Fluctuations of the overlap at low temperature in the 2-spin spherical SK model

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Abstract: We describe the fluctuations of the overlap between two replicas in the 2-spin spherical SK model about its limiting value in the low temperature phase. We show that the fluctuations are of order $N^{-1/3}$ and are given by a simple, explicit function of the eigenvalues of a matrix drawn from the Gaussian Orthogonal Ensemble. We show that this quantity converges and describe its limiting distribution in terms of the Airy$_1$ random point field (i.e., the joint limit of the extremal eigenvalues of the GOE) from random matrix theory.

1 Introduction

The 2-spin, spherical Sherrington-Kirkpatrick (SSK) model with zero magnetic field is defined by the random Hamiltonian

$$H_N(\sigma) = -\sum_{1 \leq i < j \leq N} \frac{1}{\sqrt{2N}} g_{ij}\sigma_i\sigma_j. \quad (1.1)$$

Here, $H_N$ is a function of $\sigma \in S^{N-1} := \{\sigma \in \mathbb{R}^N, ||\sigma||_2 = \sqrt{N}\}$, and the coefficients $\{g_{ij}\}_{i,j=1}^N$ are iid standard normal random variables. This model was introduced by Kosterlitz, Thouless and Jones [19], by analogy with the standard SK model where the $\sigma$ are Ising spins taking values in the hypercube $\{\pm 1\}^N$ [24, 27]. One of the interests in spherical spin glasses is the fact that the continuous nature of the phase space allows for somewhat more explicit calculations than the Ising case. We refer, for example, to the recent papers of Subag [29–31] for important work that takes advantage of the continuous geometry of spherical spin glasses.

The partition function of the SSK model is given by

$$Z_N(\beta) = \int_{S^{N-1}} e^{-\beta H_N(\sigma)} d\omega_N(\sigma), \quad (1.2)$$

where $\beta > 0$ is a parameter corresponding to the inverse temperature and $\omega_N$ is the normalized surface measure on $S^{N-1}$. The $N \to \infty$ limit of the free energy

$$F_N(\beta) = \frac{1}{N} \log Z_N(\beta)$$

was determined in [19] (with the variational Parisi formula for the spherical model being found in [10]), and a rigorous justification appeared in [32]. The model exhibits a phase transition at $\beta_c = 1$, in the sense that the limit

$$F(\beta) := \lim_{N \to \infty} F_N(\beta)$$

fails to be analytic in $\beta$ at this value.

In [3], Baik and Lee use a contour integral representation for the partition function which had previously appeared in [19] to compute the asymptotic fluctuations of $F_N$ around $F(\beta)$. They show that in the high temperature phase $\beta < \beta_c = 1$, the quantity

$$N(F_N(\beta) - F(\beta)) \quad (1.3)$$

converges to a normal random variable. This is the analog for the SSK model of the classical central limit theorem for the SK model with Ising spins of Aizenman, Lebowitz and Ruelle [1].
In the low temperature phase of the SSK, Baik and Lee proved that the free energy has asymptotic fluctuations given by the Tracy-Widom distribution (TW\textsubscript{1}) associated to the Gaussian Orthogonal Ensemble (GOE):

\[
\lim_{N \to \infty} \frac{N^{2/3}}{\beta - 1} (F_N(\beta) - F(\beta)) = \text{TW}_1. \tag{1.4}
\]

The convergence is in distribution. The TW\textsubscript{1} distribution is the limit of the re-scaled fluctuations of the largest eigenvalue of the GOE [33]. The analog of the low temperature result (1.4) for the classical SK model seems out of reach of current methods. Moreover, we are not aware of a prediction concerning the limiting distribution of the free energy of the SK model in the low temperature phase.

In a parallel development, a model related to (1.1) and (1.2) has appeared in the context of high-dimensional statistics. Onatski, Moreira and Hallin [22] obtained an analog of the high-temperature CLT in the case that the random variables \(g_{ij}\) are associated with a Wishart ensemble (as opposed to the case under consideration where they are naturally associated with a symmetric matrix of normal random variables). In this context, the Gaussian fluctuations have implications for the asymptotic power of statistical tests in detecting the presence of an unknown signal in an otherwise isotropically distributed dataset. Here, the high temperature regime corresponds to the regime of low signal-to-noise ratio.

In addition to the intrinsic interest of computing the fluctuations of \(F_N(\beta)\), the method in [3] offers a satisfying interpretation of the phase transition in the SSK in terms of random matrix theory. The argument in [3] reveals that in the high temperature phase \(F_N(\beta) - F(\beta)\) is dominated by linear statistics of eigenvalues \(\{\lambda_j(M)\}_{j=1}^N\) of the matrix

\[
M_{ij} = \frac{1}{\sqrt{2N}} (g_{ij} + g_{ji}).
\]

Linear statistics are quantities of the form

\[
\sum_{j=1}^N f(\lambda_j(M)) - N \int_{-2}^2 f(x)\rho_{sc}(x) \, dx,
\]  

where \(f\) is a regular function, and \(\rho_{sc}\) denotes the semicircle distribution, defined below. This latter quantity is the asymptotic density of states of the GOE eigenvalues, a foundational result of random matrix theory known as Wigner’s semicircle law [34]. The Gaussian behavior (1.3) then follows from the fact that the asymptotic fluctuations of (1.5) are Gaussian [26]. In the low temperature phase, \(F_N(\beta) - F(\beta)\) instead depends to leading order only on the first eigenvalue \(\lambda_1\). Thus the phase transition corresponds to a transition from a regime where all eigenvalues contribute to the limiting behavior, to one where only the leading eigenvalue does. Baik and Lee have applied their method to a number of variants of the SSK [3–6], including the bipartite SSK and models incorporating a Curie-Weiss type interaction in addition to the spin glass couplings.

The phase transition in the classical and spherical SK model, and more generally in \(p\)-spin models [24], can also be detected in the terms of the behavior of overlaps. To define these, we introduce the Gibbs measure defined by the expectation

\[
\langle f \rangle = \frac{1}{Z_N(\beta)} \int_{S^{N-1}} e^{-\beta H_N(\sigma)} f(\sigma) \, d\omega_N(\sigma). \tag{1.6}
\]

The Gibbs measure depends on \(N\) and \(\beta\) but we omit this from the notation. Let \(\sigma^{(1)}, \sigma^{(2)} \in S^{N-1}\) be two independent samples ("replicas") from the Gibbs measure (1.6). The overlap between \(\sigma^{(1)}\) and \(\sigma^{(2)}\) is the normalized inner product:

\[
R_{12} = \frac{1}{N} \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle.
\]

It is known that for \(\beta < 1\), \(R_{12}\) tends to zero as \(N \to \infty\), while in the low temperature phase \(\beta > 1\), it concentrates around the constant values \(\pm q\), where [25]

\[
q = \frac{1 - \beta}{\beta}. \tag{1.7}
\]
In [21], V.-L. Nguyen and the second author used a contour integral representation related to that to that in [4] to show that $R_{12}$ has Gaussian fluctuations in the regime $\beta \leq 1 - N^{-1/3+\varepsilon}$, for any $\varepsilon > 0$.

In the present work, we describe the annealed fluctuations of the Gibbs expectation $\langle R_{12}^2 \rangle$ about the limiting value $q^2$ in the low temperature regime. More precisely, we prove the convergence of the limit

$$
\lim_{N \to \infty} N^{1/3} \left[ \langle R_{12}^2 \rangle - \left( \frac{1 - \beta}{\beta} \right)^2 \right]
$$

in distribution to a random variable that we characterize in terms of random matrix quantities. This is the statement of Theorem 2.2 below. The limiting random variable is defined in (2.11), but we will introduce it informally in the present discussion. We pause here to comment on our choice of observable. Since we are considering Gibbs averages, it would seem more natural to consider $\langle R_{12} \rangle$; however this quantity is identically 0 due to the spin-flip symmetry of the Gibbs measure, and so the next natural quantity is instead the square $\langle R_{12}^2 \rangle$. Our methods also allow us to compute a fourth moment $\langle R_{12}^4 \rangle$ which implies a concentration result for $R_{12}^2$. From this, we also deduce that a similar result to (1.8) holds for $|\langle R_{12} |$ (or, equivalently, $\langle R_{12} 1_{|R_{12}>0} \rangle$).

The proof of our main results consists of two components:

1. An expansion of $\langle R_{12}^2 \rangle$ around its limiting value down to the scale $o(N^{-1/3})$ (we in fact expand this down to $o(N^{-2/3})$ in order to prove our additional result about $\langle |R_{12} | \rangle$) in terms of a random variable which we call $\Xi_N$ which is a simple function of the eigenvalues of $M$.

2. Convergence of $\Xi_N$ to a random variable that is defined in terms of the Airy$_1$ random point field of random matrix theory; this is the point process arising as the joint limit of the extremal eigenvalues of the GOE.

Our expansion of $\langle R_{12}^2 \rangle$ is given as Theorem 2.1 below. It shows that the leading order fluctuating term is

$$
\Xi_N := N^{1/3} \left( \frac{1}{N} \sum_{i=2}^{N} \frac{1}{\lambda_i(M) - \lambda_1(M)} + 1 \right).
$$

The proof of Theorem 2.1 is based on the method of steepest descent using a double contour integral representation of $\langle R_{12}^2 \rangle$. The works of Baik and Lee are based on a contour integral representation for the partition function, which originally appeared in the work of Kosterlitz, Thouless and Jones [19]. Nguyen and the second author of the present paper realized [21] that, at the cost of moving to a double contour integral, one could derive similar formulas for functions of the overlap $R_{12}$. The formula we use is (2.3) below.

In the low temperature regime, the steepest descent analysis is complicated by the presence of a branch point in the vicinity of the saddle. Note that the integral representation for the partition function appears in the denominator of our representation (2.3), as the normalization of the Gibbs measure. While the work of Baik and Lee [3] extends to the low temperature regime, their technique is insufficient to derive an expansion for $\langle R_{12}^2 \rangle$ that is precise enough to detect fluctuations; we need to calculate subleading fluctuating quantities for both the partition function and the quantity appearing in the numerator of (2.3). In order to do so, we need to go beyond the analysis presented in [3] and deal with the subtlety presented by the branch point.

The logarithmic singularities of our integrand are due to the eigenvalues of $M$ being close to the saddle point. The most common tool for handling the eigenvalues of $M$ are the rigidity estimates (or local semicircle law) due to Erdös, Schlein, Yau and Yin [12, 13]. These estimates have been heavily used by Baik and Lee in their works on the SSK. However, these estimates are alone insufficient for ruling out the possibility that many eigenvalues cluster near the saddle. Instead, we need to use an additional tool from random matrix theory, that of level repulsion, which is an upper bound on $\mathbb{P}[N^{2/3}(\lambda_1(M) - \lambda_2(M)) \leq s]$. The level repulsion estimate proven in a work of Knowles and Yin [18] is sufficient for our purposes; additionally, we give a (relatively) self-contained proof of a somewhat stronger estimate. We apply level repulsion to show that only $\lambda_1(M)$ is close to the saddle. We isolate its contribution explicitly, carrying the singularity through the calculation, and deal with the remaining eigenvalues via expansions.
The second major component of our work is to prove the convergence of $\Xi_N$ to a limiting random variable as $N \to \infty$. The joint limit of the largest eigenvalues of the GOE is the Airy$_1$ random point field, which we will denote by $(\chi_j)_{j=1}^\infty$ [33]. It is natural to expect that the limit of $\Xi_N$ is then the random variable,

$$
\Xi := \lim_{n \to \infty} \left( \sum_{j=2}^n \frac{1}{\chi_j - \chi_1} - \int_0^{(\Xi_n)^{2/3}} \frac{1}{\pi \sqrt{x}} \, dx \right).
$$

(1.10)

However, the convergence of $\Xi_N$ to $\Xi$ is subtle - it is nontrivial even that $\Xi$ itself is well-defined and finite almost surely. Indeed, the expected location of $\chi_j$ grows like $j^{2/3}$, and so the sum in (1.10) is expected to be infinite and requires renormalization by the deterministic integral. Some cancellation must arise for $\Xi$ to exist. Moreover, the results on the convergence of the GOE eigenvalues to the Airy$_1$ random point field concern only a fixed number of eigenvalues.

The random variable $\Xi_N$ appeared in a different context in a work of Gorin and Shkolnikov [16], where the estimate $E[\Xi_N] = o(N^{1/3})$ was proven. An application of the local semicircle law shows that $|\Xi_N| \leq C_\varepsilon N^\varepsilon$ for any $\varepsilon > 0$ with high probability. However, both of these results are insufficient for our purposes.

In order to prove the convergence of $\Xi_N$ to $\Xi$ it is necessary to truncate each random variable to involve only finitely many eigenvalues/particles; the truncated $\Xi_N$’s then converge to the truncated $\Xi$’s by the a.s. simplicity of the Airy$_1$ random point field and joint convergence. The main technical difficulty is then to truncate to a constant number of particles. For the truncation of $\Xi_N$, we turn to a result of Gustavsson [17], which provides an estimate - optimal up to constants - on the variance of the eigenvalue counting function near the spectral edge for the Gaussian Unitary Ensemble (GUE). In order to apply this, we use a coupling of Forrester and Rains [15] to derive an estimate for the GOE from the GUE result. This gives us an estimate on the deviation of an eigenvalue from its expected location due to the duality $|\{ i : \lambda_i > E \}| \geq n \iff \lambda_n > E$. The truncation of $\Xi$ (and the proof that it is well-defined) follows a similar line of reasoning relying instead on a result of Soshnikov [28] for the variance of the eigenvalue counting function of the Airy$_2$ process (the joint limit of the largest GUE eigenvalues).

The expansion we derive in Theorem 2.1 does not seem to appear in the physics literature, but while completing this work, we were informed by J. Baik that predictions close to the results we find were obtained using nonrigorous physics methods by himself and collaborators. In between the first submission of the present work to the arXiv and acceptance for publication, this work of Baik, Collins-Woodfin, Le Doussal and Wu has appeared [2]. This work also analyzes the SSK Hamiltonian with a magnetic field.

The behavior of the quantities that appear in our expansion shed light on a large deviations result of Panchenko and Talagrand [25]; we discuss this in the next section.

