Lower bounds for invariant statistical models with applications to principal component analysis

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Abstract

This paper develops nonasymptotic information inequalities for the estimation of the eigenspaces of a covariance operator. These results generalize previous lower bounds for the spiked covariance model, and they show that recent upper bounds for models with decaying eigenvalues are sharp. The proof relies on new lower bound techniques based on group invariance arguments which can also deal with a variety of other statistical models.

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1 Introduction

1.1 Motivation

In this paper we address the problem of deriving lower bounds for the estimation of derived parameters of the eigenspaces of a covariance operator. Motivated by principal component analysis (PCA), our main interest lies in the eigenspace of the $d$ leading eigenvalues.

In an asymptotic framework, such lower bounds can be obtained by existing results for statistical models satisfying the local asymptotic normality (LAN) condition, such as the local asymptotic minimax theorem due to Hájek [19]; see also the monographs by Ibragimov and Has’minskii [25] and van der Vaart [50]. Extensions are presented in Koltchinskii, Löffler and Nickl [32] and Koltchinskii [31] for the special cases of estimating linear functionals of principal components and more general smooth functionals of covariance operators, respectively. Both papers study an asymptotic scenario in which the effective rank of the covariance operator is allowed to increase with the number of observations $n$, and provide exactly matching asymptotic lower bounds based on the classical van Trees inequality.

Nonasymptotic lower bounds for the estimation of the eigenspace of the $d$ leading eigenvalues have been established in Cai, Ma and Wu [7, 8] and Vu and Lei [51] for a spiked covariance model with two different eigenvalues. This simple worst-case model can be parametrized by the Grassmann manifold, allowing to apply lower bounds under metric entropy conditions. This (Grassmann) approach has been applied to different principal subspace estimation problems and to different spiked structures; see e.g. Cai and Zhang [10] and Cai, Li and Ma [9] and the references therein.

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In this paper we derive minimax lower bounds that are nonasymptotic on the one hand (e.g. in $n$ and $d$) and can be applied to arbitrary sequences of eigenvalues on the other. To achieve this, we will apply a nonasymptotic extension of the local asymptotic minimax theorem in the context of invariant statistical models. In the important case of the eigenspace of the $d$ leading eigenvalues this will allow us to derive lower bounds that match recent nonasymptotic upper bounds for models with decaying eigenvalues derived in Mas and Ruymgaart [34], Reiß and Wahl [44] and Jirak and Wahl [27, 26].

One of our main contributions is to develop a new approach for the construction of lower bounds for statistical models equipped with an invariant group action. In particular, we provide a Chapman-Robbins inequality and a van Trees inequality in the context of equivariant statistical models. Motivated by principal component analysis, our main focus is on the orthogonal group, in which case both inequalities yield Bayes risk lower bounds when the reference measure is the Haar measure on the orthogonal group. But our lower bound techniques can also deal with a variety of other statistical models. To illustrate this further, we discuss a matrix denoising problem and the classical nonparametric density estimation problem.

The use of group invariance arguments has turned out to be crucial in multivariate statistics (see e.g. Muirhead [36], Farrell [17], Eaton [15] and Johnstone [28]), and our approach also relies on such principles. First, our approach is based on a van Trees inequality tailored for equivariant statistical models. Second, in order to derive tight nonasymptotic lower bounds, we have to study explicit prior densities with respect to the Haar measure on the orthogonal group. We propose a density based on the exponential of the trace that can be analyzed by large deviations techniques; see e.g. Meckes [35] and Hiai and Petz [22].

1.2 Main minimax lower bound

Let us present our main minimax lower bound in the context PCA. Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and let $L_1^+(\mathcal{H})$ be the class of all bounded linear operators $\Sigma : \mathcal{H} \to \mathcal{H}$ that are symmetric, positive and of trace class. For $\Sigma \in L_1^+(\mathcal{H})$, let $\lambda_1(\Sigma) \geq \lambda_2(\Sigma) \geq \cdots > 0$ be the non-increasing sequence of positive and summable eigenvalues of $\Sigma$, and let $u_1(\Sigma), u_2(\Sigma), \ldots$ be a sequence of corresponding eigenvectors. We shall assume that $u_1(\Sigma), u_2(\Sigma), \ldots$ form an orthonormal basis of $\mathcal{H}$.

For a fixed sequence $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ of positive and summable real
numbers, we define the parameter class
\[ \Theta = \{ \Sigma \in L^+_1(\mathcal{H}) : \lambda_j(\Sigma) = \lambda_j \text{ for all } j \geq 1 \}, \]
consisting of all \( \Sigma \in L^+_1(\mathcal{H}) \) with spectrum \( (\lambda_1, \lambda_2, \ldots) \), and consider the statistical model defined by the family of Gaussian measures
\[ (P_{\Sigma})_{\Sigma \in \Theta}, \quad P_{\Sigma} = \mathcal{N}(0, \Sigma)^{\otimes n}, \quad (1.1) \]
where \( \mathcal{N}(0, \Sigma) \) denotes a Gaussian measure in \( \mathcal{H} \) with expectation zero and covariance operator \( \Sigma \); see e.g. Chapter 2 in Da Prato and Zabczyk [11] for some background. An observation in this model consists of \( n \) independent \( \mathcal{H} \)-valued Gaussian random variables \( X_1, \ldots, X_n \) with expectation zero and covariance \( \Sigma \in \Theta \), and we will write \( \mathbb{E}_{\Sigma} \) to denote expectation with respect to \( X_1, \ldots, X_n \) having law \( P_{\Sigma} \).

For \( I \subseteq \{1, 2, \ldots\} \), the main parameter of interest is the orthogonal projection onto the eigenspace of the eigenvalues \( \{\lambda_i(\Sigma) : i \in I\} \) given by
\[ P_I(\Sigma) = \sum_{i \in I} u_i(\Sigma) \otimes u_i(\Sigma), \]
where \( u_i(\Sigma) \otimes u_i(\Sigma) \) is the orthogonal projection onto the span of \( u_i(\Sigma) \) defined by \( (u_i(\Sigma) \otimes u_i(\Sigma))x = \langle u_i(\Sigma), x \rangle u_i(\Sigma), \quad x \in \mathcal{H} \). Note that \( P_I(\Sigma) \) is uniquely defined in case that the eigenvalues with indices in \( I \) are separated from the rest of the spectrum, that is provided that \( \lambda_i(\Sigma) - \lambda_j(\Sigma) \neq 0 \) for every \( i \in I, \quad j \notin I \).

Our first main result provides a nonasymptotic minimax lower bound for the estimation of the parameter \( P_I(\Sigma) \) in the Hilbert-Schmidt loss.

**Theorem 1.** There exists an absolute constant \( c \in (0, 2) \) such that, for all \( J \subseteq \{1, 2 \ldots\} \), we have
\[
\inf_{P} \sup_{\Sigma \in \Theta} \mathbb{E}_{\Sigma} \| \hat{P} - P_I(\Sigma) \|_{\text{HS}} \geq c \sum_{i \in I \cap J} \sum_{j \in J \setminus I} \left( \frac{\lambda_i \lambda_j}{n(\lambda_i - \lambda_j)^2} \right) \wedge \frac{1}{|J|},
\]
where the infimum is taken over all estimators \( \hat{P} = \hat{P}(X_1, \ldots, X_n) \) with values in the space of all Hilbert-Schmidt operators, \( \| \cdot \|_{\text{HS}} \) denotes the Hilbert-Schmidt norm and \( x \wedge y = \min(x, y) \).

Theorem 1 is the consequence of a more general Bayes risk bound presented in Section 1.4. Both lower bounds follow from a van Trees inequality based on group invariance arguments. In particular, we will see that the terms \( n(\lambda_i - \lambda_j)^2/\lambda_i \lambda_j \) are the different Fisher information directions of the model (1.1), while the relaxing term \( |J| \) is the Fisher information of the prior distribution, ensuring that the lower bound does not explode for vanishing gaps.
1.3 Examples and discussion

Let us discuss some consequences of Theorem 1. For a subset $\mathcal{I} \subseteq \{1, 2, \ldots \}$ and a sequence of eigenvalues, we abbreviate the minimax risk over $\Theta_\lambda$ as follows

$$R_{n,\lambda,\mathcal{I}}^* = \inf_{\hat{P}} \sup_{\Sigma \in \Theta_\lambda} \mathbb{E}_{\Sigma} \| \hat{P} - P_{\mathcal{I}}(\Sigma) \|_{\text{HS}}^2,$$  \hspace{1cm} (1.2)

where the infimum is taken over all estimators $\hat{P} = \hat{P}(X_1, \ldots, X_n)$ taking values in the Hilbert space of all Hilbert-Schmidt operators. Note that the minimax risk increases if we restrict the infimum to all estimators taking values in the class of all orthogonal projections of rank $|\mathcal{I}|$, in which case the Hilbert-Schmidt distance $\| \hat{P} - P_{\mathcal{I}}(\Sigma) \|_{\text{HS}}$ is equal to $\sqrt{2}$ times the Euclidean norm of the sines of the canonical angles between the subspaces corresponding to $\hat{P}$ and $P_{\mathcal{I}}(\Sigma)$, see e.g. [5, Chapter VII.1].

Let us start by briefly describing the classical asymptotic scenario $n \to \infty$. Combining Theorem 1 (resp. the proof in Section 4.3, where it is shown that for each $\delta \in (0, 2)$, the constant $c$ in Theorem 1 can be replaced by $2 - \delta$ provided that $1/|\mathcal{I}|$ is replaced by $c_\delta/|\mathcal{I}|$ with a constant $c_\delta$ depending only on $\delta$) with standard perturbation bounds for the empirical covariance operator, we get

$$\lim_{n \to \infty} n \cdot R_{n,\lambda,\mathcal{I}}^* = 2 \sum_{i \in \mathcal{I}} \sum_{j \notin \mathcal{I}} \lambda_i \lambda_j \left( \lambda_i - \lambda_j \right)^2,$$  \hspace{1cm} (1.3)

where the asymptotic limit is achieved by the estimator $\hat{P} = P_{\mathcal{I}}(\hat{\Sigma})$ with empirical covariance operator $\hat{\Sigma} = n^{-1} \sum_{i=1}^n X_i \otimes X_i$; see e.g. Hsing and Eubank [24] and Jirak and Wahl [27, Equation (1.3)] for the corresponding upper bound. More specifically, our approach also implies that $P_{\mathcal{I}}(\hat{\Sigma})$ is asymptotically efficient in the sense of [25, Equation (9.4)]; see Section 3.3.1 and Corollary 4 below for more details.

Let us turn to nonasymptotic lower bounds. As stated in [7], it is highly nontrivial to obtain lower bounds which depend optimally on all parameters, in particular the eigenvalues and $\mathcal{I}$. Indeed, in contrast to the asymptotic setting, in which the above results can also be derived from the local asymptotic minimax theorem, it seems unavoidable to use some more sophisticated arguments on the underlying parameter space of all orthonormal bases in order to obtain nonasymptotic lower bounds. A fundamental result, obtained in Cai, Ma and Wu [7] and Vu and Lei [51], provides a nonasymptotic lower bound for a spiked covariance model with two groups of eigenvalues.
Corollary 1 ([7]). For $\mathcal{H} = \mathbb{R}^p$ and the sequence of eigenvalues $\lambda_1 = \cdots = \lambda_d > \lambda_{d+1} = \cdots = \lambda_p > 0$, we have

$$R_{n,\lambda,\{1,\ldots,d\}}^* \geq c \cdot \min \left( \frac{d(p-d)}{n}, \frac{\lambda_d \lambda_{d+1}}{\lambda_d - \lambda_{d+1}}^2 \right) \cdot n^{(d(p-d))/2} \cdot \min(d, p-d)$$

for some absolute constant $c$.

Corollary 1 is sharp up to a constant; see e.g. [7, Theorem 5] and [44, Section 2.4] for a corresponding upper bound. Together the lower and upper bound can be viewed as a nonasymptotic version of the phase transition phenomenon for empirical eigenvectors; see e.g. [40, 37].

Corollary 1 follows from Theorem 1 applied with $J = \{1, \ldots, p\}$, using that $2d(p-d)/p \geq \min(d, p-d)$. In [7, 51], the proof of Corollary 1 is based on the behavior of the local metric entropy of the Grassmann manifold. In fact, the parameter class can be written as $\Theta_\lambda = (\lambda_d - \lambda_{d+1})P_d + \lambda_{d+1}I_p$, with $I_p$ being the identity matrix and $P_d$ being the class of all orthogonal projections of rank $d$, for which we have the following two facts:

(i) For $\Sigma_i = (\lambda_d - \lambda_{d+1})P_i + \lambda_{d+1}I_p$, with $P_i \in P_d$, $i = 1, 2$, we have, with Kullback-Leibler divergence $K(\cdot, \cdot)$,

$$K(\mathbb{P}_{\Sigma_1}, \mathbb{P}_{\Sigma_2}) = \frac{n(\lambda_d - \lambda_{d+1})^2}{4\lambda_d \lambda_{d+1}} \|P_1 - P_2\|_{HS}.$$  

(ii) For all integers $1 \leq d \leq p$ such that $d \leq p-d$ and all real number $\delta \in (0, \sqrt{2d}]$, the covering number $\mathcal{N}(P_d, \|\cdot\|_{HS}, \delta)$ of $P_d$ with respect to the Hilbert-Schmidt norm satisfies,

$$\left( \frac{c_0 \sqrt{d}}{\delta} \right)^{d(p-d)} \leq \mathcal{N}(P_d, \|\cdot\|_{HS}, \delta) \leq \left( \frac{c_1 \sqrt{d}}{\delta} \right)^{d(p-d)}$$

for some absolute constants $c_0, c_1 > 0$. In particular, for any $\alpha \in (0, 1)$ and $\delta \in (0, \sqrt{2d}]$ there are $P_1, \ldots, P_m \in P_d$ with $m \geq (c_0/(\alpha c_1))^d(p-d)$ and $\alpha \delta \leq \|P_i - P_j\|_{HS} \leq 2\delta$ for every $i \neq j$.

For (i) see [51, Lemma A.2], (ii) can be found in [39, Proposition 8] and [7, Lemma 1]. Combining (i) and (ii), Corollary 1 follows from Fano’s lemma (cf. [55, Lemma 3]).

