On the explosion of the number of fragments in the simple exchangeable fragmentation-coalescence processes

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Abstract. We consider the exchangeable fragmentation-coagulation (EFC) processes, where the coagulations are multiple and not simultaneous, as in a Λ-coalescent, and the fragmentations dislocate at finite rate an individual block into sub-blocks of infinite size. Sufficient conditions are found for the block-counting process to explode (i.e. to reach $\infty$) or not and for infinity to be an exit boundary or an entrance boundary. In a case of regularly varying fragmentation and coagulation mechanisms, we find regimes where the boundary $\infty$ can be either an exit, an entrance or a regular boundary. In the latter regular case, the EFC process leaves instantaneously the set of partitions with an infinite number of blocks and returns to it immediately. Proofs are based on a new sufficient condition of explosion for positive continuous-time Markov chains, which is of independent interest.

Résumé. Nous considérons les processus de fragmentation-coagulation échangeables (EFC), où les coagulations sont multiples et non simultanées, comme dans un Λ-coalescent, et où les fragmentations disloquent à taux fini un bloc individuel en sous-blocs de taille infinie. Des conditions suffisantes sont trouvées pour que le processus du nombre de blocs explode (c'est-à-dire atteigne l'infini) ou non et pour que l'infini soit une frontière de sortie ou une frontière d'entrée. Lorsque les mécanismes de fragmentation et de coagulation satisfont une hypothèse de variation régulière, nous trouvons des régimes où la frontière $\infty$ peut être une sortie, une entrée ou un point régulier. Dans ce dernier cas régulier, le processus EFC quitte instantanément l'ensemble des partitions avec un nombre infini de blocs et y revient immédiatement. Les preuves sont basées sur une nouvelle condition suffisante d'explosion pour les chaînes de Markov positives en temps continu, qui peut avoir un intérêt indépendant.

MSC 2010 subject classifications: 60J80, 60J70, 60J90, 92D25.

Keywords: Λ-coalescent, fragmentation, branching process, explosion, coming down from infinity, entrance boundary, regular boundary, continuous-time Markov chains.

1. Introduction

Stochastic processes describing both coalescence and fragmentation are ubiquitous in scientific disciplines such as astrophysics, chemistry, genetics, or population dynamics. The exchangeable fragmentation-coalescence processes, called EFC processes for short, form an important class of such processes. They have been characterized and first studied by Berestycki in [Beres04]. An EFC process is a stochastic process
(\(\Pi(t), t \geq 0\)) for which at any time \(t \geq 0\), \(\Pi(t)\) stands for a collection, possibly finite, of disjoint subsets, called “fragments” or “blocks”, \((\Pi_1(t), \Pi_2(t), \cdots)\), covering the set of positive integers \(N := \{1, 2, \cdots\}\) i.e. \(\bigcup_{i \geq 1} \Pi_i(t) = \mathbb{N}\). The process is exchangeable in the sense that for any time \(t \geq 0\), the random partition \(\Pi(t)\) of \(\mathbb{N}\) has a law invariant under the action of permutations that only change finitely many integers. Last but not least, the evolution of the process is two-fold. Blocks can merge, as in an exchangeable coalescent, and can fragment as in a homogeneous exchangeable fragmentation.

This article considers the problem of classifying the nature of the boundary \(\infty\) of the continuous-time Markov chain arising as the functional of the number of blocks in EFC processes. Our main goal is to study the phenomenon of explosion in the number of blocks.

When fragmentation occurs at an infinite rate, the number of blocks is infinite at almost all times, see [Beres04, Theorem 12]. We shall not consider this case and will focus here on the class of EFC processes with fragmentations occurring at a finite rate, and in which neither simultaneous fragmentations nor simultaneous multiple coagulations can occur. We shall assume moreover that the fragmentations cannot dislocate blocks into singletons. These processes, called simple EFC processes, can therefore be seen as a generalisation of \(\Lambda\)-coalescents, as defined by Pitman [Pit99] and Sagitov [Sag99], in which “simple” fragmentations are incorporated. More precisely, each infinite block is split into \(k + 1\) sub-blocks of infinite size, (thus creating \(k\) new blocks), at rate say \(\mu(k)\), independently of each other, where \(\mu\) is a finite positive measure on \(\mathbb{N} \cup \{\infty\}\). At the level of the number of blocks, the fragmentation is therefore nothing but a discrete branching process with no death, whose offspring measure is \(\mu\). We shall call \(\mu\) the splitting measure.

EFC processes arise for instance when studying the frequency of a disadvantaged allele in certain Wright-Fisher models with selection, see e.g. Foucart and Zhou [FZ20+], González-Casanova and Spanò [GS18] and the references therein. In terms of population models, fragmentations can also be seen as reproduction events and coalescences as negative interactions between individuals in the population. We refer for instance to [Beres04] and Lambert [Lam05, Section 2.3] where it is explained that discrete logistic branching processes can be linked to the block-counting process of simple EFCs with only binary coalescences. More generally, \(\Lambda\)-coalescences can be interpreted as a competition term between multiple individuals. This point of view was chosen for instance by González-Casanova and Spanò in [GPP21]. Some continuous-state space models are also closely related to EFC processes. We refer the reader for instance to Bansaye et al. [BPMS13] and Foucart [Fou19]. We wish to mention the work of Wagner [Wag05] where the phenomenon of explosion is studied for a different family of coagulation-fragmentation particle systems. See also Bertoin and Kortchemski [BK16, Section 5.4] where scaling limits of some EFC processes are studied.

A remarkable feature of \(\Lambda\)-coalescent processes lies in the fact that under certain conditions on the coalescence, the process, started from a partition with infinitely many blocks, can instantaneously enter the set of partitions with a finite number of blocks. This phenomenon, called coming down from infinity, has been deeply studied in the 2000s, see [Sch00], [BBL10] and [LT15]. In particular, Schweinsberg [Sch00] has found a necessary and sufficient condition on the measure \(\Lambda\) for the coming down from infinity. In the pure coalescent framework, the block-counting process has decreasing sample paths and when it starts from infinity and leaves it, it stays finite at any further time. In Feller’s terminology, see [Fel59] and e.g. Anderson [And91, Chapter 8, page 262], \(\infty\) is said to be an entrance boundary.

In a symmetric way, without coalescences, when the process starts from a partition with finitely many blocks, fragmentations into finitely many sub-blocks may accumulate and push the number of blocks to \(\infty\) in finite time, which is referred to as the phenomenon of explosion. It is also well-known that \(\infty\) is an absorbing state for branching processes, so that if the process of pure fragmentation explodes then it stays infinite at any further time. In Feller’s terminology, \(\infty\) is said to be an exit boundary.

When both fragmentations and coalescences are taken into account, sample paths of the block-counting process are not monotone anymore, and some new phenomena may arise. For instance, when the pure coalescent part does come down from infinity, fragmentations may or may not prevent the coming down from infinity of the EFC process. When the pure fragmentation explodes, it is also natural to ask whether or not the coalescent part will prevent explosion. To the best of our knowledge, only few results in this direction are known for the moment.
An important step in the understanding of the possible behaviors at $\infty$ of EFC processes, has been recently made by Kyprianou et al. in [KPRS17]. They study there the “fast” fragmentation-coalescence process, in which coagulations are binary, as in a Kingman coalescent, and fragmentation dislocates at a constant rate, any individual block into its constituent elements (which creates infinitely many singleton blocks, and causes an infinite jump of the number of blocks). In [KPRS17], a phase transition is found between a regime for which the boundary is an exit and a regime where the boundary $\infty$ is regular, namely the block-counting process leaves and returns to $\infty$ almost surely. In this latter regime, it is also shown in [KPRS17] that the boundary $\infty$ is regular for itself. That is to say, when started from a partition with infinitely many blocks, the partition-valued process leaves the set of partitions with infinitely many blocks and returns to it instantaneously. This leads to many open questions for less extreme mechanisms of fragmentation and coalescences. It is natural for instance to wonder if the boundary $\infty$ can be regular for other EFC processes than the “fast” EFC process.

The class of simple EFC processes with general $\Lambda$-coalescences has been recently studied in [Fou20+]. Let $(\Pi(t), t \geq 0)$ be a simple EFC process. The block-counting process, denoted by $(\#\Pi(t), t \geq 0)$, has the following infinitesimal dynamics when it evolves in $\mathbb{N}$. Let $n \in \mathbb{N}$.

- **Coalescences**: for any $2 \leq k \leq n$, it jumps from $n$ to $n - k + 1$ at rate $\binom{n}{k} \lambda_{n,k}$, with

  $$\lambda_{n,k} := \int_{[0,1]} x^k(1-x)^{n-k}x^{-2}\Lambda(dx).$$

- **Fragmentations**: for any $k \in \mathbb{N} \cup \{\infty\}$, it jumps from $n$ to $n + k$, at rate $n\mu(k)$.

Unlike the fast EFC process, when the fragmentations cannot dislocate a block into infinitely many sub-blocks, i.e. $\mu(\infty) = 0$, we cannot immediately deduce from the dynamics above whether the boundary $\infty$ can be reached or not. The question of accessibility of $\infty$ (namely explosion) in simple EFC processes was left unaddressed in [Fou20+]. The first purpose of this article is to shed some light on the cases where fragmentations and $\Lambda$-coalescences together can allow the process to explode or not.

The coming down from infinity of simple EFC processes has been studied in [Fou20+]. A phase transition between a regime in which a simple EFC process, started from an exchangeable random partition with infinitely many blocks, comes down from infinity and one in which it stays infinite, is established in [Fou20+, Theorem 1.1]. Combining this result, recalled in Section 2.3, and our conditions for explosion/non-explosion, we will find sufficient conditions on $\Lambda$ and $\mu$ for the boundary $\infty$ to be either an exit or an entrance, see Theorem 3.1 and Theorem 3.3 respectively. We study in detail the nature of the boundary $\infty$ in two cases of regularly-varying coalescence and fragmentation mechanisms, see Theorem 3.5 and Theorem 3.9.

In particular, we shall see in Theorem 3.5, that when the coalescence and splitting measures satisfy

$$\Lambda(dx) = f(x)dx, \text{ for } x \in [0,x_0] \text{ with } f(x)x^\beta \xrightarrow{x \to 0+} c \tag{1.1}$$

and

$$\mu(n)n^{1+\alpha} \xrightarrow{n \to \infty} b \tag{1.2}$$

for some $x_0 \in (0,1]$, $\alpha, \beta \in (0,1)$ and $b, c > 0$, then indeed the boundary $\infty$ can be regular. When $\alpha + \beta = 1$, new phase transitions are found between regimes where $\infty$ is regular, an exit or an entrance. See the forthcoming Figure 1 for a summary of the possible behaviors. We will also show that when the boundary is regular, it is regular for itself.

The study of the explosion is based on a new sufficient condition for explosion of general continuous-time Markov chains, see Theorem 4.1 and Corollary 4.5. Theorem 4.1 is based on estimates for the first passage times above large levels. The main difficulties that arise when $\Lambda$-coalescences are allowed come from the fact that downwards jumps can have a size of the same order as the level of the process prior the jump. We shall see how to deal with those large jumps, and how to measure both coagulation and fragmentation strengths in order to apply our new condition of explosion.
The article is organized as follows. Our main results are stated in Section 3. In Section 2, we provide more background on EFC processes. We briefly recall their Poisson construction and some important properties of the functional of the number of blocks. We then recall some results about the coming down from infinity as well as results on explosion of pure branching processes. Our new condition for explosion is stated and established in Section 4. Proofs of the main results are given in Section 5.

Notation: For any integers \( n \leq m \), we denote by \([n,m]\) the collection of integers \( \{n, \ldots, m\} \). We shall say that a property \( P(n) \) on the integer \( n \) is true for large enough \( n \), when there exists \( n_0 \in \mathbb{N} \) such that \( P(n) \) holds for all \( n \geq n_0 \). For any \( \pi \in \mathbb{N} \) and any \( f \) a positive Borel function defined on \( (a - \epsilon, a) \) for some \( \epsilon > 0 \), we denote the integrability of \( f \) at \( a \) by \( \int_{a-}^{a+} f(x) \, dx < \infty \). Similarly, we denote \( \int_{a-}^{a+} f(x) \, dx < \infty \) for the integrability of \( f \) at \( \infty \). The left and right limit at \( a \in \mathbb{R} \) of \( f \) are denoted by \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \), respectively. For any positive functions \( f \) and \( g \) well-defined in a neighbourhood of \( a \), we set \( f(x) \sim g(x) \) when \( \frac{f(x)}{g(x)} \to 1 \). For any real number \( x \), \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \), and \( \lceil x \rceil \) the least integer greater than or equal to \( x \). Lastly, we take the convention \( \inf \emptyset = \infty \).

2. Preliminaries

2.1. Exchangeable fragmentation-coalescence processes

We denote by \( \mathcal{P}_\infty \) the space of partitions of \( \mathbb{N} \). By convention, any partition \( \pi \) of \( \mathbb{N} \) is represented by the sequence (possibly finite) of its non-empty blocks \( (\pi_i, i \geq 1) \) ordered by their least elements. For any \( n \in \mathbb{N} \), we denote by \( \pi_{[n]} \) the partition restricted to \( [n] := \{1, \ldots, n\} \): namely \( \pi_{[n]} = (\pi_1 \cap [n], \pi_2 \cap [n], \ldots) \). The space \( \mathcal{P}_\infty \) is endowed with the metric \( d \), defined by \( d(\pi, \pi') := \max\{n \geq 1; \pi_{[n]} = \pi'_{[n]}\}^{-1} \). For any partition \( \pi \in \mathcal{P}_\infty \), we denote by \( \# \pi \), its number of non-empty blocks. By convention, if \( \# \pi < \infty \), then we set \( \pi_j = \emptyset \) for any \( j \geq \# \pi + 1 \).

Exchangeable fragmentation-coalescence processes are Feller processes with state-space \( \mathcal{P}_\infty \). It is established in [Beres04] that they are characterized in law by two \( \sigma \)-finite exchangeable measures on \( \mathcal{P}_\infty \), \( \mu_{\text{Coag}} \) the measure of coagulation and \( \mu_{\text{Frag}} \), that of fragmentation. We refer the reader to Berestycki’s article for the general form that those measures can take, as well as the integrability conditions they must satisfy.

We briefly recall now the Poisson construction of EFC processes with given coagulation and fragmentation measures; see [Beres04] for details on this construction.