2 Main results

We express the asymptotic distribution of the overlap between two replicas in terms of the eigenvalue distribution generated by the Gaussian Orthogonal Ensemble (GOE). To understand the connection between the GOE and our problem, define the symmetric random matrix $M$ by

$$
M_{ij} = \begin{cases} 
\frac{g_{ij} + g_{ji}}{\sqrt{2N}}, & i \neq j \\
0, & i = j 
\end{cases}
$$

(2.1)

where the $(g_{ij})_{i,j}$ are the random variables appearing in the definition of the Hamiltonian (1.1). The distribution of $M$ is that of a normalized GOE (Gaussian Orthogonal Ensemble) matrix with the diagonal set to zero. We denote the ordered eigenvalues of $M$ by

$$
\lambda_1(M) \geq \lambda_2(M) \geq \ldots \geq \lambda_N(M).
$$
Next, note that the Hamiltonian $H_N(\sigma)$ equals
\[
-\frac{1}{\sqrt{2N}} \sum_{1 \leq i \neq j \leq N} g_{ij} \sigma_i \sigma_j = -\frac{1}{2\sqrt{N}} \sum_{1 \leq i \neq j \leq N} \frac{g_{ij} + g_{ji}}{\sqrt{2}} \sigma_i \sigma_j
= -\frac{1}{2} \langle \sigma, M \sigma \rangle.
\] (2.2)

For two vectors $\sigma_1, \sigma_2 \in \mathbb{S}^{N-1}$ we define the overlap as the normalized inner product of $\sigma_1$ and $\sigma_2$:
\[
R_{12} = \frac{1}{N} (\sigma^{(1)} \cdot \sigma^{(2)}) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{(1)} \sigma_i^{(2)},
\]
where $(\sigma^{(k)})_{1 \leq i \leq N}$, $k = 1, 2$ are the components of $\sigma^{(k)}$. For a bounded, measurable function
\[
f : (\mathbb{S}^{N-1})^k \to \mathbb{R},
\]
we denote the Gibbs expectation of $f$ by
\[
\langle f(\sigma^{(1)}, \ldots, \sigma^{(k)}) \rangle = \frac{1}{Z_N(\beta)^k} \frac{1}{|\mathbb{S}^{N-1}|^k} \int_{(\mathbb{S}^{N-1})^k} f(\sigma^{(1)}, \ldots, \sigma^{(k)}) e^{-\beta \sum_{i=1}^{k} H_N(\sigma^{(i)})} \, d\omega_N(\sigma^{(1)}) \cdots d\omega_N(\sigma^{(k)}).
\]

As a consequence of the representation (2.2), we derive the following integral formula for the Gibbs expectation $\langle R_{12}^2 \rangle$ in Section 4:
\[
\langle R_{12}^2 \rangle = \frac{\int \int e^{N(G(z)+G(w))/2} \left( \sum_{i=1}^{N} \frac{1}{\beta^2 N^2 (z-\lambda_i(M))(w-\lambda_i(M))} \right) \, dz \, dw}{(\int e^{NG(z)/2} \, dz)^2},
\] (2.3)
where the integrals are over a vertical line in the complex plane to the right of all the $\lambda_i(M)$ and
\[
G(z) := \beta z - \frac{1}{N} \sum_{i} \log(z-\lambda_i(M)).
\] (2.4)

Under the Gibbs measure, the overlap concentrates around the values $\pm q$ where [25]
\[
q := (1 - \beta^{-1})_+.
\] (2.5)

We provide an expansion of the overlap in terms of $\lambda_i(M)$, up to an error of size $o(N^{-2/3})$, around $q^2$.

**Theorem 2.1.** Assume $\beta > 1$. Let $\varepsilon, \delta > 0$ with $\frac{1}{3} > \delta$. For any $\varepsilon > 0$ and $N$ large enough, there is an event $\mathcal{F}_{\delta, \varepsilon}$ such that
\[
\mathbb{P}(\mathcal{F}_{\delta, \varepsilon}) \geq 1 - N^{-\delta + \varepsilon}
\]
on which the following estimate holds:
\[
\langle R_{12}^2 \rangle = \left( 1 - \frac{\beta}{\beta} \right)^2 + 2 \frac{\beta - 1}{\beta^2} \left( \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j(M) - \lambda_1(M)} + 1 \right)
- \frac{1}{N\beta^2} \left( \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j(M) - \lambda_1(M)} \right)^2 + \frac{1}{\beta^2} \left( \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j(M) - \lambda_1(M)} + 1 \right)^2 + O(N^{\delta + 10\varepsilon} N^{-1}).
\] (2.6)

Theorem 2.1 is a consequence of Theorem 5.9 below. The event $\mathcal{F}_{\delta, \varepsilon}$ is defined in Definition 5.3 below; it is a high probability event on which certain a-priori estimates on the eigenvalue locations $\lambda_i$ hold (the rigidity and level repulsion estimates) - these are introduced in the next section.

Define
\[
\tilde{m}_N(\lambda_1) = \frac{1}{N} \sum_{j \geq 2} \frac{1}{\lambda_j(M) - \lambda_1(M)}, \quad \tilde{m}_N'(\lambda_1) = \frac{1}{N} \sum_{j \geq 2} \frac{1}{(\lambda_j(M) - \lambda_1(M))^2}.
\] (2.7)
Note that these quantities appear on the right side of equation (2.6). The exponent \( \delta > 0 \) is associated to level repulsion, in that \( \lambda_1(M) - \lambda_2(M) \geq N^{-2/3-\delta} \) on \( F_{\delta,\epsilon} \) (by definition of \( F_{\delta,\epsilon} \)). On the event \( F_{\delta,\epsilon} \) the magnitude of \( \tilde{m}_N, \tilde{m}_N' \) will be seen to be at most

\[
\tilde{m}_N(\lambda_1) + 1 = \mathcal{O}(N^{-1/3+\delta+\epsilon_1}), \quad \frac{1}{N} \tilde{m}_N'(\lambda_1) = \mathcal{O}(N^{-2/3+2\delta+\epsilon_1}).
\]  

(2.8)

Theorem 2.1 thus identifies the overlap down to terms of order \( o(N^{-2/3}) \).

It is interesting to study the behavior of the leading order contribution to the fluctuations of \( \langle R_{12}^2 \rangle \) which is the term

\[
\tilde{m}_N(\lambda_1) + 1,
\]

(2.9)
in the context of the work [25] of Panchenko and Talagrand. They obtained an exponential estimate for the probability that \( \langle R_{12}^2 \rangle \geq q^2 + \varepsilon \) but noted that the event that \( \langle R_{12}^2 \rangle \leq q^2 - \varepsilon \) could not be ruled out at the level of large deviations.

Due to the rigidity estimates of random matrix theory which are reviewed in the next section, the quantity \( m_N(\lambda_1) + 1 \) has a light upper tail: for example the probability that it exceeds \( N^{-1/3+\varepsilon} \) goes to 0 superpolynomially, for any fixed \( \varepsilon > 0 \). On the other hand, the asymptotic density \( \mathbb{P}[N^{2/3}(\lambda_1-\lambda_2) = s] \) is expected to behave like \( s^2 \) near 0 and so (recall that the eigenvalues are ordered and so this is a positive quantity) \( \tilde{m}_N(\lambda_1) + 1 \) has a relatively heavy lower tail. Due to the somewhat large probability of the event \( F_{\delta,\epsilon} \) of Theorem 2.1, we do not attempt to make this comparison to the work [25] rigorous, settling for pointing out the heuristic agreement of the leading order term with the behavior observed in [25].

As we have noted above, the matrix \( M \) is closely related to the GOE. The GOE is the matrix ensemble with entries

\[
H_{ij} \sim \frac{1}{\sqrt{N}} N(0, 1), \quad 1 \leq i < j \leq N,
\]

\[
H_{ii} \sim \frac{1}{\sqrt{N}} N(0, 2), \quad 1 \leq i \leq N,
\]

\[
H_{ji} = H_{ij}, \quad 1 \leq i < j \leq N,
\]

(2.10)

and all the non-identical random variables are independent.

In order to apply results for GOE eigenvalues to the eigenvalues of \( M \), we need to compare them to the eigenvalues of \( H \). In Appendix A we show that if we take \( H \) so that \( H_{ij} = M_{ij} \) for \( i \neq j \) and the diagonal of \( H \) independent from \( M \), then with very high probability, \( |\lambda_i(M) - \lambda_i(H)| \leq C_\varepsilon N^{-1+\varepsilon} \) for eigenvalues near the spectral edge, and any \( \varepsilon > 0 \). Alternatively, it is possible to appeal to the literature on edge universality in random matrix theory (e.g., [8]) to obtain the results about the eigenvalues of \( M \) that we need. However, the precise statements are not written anywhere explicitly, so we have instead opted to carry out the calculations in Appendix A which are a relatively straightforward application of the resolvent method. The result derived in Proposition A.1 is in fact stronger than what could be deduced from the universality literature and may be useful for other applications.

On a related note, the most common definition in the literature of the SSK Hamiltonian is as in (1.1). A less common alternative is to include diagonal terms with additional Gaussian couplings. In the usual SK model this of course makes no difference since \( \sigma^2 = 1 \), whereas in the SSK this would result in \( M = H \) above. The inclusion of the diagonal would simplify our analysis as we could omit the calculations in Appendix A which compare the eigenvalues of \( M \) directly to those of \( H \); all else is identical. To maintain consistency with the physics literature [10,19] we have included the diagonal from the sum.

As can be seen by the expansion (2.6), the main contribution to the fluctuations of the overlap about its mean is from the extremal eigenvalues of \( M \). For any finite \( k \), the largest \( k \) eigenvalues of \( H \), \( \{\lambda_i(H)\}_{i=1}^k \) are known to converge in distribution, after a rescaling, to the first \( k \) particles of the Airy_1 random point field; we denote this latter quantity by \( \{\chi_i\}_{i=1}^\infty \). Due to the estimates proven in the appendix, the same joint convergence also holds for the largest eigenvalues of \( M \).
We therefore expect that the rescaled fluctuations of \( \langle R_{12}^2 \rangle \) converge to the random variable given by
\[
\Xi := \lim_{n \to \infty} \left( \sum_{j=2}^{n} \frac{1}{\chi_j - \chi_1} - \int_{0}^{(3n)^{2/3}} \frac{1}{\pi \sqrt{x}} dx \right). \tag{2.11}
\]
In Theorem 6.1, we show that this limit exists almost surely, and so \( \Xi \) is a well-defined random variable. The deterministic correction on the RHS of (2.11) represents the leading order term in the density of states of the Airy\(_1\) random point field. The expected location of the \( j \)th particle of the Airy\(_1\) random point field is roughly \( \chi_j \sim j^{2/3} \) and so neither the sum or the deterministic correction converge as \( n \to \infty \).

Our main result on the limiting distribution of the fluctuations of the overlap is the following.

**Theorem 2.2.** Let \( \Xi \) be the random variable in (2.11). We have the following convergence in distribution for \( \beta > 1 \):
\[
\lim_{N \to \infty} N^{1/3} \left[ \langle R_{12}^2 \rangle - \left( \frac{1 - \beta}{\beta} \right)^2 \right] = 2 \left( \frac{\beta - 1}{\beta^2} \right) \Xi. \tag{2.12}
\]

In Theorem 2.1 we introduced the square in order to study the overlap, due to the symmetry of the overlap distribution with respect to the Gibbs measure (i.e., \( \langle R_{12} \rangle = 0 \)). An alternative would be to study \( \langle |R_{12}| \rangle \); if we knew that \( |R_{12}| \) concentrated about \( q \) on the scale \( N^{-1/3} \) then this of course could be deduced from Theorem 2.1. We prove the concentration by calculating the fourth moment \( \langle (R_{12}^2 - q^2)^2 \rangle \); this is the content of the following theorem which is proven in Section 7.

**Theorem 2.3.** On the event \( \mathcal{F}_{\delta,\epsilon_{1}} \) of Theorem 2.1 we have,
\[
\langle (R_{12}^2 - q^2)^2 \rangle = \frac{8(\beta - 1)^2}{\beta^2} \frac{\tilde{m}_N(\lambda_1)}{N} + 4 \frac{(\beta - 1)^2}{\beta^4} (1 + \tilde{m}_N(\lambda_1))^2 + O(N^{-1+10\epsilon_{1}+3\delta}), \tag{2.13}
\]
and furthermore on the event \( \mathcal{F}_{\delta,\epsilon_{1}} \), the first two terms are \( O(N^{-2/3+2\delta+10\epsilon_{1}}) \). As a consequence,
\[
\langle |R_{12}| \rangle = q + \frac{1}{\beta} (\tilde{m}_N(\lambda_1) + 1) + O(N^{-2/3+2\delta+10\epsilon_{1}}) \tag{2.14}
\]
and so we have the convergence in distribution of
\[
\lim_{N \to \infty} N^{1/3} \left[ \langle |R_{12}| \rangle - \frac{1 - \beta}{\beta} \right] = \frac{1}{\beta} \Xi, \tag{2.15}
\]
where \( \Xi \) is as above.

We discuss the relation of our results to the work of Baik, Collins-Woodfin, Le Doussal and Wu [2]. They predict that the fluctuations of \( R_{12}^2 \) should be governed by
\[
R_{12}^2 - q^2 \sim \frac{2(\beta - 1)}{\beta^2} \frac{1}{N} \sum_{j=2}^{N} \frac{n_j^2}{\lambda_j - \lambda_1} + 1 \tag{2.16}
\]
where the \( \{n_j\}_j \) are independent standard normal random variables (in particular independent of the \( \lambda_j \)). The Gibbs average \( \langle \cdot \rangle \) corresponds to taking the expectation over the \( \{n_j\}_j \). It is a simple calculation to integrate out the \( \{n_j\}_j \) and find quantities agreeing with the leading order contribution in (2.6) and (2.13).

### 2.1 Outline of the paper

In Section 3, we state some basic results of random matrix theory which we will use to control the eigenvalues of the matrix \( M \). In Section 4, we obtain the representation (2.3), along the lines of similar representations in [3], [21].

Theorem 2.1 is proven in Section 5, where we analyze the representation (2.3) by the method of steepest descent. As was already noticed in [3], in the case \( \beta > 1 \) of interest here, the analysis is
between the GUE and GOE.

In Section 6, we show that the term \( \tilde{m}_N(\lambda_1) + 1 \) appearing in (2.6) of order \( N^{-1/3} \), in the sense that \( N^{1/3}(\tilde{m}_N(\lambda_1) + 1) \) converges in distribution. This involves some establishing some preliminary estimates for the GOE as well as the Airy\(_1\) random point field, which we could not locate in previous literature. We deduce using this corresponding results for the GUE and Airy\(_2\) random point field proven by Gustavsson and Soshnikov [17,28], respectively, and the Forrester-Rains coupling [15] between the GUE and GOE.

In Section 7 we calculate the fourth moment \( \langle R_{12}^4 \rangle \) and consequently extend our fluctuation result to \( \langle |R_{12}| \rangle \), proving Theorem 2.3. Appendix A collects some technical estimates comparing the eigenvalues of coupled GOE and zero-diagonal GOE (i.e., \( M \)) matrices.

### 3 Random matrix results

In this section, we summarize the results from random matrix theory we use in the rest of the paper. A central role is played by the resolvent matrix

\[
R(H, z) = \frac{1}{H - z}, \quad R(M, z) = \frac{1}{M - z}
\]

where \( H \) is the GOE matrix in (2.10), and \( M \) is the matrix ensemble given by (2.1). The spectral parameter is \( z \) is commonly denoted \( z = E + i\eta \) with \( E, \eta \in \mathbb{R} \) and \( \eta > 0 \). In the recent literature, the resolvent has customarily been denoted by \( G \), a notation we reserve for the quantity (4.3) in this paper. We also introduce the Stieltjes transform of the empirical eigenvalue distribution:

\[
m_N(H, z) = \frac{1}{N} \text{tr} R(H, z) = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\lambda_j(H) - z},
\]

and similarly for \( M \). The classical semi-circle law is then equivalent to the approximation for fixed \( z \),

\[
m_N(H, z) = m_{sc}(z) + o(1),
\]

where the semi-circle law and its Stieltjes transform are

\[
\rho_{sc}(E) = \frac{1}{2\pi} \sqrt{(4 - E^2)_+}, \quad m_{sc}(z) = \int_{\mathbb{R}} \frac{1}{x - z} \rho_{sc}(x) \, dx.
\]

We now state the local semi-circle law as it appears in [7, Theorem 2.6]. First, we introduce the notion of overwhelming probability.

**Definition 3.1.** We say that an event or family of events \( \{A_i\}_{i \in I} \) hold with overwhelming probability if for all \( D > 0 \) we have \( \sup_{i \in I} \mathbb{P}[A_i] \leq N^{-D} \) for \( N \) large enough.