The interesting point here is that we deduce Corollary 1 by an entirely different approach via invariance arguments. This will allow us to take advantage of the Fisher geometry of the statistical model (1.5) more efficiently, and leads to the improvement of Theorem 1 over Corollary 1. A first extension of Corollary 1 is as follows.
Corollary 2. For \( \mathcal{H} = \mathbb{R}^p, 1 \leq d \leq p - d \) and the sequence of eigenvalues \( \lambda_1 \geq \cdots \geq \lambda_d > \lambda_{d+1} = \cdots = \lambda_p > 0 \), we have

\[
R_{n,\lambda,\{1,\ldots,d\}}^* \geq c \cdot \min_{i \leq d} \left( \frac{p}{n} \frac{\lambda_i \lambda_{d+1}}{\lambda_i - \lambda_{d+1}}^2, 1 \right)
\]

for some absolute constant \( c \).

Corollary 2 follows from Theorem 1 applied with \( J = \{1, \ldots, p\} \), using that \( p - d \geq p/2 \). Let us give an interpretation of Corollary 2 by comparing it to upper bounds for the eigenprojections \( \hat{P} = P_{\{1,\ldots,d\}}(\hat{\Sigma}) \) of the empirical covariance operator \( \hat{\Sigma} \) established in [44]. To see this, write

\[
\|P_{\{1,\ldots,d\}}(\hat{\Sigma}) - P_{\{1,\ldots,d\}}(\Sigma)\|_{\text{HS}}^2 = 2 \sum_{i \leq d} \sum_{j > d} \|P_i \hat{P}_j\|_{\text{HS}}^2,
\]

where we abbreviated \( P_i = P_{\{i\}}(\Sigma) \) and \( \hat{P}_j = P_{\{j\}}(\hat{\Sigma}) \). Now, there are two completely different possibilities to bound the latter projector norms. First, we always have \( \sum_{j > d} \|P_i \hat{P}_j\|_{\text{HS}}^2 \leq 1 \). Second, one can apply perturbation bounds, and the first part in the minimum in Corollary 2 gives the size of the linear perturbation term of \( \sum_{j > d} \|P_i \hat{P}_j\|_{\text{HS}}^2 \); see e.g. [12, 27]. In particular, Corollary 2 implies that one can not improve upon a mixture of both inequalities.

The main focus of Theorem 1 is on PCA in infinite dimensions, typically encountered in functional data analysis and kernel methods in machine learning. For instance, Sobolev kernels usually lead to polynomially decaying eigenvalues, while smooth radial kernels have nearly exponentially decaying eigenvalues (cf. [47, 46, 4] and the references therein). In such scenarios, it is, in contrast to the spiked covariance model, no longer sufficient to study the first few principal components. Instead, it becomes more important to understand the behavior of eigenspaces for growing values of \( d \) (resp. \( d \) depending on \( n \)); see e.g. [20] and [6] for two accounts on principal component regression and more general spectral regularization methods, respectively.

Corollary 3. For each \( \alpha > 0 \), there is a constant \( c > 0 \) depending only on \( \alpha \) such that the following holds.

(i) If \( \lambda_j = j^{-\alpha-1} \) for every \( j \geq 1 \), then we have

\[
R_{n,\lambda,\{d\}}^* \geq c \cdot \min \left( 1, \frac{d^2}{n} \right),
\]
\( R^*_n,\lambda, \{1,\ldots,d\} \geq c \cdot \min \left( d, \frac{d^2}{n} \left( 1 + \log_+ \left( d \land \sqrt{\frac{n}{d}} \right) \right) \right) \)

with \( \log_+(x) = 0 \lor \log(x) \).

(ii) If \( \lambda_j = e^{-\alpha_j} \) for every \( j \geq 1 \), then we have

\[ R^*_n,\lambda, \{d\} \geq \frac{c}{n}, \quad R^*_n,\lambda, \{1,\ldots,d\} \geq \frac{c}{n}. \]

Corollary 3 follows from applying Theorem 1 with \( \mathcal{J} = \{d, d + 1\} \) and 
\( \mathcal{J} = \{j : d/2 < j \leq 3d/2\} \) respectively; see Appendix A for the details. In
the non-trivial regime \( d^2 \leq n \) (note that \( R^*_n,\lambda, \{d\} \leq 1 \), as can be seen by
considering the zero estimator), Corollary 3 (i) can be written as

\[ R^*_n,\lambda, \{d\} \geq \frac{c d^2}{n} \quad \text{and} \quad R^*_n,\lambda, \{1,\ldots,d\} \geq \frac{c d^2 \log(d)}{n}. \quad (1.4) \]

A corresponding non-asymptotic upper bound has been first obtained in Mas and Ruymgaart [34], yet with additional log factors in \( n \) and \( d \) and for the
smaller operator norm instead of the Hilbert-Schmidt norm. In case of the
Hilbert-Schmidt norm, Jirak and Wahl [26, 27] showed that the empirical
projectors \( P_{\{d\}}(\hat{\Sigma}) \) and \( P_{\{1,\ldots,d\}}(\hat{\Sigma}) \) achieve, with high probability, the upper
bounds \( Cd^2/n \) and \( Cd^2 \log(d)/n \), respectively, as long as \( d^2 \log^2(d) \leq cn \).
This implies that the lower bounds in (1.4) are, up to a constant, sharp in
the latter regime. In case of exponentially decaying eigenvalues, matching
upper bounds have been derived in Wahl [52] and Reiß and Wahl [44] for
the case of \( \mathcal{I} = \{d\} \) and \( \mathcal{I} = \{1,\ldots,d\} \), respectively.

We conclude this section by describing some extensions and open prob-
lems. Firstly, while our results are restricted to the Hilbert-Schmidt loss, we
leave it for further research to extent our bounds to other loss function such
as the operator norm and the excess risk loss. In particular, since the excess
risk can be written as a weighted squared loss (cf. [44, Lemma 2.6]), lower
bounds for the excess risk can be established by appropriate extensions of
Proposition 1. Secondly, it has been shown in [27, 26, 52] that the so-called
relative ranks are crucial to characterize the behavior of empirical eigen-
values and eigenspaces. In particular, the perturbation expansions developed in
these papers are going to break down if certain relative rank conditions are
no longer satisfied. Such conditions do not necessarily follow from Theorem
1. For instance, Corollary 3 (ii) yields the lower bound \( c/n \) for estimating
\( P_{\{d\}}(\Sigma) \) which seems to be rather sub-optimal for \( d \geq n \). A second problem
is therefore to establish lower bounds based on the relative rank that provide
an (optimal) phase transition beyond which estimating the eigenspaces is no longer possible. Thirdly, while we provide quantitative results that can be used to show that the PCA projector $P_{\{1,\ldots,d\}}(\hat{\Sigma})$ achieves the minimax risk up to a constant, it is also an interesting problem to establish some qualitative properties. Is it true, for instance, that the PCA projector is even minimax (and admissible) with respect to the Hilbert-Schmidt loss over $\Theta_\lambda$, at least over all estimators taking values in the class of all orthogonal projections of rank $d$. In fact, while it is not too difficult to show this in the setting of Corollary 1, it seems to be an open problem whether the PCA projector is minimax or not in the other examples; see also [43] for a general conjecture.

### 1.4 Main Bayes risk lower bound

We now state a stronger Bayes risk lower bound in the special case of $H = \mathbb{R}^p$. Afterwards, we show how Theorem 1 can be obtained from this result by standard decision-theoretic arguments. In what follows, we will use the following reparametrization of our model

$$((P_U)_{U \in SO(p)}, \quad P_U = N(0, U\Lambda U^T)^{\otimes n}, \quad (1.5)$$

where $SO(p)$ denotes the special orthogonal group (defined in Section 2 below) and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p)$ with $\lambda_1 \geq \cdots \geq \lambda_p > 0$ fixed. While this leads to a slight over-parametrization (each covariance matrix is now repeated at least $2^{p-1}$-times), it will allow us to invoke more directly properties of the special orthogonal group. In this reparametrization, the parameter of interest is $P_{\hat{Z}}(U) = \sum_{i \in I} u_i u_i^T$, where $u_1, \ldots, u_p$ are the columns of $U \in SO(p)$.

**Theorem 2.** Consider the statistical model (1.5). Then there are absolute constants $c, C > 0$ such that for every $h \geq C$, we have

$$\inf_{\hat{P}} \int_{SO(p)} \mathbb{E}_U \|\hat{P} - P_{\hat{Z}}(U)\|_{HS}^2 \tilde{\pi}_h(\text{tr}(U))dU \geq c \sum_{i \leq d} \sum_{j > d} \min \left( \frac{1}{n} \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2}, \frac{1}{h^2 p} \right),$$

where the infimum is taken over all $\mathbb{R}^{p \times p}$-valued estimators $\hat{P} = \hat{P}(X_1, \ldots, X_n)$, $dU$ denotes the Haar measure on $SO(p)$, $\text{tr}(U)$ denotes the trace of $U$, and $\pi_h$ is a prior density given by

$$\tilde{\pi}_h(\text{tr}(U)) = \frac{\exp(h \text{tr}(U))}{\int_{SO(p)} \exp(h \text{tr}(U))dU}, \quad h > 0.$$
While \( h = C \) provides an optimal choice from an non-asymptotic point of view, the choice \( h = o(\sqrt{n}) \) leads to a Bayesian version of the local asymptotic minimax theorem (using also that the constant \( c \) can be made more explicit, as shown in Section 4.3); see e.g. Theorem 8.11 in [50].

We conclude this section by showing how Theorem 2 with \( h = C \) implies Theorem 1. For some fixed orthonormal basis \( u_1, u_2, \ldots \) of \( H \) and a finite subset \( J \subseteq \{1, 2, \ldots \} \) with \( |J| = p \), we consider the sub-model

\[
\Theta_J^\lambda = \{ \Sigma \in \Theta_\lambda : \forall k \notin J, P_k(\Sigma) = u_k \otimes u_k \} \subseteq \Theta_\lambda,
\]

leading to covariance operators \( \Sigma \in \Theta_\lambda \) that only differ from each other on how they map \( \text{span}\{u_j : j \in J\} \) into itself. Hence, restricting the Hilbert-Schmidt norm to inner products with \( \{u_j : j \in J\} \), we have

\[
E_\Sigma[\|\hat{P} - P_I(\Sigma)\|_{\text{HS}}^2] \geq E_\Sigma[\|\hat{P}_J - P_{I\cap J}(\Sigma_J)\|_{\text{HS}}^2], \tag{1.6}
\]

with \( \hat{P}_J, \Sigma_J \in \mathbb{R}^{p \times p} \) defined by \( \hat{P}_j^J = \langle u_j, \hat{P} u_k \rangle \) and \( \Sigma_{jk}^J = \langle u_j, \Sigma u_k \rangle \) for \( j, k \in J \). Here, we used that the matrix \( P_{I\cap J}(\Sigma_J) \) has entries \( \langle u_j, P_I(\Sigma) u_k \rangle, \) \( j, k \in J \). While \( \hat{P}_J \) is based on \( X_1, \ldots, X_n \), it follows by an application of the sufficiency principle that it suffices to consider estimators based on \( X_i^J = (\langle X_i, u_j \rangle)_{j \in J}, \) \( i \leq n. \) Indeed, by the Neyman factorization theorem ([25, Theorem 1.1] or [48, Theorem 20.9]) and basic facts for Gaussian measures on separable Hilbert spaces (Chapter 2 in [11]) we know that \( (X_i^J : i \leq n) \) is a sufficient statistic for \( \{P_\Sigma : \Sigma \in \Theta_J^\lambda\} \). Hence, applying Rao-Blackwell’s theorem ([25, Theorem 2.1]) to the right-hand side of (1.6), we conclude that

\[
\inf_{\hat{P}} \sup_{\Sigma \in \Theta_J^\lambda} E_\Sigma[\|\hat{P} - P_I(\Sigma)\|_{\text{HS}}^2] \geq \inf_{\hat{P}} \sup_{\Sigma \in \Theta_J^\lambda} E_\Sigma[\|\hat{P}_J - P_{I\cap J}(\Sigma_J)\|_{\text{HS}}^2], \tag{1.7}
\]

where the infimum is taken over all estimators \( \hat{P} = \hat{P}(X_i^J : i \leq n) \) with values in \( \mathbb{R}^{p \times p} \). Now, the \( X_i^J \) are independent \( \mathcal{N}(0, \Sigma_J) \)-distributed random variables, and the class \( \{\Sigma_J : \Sigma \in \Theta_J^\lambda\} \) coincides with \( \{U \text{ diag}(\lambda_j : j \in J) U^T : U \in SO(p)\} \). Hence, using that the maximum risk over a parameter class is bounded from below by the Bayes risk over any sub-class, Theorem 1 follows from applying Theorem 2 to the right-hand side of (1.7).

Outline

The remainder of the paper is organized as follows. Section 2 provides some preliminaries on the special orthogonal group, on \( \chi^2 \)-distances and
the Fisher information, and on statistical models endowed with a group action. Section 3 develops a Chapman-Robbins inequality and a van Trees-type inequality in the context of equivariant statistical models. Section 4 is devoted to the proof of Theorem 2. Section 4.1 specializes our van Trees-type inequality to the case of eigenspaces and Section 4.2 studies the prior density from Theorem 2 based on large deviations techniques for the special orthogonal group.

2 Preliminaries

2.1 The special orthogonal group

Let us introduce some notation in connection with the (special) orthogonal group; see e.g. [35] for a detailed account. The special orthogonal group is defined by

$$SO(p) = \{ U \in \mathbb{R}^{p \times p} : UU^T = I_p, \det(U) = 1 \},$$

where $I_p$ denotes the $p \times p$ identity matrix. It is a connected and compact Lie group. The Lie algebra (i.e. the tangent space at $I_p$) is given by $\mathfrak{so}(p) = \{ \xi \in \mathbb{R}^{p \times p} : \xi + \xi^T = 0 \}$. It is generated by $L^{(ij)} = e_i e_j^T - e_j e_i^T$, $i < j$, where $e_1, \ldots, e_p$ denotes the standard basis in $\mathbb{R}^p$. Thus we have $\dim \mathfrak{so}(p) = p(p-1)/2$. More generally, the tangent space at $U$ is given by $U \mathfrak{so}(p) = \mathfrak{so}(p)U$. The exponential map is given by

$$\exp : \mathfrak{so}(p) \to SO(p), \xi \mapsto \exp(\xi) = \sum_{k \geq 0} \frac{\xi^k}{k!}$$

It has the property that $(d/dt) \exp(t\xi) = \xi \exp(t\xi) = \exp(t\xi)\xi$. Finally, there is a unique translation-invariant probability measure $\mu$ (called Haar measure) on $SO(p)$ satisfying

$$\int_{SO(p)} f(UV) \, d\mu(U) = \int_{SO(p)} f(VU) \, d\mu(U) = \int_{SO(p)} f(U) \, d\mu(U) \quad (2.1)$$

for all $V \in SO(p)$ and all integrable functions $f$. To simplify the notation, we will denote $\int_{SO(p)} f(U) \, d\mu(U)$ by $\int_{SO(p)} f(U) \, dU$.

