $\tilde{u}$, where by $\mathcal{L}[0, T]$ we denote the $\sigma$-field of Lebesgue measurable sets on the interval $[0, T]$, endowerto with the following norm

$$\|u\|_{\mathcal{M}^2(0, T)}^2 := \mathbb{E} \left[ \sup_{t \in [0, T]} |u(t)|_H^2 + \int_0^T \|\tilde{u}(s)\|^2 ds \right].$$ (2.13)

It is known that $\mathcal{M}^2(0, T)$ is a separable Banach space.

Let us observe that in a non-rigorous way

$$\mathcal{M}^2(0, T) = L^2(\Omega; \mathcal{L}[0, T] \otimes \mathbb{F}; \text{Leb} \otimes \mathbb{P}; X_T).$$

Throughout the paper, $C$ will denote a generic constant whose value may be different from line to line.

3 The existence and the uniqueness of solutions to an approximated problem

Let us recall that $D := B(0, 1)$ is the open unit ball in the separable Hilbert space $H$. We endow it with the metric inherited from $H$. Let us introduce a function $\pi : H \to \bar{D}$, called the projection onto $\bar{D}$, defined by, for $y \in H$, by

$$\pi(y) = \begin{cases} 
  y, & \text{if } |y| \leq 1, \\
  \frac{y}{|y|}, & \text{if } |y| > 1.
\end{cases}$$ (3.1)

Note that $\pi(y)$ and $y - \pi(y)$ have always the same direction as $y$. In particular,

$$y - \pi(y) = \lambda(|y|)y, \quad y \in H,$$ (3.2)

where the function $\lambda : [0, \infty) \to [0, 1]$ is given by

$$\lambda(r) = \begin{cases} 
  0, & \text{if } r \in [0, 1], \\
  1 - \frac{1}{r}, & \text{if } r > 1.
\end{cases}$$

**Remark 3.1.** It seems important to observe that $\pi - I$ (here $I$ stands for the identity) can be seen as the minus gradient of the function $\phi$ defined by the following formula:

$$\phi(y) = \frac{1}{2} \|\text{dist}(u, D)\|^2 = \begin{cases} 
  0, & \text{if } |y| \leq 1, \\
  \frac{1}{2}(|y| - 1)^2, & \text{if } |y| > 1.
\end{cases}$$ (3.3)

In other words,

$$\pi(x) - x = -\nabla \phi(x), \quad x \in H.$$ (3.4)

Strictly speaking, identity (3.4) is not true as the function $\phi$ is not differentiable for $x \in \mathbb{S} := \{x \in H : |x| = 1\}$.

Other choices of $\phi$ are possible leading of course to modifying the vector field $\pi$. 

5
The following Lemma states some straightforward properties of the projection map $\pi$ that will be used later.

**Lemma 3.2.** The function $\pi$ defined in (3.1) has the following properties.

(i) The map $\pi$ is globally Lipschitz in the sense that

$$|\pi(x) - \pi(y)| \leq 2|x - y|, \quad x, y \in H.$$  \hfill (3.5)

(ii) For all $x \in H$,

$$\langle \pi(x), x - \pi(x) \rangle = |x - \pi(x)|,$$

$$\langle x, x - \pi(x) \rangle = |x||x - \pi(x)|.$$  \hfill (3.6)

(iii) For $x \in H$ and $y \in \bar{D}$,

$$\langle x - y, x - \pi(x) \rangle \geq 0.$$  \hfill (3.7)

For every $n \in \mathbb{N}$, consider the following penalized stochastic evolution equation:

$$u^n(t) = u^n_0 - \int_0^t A\,u^n(s)\,ds + \int_0^t \sigma(u^n(s))\,dW(s)$$

$$+ \int_0^t f(u^n(s))\,ds + \int_0^t B(u^n(s), u^n(s))\,ds$$

$$- n \int_0^t (u^n(s) - \pi(u^n(s)))\,ds.$$  \hfill (3.8)

Using the function $\phi$ from Remark 3.1, the above equation can be written in the following differential form

$$du^n(t) + Au^n(t)\,dt = \sigma(u^n(t))\,dW(t)$$

$$+ [f(u^n(t)) + B(u^n(t), u^n(t)) - n\nabla \phi(u^n(t))]\,dt;$$

$$u^n(0) = u^n_0.$$  \hfill (3.10)

There exists a unique solution $u^n$ in the framework of the Gelfand triple (1.3). This can be proved along the same lines as the proof of the existence and uniqueness of solutions of stochastic Navier-Stokes equations. For the sake of completeness let us formulate the result whose proof can be traced back to many papers, see [BM, Theorem 5.1 and Corollary 7.7] and [CM] and references therein.

**Theorem 3.3.** Let us assume that assumptions (A.1)-(A.3) are satisfied. Then there exists a unique solution $u^n$ of problem (3.8) such that for $\mathbb{P}$-almost all $\omega \in \Omega$ the trajectory $u^n(\cdot, \omega)$ is equal almost everywhere to a continuous $H$-valued function defined on $[0, \infty)$ and, for every $T > 0$,

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |u^n(t)|^2_H + \int_0^T \|u^n(t)\|^2 \,dt \right] < \infty.$$  \hfill (3.11)

Moreover, this unique solution satisfies the following inequalities. For all $p \geq 2$, $T > 0$ and $n \in \mathbb{N}$, there exist positive constants $C_n(p, T)$ and $\bar{C}_n(p, T)$ such that

$$\mathbb{E} \left( \sup_{0 \leq s \leq T} |u^n(s)|^p_H \right) \leq C_n(p, T).$$  \hfill (3.12)
and

\[ \mathbb{E} \left[ \int_0^T |u^n(s)|_{H}^{2p-2} \|u^n(s)\|^2 \, ds \right] \leq \tilde{C}_n(p, T). \quad (3.13) \]

Finally, this unique solution satisfies, \( \mathbb{P} \)-almost surely, the following inequality,

\[ \langle u^n(t), u^n(t) - \pi(u^n(t)) \rangle \geq 0, \ t \geq 0. \quad (3.14) \]

Let us recall that property (3.14) follows from (3.7) with \( y = 0 \).

In Lemma 3.4 we will strengthen the assertion (3.12) from Theorem 3.3 by showing, in particular, that for every \( T > 0 \), \( \sup_n C_n(4, T) < \infty \). For that aim we will use the fact that \( C_n(8, T) < \infty \) for every \( n \in \mathbb{N} \).

In the next section we will show that the limit of \( u^n \), as \( n \to \infty \), exists and is a solution of SEE with reflection (1.1).

We begin with a number of estimates for the sequence \((u^n)_{n \geq 1}\).

**Lemma 3.4.** For every \( T > 0 \) there exist positive constants \( K_0 = K_0(T) \) and \( K_1 = K_1(T) \) such that the following estimates hold:

\[ \sup_n \mathbb{E} \left[ \sup_{t \in [0, T]} |u^n(t)|_{H}^4 \right] \leq K_0, \quad (3.15) \]

\[ \mathbb{E} \left[ \int_0^T |u^n(t)|_{H}^2 \langle u^n(t), u^n(t) - \pi(u^n(t)) \rangle \, dt \right] \leq \frac{K_1}{n}, \ n \in \mathbb{N}. \quad (3.16) \]

**Remark 3.5.** Let us recall that \( \pi(u^n(t)) \) and \( u^n(t) - \pi(u^n(t)) \) have the same directions. Thus, \( \mathbb{P} \)-almost surely, for all \( t \geq 0 \),

\[ \langle u^n(t), u^n(t) - \pi(u^n(t)) \rangle = \langle u^n(t) - \pi(u^n(t)), u^n(t) - \pi(u^n(t)) \rangle \]

\[ + \langle \pi(u^n(t)), u^n(t) - \pi(u^n(t)) \rangle \geq |u^n(t) - \pi(u^n(t))|_{H}^2. \]

Therefore, the a priori estimate (3.16) implies the following inequality which will be used later in the proof of Lemma 3.8. There exists a constant \( K_1 = K_1(T) > 0 \) such that

\[ \mathbb{E} \left[ \int_0^T |u^n(t)|_{H}^2 |u^n(t) - \pi(u^n(t))|_{H}^2 \, dt \right] \leq \frac{K_1}{n}, \ n \in \mathbb{N}. \quad (3.17) \]

The proof of inequality (3.15) is similar to the proof of the estimates (5.4) in Lemma 5.3 from [BM, Appendix A].