**Theorem 3.2 (Local semi-circle law).** Define the spectral domain \( S \) by

\[
S = \{ E + i\eta : |E| \leq 10, 0 < \eta \leq 10 \}.
\]

For any \( \varepsilon > 0 \) and all \( N \) sufficiently large, we have for both \( R(z) = R(M, z) \) and \( R(H, z) \) that the estimates,

\[
\max_{ij} |R_{ij}(z) - \delta_{ij} m_{sc}(z)| \leq \frac{\text{Im} m_{sc}(z)}{N^{1-\varepsilon}\eta} + \frac{1}{N^{1-\varepsilon}\eta},
\]

and for both \( m_N(z) = m_N(H, z) \) and \( m_N(M, z) \),

\[
|m_N(z) - m_{sc}(z)| \leq \frac{1}{N^{1-\varepsilon}\eta}
\]

hold uniformly in \( z \in S \) with overwhelming probability.
A consequence of the semi-circle we will use several times is that the eigenvalues $\lambda_i$ are close to the corresponding quantiles of the semi-circle distribution. These quantiles are known as the classical locations of the eigenvalues in random matrix theory:

$$\int_{\gamma_i}^{\gamma_{i+1}} \rho_{\text{sc}}(x) \, dx = \frac{i - 1/2}{N}. \quad (3.2)$$

**Theorem 3.3 (Eigenvalue rigidity).** For any $\varepsilon > 0$, we have that the estimates

$$|\lambda_i - \gamma_i| \leq N^{-2/3+\varepsilon} \min\{i, (N + 1 - i)\}^{-1/3}$$

hold uniformly in $i$ with overwhelming probability, for $\lambda_i$ the eigenvalues of $M$ or $H$.

We will also need some finer information concerning level repulsion. The next result shows that, up to an $O(N^\varepsilon)$ error, the distribution of the spacing between $\lambda_1$ and $\lambda_2$ has a density on scale $N^{-2/3}$. While this will be sufficient for our purposes, one instead expects that there is level repulsion, i.e. $s$ on the right side of (3.3) should be replaced by $s^2$. This has been established in great generality for the spacings $\lambda_j - \lambda_{j+1}$ where $j \gg 1$ in [8, Theorem 3.7], but has not been proven for the eigenvalues at the edge.

The following result could be deduced from Remark 1.5 of [18]. A complete proof was not explicitly given in that work and relies on asymptotics of the Hermite polynomials. For the sake of completeness, we will give a different proof which relies only on the eigenvalue rigidity and the loop equations.

**Lemma 3.4 (Existence of spacing density).** Let $\varepsilon > 0$. There is a constant $C > 0$ such that for

$$N^{-1/3+\varepsilon} \leq s \leq 1,$$

$$\mathbb{P}(N^{2/3}(\lambda_1 - \lambda_2) < s) \leq CsN^\varepsilon,$$  

(3.3)

where the $\lambda_i$ are the eigenvalues of $M$ or $H$.

**Remark.** Inspecting the proof we see that the restriction $s \geq N^{-1/3+\varepsilon}$ enters only in proving the estimate for $M$ - i.e., it holds for all $s$ for the eigenvalues of $H$.

**Proof.** By Proposition A.1 it suffices to prove the estimate for $\lambda_i := \lambda_i(H)$. We begin with the obvious estimate:

$$\frac{1}{\lambda_1 - \lambda_2} \leq \sum_{j=2}^{N^\varepsilon} \frac{1}{\lambda_1 - \lambda_j}.$$

As a consequence of Theorem 3.3 (see, for example [3, Eqn (6.3)]), we have with overwhelming probability,

$$0 \leq \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_1 - \lambda_j} - \frac{1}{N} \sum_{j=2}^{N^\varepsilon} \frac{1}{\lambda_1 - \lambda_j} = 1 - O(N^{-1/3+\varepsilon}).$$

Combining this with the weak estimate $\mathbb{E}[|\lambda_2 - \lambda_1|^{-3/2}] \leq N^C$ for some $C > 0$, which is a consequence of Section 5 of [20] we obtain the same inequality in expectation,

$$0 \leq \mathbb{E} \left[ \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_1 - \lambda_j} - \frac{1}{N} \sum_{j=2}^{N^\varepsilon} \frac{1}{\lambda_1 - \lambda_j} \right] = 1 - O(N^{-1/3+\varepsilon}).$$

Then using Markov’s inequality we have,

$$\mathbb{P}(N^{2/3}(\lambda_1 - \lambda_2) < s) \leq \mathbb{P} \left[ \sum_{j=2}^{N^\varepsilon} \frac{1}{\lambda_1 - \lambda_j} > \frac{N^{2/3}}{s} \right] \leq s N^{1/3} \mathbb{E} \left[ \frac{1}{N} \sum_{j=2}^{N^\varepsilon} \frac{1}{\lambda_1 - \lambda_j} \right] \leq s N^{1/3} \left( \mathbb{E} \left[ \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_1 - \lambda_j} - 1 \right] + O(N^{-1/3+\varepsilon}) \right).$$
By [16, Lemma 3.7], we have
\[
\mathbb{E}\left[ \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_1 - \lambda_j} \right] = \frac{\mathbb{E}[\lambda_1]}{2} = 1 + \mathcal{O}(N^{-2/3+\varepsilon})
\]
where in the last step we used again Theorem 3.3. The result follows. \(\square\)

4 Representation for the overlap

In this section, we derive a contour integral representation for Gibbs expectation \(\langle R_{12}^2 \rangle\). Throughout this section, we will denote the eigenvalues of \(M\) by
\[
\lambda_i := \lambda_i(M),
\]
for notational simplicity. We now prove the following lemma.

**Lemma 4.1.** The quantity \(\langle R_{12}^2 \rangle\) is given by
\[
\langle R_{12}^2 \rangle = \frac{\int_{\gamma-i\infty}^{\gamma+i\infty} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{z-w} \frac{1}{\beta N^2(z-\lambda_i)(w-\lambda_j)} \frac{\beta}{2} (G(z) + G(w)) \left( \sum_{i=1}^{N} \frac{1}{\beta N^2(z-\lambda_i)(w-\lambda_j)} \right) dxdy}{\left( \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\beta}{2} G(z) dz \right)^2},
\]
where
\[
G(z) = \beta z - \frac{1}{N} \sum_{i=1}^{N} \log(z - \lambda_i),
\]
for any \(\gamma \in \mathbb{R}\) so that \(\gamma > \lambda_1\).

**Remark.** Note that up to constants the quantity appearing in the denominator of (4.2) is the partition function and is the representation used by Baik and Lee [3].

**Proof.** Our starting point is the definition,
\[
\langle R_{12}^2 \rangle = \frac{1}{Z_N(\beta)^2} \int_{(\mathbb{S}^{N-1})^2} \exp \left( \frac{\beta}{2} \langle \sigma^{(1)}, M \sigma^{(1)} \rangle + \frac{\beta}{2} \langle \sigma^{(2)}, M \sigma^{(2)} \rangle \right) \left( \frac{1}{N} \langle \sigma^{(1)}, \sigma^{(2)} \rangle \right)^2 d\omega_N(\sigma^{(1)}) d\omega_N(\sigma^{(2)}).
\]

By a change of variables, we obtain
\[
Z_N(\beta) = \frac{\Gamma(N/2) \cdot 2^{N/2-1}}{2\pi i(N\beta)^{N/2-1}} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\beta}{2} G(z) dz.
\]

Let \(\mathbb{S}^{N-1} = \{ x \in \mathbb{R}^N : ||x|| = 1 \}\) be the unit sphere in \(\mathbb{R}^N\). Let \(d\Omega\) be the surface area measure on \(\mathbb{S}^{N-1}\), so that \(|\mathbb{S}^{N-1}| d\Omega\) is the uniform measure on \(\mathbb{S}^{N-1}\). By a change of variables, we obtain
\[
\int_{(\mathbb{S}^{N-1})^2} \exp \left( \frac{\beta}{2} \langle \sigma^{(1)}, M \sigma^{(1)} \rangle + \frac{\beta}{2} \langle \sigma^{(2)}, M \sigma^{(2)} \rangle \right) \left( \frac{1}{N} \langle \sigma^{(1)}, \sigma^{(2)} \rangle \right)^2 d\omega_N(\sigma^{(1)}) d\omega_N(\sigma^{(2)}) = \frac{1}{|\mathbb{S}^{N-1}|^2} \int_{(\mathbb{S}^{N-1})^2} \exp \left( \frac{\beta}{2} N \langle x, Mx \rangle + \frac{\beta}{2} N \langle y, My \rangle \right) \langle x, y \rangle^2 d\Omega(x) d\Omega(y).
\]

Let \(z, w \in \{ u \in \mathbb{C} : \text{Re} u > \lambda_1(M) \}\). In order to compute the above integral, we consider
\[
J(z, w) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^\frac{\beta}{2} N \sum_{i=1}^{N} (\lambda_i - z) x_i^2 e^\frac{\beta}{2} N \sum_{i=1}^{N} (\lambda_i - w) y_i^2 \left( \sum_{i=1}^{N} x_i y_i \right)^2 \prod_{i=1}^{N} dx_i dy_i
\]
\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^\frac{\beta}{2} N \sum_{i=1}^{N} (\lambda_i - z) x_i^2 e^\frac{\beta}{2} N \sum_{i=1}^{N} (\lambda_i - w) y_i^2 \sum_{i=1}^{N} x_i^2 y_i^2 \prod_{i=1}^{N} dx_i dy_i.
\]
We use polar coordinates, substituting \( x = r_1 x_1 \) and \( y = s_1 y_1 \) with \( r_1, s_1 > 0 \) and \( \| x_1 \| = \| y_1 \| = 1 \). We then set \((\beta/2)Nr_1^2 = r, (\beta/2)Ns_1^2 = s\) to find that

\[
J(z, w) = \frac{2^N}{(\beta N)^N} \int_0^\infty \int_0^\infty e^{-2r}e^{-us}I(r, s)s^{\frac{N-1}{2}-1}r^{\frac{N}{2}-1}drds,
\]

where

\[
I(r, s) = \int_{S^{N-1} \times S^{N-1}} e^{s(x_1,Mx_1)+s(y_1,My_1)} \frac{rs}{\beta^2 N^2} \langle x_1, y_1 \rangle^2 d\Omega(x_1)d\Omega(y_2).
\]

On the other hand, direct integration shows that the function \( J \) is given by

\[
J(z, w) = \left(\frac{2\pi}{\beta N}\right)^N N \prod_{i=1}^{N} \frac{1}{\sqrt{(z - \lambda_i)(w - \lambda_i)}} \left( \sum_{i=1}^{N} \frac{1}{\beta^2 N^2(z - \lambda_i)(w - \lambda_i)} \right).
\]

Taking the inverse Laplace transform, we obtain

\[
\frac{2^N}{(\beta N)^N} r^{\frac{N-1}{2}-1} \int_0^\infty \int_0^\infty e^{-2r}e^{-us}J(z, w)dzdw
= \frac{1}{(2\pi i)^2} \gamma_{-i\infty}^{\gamma_{+i\infty}} e^{s} e^{-us} J(z, w) dzdw
= \left(\frac{2\pi}{\beta N}\right)^N \frac{1}{(2\pi i)^2} \gamma_{-i\infty}^{\gamma_{+i\infty}} e^{s} e^{-us} \prod_{i=1}^{N} \frac{1}{\sqrt{(z - \lambda_i)(w - \lambda_i)}} \left( \sum_{i=1}^{N} \frac{1}{\beta^2 N^2(z - \lambda_i)(w - \lambda_i)} \right) dzdw
\]

where \( \gamma \) is any real number satisfying \( \gamma > \lambda_1 \). Recalling that

\[
|S^{N-1}| = \frac{2\pi^{N/2}}{\Gamma(N/2)}
\]

and letting \( r = s = \frac{\beta}{2} N \), we obtain:

\[
\frac{1}{|S^{N-1}|^2} \int_{(S^{N-1})^2} e^{N\frac{\beta}{2} \langle x_1, Mx_1 \rangle + N\frac{\beta}{2} \langle y_1, My_1 \rangle} \langle x_1, y_1 \rangle^2 d\Omega(x_1)d\Omega(y_2)
= \frac{2^{N-2}\Gamma(N/2)^2}{(2\pi i)^2(\beta N)^{N-2}} \gamma_{-i\infty}^{\gamma_{+i\infty}} e^{N\frac{\beta}{2} (z+w)} \prod_{i=1}^{N} \frac{1}{\sqrt{(z - \lambda_i)(w - \lambda_i)}} \left( \sum_{i=1}^{N} \frac{1}{\beta^2 N^2(z - \lambda_i)(w - \lambda_i)} \right) dzdw.
\]

Combining (4.12) and (4.5), we obtain (4.2).

\[\square\]

## 5 Steepest descent analysis

We proceed to the asymptotic evaluation of the integrals in (4.2). As in the previous section, we will continue to denote the eigenvalues of \( M \) by

\[
\lambda_i := \lambda_i(M).
\]

As was already noticed in [3], in the low temperature regime, the dominant contribution to the integrals comes from an \( O(N^{-1}) \) neighborhood of the saddle point \( \gamma \) which is itself distance \( N^{-1} \) from the largest eigenvalue of \( M \). The presence of the branch point due to \( \lambda_1 \) close to the saddle makes a steepest descent analysis via a direct expansion of the function \( G(z) \) untenable.

Compared to the computation in [3] and subsequent works, we must evaluate the numerator and denominator in (4.2) with greater precision. For the main result of [3], for example, it was sufficient to show that

\[
Z(\beta) = Ke^{\frac{N}{2}G(\gamma)},
\]
where $K$ satisfies $N^{-C} \leq K \leq N^{C}$ for any $C > 0$. The contribution of $K$ to the free energy is then $O(N^{-1} \log(N))$, automatically of lower order than the dominant Tracy-Widom fluctuations which are of size $O(N^{-2/3})$.

In order to evaluate the overlap, it is necessary to determine the leading order term of $Z(\beta)$, not only up to multiplicative terms. Additionally, our computation involves a more precise localization of $\gamma$ than $|\gamma - \lambda_1| \leq N^{-1+\varepsilon}$.

We now give an overview of the saddle point analysis. The saddle point of the function $G(z)$ is distance of order $N^{-1}$ from $\lambda_1$. Instead of working directly with the steepest descent contours of the function $G(z)$, we will consider the dominant contribution of $G(z)$ near the saddle which is, up to additive constants,

$$(\beta - 1)z - \frac{1}{N} \log(z - \lambda_1).$$

This function is much simpler than $G(z)$, as it involves only the eigenvalue $\lambda_1$. The contributions from other eigenvalues are replaced by their deterministic leading order term using Theorem 3.3.

The additional key input here is Lemma 3.4 which ensures that the eigenvalues $\{\lambda_j\}_{j=2}^N$ are an order of magnitude further from $\lambda_1$ than the distance between the saddle and $\lambda_1$. This allows for the localization of the function $G(z)$ near its saddle, despite the presence of the branch point due to the logarithmic singularity at $z = \lambda_1$.

The saddle point of the function (5.2) is clearly,

$$\gamma := \lambda_1 + \frac{c_\beta}{N},$$

where

$$c_\beta := \frac{1}{\beta - 1}. \quad (5.4)$$

The advantage afforded by working with the approximation (5.2) is that the behavior of the steepest descent contours of this function are relatively explicit. For the contours, we make the ansatz $z = \gamma + E + i\eta(E)$, for $E \leq 0$. Setting the imaginary part of (5.2) to zero gives a parameterization of the approximate steepest descent contour we will use. We determine properties of this parameterization in Lemma 5.1. In Lemma 5.4 we analyze the behavior of $G(z)$ along this approximate steepest descent contour.