2.2 $\chi^2$-divergence and Fisher information

In this section we provide some standard Fisher information calculations for the statistical model given in (1.5) and a related matrix denoising model. For similar results and further reading see e.g. [33, 54] and [2].
The \(\chi^2\)-divergence between two probability measures \(P\) and \(Q\) (defined on the same measurable space) is defined as
\[
\chi^2(P, Q) = \begin{cases} 
\int \left(\frac{dP}{dQ}\right)^2 dQ - 1, & \text{if } P \ll Q, \\
\infty, & \text{otherwise.}
\end{cases}
\]

If \(P = P_1 \otimes P_2\) and \(Q = Q_1 \otimes Q_2\) are product measures, then we have
\[
\chi^2(P, Q) = (1 + \chi^2(P_1, Q_1))(1 + \chi^2(P_2, Q_2)) - 1.
\]
\[\text{(2.2)}\]

The following lemma analyzes the limiting behavior of the \(\chi^2\)-divergence of the family of probability measures from (1.5), given by \(\{P_U : U \in SO(p)\}\) with \(P_U = \mathcal{N}(0, U\Lambda U^T)\otimes_n\) and \(\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p)\) being a diagonal matrix with \(\lambda_1 \geq \cdots \geq \lambda_p > 0\).

**Lemma 1.** Consider the statistical model given in (1.5). Then, for \(\xi \in so(p)\), we have
\[
\chi^2(P_{\exp(t\xi)}, P_{I_p})/t^2 \to \frac{n}{2} \sum_{i,j=1}^p \xi_{ij}^2 \left(\frac{\lambda_i - \lambda_j}{\lambda_i \lambda_j}\right)^2 \text{ as } t \to 0.
\]

**Proof.** Using (2.2), it suffices to consider the case \(n = 1\), in which case we have \(P_{\exp(t\xi)} = \mathcal{N}(0, \exp(t\xi)\Lambda \exp(-t\xi))\). Then, for all \(t\) sufficiently small, we have
\[
\chi^2(P_{\exp(t\xi)}, P_{I_p}) + 1 = 1/\sqrt{\det(2I_p - \Lambda \exp(t\xi)\Lambda^{-1}\exp(-t\xi))},
\]
as can be seen by inserting the multivariate Gaussian density function into the definition of the \(\chi^2\)-divergence. For all \(t\) sufficiently small, set \(f(t) = \det(2I_p - \Lambda \exp(t\xi)\Lambda^{-1}\exp(-t\xi))\). Then it follows from standard formulas for the derivatives of a determinant (see e.g. [33, Chapter 8.3] and also [42, Section 2.1.1]) that \(f(0) = 1, f'(0) = 0\) and
\[
f''(0) = 4 \text{tr}(\Lambda\xi^2 - \xi^2) = 2 \sum_{i,j=1}^p \xi_{ij}\xi_{ji}(\lambda_i\lambda_j^{-1} + \lambda_j\lambda_i^{-1} - 2)
\]
\[
= -2 \sum_{i,j=1}^p \xi_{ij}^2 \left(\frac{\lambda_i - \lambda_j}{\lambda_i \lambda_j}\right)^2.
\]
\[\text{(2.3)}\]

Using Taylor’s theorem we obtain that
\[
\chi^2(P_{\exp(t\xi)}, P_{I_p})/t^2 = \frac{\sqrt{f^{-1}(t)} - 1}{t^2} \to -\frac{f''(0)}{4} \text{ as } t \to 0,
\]
and the claim follows from inserting (2.3). \(\Box\)

We now give a definition of the Fisher information that is sufficient for our purposes. Let \((X, \mathcal{F}, (\mathbb{P}_\theta)_{\theta \in \Theta})\) be a statistical model, where \(\Theta\) is a manifold embedded in some Euclidean space. Suppose that the experiment is dominated by a measure \(\mu\) such that \(f(x, \theta) = \frac{d\mathbb{P}_\theta}{d\mu}(x)\) are strictly positive. Moreover, let \(l(x, \theta) = \log f(x, \theta)\). Then the Fisher information form at \(\theta\) is defined by

\[
\mathcal{I}_\theta : T_\theta \Theta \times T_\theta \Theta \to \mathbb{R}, (v, w) \mapsto \int_X dl(x, \theta) v dl(x, \theta) w d\mathbb{P}_\theta(x),
\]

provided that the last integrals exist. Here, \(T_\theta \Theta\) denotes the tangent space of \(\Theta\) at \(\theta\) and \(dl(x, \theta)\) denotes the derivative of \(l(x, \cdot)\) at \(\theta\) in the direction \(v\), defined by \(dl(x, \theta) v = \frac{d}{dt}l(x, \gamma(t))|_{t=0}\) with \(\gamma : (-\epsilon, \epsilon) \to \Theta\) such that \(\gamma(0) = \theta\) and \(\gamma'(0) = v\).

The following lemma explains the connection of the result in Lemma 1 with the Fisher information.

**Lemma 2.** Consider the statistical model given in (1.5). Then the Fisher information form at \(I_p\) is given by

\[
\mathcal{I}_{I_p} : \mathfrak{so}(p) \times \mathfrak{so}(p) \to \mathbb{R}, (\xi, \eta) \mapsto \frac{n}{2} \sum_{i,j=1}^p \xi_{ij} \eta_{ij} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i \lambda_j}.
\]

More generally, for \(U \in SO(p)\), we have \(\mathcal{I}_U(U\xi, U\eta) = \mathcal{I}_{I_p}(\xi, \eta)\).

**Remark 1.** In particular, \(\mathcal{I}_{I_p}\) is diagonalized by the basis \(\{L^{(ij)} : i < j\}\).

**Remark 2.** Using the Fisher information form from Lemma 2, Lemma 1 can be written as \(\chi^2(\mathbb{P}_{\exp(t\xi)}, \mathbb{P}_{I_p})/t^2 \to \mathcal{I}_{I_p}(\xi, \xi)\) as \(t \to 0\). For the more general concept of \(L^2\)-differentiability, see e.g. [53, Chapter 1.8].

**Proof of Lemma 2.** Without loss of generality, we may assume that \(n = 1\). It suffices to show that

\[
\mathcal{I}_{I_p}(L^{(ij)}, L^{(kl)}) = \delta_{ik} \delta_{jl} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i \lambda_j} \quad \forall i < j, k < l
\]

with \(\delta_{ik}\) equal to 1 if \(i = k\) and equal to 0 otherwise. Now, with \(\mu\) being the Lebesgue measure on \(\mathbb{R}^p\),

\[
dl(x, I_p)L^{(ij)} = -\frac{1}{d} \frac{d}{dt} \langle x, \exp(-tL^{(ij)})\Lambda^{-1} \exp(tL^{(ij)})x \rangle |_{t=0} = \frac{1}{2} \langle x, (L^{(ij)}\Lambda^{-1} - \Lambda^{-1}L^{(ij)})x \rangle = (\lambda_j^{-1} - \lambda_i^{-1})x_i x_j.
\]
Thus
\[ \mathcal{I}_p(L^{(ij)}, L^{(kl)}) = (\lambda_i^{-1} - \lambda_j^{-1}) (\lambda_k^{-1} - \lambda_l^{-1}) \int x_i x_j x_k x_l \, d\mathcal{N}(0, \Lambda)(x) \]
\[ = \delta_{ik} \delta_{jl} (\lambda_i^{-1} - \lambda_j^{-1})^2 \lambda_i \lambda_j, \]
and the claim follows. \qed

We now provide similar calculations for a related matrix denoising model. For a fixed diagonal matrix \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p) \) with \( \lambda_1 \geq \cdots \geq \lambda_p \geq 0 \), we consider the family of probability measures \( (\mathbb{P}_U)_{U \in SO(p)} \) with \( \mathbb{P}_U \) being the distribution of
\[ X = U \Lambda U^T + \sigma W, \]
where \( \sigma > 0 \) and \( W = (W_{ij})_{1 \leq i, j \leq p} \) is a GOE matrix, i.e. a symmetric random matrix whose upper triangular entries are independent zero mean Gaussian random variables with \( \mathbb{E} W_{ij}^2 = 1 \) for \( 1 \leq i < j \leq p \) and \( \mathbb{E} W_{ii}^2 = 2 \) for \( i = 1, \ldots, p \); see e.g. [38, 41, 27]. Using half-vectorization, this model can alternatively be defined on \( X = \mathbb{R}^{p(p+1)/2} \) with
\[ (\mathbb{P}_U)_{U \in O(p)}; \quad \mathbb{P}_U = \mathcal{N}(\text{vech}(U \Lambda U^T), \sigma^2 \Sigma_W), \quad (2.4) \]
where symmetric matrices \( A \in \mathbb{R}^{p \times p} \) are transformed into vectors using \( \text{vech}(A) = (A_{11}, A_{21}, \ldots, A_{p1}, A_{p2}, A_{32}, \ldots, A_{pp}) \in \mathbb{R}^{p(p+1)/2} \), and \( \Sigma_W \) is the covariance matrix of \( \text{vech}(W) \).

**Lemma 3.** Consider the statistical model given in (2.4). Then, for \( \xi \in \mathfrak{o}(p) \), we have
\[ \chi^2(\mathbb{P}_{\text{exp}(t\xi)}, \mathbb{P}_p)/t^2 \to \frac{1}{2\sigma^2} \sum_{i,j=1}^p \xi_{ij}^2 (\lambda_i - \lambda_j)^2 \quad \text{as } t \to 0. \]

**Proof.** Using the identity
\[ \chi^2(\mathcal{N}(\mu_1, \Sigma), \mathcal{N}(\mu_2, \Sigma)) = \exp(\|\Sigma^{-1/2}(\mu_1 - \mu_2)\|_2^2) - 1 \]
with Euclidean norm \( \| \cdot \|_2 \), we get
\[ \chi^2(\mathbb{P}_{\text{exp}(t\xi)}, \mathbb{P}_p) = \exp \left( \frac{1}{2\sigma^2} \|\exp(t\xi) \Lambda \exp(-t\xi) - \Lambda\|_{\text{HS}}^2 \right) - 1, \]
where we also used the identity \( \|\Sigma_W^{-1/2} \text{vech}(A)\|_2^2 = 2^{-1} \|A\|_{\text{HS}}^2 \), valid for any \( A \in \mathbb{R}^{p \times p} \) symmetric, in order to reverse the half-vectorization. From this, we get

\[
\chi^2(\mathbb{P}_{\text{exp}(t\xi)}, \mathbb{P}_{I_p})/t^2 \to \frac{1}{2\sigma^2} \|\xi\Lambda - \Lambda\xi\|_{\text{HS}}^2 \quad \text{as } t \to 0,
\]

and the claim follows. \( \square \)

The following lemma explains the connection of the result in Lemma 3 with the Fisher information.

**Lemma 4.** Consider the statistical model given in (2.4). Then the Fisher information form at \( I_p \) is given by

\[ I_{I_p} : \mathfrak{so}(p) \times \mathfrak{so}(p) \to \mathbb{R}, (\xi, \eta) \mapsto \frac{1}{2\sigma^2} \sum_{i,j=1}^p \xi_{ij} \eta_{ij} (\lambda_i - \lambda_j)^2. \]

More generally, for \( U \in SO(p) \), we have \( I_{U}(U\xi, U\eta) = I_{I_p}(\xi, \eta) \).

**Remark 3.** Again, \( I_{I_p} \) is diagonalized by the basis \( \{L^{ij} : i < j\} \).

**Proof.** A similar calculation as in the proof of Lemma 3 shows that \( I_{I_p}(\xi, \eta) = (2\sigma^2)^{-1} \text{tr}((\xi\Lambda - \Lambda\xi)(\eta\Lambda - \Lambda\eta)^T), \xi, \eta \in \mathfrak{so}(p) \), and the claim follows. \( \square \)

### 2.3 Statistical models under group action

In this section we summarize some basic facts for equivariant statistical models; see e.g. [15, 16] for two detailed accounts. Let \( G \) be a group. For every \( g \in G \) the right multiplication map \( R_g : G \to G \) is defined by \( R_g(h) = hg, h \in G \). If \( G \) acts (from the left) on a measurable space \( (\mathcal{X}, \mathcal{F}) \), then we will always assume that the map \( \mathcal{X} \to \mathcal{X}, x \mapsto gx \) is measurable for every \( g \in G \). If \( G \) itself is also a measurable space, then we will always assume that the map \( G \times \mathcal{X} \to \mathcal{X}, (g, x) \mapsto gx \) is measurable.

**Definition 1.** Suppose that \( G \) acts on \( (\mathcal{X}, \mathcal{F}) \) and \( \Theta \). Then a family \( (\mathbb{P}_\theta)_{\theta \in \Theta} \) of probability measures on \( (\mathcal{X}, \mathcal{F}) \) is called \( G \)-equivariant if

\[ \mathbb{P}_{g\theta}(gA) = \mathbb{P}_\theta(A) \quad \forall g \in G, \theta \in \Theta, A \in \mathcal{F}. \]

Definition 1 says that for a random variable \( X \) with distribution \( \mathbb{P}_\theta \) we have that \( gX \) has distribution \( \mathbb{P}_{g\theta} \).
Lemma 5. Suppose that \((P_\theta)_{\theta \in \Theta}\) is \(G\)-equivariant. For \(\theta_0, \theta_1 \in \Theta\) and \(g \in G\) the following holds:

(i) For all measurable functions \(f \geq 0\), we have
\[
\int_X f(x) \, dP_{g\theta_0}(x) = \int_X f(gx) \, dP_{\theta_0}(x).
\]

(ii) If \(P_{\theta_1} \ll P_{\theta_0}\), then \(P_{g\theta_1} \ll P_{g\theta_0}\) and
\[
\frac{dP_{g\theta_1}}{dP_{\theta_0}}(gx) = \frac{dP_{\theta_1}}{dP_{\theta_0}}(x) \quad \text{\(P_{\theta_0}\) - a.e.} \ x.
\]

(iii) We have \(\chi^2(P_{g\theta_1}, P_{g\theta_0}) = \chi^2(P_{\theta_1}, P_{\theta_0})\).

Claim (i) follows from Definition 1 and standard measure-theoretic arguments, (ii) is a consequence of (i), and (iii) is a consequence of (ii).