Consider two independent Poisson Point Processes \( \text{PPP}_C \) and \( \text{PPP}_F \) respectively on \( \mathbb{R}_+ \times \mathcal{P}_\infty \) and \( \mathbb{R}_+ \times \mathcal{P}_\infty \times \mathbb{N} \) with intensity \( dt \otimes \mu_{\text{Coag}}(d\pi) \) and \( dt \otimes \mu_{\text{Frag}}(d\pi) \otimes \# \) where \( \# \) denotes the counting measure on \( \mathbb{N} \). Let \( \pi \) be an exchangeable random partition independent of \( \text{PPP}_F \) and \( \text{PPP}_C \). For any \( m \geq 1 \), define the process \( (\Pi^m(t), t \geq 0) \) as follows: \( \Pi^m(0) = \pi_{[m]} \) and

\[
\Pi^m(t) = \begin{cases} \text{Coag} \left( \Pi^m(t^-), \pi^c_{[m]} \right) & \text{if} \ (t, \pi^c) \ \text{is an atom of} \ \text{PPP}_C, \\ \text{Frag} \left( \Pi^m(t^-), \pi^f_{[m]}, j \right) & \text{if} \ (t, \pi^f, j) \ \text{is an atom of} \ \text{PPP}_F, 
\end{cases}
\]

where for any partitions \( \pi, \pi^c, \pi^f \)

\[
\text{Coag}(\pi, \pi^c) := \{ \cup_{j \in \pi^c} \pi_i, i \geq 1 \} \quad \text{and} \quad \text{Frag}(\pi, \pi^f, j) := \{ \pi_j \cap \pi^f_i, i \geq 1; \pi_{\ell}, \ell \neq j \}^j,
\]

where \( \{\cdots\}^j \) means that we reorder the blocks by their least elements. See Bertoin’s book [Ber06] for fundamental properties of the operators Coag and Frag. The processes \( (\Pi^m(t), t \geq 0)_{m \geq 1} \) are compatible in the sense that for any \( m \geq n \geq 1 \),

\[
(\Pi^m(t)_{|[n]}), t \geq 0) = (\Pi^n(t), t \geq 0).
\]

This ensures the existence of a process \( (\Pi(t), t \geq 0) \) on \( \mathcal{P}_\infty \) such that for all \( m \geq 1 \)

\[
(\Pi(t)_{|[m]}), t \geq 0) = (\Pi^m(t), t \geq 0).
\]
The process $(\Pi(t), t \geq 0)$ is an exchangeable EFC process started from the exchangeable random partition $\Pi(0) = \pi$. Among other results, Berestycki has shown that the $\mathcal{P}_\infty$-valued process $(\Pi(t), t \geq 0)$ has càdlàg sample paths.

In this article, we will focus on simple EFC processes for which there are no multiple simultaneous mergings, as in a $\Lambda$-coalescent, fragmentations occur at finite rate and dislocate any blocks into sub-blocks of infinite size. Formally, the coagulation measure charges partitions with only one non-singleton block, and the fragmentation measure $\mu_{\text{Frag}}$ on $\mathcal{P}_\infty$ has finite mass, i.e. $\mu_{\text{Frag}}(\mathcal{P}_\infty) < \infty$, and only charges partitions whose blocks are all of infinite size. According to [Fou20+, Proposition 2.11], if $(\Pi(t), t \geq 0)$ is a simple EFC process then the process $(\#\Pi(t), t \geq 0)$ has right-continuous sample paths in $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, the one-point compactification of $\mathbb{N}$. It is important to notice that the map $\pi \in \mathcal{P}_\infty \mapsto \#\pi$ is neither continuous with respect to the metric $d$, nor injective, see e.g. [Fou20+, Remark 2.7]. This entails in particular that the process $(\#\Pi(t), t \geq 0)$ is not clearly Markovian.

Following Bertoin’s terminology, see [Ber06, Chapter 2.3], we call any exchangeable partition with no singleton blocks (namely with no dust) a proper partition. A simple application of the paint-box construction of exchangeable partitions allows one to construct a proper initial random exchangeable partition $\pi$ with $\#\pi = n$ almost surely for any $n \in \bar{\mathbb{N}}$. Since by the assumption, simple EFC processes have homogeneous fragmentations and the fragmentation measure has its support on partitions containing no singletons, the simple EFC process $(\Pi(t), t \geq 0)$, when started from a proper partition, stays proper at any time. We refer to Bertoin [Ber03] for the fact that there is no formation of dust in homogeneous fragmentation processes.

Denote by $\tau^+_n := \inf\{t > 0; \#\Pi(t) = \infty\}$ the first explosion time of $(\#\Pi(t), t \geq 0)$. Recall the convention $\inf\emptyset = \infty$. According to [Fou20+, Proposition 2.11], for any $n \in \mathbb{N}$, the process $(\#\Pi(t), t < \tau^+_n)$ started from $(\#\Pi(0) = n)$ is a Markov process, and we denote its law by $\mathbb{P}_n$. Its dynamics can be explained from those of $(\Pi(t), t \geq 0)$ as follows.

**Coalescence.** Associate to each atom of $\text{PPP}_C$, $(t, \pi^c)$, a sequence of random variables $(X_i, i \geq 1)$ such that $X_1 = 1$ if $\{i\} \notin \pi^c$ and $X_i = 0$ if $\{i\} \in \pi^c$. The random variables $(X_i, i \geq 1)$ are mixtures of i.i.d Bernoulli random variables with parameter $x$ whose “intensity” is of the form $x^{-2}\Lambda(dx)$ for some finite measure $\Lambda$ on $[0,1]$. Upon the arrival of an atom of $\text{PPP}_C$, given $\#\Pi(t) = n$, all blocks whose index $j \in [n]$ satisfies $X_j = 1$ are merged. Given the parameter $x$ of the $X_i$’s, the number of blocks that merge at time $t$ has a binomial law with parameters $(n,x)$. Therefore, for any $k \in [2,n]$ the jump such that

$$\#\Pi(t) - \#\Pi(t-) = -(k-1),$$

has rate $\binom{\pi_{n,k}}{k}$$\lambda_{n,k}$ where we recall $\lambda_{n,k} := \int_{[0,1]} x^k(1-x)^{n-k}x^{-2}\Lambda(dx)$. Binary coalescences are hidden in the description above. They are governed by the Kingman parameter $\Lambda(\{0\}) := c_k \geq 0$. We shall always assume that $\Lambda$ has no mass at 1, so that it is impossible for all the blocks to coagulate simultaneously.

**Fragmentation.** Associate to each atom of $\text{PPP}_F$, $(t, \pi^f, j)$, the random variable $k := \#\pi^f - 1$. This provides a Poisson point process on $\mathbb{R}_+ \times \bar{\mathbb{N}} \times \mathbb{N}$ with intensity $dt \otimes \mu \otimes \#$, where $\mu$ is the image of $\mu_{\text{Frag}}$ by the map $\pi \mapsto \#\pi - 1$. Upon the arrival of an atom $(t, \pi^f, j)$, given $\#\Pi(t-) = n$, if $j \leq n$, then the $j^{th}$-block is fragmented into $k + 1$ blocks. Therefore at time $t$,

$$\#\Pi(t) - \#\Pi(t-) = k,$$

(2.4)

Since there are $n$ blocks at time $t-$, the total rate at which a jump of the form (2.4) occurs is $n\mu(k)$ for any $k \in \bar{\mathbb{N}}$.

The generator $\mathcal{L}$ of $(\#\Pi(t), t < \tau^+_n)$ acts on

$$\mathcal{D} := \left\{ g : \bar{\mathbb{N}} \rightarrow \mathbb{R}; \forall n \in \bar{\mathbb{N}}, \sum_{k \in \mathbb{N} \cup \{\infty\}} |g(n+k)|\mu(k) < \infty \right\},$$

(2.5)

as follows:

$$\mathcal{L}g := \mathcal{L}^c g + \mathcal{L}^f g$$

(2.6)
with for \( n \in \mathbb{N} \)

\[
\mathcal{L}^f g(n) := \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k} [g(n-k+1) - g(n)] \quad \text{and} \quad \mathcal{L}^g g(n) := n \sum_{k \in \mathbb{N} \cup \{\infty\}} \mu(k) [g(n+k) - g(n)]
\]

where \( \mathcal{L}^g g(n) \) vanishes if \( n = 1 \).

When \( \mu(\infty) = 0 \), the generator \( \mathcal{L} \) is conservative and \( \infty \) cannot be reached by a single jump. However, it might still happen that infinitely many fragmentations have accumulated and pushed the sample path to “reach” \( \infty \) in finite time. Following the usual terminology for continuous-time Markov chains (CTMCs for short), we call this event explosion. The question whether explosion occurs or not, requires a deep study of the generator \( \mathcal{L} \). This is the main goal of the article and from now on we shall always assume \( \mu(\infty) = 0 \).

It is important to notice that since the partition-valued process \((\Pi(t), t \geq 0)\) has an infinite lifetime (namely, is defined at any time \( t \)), the process \((\#\Pi(t), t \geq 0)\) is also well-defined at any time \( t \), as a process evolving in \( \mathbb{N} \), and typically is defined after explosion (if explosion occurs). Last, in view of the possible jumps of the block-counting process, if \( \mu(\mathbb{N}) > 0 \) then \( \mathbb{N} \) is a communication class for the process \((\#\Pi(t), t \geq 0)\). Indeed, let \( n_0 \) and \( n_1 \) be two integers, if \( n_1 < n_0 \), the process started from \( n_0 \) can reach \( n_1 \) by a coalescence of \( n_0 - n_1 + 1 \) blocks. If \( n_1 > n_0 \) and there is \( k \in \mathbb{N} \) such that \( \mu(k) > 0 \), then after \( n_1 \) jumps of size \( k \), the process started at \( n_0 \) has reached the state \( n_0 + kn_1 \), and from the latter can reach \( n_1 \) by a coalescence involving \( n_0 + (k-1)n_1 + 1 \) blocks.

**Remark 2.1.** Since the map \( \pi \mapsto \#\pi \) is not continuous with respect to the metric \( d \), the Markov property of \((\#\Pi(t), t \geq 0)\) at any time \( t \) such that \( \#\Pi(t) = \infty \) is not straightforward. Indeed, the law of the process \((\#\Pi(t+s), s \geq 0)\) could depend on \( \Pi(t) \) and not only on the fact that \( \#\Pi(t) = \infty \). We refer to \([FZ20+, Remark 2.14]\) for more details. We stress that our proofs only make use of the Markov properties of the processes \((\Pi(t), t \geq 0)\) and \((\#\Pi(t), t < \tau_\infty)\). The Markov property of the block-counting process actually holds true and is established by a duality relationship with A-Wright-Fisher process with selection, see \([FZ20+, Theorem 3.7]\).

Later on, we shall also be interested in the process \((\#\Pi(t), t \geq 0)\) when \( \Pi(t), t \geq 0 \) is started from an exchangeable partition with infinitely many blocks. We recall here how to define on the same probability space as \((\Pi(t), t \geq 0)\), a monotone coupling of \((\Pi(t), t \geq 0)\) in the first initial blocks, when all blocks of \( \Pi(0) \) are infinite. Assume \( \Pi(0) \) proper and \( \#\Pi(0) = \infty \) a.s. Let \( n \in \mathbb{N} \). Set \((\Pi^{(n)}(t), t \geq 0)\) the process started from \( \Pi^{(n)}(0) := \{\Pi_1(0), \cdots, \Pi_n(0)\} \) contracted from \( \text{PPP}_C \) and \( \text{PPP}_F \) as follows:

\[
\Pi^{(n)}(t) = \begin{cases} \text{Coag} \left( \Pi^{(n)}(t^-), \pi^c \right) & \text{if} \ (t, \pi^c) \ \text{is an atom of} \ \text{PPP}_C, \\
\text{Frag} \left( \Pi^{(n)}(t^-), \pi^{f,j} \right) & \text{if} \ (t, \pi^{f,j}) \ \text{is an atom of} \ \text{PPP}_F.
\end{cases}
\]

Loosely speaking, the process \((\Pi^{(n)}(t), t \geq 0)\) follows the coalescences and fragmentations in the first \( n \) initial blocks of \( \Pi \). We refer to \([Fou20+, Lemma 3.3]\) for details on the Poisson construction.

The following lemma will play a crucial role in our proofs. See \([Fou20+, Lemma 3.4]\).

**Lemma 2.2.** Assume that \( \Pi(0) \) is proper. Then almost surely for all \( t \geq 0 \),

\[
\#\Pi^{(n)}(t) \leq \#\Pi^{(n+1)}(t) \quad \text{for all} \ n \geq 1 \quad \text{and} \quad \lim_{n \to \infty} \#\Pi^{(n)}(t) = \#\Pi(t).
\]

In addition, letting \( \tau_\infty^+ \) be the first explosion time of the process \((\#\Pi^{(n)}(t), t \geq 0)\), then for all \( n \in \mathbb{N}, \)

\[
(\#\Pi^{(n)}(t), t < \tau_\infty^+) \quad \text{has the same law as} \quad (\#\Pi(t), t < \tau_\infty^+) \quad \text{when} \quad \#\Pi(0) = n.
\]

**Remark 3.1.** It is necessary that the initial partition \( \Pi(0) \) is proper, namely that each of its blocks is infinite, for \((\#\Pi^{(n)}(t), 0 \leq t < \tau_\infty^+)\) to have the same law as \((\#\Pi(t), 0 \leq t < \tau_\infty^+)\) started from \( n \). Indeed, if a fragmentation event, at a time, say \( t > 0 \), involves a block at time \( t^- \) which is finite, then the number of sub-blocks created after dislocating this block would depend on the block shape at time \( t^- \), and the Markov property would be lost. See Proposition 2.11 in \([Fou20+]\) and its proof.
2.2. Coming down from infinity for $\Lambda$-coalescent processes

Recall the Poisson description given in Section 2.1 and consider a pure $\Lambda$-coalescent process $(\Pi(t), t \geq 0)$. Pitman [Pit99, Proposition 23] has established a zero-one law for the coming down from infinity of $\Lambda$-coalescents. Under the assumption $\Lambda(\{1\}) = 0$, when started from a partition with infinitely many blocks, either the process comes down from infinity a.s., or it stays infinite a.s.:

$$\exists t > 0, \# \Pi(t) < \infty \text{ a.s. or } \forall t \geq 0, \# \Pi(t) = \infty \text{ a.s.}$$ (2.7)

Pitman [Pit99] has also shown that when a $\Lambda$-coalescent comes down from infinity, then provided $\Lambda(\{1\}) = 0$, it does it instantaneously a.s. that is to say if we set $\tau^{-\infty} := \inf\{t > 0; \# \Pi(t) < \infty\}$, then $\tau^{-\infty} = 0$ a.s. The following necessary and sufficient condition for coming down from infinity of $\Lambda$-coalescents was discovered by Schweinsberg [Sch00]. Define for any $n \geq 2$,

$$\Phi(n) := \sum_{k=2}^{n} (k-1) \binom{n}{k} \lambda_{n,k}. \quad (2.8)$$

The $\Lambda$-coalescent $(\Pi(t), t \geq 0)$ comes down from infinity if and only if

$$\sum_{n=2}^{\infty} \frac{1}{\Phi(n)} < \infty \quad \text{(Schweinsberg’s condition).} \quad (2.9)$$

The map $\Phi$ will play an important role in the sequel and we recall some of its properties. For any $n \geq 2$, $\Phi(n)$ represents the rate of the total reduction of the block-counting process $(\# \Pi(t), t \geq 0)$, when it starts from $n$. We stress first that simple binomial calculations entail that for any $n \geq 2$

$$\Phi(n) = \frac{c_k}{2} n(n-1) + \int_{(0,1)} ((1-x)^n + nx - 1)x^{-2}\Lambda(dx). \quad (2.10)$$

One can also check that the map $n \mapsto \Phi(n)/n$ is non-decreasing and by the inequalities

$$(1-x)^n + nx - 1 \leq e^{-nx} + nx - 1 \leq \frac{n^2}{2} x^2$$

for any $x \in [0,1]$, we see that $\Phi(n) \leq \Psi(n) \leq \frac{\Lambda(0,1)}{2} n^2$ for all $n \geq 2$, with

$$\Psi(n) = \frac{c_k}{2} n^2 + \int_{(0,1)} \left(e^{-nx} - 1 + nx\right)x^{-2}\Lambda(dx). \quad (2.11)$$

We mention also the equivalence $\Phi(n) \sim_{n \to \infty} \Psi(n)$. We refer for these properties of the function $\Phi$ to Berestycki’s book [Beres09, Chapter 4] and Limic and Talarczyk [LT15, Lemma 2.1]. In particular, when the measure $x^{-2}\Lambda(dx)$ is regularly varying near 0, the Tauberian theorem and the equivalence above ensure that $\Phi$ is regularly varying at $\infty$. For instance, if $\Lambda$ satisfies (1.1) then $\Phi(n) \sim_{n \to \infty} dm^{1+\beta}$ with $d := \frac{\Gamma(1-\beta)}{\beta (\Gamma(3+\beta))} > 0$ where $\Gamma$ is the Gamma function. This holds for instance in the case of the Beta-coalescent, see e.g. [Beres09].