**Lemma 5.1.** For $z \in \mathbb{C}\setminus(-\infty, 0]$, let $-\pi < \arg(z) \leq \pi$ be the standard determination of the argument. For $E < 0$ the equation

$$\eta(\beta - 1) = \frac{1}{N} \arg(E + i\eta + c_\beta/N) \quad (5.5)$$

has a unique strictly positive solution which we denote $\eta(E)$. Furthermore, there is a constant $c_1 > 0$ so that if $0 \geq E \geq -c_1/N$,

$$N\eta(E) = \sqrt{3c_\beta|NE|(1 + O(|NE|))}. \quad (5.6)$$

For any $c_2 > 0$ there is a $c_3 > 0$ depending on $c_2 > 0$ so that if $E \leq -\frac{c_2}{N}$, we have

$$\frac{c_3}{N} \leq \eta \leq \frac{\pi}{N(\beta - 1)}. \quad (5.7)$$

Before proceeding to the proof of Lemma 5.1, we note that if $f$ is analytic with real and imaginary parts denoted by

$$f(x + iy) = u(x, y) + iv(x, y), \quad (5.8)$$

then, with $\partial_z = (\partial_x - i\partial_y)/2$, the Cauchy-Riemann equations imply

$$f'(z) = u_x - iv_y = v_y + iv_x \quad (5.9)$$

and so

$$\partial_E \text{Re}[f] = \text{Re}[f'], \quad \partial_y \text{Re}[f] = -\text{Im}[f'], \quad \partial_E \text{Im}[f] = \text{Im}[f'], \quad \partial_y \text{Im}[f] = \text{Re}[f'], \quad (5.10)$$
using the notation $z = E + i\eta$. We will denote by $\log(z)$ denote the principal determination of the logarithm so that $\text{Im}[\log(z)] = \arg(z)$.

**Proof of Lemma 5.1.** For uniqueness we note that if $E < -c_\beta/N$, then the left side of (5.5) is increasing, whereas the right side is decreasing. If $E > -c_\beta/N$ we calculate the derivative of the right side,

$$\frac{\partial \eta}{\partial E} \left( \frac{1}{N} \arg(E + i\eta + c_\beta/N) \right) = \frac{1}{N} \left( \frac{E + c_\beta/N}{N(E + c_\beta/N)^2 + \eta^2} \right).$$

(5.11)

This is a decreasing function of $\eta$, and so the right side of (5.5) is a concave function of $\eta$. Its derivative at $\eta = 0$ is strictly greater than $(\beta - 1)$, so we get the uniqueness as the left side is a linear function with slope $(\beta - 1)$. Differentiating the equation (5.5), we find using (5.10)

$$\frac{d\eta}{dE} \left( \beta - 1 - \frac{1}{N(E + c_\beta/N)^2 + \eta^2} \right) = \frac{1}{N} \frac{-\eta}{(E + c_\beta/N)^2 + \eta^2}.$$ 

(5.12)

Note that the second factor on the left is positive (for $E > -c_\beta/N$ it is the difference of the slopes of the tangent lines of the functions on either side of (5.5) at the point $\eta(E)$), so

$$\frac{d\eta}{dE} \leq 0,$$ 

(5.13)

and the lower bound of (5.7) will follow once we establish (5.6). The upper bound is immediate.

Expanding this function in a power series around $c_\beta/N$, we have for some $c > 0$ and $|z| \leq c/N$,

$$\log(z + c_\beta/N) - \log(c_\beta/N) = z - \frac{2N^2}{c_\beta} + \frac{3N^3}{6c_\beta^2} f(z)$$ 

where $f(z)$ is an analytic function in the disc $|z| \leq c/N$, obeying the estimates

$$|\text{Im}[f(z)]| \leq C N |\text{Im}[z]|, \quad |f(z)| \leq C.$$ 

(5.15)

These estimates follow from the fact that all the coefficients in the power series expansion of the logarithm are real. The imaginary part of log is the argument function appearing on the right side of (5.5). Taking imaginary parts on both sides of (5.14) and using (5.5), we find (denoting $\eta = \eta(E)$ for brevity)

$$0 = -E\eta \frac{N^2}{c_\beta} + E\eta \frac{N^3}{3c_\beta^2} - \eta^2 \frac{N^3}{3c_\beta^2} + O(N^4E^3 + N^4E\eta^3)$$

(5.16)

Dividing by $N\eta$ and using the fact that $N\eta \leq (\beta - 1)^{-1}\pi$ for all $E$ we obtain,

$$N\eta \leq C \sqrt{N}|E|$$

(5.17)

after possibly making $c > 0$ smaller. Using this to estimate the higher order terms in (5.16) yields (5.6) after solving (5.16) for $N\eta$ as a function of $NE$.

We also require the following elementary lemma.

**Lemma 5.2.** Let $\eta(E)$ be as above. For any $c_1 > 0$, there is a $c_2 > 0$ so that if $E \leq -c_1/N$ then,

$$\beta - 1 - \frac{1}{N(E + c_\beta/N)^2 + \eta(E)^2} \geq c_2.$$ 

(5.18)

**Proof.** The case $E + c_\beta/N \leq 0$ is trivial so we may assume $E \geq -c_\beta/N$. Recall that $\eta(E)$ is the unique positive solution to $(N\eta)c_\beta = \text{arctan}(\eta/E(c_\beta + N))$. If $NE/c_\beta + 1 \leq \frac{1}{10}$, then $\text{arctan}(1/NE/c_\beta + 1) \geq \pi/3 > 1$ and so for such $E$, we have $N\eta(E)/c_\beta \geq 1$ (the function arctan is concave in $\eta$ so the solution must occur at larger $\eta$). We have,

$$\frac{1}{c_\beta} - \frac{1}{N(E + c_\beta/N)^2 + (\eta(E))^2} = \frac{(N\eta(E))^2 + (NE + c_\beta)(NE)}{c_\beta((NE + c_\beta)^2 + (N\eta(E))^2)}$$ 

(5.19)
The denominator is bounded above, and the numerator is bounded below by \(c_0^2/2\) for \(NE/c_\beta + 1 \leq \frac{1}{N}\), by our lower bound for \(\eta\) that we just derived. It remains to consider the case where \(NE/c_\beta + 1 \in (c_3, 1 - c_3)\) for fixed \(c_3 > 0\). Consider the concave function,

\[
f(\eta) = \frac{1}{N} \arctan(\eta/(E + c_\beta/N)) - \eta/c_\beta.
\]  
(5.20)

This function has zeros at \(\eta = 0\) and \(\eta = \eta(E)\) and is strictly positive in between these points. We need to prove that there is a constant \(c'\) depending on \(c_3\) so that \(f'(\eta(E)) < -c'\). By direct calculation, it has a local maximum in between these two points at

\[
N\eta^* = \sqrt{(NE + c_\beta)(-NE)}.
\]  
(5.21)

From the formula for \(f'\) and \(f''\) we see that \(f'(0) > c\) for a \(c > 0\) and that \(0 \geq f''(\eta) \geq -CN\) for \(\eta \geq 0\) and constants \(c, C\) depending on \(c_3\). Since \(f''(\eta^*) = 0\) it follows from the bound on \(f''\) (or the explicit form (5.21)) that \(\eta^* \geq c/N\) for some new \(c > 0\) depending on \(c_3\), and then that \(f(\eta^*) \geq c'/N\) for some \(c' > 0\) (the latter estimate being a consequence of the fact that \(f'(0) > c\) and the bound on \(f''\)). Since \(f'(\eta)\) is bounded, we then see that \(\eta(E) - \eta^* \geq c''/N\) for some \(c'' > 0\), from the lower bound on \(f(\eta^*)\) and the definition \(f(\eta(E)) = 0\). Since \(C'/N \geq \eta(E) \geq \eta^* \geq c/N\) we see from the explicit form of \(f''\) that \(f''(\eta) \leq -Nc''/\eta(E) \geq \eta^* \geq c''/N\) for some \(c'' > 0\). This then implies that \(f'(\eta(E)) \leq -c_4\) for some \(c_4 > 0\), which is what we needed to prove.

We now define the event \(F_{\delta, \epsilon_1}\) of Theorem 2.1.

**Definition 5.3.** Let \(\frac{1}{3} > \delta > 0\) and \(\epsilon_1 > 0\). Let \(F_{\delta, \epsilon_1}\) be the following event:

\[
F_{\delta, \epsilon_1} = \left\{ N^{2/3}(\lambda_1 - \lambda_2) > N^{-\delta} \right\} \cap \left\{ |\lambda_i - \gamma_i| \leq \frac{N^{\epsilon_1/10}}{\min \{ i^{1/3}, (N + 1 - i)^{1/3} \} N^{2/3}, i = 1, \ldots, N \right\},
\]  
(5.22)

where \(\gamma_i\) are the classical eigenvalue locations defined in (3.2).

As stated above, this is the event in the statement of Theorem 2.1. By Theorem 3.3 and Lemma 3.4 we have that

\[
\mathbb{P}[F_{\delta, \epsilon_1}] \geq 1 - N^{-\delta + \epsilon'}
\]  
(5.23)

for any \(\epsilon' > 0\) and \(N\) large enough. Fix a sufficiently small \(\kappa > 0\). In particular, we take

\[
\kappa < \frac{1/3 - \delta}{10}
\]  
(5.24)

Define the contours

\[
\Gamma_1 := \{ E + \text{i}\eta(E) : 0 \geq E \geq -N^{-1+\kappa} \}
\]  
(5.25)

and

\[
\Gamma_2 := \{-N^{-1+\kappa} \pm \text{i}\eta : \eta \geq \eta(-N^{-1+\kappa}) \}
\]  
(5.26)

The contour \(\Gamma_1\) is a U-shaped contour symmetric along the negative real axis, and \(\Gamma_2\) are vertical lines at the ends of \(\Gamma_1\) going to \(\pm \text{i}\infty\).

**Lemma 5.4.** Assume that \(F_{\delta, \epsilon_1}\) holds with \(\epsilon_1\) sufficiently small. The following estimates hold. There is a \(c > 0\) so that

\[
\text{Re}[G(z + \gamma)] - \text{Re}[G(\gamma)] \leq -cn^{-1+\kappa}, \quad z = -n^{-1+\kappa} + \text{i}\eta, \quad \eta \geq 0.
\]  
(5.27)

For any \(c_1 > 0\) there are \(c_2 > 0\) and \(C_1\) so that the following holds for \(z \in \Gamma_1\) and large enough \(N\):

\[
\text{Re}[G(z + \gamma)] - \text{Re}[G(\gamma)] \leq 1_{\{|E| \leq c_1/N\}} |E| N^{-1/3+\epsilon_1+\delta} - 1_{\{|E| \geq c_1/N\}} \left( |E| - c_1/N \right) c_2 - C_1 N^{-1-1/3+\epsilon_1+\delta},
\]  
(5.28)

Finally, there is the following estimate for \(-N^{-1+\kappa} \leq E \leq 0\) and \(\eta \geq 10\),

\[
\text{Re}[G(z + \gamma)] - \text{Re}[G(\gamma)] \leq -\frac{1}{3} \log(1 + \eta).
\]  
(5.29)
Proof. We calculate some derivatives of $\text{Re}[G]$ along the contours, freely using (5.10). In this proof, $z$ will be restricted to lie on the various contours and so we will generally denote $z = E + i\eta(E)$. First, along $\Gamma_1$,

$$
\frac{d}{dE}\text{Re}G(\gamma + E + i\eta(E)) = \beta + \text{Re}[m_N(\gamma + z)] - \text{Im}[m_N(\gamma + z)]\frac{d\eta}{dE}
$$

$$
\geq \beta + \text{Re}[m_N(\gamma + z)],
$$

(5.30)

where we used that $d\eta/dE$ is negative. We write,

$$
\text{Re}[m_N(\gamma + z)] = \frac{1}{N} \frac{c_\beta/N + E}{(c_\beta/N + E)^2 + \eta^2} - \frac{1}{N} \sum_{j=2}^{N} \frac{E + \gamma - \lambda_j}{(E + \gamma - \lambda_j)^2 + \eta^2}.
$$

(5.31)

We need to estimate the second term. We write,

$$
\frac{1}{N} \sum_{j=2}^{N} \frac{E + \gamma - \lambda_j}{(E + \gamma - \lambda_j)^2 + \eta^2} = \frac{1}{N} \sum_{j=2}^{N} \frac{E + \gamma - \lambda_j}{(E + \gamma - \lambda_j)^2 + \eta^2} + \frac{1}{N} \sum_{j=N+1}^{N} \frac{E + \gamma - \lambda_j}{(E + \gamma - \lambda_j)^2 + \eta^2}.
$$

(5.32)

From the level repulsion assumption and choice of $\kappa$,

$$
|E + \gamma - \lambda_j| \geq N^{-2/3-\delta} - N^{1+\kappa} \geq c N^{-2/3-\delta},
$$

for all $j \geq 2$, and so

$$
\left| \frac{1}{N} \sum_{j=2}^{N} \frac{E + \gamma - \lambda_j}{(E + \gamma - \lambda_j)^2 + \eta^2} \right| \leq C N^{\varepsilon_1 + \delta - 1/3}.
$$

(5.34)

For the second term in (5.32), rigidity gives

$$
\left| \frac{1}{N} \sum_{j=N+1}^{N} \frac{E + \gamma - \lambda_j}{(E + \gamma - \lambda_j)^2 + \eta^2} - \text{Re} \left[ \int_{2}^{\gamma+N} \frac{\rho_{\text{sc}}(x)dx}{(\gamma + z) - x} \right] \right| \leq C N^{-1/3+\varepsilon_1/10}.
$$

(5.35)

Now, since $|\gamma + z - 2| \leq C N^{\varepsilon_1/10-2/3},$

$$
\left| \int_{-2}^{\gamma+N} \frac{\rho_{\text{sc}}(x)}{x - (\gamma + z)} - \int_{-2}^{\gamma+N} \frac{\rho_{\text{sc}}(x)}{x - 2} \right| \leq N^{\varepsilon_1/10-2/3} C \int_{N^{-2/3}}^{2} \frac{\sqrt{x}}{x^2} \leq C N^{\varepsilon_1 - 1/3}.
$$

(5.36)

Finally,

$$
\left| \int_{-2}^{\gamma+N} \frac{\rho_{\text{sc}}(x)}{x - 2} dx + 1 \right| = \left| \int_{-2}^{\gamma+N} \frac{\rho_{\text{sc}}(x)}{x - 2} dx - m_{\text{sc}}(2) \right| \leq N^{\varepsilon_1 - 1/3}.
$$

(5.37)

Therefore,

$$
\beta + \text{Re}[m_N(\gamma + z)] \geq \left( \beta - 1 - \frac{1}{N} \frac{c_\beta/N + E}{(c_\beta/N + E)^2 + \eta^2} \right) - C N^{\varepsilon_1 + \delta - 1/3}.
$$

(5.38)

As observed in the proof of Lemma 5.1, the first term on the RHS (in the brackets) is positive. If $E \leq -c_1/N$ for a $c_1 > 0$ then by Lemma 5.2 we conclude that there is a $c_2 > 0$ depending on $c_1$ so that

$$
\beta + \text{Re}[m_N(\gamma + z)] \geq c_2.
$$

(5.39)

We have therefore proven that for $-c_1/N \leq E \leq 0,$

$$
\frac{d}{dE}\text{Re}[G(E + i\eta(E) + c_\beta/N)] \geq -C N^{\varepsilon_1 + \delta - 1/3}
$$

(5.40)

and for $-N^{-1+\kappa} \leq E \leq -c_1/N,$

$$
\frac{d}{dE}\text{Re}[G(E + i\eta(E) + c_\beta/N)] > c_2/2.
$$

(5.41)
The estimate (5.28) follows from the previous two estimates and integration. We consider \( z \) of the form \( z = -N^{-1+\kappa} + i\eta \) with \( \eta \) varying over \( \mathbb{R} \) in order to prove (5.27). We consider the behavior of \( \text{Re}[G(z)] \) as \( \eta \) varies. We calculate,

\[
\partial_{\eta} \text{Re}[G(\gamma + z)] = -\text{Im}[m_N(\gamma + z)].
\] (5.42)

This is decreasing, so we immediately get, using (5.28), the estimate (5.27) in the region \( \eta \geq \eta(-N^{-1+\kappa}) \). For smaller \( \eta \leq \eta(-N^{-1+\kappa}) \), note that

\[
|\gamma - N^{-1+\kappa} - \lambda_1| \geq cN^{-1+\kappa}, \quad |\gamma - N^{-1+\kappa} - \lambda_2| \geq cN^{-2/3-\delta} \geq cN^{-1+\kappa}.
\] (5.43)

Hence,

\[
\text{Im}[m_N] \leq N^{\varepsilon_1} \frac{N\eta}{N^{2\kappa}} + \frac{1}{N} \sum_{j \geq N^{\varepsilon_1}} |\eta + j - \lambda_j|^2 \leq N^{\varepsilon_1} \frac{N\eta}{N^{2\kappa}} + \eta N^{1/3}.
\] (5.44)

Since \( \eta(-N^{-1+\kappa}) \leq C/N \), we get (5.27) for the rest of the possible values of \( \eta \), integrating \( \text{Re}[G] \) from \( z = -N^{-1+\kappa} + i\eta(N^{-1+\kappa}) \) to a smaller \( \eta \leq \eta(N^{-1+\kappa}) \), using (5.42) and the above estimate on the derivative.