Given an action \(G\) on a measurable space \((X, \mathcal{F})\), there is an induced action on the set of all probability measures on \((X, \mathcal{F})\) defined by
\[
g P(A) = P(g^{-1}A)
\]
for every \(g \in G\), \(P\) probability measure on \((X, \mathcal{F})\), and \(A \in \mathcal{F}\). Using this action, a family of probability measures \(P\) on \((X, \mathcal{F})\) is called \(G\)-invariant if for each \(P \in P\), we have \(gP \in P\) for all \(g \in G\). On the other hand, the term equivariant is used when the group actions on the sample space and the parameter class lead to the specific relation given in Definition 1.

A more general definition that also applies to estimators is as follows.

Definition 2. Suppose that \(G\) acts on both \(X\) and \(Y\). Then a function \(\psi : X \to Y\) is called \(G\)-equivariant if \(\psi(gx) = g\psi(x)\) for every \(g \in G, x \in X\).

The following lemma collects some properties on the minimax and Bayes risk of equivariant statistical models. Let \((X, \mathcal{F}, (P_\theta)_{\theta \in \Theta})\) be a statistical model and \(\psi : \Theta \to \mathbb{R}^m\) be a derived parameter. For a loss function \(L : \mathbb{R}^m \times \mathbb{R}^m \to [0, \infty)\) and an estimator \(\hat{\psi} : X \to \mathbb{R}^m\), the risk function is given by
\[
E_\theta L(\hat{\psi}(X), \psi(\theta)) = \int L(\hat{\psi}(x), \psi(\theta)) \, dP_\theta(x),
\]
where \(X\) is an observation from the model with distribution \(P_\theta\).

Lemma 6. Suppose that \((P_\theta)_{\theta \in \Theta}\) and \(\psi\) are \(G\)-equivariant and that the loss function \(L\) is convex in the first argument and satisfies \(L(ga, gb) = L(a, b)\) for every \(g \in G, a, b \in \mathbb{R}^m\). Then, for any \(G\)-equivariant estimator \(\hat{\psi}\),
\[
E_\theta L(\hat{\psi}(X), \psi(\theta)) = E_{g\theta} L(\hat{\psi}(X), \psi(g\theta)) \quad \forall \theta \in \Theta, g \in G.
\]
Suppose additionally that $G$ is a compact group with Haar measure $\mu$. Then, for any estimator $\hat{\psi}$, the estimator $\hat{\psi}$ defined by $\hat{\psi}(x) = \int_G g^{-1} \tilde{\psi}(gx) \mu(\text{d}g)$, $x \in \mathcal{X}$ (provided that it exists) is $G$-equivariant with

$$
\mathbb{E}_\theta L(\hat{\psi}(X), \psi(\theta)) \leq \int_G \mathbb{E}_{\text{g}^g} L(\tilde{\psi}(X), \psi(\text{g}^g)) \mu(\text{d}g) \quad \forall \theta \in \Theta.
$$

Let us conclude this section by showing how the two statistical models discussed in Section 2.2 can be realized as equivariant models.

**Example 1.** For $\mathcal{X} = (\mathbb{R}^p)^n$ equipped with its Borel $\sigma$-algebra $\mathcal{B}_{\mathbb{R}^p}^n$, consider the statistical model given in (1.5)

$$
\{ P_U : U \in SO(p) \}, \quad P_U = \mathcal{N}(0, U \Lambda U^T)_{\otimes n}
$$

with $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p)$ and $\lambda_1 \geq \cdots \geq \lambda_p > 0$. In this case we have that if $(X_1, \ldots, X_n)$ has distribution $P_U$, then $(VX_1, \ldots, VX_n)$ has distribution $P_{UV}, U, V \in SO(p)$. Hence, letting $SO(p)$ act coordinate-wise on $(\mathbb{R}^p)^n$, the latter property translates to $P_{UV}(VA) = P_U(A)$ for every $U, V \in SO(p)$ and every $A \in \mathcal{B}_{\mathbb{R}^p}^n$, meaning that the family is indeed $SO(p)$-equivariant.

**Example 2.** For $\mathcal{X} = \mathbb{R}^{p \times p}$ equipped with its Borel $\sigma$-algebra $\mathcal{B}_{\mathbb{R}^{p \times p}}$, consider the family of probability distributions $\{ P_U : U \in SO(p) \}$ with $P_U$ being the distribution of $X = U \Lambda U^T + \sigma W$, where $\sigma > 0$, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p)$ with $\lambda_1 \geq \cdots \geq \lambda_p \geq 0$ and $W$ is drawn from the $p \times p$ GOE (Gaussian orthogonal ensemble). Since the GOE ensemble is invariant under orthogonal conjugation, we have for $X = U \Lambda U^T + \sigma W$ that $VXV^T$ is equal to $VU\Lambda(VU)^T + \sigma W$ in distribution, $U, V \in SO(p)$. Hence, letting $SO(p)$ act on $\mathbb{R}^{p \times p}$ by conjugation, the latter property translates to the fact that the family is $SO(p)$-equivariant.

3 Information inequalities under group action

3.1 An equivariant Chapman-Robbins inequality

The one-dimensional Chapman-Robbins inequality is a simple lower bound for the variance of an unbiased estimator of a real-valued parameter. Letting $(\mathcal{X}, \mathcal{F}, (P_\theta)_{\theta \in \Theta})$ be a statistical model, $\psi : \Theta \to \mathbb{R}^m$ be a derived parameter, $\hat{\psi} : \mathcal{X} \to \mathbb{R}^m$ be an unbiased estimator (i.e. $\mathbb{E}_\theta \hat{\psi}(X) = \psi(\theta)$ for all $\theta \in \Theta$ with $X$ being an observation from the model), a multidimensional version says that

$$
\mathbb{E}_\theta \| \hat{\psi}(X) - \psi(\theta) \|^2 \geq \frac{\left( \sum_{j=1}^m (\psi_j(\theta_j) - \hat{\psi}_j(\theta)) \right)^2}{\sum_{j=1}^m \chi^2(P_{\psi_j}, P_{\theta})} \quad (3.1)
$$
for every \( \theta, \theta_1, \ldots, \theta_m \in \Theta \) and \( \| \cdot \|_2 \) being the Euclidean norm. The proof is simple and based on the identity

\[
\sum_{j=1}^{m} \int_X (\hat{\psi}_j(x) - \psi_j(\theta)) (d\mathbb{P}_{\theta_j}(x) - d\mathbb{P}_\theta(x)) = \sum_{j=1}^{m} (\psi_j(\theta_j) - \psi_j(\theta))
\]

in combination with the Cauchy-Schwarz inequality (applied twice). In the case that \( \Theta \subseteq \mathbb{R}^d \) and under additional regularity conditions (e.g. the differentiability of \( \psi \) at \( \theta \) and the \( L^2 \)-differentiability of the model), (3.1) implies the classical Cramér-Rao lower bound (in the case \( m > 1 \) the quadratic risk corresponds to the trace of the covariance of the estimator, and we get the Cramér-Rao lower bound when the trace is applied to both sides). This can be easily seen by setting \( \theta_j = \theta + tv_j, t \to 0 \), and optimizing in the \( v_j \); see e.g. [45] for the case \( m = 1 \).

If \( \hat{\psi} \) is biased, then the above approach does not work anymore. Based on a variation of (3.2), the following proposition provides a Bayesian version of the Chapman-Robbins inequality for equivariant statistical models.

**Proposition 1.** Let \((X, \mathcal{F}, (\mathbb{P}_\theta)_{\theta \in \Theta})\) be a statistical model. Let \( G \) be a topological group acting on \((X, \mathcal{F})\) and \( \Theta \) such that \((\mathbb{P}_\theta)_{\theta \in \Theta}\) is \( G \)-equivariant. Suppose that \( g \mapsto \mathbb{P}_{g\theta}(A) \) is measurable for every \( A \in \mathcal{F} \), \( \theta \in \Theta \). Let \( \Pi \) be a Borel probability measure on \( G \). Let \( \hat{\psi} : \Theta \to \mathbb{R}^m \) be a derived parameter, \( \hat{\psi} : X \to \mathbb{R}^m \) be an estimator and \( X \) be an observation from the model. Then, for each \( \theta \in \Theta \) and each \( h_1, \ldots, h_m \in G \), we have

\[
\int_G \mathbb{E}_{g\theta} \| \hat{\psi}(X) - \psi(g\theta) \|_2^2 \, d\Pi(g) \geq \frac{\left( \sum_{j=1}^{m} \int_G (\psi_j(gh_j^{-1}\theta) - \psi_j(g\theta)) \, d\Pi(g) \right)^2}{\sum_{j=1}^{m} \chi^2(\mathbb{P}_{h_j\theta}, \mathbb{P}_\theta) + \chi^2(\Pi \circ R_{h_j}, \Pi) + \chi^2(\mathbb{P}_{h_j\theta}, \mathbb{P}_\theta)\chi^2(\Pi \circ R_{h_j}, \Pi)},
\]

with \( \Pi \circ R_{h_j} \) defined by \( \Pi \circ R_{h_j}(B) = \Pi(Bh_j) \) for every Borel set \( B \) in \( G \).

**Remark 4.** If \( G \) is locally compact, then it is natural to choose a probability density function \( \pi \) with respect to the right Haar measure \( \mu \) on \( G \). Then we have \( \Pi \circ R_h(A) = \int_A \pi(g) \mu(dg) = \int_A \pi(gh) \mu(dg) \). Note that the choice \( \pi \equiv 1 \) leads to the trivial lower bound zero, meaning that in applications we have to construct non-uniform prior densities.

**Proof of Proposition 1.** We may assume that \( \mathbb{P}_{h_j\theta} \ll \mathbb{P}_\theta \) and \( \Pi \circ R_{h_j} \ll \Pi \) for every \( j = 1, \ldots, m \) since the claim is trivial otherwise. Consider the
expression
\[
\sum_{j=1}^{m} \int_{G} \int_{X} (\hat{\psi}_{j}(x) - \psi_{j}(g\theta))(d\mathbb{P}_{g\theta}(x)d\Pi(g) - d\mathbb{P}_{gh_{j}\theta}(x)d\Pi \circ R_{h_{j}}(g)) \quad (3.3)
\]
Applying the transformation formula ([14, Theorem 4.1.11]), the terms involving \( \hat{\psi}_{j} \) cancel each other, and (3.3) is equal to
\[
-\sum_{j=1}^{m} \int_{G} \psi_{j}(g\theta)(d\Pi(g) - d\Pi \circ R_{h_{j}}(g)) = \sum_{j=1}^{m} \int_{G} (\hat{\psi}_{j}(gh_{j}^{-1}\theta) - \psi_{j}(g\theta))d\Pi(g).
\]
On the other hand, by the Cauchy-Schwarz inequality, the absolute value of (3.3) is bounded by
\[
\sum_{j=1}^{m} \left( \int_{G} \int_{X} (\hat{\psi}_{j}(x) - \psi_{j}(g\theta))^{2}d\mathbb{P}_{g\theta}(x)d\Pi(g) \right)^{1/2}
\cdot \left( \int_{G} \int_{X} \left( 1 - \frac{d\mathbb{P}_{gh_{j}\theta}}{d\mathbb{P}_{g\theta}}(x) \frac{d\Pi \circ R_{h_{j}}(g)}{d\Pi}(g) \right)^{2}d\mathbb{P}_{g\theta}(x)d\Pi(g) \right)^{1/2}.
\]
Since \((\mathbb{P}_{\theta})_{\theta \in \Theta}\) is \(G\)-equivariant, Lemma 5 (ii) yields that the last term is equal to
\[
\sum_{j=1}^{m} \left( \int_{G} \int_{X} (\hat{\psi}_{j}(x) - \psi_{j}(g\theta))^{2}d\mathbb{P}_{g\theta}(x)d\Pi(g) \right)^{1/2}
\cdot \left( \int_{G} \int_{X} \left( 1 - \frac{d\mathbb{P}_{h_{j}\theta}}{d\mathbb{P}_{g\theta}}(x) \frac{d\Pi \circ R_{h_{j}}(g)}{d\Pi}(g) \right)^{2}d\mathbb{P}_{g\theta}(x)d\Pi(g) \right)^{1/2}.
\]
The term in the second brackets is a \(\chi^{2}\) divergence of two product measures. Thus, by (2.2), the last term is equal to
\[
\sum_{j=1}^{m} \left( \int_{G} \int_{X} (\hat{\psi}_{j}(x) - \psi_{j}(g\theta))^{2}d\mathbb{P}_{g\theta}(x)d\Pi(g) \right)^{1/2}
\cdot \left( (\chi^{2}(\mathbb{P}_{h_{j}\theta}, \mathbb{P}_{\theta}) + 1)(\chi^{2}(\Pi \circ R_{h_{j}}, \Pi) + 1) - 1 \right)^{1/2}.
\]
Applying the Cauchy-Schwarz inequality in \(\mathbb{R}^{m}\), this is bounded by
\[
\left( \int_{G} \mathbb{E}_{g\theta} ||\hat{\psi}(X) - \psi(g\theta)||_{2}^{2}d\Pi(g) \right)^{1/2}
\cdot \left( \sum_{j=1}^{m} ((\chi^{2}(\mathbb{P}_{h_{j}\theta}, \mathbb{P}_{\theta}) + 1)(\chi^{2}(\Pi \circ R_{h_{j}}, \Pi) + 1) - 1) \right)^{1/2},
\]
and the claim follows. \(\square\)
3.2 An equivariant van Trees inequality

Under additional regularity conditions, Proposition 1 implies a Bayesian version of the Cramér-Rao inequality (similarly as the classical Chapman-Robbins inequality implies a Cramér-Rao inequality). Yet, since Proposition 1 involves integrals with respect to the prior distribution, this requires arguments on the differentiation of integrals. Such justifications are more simple in the case of compact groups and in this section, we illustrate this for the special case where both \( \Theta \) and \( G \) coincide with the special orthogonal group \( SO(p) \). More precisely, we consider a statistical model \((X, \mathcal{F}, (P_U)_{U \in SO(p)})\) satisfying the following two assumptions.

**Assumption 1.** The family \( \{P_U: U \in SO(p)\} \) is \( SO(p) \)-equivariant in the sense of Definition 1, that is \( SO(p) \) acts on \( X \) such that \( P_{VU}(VA) = P_U(A) \) for all \( U, V \in SO(p) \) and all \( A \in \mathcal{F} \).

Assumption 1 implies that the family \( \{P_g: g \in G\} \) is \( G \)-equivariant for any closed subgroup \( G \) of \( SO(p) \), and our goal is to apply Proposition 1 combined with a limiting argument in this scenario.