2.3. Coming down from infinity for simple EFC processes

The coming down from infinity of simple EFC processes, namely the possibility to visit partitions with finitely many blocks, when started from a partition with an infinite number of blocks, has been studied in [Fou20+]. As noticed in Berestycki [Beres04] and in [Fou20+, Lemma 2.5], the dichotomy (2.7) also holds when fragmentations are added. Similar to a pure $\Lambda$-coalescent (under the assumption $\Lambda(\{1\}) = 0$), an EFC either comes down from infinity instantaneously or stays infinite.
When the pure A-coalescent process stays infinite, i.e., under the condition $\sum_{n=2}^{\infty} \frac{1}{\Phi(n)} = \infty$, any EFC process with same coalescence measure $\Lambda$ also stays infinite. In order to deal with the case $\sum_{n=2}^{\infty} \frac{1}{\Phi(n)} < \infty$, the two following parameters have been introduced in [Fou20+],

$$\theta^* := \limsup_{n \to \infty} \sum_{k=1}^{\infty} \frac{n \bar{\mu}(k)}{\Phi(n + k)} \in [0, \infty] \quad \text{and} \quad \theta_* := \liminf_{n \to \infty} \sum_{k=1}^{\infty} \frac{n \bar{\mu}(k)}{\Phi(n + k)} \in [0, \infty].$$

(2.12)

**Theorem 2.4** (Theorem 1.1 in [Fou20+]). Let $(\Pi(t), t \geq 0)$ be a simple EFC process started from an exchangeable random partition such that $\#\Pi(0) = \infty$.

Assume $\sum_{n=2}^{\infty} \frac{1}{\Phi(n)} < \infty$.

- If $\theta^* < 1$, then $(\Pi(t), t \geq 0)$ comes down from infinity a.s.
- If $\theta_* > 1$, then $(\Pi(t), t \geq 0)$ stays infinite a.s.

The parameters $\theta^*$ and $\theta_*$ are somewhat intricate since both arguments $n$ and $k$ are not separated in the sum. However they coincide in many regular cases (if so, we denote the value by $\theta$) and can be computed explicitly when both $\mu$ and $\Phi$ are regularly varying.

**Proposition 2.5** (Regularly-varying cases, Proposition 1.6 in [Fou20+]).

If $\Phi(n) \sim n^{-\alpha}$, $\beta \in (0, 1)$ and $\mu(n) \sim n^{-\alpha}$ with $\alpha \in (0, 1)$ and $b > 0$, then

1. $\theta = \infty$ for $\beta < 1 - \alpha$,
2. $\theta = 0$ for $\beta > 1 - \alpha$,
3. $\theta = b \frac{1}{\alpha(1-\alpha)} \in (0, \infty)$ for $\beta = 1 - \alpha$.

Note that when $\mu(n) \sim n^{-\alpha}$, one has $\bar{\mu}(n) \sim \frac{\lambda}{n^{\alpha}}$ with $\lambda = b/\alpha$. According to Theorem 2.4, if $b/d < \alpha(1-\alpha)$, i.e., $\theta < 1$, then the process comes down from infinity. We shall see later that it does not always entail the non-explosion of the process $(\#\Pi(t), t \geq 0)$, see the last statement in Theorem 3.5.

In the next section, we briefly summarize some results on the explosion of pure branching processes.

### 2.4. Explosion in branching processes

Consider an immortal pure branching process $(N_t, t \geq 0)$ with offspring measure $\mu$, namely a process whose generator is $\mathcal{L}^f$. It is well-known that some of these processes can explode in finite time even though the generator is conservative, i.e., $\mu(\infty) = 0$. If one denotes by $\varphi$ the generating function of the renormalized measure $\mu(\cdot)/\mu(N)$, then explosion occurs if and only if

$$\int_{1}^{\infty} \frac{dx}{x^\alpha - \varphi(x)} < \infty \quad \text{Dynkin’s condition}.$$

We refer the reader to Harris’ book [Har63, Chapter V, Section 9, Theorem 9.1]. We now recall a necessary and sufficient condition due to Doney [Don84] and Schuh [Sch82]. For any $n \geq 1$,

$$\ell(n) := \sum_{k=1}^{n} \bar{\mu}(k),$$

(2.13)

where for any $k \in \mathbb{N}$, $\bar{\mu}(k) := \mu(\{k, k + 1, \ldots\})$. The process $(N_t, t \geq 0)$ whose generator is $\mathcal{L}^f$, see (2.6), explodes if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n\ell(n)} < \infty \quad \text{Doney’s condition}.$$

(2.14)

We mention that Doney’s condition cannot be simplified, as Grey [Gre89] has shown that explosion of a pure branching process cannot be expressed in terms of a moment condition on $\mu$. A simple application
of Fubini’s theorem shows that
\[
\ell(n) = \sum_{k=1}^{\infty} (k \wedge n) \mu(k) = \sum_{k=1}^{n} k \mu(k) + n \bar{\mu}(n+1).
\]

(2.15)

Doney’s condition does not require us to work with the generating function of \( \mu \) and can be used for building many examples of explosive branching processes. If for instance, \( \ell(n) \geq (\log n)^r \) for large enough \( n \), with \( r > 1 \), then (2.14) is satisfied and the branching process \((N_t, t \geq 0)\) explodes almost surely. Similarly if \( \ell(n) \leq (\log n)^r \) for large enough \( n \) with \( r \leq 1 \), then (2.14) is not satisfied and the process does not explode.

We now introduce a condition on the map \( \ell \).

**Condition \( H \):** there exists an eventually non-decreasing positive function \( g \) such that \( \int_{x_0}^{\infty} \frac{dx}{xg(x)} < \infty \) and
\[
\ell(n) \geq g(\log n) \log n \text{ for large enough } n.
\]

(\( \mathbb{H} \))

This latter condition covers a rather broad class of splitting measures since for instance, all measures \( \mu \) for which, for large enough \( n \)
\[
\ell(n) \geq (\log^k n)^r \log^{k-1} n \times \cdots \times \log^2 n \log n,
\]
with \( k \geq 1 \) and \( r > 1 \) (where \( \log^k \) denotes the \( k \)-iterated logarithm), satisfy \( \mathbb{H} \). We also stress that if \( \mu \) satisfies (1.2) with \( \alpha \in (0, 1) \) then
\[
\ell(n) \underset{n \to \infty}{\sim} \frac{b}{\alpha(1-\alpha)} n^{1-\alpha} \text{ for some } b > 0.
\]

In particular, if \( \mu \) satisfies (1.2), then condition \( \mathbb{H} \) is satisfied.

A simple comparison of the series \( \sum_{n \geq 1} \frac{1}{n \ell(n)} \) with the integral \( \int_{x_0}^{\infty} \frac{dx}{xg(x)} \) shows that if \( \mathbb{H} \) holds then Doney’s condition for explosion (2.14) is satisfied. Condition \( \mathbb{H} \) will enable us to find rather sharp conditions for explosion when coalescences are taken into account.

**Remark 2.6.**
1. Doney’s proof is based on the representation of the explosion time of a branching process as a perpetual integral for a continuous-time left-continuous random walk. Such approach for studying explosion is classical for processes that are obtained through random time changes of other processes. We refer for instance to the recent works of Kühn [Kuh19] and Li and Zhou [LZ18]. We shall not follow this approach here but will look for sufficient conditions based on “local” estimates on the generator. We will establish in the forthcoming Section 4, a new sufficient condition for explosion of continuous-time Markov chains, see Theorem 4.1. This condition can be seen as belonging to the methods of Lyapunov functions, see e.g. Chow and Khasminskii [CK11] and Menschikov and Petretis [MP14] for recent works on this approach.

2. We are not aware of any proof of Doney’s result based on Lyapunov functions. However, by setting \( \ell(x) = \ell([x]) \) for all \( x \geq 1 \), we observe that if \( x \mapsto \ell(e^x)/x \) is eventually non-decreasing, then \( \ell \) satisfies \( \mathbb{H} \) with \( g(x) := \ell(e^x)/x \) as soon as Doney’s condition holds : \( \sum_{n \geq 1} \frac{1}{n \ell(n)} < \infty \). There are nevertheless examples of measures \( \mu \), satisfying Doney’s condition (2.14) for which this function \( g \) is not eventually monotone.

**3. Main results**

Consider a simple EFC process \((\Pi(t), t \geq 0)\) with coalescence measure \( \Lambda \) and splitting measure \( \mu \). We recall that the boundary \( \infty \) is said to be an exit (respectively, an entrance), if the process \((\#\Pi(t), t \geq 0)\) can reach \( \infty \) but can not leave from \( \infty \) (respectively, can leave from \( \infty \) but can not reach \( \infty \)). The boundary \( \infty \) is called regular if the process \((\#\Pi(t), t \geq 0)\) can both enter \( \infty \) and leave \( \infty \). By convention, when we say that \( \infty \) is an entrance (respectively an exit), it is always implicitly assumed that \( \#\Pi(0) = \infty \) (respectively \( \#\Pi(0) < \infty \)).

We first provide some general conditions on the coalescence measure \( \Lambda \) and the splitting measure \( \mu \) ensuring that the boundary \( \infty \) is either an exit or an entrance.
Theorem 3.1 (Explosion and exit). Assume condition $H$, if $\rho := \limsup_{n \to \infty} \Phi(n) / n(\Phi(n)) < 1/2$, then the process $(\#\Pi(t), t \geq 0)$ explodes almost surely. If furthermore, $\sum_{n=2}^{\infty} \frac{1}{\Phi(n)} < \infty$ and $\rho < 1/4$, then $\infty$ is an exit boundary.

Remark 3.2. For any simple EFC process $(\Pi(t), t \geq 0)$, if $(\#\Pi(t), t \geq 0)$ explodes, then $(\Pi(t), t \geq 0)$ reaches in a finite time almost surely a proper partition with infinitely many blocks. Note also that when $\sum_{n=2}^{\infty} \frac{1}{\Phi(n)} = \infty$, since the EFC process stays infinite, if the process explodes then $\infty$ is an exit.

Theorem 3.3 (Non-explosion and entrance). If $\sum_{n=2}^{\infty} \frac{n}{\Phi(n)} \mu(n) < \infty$, then $(\#\Pi(t), t \geq 0)$ does not explode almost surely. If furthermore, $\sum_{n=2}^{\infty} \frac{1}{\Phi(n)} < \infty$, then $\infty$ is an entrance boundary.

Remark 3.4. When the coalescences are driven by a pure Kingman coalescent, $\Lambda = c_k \delta_0$ with $c_k > 0$, $\Phi(n) = c_k n^{-\alpha} \sim \frac{\Phi(n)}{n} n^2$ and by Theorem 3.3, we see that if $\sum_{n \geq 1} \frac{\mu(n)}{\Phi(n)} < \infty$, then $\infty$ is an entrance boundary. It is equivalent to a log-moment assumption on $\mu$, i.e. $\sum_{k \geq 1} \mu(k) \log k < \infty$, and we recover with a different method a result of Lambert; see [Lam05, Theorem 2.3].

Our next results concern simple EFC processes with regularly varying coagulation/splitting measures. Examples of simple EFC processes for which the boundary is regular are exhibited.

Theorem 3.5. Assume that $\Phi(n) \sim \frac{d n^{1+\beta}}{b}$ with $d > 0$ and $\beta \in (0, 1)$ and $\mu(n) \sim \frac{b}{n^{\alpha+\beta}}$ with $b > 0$ and $\alpha \in (0, \infty)$. Then $n(\mu(n)) \sim n(\Phi(n))^{2-\alpha}$ and the boundary $\infty$ of $(\#\Pi(t), t \geq 0)$ is classified as follows:

- if $\alpha + \beta < 1$, then $\infty$ is an exit boundary;
- if $\alpha + \beta > 1$, then $\infty$ is an entrance boundary;
- if $\alpha + \beta = 1$ and further,
  - if $b/d > \alpha(1 - \alpha)$, then $\infty$ is an exit boundary,
  - if $\frac{\alpha \sin(\pi \alpha)}{\pi} < b/d < \alpha(1 - \alpha)$, then $\infty$ is a regular boundary,
  - if $b/d < \frac{\alpha \sin(\pi \alpha)}{\pi}$, then $\infty$ is an entrance boundary.

Remark 3.6. The first two statements of Theorem 3.5 are consequences of Theorem 3.1 and Theorem 3.3, respectively. The third statement is shown in Section 5.3.

The figure below represents the different possible regimes for the boundary $\infty$, found in Theorem 3.5 when $\alpha + \beta = 1$, according to the location of ratio $b/d$.

![Diagram](Fig 1. Boundary classification when $\Phi(n) \sim \frac{d n^{1+\beta}}{b}$ and $\mu(n) \sim \frac{b}{n^{\alpha+\beta}}$).

The next proposition describes more precisely the behavior of the EFC process $(\Pi(t), t \geq 0)$ with regularly varying coalescence-splitting measures, when its block-counting process has $\infty$ as regular boundary.
We establish that the boundary $\infty$ is regular for itself, in the sense that the block-counting process returns to $\infty$ immediately after having left it.

**Proposition 3.7.** Suppose that the assumptions of Theorem 3.5 hold. If $\beta = 1 - \alpha$ and $\frac{\alpha \sin(\pi \alpha)}{\pi} b/d < \alpha(1 - \alpha)$, then the process $(\Pi(t), t \geq 0)$ started from a partition with infinitely many blocks comes down from infinity and returns instantaneously to a proper partition with infinitely many blocks almost surely.

The critical cases in the last statement of Theorem 3.5 for which the ratio $\frac{b}{d}$ equals $\frac{\alpha \sin(\pi \alpha)}{\pi}$ or $\alpha(1 - \alpha)$ seem to require finer arguments. We find in the next proposition a class of coalescence and splitting measures for which the critical case of $\frac{b}{d} = \frac{\alpha \sin(\pi \alpha)}{\pi}$ can be handled.