Finally, we turn to (5.29). We have,

\[
\text{Re}[G(\gamma + z)] - \text{Re}[G(\gamma)] = \beta E - \frac{1}{N} \sum_j \log \left| 1 + \frac{z}{\gamma - \lambda_j} \right|.
\] (5.45)

For \( j \leq N^{\varepsilon_1} \),

\[
\left| 1 + \frac{z}{\gamma - \lambda_j} \right| \geq \eta N^{1/3}
\] (5.46)

and for the rest of \( j \),

\[
\left| 1 + \frac{z}{\gamma - \lambda_j} \right| \geq \frac{\eta}{3}
\] (5.47)

This yields (5.29) (recall \( \eta \geq 10 \) for this estimate).

Now introduce the contour

\[
\hat{\Gamma} = \Gamma_1 \cup \Gamma_3,
\] (5.48)

where

\[
\Gamma_3 = \{ -N^{-1+\kappa} + i\eta : 0 \leq \eta \leq \eta(-N^{-1+\kappa}) \}.
\] (5.49)

Recall that \( \Gamma_1 \) is a U-shaped contour symmetric about the negative real axis. The contour \( \Gamma_3 \) connects the ends of the \( U \) to the real axis.

The following lemma shows that we can deform the contours in (4.2) into \( \hat{\Gamma} \).

**Lemma 5.5.** Suppose the event \( \mathcal{F}_{\delta,\varepsilon_1} \) holds with \( \varepsilon_1 \) sufficiently small. Then the following estimates hold. There is a \( c > 0 \) so that

\[
\int_{\gamma-i\infty}^{\gamma+i\infty} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{N}{2}(G(z)+G(w)-2G(\gamma))} \left( \sum_{i=1}^{N} \frac{1}{\beta^2(z + \gamma - \lambda_i)(w + \gamma - \lambda_i)} \right) \ d\gamma \ d\omega = \int_{\hat{\Gamma} \times \hat{\Gamma}} e^{\frac{N}{2}(G(z)+G(w)-2G(\gamma))} \left( \sum_{i=1}^{N} \frac{1}{\beta^2(z + \gamma - \lambda_i)(w + \gamma - \lambda_i)} \right) + O\left(e^{-N^c}\right)
\] (5.50)

and

\[
\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{N}{2}(G(z)-G(\gamma))} = \int_{\hat{\Gamma}} e^{\frac{N}{2}(G(z)-G(\gamma))} + O\left(e^{-N^c}\right).
\] (5.51)
Proof. By analyticity and the absolute convergence guaranteed by (5.29), all of the contours can be moved from the vertical lines appearing in (4.2) to the contour $\Gamma_1 \cup \Gamma_2$. It just remains to replace $\Gamma_2$ by $\Gamma_3$. The replacement for the integral appearing on the LHS of (5.51) is immediate from (5.27) and (5.29), and so (5.51) follows.

For (5.50) we also have to deal with cross terms (e.g., the integral over $\Gamma_2$ or $\Gamma_3$ in the $z$ variable times the integral over $\Gamma_1$ in the $w$ variable) and the extra terms in the integrand. Note that along all of the contours under consideration we always have

$$|z + \gamma - \lambda_j| \geq N^{-1}. \tag{5.52}$$

Note furthermore that for $z \in \Gamma_1$, the estimate (5.28) gives

$$N(\text{Re}G(z) - \text{Re}G(\gamma)) \leq C, \tag{5.53}$$

and that the arc length of $\Gamma_1$ satisfies

$$|\Gamma_1| \leq CN^{-1+\kappa} \tag{5.54}$$
as $\eta(E)$ is monotonic. These observations together with (5.29) and (5.27) yield (5.50).

In the following lemma we Taylor expand $G(z)$ (or at least all of the terms appearing in its definition except the one with $\lambda_1$) around the saddle $\gamma$, arriving at a form of the integrands which we will be able to calculate. We first define a few functions which naturally appear in the Taylor expansion. Let,

$$\tilde{m}_N(w) := \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j - w} \tag{5.55}$$

and

$$g(z) := (\beta + \tilde{m}_N(\gamma))z - \frac{1}{N} \log(1 + Nz/c_\beta) \tag{5.56}$$

Lemma 5.6. The following holds on the event $\mathcal{F}_{\delta,\varepsilon_1}$ for sufficiently small $\varepsilon_1 > 0$. First, we have

$$\int_{\Gamma} e^{\frac{N}{2}(G(z+\gamma) - G(\gamma))} dz = \int_{\Gamma} e^{\frac{N}{2}g(z)} \left(1 + Nz^2\frac{\tilde{m}_N'(\gamma)}{4}\right) dz + \mathcal{O}\left(N^{-2+4\kappa+3\delta+\varepsilon_1}\right) \tag{5.57}$$

Second,

$$\int_{\Gamma \times \Gamma} e^{\frac{N}{2}(G(z+\gamma) + G(w+\gamma) - 2G(\gamma))} \frac{1}{N^2(z + c_\beta/N)(w + c_\beta/N)} dz dw$$

$$= \int_{\Gamma \times \Gamma} e^{\frac{N}{2}g(z) + g(w)} \left(1 + N\frac{z^2\tilde{m}_N'(\gamma)}{4}\right) \left(1 + \frac{w^2\tilde{m}_N'(\gamma)}{4}\right) \frac{1}{N^2(z + c_\beta/N)(w + c_\beta/N)} dz dw$$

$$+ \mathcal{O}\left(\frac{N^{5\kappa+\varepsilon_1+3\delta}}{N^3}\right). \tag{5.58}$$

Finally,

$$\int_{\Gamma \times \Gamma} e^{\frac{N}{2}(G(z+\gamma) + G(w+\gamma) - 2G(\gamma))} \sum_{j=2}^{N} \frac{1}{N^2(z + \gamma - \lambda_j)(w + \gamma - \lambda_j)} dz dw$$

$$= \frac{1}{N^2} \sum_{j=2}^{N} \frac{1}{(\lambda_1 - \lambda_j)^2} \left(\int_{\Gamma} e^{\frac{N}{2}g(z)} \left(1 + N\frac{z^2\tilde{m}_N'(\gamma)}{4}\right) dz\right)^2 + \mathcal{O}\left(\frac{N^{5\kappa+\varepsilon_1+3\delta}}{N^3}\right). \tag{5.59}$$

Proof. We write

$$G(z + \gamma) - G(\gamma) = \beta z - \frac{1}{N} \log\left(1 + \frac{Nz}{c_\beta}\right)$$

$$- \frac{1}{N} \sum_{j=2}^{N} \left(\log(z + \gamma - \lambda_j) - \log(\gamma - \lambda_j)\right) \tag{5.60}$$
We Taylor expand the second term. For $z \in \hat{\Gamma}$,
\[
\frac{1}{N} \sum_{j \geq 2} \log(z + \gamma - \lambda_j) - \log(\gamma - \lambda_j) = -z \tilde{m}_N(\gamma) - \frac{z^2}{2} \tilde{m}_N'(\gamma) + \mathcal{O}\left(\frac{\lambda^{3\kappa+\varepsilon_1+3\delta}}{N^2}\right). \tag{5.61}
\]
From the fact that $z^2 \tilde{m}_N'(\gamma) = \mathcal{O}(N^{-5/3+5\kappa+2\delta+\varepsilon_1})$ for $z \in \hat{\Gamma}$ we first conclude that
\[
\text{Re}[g(z)] \leq \frac{C}{N}, \tag{5.62}
\]
where we used Lemma 5.4 (i.e., the corresponding estimate for $\text{Re}[G(z+\gamma)] - \text{Re}[G(\gamma)]$ and the above two estimates which relate this quantity to $g(z)$). Then, for $z \in \hat{\Gamma}$,
\[
e^{-\frac{N}{2}(G(z+\gamma)-G(\gamma))} = e^{\frac{N}{2}g(z)} \left(1 + N\frac{z^2}{4} \tilde{m}_N'(\gamma)\right) + \mathcal{O}\left(\frac{\lambda^{3\kappa+\varepsilon_1+3\delta}}{N}\right). \tag{5.63}
\]
Note that
\[
\left|\frac{1}{N^2} \sum_{j=1}^{N} \frac{1}{(z - \lambda_j)(w - \lambda_j)}\right| \leq C, \tag{5.64}
\]
and
\[
\left|\frac{1}{N^2} \sum_{j=2}^{N} \frac{1}{(\lambda_j - z - \gamma)(\lambda_j - w - \gamma)} - \frac{1}{N^2} \sum_{j=2}^{N} \frac{1}{(\lambda_j - \lambda_1)^2}\right| \leq \frac{C \lambda^{\kappa+3\delta+\varepsilon_1}}{N}. \tag{5.65}
\]
The first and second estimates of the lemma follows from (5.63) and the fact that $|\hat{\Gamma}| \leq C N^{1+\kappa}$. For the third estimate, one first uses (5.64) and (5.63) to arrive at an integral in terms of $g(z), g(w)$ and the quantity (inside the absolute value) on the LHS of (5.64). The error is $\mathcal{O}(N^{5\kappa+\varepsilon_1+3\delta-3})$. The next replacement uses (5.65), and one arrives at the final estimate of the lemma.

We now rescale and shift the contour of integration to lie along the real axis. Let $\Gamma_r$ be the following keyhole contour around the point $-c_\beta$, for $r < c_\beta/10$:
\[
\Gamma_r := \{ E \pm i0 : E < -c_\beta - r \} \cup \{ z : |z - c_\beta| = r \}. \tag{5.66}
\]

**Lemma 5.7.** On the event $\mathcal{F}_{\delta,\varepsilon_1}$ we have, for sufficiently small $\varepsilon_1 > 0$, the following estimates for some $c > 0$.
\[
\int_{\Gamma_r} e^{\frac{N}{2}g(z)} \left(1 + N\frac{z^2}{4} \tilde{m}_N'(\gamma)\right) \frac{dz}{N} = \frac{1}{N} \int_{\Gamma_r} e^{(\beta + \tilde{m}_N(\gamma))u/2} \left(1 + \frac{u^2 \tilde{m}_N'(\gamma)}{4N}\right) du + \mathcal{O}(e^{-N^c}) \tag{5.67}
\]
and
\[
\int_{\Gamma_r} e^{\frac{N}{2}g(z)} \left(1 + N\frac{z^2}{4} \tilde{m}_N'(\gamma)\right) \frac{1}{N(z + c_\beta/N)} dz = \frac{1}{N} \int_{\Gamma_r} e^{(\beta + \tilde{m}_N(\gamma))u/2} \left(1 + \frac{u^2 \tilde{m}_N'(\gamma)}{4N}\right) \frac{du}{u + c_\beta} + \mathcal{O}(e^{-N^c}). \tag{5.68}
\]

**Proof.** First we make the substitution $u = z/N$. As the integrand is analytic on $\mathbb{C}\backslash \{ E \leq -c_\beta \}$ we see that all of the contours may be shifted from $N\hat{\Gamma}$ to $\Gamma_r \backslash \{ E \pm i0 : E \leq -N^\kappa \}$. The rest of $\Gamma_r$ may be added to the integral as $\beta + \tilde{m}_N(\gamma) > c_1$ for some $c_1 > 0$, at only an error exponentially small in $N$.

We collect some explicit integrals in the next lemma.
Lemma 5.8. Let $a, b > 0$, and $\Gamma_{r,b}$ be a keyhole contour around $-b$ as above. Then,

$$\int_{\Gamma_{r,b}} \frac{e^{az}}{\sqrt{z+b}} = 2ie^{-ab} \sqrt{\pi} \tag{5.69}$$

$$\int_{\Gamma_{r,b}} e^{az} \sqrt{z+b} = -ie^{-ab} a^{-3/2} \sqrt{\pi} \tag{5.70}$$

$$\int_{\Gamma_{r,b}} e^{az} (z+b)^{3/2} = \frac{3}{2} ie^{-ab} a^{-5/2} \sqrt{\pi} \tag{5.71}$$

$$\int_{\Gamma_{r,b}} \frac{e^{az}}{\sqrt{z+b}} z^2 = \sqrt{\pi} e^{-ab} 4u \sqrt{\pi} \tag{5.72}$$

$$\int_{\Gamma_{r,b}} \frac{e^{az}}{(z+b)^{3/2}} = \sqrt{\pi} e^{-ab} \beta^2 \tag{5.73}$$

$$\int_{\Gamma_{r,b}} e^{az} (z+b)^{3/2} z^2 = \sqrt{\pi} e^{-ab} \beta^2 \left( -\frac{1}{a} - 4b + 4b^2 a \right) \tag{5.74}$$

All of the above calculations can be done by considering the contributions from the integral along the real axis and the circle around $z = b$ as $r \to 0$. In the cases where these contributions are diverging, one treats the circular integral by Taylor expansion (i.e., expanding the exponential around $z = b$), and integrates by parts the integral along the real axis. One finds that the diverging quantities cancel, and is left with a real integral which can be calculated explicitly. Alternatively, starting from (5.69), all of the remaining formulas may be derived via differentiating both sides wrt $a$ or $b$, or straightforward algebraic manipulations.

We finally arrive at the following, from which Theorem 2.1 follows.