**Assumption 2.** There is a bilinear form \( I_p: so(p) \times so(p) \to \mathbb{R} \) such that for all \( \xi \in so(p) \),

\[
\chi^2(P_{\exp(t\xi)} P_I_p) / t^2 \to I_p(\xi, \xi) \quad \text{as} \quad t \to 0. \tag{3.4}
\]

In Sections 2.2 and 2.3 we have seen that both assumptions are satisfied in the context of principal component analysis and in a matrix denoising model. In particular, the following theorem includes the problem of estimating general derived parameters of the eigenspaces of a covariance matrix.

**Proposition 2.** Suppose that Assumptions 1 and 2 are satisfied. Let \( G \) be a closed subgroup of \( SO(p) \) with Lie algebra \( g \). Let \( \psi: G \to \mathbb{R}^m \) be a continuous function, \( \pi: G \to [0, \infty) \) be a continuous probability density function with respect to the Haar measure \( dg \) on \( G \) and \( \xi_1, \ldots, \xi_m \in g \). Suppose that \( \psi \) and \( \pi \) are differentiable with bounded derivatives in the sense that

\[
\forall g \in G, \forall j \in \{1, \ldots, m\}, \quad |d\psi_j(g) g\xi_j|, |d\pi(g) g\xi_j| \leq M
\]

for some constant \( M > 0 \). Then, for any estimator \( \hat{\psi} = \hat{\psi}(X_1, \ldots, X_n) \) taking values in \( \mathbb{R}^m \), we have

\[
\int_G \mathbb{E}_g \|\hat{\psi} - \psi(g)\|^2_2 \pi(g) dg \geq \frac{\left( \int_G \sum_{j=1}^m d\psi_j(g) g\xi_j \pi(g) dg \right)^2}{\sum_{j=1}^m \left( I_p(\xi_j, \xi_j) + \int_G \frac{(d\pi(g) g\xi_j)^2}{\pi(g)} dg \right)^2}.
\]
where $\mathcal{I}_p(\cdot, \cdot)$ is the Fisher information form given in Lemma 2 and $\| \cdot \|_2$ denotes the Euclidean norm in $\mathbb{R}^m$.

Remark 5. Here, $d\psi_j(g)g\xi_j$ and $d\pi(g)g\xi_j$ denotes the derivative of $\psi_j$ and $\pi$ at the point $g \in G$ in the direction $g\xi_j$ (cf. Section 2.2).

Remark 6. The resulting information inequality involves only the Fisher information at $I_p$. This is due to the equivariance of the statistical model in (1.5), leading to constant Fisher information form.

Remark 7. Proposition 2 provides an extension of the van Trees inequality in the context of equivariant statistical models. The van Trees inequality is a well-known lower bound technique that has been applied in a variety of problems; see e.g. Tsybakov [49] and the references therein for the classical one-dimensional inequality, Gill and Levit [18] for multidimensional extensions, and Jupp [29] for the case of smooth loss functions on manifolds.

Proof of Proposition 2. Proposition 2 follows from Proposition 1 applied with $h_j = \exp(t\xi_j)$, $t \to 0$, by standard measure-theoretic arguments on the differentiation of integrals where the integrand depends on a real parameter (cf. [3, Corollary 5.9]).

Without loss of generality we may restrict ourselves to estimators with bounded Hilbert-Schmidt norm $\sup_{x \in \mathcal{X}} \| \hat{\psi}(x) \|_{\text{HS}} < \infty$, such that the risk of the estimator is bounded. We first assume that $\pi(g) > 0$ for all $g \in G$, which implies that that $\min_{g \in G} \pi(g) > 0$ by a compactness argument, using that $G$ is compact and $\pi$ is continuous. By Assumption 2, we have, for every $j = 1, \ldots, m$,

$$
\chi^2(\mathbb{P}_{\exp(t\xi_j)}, \mathbb{P}_{I_p})/t^2 \to \mathcal{I}_p(\xi_j, \xi_j) \quad \text{as} \quad t \to 0.
$$

Next, consider the term

$$
\chi^2(\Pi \circ R_{\exp(t\xi_j)}, \Pi) / t^2 = \int_G \frac{(\pi(g \exp(t\xi_j)) - \pi(g))^2}{\pi(g)t^2} dg.
$$

By assumption, the function $s \mapsto \pi(g \exp(s\xi_j))$ is differentiable with derivative $s \mapsto d\pi(g \exp(s\xi_j))g \exp(s\xi_j)\xi_j$. By the boundedness assumption this derivative is bounded in $g \in G$ and $s \in \mathbb{R}$. Thus, we get that the difference quotient

$$
\frac{\pi(g \exp(t\xi_j)) - \pi(g)}{t}
$$

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is bounded in $g \in G$ and $t \in \mathbb{R}$. Moreover, it converges to $d\pi(g)g\xi_j$ as $t \to 0$.

Thus the dominated convergence theorem (noting that $\pi$ is bounded away from zero) implies that for every $j = 1, \ldots, m$,

$$
\int_G \frac{(\pi(g \exp(t\xi_j)) - \pi(g))^2}{\pi(g) t^2} \, dg \to \int_G \frac{(d\pi(g)g\xi_j)^2}{\pi(g)} \, dg \quad \text{as} \quad t \to 0.
$$

Similarly, using the boundedness assumption on the derivatives $\psi_j$ this time, we get, for every $j = 1, \ldots, m$,

$$
\int_G \frac{\psi_j(g \exp(-t\xi_j)) - \psi_j(g)}{t} \pi(g) \, dg \to -\int_G d\psi_j(g)g\xi_j \pi(g) \, dg \quad \text{as} \quad t \to 0.
$$

Finally, the third term in the denominator in Proposition 1 divided by $t^2$ vanishes as $t \to 0$. Hence, for positive $\pi$, the claim follows from applying Proposition 1 with $h_j = \exp(t\xi_j)$ and letting $t \to 0$, recalling that $\{\mathbb{P}_g : g \in G\}$ is indeed $G$-equivariant.

It remains to consider the case that $\pi$ is not necessarily positive. Then we can consider $\pi_\epsilon = (\pi + \epsilon)/(1 + \epsilon)$, for which the lower bound in Proposition 2 holds by what we have shown so far. Letting $\epsilon$ go to zero, the Bayes risk of the prior $\pi_\epsilon$ converges to the Bayes risk of the prior $\pi$, and by the monotone convergence theorem, we have

$$
\int_G \frac{(d\pi_\epsilon(g)g\xi_j)^2}{\pi_\epsilon(g)} \, dg = \frac{1}{1 + \epsilon} \int_G \frac{(d\pi(g)g\xi_j)^2}{\pi(g) + \epsilon} \, dg \to \int_G \frac{(d\pi(g)g\xi_j)^2}{\pi(g)} \, dg
$$

as $\epsilon \downarrow 0$. The numerator is treated similarly.

If $\pi$ is radially symmetric around $I_p$, i.e. does only depend on the Hilbert-Schmidt distance $\|I_p - g\|^2_{HS} = 2(p - \text{tr}(g))$, then can write $\pi(g) = \tilde{\pi}(\text{tr}(g))$ for some function $\tilde{\pi} : [-q, q] \to \mathbb{R}_{\geq 0}$.

**Proposition 3.** Suppose that Assumptions 1 and 2 are satisfied. Let $G$ be a closed subgroup of $SO(p)$ with Lie algebra $\mathfrak{g}$. Let $\psi : G \to \mathbb{R}^m$ be a continuous function and $\pi : G \to [0, \infty)$ be a continuous probability density function with respect to the Haar measure $dg$ on $G$ of the form $\pi(g) = \tilde{\pi}(\text{tr}(g))$ with $\tilde{\pi} : [-p, p] \to [0, \infty)$ continuously differentiable, and let $\xi_1, \ldots, \xi_m \in \mathfrak{g}$. Suppose that for some $M > 0$, $\|d\psi(g)g\xi_j\|_2 \leq M$ for every $g \in G$. Then, for any estimator $\hat{\psi} = \hat{\psi}(X_1, \ldots, X_n)$ taking values in $\mathbb{R}^m$, we have

$$
\int_G \mathbb{E}_g \|\hat{\psi} - \psi(g)\|^2 \tilde{\pi}(\text{tr}(g)) \, dg \geq \frac{\left( \int \sum_{j=1}^m d\psi_j(g)g\xi_j \tilde{\pi}(\text{tr}(g)) \, dg \right)^2}{\sum_{j=1}^m \left( \mathcal{I}_p(\xi_j, \xi_j) + \int_G \frac{\tilde{\pi}(\text{tr}(g)) \text{tr}(g\xi_j)^2}{\tilde{\pi}(\text{tr}(g))} \, dg \right)}.
$$

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Remark 8. The condition that $\tilde{\pi}$ is continuously differentiable can be weakened to $\tilde{\pi}$ absolutely continuous with bounded derivative $\tilde{\pi}'$.

Proof of Proposition 3. We first check that $\pi$ from Theorem 3 satisfies the condition of Proposition 2. By definition, for $g \in G$ and $j = 1, \ldots, m$, we have $d\pi(g)g\xi_j = f'(0)$ with $f(t) = \tilde{\pi}(\text{tr}(g\exp(t\xi_j)))$. Hence,

$$d\pi(g)g\xi_j = \text{tr}(g\xi_j)\tilde{\pi}'(\text{tr}(g)).$$

From the assumptions it follows that $\tilde{\pi}'$ is bounded (say, by $C > 0$). Using this and the Cauchy-Schwarz inequality, we conclude that

$$\sup_{g \in G} |d\pi(g)g\xi_j| \leq C \sqrt{\|\xi_j\|_{\text{HS}}}$$

Hence the assumptions of Proposition 2 are satisfied and the claim follows from Proposition 3 and the identities $d\pi(g)g\xi_j = \text{tr}(g\xi_j)\tilde{\pi}'(\text{tr}(g))$, $j = 1, \ldots, m$. \hfill $\square$

We conclude this section by providing a matrix representation of our lower bound. To achieve this, suppose that $\pi$ has Fisher information form

$$I_\pi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, (\xi, \eta) \mapsto \int_G d\pi(g)g\xi g\eta \pi(g) dg$$

satisfying the assumptions of Proposition 2. For $d = \dim \mathfrak{g}$ and a basis $L_1, \ldots, L_d$ of $\mathfrak{g}$, we consider the following matrix representations

$$M_I = (I_{\pi}(L_k, L_l))_{1 \leq k, l \leq d}, \quad M_{I*} = (I_{\pi}(L_k, L_l))_{1 \leq k, l \leq d},$$

and

$$M_{d\psi, \pi} = \int_G (d\psi(g)gL_1, \ldots, d\psi(g)gL_d) \pi(g) dg \in \mathbb{R}^{m \times d}.$$ 

Then, applying Proposition 2 with the choices $\xi_j = \sum_{k=1}^d x_{jk}L_k \ x_j \in \mathbb{R}^d$, $1 \leq j \leq m$, the right-hand side of the lower bound can be written as

$$\left( \sum_{j=1}^m e_j^T M_{d\psi, \pi} x_j \right)^2 \over \sum_{j=1}^m x_j^T (M_I + M_{I*}) x_j,$$

where $e_1, \ldots, e_m$ denotes the standard basis in $\mathbb{R}^m$. Optimizing in the $x_j$ leads to the choices $x_j = (M_I + M_{I*})^{-1} M_{d\psi, \pi} e_j$ and to the lower bound

$$\int_G \mathbb{E}_g \|\hat{\psi} - \psi(g)\|_2^2 \pi(g) dg \geq \text{tr} \left( M_{d\psi, \pi} (M_I + M_{I*})^{-1} M_{d\psi, \pi}^T \right).$$

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This provides a representation of our equivariant van Trees inequality that is closer to the classical Cramér-Rao inequality briefly discussed before Proposition 1. In Section 4.1, we will use the above optimization strategy in the case of estimating the eigenspaces using the (diagonalizing) basis $L^{(ij)}$ (cf. Lemma 2). The main remaining difficulty will be to construct optimal prior densities.

3.3 Two simple applications

In this section, we provide two simple applications of the Chapman-Robbins inequality and the van Trees inequality for equivariant models.

3.3.1 Linear functional of principal components

Before we turn to the proof of Theorem 2 and thus to high-dimensional derived parameters, let us illustrate Proposition 2 in the simple case of functionals (i.e. one-dimensional derived parameters). This case can also be treated with a classical one-dimensional van Trees inequality, mainly due to the existence of simple one-parameter subgroups of the form $G = \{\exp(t\xi) : t \in \mathbb{R}\}$, $\xi \in \mathfrak{so}(p)$. The full strength of our approach becomes apparent in the next section by considering high-dimensional eigenprojections.

For simplicity let us consider the problem of estimating linear functionals of the principal components (cf. [32]), the general case can be treated similarly. More precisely, for $i \leq p$ and a fixed and known $\alpha \in \mathbb{R}^p$, the parameter of interest is $\psi(U) = \langle u_i, \alpha \rangle$, $U \in SO(p)$, with $u_i = Ue_i$ being the $i$-th column of $U$. To obtain non-trivial bounds we assume that $\min(\lambda_i - 1 - \lambda_i, \lambda_i - \lambda_{i+1}) > 0$. Let us also mention that eigenvectors can only be estimated up to a sign in general. In order to take this into account, we will consider parameter classes for which the sign of $u_i$ is uniquely determined.