**Proposition 3.8.** Let $b, d > 0$, $\alpha \in (0, 1)$ and $h$ be a measurable function on $[0, 1]$ such that $h \geq 1$. Set $\beta = 1 - \alpha$. Assume that $\Lambda(dx) = \frac{d}{c_\beta}x^{-\beta}h(x)dx$ with $c_\beta := \Gamma(1 - \beta)/\beta^{\beta+1}$ and that $\mu(n) = \frac{b}{n^{1+\alpha}}$ for all integer $n \geq 1$. Then, when $\frac{b}{d} = \frac{\alpha \sin(\pi \alpha)}{\pi}$, the boundary $\infty$ is an entrance.

We study now the slower regime of coalescences for which for some $\beta > 0$

$$\Phi(n) \sim \frac{dn(\log n)^\beta}{n^{1+\alpha}}.$$  

(3.16)

We refer the reader to [Fou20+, Section 2.2] for conditions on the coalescence measure $\Lambda$ entailing that the function $\Phi$ has these asymptotics. As we shall see in the next theorem, when (3.16) holds, there is no regular regime when $\eta(n)$ is of the same order as $\Phi(n)$.

**Theorem 3.9.** Let $\beta > 0$, $d > 0$, $b > 0$ and $\alpha > 0$. Assume that $\Phi(n) \sim \frac{dn(\log n)^\beta}{n^{1+\alpha}}$ and $\mu(n) \sim \frac{b(\log n)^\alpha}{n^{1+\alpha}}$. Then $\eta(n) \sim \frac{b}{1+\alpha} n(\log n)^{\alpha+1}$ and

- if $\beta < 1 + \alpha$, then $\infty$ is an exit boundary;
- if $\beta > 1 + \alpha$, then $\infty$ is an entrance boundary;
- if $\beta = 1 + \alpha$,
  - if $b/d > 1 + \alpha$, then $\infty$ is an exit boundary,
  - if $b/d < 1 + \alpha$, then $\infty$ is an entrance boundary.

**Remark 3.10.** The first two statements of Theorem 3.9 are consequences of Theorem 3.1 and Theorem 3.3, respectively. The proof of the third statement is deferred to Section 5.4. The boundary behaviors for the critical case $b/d = 1 + \alpha$ in the last statement of Theorem 3.9 remains unsolved.

### 4. Explosion of a general CTMC on $\mathbb{N}$

We state in this section sufficient conditions for explosion and non-explosion of a general continuous-time Markov chain taking values in $\mathbb{N}$. Explosion in this setting corresponds to accumulations of large jumps in compact intervals of time that are pushing the process to $\infty$. We believe the results of Section 4.1 are of independent interest.

Consider an infinitesimal generator $\mathcal{L} = \mathcal{L}^- + \mathcal{L}^+$ acting on all bounded function $g : \mathbb{N} \to \mathbb{R}_+$ and all $n \in \mathbb{N}$ as follows

$$\mathcal{L}^- g(n) = \sum_{k=1}^{n-1} (g(n-k) - g(n))p^-_{n,k}$$
and $\mathcal{L}^+ g(n) = \sum_{k=1}^{\infty} (g(n+k) - g(n))p^+_{n,k}$

(4.17)

where $p^+_{n,k} \in [0, \infty)$ and $p^-_{n,k} \in [0, \infty)$ are respectively the rates of positive and negative jumps.

Standard theory of Markov processes, see e.g. [And91], ensures that there exists a unique continuous-time Markov chain $(N_t, t \geq 0)$, taking values in $\mathbb{N} \cup \{\infty\}$, with generator $\mathcal{L}$, absorbed at $\infty$ (viewed as a cemetery point) after explosion. Recall the first explosion time denoted by $\tau^+_\infty := \inf\{t > 0 : N_t = \infty\}$. 

4.1. Explosion

The following theorem provides a sufficient condition for explosion to occur with positive probability. Theorem 4.1 has several precursors in the literature for positive real-valued Markov processes without negative jumps, see [LYZ19] and the references therein.

For any $a > 0$ and for any $n \in \mathbb{N}$, set $g_a(n) := n^{1-a}$ and $G_a(n) := -\frac{1}{n^{1-a}} \mathcal{L} g_a(n)$.

**Theorem 4.1.** If there exist $a > 1$ and an eventually non-decreasing positive function $g$ satisfying $\int_{\mathbb{R}} \frac{dx}{x g(x)} < \infty$ such that for all large enough $n$

\[ G_a(n) \geq g(\log n) \log n, \quad (4.18) \]

then $\mathbb{P}_n(\tau^+_{\infty} < \infty) > 0$ for all large enough $n \in \mathbb{N}$.

If moreover, the process is irreducible in $\mathbb{N}$, then $\mathbb{P}_n(\tau^+_{\infty} < \infty) > 0$ for all $n \in \mathbb{N}$.

We adapt the method of Li et al. in [LYZ19, Section 5] where the explosion of nonlinear branching processes is studied. We stress that in our framework, the process has both positive and negative jumps, moreover large negative jumps may occur along large coalescence events. We establish Theorem 4.1 with the help of several lemmas.

For any integers $n < m$, set $\tau_n^- := \inf\{t > 0; N_t \leq n\}$ and $\tau_m^+ := \inf\{t > 0; N_t \geq m\}$. The proof of Theorem 4.1 relies on some estimates on exit probabilities from the interval $[n, m]$ for the process $(N_t, t \geq 0)$. The latter will be obtained using a martingale appearing in the following lemma.

**Lemma 4.2 (Martingale).** Let $n_0$ be a fixed integer and let $n$ and $m$ be two integers such that $n > n_0 > m$. Set $T := \tau_n^- \wedge \tau_m^+$. For $a > 1$, the process

\[ \left( N^{1-a}_{t \wedge T} \exp \left( \int_0^{t \wedge T} G_a(N_s) ds \right), t \geq 0 \right) \]

is a bounded $(\mathcal{F}_t)$-martingale, and

\[ \mathbb{E}_{n_0} \left[ N^{1-a}_{T} \exp \left( \int_0^{T} G_a(N_s) ds \right) \right] \leq n_0^{1-a}. \]

**Proof.** Recall $g_a(n) := n^{1-a}$ and $G_a(n) := -\frac{1}{n^{1-a}} \mathcal{L} g_a(n)$ for all $n \geq 1$. By Dynkin’s formula, the process $(N^{1-a}_{t \wedge T} - \int_0^{t \wedge T} \mathcal{L} g_a(N_s) ds, t \geq 0)$ is a local martingale. Since the quadratic variation process vanishes, i.e. $\langle N^{1-a}_{t \wedge T}, \int_0^{t \wedge T} G_a(N_s) ds \rangle = 0$, by the product rule of Itô’s formula, $(N^{1-a}_{t \wedge T} \exp \left( \int_0^{t \wedge T} G_a(N_s) ds \right), t \geq 0)$ is a local martingale. Observe that for each $t > 0$, $\int_0^{t \wedge T} G_a(N_s) ds$ is bounded. Since $a > 1$, $N^{1-a}_{t \wedge T}$ is bounded from above uniformly for all $t > 0$, and by [Prot05, Theorem I.51], the process

\[ \left( N^{1-a}_{t \wedge T} \exp \left( \int_0^{t \wedge T} G_a(N_s) ds \right), t \geq 0 \right) \]

is a martingale. The inequality (4.19) follows from Fatou’s lemma.

**Lemma 4.3 (Estimates on exit probabilities).** Under the assumption of Theorem 4.1, for $a > 1$, any $n$ large enough and any $m > n_0 > n$, we have

\[ \mathbb{P}_{n_0}(\tau_n^- < \tau_m^+) \leq \left( \frac{n}{n_0} \right)^{a-1}, \quad (4.20) \]

\[ \mathbb{P}_{n_0}(\tau_n^- = \tau_m^+ = \infty) = 0. \quad (4.21) \]
For any $u > 0$, we have
\[ P_{n_0}(\tau^-_n > \tau^+_m > u) \leq \left( \frac{n_0}{m} \right)^{1-a} e^{-ug(\log n)\log n}. \] (4.22)

For $0 < \delta < 1/(2a-1)$, define $t(y) := \frac{1}{g(\log y^{1-a})}$ for any $y \geq 1$. We have
\[ P_{n_0}(\tau^+_{[n_0+1]} > t(n_0)) \leq (2 + 2^{a-1})n_0^{\delta(1-a)}. \] (4.23)

**Proof.** By the assumption, $G_\alpha(n) \geq g(\log n)\log n$ for large enough $n$. In particular, for any $s < \tau_n^-$, $N_s \geq n$ and if $n$ is large enough then $G_\alpha(N_s) \geq 0$. By Lemma 4.2,
\[ n_0^{1-a} \geq E_{n_0} \left( N_1^{1-a} \exp \left( \int_0^T G_\alpha(N_s)ds \right) \mathbb{1}_{\{\tau^-_n < \tau^+_m\}} \right) \geq n^{1-a}P_{n_0}(\tau^-_n < \tau^+_m). \]

Thus, (4.20) is established.

Applying Lemma 4.2 to the bounded stopping time $T \wedge u$, we get
\[ n_0^{1-a} \geq E_{n_0} \left( N_1^{1-a} \exp \left( \int_0^u G_\alpha(N_s)ds \right) \mathbb{1}_{\{\tau^-_n \wedge \tau^+_m > u\}} \right). \] (4.24)

Moreover, when $s \leq u < \tau^-_n \wedge \tau^+_m$, $N_s \in [n, m]$ and thus $G_\alpha(N_s) \geq g(\log n)\log n$. The inequality (4.24) provides (4.22).
\[ P_{n_0}(\tau^-_n \wedge \tau^+_m > u) \leq e^{-ug(\log n)\log n} \left( \frac{n_0}{m} \right)^{1-a}. \]

Letting $u$ go to $\infty$ yields (4.21).

For $\delta \in (0, 1)$, put $n = \lceil n_0^{1-\delta} \rceil$ and $m = \lceil n_0^{1+\delta} \rceil$. We show now that $P_{n_0}(\tau^-_n > \tau^+_m > t(n_0)) \leq 2n_0^{a\delta-1}$. By (4.22),
\[ P_{n_0}(\tau^-_n > \tau^+_m > t(n_0)) \leq \left( \frac{n_0}{m} \right)^{1-a} e^{-t(n_0)g(\log n)\log n} \leq \left( \frac{n_0}{m} \right)^{1-a} e^{-g(\log n_0^{1-\delta})g(\log n)\log n}. \]

Since $n \geq n_0^{1-\delta}$, we have that
\[ \log n \geq \log n_0^{1-\delta} \quad \text{and} \quad (g(\log n_0^{1-\delta}))^{-1}g(\log n)\log n \geq \log (n_0^{1-\delta}). \]

Recall $a > 1$ and $m = \lceil n_0^{1+\delta} \rceil \leq n_0^{1+\delta} + 1$. This entails
\[ P_{n_0}(\tau^-_n > \tau^+_m > t(n_0)) \leq \left( \frac{n_0}{m} \right)^{1-a} e^{-\log(n_0^{1-\delta})} \leq \left( \frac{n_0}{m} \right)^{1-a} n_0^{1-\delta} \leq 2n_0^{(a-1)\delta}n_0^{-1} = 2n_0^{a\delta-1}. \] (4.25)

One thus has
\[ P_{n_0}(\tau^+_m > t(n_0)) \leq P_{n_0}(\tau^-_n > \tau^+_m > t(n_0)) + P(\tau^-_n < \tau^+_m) + P(\tau^-_n = \tau^+_m = \infty). \]

Combining (4.20), (4.21) and (4.25), we finally get
\[ P_{n_0}(\tau^+_m > t(n_0)) \leq 2n_0^{a\delta-1} + \left( \frac{n}{n_0} \right)^{a-1}. \] (4.26)
Since $n = [n_0^{1-\delta}]$, then 
\[
\left( \frac{n}{n_0} \right)^{a-1} \leq \left( \frac{n_0^{1-\delta} + 1}{n_0} \right)^{a-1} = n_0^{-\delta(a-1)} \left( 1 + n_0^{\delta(a-1)} \right)^{a-1}.
\]

Since $a > 1$ and $\delta < \frac{1}{2a-1} < 1$, $n_0^{\delta(a-1)} \leq 1$ and we get $\left( \frac{n}{n_0} \right)^{a-1} \leq 2^{a-1}n_0^{-\delta(a-1)}$. Moreover, $\delta < \frac{1}{2a-1}$, thus $a\delta - 1 \leq \delta(1 - a)$ and we deduce from (4.26) that
\[
\mathbb{P}_{n_0}(\tau_m^+ > t(n_0)) \leq 2n_0^{a\delta-1} + 2^{a-1}n_0^{-\delta(a-1)} \leq (2 + 2^{a-1})n_0^{\delta(1-a)} = (2 + 2^{a-1})n_0^{\delta(1-a)}.
\]

\hfill \Box

Using the estimates on the exit probabilities given in Lemma 4.3, we will show that there is an accumulation of positive jumps pushing up the process to infinity in finite time with positive probability. This will finish the proof of Theorem 4.1.

**Lemma 4.4.** Under the assumptions of Theorem 4.1, for any $a > 1$ and $0 < \delta < \frac{1}{2a-1}$, we have for large enough $n_0$,
\[
\mathbb{P}_{n_0}(\tau_{\infty}^+ \leq \int_{\frac{m}{\log n_0^{1-\delta}}}^{\infty} \frac{dv}{vg(v)}) \geq \prod_{k=0}^{\infty} h_a(k, n_0) > 0 \tag{4.27}
\]

with $h_a(k, n_0) := 1 - (2 + 2^{a-1}) \left( \frac{1}{n_0^{(a-1)}} \right)^{(1+\delta)k}$ for all $k \in \mathbb{Z}_+$.

**Proof.** Define $(\tilde{\tau}_k)$ recursively by $\tilde{\tau}_0 := 0$ and
\[
\tilde{\tau}_{k+1} := \tau_{\left[ N_{\tilde{\tau}_k}^{1+\delta} \right]} \circ \theta_{\tilde{\tau}_k} + \tilde{\tau}_k,
\]
where $\theta_t$ is the shift operator and with the convention that on the event $\tilde{\tau}_k = \infty$ we set $N_{\tilde{\tau}_k} = 1$. By the construction, under $\mathbb{P}_{n_0}$ for any $k \geq 0$, $N_{\tilde{\tau}_k} \geq n_0^{(1+\delta)k}$ a.s. on the event $\{\tilde{\tau}_k < \infty\}$. Therefore, if $k \to \infty$, then
\[
N_{\tilde{\tau}_k} = \lim_{k \to \infty} N_{\tilde{\tau}_k} \geq \lim_{k \to \infty} n_0^{(1+\delta)k} = \infty.
\]

Hence, recalling that $\tau_{\infty}^+ := \inf\{ t > 0; N_t = \infty \}$, we have $\tilde{\tau}_{\infty} = \tau_{\infty}^+$ a.s. Intuitively, the event $\{\tilde{\tau}_{\infty} < \infty\}$ corresponds to having an accumulation of positive jumps (in our case fragmentation events) in which the number of fragments increases by a factor at least of order $N_{\tilde{\tau}_k}^k$ between times $\tilde{\tau}_k$ and $\tilde{\tau}_{k+1}$. We study $\tilde{\tau}_{\infty}$.