**Proof of Lemma 3.4.** Let us choose and fix \( T > 0 \). Let us fix \( n \in \mathbb{N} \) and \( u_0 \in H \). Let \( \psi(z) = |z|_{H}^4, z \in H \). It is easy to see that \( \psi \) is of \( C^2 \) class and

\[ \psi'(z) = 4|z|_{H}^3 z \] and \( \psi''(z) = 8z \otimes z + 4|z|_{H}^2 I_{H}, z \in H, \]

where \( I_{H} \) stands for the identity operator on \( H \). Applying the Itô formula from [P] and assumption (2.10) we have

\[ |u^n(t)|_{H}^4 = |u^n(0)|_{H}^4 - 4 \int_0^t |u^n(s)|_{H}^2 \langle u^n(s), Au^n(s) \rangle \, ds \]

\[ + 4 \int_0^t |u^n(s)|_{H}^2 \langle u^n(s), \sigma(u^n(s)) \rangle \, dW(s) + 4 \int_0^t |u^n(s)|_{H}^2 \langle u^n(s), f(u^n(s)) \rangle \, ds \]

\[ - 4n \int_0^t |u^n(s)|_{H}^2 \langle u^n(s), u^n(s) - \pi(u^n(s)) \rangle \, ds + 4 \int_0^t \langle u^n(s), \sigma(u^n(s)) \rangle^2 \, ds \]

\[ + 2 \int_0^t |u^n(s)|_{H}^2 \langle \sigma(u^n(s)), \sigma(u^n(s)) \rangle \, ds. \quad (3.18) \]
In the first instance, we do not know whether the process \( M \) defined by

\[
M(t) := \int_0^t |u^n(s)|^2_H \langle u^n(s), \sigma(u^n(s)) \rangle dW(s), \quad t \geq 0,
\]

is a martingale. Fortunately, this is the case as it follows from inequality (3.13).

Observe that by (3.14) we have \( \langle u^n(s), u^n(s) - \pi(u^n(s)) \rangle \geq 0 \) for all \( s \in [0,T] \). We also note that

\[
4 \int_0^t |u^n(s)|^2_H \langle u^n(s), Au^n(s) \rangle ds \geq 0.
\]

As the function \( f : H \rightarrow V^* \) is of linear growth, there exists a constant \( C \) such that

\[
| \langle f(u), u \rangle | \leq C \| u \|_H (1 + \| u \|_H), \quad u \in V.
\]  

By (3.19) we have

\[
\langle u^n(s), u^n(s) \rangle \leq \| u^n(s) \|_H^2 (1 + \| u^n(s) \|_H^2)
\]

so that

\[
\| u^n(s) \|_H^2 \langle u^n(s), f(u^n(s)) \rangle \leq C \| u^n(s) \|_H (1 + \| u^n(s) \|_H) \| u^n(s) \|_H^2
\]

Rearranging terms and taking the expectation in (3.18) yield

\[
\mathbb{E}[ \sup_{0 \leq r \leq t} |u^n(r)|^4_H ] + 4n \mathbb{E} \left[ \int_0^t |u^n(s)|^2_H \langle u^n(s), u^n(s) - \pi(u^n(s)) \rangle dW(s) \right]
\]

\[
\leq |u^n(0)|^4_H + 4 \mathbb{E} \left[ \sup_{0 \leq r \leq t} |u^n(s)|^2_H \langle u^n(s), (u^n(s)) \rangle dW(s) \right]
\]

\[
+ 4 \mathbb{E} \left[ \int_0^t |u^n(s)|^2_H \langle u^n(s), f(u^n(s)) \rangle |dW(s)| + 4 \mathbb{E} \left[ \int_0^t \langle u^n(s), \sigma(u^n(s)) \rangle^2 ds \right]
\]

\[
+ 2 \mathbb{E} \left[ \int_0^t |u^n(s)|^2_H \langle \sigma(u^n(s)), \sigma(u^n(s)) \rangle ds \right].
\]  

By the Burkholder inequality,

\[
\mathbb{E} \left[ \sup_{0 \leq r \leq t} \left| \int_0^r |u^n(s)|^2_H \langle u^n(s), \sigma(u^n(s)) \rangle dW(s) \right| \right]
\]

\[
\leq C_1 \mathbb{E} \left[ \left( \int_0^t |u^n(s)|^4_H \langle u^n(s), \sigma(u^n(s)) \rangle^2 ds \right)^{\frac{1}{2}} \right]
\]

\[
\leq C_1 \mathbb{E} \left[ \left( \sup_{0 \leq r \leq t} |u^n(r)|^4_H \right) \left( \int_0^t \langle u^n(s), \sigma(u^n(s)) \rangle^2 ds \right)^{\frac{1}{2}} \right]
\]

\[
\leq \frac{1}{2} \mathbb{E} \left[ \left( \sup_{0 \leq r \leq t} |u^n(r)|^4_H \right) \right] + C_2 \mathbb{E} \left[ \int_0^t \langle u^n(s), \sigma(u^n(s)) \rangle^2 ds \right].
\]  

By the linear growth of \( f, \sigma, \) and substituting (3.22) into (3.21) we obtain that

\[
\mathbb{E} \left[ \sup_{0 \leq r \leq t} |u^n(r)|^4_H \right] + 4n \mathbb{E} \left[ \int_0^t |u^n(s)|^2_H \langle u^n(s), u^n(s) - \pi(u^n(s)) \rangle dW(s) \right]
\]

\[
\leq C |u^n(0)|^4_H + C \mathbb{E} \left[ \int_0^t (1 + |u^n(s)|^2_H) ds \right].
\]  

\[\text{Page 8}\]
By applying the Gronwall Lemma the above inequality implies inequality (3.15). Finally, combination of inequalities (3.23) (3.15) implies inequality (3.16). Hence, the proof of Lemma 3.4 is complete.

Lemma 3.6. For every $T > 0$ there exist constants $M_1 = M_1(T)$ and $M_2 = M_2(T)$ such that
\[
\sup_n \mathbb{E} \left[ \left( n \int_0^T |u^n(s) - \pi(u^n(s))| ds \right)^2 \right] \leq M_1,
\]
and
\[
\sup_n \mathbb{E} \left[ \int_0^T \|u^n(s)\|^2 ds \right] \leq M_2.
\]

Proof of Lemma 3.6. Let us choose and fix $T > 0$. By the Itô formula, see [P], we have
\[
|u^n(t)|^2_H = |u(0)|^2_H - 2 \int_0^t \langle u^n(s), Au^n(s) \rangle ds
+ 2 \int_0^t \langle u^n(s), \sigma(u^n(s)) \rangle dW(s) + 2 \int_0^t \langle u^n(s), f(u^n(s)) \rangle ds
- 2n \int_0^t \langle u^n(s), u^n(s) - \pi(u^n(s)) \rangle ds + \int_0^t \|\sigma(u^n(s))\|^2_H ds.
\]
Note that $\langle u^n(s), Au^n(s) \rangle = \|u^n(s)\|^2$, $s \in [0, T]$ and
\[
|\sigma(u^n(s))|_H \leq C(1 + |u^n(s)|_H), \quad s \in [0, T].
\]
By Young’s inequality and (3.19) we have
\[
2 \int_0^t \langle u^n(s), f(u^n(s)) \rangle ds \leq \int_0^t \|u^n(s)\|^2 ds + C \int_0^t (1 + |u^n(s)|_H^2) ds
\]
\[
= \int_0^t \langle u^n(s), Au^n(s) \rangle ds + C \int_0^t (1 + |u^n(s)|_H^2) ds.
\]
In view of property (ii) in Lemma 3.2,
\[
2n \int_0^t \langle u^n(s), u^n(s) - \pi(u^n(s)) \rangle ds
\]
\[
= 2n \int_0^t |u^n(s) - \pi(u^n(s))|^2_H ds + 2n \int_0^t \langle \pi(u^n(s)), u^n(s) - \pi(u^n(s)) \rangle ds
\]
\[
= 2n \int_0^t |u^n(s) - \pi(u^n(s))|^2_H ds + 2n \int_0^t |u^n(s) - \pi(u^n(s))|_H ds, \quad t \in [0, T].
\]
By Lemma 3.4, inequalities (3.26), (3.27), (3.28) and the Burkholder inequality we arrive at
\[
\sup_n \mathbb{E} \left[ \left( n \int_0^T |u^n(s) - \pi(u^n(s))| ds \right)^2 \right] \leq C + C \sup_n \mathbb{E} \left[ \sup_{t \in [0, T]} |u^n(t)|^2_H \right] \leq M_1(T)
\]
and
\[
\sup_n \mathbb{E} \left[ \int_0^T \|u^n(s)\|^2 ds \right] \leq M_2(T).
\]
Hence the proof of Lemma 3.6 is complete.
In the proof of the next Lemma 3.8 we will use two auxiliary functions. Let us list them for the convenience of the reader in a separate item.