For a fixed $V \in SO(p)$, a subgroup of the form $G = \{\exp(t\xi) : t \in \mathbb{R}\}$ with $\xi \in \mathfrak{so}(p)$ and a probability density function $\pi$ on $G$ with respect to the Haar measure $dg$ on $G$, we consider the Bayes risk

$$R_{V;\xi,\pi}(\alpha) = \inf_\psi \int_G E_{Vg}(\psi - \langle Vge_i, \alpha \rangle)^2 \pi(g) dg.$$ 

Since $R_{V;\xi,\pi}(\alpha) = R_{I_p;\xi,\pi}(V^T\alpha)$, let us focus on the case $V = I_p$. Moreover, since we consider the $i$-th eigenvector it turns out to be sufficient to consider

$$\xi = \sum_{j \neq i} x_j L^{(ij)}, \quad \sum_{j \neq i} x_j^2 = 1.$$
The normalization ensures that $\exp((t + 2\pi)\xi) = \exp(t\xi)$. In particular, in this parametrization, the Haar measure on $G$ is given by $1_{[-\pi,\pi]}(t)dt/(2\pi)$. Moreover, it is shown in Appendix A that

\[
\exp(t\xi)_{ii} = \cos(t), \quad \text{while} \quad \exp(t\xi)_{ji} = -x_j \sin(t) \quad \forall j \neq i, \quad (3.5)
\]

meaning that $\langle \exp(t\xi), \alpha \rangle = \alpha_i \cos(t) - \sum_{j \neq i} \alpha_j x_j \sin(t)$. In addition, for $k \geq 1$, we choose

\[
\pi_k(\exp(t\xi)) = 4k1_{[-\frac{\pi}{2k}, \frac{\pi}{2k}]}(t) \cos^2(kt).
\]

Applying Proposition 2 with these choices, using also that the directional derivatives coincide with the usual derivative with respect to $t$, we get

\[
R_{1,p,\xi,\pi_k}(\alpha) = \inf_{\psi} \int_{\frac{-\pi}{2k}}^{\frac{\pi}{2k}} \mathbb{E}_{\exp(t\xi)}(\psi - \langle \exp(t\xi), \alpha \rangle)^2 4k \cos^2(kt) \frac{dt}{2\pi} 
\geq \left( \sum_{j \neq i} \alpha_j x_j \int_{\frac{-\pi}{2k}}^{\frac{\pi}{2k}} 4k \cos^2(kt) \cos(t) \frac{dt}{2\pi} \right)^2 Σ_{j \neq i} \alpha_j x_j^2 \left( \frac{n(\lambda_i - \lambda_j)^2}{\lambda_i \lambda_j} + \frac{\pi^2}{4c^2} \right).
\]

Optimizing in the $x_j$ we conclude that

\[
R_{1,p,\xi,\pi_k}(\alpha) \geq \cos^2(\pi/(2k)) \sum_{j \neq i} \alpha_j^2 \left( \frac{n(\lambda_i - \lambda_j)^2}{\lambda_i \lambda_j} + \frac{\pi^2}{4c^2} \right)^{-1}.
\]

In particular, if we choose $k_n = (\pi/2)(\sqrt{n}/c)$ with $c > 0$ and let $n \to \infty$, then we get

\[
\liminf_{n \to \infty} n \cdot R_{1,p,\xi,\pi_{kn}}(\alpha) \geq \sum_{j \neq i} \alpha_j^2 \left( \frac{(\lambda_i - \lambda_j)^2}{\lambda_i \lambda_j} + \frac{\pi^2}{4c^2} \right)^{-1}. \quad (3.6)
\]

We thus obtain a slightly more precise version of a similar result derived in [32]. One advantage of our approach is that we directly perturb the eigenspaces, leading to lower bounds that are already quite precise for finite samples (in contrast, [32, 31] consider additive perturbations of the form $\Sigma + tH$, $t \in (-\delta_n, \delta_n)$). Clearly, (3.6) also gives a lower bound for the minimax risk over the support of $\pi_k$. Let us show that (3.6) implies a local asymptotic minimax theorem (cf. [50, Theorem 8.11]). Setting

\[
\Theta_\delta = \Theta_\delta(\Lambda, \xi) = \{ \Sigma = \exp(t\xi)\Lambda \exp(-t\xi) : t \in (-\delta, \delta) \},
\]

relation. The set $\Theta_\delta$ then has density $1$ in $G$.
it follows from (3.6), using the fact that the prior \( \pi_{kn} \) has support \([-c/\sqrt{n}, c/\sqrt{n}]\),

\[
\lim_{c \to \infty} \lim_{n \to \infty} \inf_{\psi} \sup_{\Sigma \in \Theta} n \mathbb{E}_\Sigma (\hat{\psi} - \langle u_i(\Sigma), \alpha \rangle)^2 \geq \sum_{j \neq i} \alpha_j^2 \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2},
\]

where we translated (3.6) to the notation of Section 1.2. For a matching upper bound, see [32].

### 3.3.2 Nonparametric density estimation

Our main focus is on statistical models with group valued parameters. Yet, it is also possible to apply the equivariant Chapman-Robbins inequality from Proposition 1 to statistical models not directly related to groups. In this section, we illustrate this in the simple case of density estimation. For simplicity, we only explain how Proposition 1 implies the standard nonparametric \( n^{-\beta/(2\beta+1)} \)-rate for the pointwise risk, but similar considerations (mentioned briefly at the end of this section) also yield corresponding result for the \( L^2 \) risk. For \( g \in \{\pm 1\} \), we consider

\[
f_g(x) = \frac{1}{2} + c_0 h^\beta \{ K\left(\frac{x-1/2}{h}\right) - K\left(\frac{x+1/2}{h}\right) \}, \quad x \in [-1,1],
\]

where \( h \in (0,1] \), \( \beta > 0 \), \( K \) is a symmetric and continuous probability density function with respect to the Lebesgue measure with support in \([-1/2,1/2]\) and \( K(0) > 0 \), and \( c_0 > 0 \) is a sufficiently small constant such that \( f_g(x) \geq 1/4 \) for all \( x \in [-1,1] \), \( h \in (0,1] \) and \( g \in \{\pm 1\} \). This provides a slight variation of a standard lower bound construction; see e.g. [50, 49]. By construction, \( f_1 \) and \( f_{-1} \) are probability densities on \([-1,1]\). Moreover, it is possible to choose the kernel such that \( f_1 \) and \( f_{-1} \) are contained in the Hölder class on \([-1,1]\) of smoothness \( \beta \) for all \( h \in (0,1]\); see [49, Chapter 2.5] and [50, Chapter 24].

We now consider the statistical model \(([[-1,1]^n, \mathcal{B}_{[-1,1]}^n, (\mathbb{P}_g)_{g \in \{\pm 1\}}])\) with \( \mathbb{P}_g \) being the probability measure associated with a sample of \( n \) independent random variables \( X_1, \ldots, X_n \) when the density of \( X_i \) is \( f_g \). By construction, we have \( f_g(-x) = f_{-g}(x) \) for all \( x \in [-1,1] \) and \( g \in \{\pm 1\} \), where we used the assumption that \( K(x) = K(-x) \) for all \( x \in [-1,1] \). Hence, letting the group \( G = \{\pm 1\} \) act on \([-1,1]^n \) coordinate-wise by multiplication (sign change), the transformation formula implies that for the random variable \( X_i \) with density \( f_g \), the random variable \(-X_i\) has density \( f_{-g} \), meaning that this statistical model is indeed \( G \)-equivariant. In order to apply Proposition 1, we choose \( h_1 = -1 \) and the prior \( \Pi \) given by \( \Pi(1) = 1 - \Pi(-1) = q \in (0,1) \).
With these choices, we have
\[ \chi^2(\Pi \circ R_{-1}, \Pi) = \frac{(1 - 2q)^2}{q(1 - q)}, \quad \chi^2(\mathbb{P}_{-1}, \mathbb{P}_1) \leq e^{25c_2^2h^{2\beta+1}n\|K\|^2_{L^2}}, \tag{3.7} \]
see Appendix A for the standard calculations. Now, considering the derived parameter \( \psi(g) = f_g(1/2), \ g \in \{\pm 1\} \), we have
\[ \sum_{g \in \{\pm 1\}} (\psi(-g) - \psi(g))\Pi(g) = \sum_{g \in \{\pm 1\}} (f_{-g}(1/2) - f_g(1/2))\Pi(g) = -(q - (1 - q))2h^\beta c_0 K(0). \]
Choosing e.g. \( q = 3/4 \) (that is a non-uniform prior) and \( h = c_1 n^{-1/(2\beta+1)} \), we obtain that
\[ \inf_{\hat{f}} \left\{ \frac{1}{2} \mathbb{E}_1(\hat{f} - f_1(1/2))^2 + \frac{1}{2} \mathbb{E}_{-1}(\hat{f} - f_{-1}(1/2))^2 \right\} \geq c_2 n^{-\frac{2\beta}{2\beta+1}}, \]
where \( c_1, c_2 \) are two absolute constants and where the infimum is taken over all estimators \( \hat{f} = \hat{f}(X_1, \ldots, X_n) \) with values in \( \mathbb{R} \).
Finally, in order to deal the estimation of more than one point or the \( L^2 \) risk, one can consider for \( g \in G = \{\pm 1\}^m \),
\[ f_g(x) = \frac{1}{2} + c_0 h^\beta \sum_{k=1}^m g_k \left\{ K\left( \frac{x - x_k}{h} \right) - K\left( \frac{x + x_k}{h} \right) \right\}, \quad x \in [-1,1] \]
with \( x_k = (2k - 1)/m, \ k = 1, \ldots, m, \) and \( h \in (0,1/m] \). The key observation is that the resulting family of probability measures \( \{\mathbb{P}_g : g \in G\} \) can again be realized as an equivariant statistical model. For this, let \( G \) act on \( [-k/m, -(k - 1)/m] \cup ((k - 1)/m, k/m] \) by multiplication with \( g_k, \ k = 2, \ldots, m, \) and on \( [-1/m, 1/m] \) by multiplication with \( g_1 \).

4 Lower bounds for the estimation of eigenspaces
The purpose of this section is to apply Proposition 3 in the case of PCA and the matrix denoising model.

4.1 Invoking the van Trees-type inequality
We first specialize Proposition 3 to the case of eigenspaces, that is to the case where
\[ \psi(U) = P_I(U) = \sum_{i \in I} (U e_i)(U e_i)^T, \quad U \in SO(p). \]
where $e_1, \ldots, e_p$ denotes the standard basis in $\mathbb{R}^p$ and $\mathcal{I} \subseteq \{1, \ldots, p\}$. The following corollary applies Proposition 3 in the above setting and the choice $G = SO(p)$, and proposes a density for which the Fisher information of $\pi$ becomes tractable.

**Corollary 4.** Suppose that Assumptions 1 and 2 are satisfied. Let $\tilde{\pi} : [-p, p] \to [0, \infty)$ be continuously differentiable with $\int_{SO(p)} \tilde{\pi}(\text{tr}(U)) \, dU = 1$. Then, for any estimator $\hat{P} = \hat{P}(X_1, \ldots, X_n)$ with values in $\mathbb{R}^{p \times p}$, we have

$$\int_{SO(p)} \mathbb{E}_U \|\hat{P} - P_{I^p}(U)\|_{HS}^2 \tilde{\pi}(\text{tr}(U)) \, dU \geq 2 \left( \int_{SO(p)} (U_{11} U_{22} + U_{12} U_{21}) \tilde{\pi}(\text{tr}(U)) \, dU \right)^2 \cdot \sum_{i \in \mathcal{I}} \sum_{j \not\in \mathcal{I}} (I_{I^p}(L^{(ij)}, L^{(ij)}) + \int_{SO(p)} (U_{12} - U_{21})^2 \frac{\tilde{\pi}'(\text{tr}(U))}{\tilde{\pi}(\text{tr}(U))} \, dU)^{-1}.$$

In particular, if $\tilde{\pi}$ is given by $\tilde{\pi}_h(\cdot) = \exp(h \cdot \cdot) / Z_h$ with $h > 0$ and normalizing constant $Z_h = \int_{SO(p)} \exp(h \text{tr}(U)) \, dU$, then we have

$$\int_{SO(p)} \mathbb{E}_U \|\hat{P} - P_{I^p}(U)\|_{HS}^2 \tilde{\pi}_h(\text{tr}(U)) \, dU \geq 2 \left( \int_{SO(p)} (U_{11} U_{22} + U_{12} U_{21}) \tilde{\pi}_h(\text{tr}(U)) \, dU \right)^2 \cdot \sum_{i \in \mathcal{I}} \sum_{j \not\in \mathcal{I}} (I_{I^p}(L^{(ij)}, L^{(ij)}) + 8 h^2 p)^{-1}.$$

Obviously, the two terms involving the prior density are competing: if the prior approximates the Dirac measure at $I^p$, then the prefactor tends to one, but the Fisher information of the prior explodes at the same time. On the other hand, if we choose $\tilde{\pi} \equiv 1$, then the Fisher information of the prior is zero, but the average in the prefactor as well.

At this point the difference between the asymptotic and the nonasymptotic framework becomes apparent. To see this, consider the statistical model in (1.5) in which case we have $I_{I^p}(L^{(ij)}, L^{(ij)}) = n(\lambda_i - \lambda_j)^2 / \lambda_i \lambda_j$. For an asymptotic lower bound it suffices to know that the prior approximates the Dirac measure at $I^p$ (e.g. as $h$ tend to infinity). In contrast, in order to obtain a nonasymptotic lower bound we need a corresponding quantitative statement, which requires some more sophisticated analysis on $SO(p)$; see Section 4.2 for the details.
More precisely, in an asymptotic setting \( n \to \infty \), the second lower bound implies a Bayesian version of the local asymptotic minimax theorem; see e.g. Theorem 8.11 in [50]. Indeed, as long as \( h \to \infty \) such that \( h = o(\sqrt{n}) \), the term related to the Fisher information of the prior asymptotically vanishes, and the asymptotic Bayes risk multiplied with \( n \) is bounded from below by
\[
2 \sum_{i \in \mathcal{I}} \sum_{j \not\in \mathcal{I}} \lambda_i \lambda_j (\lambda_i - \lambda_j)^{-2}.
\]