Recall $t(y) = g(y)g^{-1}(y)$. Since the function $g$ is eventually nondecreasing, if $\tilde{\tau}_k < \infty$, we have
\[
t(N_{\tilde{\tau}_k}) \leq (g(\log n_0^{(1+\delta)k} - (1-\delta)))^{-1} = (g((1+\delta)^k \log n_0^{\delta}))^{-1}.
\]

Therefore, by the assumption on $g$, for any large $m \geq 0$
\[
\sum_{k=m}^{\infty} t(N_{\tilde{\tau}_k}) \leq \sum_{k=m}^{\infty} \frac{1}{g((1+\delta)^k \log n_0^{1-\delta})}
\]
\[
= \frac{1 + \delta}{\delta} \sum_{k=m}^{\infty} \frac{(1+\delta)^k - (1+\delta)^{k-1}}{g((1+\delta)^k \log n_0^{1-\delta})}
\]
\[
\leq \frac{1 + \delta}{\delta} \sum_{k=m}^{\infty} \int_{(1+\delta)^{k-1}}^{(1+\delta)^k} \frac{1}{ug(u \log n_0^{1-\delta})} du
\]
\[
\leq \frac{1 + \delta}{\delta} \int_{(1+\delta)^{m-1}}^{\infty} \frac{1}{ug(u \log n_0^{1-\delta})} du
\]
\[
= \frac{1 + \delta}{\delta} \int_{(1+\delta)^{m-1} \log n_0^{1-\delta}}^{\infty} \frac{dv}{vg(v)} < \infty \quad \text{a.s.} \tag{4.28}
\]
Conditionally on $N_{\tilde{\tau}_k}$, one has using (4.23),
\[ \mathbb{P}_{N_{\tilde{\tau}_k}} \left( \tau^+_\infty \leq t(N_{\tilde{\tau}_k}) \right) \geq 1 - (2 + 2^{a-1}) \leq \left( \frac{1}{n_0 \delta(a-1)} \right)^{(1+\delta)k} =: h_a(k, n_0). \]

Using the strong Markov property in the second inequality below, we get for $n_0$ large enough,
\[ \mathbb{P}_{n_0} \left( \tilde{\tau}^+_\infty \leq \sum_{k=0}^{\infty} t(N_{\tilde{\tau}_k}) \right) \geq \mathbb{P}_{n_0} \left( \tilde{\tau}_{k+1} - \tilde{\tau}_k < t(N_{\tilde{\tau}_k}), \forall k \geq 0 \right) \geq \prod_{k=0}^{\infty} h_a(k, n_0) > 0. \]

Combining (4.28) and (4.29) and using the fact that $\tilde{\tau}^+_\infty = \tau^+_\infty$ a.s., we get
\[ \mathbb{P}_{n_0} \left( \tau^+_\infty \leq \int_{\mathbb{R}^+} \log n \frac{dv}{vg(v)} \right) \geq \prod_{k=0}^{\infty} \left( 1 - (2 + 2^{a-1}) \left( \frac{1}{n_0 \delta(a-1)} \right)^{(1+\delta)k} \right) > 0 \]

which ensures that explosion has a positive probability when the process starts from a large enough $n_0 \in \mathbb{N}$. \hfill \Box

**Proof of Theorem 4.1.** This is a consequence of Lemma 4.4. \hfill \Box

In order to see how the positive and negative jumps interplay in the condition (4.18), we introduce the following functions. For any $a > 0$,
\[ G^+_a(n) := \frac{1}{n^{1-a}} \mathcal{L}^- g_a(n), \quad G^-_a(n) := \frac{1}{n^{1-a}} \mathcal{L}^+ g_a(n). \]

Note that for any $a$, since the sum in the expression of $\mathcal{L}^-$ is finite, then $G^-_a$ is well-defined. When $a \geq 1$, since $g_a$ is bounded, the infinite sum in the expression of $G^+_a$ is meaningful. Moreover, when $a > 1$, $G^+_a(n) \geq 0$ and $G^-_a(n) \leq 0$.

The next corollary states a simpler condition for explosion in terms of $G^+_a$ and $G^-_a$.

**Corollary 4.5.** Assume that there are $a > 1$ and a non-decreasing positive function $g$ such that $\int_{\mathbb{R}^+} \frac{dx}{vg(x)} < \infty$, and for large enough $n$, $G^+_a(n) \geq g(\log n) \log n$. If $\gamma_a := \limsup_{n \to \infty} \frac{-G^-_a(n)}{G^+_a(n)} < 1$, then Condition (4.18) holds and $\infty$ is accessible from any large enough initial state. If moreover, the process is irreducible in $\mathbb{N}$ then $\mathbb{P}_n(\tau^+_\infty < \infty) > 0$ for all $n \in \mathbb{N}$.

**Proof.** By definition of $\gamma_a$ and the assumption $\gamma_a < 1$, there is an integer $n_0$ such that for all $n \geq n_0$,
\[ \frac{-G^-_a(n)}{G^+_a(n)} \leq \gamma_a + (1 - \gamma_a)/2 = (\gamma_a + 1)/2 < 1. \]

Hence, by definition of $G^-_a$ and $G^+_a$, for all $n \geq n_0$,
\[ -G_a(n) = -G^-_a(n) - G^+_a(n) = -G^+_a(n) \left( 1 - \frac{-G^-_a(n)}{G^+_a(n)} \right) \leq -cg(\log n) \log n \]

where $c := \frac{1 - \gamma_a}{2} \in (0, 1)$, and Condition (4.18), for Theorem 4.1 to apply, holds with the function $x \mapsto cg(x)$.

\hfill \Box
4.2. Non-explosion

We recall here a classical Foster-Lyapunov’s sufficient condition for non-explosion. We refer for instance to Chow and Khasminskii [CK11] and the references therein. If \( f : n \mapsto f(n) \) is non-decreasing, \( f(n) \xrightarrow{n \to \infty} \infty \) and

\[ \mathcal{L}f(n) \leq cf(n) \text{ for all } n \geq 1 \]  

(4.31)

for some \( c > 0 \), then the (minimal) continuous-time Markov chain \((N_t, t \geq 0)\) with generator \( \mathcal{L} \) does not explode from all initial states.

Choose the Lyapunov function \( f \) in (4.31) as the function \( g_a : n \mapsto n^{1-a} \) with \( 0 < a < 1 \). For any \( 0 < a < 1 \), \( g_a(n) \xrightarrow{n \to \infty} \infty \) and a simple sufficient condition entailing (4.31) and thus non-explosion is \( \mathcal{L}g_a(n) \leq 0 \) for large enough \( n \). Recall \( \mathcal{L}^{\pm}g_a(n) =: -n^{1-a}G_a^{\pm}(n) \). Note that when \( a < 1 \), \( G_a^{\pm}(n) \leq 0 \) and \( G_a^{-}(n) \geq 0 \) for all \( n \). One has the following proposition.

**Proposition 4.6.** If there exists \( a < 1 \) such that \( \mathcal{L}^{\pm}g_a(n) < \infty \) for all \( n \in \mathbb{N} \), and

\[ \limsup_{n \to \infty} -\frac{G_a^{+}(n)}{G_a^{-}(n)} < 1, \]  

(4.32)

then \( \mathcal{L}g_a(n) \leq 0 \) for large enough \( n \) and Condition (4.31) holds with \( f = g_a \).

**Proof.** For any \( n \in \mathbb{N} \), \( \mathcal{L}g_a(n) = -n^{1-a}G_a^{-}(n) \left( 1 - \frac{G_a^{+}(n)}{G_a^{-}(n)} \right) \). If (4.32) holds, then there is a large \( n_0 \) such that for all \( n \geq n_0 \), \( -\frac{G_a^{+}(n)}{G_a^{-}(n)} < 1 \). Since for all \( n \geq n_0 \), \( -n^{1-a}G_a^{-}(n) \leq 0 \) and \( 1 - \frac{G_a^{+}(n)}{G_a^{-}(n)} \geq 0 \), we have that \( \mathcal{L}g_a(n) \leq 0 \) for all \( n \geq n_0 \). Set \( c := \max_{1 \leq n \leq n_0 - 1} \left| \frac{\mathcal{L}g_a(n)}{g_a(n)} \right| \), then we have \( \mathcal{L}g_a(n) \leq cg_a(n) \) for all \( n \geq 1 \). \( \Box \)

5. Proofs of the main results

5.1. A sufficient condition for explosion of \((\#\Pi(t), t \geq 0)\)

Recall the condition \( H \) and the statement of Theorem 3.1. We will study the explosion by applying Corollary 4.5 and Proposition 4.6. Recall the functions \( \ell \) and \( \Phi \) governing the fragmentations and the coagulations defined respectively in (2.13) and (2.8) and the parameter \( \rho = \limsup_{n \to \infty} \frac{\Phi(n)}{n^2} \).

We start by explaining the main idea of the proof of the explosion when \( \rho < 1/2 \). We shall follow a similar route in order to show Theorem 3.5 and Theorem 3.9. Fix \( p \in (0, 1) \) and \( n_0 \in \mathbb{N} \). Recall \((\#\Pi^{(n_0)}(t), t \geq 0)\) defined in Section 2, Lemma 2.2 and the generator \( \mathcal{L} \) in (2.6). Consider the process \((\#\Pi^{(n_0)}(t), t \geq 0)\) stopped at the first coalescence time at which the number of blocks decreases by a proportion larger than \( p \). Set

\[ \sigma_p^{(n_0)} := \inf\{ t \geq 0 ; \#\Pi^{(n_0)}(t) \leq (1-p)\#\Pi^{(n_0)}(t-) \}. \]  

(5.33)

The process \((\#\Pi^{(n_0)}(t), t \leq \sigma_p^{(n_0)})\), which appears also in [Fou20+, Section 3.2], is a Markov process with generator \( \mathcal{L}^p \) defined by

\[ \mathcal{L}^p g(n) = \mathcal{L}^{\alpha^p} g(n) + \mathcal{L}^f g(n) \]

with

\[ \mathcal{L}^{\alpha^p} g(n) := \sum_{k=2}^{[np]} \binom{n}{k} \lambda_{n,k} (g(n-k+1) - g(n)). \]
The assumption $\rho < 1/2$ will allow us to choose $p$ small enough and $a$ close enough to 1, in order to be able to apply the results of Section 4.1, in particular Corollary 4.5, to the minimal (unstopped) Markov process with generator $L^p$. The latter will therefore explode with a positive probability. Estimates on the jump-time $\sigma_p^{(n_0)}$, found in [Fou11] and [Fou20+], will ensure that the process $(\#\Pi^{(n_0)}(t \wedge \sigma_p^{(n_0)}), t \geq 0)$ explodes also with positive probability for large enough $n_0$. The fact that the process can explode with positive probability when started from any initial point $n_0 \in \mathbb{N}$ is a consequence of the irreducibility of the process. Lastly, the Markov property will entail that explosion happens actually almost surely.

Call respectively $G^f_a$, $G^c_a$ and $G^{c,p}_a$ the functions $G^+_a$ and $G^-_a$ in (4.30) associated to the generators $L$, $L^p$ and function $g_a$. For any $n \geq 2$ and any $p \in (0, 1)$

$$
G^c_a(n) := -\sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k} \left[ \left(1 - \frac{k-1}{n}\right)^{1-a} - 1 \right],
$$

$$
G^{c,p}_a(n) := -\sum_{k=2}^{\lfloor np \rfloor} \binom{n}{k} \lambda_{n,k} \left[ \left(1 - \frac{k-1}{n}\right)^{1-a} - 1 \right].
$$

For any $n \geq 1$

$$
G^f_a(n) := -n \sum_{k=1}^{\infty} \mu(k) \left[ \left(1 + \frac{k}{n}\right)^{1-a} - 1 \right].
$$

We need the following estimates.

**Lemma 5.1.** Given $a > 1$,

(i) for all $n \geq 2$ we have

$$
-G^{c,p}_a(n) \leq \frac{\Phi(n)}{n} (a - 1) (1 - p)^{-a};
$$

(ii) for all $n \geq 1$ we have

$$
G^f_a(n) \geq 2^{-a} (a - 1) \ell(n).
$$

**Proof.** For any $a > 1$ and any $x \in (0, 1)$, by the mean-value theorem

$$(1 - x)^{1-a} - 1 \leq (a - 1)(1 - x)^{-a}x.$$

Hence, for all $n \geq 2$

$$
-G^{c,p}_a(n) = \sum_{k=2}^{\lfloor np \rfloor} \binom{n}{k} \lambda_{n,k} \left[ \left(1 - \frac{k-1}{n}\right)^{1-a} - 1 \right]
\leq (a - 1) \left(1 - \frac{\lfloor np \rfloor}{n}\right)^{-a} \sum_{k=2}^{\lfloor np \rfloor} \frac{k-1}{n} \lambda_{n,k} \binom{n}{k}
\leq (a - 1) \left(1 - \frac{\lfloor np \rfloor}{n}\right)^{-a} \frac{\Phi(n)}{n} \leq (a - 1) (1 - p)^{-a} \frac{\Phi(n)}{n}.
$$

Recall now $G^f_a(n)$. A simple study of the function $g(x) := 1 - (1 + x)^{1-a} - cx$ shows that if $c = (a - 1)2^{-a}$
then $1 - (1 + x)^{1-a} \geq cx$ for all $x \leq 1$. For all $n$,

$$G_a^r(n) = n \sum_{k=1}^{\infty} \mu(k) \left[ 1 - \left( 1 + \frac{k}{n} \right)^{1-a} \right]$$

$$= n \sum_{k=1}^{n} \mu(k) \left[ 1 - \left( 1 + \frac{k}{n} \right)^{1-a} \right] + n \sum_{k=n+1}^{\infty} \mu(k) \left[ 1 - \left( 1 + \frac{k}{n} \right)^{1-a} \right]$$

$$\geq (a-1)2^{-a} \sum_{k=1}^{n} k\mu(k) + (1 - 2^{1-a})n\bar{\mu}(n+1)$$

$$\geq (a-1)2^{-a} \left( \sum_{k=1}^{n} k\mu(k) + n\bar{\mu}(n+1) \right) = (a-1)2^{-a} \ell(n)$$

where we have used (2.15) in the last equality.

We now deal with the proof of Theorem 3.1.

**Proof of Theorem 3.1.** Denote by $(N^{(p)}_t, t \geq 0)$ the minimal Markov process with generator $\mathcal{L}^p$. We first establish that the process $(N^{(p)}_t, t \geq 0)$ explodes with positive probability by applying Corollary 4.5. Let $c(a) = (a-1)2^{-a}$. By assumption $\mathbb{H}$, there is $n_0$ and a positive function $g$ such that for all $n \geq n_0$, $\ell(n) \geq g(\log n) \log n$. By Lemma 5.1-(ii), $G_a^r(n) \geq c(a)g(\log n) \log n$, so that, the first assumption of Corollary 4.5 holds. Recall our assumption $\rho := \limsup_{n \to \infty} \frac{\Phi(n)}{n\ell(n)} < \frac{1}{2}$. Choose $p$ satisfying $\frac{1}{1-p}\rho < \frac{1}{2}$.