**Lemma 3.7.** Let us define two functions $G, g : H \to [0, \infty)$ by, for $y \in H$,

$$G(y) = d(y, \bar{D})^4, \quad g(y) = d(y, \bar{D})^2.$$

Then for all $y, v, h \in H$, the following identities hold

$$\nabla G(y) = 4 g(y) (y - \pi(y))$$

$$g(y) |y - \pi(y)|^2 = G(y), \quad y \in H,$$  \hfill (3.31)

$$g(y) \leq |y - \pi(y)|^2_H, \quad y \in H,$$  \hfill (3.32)

$$G'(y)(v) = 4g(y) \langle y - \pi(y), v \rangle,$$  \hfill (3.33)

$$G''(y)(h, v) = 8 \langle y - \pi(y), h \rangle \langle y - \pi(y), v \rangle + 4g(y) |y|_{H}^2 \left[ \langle h, v \rangle (1 - \frac{1}{|y|_H^2}) + \frac{1}{|y|_H^2} \langle h, y \rangle \langle y, v \rangle \right].$$  \hfill (3.34)

**Proof.** This Lemma follows by simple calculations by using the following formula:

$$g(y) = |y - \pi(y)|^2 = \begin{cases} 0, & \text{if } y \in H, |y| < 1, \\ |y - \frac{y}{|y|}|^2 = (|y| - 1)^2, & \text{if } y \in H, |y| \geq 1. \end{cases} \hfill (3.35)$$

\[\square\]

**Lemma 3.8.** In the above framework, the following holds for every $T > 0$

$$\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |u^n(t) - \pi(u^n(t))|^4 \right] = 0.$$  \hfill (3.36)

**Proof of Lemma 3.8.** The proof of this Lemma is somehow similar to a proof of a Burkholder inequality for stochastic evolution equations, see for instance [BP],

Let us choose and fix $T > 0$. Let $G$ and $g$ be the functions from Lemma 3.7. Applying
the Itô formula, see [P], since $G(u^n(0)) = 0$ we have

$$G(u^n(t)) = -4 \int_0^t g(u^n(s)) \langle u^n(s) - \pi(u^n(s)), Au^n(s) \rangle \, ds$$

$$+ 4 \int_0^t g(u^n(s)) \langle u^n(s) - \pi(u^n(s)), \sigma(u^n(s)) \rangle \, dW(s)$$

$$+ 4 \int_0^t g(u^n(s)) \langle u^n(s) - \pi(u^n(s)), f(u^n(s)) \rangle \, ds$$

$$+ 4 \int_0^t g(u^n(s)) \langle u^n(s) - \pi(u^n(s)), B(u^n(s), u^n(s)) \rangle \, ds$$

$$- 4n \int_0^t g(u^n(s)) |u^n(s) - \pi(u^n(s))|^2_H \, ds$$

$$+ 4 \int_0^t \langle u^n(s) - \pi(u^n(s)), \sigma(u^n(s)) \rangle^2 ds$$

$$+ 2 \int_0^t g(u^n(s)) 1_{|u^n(s)| > 1} \left[ \frac{1}{|u^n(s)|^2} \left( 1 - \frac{1}{|u^n(s)|_H} \right) \right.$$

$$+ \left. \frac{1}{|u^n(s)|^2} \langle u^n(s), \sigma(u^n(s)) \rangle^2 \right] \, ds$$

$$:= I^n_1(t) + I^n_2(t) + I^n_3(t) + I^n_4(t) + I^n_5(t) + I^n_6(t) + I^n_7(t), \quad t \in [0,T]. \quad (3.37)$$

We now look at each of the seven terms separately. Clearly $I^n_5(t) \leq 0$, for $t \in [0,T]$. For the term $I^n_1$ we have, for $t \in [0,T]$,

$$- \frac{1}{4} I^n_1(t) = \int_0^t g(u^n(s)) \langle u^n(s) - \pi(u^n(s)), Au^n(s) \rangle \, ds \quad (3.38)$$

$$= \int_0^t g(u^n(s)) \lambda(|u^n(s)|_H) \langle u^n(s), Au^n(s) \rangle \, ds$$

$$= \int_0^t g(u^n(s)) \lambda(|u^n(s)|_H) \|u^n(s)\|^2 \, ds \geq 0.$$

For the term $I^n_4$, by assumption (2.10) we have, for $t \in [0,T]$,

$$I^n_4(t) = 4 \int_0^t g(u^n(s)) \langle u^n(s) - \pi(u^n(s)), B(u^n(s), u^n(s)) \rangle \, ds$$

$$= 4 \int_0^t g(u^n(s)) \lambda(|u^n(s)|_H) \langle u^n(s), B(u^n(s), u^n(s)) \rangle \, ds = 0. \quad (3.39)$$
By the Burkholder inequality and inequality (3.31) we infer that for \( t \in [0, T] \),

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |I_2^n(s)| \right] \leq C \mathbb{E} \left[ \int_0^t g(u^n(s))^2 (u^n(s) - \pi(u^n(s)), \sigma(u^n(s)))^2 ds \right] \tag{3.40}
\]

\[
\leq C \mathbb{E} \left[ \int_0^t g(u^n(s))^2 (u^n(s) - \pi(u^n(s)))^2 g(u^n(s)) \sigma(u^n(s))^2 ds \right] \leq C \mathbb{E} \left[ \sup_{0 \leq s \leq t} g(u^n(s)) \right]^{\frac{1}{2}} \left( \int_0^t (u^n(s) - \pi(u^n(s)), \sigma(u^n(s)))^2 g(u^n(s)) \sigma(u^n(s))^2 ds \right) \leq \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq s \leq t} g(u^n(s)) \right] + C_4 \mathbb{E} \left[ \int_0^t (u^n(s) - \pi(u^n(s)), Au^n(s)) ds \right]
\]

By Young’s inequality and (3.19) we infer that

\[
I_3^n = 4 \int_0^t g(u^n(s)) \lambda(|u^n(s)|) \langle u^n(s), f(u^n(s)) \rangle ds \leq 2 \int_0^t g(u^n(s)) \lambda(|u^n(s)|) \|u^n(s)\|^2 ds + C \int_0^t g(u^n(s)) \lambda(|u^n(s)|) (1 + |u^n(s)|^2) ds \leq 2 \int_0^t g(u^n(s)) (u^n(s) - \pi(u^n(s)), Au^n(s)) ds + C \int_0^t g(u^n(s)) (1 + |u^n(s)|^2) ds.
\tag{3.41}
\]

Let us now observe that by (3.32) we deduce that

\[
I_3^n(t) \leq 2 \int_0^t g(u^n(s)) (u^n(s) - \pi(u^n(s)), Au^n(s)) ds + C \int_0^t |u^n(s) - \pi(u^n(s))|^2 (1 + |u^n(s)|^2) ds.
\tag{3.42}
\]

Using the linear growth of \( \sigma \) we similarly have

\[
I_6^n(t) \leq \int_0^t |u^n(s) - \pi(u^n(s))|^2 (1 + |u^n(s)|^2) ds,
\tag{3.43}
\]

and

\[
I_7^n(t) \leq \int_0^t |u^n(s) - \pi(u^n(s))|^2 (1 + |u^n(s)|^2) ds.
\tag{3.44}
\]

Hence, from some of the above estimates we infer that, for \( t \in [0, T] \),

\[
\sum_{j \neq 2} I_j^n(t) \leq C \int_0^t |u^n(s) - \pi(u^n(s))|^2 (1 + |u^n(s)|^2) ds.
\tag{3.45}
\]

Combining the above inequality with inequality (3.40) we deduce that, for \( t \in [0, T] \),

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} G(u^n(s)) \right] \leq \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq s \leq t} G(u^n(s)) \right] + C_3 \mathbb{E} \left[ \int_0^t (1 + |u^n(s)|^2) |u^n(s) - \pi(u^n(s))|^2 ds \right].
\]

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Thus, by exploiting the constant $\frac{1}{2}$ on the RHS above we infer that
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} G(u^n(s)) \right] \leq 2C_3 \mathbb{E} \left[ \int_0^t (1 + |u^n(s)|_H^2)|u^n(s) - \pi(u^n(s))|_H^2 \, ds \right].
\]

Thus, by (3.17), we infer that as $n \to \infty$,
\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} |u^n(t) - \pi(u^n(t))|_H^4 \right] = \lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} G(u^n(t)) \right] = 0. \tag{3.46}
\]

This concludes the proof of Lemma 3.8. \hfill \square

4 The existence and the uniqueness of solutions to the reflected problem

The aim of this section is to formulate and prove the following main result of the paper.

**Theorem 4.1.** Let us assume that assumptions (A.1)-(A.3) are satisfied. The reflected stochastic evolution equation (1.1) admits a unique solution $(u, L)$ that satisfies, for $T > 0$,
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |u(t)|_H^2 + \int_0^T \|u(t)\|^2 \, dt \right] < \infty. \tag{4.1}
\]

**Remark 4.2.** Let us recall, see [D, §17 and Theorem III.2.1, p. 358], that if a function $L : [0, \infty) \to H$ is of locally bounded variation, then there exists a unique function $\mu : \bigcup_{t>0} \mathcal{B}([0, t]) \to H$ such that $\mu((s, t]) = L(t+) - L(s+)$. This measure is denoted by $dL(t)$.