**Proof of Corollary 4.** We apply Proposition 3 with \( G = SO(p) \). We start by checking that \( P_\mathcal{I} : SO(p) \to \mathbb{R}^{p \times p}, P_\mathcal{I}(U) = \sum_{i \in \mathcal{I}} (Ue_i)(Ue_i)^T \) satisfies the boundedness assumption in Proposition 3. By definition, for \( U \in SO(p) \) and \( \xi \in so(p) \), we have \( dP_\mathcal{I}(U) = f'(0) \) with \( f : \mathbb{R} \to \mathbb{R}^{p \times p}, t \mapsto \sum_{i \in \mathcal{I}} U \exp(t\xi)e_i(U \exp(t\xi)e_i)^T \). Hence,
\[
dP_\mathcal{I}(U)\xi = U \left( \sum_{i \in \mathcal{I}} e_i e_i^T - \sum_{i \in \mathcal{I}} e_i e_i^T \xi \right) U^T \tag{4.1}
\]
and, using that the operator norm of an orthogonal matrix and an orthogonal projection is bounded by 1, we get
\[
\sup_{U \in SO(p)} \|dP_\mathcal{I}(U)\xi\|_{HS} \leq 2\|\xi\|_{HS}.
\]
Hence, the assumptions of Proposition 3 are satisfied for the choices
\[
\xi^{(jk)} = \begin{cases} 
 c_{jk}L^{(jk)}, & j \in \mathcal{I}, k \not\in \mathcal{I}, \\
 c_{kj}L^{(kj)}, & k \in \mathcal{I}, j \not\in \mathcal{I}, \\
 0, & \text{otherwise}.
\end{cases}
\]
with real numbers \( c_{jk} \) to be chosen later. Then it follows from (4.1) that
\[
d(P_\mathcal{I})^{jk}(U)\xi^{(jk)} = -c_{jk}e_j^T U(e_k e_j^T + e_j e_k^T)U^T e_k = -c_{jk}(U_{jj}U_{kk} + U_{jk}U_{kj}),
\]
for \( j \in \mathcal{I}, k \not\in \mathcal{I} \) or \( k \in \mathcal{I}, j \not\in \mathcal{I} \). Moreover, we have \( \text{tr}(U\xi^{(jk)}) = c_{jk}(U_{jk} - U_{kj}) \). Hence, Proposition 3 and the fact that \( \mathcal{I}_p \) is a bilinear form yield
\[
\int_{SO(p)} \mathbb{E}_U \|\hat{P} - P_\mathcal{I}(U)\|_{HS}^2 \hat{\pi}(\text{tr}(U))dU \\
\geq \frac{2 \left( \sum_{j \in \mathcal{I}} \sum_{k \not\in \mathcal{I}} c_{jk} \int_{SO(p)} (U_{jj}U_{kk} + U_{jk}U_{kj})\hat{\pi}(\text{tr}(U))dU \right)^2}{\sum_{j \in \mathcal{I}} \sum_{k \not\in \mathcal{I}} c^2_{jk} (\mathcal{I}_p(L^{(jk)}, L^{(jk)}) + \int_{SO(p)} (U_{jk} - U_{kj})^2 \hat{\pi}(\text{tr}(U))^2 dU)}
\]
29
Similarly from the transformation and leaves the trace fixed, the first claim follows. The other claims follow from optimizing in the $c_{jk}$, leading to the choices

$$c_{jk} = \sum_{i \in I} \sum_{k \notin I} (L_{(ij)}, L_{(jk)}) + \int_{SO(p)} (U_{12} - U_{21})^2 \frac{\left(\frac{\text{tr}(U)}{\pi(\text{tr}(U))}\right)^2}{\pi(\text{tr}(U))} dU,$$

where the equality follows from Lemma 7 below. The lower bound now follows from optimizing in the $c_{jk}$, leading to the choices

$$c_{jk} = \sum_{i \in I} \sum_{k \notin I} (L_{(ij)}, L_{(jk)}) + \int_{SO(p)} (U_{12} - U_{21})^2 \frac{\left(\frac{\text{tr}(U)}{\pi(\text{tr}(U))}\right)^2}{\pi(\text{tr}(U))} dU.$$

In special case that $\tilde{\pi}$ is given by $\tilde{\pi}(s) = \exp(hps)/Z_h$ with $h > 0$ and normalizing constant $Z_h$, we have the identity $(\tilde{\pi}(\text{tr}(U)))^2/\tilde{\pi}(\text{tr}(U)) = (hp)^2 \tilde{\pi}(\text{tr}(U))$. Hence, in this case, it holds that

$$\int_{SO(p)} (U_{12} - U_{21})^2 \frac{\left(\frac{\text{tr}(U)}{\pi(\text{tr}(U))}\right)^2}{\pi(\text{tr}(U))} dU = (hp)^2 \int_{SO(p)} (U_{12} - U_{21})^2 \tilde{\pi}(\text{tr}(U)) dU.$$

Moreover, we have

$$\int_{SO(p)} (U_{12} - U_{21})^2 \tilde{\pi}(\text{tr}(U)) dU \leq 2 \int_{SO(p)} (U_{12}^2 + U_{21}^2) \tilde{\pi}(\text{tr}(U)) dU$$

$$= \frac{2}{p - 1} \int_{SO(p)} \sum_{j=1}^{p} (U_{12}^2 + U_{21}^2) \tilde{\pi}(\text{tr}(U)) dU \leq \frac{4}{p - 1} \leq \frac{8}{p},$$

where the equality follows from Lemma 7 below.

**Lemma 7.** The maps $i \mapsto \int U_{ii} \tilde{\pi}(\text{tr}(U)) dU$, $(i, j) \mapsto \int U_{ij} U_{ji} \tilde{\pi}(\text{tr}(U)) dU$, $(i, j) \mapsto \int U_{ij} U_{ji} \tilde{\pi}(\text{tr}(U)) dU$, and $(i, j) \mapsto \int (U_{ij} - U_{ji})^2 \tilde{\pi}(\text{tr}(U)) dU$ are constant in $i, j = 1, \ldots, p$.

**Proof.** For a permutation $\sigma \in S_q$, let $P_\sigma = (e_{\sigma(1)}, \ldots, e_{\sigma(p)})^T \in O(p)$ be the associated permutation matrix. Then multiplication from the left with $P_\sigma$ permutes the rows by $\sigma$ and multiplication from the right permutes the columns by $\sigma^{-1}$. By the properties of the Haar measure on the orthogonal group applied to the transformation $U \mapsto P_{(1)} U P_{(1)}$, which sends $U_{ii}$ to $U_{11}$ and leaves the trace fixed, the first claim follows. The other claims follow similarly from the transformation $U \mapsto P_{(2)} P_{(1)} U P_{(1)} P_{(2)}$.

**4.2 Designing optimal prior densities**

In this section, we analyze the density function on $SO(p)$ defined by

$$\tilde{\pi}_h(\text{tr}(U)) = \frac{\exp(h p \text{tr}(U))}{\int_{SO(p)} \exp(h p \text{tr}(U)) dU}, \quad U \in SO(p), \quad (4.2)$$
where $h > 0$ is a real parameter. It is clear that the resulting probability measure on $SO(p)$ approximates the Dirac measure at the identity matrix $I_p$ as $h \to \infty$. The following proposition gives a quantitative statement needed to deduce Theorem 2 from the second part of Corollary 4.

**Proposition 4.** For each $\delta \in (0, 1)$, there is a real number $h_\delta > 0$ depending only on $\delta$ such that the density function given in (4.2) satisfies

$$\int_{SO(p)} \tr(U) \tilde{\pi}_h(\tr(U)) \, dU \geq (1 - \delta)p \quad \forall p \geq 2, h \geq h_\delta. \quad (4.3)$$

While Proposition 4 can be deduced from Varadhan’s lemma using that the Jacobi ensemble satisfies a large deviation principle, we present a more direct proof based on Weyl’s integration formula in combination with some elementary large deviations arguments from [23] (see also [22] or [1]). After that we explain how the claim can be alternatively obtained by more general large deviations arguments.

**Proof of Proposition 4.** Let us start with proving a lower bound for the exponential moment $\int_{SO(p)} \exp(h \tr(U)) \, dU$. We first consider the case $p = 2m$ even. Then Weyl’s integration formula (see e.g. Theorem 3.5 in [35]) in combination with the change of variables $y_i = \cos(\theta_i)$ gives

$$\int_{SO(p)} \exp(h \tr(U)) \, dU = \frac{2m^{2-m} + 1}{m!(2\pi)^m} \int_{[-1,1]^m} e^{4hm \sum_{i=1}^m y_i} \prod_{1 \leq i < j \leq m} (y_j - y_i)^2 \prod_{i=1}^m (1 - y_i^2)^{-1/2} \prod_{i=1}^m dy_i.$$ 

Now, let $0 < \delta < 1$ be a real number to be chosen later. Restricting the integral to the set

$$\Delta_m := \left\{ y \in [-1,1]^m : 1 - \frac{\delta i}{m+1} \leq y_i \leq 1 - \delta + \frac{\delta(i + 1/2)}{m+1}, 1 \leq i \leq m \right\},$$

we obtain

$$\int_{SO(p)} \exp(h \tr(U)) \, dU \geq \frac{2m^{2-m} + 1}{m!(2\pi)^m} \int_{\Delta_m} e^{4hm \sum_{i=1}^m y_i} \prod_{1 \leq i < j \leq m} (y_j - y_i)^2 \prod_{i=1}^m dy_i \geq \frac{2m^{2-m} + 1}{m!(2\pi)^m} \left( \frac{\delta/2}{m+1} \right)^m e^{4hm^2(1-\delta/2)} \delta^{m^2 - m} \prod_{1 \leq i < j \leq m} \left( \frac{j - i - 1/2}{m+1} \right)^2. \quad (4.4)$$
Considering the last product term, we have the following bound
\[
\prod_{1 \leq i < j \leq m} \frac{j - i - 1/2}{m + 1} = \prod_{k=1}^{m-1} \left( \frac{k - 1/2}{m + 1} \right)^{m-k} \geq \prod_{k=1}^{m} \left( \frac{k}{m + 1} \right)^{m+1-k} \prod_{k=1}^{m} \left( \frac{k - 1/2}{k} \right)^{m-k}.
\]

On the one hand, using that the function \( t \mapsto (1 - t) \log t \) is monotone increasing in \( t \in (0, 1] \) and equal to zero for \( t = 1 \), we have
\[
\frac{1}{(m + 1)^2} \sum_{k=1}^{m} (m + 1 - k) \log \left( \frac{k}{m + 1} \right) \geq \int_0^1 (1 - t) \log t \, dt = \frac{3}{4},
\]
and thus
\[
\prod_{k=1}^{m} \left( \frac{k}{m + 1} \right)^{m+1-k} \geq e^{-\frac{2(m+1)^2}{4}}.
\]

On the other hand, using the concavity of the function \( t \mapsto \log t \), we have
\[
\sum_{k=1}^{m} (m - k)(\log(k) - \log(k - 1/2)) \leq \sum_{k=1}^{m} \frac{m - k}{2k - 1} \leq \sum_{k=1}^{m} \frac{m}{k} \leq m \log(em),
\]
and thus
\[
\prod_{k=1}^{m} \left( \frac{k - 1/2}{k} \right)^{m-k} \geq e^{-m \log(em)}.
\]

Inserting these lower bounds into (4.4), we get that the leading exponent with respect to \( m \) is \( m^2 \). This allows us to conclude that there is a real number \( h_\delta > 0 \) such that for every \( h \geq h_\delta \) and every \( p \geq 2 \) even,
\[
\int_{SO(p)} \exp(h p \text{ tr}(U)) \, dU \geq \exp(hp^2(1 - \delta)). \quad (4.5)
\]

The same bound also holds for \( p \) odd, in which case Weyl’s integration formula has a slightly different form (see e.g. Theorem 3.5 in [35]). Next, using that the logarithmic moment-generating function
\[
\psi(h) = \log \left( \int_{SO(p)} \exp(h p \text{ tr}(U)) \, dU \right), \quad h \in \mathbb{R},
\]
is convex, we get
\[
\int_{SO(p)} p \text{tr}(U) \tilde{\pi}_h(\text{tr}(U)) dU = \psi'(h) \geq h^{-1} \psi(h) \geq p^2(1 - \delta)
\]
for every \( h \geq h_\delta \) and every \( p \geq 2 \), where we used convexity and \( \psi(0) = 0 \) in the first inequality and (4.5) in the last inequality. Dividing both sides by \( p \), the claim follows. \( \square \)

The following corollary deals with the prefactor from the second lower bound in Corollary 4.

**Corollary 5.** For each \( \delta \in (0, 1) \) there is a real number \( h_\delta > 0 \) such that the density function given in (4.2) satisfies, for every \( p \geq 2 \) and every \( h \geq h_\delta \),
\[
\int_{SO(p)} (U_{11}U_{22} + U_{12}U_{21}) \tilde{\pi}_h(\text{tr}(U)) dU \geq (1 - \delta)^2 - \frac{2(1 - (1 - \delta)^2)}{p - 1}.
\]

**Proof.** Consider \( h_\delta \) from Proposition 4 and let \( h \geq h_\delta \). By the Cauchy-Schwarz inequality and Proposition 4, we have
\[
\int_{SO(p)} (\text{tr}(U))^2 \tilde{\pi}_h(\text{tr}(U)) dU \geq \left( \int_{SO(p)} \text{tr}(U) \tilde{\pi}_h(\text{tr}(U)) dU \right)^2 \geq (1 - \delta)^2 p^2.
\]
On the other hand, by Lemma 7, we have
\[
\int_{SO(p)} (\text{tr}(U))^2 \tilde{\pi}_h(\text{tr}(U)) dU = \sum_{i=1}^p \int_{SO(p)} U_{ii}^2 \tilde{\pi}_h(\text{tr}(U)) dU + \sum_{i \neq j} \int_{SO(p)} U_{ii}U_{jj} \tilde{\pi}_h(\text{tr}(U)) dU
\]
\[
= p \int_{SO(p)} U_{11}^2 \tilde{\pi}_h(\text{tr}(U)) dU + p(p - 1) \int_{SO(p)} U_{11}U_{22} \tilde{\pi}_h(\text{tr}(U)) dU.
\]
Now, \( \int U_{11}^2 \tilde{\pi}_h(\text{tr}(U)) dU \leq 1 \) and thus
\[
\int_{SO(p)} U_{11}U_{22} \tilde{\pi}_h(\text{tr}(U)) dU \geq \frac{(1 - \delta)^2 p - 1}{p - 1} = (1 - \delta)^2 - \frac{1 - (1 - \delta)^2}{p - 1}.
\]
(4.6)
This gives the first part of the integral. For the second part, Lemma 7 gives
\[
\int_{SO(p)} U_{12}U_{21} \tilde{\pi}_h(\text{tr}(U)) dU = \frac{1}{p - 1} \sum_{k=2}^p \int_{SO(p)} U_{1k}U_{k1} \tilde{\pi}_h(\text{tr}(U)) dU.
\]
Hence
\[ \left| \int_{SO(p)} U_{12} U_{21} \tilde{\pi}_h(\text{tr}(U)) \, dU \right| \leq \frac{1}{p-1} \sum_{k=2}^{p} \int_{SO(p)} |U_{1k} U_{k1}| \tilde{\pi}_h(\text{tr}(U)) \, dU \]
\[ \leq \frac{1}{2(p-1)} \sum_{k=2}^{p} \int_{SO(p)} (U_{1k}^2 + U_{k1}^2) \tilde{\pi}_h(\text{tr}(U)) \, dU \]
\[ = \frac{1}{p-1} \int_{SO(p)} (1 - U_{11}^2) \tilde{\pi}_h(\text{tr}(U)) \, dU. \quad (4.7) \]

By the Cauchy-Schwarz inequality, Lemma 7 and Proposition 4, we have
\[ \int_{SO(p)} U_{11}^2 \tilde{\pi}_h(\text{tr}(U)) \, dU \geq \left( \int_{SO(p)} U_{11} \tilde{\pi}_h(\text{tr}(U)) \, dU \right)^2 \geq (1 - \delta)^2, \]
and inserting this into (4.7), we get
\[ \left| \int_{SO(p)} U_{12} U_{21} \tilde{\pi}_h(\text{tr}(U)) \, dU \right| \leq \frac{1 - (1 - \delta)^2}{p-1}. \quad (4.8) \]
The claim now follows from combining (4.6) and (4.8).