Then by Lemma 5.1, for all $n$ and all $a > 1$

$$-\frac{G_a^r(n)}{G_a^l(n)} \leq 2^a (1 - p)^{-a} \frac{\Phi(n)}{n\ell(n)},$$

and thus,

$$\limsup_{n \to \infty} -\frac{G_a^r(n)}{G_a^l(n)} \leq 2^a (1 - p)^{-a} =: \gamma_a.$$ (5.37)

By the assumption $\rho < \frac{1}{2}$, take the constant $a$ close enough to 1 such that the upper bound $\gamma_a$ in (5.37) above is strictly smaller than 1. Finally, Corollary 4.5 applies and the process $(N^{(p)}_t, t \geq 0)$ explodes with positive probability when starting from a large enough initial value. More precisely, if one denotes by $\tau^{\infty, p}$ the explosion time of $(N^{(p)}_t, t \geq 0)$ and set $c = 1-\gamma_a \in (0, \infty)$, then by applying the estimate (4.27), we get that if $n_0$ is large enough,

$$\mathbb{P}_{n_0} \left( \tau^{\infty, p}_{\infty} \leq \int_{\frac{1}{\ell_t} \log n_0^{1-a}}^{\infty} \frac{dv}{c\gamma(v)} \right) \geq \prod_{k=0}^{\infty} h_a(k, n_0) > 0,$$

with $h_a(k, n_0) := 1 - (1 + 2^{a-1}) \left( \frac{1}{n_0^{1-a}} \right)^{(1+\delta)k}$. Simple calculations show that $\prod_{k=0}^{\infty} h_a(k, n_0)$ converges to 1 as $n_0$ goes to $\infty$. Since $\prod_{k=0}^{\infty} h_a(k, n_0) \to 0$, for all $t > 0$

$$\mathbb{P}_{n_0} \left( \tau^{\infty, p}_{\infty} \leq t \right) \to 0 \quad \text{as} \quad n_0 \to \infty.$$ (5.38)

Recall $\sigma^{(n_0)}_p$ defined in (5.33). We now show that $\mathbb{P}_{n_0} (\tau^{\infty, p}_t < \sigma^{(n_0)}_p) > 0$ for large enough $n_0$, where $\tau^{\infty, p}_t$ denotes the first explosion time of $(\#\Pi^{(p)}(t), t \geq 0)$. Recall the Poisson construction of $(\Pi^{(p)}(t), t \geq 0)$ in Section 2.1. For each atom $(t, \pi^t)$ of PPPC, we associate the i.i.d Bernoulli random variables $(X_i, i \geq 1)$ defined by $X_i = 1$, if $\{i\} \notin \pi^t$, that is to say if the $i^{th}$ block of $\Pi^{(p)}(t-)$ takes part to the merging at
time \( t \), and \( X_i = 0 \) if \( \{i\} \in \pi^c \), which means that the \( i \)th block of \( \Pi^{(n_0)}(t-) \) does not take part to the merging at time \( t \). By definition, the jump time \( \sigma_p^{(n_0)} \) belongs to the set of atoms of coalescence

\[
J_p := \left\{(t, \pi^c) \text{ atom of } \text{PPP}_C; \exists n \geq 2 \sum_{k=1}^{n} X_k \geq np \right\}.
\]

Applying [Fou20+, Lemma 3.14], we see from the calculations in the proof of [Fou20+, Lemma 3.15] that \( \mathbb{E}(\text{PPP}_C(J_p)) < \infty \), hence \( J_p \) is locally finite. Moreover \( \Pi^{(n_0)}(0) = \Pi^{(n_0)}(0+) \) for any \( n_0 \geq 1 \), therefore \( 0 \) is neither a coalescence time nor an accumulation point of \( J_p \) and we have \( \inf \{t > 0; (t, \pi^c) \in J_p \} > 0 \) a.s. This ensures that \( \inf \sigma_p^{(n)} > 0 \) a.s. Note that

\[
P_{n_0}(\tau_{\infty} < \sigma_p^{(n_0)}) = P_{n_0}(\tau_{\infty}^+ < \sigma_p^{(n)}) \geq P_{n_0}(\tau_{\infty}^+ < \inf \sigma_p^{(n)})
\]

For any \( t > 0 \), one has

\[
P_{n_0}(\tau_{\infty}^+ < \sigma_p^{(n_0)}) \geq P_{n_0}(\tau_{\infty}^+ < t, \inf \sigma_p^{(n)} > t)
\]

\[
= P_{n_0}(\tau_{\infty}^+ < t) + P(\inf \sigma_p^{(n)} > t) - P_{n_0}(\{\tau_{\infty}^+ < t \} \cup \{\inf \sigma_p^{(n)} > t\})
\]

\[
\geq P_{n_0}(\tau_{\infty}^+ < t) + P(\inf \sigma_p^{(n)} > t) - 1.
\]

By letting \( n_0 \) to \( \infty \) and applying (5.38), we get

\[
\lim_{n_0 \to \infty} P_{n_0}(\tau_{\infty} < \sigma_p^{(n_0)}) \geq P(\inf n_{\geq 2} \sigma_p^{(n)} > t).
\]

Recall that \( \inf \sigma_p^{(n)} > 0 \) a.s. By letting \( t \) towards \( 0 \), we have \( \lim_{n_0 \to \infty} P_{n_0}(\tau_{\infty} < \sigma_p^{(n_0)}) = 1 \). Finally, we see that the stopped process \( (#\Pi(t \wedge \sigma_p^{(n_0)}), t \geq 0 \) explodes with a positive probability under \( P_{n_0} \) for large enough \( n_0 \). Hence \( (#\Pi(t), t \geq 0 \) started from \( #\Pi(0) = n_0 \), has also a positive probability to explode for large enough \( n_0 \). Since \( (#\Pi(t \wedge \tau_{\infty}^+), t \geq 0 \) is irreducible in \( \mathbb{N} \), its probability of explosion starting from \( n = 1 \) is also positive, namely \( P_1(\tau_{\infty}^+ < \infty) > 0 \). We now establish that the process \( (#\Pi(t), t \geq 0 \) started from any integer \( n_0 \), explodes almost surely. Pick \( t > 0 \) such that \( P_1(\tau_{\infty}^+ < t) = 0 \). The stochastic monotonicity in the initial states, see Lemma 2.2, ensures that for any \( n_0 \geq 1 \), \( P_{n_0}(\tau_{\infty}^+ > t) \leq P_1(\tau_{\infty}^+ > t) > 0 \). Let \( n \geq 2 \), by the Markov property at time \( (n-1)t \), we have

\[
P_{n_0}(\tau_{\infty}^+ > nt) = P_{n_0}(\tau_{\infty}^+ > (n-1)t) \mathbb{E}(P_{\Pi((n-1)t)}(\tau_{\infty}^+ > t))
\]

\[
\leq P_{n_0}(\tau_{\infty}^+ > (n-1)t) P_1(\tau_{\infty}^+ > t).
\]

By induction,

\[
P_{n_0}(\tau_{\infty}^+ > nt) \leq P_1(\tau_{\infty}^+ > t)^n \to 0.
\]

Therefore,

\[
P_{n_0}(\tau_{\infty}^+ < \infty) = 1.
\]

Recall that if the pure coalescent part does not come down from infinity, i.e. when \( \sum_{n \geq 2} \frac{1}{\mathbb{P}(n)} = \infty \), then \( \infty \) is accessible it is necessarily an exit. It remains to justify that \( \infty \) is an exit when \( \sum_{n \geq 2} \frac{1}{\mathbb{P}(n)} < \infty \) and \( \rho < \frac{1}{4} \). Recall \( \theta_* \) defined in Section 2.3. By [Fou20+, Lemma 4.1-(3)], we get \( \theta_* \geq \frac{1}{4} \lim_{n \to \infty} \frac{\mathbb{P}(n)}{\mathbb{P}(\infty)} = \frac{1}{4} \rho \).

Finally, if \( \rho < \frac{1}{4} \), then \( \theta_* > 1 \) and by Theorem 2.4, \( (\Pi(t), t \geq 0 \) does not come down from infinity.

We stress here on some regularity of the explosion time with respect to the initial state of the process. For any \( n_0 \in \mathbb{N} \), set \( \tau_{\infty}^{+(n_0)} := \inf\{t > 0; #\Pi^{(n_0)}(t-) = \infty \} \).

**Proposition 5.2.** Assume condition \( \mathbb{H} \). If \( \rho < \frac{1}{4} \), then \( \tau_{\infty}^{+(n_0)} \to 0 \) a.s.

**Remark 5.3.** In particular, when \( \rho < 1/4 \), the boundary \( \infty \) is an instantaneous exit, in the sense that the time of getting absorbed at \( \infty \), \( \tau_{\infty}^{+(n_0)} \), goes almost surely towards \( 0 \) as the initial state \( n_0 \) goes to \( \infty \).
Proof. Recall $\tau_{\infty}^+$ the first explosion time of the unstopped process $(N^p_t, t \geq 0)$ with generator $L^p$, Lemma 4.4 and the estimates (4.27) with $\delta < 1$. Set $\varphi(n) = \int_{\mathbb{R}^+} \log n_1^{1-\delta} e^{\nu(y)} dy$. One has $\varphi(n) \to 0$ and one easily checks from (4.27) that $\mathbb{P}_{n_0}(\tau_{\infty}^+ \leq \varphi(n_0)) \to 1$. Recall $\sigma_p := \inf_{n \geq 1} \sigma^n_p > 0$ a.s. Fix $t > 0$, and take $n_0$ large enough such that $\varphi(n_0) \leq t$. Then

$$
\mathbb{P}_{n_0}(\tau_{\infty}^+ \leq t) \geq \mathbb{P}_{n_0}(\tau_{\infty}^+ \leq \varphi(n_0), \varphi(n_0) < \sigma_p) = \mathbb{P}_{n_0}(\tau_{\infty}^+ \leq \varphi(n_0)) + P(\varphi(n_0) < \sigma_p) - 1.
$$

Since $\sigma_p > 0$ a.s, $P(\varphi(n_0) < \sigma_p) \to 1$ and $\mathbb{P}_{n_0}(\tau_{\infty}^+ \leq t) \to 1$ for any fixed $t > 0$. \hfill \Box

The conditions $\rho < \frac{1}{2}$ and $\rho < \frac{1}{4}$ are not very sharp but they hold for quite general splitting measures. We shall see how to improve them for coagulation and splitting measures with some regular variation properties in Section 5.3.

5.2. Sufficient conditions for non-explosion

We first establish Theorem 3.3 and then design, in a similar way as what has been done for the explosion, some sufficient conditions for non-explosion using the Lyapunov function $g_\alpha(n) = n^{1-\alpha}$ with $0 < \alpha < 1$.

Proof of Theorem 3.3. Recall that the sequence $(\Phi(n)/n, n \geq 2)$ is non-decreasing; see Section 2.2. Assume first that $\lim_{n \to \infty} \frac{n}{\Phi(n)} = 0$. By assumption $\sum_{n \geq 2} \frac{n}{\Phi(n)} f(n) < \infty$ ensures that the function $f$, defined by $f(n) := \sum_{k=2}^{n} \frac{k}{\Phi(k)}$ for all $n \geq 2$ and $f(1) = \frac{2}{\Phi(1)}$, belongs to $\mathcal{D}$. Since $\Phi(k) \leq \frac{n(0,1)}{2} k^2$ for all $k \geq 2$, see Section 2.2, one has $\frac{k}{\Phi(k)} \geq \frac{1}{\Phi(1)}$ for all $k \geq 2$. Then $f(n) \to \infty$. For any $n \geq 2$,

$$
L^f f(n) = n \sum_{k=2}^{n} \mu(k) \sum_{j=n-k+2}^{n} \frac{j}{\Phi(j)} = n \sum_{j=n+1}^{\infty} \sum_{k=j-n}^{\infty} \frac{j}{\Phi(j)} \mu(k) = n \sum_{i=1}^{\infty} \frac{i+n}{\Phi(i+n)} \bar{\mu}(i).
$$

For the coalescent part, since $\frac{n}{\Phi(n)} \leq \frac{j}{\Phi(j)}$ for $j \leq n$,

$$
\mathcal{L} c f(n) = -\sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k} \sum_{j=n-k+2}^{n} \frac{j}{\Phi(j)} \leq -\frac{n}{\Phi(n)} \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k} (k-1) = -n.
$$

We now check that there exists $c$ such that $L f(n) \leq c f(n)$ for all $n \geq 1$. By combining (5.39) and (5.40) one gets for any $n \geq 2$

$$
L f(n) \leq n \left( -1 + \sum_{i=1}^{\infty} \frac{i+n}{\Phi(i+n)} \bar{\mu}(i) \right).
$$

Recall the assumptions $\sum_{i=2}^{\infty} \frac{1}{\Phi(i)} \bar{\mu}(i) < \infty$ and $\lim_{n \to \infty} \frac{n}{\Phi(n)} = 0$. Since $\frac{i+n}{\Phi(i+n)} \leq \frac{1}{\Phi(i)}$ for any $n, i \geq 2$, by Lebesgue’s theorem, $\sum_{n=1}^{\infty} \frac{i+n}{\Phi(i+n)} \bar{\mu}(i) \to 0$. Therefore, there exists $n_0$, such that $L f(n) \leq 0$ for all $n \geq n_0$. This entails that for all $n \geq 1$, $L f(n) \leq c_0$ with $c_0 := \max_{k \in [1,n_0]} (\|L f(k)\|)$. By setting $c = \frac{\Phi(2)}{2} c_0$, since $f(n) \geq \frac{2}{\Phi(2)}$ for all $n \geq 1$, we get finally $L f(n) \leq c f(n)$ for all $n \geq 1$. As recalled in Section 4.2, this entails that the process does not explode.

According to [Fou20+, Corollary 4-2], if $\sum_{n=1}^{\infty} \frac{1}{\Phi(n)} < \infty$ and $\sum_{n=1}^{\infty} \frac{n}{\Phi(n)} \bar{\mu}(n) < \infty$, then $\theta = 0$ and by Theorem 2.4 the process comes down from infinity. We conclude that $\infty$ is an entrance. \hfill \Box
We establish now sufficient conditions for $\infty$ to be inaccessible, based on Proposition 4.6. We assume in this section that $\sum_{k=1}^{\infty} k^{1-a}\mu(k) < \infty$ for some $a < 1$. This ensures that $G^f_a$ in (5.36) is well-defined. We first find some estimates of the functions $G^f_a$ and $G^c_a$. Recall that for all $a < 1$ and all $n \geq 1$, $G^c_a(n) \geq 0$ and $G^f_a(n) \leq 0$.