**Proof of Theorem 4.1.** Recall that $u_0 \in \bar{D}$.

Let us observe that without a loss of generality we can fix $T > 0$ for the remainder of the proof.

For $\lambda > 0$ and $n \in \mathbb{N}$ let us define a process $f_n$ by the following formula.
\[
f_n(t) = \exp(-\lambda \int_0^t \|u^n(s)\|^2 \, ds), \quad t \geq 0.
\]

Our proof here has some common features with the proof of the uniqueness to the 2D stochastic Navier-Stokes Equations, see [S] and/or [BM].

**Step 1** We will show the following auxiliary result.

**Lemma 4.3.** There exists an adapted process $u$ with trajectories in the space $C([0, T], H) \cap L^2([0, T], V)$ such that
\[
\lim_{n \to \infty} \left\{ \sup_{s \in [0, T]} |u^n(s) - u(s)|_H^2 + \int_0^T \|u^n(s) - u(s)\|^2 \, ds \right\} = 0 \text{ in probability } \mathbb{P}. \tag{4.2}
\]
Proof of Lemma 4.3. Let us choose and fix natural numbers \( m \geq n \).

Applying the Itô formula from [P] we infer that

\[
\begin{align*}
    f_n(t) |u^n(t) - u^m(t)|_H^2 & = -\lambda \int_0^t f_n(s) \|u^n(s)\|^2 |u^n(t) - u^m(t)|_H^2 \, ds \\
    & - 2 \int_0^t f_n(s) (u^n(s) - u^m(s), A(u^n(s) - u^m(s))) \, ds \\
    & + 2 \int_0^t f_n(s) \langle u^n(s) - u^m(s), \sigma(u^n(s)) - \sigma(u^m(s)) \rangle \, dW(s) \\
    & + 2 \int_0^t f_n(s) \langle u^n(s) - u^m(s), f(u^n(s)) - f(u^m(s)) \rangle \, ds \\
    & + 2 \int_0^t f_n(s) \langle u^n(s) - u^m(s), B(u^n(s), u^n(s)) - B(u^m(s), u^m(s)) \rangle \, ds \\
    & - 2m \int_0^t f_n(s) \langle u^n(s) - u^m(s), u^n(s) - \pi(u^n(s)) \rangle \, ds \\
    & + 2m \int_0^t f_n(s) \langle u^n(s) - u^m(s), u^m(s) - \pi(u^m(s)) \rangle \, ds \\
    & + \int_0^t f_n(s) \sigma(u^n(s)) - \sigma(u^m(s)) \rangle_H^2 \, ds \\
    & := I_1^{n,m}(t) + I_2^{n,m}(t) + I_3^{n,m}(t) + I_4^{n,m}(t) + I_5^{n,m}(t) + I_6^{n,m}(t) + I_7^{n,m}(t) + I_8^{n,m}(t). (4.3)
\end{align*}
\]

Observe that

\[
I_2^{n,m}(t) = -2 \int_0^t f_n(s) \|u^n(s) - u^m(s)\|^2 \, ds. \tag{4.4}
\]

By the assumption on \( f \) and Young’s inequality, we have

\[
I_4^{n,m}(t) \leq C \int_0^t f_n(s) \|u^n(s) - u^m(s)\| \|f(u^n(s)) - f(u^m(s))\|_H \, ds \\
\leq C \int_0^t f_n(s) \|u^n(s) - u^m(s)\| \|u^n(s) - u^m(s)\|_H \, ds \\
\leq \frac{1}{2} \int_0^t f_n(s) \|u^n(s) - u^m(s)\|^2 \, ds + C \int_0^t f_n(s) \|u^n(s) - u^m(s)\|_H^2 \, ds. \tag{4.5}
\]

Now,

\[
I_5^{n,m}(t) = 2 \int_0^t f_n(s) \left( \tilde{b}(u^n(s), u^n(s), u^n(s) - u^m(s)) - \tilde{b}(u^m(s), u^m(s), u^m(s) - u^m(s)) \right) \right) \right) \tag{4.6}
\]

By assumption (2.10), we have

\[
\tilde{b}(u^n(s), u^n(s), u^n(s) - u^m(s)) = \tilde{b}(u^m(s), u^n(s), u^n(s) - u^m(s)) \tag{4.7}
\]

\[
- \tilde{b}(u^m(s), u^n(s) - u^m(s), u^n(s) - u^m(s)) = \tilde{b}(u^m(s), u^n(s), u^n(s) - u^m(s)).
\]

Therefore, in view of assumption (2.11) we infer that

\[
\begin{align*}
    \|\tilde{b}(u^n(s), u^n(s), u^n(s) - u^m(s)) - \tilde{b}(u^m(s), u^m(s), u^n(s) - u^m(s))\| \\
    & \leq 2 \|u^n(s)\| \|u^n(s) - u^m(s)\|_H \|u^n(s) - u^m(s)\|. \tag{4.8}
\end{align*}
\]
It follows from (4.6), (4.8) that

\[ I_6^{n,m}(t) \leq 4 \int_0^t f_n(s) \| u^n(s) \|_H^2 \| u^n(s) - u^m(s) \|_H^2 ds \]

\[ \leq \int_0^t f_n(s) \| u^n(s) - u^m(s) \|^2 ds + 4 \int_0^t f_n(s) \| u^n(s) \|_H^2 \| u^n(s) - u^m(s) \|^2_H ds, \]

where the inequality \( 4ab \leq a^2 + 4b^2 \) has been used.

As \( \pi(u^n(s)) \in \tilde{D} \) and \( \pi(u^m(s)) \in \tilde{D} \), it follows from (3.7) in part (iii) of Lemma 3.2 that

\( \langle u^n(s) - \pi(u^m(s)), u^n(s) - \pi(u^m(s)) \rangle \geq 0 \) and \( \langle u^m(s) - \pi(u^n(s)), u^m(s) - \pi(u^n(s)) \rangle \geq 0 \).

Hence,

\[ I_6^{n,m}(t) = -2n \int_0^t f_n(s) \langle u^n(s) - \pi(u^m(s)), u^n(s) - \pi(u^n(s)) \rangle ds \]

\[ + 2n \int_0^t f_n(s) \langle u^m(s) - \pi(u^m(s)), u^m(s) - \pi(u^n(s)) \rangle ds \]

\[ \leq 2n \int_0^t f_n(s) \langle u^m(s) - \pi(u^n(s)), u^m(s) - \pi(u^n(s)) \rangle ds \]

\[ \leq 2n \| u^n(s) - \pi(u^n(s)) \|_H ds \sup_{0 \leq s \leq t} | u^m(s) - \pi(u^m(s)) |_H, \]

as \( f_n(s) \leq 1 \). A similar calculation also yields

\[ I_7^{n,m}(t) \leq (2m \int_0^t | u^m(s) - \pi(u^m(s)) |_H ds) \sup_{0 \leq s \leq t} | u^n(s) - \pi(u^n(s)) |_H \]

Substituting (4.4), (3.41), (4.6), (4.11), (4.10) into (4.3), choosing \( \lambda > 4 \), as in the proof of Lemma 3.8, using the Burkholder and the Hölder inequalities, as well as the Lipschitz continuity of the maps \( \sigma \), we obtain that

\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} f_n(s) \| u^n(s) - u^m(s) \|_H^2 \right] + \mathbb{E} \left[ \int_0^t f_n(s) \| u^n(s) - u^m(s) \|^2 ds \right] \]

\[ \leq \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq s \leq t} f_n(s) \| u^n(s) - u^m(s) \|_H^2 \right] + C \mathbb{E} \left[ \int_0^t f_n(s) \| u^n(s) - u^m(s) \|^2_H ds \right] \]

\[ + C (\mathbb{E} \left[ 2m \int_0^t | u^n(s) - \pi(u^n(s)) |_H ds \right]^2 )^{\frac{1}{2}} \left( \mathbb{E} \left[ \sup_{0 \leq s \leq t} | u^m(s) - \pi(u^m(s)) |_H^2 \right] \right)^{\frac{1}{2}} \]

\[ + C (\mathbb{E} \left[ 2m \int_0^t | u^m(s) - \pi(u^m(s)) |_H ds \right]^2 )^{\frac{1}{2}} \left( \mathbb{E} \left[ \sup_{0 \leq s \leq t} | u^n(s) - \pi(u^n(s)) |_H^2 \right] \right)^{\frac{1}{2}}. \]

Remark that the first term on the right side of (4.12) comes from the estimate of \( I_3^{n,m} \) and Young’s inequality. By the Gronwall Lemma and also Lemma 3.6, we get

\[ \mathbb{E} \left[ \sup_{s \in [0,T]} f_n(s) \| u^n(s) - u^m(s) \|_H^2 \right] + \mathbb{E} \left[ \int_0^T f_n(s) \| u^n(s) - u^m(s) \|^2 ds \right] \]

\[ \leq C(M_T)^\frac{1}{2} \left( \mathbb{E} \left[ \sup_{s \in [0,T]} | u^m(s) - \pi(u^m(s)) |_H^2 \right] \right)^\frac{1}{2} \]

\[ + C(M_T)^\frac{1}{2} \left( \mathbb{E} \left[ \sup_{s \in [0,T]} | u^n(s) - \pi(u^n(s)) |_H^2 \right] \right)^\frac{1}{2}. \]
By Lemma 3.8, the above yields that
\[
\lim_{n,m \to \infty} \{ \mathbb{E} \left[ \sup_{s \in [0,T]} |f_n(s)|^2 \right] + \mathbb{E} \left[ \int_0^T f_n(s) \|u^n(s) - u^m(s)\|^2 ds \right] \} = 0. \tag{4.14}
\]

Next we will show the following auxiliary result in which we use a classical concept, see e.g. Definition 1.2.6 in [Geis].