We conclude this section by outlining a slightly different proof of Proposition 4. Making the change of variables \( x_j = (1 + y_j)/2 \) at the beginning of the above proof leads to
\[ \int_{SO(p)} \exp \left( h p \text{tr}(U) \right) \, dU = \int_{[0,1]^m} e^{8hm \sum_{i=1}^{m} x_i - 4hm^2} d\nu_m(x_1, \ldots, x_m) \quad (4.9) \]
with probability measure \( \nu_m \) on \([0,1]^m\) defined by
\[ d\nu_m(x_1, \ldots, x_m) = \frac{2^{2m^2-2m+1}}{m!(2\pi)^m} \prod_{i=1}^{m} (x_i(1-x_i))^{-1/2} \prod_{1 \leq i < j \leq m} (x_j - x_i)^2 \prod_{i=1}^{m} dx_i. \]
This measure corresponds to a Jacobi ensemble and we next state a large deviation principle proved in [23]. For this let \( M_1([0,1]) \) be the space of all probability measures on \([0,1]\) endowed with the usual weak topology (metrizable by the bounded Lipschitz metric), and let \( \Sigma : M_1([0,1]) \to [-\infty, -2 \log 2] \) be the noncommutative entropy define by
\[ \Sigma(\mu) = \int_0^1 \int_0^1 \log |x - y| \, d\mu(x) d\mu(y), \quad \mu \in M_1([0,1]). \]
Assume now that the random vector $x^{(m)}$ is distributed according to the probability measure $\nu_m$ on $[0,1]^m$. Then it follows from Proposition 2.1 in [23] (resp. a slight variation of it with $\kappa(N)$ and $\lambda(N)$ allowed to be negative such that $\kappa(N)/N, \lambda(N)/N \to 0$) that the empirical measure

$$\frac{1}{m} \delta_{x_1^{(m)}} + \cdots + \frac{1}{m} \delta_{x_m^{(m)}}$$

satisfies the large deviation principle with speed $m^2$ and (good) rate function $I(\mu) = -\Sigma(\mu) - 2\log 2$, $\mu \in M_1([0,1])$. We now apply Varadhan’s integral lemma to the function $f(\mu) = 8h \int_0^1 y d\mu(y)$. This function is indeed continuous and bounded and it follows from [30, Theorem 27.10(i)] or [13, Theorem 4.3.1] that for every $h > 0$,

$$\lim_{m \to \infty} \frac{1}{m^2} \log \mathbb{E} \exp \left( 8hm \sum_{i=1}^m x_i^{(m)} \right) = \sup_{\mu \in M_1([0,1])} \left( 8h \int_0^1 y d\mu(y) - I(\mu) \right).$$

(4.10)

From this point we can argue as above. Note that the above proof corresponds to choosing $\mu$ as the uniform measure on $[1 - \delta/2, 1]$. Note that it is possible to show that the right-hand side of (4.10) is equal to $4h + 2h^2$ if $2h \leq 1$ and equal to $4h - \log(2h) + 4h - 3/2$ if $2h > 1$. This can be achieved by relating the right-hand side of (4.10) to the rate function in a large deviation result for the Gross-Witten-Wadia model [21, p. 82] and [22, Proposition 5.3.10]. We omit here the details. Translating this back to the left-hand side of (4.9), we get

$$\lim_{p \to \infty} \frac{1}{p^2} \log \left( \int_{SO(p)} \exp \left( hp \text{tr}(U) \right) dU \right) \to \begin{cases} \frac{h^2}{2}, & 2h \leq 1, \\ h - \frac{1}{4} \log(2h) - \frac{3}{8}, & 2h \geq 1. \end{cases}$$

In this paper the main focus is on nonasymptotic lower bounds in which case the exact limit from above has no special meaning. Yet, let us mention that in the asymptotic scenario $n,p \to \infty$ with $p/n \to \gamma \in (0,\infty)$, it allows to obtain more precise constants than in Corollary 5 (although the result is not enough to exactly compute the asymptotic value of the integral in Corollary 5). All this leads to the interesting question to prove an (asymptotic) version of Corollary 1 with constants as sharp as possible.

### 4.3 Application: principal component analysis

*End of proof of Theorem 2.* Consider the statistical model $\{P_U : U \in SO(p)\}$ from (1.5) with $P_U = \mathcal{N}(0, U\Lambda U^T)^{\otimes n}$. By Example 1 and Lemma 1,
we know that Assumptions 1 and 2 are satisfied with $I_p(L^{(ij)}, L^{(ij)}) = n(\lambda_i - \lambda_j)^2 / \lambda_i \lambda_j$. Hence, combining the second claim in Corollary 4 with Corollary 5, we conclude that for every $\delta \in (0, 1)$, there is a constant $C_\delta > 0$ depending only on $\delta$ such that for every $h \geq C_\delta$,

$$\inf_{P} \int_{SO(p)} \mathbb{E}_U \| \hat{P} - P_I(U) \|_{HS}^2 \bar{\pi}_h(\text{tr}(U)) dU \geq 2(1 - \delta) \sum_{i \in I} \sum_{j \notin I} \left( \frac{n(\lambda_i - \lambda_j)^2}{\lambda_i \lambda_j} + 8h^2 p \right)^{-1}.$$ 

Inserting the inequality

$$(x + y)^{-1} \geq (1 - \delta)x^{-1} \wedge \delta y^{-1}, \quad x, y \geq 0,$$ 

the claim follows.

### 4.4 Application: matrix denoising

Consider the statistical model $\{P_U : U \in SO(p)\}$ from (2.4) with $P_U$ being the being the distribution of $X = U \Lambda U^T + \sigma W$, where $\sigma > 0$, $W$ is a GOE matrix, and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p)$ is a fixed diagonal matrix with $\lambda_1 \geq \cdots \geq \lambda_p \geq 0$. By Example 2 and Lemma 3, we know that Assumptions 1 and 2 are satisfied with $I_p(L^{(ij)}, L^{(ij)}) = (\lambda_i - \lambda_j)^2 / \sigma^2$. Hence, combining the second claim in Corollary 4 with Corollary 5, we conclude that for every $\delta \in (0, 1)$, there is a constant $C_\delta > 0$ depending only on $\delta$ such that for every $h \geq C_\delta$,

$$\inf_{P} \int_{SO(p)} \mathbb{E}_U \| \hat{P} - P_I(U) \|_{HS}^2 \bar{\pi}_h(\text{tr}(U)) dU \geq 2(1 - \delta) \sum_{i \in I} \sum_{j \notin I} \left( \frac{\lambda_i - \lambda_j}{\sigma^2} + 8h^2 p \right)^{-1}.$$ 

Hence, applying (4.11), we get that for every $h \geq C_\delta$,

$$\inf_{P} \int_{SO(p)} \mathbb{E}_U \| \hat{P} - P_I(U) \|_{HS}^2 \bar{\pi}_h(\text{tr}(U)) dU \geq 2(1 - \delta)^2 \sum_{i \in I} \sum_{j \notin I} \min \left( \frac{\sigma^2}{(\lambda_i - \lambda_j)^2}, \frac{\delta}{8h^2 p} \right).$$ 

(4.12)
A  Additional proofs

Proof of Corollary 3. By convexity, we have \((j^\alpha - i^\alpha)/(j-i) \leq \alpha j^{\alpha-1}\) for every \(j>i\) and thus

\[
\frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} = \frac{i^\alpha j^\alpha}{(j^\alpha - i^\alpha)^2} \geq \frac{\alpha^{-2} i^\alpha j^{2-\alpha}}{(j-i)^2} \quad \forall j>i.
\]

We now assume for simplicity that \(d\) is even and choose \(J = \{d/2 + 1, d/2 + 2, \ldots, d + d/2\}\) such that \(|J| = d\). Then

\[
\sum_{i \in I \cap J} \sum_{j \in J \setminus I} \left( \frac{\lambda_i \lambda_j}{n(\lambda_i - \lambda_j)^2} \wedge \frac{1}{|J|} \right) = \sum_{d/2 < i \leq \frac{d}{d} \leq \frac{d}{d} \leq \frac{d}{d}} \sum \left( \frac{\lambda_i \lambda_j}{n(\lambda_i - \lambda_j)^2} \wedge \frac{1}{d} \right)
\]

\[
\geq \sum_{d/2 < i \leq \frac{d}{d} \leq \frac{d}{d}} \sum \left( \frac{\alpha^{-2} 3^{-\alpha} d^2}{(j-i)^2} \wedge \frac{1}{d} \right) \geq \sum_{k=1}^{\frac{d}{d}} \left( \frac{\alpha^{-2} 3^{-\alpha} d^2}{nk} \wedge \frac{k}{d} \right),
\]

where we used in the last inequality that for \(k \leq d/2\) the number of indices \((i,j)\) in the double sum satisfying \(j-i = k\) is equal to \(k\). We conclude that

\[
\sum_{i \in I \cap J} \sum_{j \in J \setminus I} \left( \frac{\lambda_i \lambda_j}{n(\lambda_i - \lambda_j)^2} \wedge \frac{1}{|J|} \right) \geq c \sum_{k=1}^{\frac{d}{d}} \left( \frac{(d/2)^2}{nk} \wedge \frac{k}{d} \right)
\]

with \(c = \alpha^{-2} 3^{-\alpha} 2^{-1}\). Hence Corollary 3 follows from Theorem 1 and the following lemma applied with \(x = (d/2)^2/n\) and \(m = d/2\).

Lemma 8. For every \(m \geq 1\) and every \(x \geq 0\), we have

\[
\sum_{k=1}^{m} \left( \frac{x}{k} \wedge \frac{k}{m} \right) \geq c \cdot \min \left( m, x + x \log_+ \left( m \wedge \sqrt{m/x} \right) \right)
\]

for some absolute constant \(c > 0\).

It remains to prove Lemma 8. If \(x \geq m\), then we have

\[
(I) := \sum_{k=1}^{m} \left( \frac{x}{k} \wedge \frac{k}{m} \right) = \sum_{k=1}^{m} \frac{k}{m} = \frac{m + 1}{2}.
\]

On the other hand if \(x \leq 1/m\), then

\[
(I) = \sum_{k=1}^{m} \frac{x}{k} \geq x \int_{1}^{m+1} \frac{1}{t} \, dt = x \log(m + 1).
\]
Finally, if $1/m < x < m$, then let $k_0 \geq 2$ be the smallest natural number in \{1, \ldots, d\} such that $x/k_0 \leq k_0/m$. Then we have

$$ (I) = \sum_{k=1}^{k_0-1} \frac{k}{m} + \sum_{k=k_0}^{m} \frac{x}{k} \geq \frac{k_0(k_0-1)}{2m} + x \log \left( \frac{m+1}{k_0} \right), $$

where the second sum is treated as in (A.2). Using the definition of $k_0$, we get

$$ (I) \geq x \frac{k_0 - 1}{2k_0} + x \log \left( \frac{m+1}{\sqrt{xm} + 1} \right) \geq \frac{x}{4} + x \log \left( \frac{m+1}{\sqrt{xm} + 1} \right). $$

Hence, for $1/m < x < m$, we get

$$ (I) \geq \frac{x}{8} + \frac{x}{8} \log \left( \frac{e(m+1)}{\sqrt{xm} + 1} \right) \geq \frac{x}{8} + \frac{x}{8} \log \left( \sqrt{\frac{m}{x}} \right). \quad (A.3) $$

Collecting (A.1)–(A.3), the claim follows. \qed

**Proof of (3.5).** The second and third power of

$$ \xi = \sum_{j \neq i} x_j (e_i e_j^T - e_j e_i^T), \quad \sum_{j \neq i} x_j^2 = 1, $$

are given by

$$ \xi^2 = -e_i e_i^T - \sum_{j \neq i, k \neq i} x_k x_j e_i e_j e_k^T, \quad \xi^3 = -\xi, $$

as can be seen from the fact that $e_1, \ldots, e_p$ is an orthonormal basis and the relation $\sum_{j \neq i} x_j^2 = 1$. Hence, we have $\xi^{2n-1} = (-1)^{n-1} \xi$ and $\xi^{2n} = (-1)^{n-1} \xi^2$ for all $n \geq 1$. Combining these relations with $\exp(t\xi) = \sum_{n \geq 0} (t\xi)^n / n!$ and $\xi^0 = I_p$, we get

$$ \exp(t\xi)_{ii} = e_i^T \exp(t\xi) e_i = 1 - \sum_{n \geq 1} (-1)^{n-1} \frac{t^{2n}}{(2n)!} = \cos(t), $$

$$ \exp(t\xi)_{ji} = e_j^T \exp(t\xi) e_i = -x_j \sum_{n \geq 1} (-1)^{n-1} \frac{t^{2n-1}}{(2n-1)!} = -x_j \sin(t), \quad j \neq i, $$

which gives the claim. \qed

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Proof of (3.7). For the first claim, note that we have $\Pi \circ R_{-1}(1) = 1 - \Pi \circ R_{-1}(-1) = 1 - q$ and thus
\[
\chi^2(\Pi \circ R_{-1}, \Pi) = (1 - q)\left(\frac{q}{1-q}\right)^2 + q\left(\frac{1-q}{q}\right)^2 - 1 = \frac{(1 - 2q)^2}{q(1-q)}.
\]
To see the second claim, let $P_g$ be the distribution of an observation $X$ with density $f_g$. Then we have
\[
\chi^2(P_{-1}, P_1) = \int_{-1}^{1} (f_{-1}(x) - f_1(x))^2 \frac{1}{f_1(x)} \, dx
\leq 4 \int_{-1}^{1} (f_{-1}(x) - f_1(x))^2 \, dx = 25c_3^2h^{2\beta+1}\|K\|^2_{L^2},
\]
where the inequality follows from the fact that $f_1(x) \geq 1/4$ for all $x \in [-1, 1]$ by construction. Hence, by (2.2), we have
\[
\chi^2(P_{-1}, P_1) = (1 + \chi^2(P_{-1}, P_1))^\alpha - 1 \leq e^{25c_3^2h^{2\beta+1}\|K\|^2_{L^2}} - 1,
\]
which gives the second claim. \(\Box\)

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References


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