Theorem 5.9. On the event $\mathcal{F}_{\delta,\varepsilon_1}$ we have with sufficiently small $\varepsilon_1$ that,

$$\langle R_{12}^2 \rangle = (1 - \beta^{-1})^2 + 2\frac{\beta-1}{\beta^2} (\bar{m}_N(\lambda_1) + 1) - \frac{\bar{m}_N(\lambda_1) + 1}{\beta^2} + O \left( \frac{N^{3\delta+10\varepsilon_1}}{N} \right). \tag{5.75}$$

Proof. We first use Lemma 4.1 to arrive at the formula (4.2) for the overlap. The results of the present section are used to analyze the contour integrals appearing in the numerator and denominator of (4.2). From (5.50), (5.58), (5.59) and Lemma 5.7 we arrive at the following expression for the numerator:

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{x}{2} (G(z)+G(w))} \left( \sum_{j=1}^{N} \frac{1}{\beta^2 N^2 (z - \lambda_j)(w - \lambda_j)} \right) dz dw$$

$$= \left( \frac{1}{N \beta} \int_{\Gamma_r} e^{(\beta + \bar{m}_N(\gamma))u/2} \left( 1 + \frac{u^2 \bar{m}_N(\gamma)}{4N} \right) \frac{du}{\sqrt{1 + u/c_\beta}} \right)^2 + O(N^{5\varepsilon_1+3\delta-3}) \tag{5.76}$$

for $\kappa$ as above. For the integral in the denominator we use (5.51), (5.57) and Lemma 5.7 to find,

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{x}{2} G(z)} dz = \frac{1}{N} \int_{\Gamma_r} e^{(\beta + \bar{m}_N(\gamma))u/2} \left( 1 + \frac{u^2 \bar{m}_N(\gamma)}{4N} \right) du + O(N^{3\varepsilon_1+3\delta-2}). \tag{5.77}$$

We now use Lemma 5.8 with $a = \frac{1}{2} (\beta + \bar{m}_N(\gamma))$ and $b = c_\beta$. For the numerator, the two terms of (5.76) equal

$$\left( \frac{2 \sqrt{\beta} e^{-a \beta \gamma} i \sqrt{\pi}}{N \beta} \right)^2 \left( 2a^2 - \frac{\bar{m}_N(\gamma)}{N} + \frac{\bar{m}_N(\lambda_1)}{N} \right) + O(N^{5\varepsilon_1+3\delta-3}). \tag{5.78}$$
whereas the denominator equals
\[
\left( \frac{2 \sqrt{\beta} e^{-ac_\beta i \sqrt{\pi} N}}{N} \right)^2 \left( 1 + \frac{3m_N'(\gamma)}{4Na^2} \right) + \mathcal{O}(N^{5\varepsilon_1+3\delta-3}).
\]  
(5.79)

Note we used \(2a = \beta - 1 + \mathcal{O}(N^{-1/3+\delta+\varepsilon_1/5})\). We get the claim from these two calculations as well as,
\[
m_N(\gamma) = \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j - \lambda_1 - c_\beta/N} = \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j - \lambda_1} + \frac{c_\beta}{N^2} \sum_{j=2}^{N} \frac{1}{(\lambda_j - \lambda_1)^2} + \mathcal{O}\left(N^{\varepsilon_1+3\delta-1}\right). \quad (5.80)
\]

6 Existence of limit

In this section we consider the limit of the random variables
\[
N^{1/3}(\tilde{m}_N(\lambda_1) + 1) = N^{1/3} \left( \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_1 - \lambda_j - 1} \right)
\]
as \(N \to \infty\). The limit will be characterized in terms of the Airy\(_1\) random point field. Our convention is so that if \(\lambda_1 \geq \lambda_2 \geq \ldots\) are the largest eigenvalues of the GOE, then for every finite \(k\),
\[
\{N^{2/3}(2 - \lambda_j)\}_{j=1}^{k} \to \{\chi_{j1}\}_{j=1}^{k},
\]
so that the ensemble \(\{\chi_j\}_{j=1}^{\infty}\) has finitely many particles located on the negative real line. We will prove the following theorem.

**Theorem 6.1.** Let \(\lambda_1 \geq \lambda_2 \geq \ldots\) denote the largest eigenvalues of the GOE, and \(\{\chi_j\}_{j=1}^{\infty}\) the Airy\(_1\) random point field. The sequences of random variables
\[
N^{1/3} \left( \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_1 - \lambda_j - 1} \right)
\]
converges in distribution to a random variable \(\Xi\) which is given by
\[
\Xi = \lim_{n \to \infty} \left( \sum_{j=2}^{n} \frac{1}{\chi_j - \chi_1} - \frac{1}{\pi} \int_{0}^{(\frac{3\pi n}{2})^{3/2}} \frac{dx}{\sqrt{x}} \right),
\]
where the limit on the RHS of (6.4) exists almost surely.

**Remark.** We do not determine whether or not the distribution of \(\Xi\) is non-trivial.

6.1 Preliminary estimates

We will need an estimate for the variance of the number of eigenvalues of the GOE in an interval as well as the corresponding estimate for the Airy\(_1\) random point field. We will deduce these from the corresponding results for the GUE and Airy\(_2\) random point field and the coupling of Forrester and Rains between the GUE and the GOE [15].

**Theorem 6.2** (Soshnikov [28]). Let \(\chi_{i(2)}^{(2)}\) be the particles of the Airy\(_2\) point process and let \(T > 0\). We have the following estimates for some \(C > 0\) and any \(T > 0\).
\[
\left| \mathbb{E} \left( \left\{ i : \chi_{i(2)}^{(2)} \leq T \right\} \right) - \frac{2}{3\pi} T^{3/2} \right| \leq C,
\]
and
\[
\left| \text{Var} \left( \left\{ i : \chi_{i(2)}^{(2)} \leq T \right\} \right) - \frac{3}{4\pi^2} \log T \right| \leq C.
\]
Remark. Soshnikov states the variance asymptotics (6.6) with the constant \( \frac{1}{\pi^2} \) instead of \( \frac{3}{4\pi^2} \). This appears to be due to a mistake in the computation of the quantity \( I_3(u) \) in [28, Lemma 5]. Note also that the factor 3/4 is consistent with the variance asymptotics for the counting function in the GUE, in (6.9) below. In any case, the value of the constant does not concern us as we will use only that the variance grows logarithmically in \( T \).

Theorem 6.3 (Gustavsson [17]). Let \( \{\mu_i\}_{i=1}^N \) be the eigenvalues of the GUE. Let \( \varepsilon > 0 \). There is a \( C > 0 \) so that the following holds. For any \( 0 \leq s \leq N^{2/3-\varepsilon} \), we have

\[
\left| \mathbb{E}\left[ \left| \{i : \mu_i \geq 2 - sN^{-2/3}\} \right| \right] - N \int_{2-sN^{-2/3}} \frac{\sqrt{4-E^2(dE)}}{2\pi} \right| \leq C
\]

(6.7)

and so for \( s \leq N^{4/15} \),

\[
\left| \mathbb{E}\left[ \left| \{i : \mu_i \geq 2 - sN^{-2/3}\} \right| \right] - \frac{2}{3\pi} s^{3/2} \right| \leq C.
\]

(6.8)

Furthermore, there is a \( C_1 > 0 \) so that if \( C_1 \leq s \leq N^{2/3-\varepsilon} \), then

\[
\left| \text{Var}\left( \left| \{i : \mu_i \geq 2 - sN^{-2/3}\} \right| \right) - \frac{3}{4\pi^2} \log(s) \right| \leq C(1 + \log \log(s))
\]

(6.9)

Remark. Gustavsson only claimed the result (6.9) in the case that \( s \to \infty \) as \( N \to \infty \) at any arbitrarily slow rate (his interest was in the case that the variance tends to infinity, a necessary condition for applying a theorem of Costin and Lebowitz [9]). Inspecting his proof yields the estimate (6.9) for fixed but large enough \( s \).

We need also the following result of Forrester and Rains [15, Theorem 5.2]. It is stated in terms of unscaled versions of the Gaussian ensembles which we now introduce. For each \( n \), consider the real symmetric matrix \( A^{(n)} \) whose upper triangular part are independent Gaussian variables with variance \( 1 + \delta_{ij} \). Furthermore, consider the complex Hermitian matrix \( B^{(n)} \) whose diagonal entries are independent real Gaussians with variance 1 and whose off-diagonal elements are independent complex Gaussians with independent real and imaginary part, each with variance \( 1/2 \). Then \( A^{(n)} \) and \( B^{(n)} \) are respectively the GOE and GUE scaled so that the extremal eigenvalues sit at \( \pm 2\sqrt{n} \). We will denote their eigenvalue point processes by \( \tilde{\text{GOE}}_n \) and \( \tilde{\text{GUE}}_n \), respectively.

Theorem 6.4 (Forrester, Rains, [15]). Let \( \tilde{\text{GOE}}_n \) and \( \tilde{\text{GUE}}_n \) be as above. Then,

\[
\tilde{\text{GUE}}_n \overset{d}{=} \text{Even}\left( \tilde{\text{GOE}}_n \cup \tilde{\text{GOE}}_{n+1} \right)
\]

(6.10)

where the RHS is the set formed by the second largest, fourth largest, sixth largest, etc. elements of \( \tilde{\text{GOE}}_n \cup \tilde{\text{GOE}}_{n+1} \), and \( \tilde{\text{GOE}}_n \) and \( \tilde{\text{GOE}}_{n+1} \) are independent.

From the above results we deduce the following. Similar arguments appeared in the work of O’Rourke [23].

Proposition 6.5. Let \( \{\lambda_i\}_{i=1}^N \) denote the eigenvalues of the GOE and let \( \varepsilon > 0 \). There is a \( C > 0 \) so that the following holds. For any \( 0 \leq s \leq N^{4/15} \), we have

\[
\left| \mathbb{E}\left[ \left| \{i : \lambda_i \geq 2 - sN^{-2/3}\} \right| \right] - \frac{2}{3\pi} s^{3/2} \right| \leq C.
\]

(6.11)

There is a \( C_1 > 0 \) so that if \( C_1 \leq s \leq N^{2/3-\varepsilon} \), then

\[
\text{Var}\left( \left| \{i : \lambda_i \geq 2 - sN^{-2/3}\} \right| \right) \leq C \log(s)
\]

(6.12)

Proof. For any \( n \) and \( s \) let \( X_n^{(1)}(s) \) denote the number of particles of \( \tilde{\text{GOE}}_n \) lying in \( [2\sqrt{N} - sN^{-1/6}, \infty) \), and couple all of these processes together so they are independent for every \( n \) (note that the interval involves the parameter \( N \) and is fixed for every \( n \); we consider only \( n = N \) and \( n = N + 1 \)).
Then, the random variable \( X_N^{(1)}(s) \) has the same distribution as \( \{i : \lambda_i \geq 2 - sN^{-2/3}\} \). Let \( X_N^{(2)}(s) \) denote the number of particles of \( \tilde{\text{GUE}}N \) lying in \( [2\sqrt{N} - sN^{-1/6}, \infty) \). The coupling of Theorem 6.4 implies that there is a random variable \( Y \) and a bounded random variable \( Z \) so that,

\[
X_N^{(2)}(s) \equiv Y, \quad Y - Z = \frac{1}{2} \left( X_N^{(1)}(s) + X_{N+1}^{(1)}(s) \right).
\] (6.13)

That is, \( Z \) is \(-1/2\) if the quantity \( X_N^{(1)}(s) + X_{N+1}^{(1)}(s) \) is odd and 0 otherwise. The random variable \( Y \) is the random variable for which the estimates of Theorem 6.3 hold. The estimate (6.12) follows from \( X \) has the same distribution as the number of eigenvalues of the minor interlace those of \( A \). That is, \( Z \) is odd and 0 otherwise. The random variable \( Y \) is the random variable for which the estimates of Theorem 6.3 hold. The estimate (6.12) follows from the facts that that \( X_N^{(1)}(s) \) and \( X_{N+1}^{(1)}(s) \) are independent, that \( Z \) is bounded, and the estimate (6.9). Taking expectations we see that

\[
\mathbb{E}[X_N^{(2)}(s)] = \frac{1}{2} \left( \mathbb{E}[X_N^{(1)}(s)] + \mathbb{E}[X_{N+1}^{(1)}(s)] \right) + \mathcal{O}(1).
\] (6.14)

Recall that \( X_{N+1}^{(1)}(s) \) is the number of eigenvalues of the matrix \( A^{(N+1)} \) lying in the interval \([2\sqrt{N} - sN^{-1/6}, \infty)\). The top left \( N \times N \) minor of \( A^{(N+1)} \) has the same distribution of \( A^{(N)} \). Since the eigenvalues of the minor interlace those of \( A^{(N+1)} \) we see that there is a random variable \( \tilde{X}_N^{(1)}(s) \) that has the same distribution as \( X_N^{(1)}(s) \) such that

\[
|\tilde{X}_N^{(1)}(s) - X_N^{(1)}(s)| \leq 1.
\] (6.15)

We conclude (6.11). \( \square \)

The following is Proposition 9.7.2 of [14], proven by taking the limit \( N \to \infty \) in Theorem 6.4.

**Lemma 6.6.** Let \( \{\chi_i\}_i \) and \( \{\chi'_i\}_i \) be two independent Airy\(_1\) random point fields. Let \( \{\zeta_{ij}\}_i \) be an Airy\(_2\) random point field. Let \( T \in \mathbb{R} \). Then,

\[
\mathbb{P} \left[ |\{i : \zeta \leq T\}| = k \right] \quad \text{(6.16)}
\]

is equal to the probability that there are either \( 2k \) or \( 2k + 1 \) particles from the superimposed point process \( \{\chi_i\}_i \cup \{\chi'_i\}_i \) below \( T \).

From the above we easily deduce the following.

**Proposition 6.7.** Let \( \{\chi_i\}_i \) be the Airy\(_1\) random point field. Then, for \( T > 0 \),

\[
\left| \mathbb{E}[|\{i : \chi_i \leq T\}|] - \frac{2}{3} T^{3/2} \right| \leq C,
\] (6.17)

and

\[
\text{Var}(\{i : \chi_i \leq T\}) \leq C(|\log(T)| + 1)
\] (6.18)

**Proof.** Let \( \chi_i \) and \( \chi'_i \) be two independent Airy\(_1\) random point fields. Let \( Y \) be the random variable that is \( k \) if there are \( 2k \) or \( 2k + 1 \) particles in the superposition \( \{\chi_i\}_i \cup \{\chi'_i\}_i \) below \( T \). Then by the previous lemma, \( Y \) has the same distribution as the number of particles in an Airy\(_2\) random point field below \( T \). If \( Z = \frac{1}{2}(\{|i : \chi_i \leq T\}| + \{|i : \chi'_i \leq T\}|) - Y \), then \( |Z| \leq C \). The claim now follows. \( \square \)

### 6.2 Proof of Theorem 6.1

Let us denote by \( N_T \) the random variable,

\[
N_T = \left| \{i : \lambda_i \geq 2 - TN^{-2/3}\} \right|.
\] (6.19)