**Lemma 4.4.** The sequence \( \{u^n : n \geq 1\} \) of \( C([0,T], H) \cap L^2([0,T], V) \)-valued random variables is Cauchy in probability.

**Proof of Lemma 4.4.** Indeed, given \( \delta > 0 \), for any \( M > 0 \) we have
\[
\mathbb{P}( \sup_{s \in [0,T]} |u^n(s) - u^m(s)|^2_H + \int_0^T \|u^n(s) - u^m(s)\|^2 ds \geq \delta ) \leq \mathbb{P}( \sup_{s \in [0,T]} |f_n(s)|^2 + \int_0^T f_n(s) \|u^n(s) - u^m(s)\|^2 ds \geq e^{-\lambda TM} \delta ) \tag{4.15}
\]
\[
+ \frac{1}{M} \mathbb{E} \left[ \int_0^T \|u^n(s)\|^2 ds \right]
\]
\[
\leq e^{\lambda TM} \frac{1}{\delta} \left\{ \mathbb{E} \left[ \sup_{s \in [0,T]} f_n(s) \|u^n(s) - u^m(s)\|^2_H \right] + \mathbb{E} \left[ \int_0^T f_n(s) \|u^n(s) - u^m(s)\|^2 ds \right] \right\}
\]
\[
+ \frac{1}{M} \mathbb{E} \left[ \int_0^T \|u^n(s)\|^2 ds \right]. \tag{4.16}
\]

Now, for any \( \varepsilon > 0 \), by Lemma 3.6 we can choose \( M > 0 \) such that
\[
\frac{1}{M} \mathbb{E} \left[ \int_0^T \|u^n(s)\|^2 ds \right] \leq \varepsilon, \quad \text{for all} \quad n \geq 1. \tag{4.17}
\]

Then by letting \( n, m \to \infty \) in (4.15) we obtain
\[
\limsup_{n,m \to \infty} \mathbb{P}( \sup_{s \in [0,T]} |u^n(s) - u^m(s)|^2_H + \int_0^T \|u^n(s) - u^m(s)\|^2 ds \geq \delta ) \leq \varepsilon.
\]

As \( \varepsilon \) is arbitrary, we infer that
\[
\lim_{n,m \to \infty} \mathbb{P}( \sup_{s \in [0,T]} |u^n(s) - u^m(s)|^2_H + \int_0^T \|u^n(s) - u^m(s)\|^2 ds \geq \delta ) = 0.
\]

This completes the proof of Lemma 4.4. \( \square \)

Since the space \( C([0,T], H) \cap L^2([0,T], V) \) is complete, we infer that there exists an adapted process \( u \) with trajectories in the space \( C([0,T], H) \cap L^2([0,T], V) \) such that
\[
\lim_{n \to \infty} \left[ \sup_{s \in [0,T]} |u^n(s) - u(s)|^2_H + \int_0^T \|u^n(s) - u(s)\|^2 ds \right] = 0 \quad \text{in probability}. \tag{4.18}
\]

Hence the proof of Lemma 4.3 is complete. \( \square \)
Step 2. Applying [R, the Fatou Lemma I.28] in view Lemma 3.6 we infer that the process \( u \) satisfies inequality (4.1). Let us stress that we do not claim that the convergence of \( u_n \) to \( u \) is in the space \( L^2(\Omega, C([0,T], H) \cap L^2([0,T], V)) \).

Step 3. Our final task is to show that the process \( u(t), t \geq 0 \) is a solution to equation (1.1). To make this precise we need to construct an appropriate process \( L \).

For this aim let us consider a sequence \((L^n)_{n \in \mathbb{N}}\) of stochastic processes defined by the following formulae

\[
L^n(t) = -n \int_0^t (u^n(s) - \pi(u^n(s))) \, ds, \quad t \geq 0. \tag{4.19}
\]

Let us observe that according to inequality (3.24) in Lemma 3.6,

\[
\sup_n \mathbb{E}[\operatorname{Var}_H(L^n)([0,T])^2] \leq \sup_n \mathbb{E}[ (n \int_0^T |u^n(t) - \pi(u^n(t))|^H dt)^2 ] < \infty. \tag{4.20}
\]

Next we will show the following auxiliary result.

Lemma 4.5. The sequence \((L^n)_{n=1}^{\infty}\) of \(C([0,T], V^*)\)-valued random variables is convergent (in \(C([0,T], V^*)\)) in probability. Moreover, if a process \( L \) is the limit of \((L^n)\), then it is a \(H\)-valued adapted process of bounded variation and

\[
\mathbb{E}[\operatorname{Var}_H(L([0,T])^2] < \infty.
\]

Proof of Lemma 4.5. Let us recall from (3.8) that the process \( L^n \) is defined by

\[
L^n(t) = u^n(t) + \int_0^t Au^n(s) \, ds - \int_0^t \sigma(u^n(s)) \, dW(s) - \int_0^t f(u^n(s)) \, ds - \int_0^t B(u^n(s)) \, ds - u(0), \quad t \in [0,T]. \tag{4.21}
\]

Let us also define a process \( L = (L(t), t \in [0,T]) \), by

\[
L(t) = u(t) + \int_0^t Au(s) \, ds - \int_0^t \sigma(u(s)) \, dW(s) - \int_0^t f(u(s)) \, ds - \int_0^t B(u(s)) \, ds - u(0), \quad t \in [0,T]. \tag{4.22}
\]

Let us observe that \( L^n(0) = 0 \) and \( L(0) = 0 \).

Since \( u \) is an adapted process with trajectories in the space \( C([0,T], H) \cap L^2([0,T], V) \), one can easily show, see for instance properties (i)-(iv) listed below, that \( L \) is a \(V^*\)-valued continuous and adapted process.

Moreover, keeping in mind that

(i) the operator \( A : V \to V^* \) is continuous,
(ii) \( B \) induces a continuous bilinear map,

\[
[C([0,T], H) \cap L^2([0,T], V)]^2 \ni (u,v) \mapsto \{[0,T] \ni s \mapsto B(u(s), v(s)) \in V^* \}
\]

see, for instance, Lemma III.3.4 in [T],
(iii) $\sigma : H \to H$ is globally Lipschitz continuous,
(iv) $f : H \to V^*$ is globally Lipschitz continuous,

in view of (4.18) we infer that as $n \to \infty$, the sequence $L^n$ converges to $L$ in $C([0, T], V^*)$ in probability. In particular, we use [F, Theorem 4.3.4] to deduce from Lemma 4.3 that

$$\lim_{n \to \infty} \left\{ \sup_{t \in [0, T]} \left| \int_0^t \sigma(u^n(s)) \, dW(s) - \int_0^t \sigma(u(s)) \, dW(s) \right|^2_H \right\} = 0 \text{ in probability.} \quad (4.23)$$

It is easy to see that the mapping $| \cdot |_H : V^* \to \mathbb{R}^+$ (with the convention $|h|_H = \infty$ for $h \in V^* \setminus H$) is lower semi-continuous. Now we will provide an elementary proof for the fact that the mapping of the total variation norm in $H$, i.e. the function

$$\text{Var}_H(\cdot)([0, T]) : C([0, T], V^*) \ni v \mapsto \text{Var}_H(v)([0, T]) \in [0, \infty] \quad (4.24)$$

is lower semi-continuous. Suppose $v_n \to v$ in $C([0, T], V^*)$ as $n \to \infty$. We will show

$$\text{Var}_H(v)([0, T]) \leq \liminf_{n \to \infty} \text{Var}_H(v_n)([0, T]). \quad (4.25)$$

Without loss of generality we assume $\liminf_{n \to \infty} \text{Var}_H(v_n)([0, T]) < \infty$. Taking any finite partition $0 = t_1 < t_2 < \cdots < t_m = T$ of the interval $[0, T]$, we have

$$\sum_{i=1}^{m-1} |v(t_{i+1}) - v(t_i)|_H \leq \sum_{i=1}^{m-1} \liminf_{n \to \infty} |v_n(t_{i+1}) - v_n(t_i)|_H$$

$$= \sum_{i=1}^{m-1} \liminf_{n \to \infty} |v_k(t_{i+1}) - v_k(t_i)|_H$$

$$= \liminf_{n \to \infty} \sum_{i=1}^{m-1} |v_k(t_{i+1}) - v_k(t_i)|_H$$

$$\leq \liminf_{n \to \infty} \{ \sum_{i=1}^{m-1} |v(t_{i+1}) - v(t_i)|_H \}$$

$$\leq \liminf_{n \to \infty} \text{Var}_H(v_k)([0, T])$$

$$= \liminf_{n \to \infty} \text{Var}_H(v_n)([0, T]). \quad (4.26)$$

Now take the supremum of the left side over all the finite partitions of the interval $[0, T]$ to obtain (4.25).