**Lemma 5.4.** Given $a < 1$ such that $\sum_{n=1}^{\infty} k^{1-a}\mu(k) < \infty$,

1. for all $n \geq 2$ we have

   \[ G^c_a(n) \geq (1-a)\frac{\Phi(n)}{n} \]

2. for all $n \geq 1$ we have

   \[ -G^f_a(n) \leq (1-a)\sum_{k=1}^{n} k\mu(k) + \sum_{k=n+1}^{\infty} k^{1-a}\mu(k). \]

**Proof.** Set $g(x) = 1 - (1-x)^{1-a} - (1-a)x$ for all $x \in (0,1)$. A simple study of the function $g$ yields that $g(x) \geq 0$ and thus $1 - (1-x)^{1-a} \geq (1-a)x$ for all $x \in (0,1)$. Hence, by definition of $\Phi$, for all $n \geq 2$

\[ G^c_a(n) = \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k} \left[ 1 - \left( \frac{k-1}{n} \right) \right]^{1-a} \geq (1-a)\frac{\Phi(n)}{n}. \]

Recall $G^f_a$ in (5.36). Note that for any $x \in (0,1)$, $(1+x)^{1-a} - 1 \leq (1-a)x$ and for any $x \in (1,\infty)$, $(1+x)^{1-a} - 1 \leq x^{1-a}$. We get for all $n \geq 2$

\[ -G^f_a(n) = n \sum_{k=1}^{\infty} \mu(k) \left[ \left( 1 + \frac{k}{n} \right)^{1-a} - 1 \right] \leq (1-a)\sum_{k=1}^{n} k\mu(k) + n \sum_{k=n+1}^{\infty} \mu(k) \left( \frac{k}{n} \right)^{1-a}. \]

\[ \square \]

**Proposition 5.5.** Assume that there exists $a \in (0,1)$ such that $\sum_{n=1}^{\infty} n^{1-a}\mu(n) < \infty$. Set the condition

\[ \lim_{n \to \infty} n^{1+a}\Phi(n) \sum_{k=n}^{\infty} k^{1-a}\mu(k) = 0. \]  

(5.41)

If (5.41) holds and \( \limsup_{n \to \infty} \frac{n}{\Phi(n)} \sum_{k=1}^{n} k\mu(k) < 1 \), then the process \((\#\Pi(t), t \geq 0)\) does not explode.

We shall see an example where Proposition 5.5 applies when establishing Theorem 3.9 in Section 5.4.

**Proof.** By Lemma 5.4,

\[ \limsup_{n \to \infty} -\frac{G^f_a(n)}{G^c_a(n)} \leq \limsup_{n \to \infty} \frac{n}{\Phi(n)} \sum_{k=1}^{n} k\mu(k) + \frac{1}{1-a} \limsup_{n \to \infty} \frac{n^{a+1}}{\Phi(n)} \sum_{k=n+1}^{\infty} k^{1-a}\mu(k). \]  

(5.42)

By the assumption (5.41), the second term on the right-hand side of (5.42) vanishes and we get

\[ \limsup_{n \to \infty} -\frac{G^f_a(n)}{G^c_a(n)} \leq \limsup_{n \to \infty} \frac{n}{\Phi(n)} \sum_{k=1}^{n} k\mu(k). \]

Proposition 4.6 applies and yields that the process does not explode.

\[ \square \]
5.3. Regularly varying coagulation/fragmentation mechanisms

Let $\beta \in (0, 1)$ and $\alpha \in (0, \infty)$. In this section, we will study in more details the cases
\[
\Phi(n) \sim \frac{b}{n^{\alpha+1}} \text{ and } \mu(n) \sim \frac{b}{n^{\alpha+1}}.
\]
The study of the cases $\alpha + \beta < 1$ and $\alpha + \beta > 1$ are applications of Theorem 3.1 and Theorem 3.3. For the case $\alpha + \beta < 1$, note that $\ell(n) \sim \frac{b}{n^{\alpha+1}}$, so that in particular condition $H$ is fulfilled and $\Phi(n) \sim \frac{d}{\beta n^{\alpha+1}} n^{\beta+\alpha-1}$, and by Theorem 3.1, $\infty$ is an exit. For the case $\alpha + \beta > 1$, note that $\Phi(n) \sim \frac{d}{\beta n^{\alpha+1}} n^{\beta+\alpha}$. Hence $\sum_{n=2}^{\infty} \frac{n^{b}}{\Phi(n)} \mu(n) < \infty$ and by Theorem 3.3, $\infty$ is an entrance. We focus on the case $\alpha + \beta = 1$ for which these sufficient conditions for explosion or non explosion can be sharpened. We provide a finer study of $G^f$ for this case in the following lemma.

**Lemma 5.6.** Let $\alpha \in (0, 1)$ and $b_2 > b_1 > 0$. Assume that the splitting measure $\mu$ satisfies for all large enough $k$,
\[
\frac{b_1}{k^{\alpha+1}} \leq \mu(k) \leq \frac{b_2}{k^{\alpha+1}}.
\]  
(5.43)

For any $a > 1$ and for any $\epsilon > 0$, there is $n_0$ such that if $n \geq n_0$,
\[
G^f_a(n) \geq \frac{1}{1+\epsilon} i_\alpha(a) b_1 n^{1-\alpha}
\]  
(5.44)

where $i_\alpha(a) := \int_0^1 \frac{1-(1+u)^{1-a}}{u^{\alpha+1}} \, du < \infty$.

For any $a \in (1-\alpha, 1)$ and for any $\epsilon > 0$, there is $n_0$ such that if $n \geq n_0$,
\[
-G^f_a(n) \leq \frac{1}{1-\epsilon} j_\alpha(a) b_2 n^{1-\alpha}
\]  
(5.45)

where $j_\alpha(a) := \int_0^1 \frac{(1+u)^{1-a} - 1}{u^{\alpha+1}} \, du < \infty$.

**Proof.** Let $a > 1$. For any $n \geq 1$,
\[
\frac{1}{k^{\alpha+1}} \left[ 1 - \left( 1 + \frac{k}{n} \right)^{1-a} \right] = \frac{1}{k^{\alpha+1}} \left[ 1 - \left( 1 + \frac{k}{n} \right)^{1-a} \right] + n \left( 1 - (1 + 1/n)^{1-a} \right)
\]
and
\[
\sum_{k=2}^{\infty} \frac{1}{k^{\alpha+1}} \left[ 1 - \left( 1 + \frac{k}{n} \right)^{1-a} \right] \sim \frac{1}{k^{\alpha+1}} \sum_{k=2}^{\infty} \left[ 1 - \left( 1 + \frac{k}{n} \right)^{1-a} \right] \, dx
\]
\[
\sim \frac{1}{k^{\alpha+1}} \left[ 1 - \left( 1 + \frac{x}{n} \right)^{1-a} \right] \, dx
\]
\[
\sim \frac{1}{\alpha+1} \int_0^1 \left[ 1 - (1+u)^{1-a} \right] \, du \sim i_\alpha(a) n^{1-\alpha}.
\]
In particular, we see that for any $\epsilon > 0$, there is $n_0$ such that if $n \geq n_0$, (5.44) holds. The proof is similar for the case $a < 1$.

We need the following analytical lemma. Recall the functions $i_\alpha$ and $j_\alpha$ defined in Lemma 5.6.

**Lemma 5.7.** Set $I(\alpha) := \int_0^\infty \frac{\log(1+u)}{u^{\alpha+1}} \, du$ for all $\alpha \in (0, 1)$. Then $I(\alpha) = \frac{\pi}{\alpha \sin(\pi \alpha)} > \frac{1}{\alpha(1-\alpha)}$. Further,
\[
\frac{i_\alpha(a)}{a-1} \xrightarrow{a \to 1^+} I(\alpha) \quad \text{and} \quad \frac{j_\alpha(a)}{a-1} \xrightarrow{a \to 1^-} I(\alpha).
\]
Thus, under \( P \) checked by showing that the function \( f(1 + u)^{1-a} \leq (a - 1) \log(1 + u) \).

By Lebesgue’s theorem, we have \( j_n(a) \rightarrow I(\alpha) := \int_0^\infty \frac{\log(1 + u)}{u^{a+\alpha}} \, du \).

Recall \( j_n(a) = \int_0^\infty \frac{(1 + u)^{1-a} - 1}{u^{a+\alpha}} \, du < \infty \), for any \( 1 - \alpha < a < 1 \). Let \( \epsilon > 0 \) and assume \( a > 1 - \alpha + \epsilon \). By applying the mean value theorem to the function \( a \mapsto (1 + u)^{1-a} - 1 = e^{(1-a) \log(1+u)} - 1 \), we get that for any \( u > 0 \)

\[
(1 + u)^{1-a} - 1 \leq (1 - a) \log(1 + u)(1 + u)^{1-a} \leq (1 - a) \log(1 + u)(1 + u)^{a-\epsilon}.
\]

Thus, \( \frac{1}{1-a} \log(1 + u)(1 + u)^{a-\epsilon} \leq \log(1 + u)(1 + u)^{a-\epsilon} \). The function \( u \mapsto \log(1 + u)(1 + u)^{a-\epsilon} \) is integrable on \((0, \infty)\) and by Lebesgue’s theorem, we have \( j_n(a) \rightarrow I(\alpha) \).

We now show the identity \( I(\alpha) = \frac{\pi}{\alpha \sin(\pi \alpha)} \) for any \( \alpha \in (0, 1) \). One has, by integration by parts and the change of variable \( v = \frac{1}{1+u} \)

\[
I(\alpha) := \int_0^\infty \frac{\log(1 + u)}{u^{a+\alpha}} \, du = \frac{1}{\alpha} \int_0^\infty \frac{1}{1+u} u^{-\alpha} \, du = \frac{1}{\alpha} \int_0^1 \frac{1}{v} \left( \frac{1}{v} - 1 \right)^{-\alpha} \, dv = \frac{1}{\alpha} \int_0^1 v^{\alpha-1} (1 - v)^{-\alpha} \, dv = \frac{1}{\alpha} \Gamma(\alpha, 1 - \alpha) \Gamma(\alpha) = \frac{\pi}{\alpha \sin(\pi \alpha)},
\]

where Beta(\( \cdot, \cdot \)) is the Beta function. We have used Beta-Gamma relation and Euler’s reflection formula in the two last equalities, see e.g. [Sch66, VIII.3]. The strict inequality \( \frac{\pi}{\alpha \sin(\pi \alpha)} > \frac{1}{\alpha (1-\alpha)} \) can be easily checked by showing that the function \( a \mapsto (1 - \alpha) \pi - \sin(\pi \alpha) \) is strictly decreasing on \((0, 1)\).

**Lemma 5.8.** Assume that \( \mu(n) \sim \frac{b}{n^{\alpha+\epsilon}} \), \( \Phi(n) \sim \frac{\Phi(n)}{n^{\beta+1}} \) and \( \alpha + \beta = 1 \). For any \( n \in \mathbb{N} \), we have under \( P_n \),

1. if \( \frac{b}{n^{\alpha+\epsilon}} \geq \frac{\alpha \sin(\pi \alpha)}{\pi} \), then process \( \#\Pi(t), t \geq 0 \) explodes almost surely;
2. if \( \frac{b}{n^{\alpha+\epsilon}} < \frac{\alpha \sin(\pi \alpha)}{\pi} \), then process \( \#\Pi(t), t \geq 0 \) does not explode almost surely.

**Proof.** We first establish assertion (1). Assume that \( \frac{1}{1-p} \frac{d}{d} \frac{1}{I(\alpha)} < 1 \). Let \( p \) be small enough such that \( \frac{1}{1-p} \frac{d}{d} \frac{1}{I(\alpha)} < 1 \). Recall that \( (N_t^{(p)}, t \geq 0) \) denotes the process \( \#\Pi(t), t \geq 0 \) stopped at \( \sigma_p \). We show that \( (N_t^{(p)}, t \geq 0) \) explodes with positive probability by using Corollary 4.5. Using Lemma 5.6, we see that the first condition of Corollary 4.5 on \( G^{c-p}_a \) for \( a > 1 \) is plainly satisfied with for instance \( g(n) := c e^{(1-\alpha)n} / n \) for some constant \( c > 0 \). According to Lemma 5.1-(i),

\[
-G^{c-p}_a \leq \frac{\Phi(n)}{n^2} (a - 1) (1 - p)^{-a}.
\]

Combining this latter bound with Lemma 5.6, we get that

\[
\limsup_{n \to \infty} \frac{-G^{c-p}_a(n)}{G_a(n)} \leq (1 - p)^{-a} \frac{a - 1}{i_n(a)} \limsup_{n \to \infty} \frac{\Phi(n)}{bn^{2-a}}.
\]

Since \( \Phi(n) \sim \frac{\Phi(n)}{n^{\beta+1}} \), we have

\[
\limsup_{n \to \infty} \frac{-G^{c-p}_a(n)}{G_a(n)} \leq \frac{1}{(1 - p)^a i_n(a)} b =: \gamma_{a,p}.
\]
The upper bound in (5.46), $\gamma_{a,p}$, converges towards $\frac{1}{1-p} \frac{1}{I(\alpha)} \frac{d}{b} < 1$ as $a$ goes towards $1^+$. We can therefore find $a > 1$ close enough to $1$ such that $\limsup G^c_a(n) < 1$. The fact that the unstopped process $(\#\Pi(t), t \geq 0)$ explodes almost surely is proven by the same argument as in the proof of Theorem 3.1.

We now establish assertion (2). Recall Lemma 5.4 and the bound $G^c_a(n) \geq (1 - a) \frac{\Phi(n)}{n}$ for all $n$ for $a < 1$. Then

$$\limsup_{n \to \infty} \frac{G^f_a(n)}{G^c_a(n)} \leq j_0(a) \frac{b}{1 - a d}.$$  

The upper bound goes to $\frac{b}{d} I(\alpha)$ as $a$ goes to $1$. Thus, if $\frac{b}{d} I(\alpha) < 1$, one can find $a < 1$ close enough to $1$ such that

$$\limsup_{n \to \infty} \frac{G^f_a(n)}{G^c_a(n)} < 1.$$  

By Proposition 4.6, the process $(\#\Pi(t), t \geq 0)$ does not explode.

We can now classify the possible behaviors of the process $(\Pi(t), t \geq 0)$ on the boundary of partitions with infinitely many blocks, by combining the properties of explosion and of coming down from infinity of the process $(\#\Pi(t), t \geq 0)$.

**Proof of Theorem 3.5.** We are left to show the results for $\alpha + \beta = 1$. Let $\alpha \in (0, 1)$. First observe that if $\mu(n) \sim n^{-\gamma}$, then $\bar{\mu}(n) \sim \frac{b}{\alpha n^\gamma}$. Recall Proposition 2.5 and Theorem 2.4. When $\theta = \frac{b}{d\alpha(1-\alpha)} > 1$ (respectively, $\theta < 1$), the process stays infinite (respectively, comes down from infinity). The strict inequality in Lemma 5.7 ensures that $\sigma := \frac{b}{d \sin(\pi \alpha)} > \theta$. In particular, we see that if $\theta > 1$, then $\sigma > 1$ which entails on the one hand that the process cannot leave infinity, and on the other hand, by Lemma 5.8, that started from a finite state, it explodes almost surely. When $\sigma > 1$, by Lemma 5.8, $(\#\Pi(t), t \geq 0)$ explodes a.s and by Proposition 2.5, it comes down from infinity a.s, thus $\infty$ is regular.