Let \( N^{4/15} \geq T \geq C_1 \) where \( C_1 \) is the constant appearing in the statement of Proposition 6.5. We have by (6.12),

\[
\mathbb{P}[\lambda_k \geq 2 - TN^{-2/3}] = \mathbb{P}[N_T \geq k] \leq C \frac{\log(T)}{(\mathbb{E}[N_T] - k)^2},
\] (6.20)
as long as $k \geq \mathbb{E}[N_T]$. Note that if $\gamma$ solves
\[ \mathbb{E}[N_\gamma] = x \] (6.21)
and $x \leq N^{2/5}$ then by (6.11),
\[ \gamma = \left(\frac{3\pi x}{2}\right)^{2/3} + \mathcal{O}(1 + x^{1/3})^{-1}. \] (6.22)
Assume that $k \leq N^{2/5}$. Choosing now
\[ T = \left(\frac{3\pi k}{2}\right)^{2/3} - s \] (6.23)
we see by (6.11) that for some $c > 0$,
\[ k - \mathbb{E}[N_T] \geq c k^{1/3} - C, \] (6.24)
as long as $s \leq \left(\frac{3\pi k}{2}\right)^{2/3} - C_1$. Therefore, we have that
\[ \mathbb{P}\left[N^{2/3}(\lambda_k - 2) \geq -\left(\frac{3\pi k}{2}\right)^{2/3} + s\right] \leq C' \frac{1 + \log(k)}{(sk^{1/3} - C)^2}, \] (6.25)
as long as $0 \leq s \leq \left(\frac{3\pi k}{2}\right)^{2/3} - C_1$. In particular, we see that there is a $K_1 > 0$ so that for all $k \geq K_1$,
\[ \mathbb{P}\left[\bigcap_{N^{2/5} \geq j \geq k} \left\{ N^{2/3}(\lambda_j - 2) \leq -\left(\frac{3\pi j}{2}\right)^{2/3} + \frac{1}{10} j^{2/3}\right\}\right] \geq 1 - \frac{1}{k^{1/2}}. \] (6.26)
A similar argument gives
\[ \mathbb{P}\left[ N^{2/3}(\lambda_k - 2) \leq -\left(\frac{3\pi k}{2}\right)^{2/3} - s \right] \leq C' \frac{1 + \log(k) + \log(1 + s)}{(sk^{1/3} - C)^2}, \] (6.27)
for $0 \leq s \leq N^{4/15}$ and $k \leq N^{2/5}$. From all of these estimates we find,
\[ \mathbb{E}\left[1_{\{N^{2/3}(\lambda_k - 2) \leq -C_1\}} \left| N^{2/3}(\lambda_k - 2) + \left(\frac{3\pi k}{2}\right)^{2/3}\right| \right] \leq \frac{C\log(k)^2}{k^{1/3}}, \quad k \leq N^{2/5}. \] (6.28)
Denote by $G_k$ the event on the left side of (6.26). Let $\varepsilon > 0$ and choose constants $C_\varepsilon$ and $k_0$ so that for all large $N$,
\[ \mathbb{P}[G_{k_0}] \geq 1 - \varepsilon, \quad \mathbb{P}[\{|N^{2/3}(\lambda_1 - 2)| \leq C_\varepsilon\}] \geq 1 - \varepsilon. \] (6.29)
Let $\mathcal{F}$ denote the intersection of these two events. Choose $K_2 \geq k_0$ so that $K_2 \geq 100(C_\varepsilon)^3/2$. Fix also
\[ \delta_0 = \frac{1}{20}. \] (6.30)
By the choice of $K_2$ and the definition of $\mathcal{F}$ we have for any $k \geq K_2$ that on the event $\mathcal{F}$ for some $c > 0$,
\[ N^{2/3}(\lambda_1 - \lambda_k) \geq ck^{2/3}. \] (6.31)
For any $k \geq K_2$ we then have,
\[ \mathbb{E}\left[1_{\mathcal{F}} \left| \sum_{j=k}^{N_{k_0}} \frac{1}{N^{2/3}(\lambda_1 - \lambda_j)} - \sum_{j=k}^{N_{k_0}} \frac{\left(\frac{3\pi j}{2}\right)^{2/3}}{\pi \sqrt{x}} \right| \right] \leq C \sum_{j=k}^{N_{k_0}} C_\varepsilon + C j^{-1/3} + \mathbb{E}[1_{\mathcal{F}} | N^{2/3}(\lambda_j - 2) + \left(\frac{3\pi j}{2}\right)^{2/3} |] \leq C(C_\varepsilon + 1) \log(k) k^{-1/3}. \] (6.32)
The outcome of all of this is that there for any \( \varepsilon > 0 \), there is a \( K_3 > 0 \) so that for any \( k \geq K_3 \) we have the estimate,

\[
P \left[ \sum_{j=k}^{N^6_0} \frac{1}{N^{2/3}(\lambda_1 - \lambda_j)} \right] \leq \frac{1}{N^{2/3}} \sum_{j=2}^{N^6_0} \frac{1}{\lambda_1 - \lambda_j} - \int_0^{(3\pi k)^{2/3}} \frac{1}{\pi \sqrt{x}} \, dx > \varepsilon \right] \leq \varepsilon. \tag{6.33}
\]

By Theorem 3.3,

\[
\frac{1}{N} \sum_{j=1}^{N} \frac{1}{\lambda_1 - \lambda_j} - \int_{-2}^{\gamma N^6_0} \rho_{sc}(x) \, dx \leq N^{-c_0-1/3}
\]

with overwhelming probability. We write, for \( k \geq K_3 \),

\[
\frac{1}{N^{2/3}} \sum_{j=2}^{N} \frac{1}{\lambda_1 - \lambda_j} - N^{1/3} = \frac{1}{N^{2/3}} \sum_{j=2}^{k} \frac{1}{\lambda_1 - \lambda_j} - \int_0^{(3\pi k)^{2/3}} \frac{1}{\pi \sqrt{x}} \, dx
\]

\[
+ \frac{1}{N^{2/3}} \sum_{j=k+1}^{N^6_0} \frac{1}{\lambda_1 - \lambda_j} - \int_{(3\pi k)^{2/3}}^{(3\pi N^6_0)^{2/3}} \frac{1}{\pi \sqrt{x}} \, dx
\]

\[
+ \frac{1}{N^{2/3}} \sum_{j=N^6_0+1}^{N} \frac{1}{\lambda_1 - \lambda_j} - N^{1/3} \int_{-2}^{\gamma N^6_0} \rho_{sc}(x) \, dx
\]

\[
+ \int_{0}^{(3\pi N^6_0)^{2/3}} \frac{1}{\pi \sqrt{x}} \, dx - N^{1/3} \int_{\gamma N^6_0}^{2} \rho_{sc}(x) \, dx
\]

A calculation shows that the term on the last line is \( \mathcal{O}(N^{7\delta_0/6-2/3}) = o(1) \) by our assumption on \( \delta_0 \). Hence, for any bounded Lipschitz \( F \) we see that for any \( \varepsilon > 0 \), there is a \( k_1 = k_1(\varepsilon) \) so that for any fixed \( k > k_1 \),

\[
\limsup_{N \to \infty} \mathbb{E} \left[ F \left( N^{-2/3} \sum_{j=2}^{N} \frac{1}{\lambda_1 - \lambda_j} - N^{1/3} \right) \right] - F \left( N^{-2/3} \sum_{j=2}^{k} \frac{1}{\lambda_1 - \lambda_j} - \int_0^{(3\pi k)^{2/3}} \frac{1}{\pi \sqrt{x}} \, dx \right) \]

\[
\leq C\varepsilon \| F \|_{\text{Lip}}. \tag{6.39}
\]

With \( \{ \chi_i \}_i \) denoting the particles of the Airy\(_1\) process, much of the same calculations as above show that

\[
P \left[ \chi_k \leq \left( 3\pi k \right)^{2/3} \right] \leq \frac{C \log(k)}{(k^{1/3} \epsilon - C)^2}
\]

for \( 0 \leq s \leq k^{2/3} - C' \) some \( C' > 0 \), and

\[
P \left[ \chi_k \geq \left( 3\pi k \right)^{2/3} + s \right] \leq \frac{C \log(k) \log(1 + s)}{(k^{1/3} \epsilon - C)^2}.
\]

Arguing as above, we see that for any \( \varepsilon > 0 \) there is an event \( \mathcal{F}' \) with probability at least \( 1 - \varepsilon \) and a \( K'_1 > 0 \) so that for all \( k > K'_1 \), we have

\[
\mathbb{E} \left[ \frac{1}{\lambda_1 - \chi_j} + \int_0^{(3\pi j)^{2/3}} \frac{1}{\pi \sqrt{x}} \, dx \right] \leq C \log(k)^2 k^{-1/3}.
\]

From this we see that,

\[
\limsup_{n \to \infty} \sum_{j \geq n} \left| \frac{1}{\lambda_1 - \chi_j} + \int_0^{(3\pi j)^{2/3}} \frac{1}{\pi \sqrt{x}} \, dx \right| = 0
\]

(6.43)
almost surely which proves that the limiting random variable $\Xi$ exists. Moreover, we see that for any $\varepsilon > 0$ there is a $k_2$ so that for all $k > k_2$ and any bounded Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ we have,

$$\left| \mathbb{E} \left[ F \left( \sum_{j=2}^{\infty} \frac{1}{\lambda_j} \right) + \int_{\mathbb{R}}^{(\frac{3\pi}{2})^{2/3}} \frac{1}{\pi \sqrt{x}} \, dx \right] \right| - \mathbb{E} \left[ F \left( \sum_{j=2}^{k} \frac{1}{\lambda_j} \right) + \int_{\mathbb{R}}^{(\frac{3\pi}{2})^{2/3}} \frac{1}{\pi \sqrt{x}} \, dx \right] \leq C\varepsilon \| F \|_{\text{Lip}}.$$  (6.44)

On the other hand we know that for any finite $k$,

$$\frac{1}{N^{2/3}} \sum_{j=2}^{k} \frac{1}{\lambda_j} \to -\frac{k}{\lambda_1},$$  (6.45)

where the convergence is in distribution as $N \to \infty$. This yields the claim.  

### 6.3 Proof of Theorem 2.2

Let $H$ be the GOE matrix given by (2.10) and $M$ be the matrix (2.1). We choose a coupling so that $M_{ij} = H_{ij}$ for $i \neq j$. By Proposition A.1, Theorem 3.3 and Lemma 3.4 there is an event of probability at least $1 - N^{-1/20}$ on which,

$$N^{1/3} \left| \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j(H)} - \frac{1}{\lambda_1(M)} \right| \leq N^{-c},$$  (6.46)

for some $c > 0$. The result follows from this and Theorems 2.1 and 6.1. 

### 7 Extension to $\langle |R_{12}| \rangle$: Proof of Theorem 2.3

In this section we extend our results to the quantity $\langle |R_{12}| \rangle$ and prove Theorem 2.3. Recall the notation $q = 1 - \beta^{-1}$. Our starting point is the following elementary calculation,

$$\langle |R_{12}| - q \rangle = \langle \frac{R_{12}^2 - q^2}{|R_{12}| + q} \rangle = \frac{1}{2q} \langle R_{12}^2 - q^2 \rangle + \langle \frac{R_{12}^2 - q^2}{|R_{12}| + q} \left( \frac{q - |R_{12}|}{|R_{12}| + q} \right) \rangle.$$  (7.1)

For the second term we have,

$$\left| \langle \frac{R_{12}^2 - q^2}{|R_{12}| + q} \langle |R_{12}| - q \rangle \rangle \right| = \left| \langle \frac{R_{12}^2 - q^2}{|R_{12}| + q} \rangle \frac{1}{|R_{12}| + q} \right| \leq \frac{1}{2q^3} \langle (R_{12}^2 - q^2)^2 \rangle.$$  (7.2)

Hence, if we can show that $\langle (R_{12}^2 - q^2)^2 \rangle = o(N^{-1/3})$ with probability $1 - o(1)$, then the convergence of Theorem 2.2 extends to $\langle |R_{12}| - q \rangle$; i.e., this shows how to deduce (2.14) and (2.15) from (2.13).

We expand,

$$\langle (R_{12}^2 - q^2) \rangle = \langle R_{12}^4 \rangle - 2q^2 \langle R_{12}^2 \rangle + q^4.$$  (7.3)

We already calculated $\langle R_{12}^4 \rangle$ in Theorem 5.9 down to $o(N^{-2/3})$. It remains to calculate the first term $\langle R_{12}^4 \rangle$. The modification of the representation formula is,

$$\langle R_{12}^4 \rangle = \frac{\sum_{i=1}^{N} \frac{1}{(\lambda_{i-1} - \lambda_i)^2} \langle G(z) \rangle \langle G(w) \rangle}{\left( \sum_{i=1}^{N} \frac{1}{(\lambda_{i-1} - \lambda_i)^2} \right)^2} \left( \sum_{i=1}^{N} \frac{1}{(\lambda_{i-1} - \lambda_i)^2} \right)^2.$$  (7.4)

Since the function $G(z)$ appearing in the exponential is identical to what we encountered in considering $\langle R_{12}^2 \rangle$, the steepest descent analysis of the numerator is very similar to that in Section 5. In light of this, we will use the same notation as in Section 5.

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Following along the argument of Section 5 we see that, analogously to Lemma 5.5, we can change the contour from the vertical line through $\gamma$ to $\tilde{\Gamma}$ at an error of size $O(e^{-N^c})$, for some $c > 0$, on the event $F_{\delta,\varepsilon_1}$ for sufficiently small $\varepsilon_1$.

For the analog of Lemma 5.6 we similarly derive the following estimates which all hold on the event $F_{\delta,\varepsilon_1}$ for $\varepsilon_1 > 0$ sufficiently small. First,

$$
\int_{\Gamma \times \tilde{\Gamma}} e^{\frac{N}{2}(G(z+\gamma)+G(w+\gamma)-2G(\gamma))} \frac{1}{N^4(z + c_\beta/N)^2(w + c_\beta/N)^2} \, dz \, dw
= \int_{\Gamma \times \tilde{\Gamma}} e^{\frac{N}{2}(g(z)+g(w))} \left(1 + \frac{1}{N} z^2 \tilde{m}'_N(\gamma) \right) \left(1 + \frac{1}{N} w^2 \tilde{m}'_N(\gamma) \right) \frac{1}{N^4(z + c_\beta/N)^2(w + c_\beta/N)^2} \, dz \, dw
+ O\left(\frac{N^{5\varepsilon + \varepsilon_1 + 3\delta}}{N^3}\right).
$$

(7.5)

Second,

$$
\int_{\Gamma \times \tilde{\Gamma}} e^{\frac{N}{2}(G(z+\gamma)+G(w+\gamma)-2G(\gamma))} \sum_{j=2}^{N} \frac{1}{N^4(z + \gamma - \lambda_j)^2(w + \gamma - \lambda_j)^2} \, dz \, dw = O\left(\frac{N^{5\varepsilon + \varepsilon_1 + 3\delta}}{N^3}\right).
$$

(7.6)

Third,

$$
\int_{\Gamma \times \tilde{\Gamma}} e^{\frac{N}{2}(G(z+\gamma)+G(w+\gamma)-2G(\gamma))} \frac{1}{N^2(z + c_\beta/N)(w + c_\beta/N)} \sum_{j=2}^{N} \frac{1}{N^2(z + \gamma - \lambda_j)(w + \gamma - \lambda_j)} \, dz \, dw
= \left(\frac{1}{\sum_{j=2}^{N} \frac{1}{N^2(\lambda_1 - \lambda_j)^2}}\right)
\int_{\Gamma \times \tilde{\Gamma}} e^{\frac{N}{2}(g(z)+g(w))} \left(1 + \frac{1}{N} z^2 \tilde{m}'_N(\gamma) \right) \left(1 + \frac{1}{N} w^2 \tilde{m}'_N(\gamma) \right) \frac{1}{N^2(z + c_\beta/N)(w + c_\beta/N)} \, dz \, dw
+ O\left(\frac{N^{5\varepsilon + \varepsilon_1 + 3\delta}}{N^3}\right).
$$

(7.7)

Finally,

$$
\int_{\Gamma \times \tilde{\Gamma}} e^{\frac{N}{2}(G(z+\gamma)+G(w+\gamma)-2G(\gamma))} \left(\sum_{j=2}^{N} \frac{1}{N^2(z + \gamma - \lambda_j)(w + \gamma - \lambda_j)}\right)^2 \, dz \, dw = O\left(\frac{N^{5\varepsilon + \varepsilon_1 + 3\delta}}{N^3}\right).
$$

(7.8)