By the lower semi-continuity of the total variation norm just proved, it follows from (4.19) that $L$ is a $H$-valued process of bounded variation and $\mathbb{P}$-almost surely

$$\text{Var}_H(L)([0, T]) \leq \liminf_{n \to \infty} \text{Var}_H(L^n)([0, T]) \leq \liminf_{n \to \infty} \int_0^T n|u^n(t) - \pi(u^n(t))|_H \, dt.$$

Hence, by the Fatou Lemma and inequality (4.20), we infer that

$$\mathbb{E}[\text{Var}_H(L)([0, T])^2] \leq \sup_n \mathbb{E}[\int_0^T n|u^n(t) - \pi(u^n(t))|_H \, dt]^2 < \infty. \quad (4.27)$$

Hence the proof of Lemma 4.5 is complete. \qed
To complete the proof of Theorem 4 it is necessary to show that \((u, L)\) is a solution to equation 1.6. To this aim we need to verify the remaining conditions of Definition 1.1. By the Fatou Lemma and Lemma 3.8 we have
\[
\mathbb{E} \left[ \sup_{s \in [0,T]} |u(s) - \pi(u(s))|^2_H \right] \leq \lim_{n \to \infty} \mathbb{E} \left[ \sup_{s \in [0,T]} |u^n(s) - \pi(u^n(s))|^2_H \right] = 0. \tag{4.28}
\]
This implies that \(\mathbb{P}\)-almost surely, for every \(t > 0\), \(u(t) = \pi(u(t)) \in \bar{D}\).

Let us choose and fix function \(\phi \in C([0,T], \bar{D})\). Since by property (3.7), for every \(n \in \mathbb{N}\), \(\langle u^n(t) - \phi(t), u^n(t) - \pi(u^n(t)) \rangle \geq 0\), we deduce by (4.19) that almost surely
\[
\int_0^T (\phi(t) - u^n(t), L^n(dt)) = -n \int_0^T (\phi(t) - u^n(t), u^n(s) - \pi(u^n(t))) dt \geq 0, \tag{4.29}
\]
where the expression on the LHS is the Riemann-Stieltjes integral of the \(H\)-valued function \(\phi(t) - u^n(t)\) with respect to an \(H\)-valued bounded-variation function \(L^n\). This will imply that
\[
\int_0^T (\phi(t) - u(t), L(dt)) \geq 0 \tag{4.30}
\]
provided we can show that in probability
\[
\int_0^T (\phi(t) - u(t), L(dt)) = \lim_{n \to \infty} \int_0^T (\phi(t) - u^n(t), L^n(dt)). \tag{4.31}
\]
Let us observe that
\[
\int_0^T (\phi(t) - u^n(t), L^n(dt)) - \int_0^T (\phi(t) - u(t), L(dt)) \tag{4.32}
\]
\[
= \int_0^T (u(t) - u^n(t), L^n(dt)) + \left( \int_0^T (\phi(t) - u^n(t), L^n(dt)) - \int_0^T (\phi(t) - u(t), L^n(dt)) \right)
\]
\[
=: C^n_1 + C^n_2, \quad n \in \mathbb{N}.
\]
In view of (4.18) and (4.20) we infer that in probability
\[
|C^n_1| \leq \sup_{t \in [0,T]} |u^n(t) - u(t)|_H \text{Var}_H(L^n)(T) \to 0, \quad \text{as } n \to \infty. \tag{4.33}
\]
As we only know that \(L^n \to L\) in the space \(C([0,T], V^*)\), it is not evident that \(C^n_2 \to 0\). However, we can prove this by employing the classical density argument. Let us put \(v = \phi - u\). Since both \(\phi\) and \(u\) belong to \(C([0,T], H)\), so does \(v\). By the density of the space \(C([0,T], V)\) in the space \(C([0,T], H)\), for every \(\varepsilon > 0\), we can choose \(v_\varepsilon \in C([0,T], V)\) such that \(|v_\varepsilon - v|_{C([0,T], H)} = \sup_{t \in [0,T]} |v_\varepsilon(t) - v(t)|_H < \varepsilon\). Then \(C^n_2\) can be bounded as
\[
|C^n_2| = |\int_0^T (v(t), L^n(dt)) - \int_0^T (v(t), L(dt))| \leq |\int_0^T (v(t) - v_\varepsilon(t), L^n(dt))| + |\int_0^T (v(t) - v_\varepsilon(t), L(dt))| \tag{4.34}
\]
\[
+ |\int_0^T (v_\varepsilon(t), L^n(dt)) - \int_0^T (v_\varepsilon(t), L(dt))|.
\]
In view of the uniform bounds (4.20) and (4.27) on the total variation of $L^n$ and $L$, the expectation of the first two terms on the RHS of (4.34) can be bounded by $C\varepsilon^{2}$, while the expectation of the third term tends to zero as $n \to \infty$. As $\varepsilon$ is arbitrary, we conclude that

$$\lim_{n \to \infty} C_2^n = 0.$$  \hfill (4.35)

Let us emphasize that the equality (4.35) can be seen as a version of the so called Helly’s Second Theorem, see [L, Theorem 1.6.10 p. 29].

Putting the above estimates together we infer equality (4.31). Thus we have shown that $(u, L)$ is a solution to equation (1.1).

**Proof of the uniqueness part of Theorem 4.1.** Our proof is similar to the proof of Lemma 7.3 from [BM] and uses the Schmalfuss idea of application of the Itô formula for appropriate function, see [S].

Let $(u, L)$ be the solution to the reflected SEE (1.1) constructed above. Let $(v, L_1)$ be another solution to the reflected SEE (1.1). Set $h(t) = e^{-4\int_0^t \|u(s)\|^2 ds}$, $t \geq 0$. By the Itô formulae from [P] and for real-valued processes it follows that

$$h(t)|u(t) - v(t)|_H^2 = -4\int_0^t h(s)\|u(s)\|^2|u(s) - v(s)|_H^2 ds$$  \hfill (4.36)

$$- 2\int_0^t h(s)\|u(s) - v(s)\|^2 ds$$

$$+ 2\int_0^t h(s)(u(s) - v(s), \sigma(u(s)) - \sigma(v(s))) dW(s)$$

$$+ 2\int_0^t h(s)(u(s) - v(s), f(u(s)) - f(v(s))) ds$$

$$+ 2\int_0^t h(s)(u(s) - v(s), B(u(s), u(s)) - B(v(s), v(s))) ds$$

$$+ \int_0^t h(s)|\sigma(u(s)) - \sigma(v(s))|_H^2 ds$$

$$+ 2\int_0^t h(s)(u(s) - v(s), L(ds)) - 2\int_0^t h(s)(u(s) - v(s), L_1(ds)).$$

As $v(t), u(t) \in \bar{D}$, for all $t \geq 0$, we infer from the definition of the solution that

$$\int_0^t h(s)(u(s) - v(s), L(ds)) \leq 0, \quad \int_0^t h(s)(u(s) - v(s), L_1(ds)) \geq 0$$
Hence, it follows that
\[
\begin{align*}
&h(t)|u(t) - v(t)|_H^2 \leq -4 \int_0^t h(s)\|u(s)\|^2|u(s) - v(s)|_H^2 ds \\
&- 2 \int_0^t h(s)\|u(s) - v(s)\|^2 ds \\
&+ 2 \int_0^t h(s)(u(s) - v(s), \sigma(u(s)) - \sigma(v(s))) dW(s) \\
&+ 2 \int_0^t h(s)(u(s) - v(s), f(u(s)) - f(v(s))) ds \\
&+ 2 \int_0^t h(s) \langle u(s) - v(s), B(u(s), u(s)) - B(v(s), v(s)) \rangle ds \\
&+ \int_0^t h(s)\|\sigma(u(s)) - \sigma(v(s))\|_H^2 ds.
\end{align*}
\] (4.37)

Now, following the same argument as in the proof of (4.12) and (4.13), by Gronwall inequality, it can be shown that $E[h(t)|u(t) - v(t)|_H] = 0$ for all $t \geq 0$ proving part of Theorem 4.1. □

In order to formulate our next result let us recall that $L$ is an $H$-valued process whose trajectories are locally of bounded variation. We introduce the following notation
\[
|L|_t := \text{Var}([0,t])(L), \ t \in [0,\infty).
\] (4.38)

For each $\omega \in \Omega$, the function $\mathbb{R}_+ \ni t \mapsto |L|_t$ is increasing. Hence, one can associate with it a unique measure $m_{|L|}$, called the Lebesgue-Stieltjes measure. This random family of Lebesgue-Stieltjes measures is traditionally denoted by $d|L|_t$.