We now establish that when the boundary $\infty$ is regular, it is regular for itself.

**Proof of Proposition 3.7.** Consider a simple EFC process $(\Pi(t), t \geq 0)$ with coagulation and splitting measures satisfying the assumptions of Theorem 3.5 with $\alpha + \beta = 1$. Recall the first explosion time $\tau^+_\infty := \inf\{t > 0; \#\Pi(t^-) = \infty\}$. We show that when $\#\Pi(0) = \infty$ and $\frac{\pi}{2} \sin(\pi \alpha) < b/d < \alpha(1-\alpha)$, $\mathbb{P}(\tau^+_\infty = 0) = 0$. Assume first that $\#\Pi(0)$ is proper. We follow the same arguments as in the proof of Proposition 5.2 by studying more precisely the estimate (4.27) on the first passage times above large levels. We have seen in the proof of Lemma 5.8 that when $\sigma := \frac{b}{d \sin(\pi \alpha)} > 1$, one can find $p$ small enough and $a$ close enough to $1$, such that $\gamma_{a,p} < 1$, where $\gamma_{a,p}$ is defined in (5.46). By setting $c := \frac{1 - \gamma_{a,p}}{2} > 0$ and applying the estimate (4.27) to the stopped process $(\#\Pi(t \wedge \sigma_p(n_0)), t \geq 0)$, we obtain that for any $0 < \delta < \frac{1}{2a-1}$,

$$\mathbb{P}_{n_0}(\tau^+_{\infty}(n_0) \leq \varphi(n_0), \tau^+_{\infty}(n_0) \leq \sigma_p(n_0)) \to 1,$$

with $\tau^+_{\infty}(n_0) := \inf\{t > 0; \#\Pi(n_0)(t^-) = \infty\}$ and $\varphi(n_0) = \int_{\frac{1}{2a-1} \log n_0^{-1-s}} d\gamma(y)$. According to Lemma 2.2, for all $n_0 \geq 1$ and all $t \geq 0$, $\#\Pi(n_0)(t) \leq \#\Pi(t)$ almost surely, thus $\mathbb{P}(\tau^+_{\infty} \leq \tau^+_{\infty}(n_0)) = 1$. Since $\varphi(n_0) \to 0$, we have that $\mathbb{P}(\tau^+_{\infty} = 0) \geq \lim_{n_0 \to \infty} \mathbb{P}(\tau^+_{\infty} \leq \varphi(n_0)) = 1$.

If $\#\Pi(0)$ is improper, then according to [Fou20+, Lemma 3.1], since there is no dust in the pure coalescent process and no formation of dust in the fragmentation one, for any $t > 0$, $\#\Pi(t)$ is a proper partition. This allows one to apply the previous result to the EFC process $(\Pi(t+s), s \geq 0)$. Denote its first explosion time by $\tau^+_{\infty}(t) := \inf\{s > 0; \#\Pi(t+s^-) = \infty\}$. One has $\tau^+_{\infty}(t) = \tau^+_{\infty}(\#\Pi(t))$ in law and since $\#\Pi(t) \to \infty$ a.s. our previous argument entails that $\tau^+_{\infty}(t)$ goes to $0$ in probability. Since $\tau^+_{\infty} \leq \tau^+_{\infty}(t) + t$, we finally get $\tau^+_{\infty} = 0$ a.s.
We study now the critical case for which \( \beta = 1 - \alpha \) and \( \sigma = 1 \) i.e. \( \frac{b}{\sigma} = \frac{\alpha \sin(\pi \alpha)}{\pi} \) under additional assumption on the measure \( \Lambda \) and establish Proposition 3.8.

**Proof of Proposition 3.8.** Recall the assumptions on \( \mu \) and \( \Lambda \) and the maps \( \Phi \) and \( \Psi \) in (2.10) and (2.11). We first verify that the process comes down from infinity. Set \( \tilde{\Lambda}(dx) := \frac{d}{c_\beta} x^{2-\beta} dx \) with \( c_\beta = \frac{\Gamma(1-\beta)}{\beta \Gamma(\beta+1)} \) and \( \tilde{\Phi} \) the function defined in (2.10) with coalescence measure \( \tilde{\Lambda} \). By assumption \( \Lambda(dx) = h(x) \tilde{\Lambda}(dx) \) with \( h \geq 1 \). Thus, \( \Phi(n) \geq \tilde{\Phi}(n) \) for all \( n \geq 2 \). A Tauberian theorem, see e.g. [Fou20+, Section 2.2] for details, ensures that \( \tilde{\Phi}(n) \sim n^{1+\beta} \). Therefore by Proposition 2.5, \( \tilde{\theta} := \lim_{n \to \infty} \sum_{k=1}^{n} \frac{n \mu(k)}{\tilde{\Phi}(n+k)} = \frac{b}{\sigma} \frac{1}{\alpha(1-\alpha)} \) where we recall \( \beta = 1 - \alpha \). From the definition of \( \theta^* \), see (2.12), and the inequality \( \hat{\Phi} \geq \tilde{\Phi} \), we get \( \theta^* \leq \hat{\theta} = \frac{\beta}{\sigma} \frac{1}{\alpha(1-\alpha)} \). On the other hand, Lemma 5.7 ensures that \( \frac{b}{\sigma} \frac{1}{\alpha(1-\alpha)} < \sigma := \frac{b}{\sigma} \frac{1}{\alpha \sin(\pi \alpha)} \), so that if \( \frac{b}{\sigma} = 1/I(\alpha) = \frac{\alpha \sin(\pi \alpha)}{\pi} \), then \( \sigma = 1 \) and \( \theta^* \leq \frac{b}{\sigma} \frac{1}{\alpha(1-\alpha)} < 1 \). By Theorem 2.4, \( \#P(t), t \geq 0 \) comes down from infinity.

We now show that \( \infty \) is inaccessible. We check that under our assumptions \( \Phi(n) \geq n^{1+\beta} - Cn \) for large \( n \) and some constant \( C > 0 \). Note that for any \( n \geq 2 \), \( \Psi(n) = \Phi(n) \leq C' n \) for some constant \( C' > 0 \), see e.g. [LT15, Lemma 2.1]. Moreover, recalling that for any \( n \geq 0 \), \( \int_{0}^{\infty} (e^{-nx} - 1 + nx)x^{-2-\beta} dx = n^{1+\beta}/c_\beta \), we get

\[
\tilde{\Psi}(n) := dc_\beta \int_{0}^{\infty} (e^{-nx} - 1 + nx)x^{-2-\beta} \left( h(x)1_{[0,1]}(x) + 1_{1,\infty}(x) \right) dx \geq n^{1+\beta}.
\]

For any \( n \geq 2 \),

\[
d^{1+\beta} - \Phi(n) \leq \tilde{\Psi}(n) - \Phi(n) = \Psi(n) - \Psi(n) + \Psi(n) - \Phi(n) = dc_\beta \int_{1}^{\infty} (e^{-nx} - 1 + nx)x^{-2-\beta} dx + \Psi(n) - \Phi(n) \leq \left( dc_\beta \int_{1}^{\infty} x^{-1-\beta} dx + C' \right) n \leq Cn.
\]

We show now that \( \mathcal{L} \log(n+1) \leq c \log(n+1) \) for some \( c > 0 \) and \( n \geq 1 \). Non-explosion will follow by applying the Foster-Lyapunov criterion (4.31). Let \( n \geq 2 \),

\[
\mathcal{L}^c \log n = \sum_{k=2}^{n} \log \left( \frac{n-k+1}{n} \right) \left( \frac{n}{k} \right) \lambda_{n,k} = \sum_{k=2}^{n} \log \left( 1 - \frac{k-1}{n} \right) \left( \frac{n}{k} \right) \lambda_{n,k} \leq - \sum_{k=2}^{n} \frac{k-1}{n} \left( \frac{n}{k} \right) \lambda_{n,k} = - \frac{\Phi(n)}{n} \leq -dn^{1+\beta} + C.
\]

Recall \( I(\alpha) := \int_{0}^{\infty} \frac{\log(1+x)}{x^{1+\alpha}} dx = \frac{\pi}{\alpha \sin(\pi \alpha)} \). For any \( n \geq 1 \),

\[
\mathcal{L}^J \log n = n \sum_{k=1}^{\infty} \frac{b}{k^{1+\alpha}} \log \left( 1 + \frac{k}{n} \right) \leq nb \log \left( 1 + \frac{1}{n} \right) + n \sum_{k=2}^{\infty} \int_{1}^{\infty} \frac{b}{x^{1+\alpha}} \log \left( 1 + \frac{x}{n} \right) dx \leq b + n \int_{1}^{\infty} \frac{b}{x^{1+\alpha}} \log \left( 1 + \frac{x}{n} \right) dx \leq b + bn^{1-\alpha} I(\alpha).
\]

Finally, since \( \frac{b}{\sigma} = 1/I(\alpha) \), we obtain for any \( n \geq 1 \),

\[
\mathcal{L} \log(n+1) \leq b + C + (bI(\alpha) - d)(n+1)^{1-\alpha} = b + C \leq c \log(n+1)
\]

with \( c := \frac{b+C}{\log 2} \). \( \square \)

### 5.4. Slower coalescence

Let \( d > 0 \), \( \beta > 1 \), \( b > 0 \) and \( \alpha > 0 \). We study in this section the case of EFCs with coagulation and splitting measures satisfying \( \Phi(n) \sim n \log(n)^{\beta} \) and \( \mu(n) \sim \frac{b(\log n)^{n}}{n^{2}} \) and establish Theorem 3.9.
Note that the coalescences occur slower than that in the previous section. Moreover the tail of the splitting measure has for asymptotics \( \mu(n) \sim b(\log n)^{\alpha} \) and \( \sum_{k=1}^{n} k \mu(k) \sim b \frac{d}{\alpha+1} (\log n)^{\alpha+1} \). Recalling formula (2.15), we get \( \ell(n) \sim \frac{b}{\alpha+1} (\log n)^{\alpha+1} \).

**Proof of Theorem 3.9.** The cases \( \beta < 1 + \alpha \) and \( \beta > 1 + \alpha \) are covered by Theorem 3.1 and Theorem 3.3. We thus focus on the setting \( \beta = 1 + \alpha \). Set \( \theta = \frac{b}{d(1+\alpha)} \). According to [Fou20+, Proposition 1.8], the process \((\# \Pi(t), t \geq 0)\) either comes down from infinity or stays infinite depending on \( \theta < 1 \) or \( \theta > 1 \), respectively. Only remains the study of the accessibility of the boundary \( \infty \). For simplicity, assume that \( \mu(n) = b(\log n)^{\alpha} \) for all \( n \geq 1 \). The case for which only the equivalence \( \mu(n) \sim b \frac{(\log n)^{\alpha}}{n^2} \) holds, follows from an easy adaptation.

Assume \( \theta > 1 \). Let \( a > 1 \) and \( \epsilon > 0 \). One can check that for any \( 0 \leq x \leq \epsilon \),

\[
1 - (1+x)^{1-a} - (a-1)(1+\epsilon)^{-a} x \geq 0.
\]

Hence for any \( n \geq 1 \),

\[
G^a(n) = bn \frac{1}{a} \sum_{k=1}^{\infty} \left( \frac{\log k}{k^2} \right)^{\alpha} \left[ 1 - (1+k/n)^{1-a} \right] \geq bn \int_{1}^{\infty} \frac{(\log x)^{\alpha}}{x^2} \left[ 1 - (1+x/n)^{1-a} \right] dx
\]

\[
\geq b(a-1)(1+\epsilon)^{-a} \int_{1}^{\infty} \frac{(\log x)^{\alpha}}{x} dx = b(a-1)(1+\epsilon)^{-a} \frac{(\log x)^{\alpha+1}}{\alpha+1}.
\]

By Lemma 5.1-(i), for any \( p > 0 \), \(-G^a_p(n) \leq d(\log n)^{\alpha+1}(a-1)(1-p)^{-a}\). Thus, for all \( n \),

\[
-\frac{G^a_p(n)}{G^a(n)} \leq \frac{d(a+1)}{b} \frac{1}{1-p} \left( \frac{\log n}{\log \epsilon n} \right)^{\alpha+1}.
\]

Recall that \( \frac{d(a+1)}{b} = \frac{1}{b} < 1 \) and choose both \( \epsilon \) and \( p \) small enough such that \( \frac{d(a+1)}{b} \frac{1+\epsilon}{1-p} < 1 \). We see that there is \( a > 1 \) such that \( \limsup_{n \to \infty} \frac{-G^a_p(n)}{G^a(n)} < 1 \). By Corollary 4.5, we see that when \( \theta > 1 \), the process \((\# \Pi(t \wedge \sigma_p), t \geq 0)\) explodes with positive probability. Following the same argument as in the proof of Theorem 3.1, we get that the process explodes almost surely.

Assume now \( \theta < 1 \). Let \( a < 1 \). Recall \( \sum_{k=1}^{n} k \mu(k) \sim \frac{b}{\alpha+1} (\log n)^{\alpha+1} \) and \( \Phi(n) \sim \frac{d(\log n)}{b} n(k) \). We apply Proposition 5.5. By comparison with an integral and by applying Karamata’s theorem, we obtain

\[
\sum_{k=1}^{\infty} k^{1-a} \mu(k) \sim b \int_{n+1}^{\infty} x^{-a-1} (\log x)^{\alpha} dx \sim \frac{b}{a} n^{-a} (\log n)^{\alpha}.
\]

Therefore, \( \frac{\alpha+1}{\Phi(n)} \sum_{k=1}^{n} k^{1-a} \mu(k) \sim \frac{b}{d(\log n)} n \to 0 \), and (5.41) holds. Moreover,

\[
\limsup_{n \to \infty} \frac{n}{\Phi(n)} \sum_{k=1}^{n} k \mu(k) = \frac{b}{d(1+\alpha)} = \theta.
\]

Applying Proposition 5.5, we see that if \( \theta < 1 \), then the process does not explode. \( \square \)

We conclude this article by highlighting that the nature of the boundary \( \infty \) is not known in general for the critical cases \( \sigma = 1 \) and/or \( \theta = 1 \). In view of the proof of Proposition 3.8, this may require finer estimates that are unavailable with the functions \( (G_n, a > 0) \). This is left for potential future studies.

Last, the results presented here and in [Fou20+] have counterparts for certain processes in duality, called \( \Lambda \)-Wright-Fisher processes with frequency-dependent selection. We refer the interested reader to [FZ20+] where further properties of simple EFC are also established.

**Acknowledgements:** The authors are grateful to two anonymous referees for very detailed suggestions and comments. C.F.’s research is partially supported by LABEX MME-DII (ANR11-LBX-0023-01). X.Z.’s research is supported by Natural Sciences and Engineering Research Council of Canada (RGPIN-2016-06704) and by National Natural Science Foundation of China (No. 11731012).
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