If $m_L$ denotes the unique $H$-valued Lebesgue-Stieltjes measure associated with the $H$-valued process $L$, see [D, Theorem III.2.1, p. 358], then the measure $m_{|L|}$ is equal to the variation of the measure $m_L$, see [D, pp. 361 and 363].

**Proposition 4.6.** Let $(u,L)$ be the solution to equation (1.1). Then $\mathbb{P}$-almost surely the measure $d|L|_t$ is supported on the set
\[
\{ t \in [0,\infty) : u(t) \in \partial D \} = \{ t \in [0,\infty) : |u(t)| = 1 \}.
\] (4.39)

**Proof of Proposition 4.6.** The assertion of the proposition is equivalent to $\mathbb{P}$-almost surely
\[
\int_0^T 1_{[0,1]}(|u(t)|_H) d|L|_t = 0,
\] (4.40)
where on the LHS we have the Lebesgue-Stieltjes integral.

To this end, by an approximation argument it suffices to prove that for every $C^1$ function with compact support $\psi : [0,1) \to [0,\infty)$, the following equality holds
\[
\int_0^T \psi(|u(t)|_H) d|L|_t = 0, \ a.s.,
\] (4.41)
or equivalently,
\[
E\left[ \int_0^T \psi(|u(t)|_H) d|L|_t \right] = 0.
\] (4.42)
Let us observe that since the function \( \psi([u(\cdot)]_H) \) is continuous, the Lebesgue-Stieltjes integral on the LHS of (4.41) is equal to the Riemann-Stieltjes integral.

Recall that

\[
L^n_t = -n \int_0^t (u^n(s) - \pi(u^n(s))) \, ds, \quad t \in [0, T].
\]

As \( L^n \to L \) in \( C([0, T], V^*) \) in probability, by the lower semi-continuity of the function \( \text{Var}_H(\cdot)((s, t]) \) defined in (4.24) (with \([0, T]\) replaced by \((s, t]\)) we conclude that

\[
\text{Var}_H(L)((s, t]) \leq \liminf_{n \to \infty} \text{Var}_H(L^n)((s, t]), \quad \text{for any } s < t. \tag{4.43}
\]

This implies that for any non-negative, continuous function \( \eta(\cdot) : [0, T] \to [0, \infty) \), it holds that

\[
\int_0^T \eta(t) \, d|L|_t \leq \liminf_{n \to \infty} \int_0^T \eta(t) \, d|L^n|_t. \tag{4.44}
\]

Indeed, since \( \eta \) is uniformly continuous on \([0, T]\), for any given \( \varepsilon > 0 \) there exist a partition \( 0 = t_0 < t_1 < \cdots < t_m = T \) and a simple function \( \eta_m(t) = \sum_{k=0}^{m-1} a_k \chi_{(t_k, t_{k+1}]}(t) \) such that \( \eta_m(t) \leq \eta(t), t \in [0, T], \) and \( \sup_{0 \leq t \leq T} |\eta_m(t) - \eta(t)| \leq \varepsilon. \) For example, take \( a_k = \min_{t \in [t_k, t_{k+1}]} \eta(t) \). For the simple function \( \eta_m \), using the Riemann sum it follows from (4.43) that

\[
\int_0^T \eta_m(t) \, d|L|_t \leq \liminf_{n \to \infty} \int_0^T \eta_m(t) \, d|L^n|_t. \tag{4.45}
\]

Therefore we have

\[
\int_0^T \eta(t) \, d|L|_t \leq \int_0^T \eta_m(t) \, d|L|_t + |L|_T \varepsilon
\]

\[
\leq \liminf_{n \to \infty} \int_0^T \eta_m(t) \, d|L^n|_t + |L|_T \varepsilon
\]

\[
\leq \liminf_{n \to \infty} \int_0^T \eta(t) \, d|L^n|_t + |L|_T \varepsilon, \tag{4.46}
\]

here we have used the fact that \( \eta_m(t) \leq \eta(t), t \in [0, T]. \) Since \( \varepsilon \) is arbitrary, (4.44) is proved. In particular, letting \( \eta(t) = \psi([u(t)]_H) \) and using the Fatou Lemma, we obtain

\[
E \left[ \int_0^T \psi([u(t)]_H) \, d|L|_t \right] \leq \liminf_{n \to \infty} E \left[ \int_0^T \psi([u^n(t)]_H) \, d|L^n|_t \right]. \tag{4.47}
\]

On the other hand, recalling \( C = \sup_n E[||L^n|_T]^2] < \infty \), by the dominated convergence theorem, we infer that

\[
E \left[ \left| \int_0^T \psi([u(t)]_H) \, d|L^n|_t - \int_0^T \psi([u^n(t)]_H) \, d|L^n|_t \right| \right]
\]

\[
\leq E \left[ \sup_{t \in [0, T]} |\psi([u(t)]_H) - \psi([u^n(t)]_H)||L^n|_T \right]
\]

\[
\leq C \left( E \left[ \sup_{t \in [0, T]} |\psi([u(t)]_H) - \psi([u^n(t)]_H)|^2 \right] \right)^{1/2} \to 0 \text{ in probability.}
\]
Together with (4.47) we deduce that

\[
E \left[ \int_0^T \psi(|u(t)|_H) d|L|_t \right] \leq \liminf_{n \to \infty} E \left[ \int_0^T \psi(|u^n(t)|_H) d|L^n|_t \right] \\
= \liminf_{n \to \infty} E \left[ n \int_0^T \psi(|u^n(s)|_H)|u^n(s) - \pi(u^n(s)|_H ds \right] = 0.
\]

This completes the proof of Proposition 4.6.

Therefore, the proof of Theorem 4.1 is now complete. □

5 Reflected Stochastic Navier Stokes Equations

The main motivation of this paper is to treat the stochastic Navier Stokes Equations on a two dimensional domains \(D\). Since we do not require any compactness of the embeddings, our domain can be unbounded, for example, the whole Euclidean space \(\mathbb{R}^2\). We will concentrate on this case and mostly follow a recent paper [BF2] by the first named author and Ferrario.

For a natural number \(d\) and \(p \in [1, \infty)\), let \(L^p = L^p(\mathbb{R}^d, \mathbb{R}^d)\) be the classical Lebesgue space of all \(\mathbb{R}^d\)-valued Lebesgue measurable functions \(v = (v^1, \ldots, v^d)\) defined on \(\mathbb{R}^d\) endowed with the following classical norm

\[
\|v\|_{L^p} = \left( \sum_{k=1}^d \|v^k\|_{L^p(\mathbb{R}^d)}^p \right)^{\frac{1}{p}}.
\]

For \(p = \infty\), we set \(\|v\|_{L^\infty} = \max_{k=1}^d \|v^k\|_{L^\infty(\mathbb{R}^d)}\).

Set \(J^s = (I - \Delta)^{\frac{s}{2}}\). We define the generalized Sobolev spaces of divergence free vector distributions, for \(s \in \mathbb{R}\), as

\[
H^{s,p} = \left\{ u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^d) : \|J^s u\|_{L^p} < \infty \right\},
\]

\[
H^{s,\text{sol}} = \left\{ u \in H^{s,p} : \text{div } u = 0 \right\}.
\]

It is well known that \(J^s\) is an isomorphism between \(H^{s,p}\) and \(H^{s-\sigma,p}\) for \(s \in \mathbb{R}\) and \(1 < p < \infty\). Moreover \(H^{s_2,p} \subset H^{s_1,p}\) when \(s_1 < s_2\). In particular, for the Hilbert case \(p = 2\) we set \(H = H^{0,2}\) and, for \(s \neq 0\), \(H^s = H^{s,2}_{\text{sol}}\), so that (Warning!), \(H^s\) is a proper closed subspace of the classical Sobolev space usually denoted by the same symbol. In particular, we put

\[
H = \left\{ v \in L^2(\mathbb{R}^d, \mathbb{R}^d) : \text{div } v = 0 \right\}
\]

with scalar product inherited from \(L^2(\mathbb{R}^d, \mathbb{R}^d)\).

We will also denote by \(\langle \cdot, \cdot \rangle\) the duality bracket between \((H^{s,p})'\) and \(H^{s,p}\) spaces. Note that for \(p \in [1, \infty)\), the space \((H^{s,p})'\) can be identified with \((H^{-s,-p})\), where \(\frac{1}{p} + \frac{1}{-p} = 1\).

Now we define the operators appearing in the abstract formulation. Assume that \(s \in \mathbb{R}\) and \(1 \leq p < \infty\). Let \(A_0 = -\Delta\); then \(A_0\) is a linear unbounded operator in \(H^{s,p}\) and bounded from \(H^{s+2,p}\) to \(H^{s,p}\). Moreover, the spaces \(H^{s,\text{sol}}\) are invariant w.r.t. \(A_0\) and the corresponding operator will be denoted by \(A\). Let us observe that \(A\) is a linear unbounded operator in \(H^{s,p}\) and bounded from \(H^{s+2,\text{sol}}\) to \(H^{s,\text{sol}}\). The operator \(-A_0\) generates a contractive and analytic \(C_0\)-semigroup \(e^{-tA}\) on \(H^{s,p}\) and therefore, the operator \(-A\) generates a contractive and
analytic $C_0$-semigroup \( \{e^{-tA}\}_{t \geq 0} \) on \( H^{s,p} \). Moreover, for \( t > 0 \) the operator \( e^{-tA} \) is bounded from \( H^{s,p}_0 \) into \( H^{s',p}_0 \) with \( s' > s \) and there exists a constant \( M \) (depending on \( s' - s \) and \( p \)) such that
\[
\|e^{-tA}\|_{\mathcal{L}(H^{s,p}_0; H^{s',p}_0)} \leq M(1 + t^{-(s'-s)/2}).
\] (5.3)
We have \( A : H^1 \to H^{-1} \) as a linear bounded operator and
\[
\langle Av, v \rangle = \|\nabla v\|_{L^2}^2, \quad v \in H^1,
\]
where
\[
\|\nabla v\|_{L^2}^2 = \sum_{k=1}^d \|\nabla v^k\|_{L^2}^2, \quad v \in H^1.
\]
Moreover we have
\[
\|v\|_{H^1}^2 = \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2. \quad (5.4)
\]
We define a bounded trilinear form \( b : H^1 \times H^1 \times H^1 \to \mathbb{R} \) by
\[
b(u,v,z) = \int_{\mathbb{R}^d} (u(\xi) \cdot \nabla)(\xi) \cdot z(\xi) \, d\xi, \quad u,v,z \in H^1.
\]
and the corresponding bounded bilinear operator \( B : H^1 \times H^1 \to H^{-1} \) via the trilinear form
\[
\langle B(u,v), z \rangle = b(u,v,z), \quad u,v,z \in H^1.
\]
This operator satisfies, for all \( u,v,z \in H^1 \),
\[
\langle B(u,v), z \rangle = -\langle B(u,z), v \rangle, \quad \langle B(u,v), v \rangle = 0. \quad (5.5)
\]
Note that (5.5) implies a weaker version of it, i.e.
\[
\langle B(u,u), u \rangle = 0, \quad \text{for all } u \in H^1. \quad (5.6)
\]
\( B \) can be extended to a bounded bilinear operator from \( H^{0,1}_0 \times H^{0,1}_0 \) to \( H^{-1} \) with
\[
\|B(u,v)\|_{H^{-1}} \leq \|u\|_{L^4}\|v\|_{L^4}, \quad (5.7)
\]
see [T1, (2.29)]. Moreover, for any \( a > \frac{d}{2} + 1 \), \( B \) can be extended to a bounded bilinear operator from \( H \times H \) to \( H^{-a} \) with
\[
\|B(u,v)\|_{H^{-a}} \leq C\|u\|_{L^2}\|v\|_{L^2}, \quad (5.8)
\]
see [T1, Lemma 2.1]. Let us also observe that in view of the so called Ladyzhenskaya inequality, see [T, Lemma 3.3.3], inequality (5.7) implies that there exists \( C > 0 \) such that for all \( u \in H^1 \)
\[
\|B(u,u)\|_{H^{-1}} \leq \sqrt{2}\|u\|_{L^2}\|\nabla u\|_{L^2} + \|u\|_{L^2}\|\nabla u\|_{L^2} + \|u\|_{L^2}\|\nabla u\|_{L^2}. \quad (5.9)
\]
In particular, this proves that inequality (2.9) in part (b) of Assumption (A.2), or equivalently, inequality (2.11).

Once and for all we denote by \( C \) a generic constant, which may vary from line to line; we number it if we need to identify it.

Finally, we define the noise forcing term. Given a real separable Hilbert space \( K \) we consider a \( K \)-cylindrical Wiener process \( \{W(t)\}_{t \geq 0} \) defined on a stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) satisfying the usual conditions. For the covariance \( \sigma \) of the noise we make the following assumptions.
The mapping $\sigma : H \to \gamma(K;H)$ is well-defined and is a Lipschitz continuous map $G : H \to \gamma(K;H)$, i.e.,

$$\exists L > 0 : \|\sigma(v_1) - \sigma(v_2)\|_{\gamma(K;H)} \leq L\|v_1 - v_2\|_H$$

for all $v_1, v_2 \in H$.

We consider the stochastic damped Navier-Stokes equations, that is the equations of motion of a viscous incompressible fluid with two forcing terms, one is random and the other one is deterministic. These equations are

$$\begin{cases}
\partial_t u + [-\nu \Delta u + \gamma u + (u \cdot \nabla)u + \nabla p] dt = \sigma(u) \partial_t W + f dt \\
\text{div } u = 0
\end{cases}$$

(5.10)

where the unknowns are the vector velocity $u = u(t, \xi)$ and the scalar pressure $p = p(t, \xi)$ for $t \geq 0$ and $\xi \in \mathbb{R}^d$. By $\nu > 0$ we denote the kinematic viscosity and by $\gamma > 0$ the sticky viscosity, see for instance [Gal] and [CR]. When $\gamma = 0$ (5.10) reduce to the classical stochastic Navier-Stokes equations. The notation $\partial_t W$ on the right hand side is for the space correlated and white in time noise and $f$ is a deterministic forcing term. We consider a multiplicative term $\sigma(u)$ keeping track of the fact that the noise may depend on the velocity.

Projecting equations (5.10) onto the space $H$ of divergence free vector fields, we get the abstract form of the stochastic damped Navier-Stokes equations (5.10)

$$du(t) + [Au(t) + \gamma u(t) + B(u(t), u(t))] dt = \sigma(u(t)) dW(t) + f(t) dt$$

(5.11)

with the initial condition

$$u(0) = u_0$$

(5.12)

where the initial velocity $u_0 : \Omega \to H$ is an $\mathcal{F}_0$-measurable random variable. Here $\gamma > 0$ is fixed and for simplicity we have put $\nu = 1$. The case $\gamma = 0$ was considered in, e.g. [BF1]. We assume that $\gamma > 0$ so that the operator $A + \gamma I$ satisfies the assumption (2.1). However, one can slightly modify the proofs in the paper in such a way that assumption (2.1) is replaced by the following one. There exists $\alpha > 0$ such that

$$(Au, u) + \alpha(u, u) \geq 0, \quad u \in D(A).$$

(5.13)

Here we consider classical strong solutions.

**Definition 5.1** (strong solution). We say that a progressively measurable process $u : [0, T] \times \Omega \to H$ with $\mathbb{P}$-almost all paths satisfying

$$u \in C([0, T]; H) \cap L^2(0, T; H^1)$$

is a strong solution to problem (5.11) if and only if

- $u(0) = u_0$
- for any $t \in [0, T], \psi \in H^2, \mathbb{P}$-a.s.,

$$\begin{align*}
(u(t), \psi)_H &+ \int_0^t (Au(s), \psi) ds + \gamma \int_0^t (u(s), \psi)_H ds \\
&+ \int_0^t (B(u(s), u(s)), \psi) ds \\
&= (u(0), \psi)_H + \int_0^t (f(s), \psi) ds + \int_0^t \sigma(u(s)) dW(s), \psi).
\end{align*}$$

(5.14)
Now consider the reflected stochastic Navier-Stokes equation:

$$du(t) + [Au(t) + \gamma u(t) + B(u(t), u(t))] dt = \sigma(u(t)) dW(t) + f(t) dt + dL(t)$$

(5.15)

with the initial condition

$$u(0) = u_0 \in \bar{D}$$

(5.16)

Our main result, Theorem 4.1, applies in this setting. We have the following

**Theorem 5.2.** The reflected stochastic evolution equation (5.15) admits a unique solution \((u, L)\) that satisfies, for \(T > 0\),

$$\mathbb{E} \left[ \sup_{t \in [0,T]} |u(t)|_H^2 + \int_0^T ||u(t)||^2 dt \right] < \infty.$$  

(5.17)

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