In order to calculate the numerator of (7.4) up to errors that are $o(N^{-2/3})$ we see that it suffices to compute the integrals in (7.5) and (7.7). We can proceed identically to Lemma 5.7 and pass to the rescaled variable $u$ being integrated over $\Gamma_r$ up to again an error exponential in $-N^c$ for some $c > 0$. The integral resulting from (7.7) is identical to (5.68), whereas the integral coming from (7.5) is

$$
\frac{1}{N} \int_{\Gamma_r} e^{(\beta + \tilde{m}_N(\gamma))u/2} \left(1 + \frac{u^2 \tilde{m}'_N(\gamma)}{4N}\right) \frac{du}{(u + c_\beta)^2}.
$$

(7.9)

In order to calculate this, we note the identities

$$
\int_{\Gamma_{r,b}} \frac{e^{az}}{(z + b)^{3/2}} \, dz = e^{-ab} \frac{3^{1/2}}{3} \sqrt{\pi}
$$

(7.10)

and

$$
\int_{\Gamma_{r,b}} \frac{e^{az}}{(z + b)^{5/2}}z^2 \, dz = e^{-ab} \frac{\sqrt{a}}{\sqrt{\pi}} \left(\frac{8a^2b^2}{3} - 8ba + 2\right).
$$

(7.11)
From all of this, we see that we have derived the following estimate for the numerator of \((7.4)\), with \(a = (\beta + \tilde{m}_N(\gamma))/2\),
\[
\int_{\gamma-i\infty}^{\gamma+i\infty} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{N}{2}(G(z)+G(w)-2G(\gamma))} \times \left[ 6 \sum_{i=1}^{N} \frac{1}{(N\beta)^4(\lambda_i - w)^2(\lambda_i - z)^2} + 3 \left( \sum_{i=1}^{N} \frac{1}{\beta^2 N^2(\lambda_i - z)(\lambda_i - w)} \right)^2 \right] \,dz \,dw 
= \left( \frac{2}{\sqrt{a}} \sqrt{\pi e^{-ac\beta_i^2}} \right)^2 \left[ 16a^4 - 4 \tilde{m}_N(\gamma)^2 + 24a^2 \tilde{m}_N(\lambda_1) \right] + \mathcal{O}\left( \frac{N^{5\kappa + 3\delta + \varepsilon_1}}{N^3} \right) \tag{7.12}
\]
whereas for the denominator we have
\[
\left( \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{N}{2}G(z)} \right)^2 = \left( \frac{2}{\sqrt{a}} \sqrt{\pi e^{-ac\beta_i^2}} \right)^2 \left( 1 + \frac{3\tilde{m}_N(\lambda_1)}{4Na^2} \right) + \mathcal{O}\left( \frac{N^{5\kappa + 3\delta + \varepsilon_1}}{N^3} \right). \tag{7.13}
\]
From this and \((5.80)\) we see that
\[
\langle R_{12}^2 \rangle = (1 - \beta^{-1})^4 + 4 \left( \frac{\beta - 1}{\beta^4} \right) (1 + \tilde{m}_N(\lambda_1)) + 6 \left( \frac{\beta - 1}{\beta^4} \right)^2 \bigl( 1 + \tilde{m}_N(\lambda_1) \bigr)^2 + 6 \left( \frac{\beta - 1}{\beta^4} \right) \tilde{m}_N(\lambda_1) \
+ \mathcal{O}(N^{-1+5\kappa+\varepsilon_1+3\delta}), \tag{7.14}
\]
and furthermore that
\[
\langle (R_{12}^2 - q^2) \rangle = \frac{8(\beta - 1)^2 \tilde{m}_N(\lambda_1)}{\beta^2} + 4 \left( \frac{\beta - 1}{\beta^4} \right) \bigl( 1 + \tilde{m}_N(\lambda_1) \bigr)^2 + \mathcal{O}(N^{-1+5\kappa+\varepsilon_1+3\delta}). \tag{7.15}
\]
On the event \(\mathcal{F}_{\delta,\varepsilon_1}\) the first two terms are \(\mathcal{O}(N^{-2/3+2\delta+\varepsilon_1})\). This yields the estimate \((2.13)\) and completes the proof of Theorem 2.3.

## A Zero-diagonal GOE

Let \(H\) be a GOE matrix as in \((2.10)\), and let \(V\) be its diagonal and let
\[
H = M + V \tag{A.1}
\]
so that \(M\) is as in \((2.1)\). In this section we prove that with overwhelming probability, the extremal eigenvalues of \(H\) and \(M\) are close.

**Proposition A.1.** Let \(\varepsilon > 0\). The following estimate holds for \(i \leq N^{1/20}\) with overwhelming probability:
\[
|\lambda_i(H) - \lambda_i(M)| \leq \frac{N\varepsilon}{N}. \tag{A.2}
\]
The proof follows from the Helffer-Sjöstrand formula, which we recall in \((A.30)\), and the following lemma providing control over the difference of Stieltjes transforms. We recall the notation,
\[
z = E + i\eta, \quad E, \eta \in \mathbb{R}. \tag{A.3}
\]
We will consider for the most part only the Stieltjes transform for \(\eta > 0\), as the value in the lower half-plane is related by conjugation.

**Lemma A.2.** Denote by \(m_M\) and \(m_H\) the empirical Stieltjes transforms of \(M\) and \(H\). Let \(\varepsilon > 0\) and \(\delta > 0\). With overwhelming probability, for any \(N^{-\delta} \geq \eta \geq N^{\delta}/N\), and \(|E| \leq 10\), we have
\[
|m_M(z) - m_H(z)| \leq \frac{N\varepsilon}{N\eta} \left( \frac{1}{N\eta} + \text{Im}[m_{sc}] \right) \tag{A.4}
\]
The proof of the above lemma is based on the following resolvent expansion as well as two moment estimates which are the content of Lemmas A.3 and A.4 below. We have,
\[
\frac{1}{M-z} - \frac{1}{H-z} = \frac{1}{M-z} \sum_{k=1}^{m} (-1)^{k+1} (V(M-z)^{-1})^k + \frac{1}{H-z} (V(M-z))^{-(m+1)} (-1)^m \quad (A.5)
\]
Denote,
\[
A_k := \frac{1}{M-z} \left( V \frac{1}{M-z} \right)^k, \quad R(z) := \frac{1}{M-z}. \quad (A.6)
\]
Note that \( R \) is independent of \( V \). We first prove,

**Lemma A.3.** Let \( C > 0 \) be a constant. On the event
\[
\max_{i,j} |R_{ij}| \leq C, \quad (A.7)
\]
we have for even \( p \)
\[
\mathbb{E}_V \left| \frac{1}{N} \text{tr} A_k \right|^p \leq C(k,p) \left[ \frac{1}{N\eta} \max_a \text{Im}[R_{aa}] \right]^p, \quad (A.8)
\]
where \( \mathbb{E}_V \) denotes the expectation over \( V \).

**Remark.** This estimate is sub-optimal for \( k \geq 2 \) but we will not need a better estimate. The next lemma below deals with the error term in the resolvent expansion.

Before embarking on the proof, we record here the Ward identity,
\[
\sum_{a=1}^{N} \left| \left( \frac{1}{A-z} \right)_{ab} \right|^2 = \frac{1}{\eta} \text{Im} \left( \frac{1}{A-z} \right)_{bb} \quad (A.9)
\]
for any self-adjoint matrix \( A \). This is a consequence of the spectral theorem (see Section 3 of \([7]\)).

**Proof.** Denote by \( j = (j_1, \ldots, j_{kp}) \) and \( i = (i_1, \ldots, i_p) \) multi-indices in \( kp \) and \( p \) variables respectively, with \( p \) even. We will use the \( i \)-indices to denote the summation coming from the trace, and the \( j \)-indices to be the summations coming from matrix multiplication. Roughly, we are writing the trace as,
\[
\frac{1}{N} \text{tr} A_2 = \frac{1}{N} \sum_{i,j_1,j_2} R_{ij_1} V_{j_1} R_{j_1j_2} V_{j_2} R_{j_2i}. \quad (A.10)
\]
With this convention we then have,
\[
\mathbb{E}_V \left| \frac{1}{N} \text{tr} A_k \right|^p = \frac{1}{N^p} \sum_{i} \sum_{\bar{j}} \sum_{\bar{j}} R_{i_1,j_1}^* R_{i_2,j_2}^* \cdots R_{i_{p-1},j_{p-1}}^* \cdot \cdot \cdot R_{i_{kp},j_{kp}}^* M(\bar{j}) \mathbb{E}_V[V_{j_1} \ldots V_{j_{kp}}]. \quad (A.11)
\]
where \( R^* \) denotes either \( R \) or \( \bar{R} \) as appropriate, and \( M(\bar{j}) \) is a monomial which contains all of the Green’s function elements \( R_{j_a,j_{a+1}} \) that has indices only in \( \bar{j} \) (we separate out the matrix elements of \( R \) that have an index in \( j \)). The choice of \( R \) or \( \bar{R} \) or the form of \( M \) will not be important for the calculation done here. The \( \bar{j} \) can be grouped into partitions of the \( kp \) indices into coincidences. That is,
\[
\sum_{\bar{j}} = \sum_{\mathcal{P}} \sum_{j \in \mathcal{P}} \sum_{\bar{\mathcal{P}}} \sum_{\bar{j} \in \bar{\mathcal{P}}} \quad (A.12)
\]
where the first sum is over partitions \( \mathcal{P} \) on \( kp \) elements, and the second summation means the sum over all \( j \) so that if \( j_a = j_b \) whenever \( a \) and \( b \) are in the same block of \( \mathcal{P} \) and \( j_a \neq j_b \) whenever \( a \) and \( b \) are in distinct blocks of the partition. The independence of the \( V_{j} \) implies that unless the size of each block of the partition \( \mathcal{P} \) is at least 2, then the expectation vanishes. Denote by \( \mathcal{P}_2 \) the set of such partitions. Estimating \( |M(\bar{j})| \leq C(k,p) \) (using the assumption (A.7)) we see that from this discussion,
\[
\mathbb{E}_V \left| \frac{1}{N} \text{tr} A_k \right|^p \leq \sum_{\mathcal{P} \in \mathcal{P}_2} \sum_{j \in \mathcal{P}} \sum_{\bar{\mathcal{P}}} \frac{C}{N^{kp/2}} \frac{1}{N^p} \sum_{i} |R_{i_1,j_1} R_{i_2,j_2} \cdots R_{i_{kp},j_{kp}}| \quad (A.13)
\]
From the Ward identity (A.9), for any index $a, b$ we have
\[
\frac{1}{N} \sum_{k=1}^{N} |R_{ia}R_{kb}| \leq \frac{1}{N^2} \sum_{k=1}^{N} |R_{ia}|^2 + |R_{kb}|^2 \leq \frac{1}{N\eta} \sup_{k} \text{Im}[R_{kk}] \tag{A.14}
\]
and so
\[
\frac{1}{N^p} \sum_{\ell=1}^{N} |R_{i,j_1}R_{i_1,j_k} \cdots R_{i_p,j_p}| \leq \left( \frac{\sup_{k} \text{Im}[R_{aa}]}{N\eta} \right)^p. \tag{A.15}
\]

The summation over $j \in \mathcal{P}$ for $\mathcal{P} \in \mathcal{P}_2$ has at most $N^{kp/2}$ terms, and $\mathcal{P}_2$ has cardinality bounded in terms of only $k$ and $p$, so we get the claim.

**Lemma A.4.** Let $C > 0$. On the event that $\max_{i,j} |R_{ij}| \leq C$ we have for even $p$ that,
\[
\mathbb{E}_V[|(RVR)^k_{ab}|^p] \leq C(k, p) \left( \frac{1}{N^{p(k/2-1)}} + \max_{i \neq j} |R_{ij}|^{p(k/2-1)} \right), \tag{A.16}
\]
where $\mathbb{E}_V$ denotes the expectation over $V$.

**Proof.** We expand out the expectation similar to the proof of Lemma A.3 and use similar notation. We estimate $\sup_j |R_{aj}| \leq C$, and $\sup_j |R_{bj}| \leq C$ and obtain,
\[
\mathbb{E}_V[|(RVR)^k_{ab}|^p] \leq C(k, p) \sum_{\mathcal{P} \in \mathcal{P}_2} \sum_{j \in \mathcal{P}} |M(j)| \tag{A.17}
\]
where $j$ is the following monomial in Green’s function elements,
\[
M(j) = R_{j_1,j_2}R_{j_2,j_3} \cdots R_{j_{k-1}j_k}R_{j_{k+1}j_{k+2}} \cdots R_{j_{kp-1}j_{kp}}, \tag{A.18}
\]
i.e., it is the product of $R_{j_1,j_{k+1}}$ except when $i = nk$ for any $n$. Note that we have dropped any Green’s function elements that involve the index $a$ or $b$, and kept the ones involving only the $j_i$ indices. We will use the estimate,
\[
|M(j)| \leq C(k, p) \left( \max_{i \neq j} |R_{ij}| \right)^\# \text{ off-diagonal}, \tag{A.19}
\]
and so we need to count how many off-diagonal entries appear in $M(j)$ when $j$ is in a specific partition $\mathcal{P} \in \mathcal{P}_2$. Suppose that $\mathcal{P} \in \mathcal{P}_2$ has $\ell$ blocks. Recall that $j \in \mathcal{P}$ means that $j_a = j_b$ if and only if $a$ and $b$ are in the same block in the partition $\mathcal{P}$. Note that $M(j)$ contains $p(k-1)$ Green’s function entries. Denote the size of the $i$th block of $\mathcal{P}$ by $n_i \geq 2$. There can be at most $n_i - 1$ Green’s function entries in the monomial $M(j)$ whose indices $j_k$ and $j_{k+1}$ both appear in the $i$th block. Therefore, there are at least
\[
p(k-1) - \sum_{i=1}^{\ell} (n_i - 1) = pk - p - pk + \ell = \ell - p \tag{A.20}
\]
off-diagonal Green’s function entries in the monomial $M(j)$. Hence, for $j \in \mathcal{P}$ where $\mathcal{P}$ has $\ell$ blocks,
\[
|M(j)| \leq C(k, p) \left( \max_{i \neq j} |R_{ij}| \right)^{(\ell-p)+}, \tag{A.21}
\]
The summation over $j \in \mathcal{P}$ has less than $N^\ell$ terms, and so
\[
\mathbb{E}_V[|(RVR)^k_{ab}|^p] \leq C(k, p) \max_{1 \leq \ell \leq kp/2} \frac{N^\ell}{N^{kp/2}} \left( \max_{i \neq j} |R_{ij}| \right)^{(\ell-p)+} \leq C(k, p) \left( \frac{1}{N^{p(k/2-1)}} + \left( \max_{i \neq j} |R_{ij}|^{p(k/2-1)} \right) \right). \tag{A.22}
\]
This is the claim. 

\[\square\]
Proof of Lemma A.2. By the local semi-circle law, we have the estimates for \( N^{\delta}/N \leq \eta \leq N^{-\delta}, \)
\[
\max_{i,j} |R_{ij}| \leq 2, \quad \max_{i \neq j} |R_{ij}| \leq N^{-\delta/4}
\]
(A.23)
and
\[
\max_a \text{Im}[R_{aa}] \leq N^\delta \left( \frac{1}{N\eta} + \text{Im}[m_{ac}] \right)
\]
(A.24)
with overwhelming probability. Choose \( m \) large enough so that \((m/2 - 1)\delta > 1000\). Then by Lemma A.4, the final term in the resolvent expansion (A.5) is less than terms in the resolvent expansion are bounded using Lemma A.3.

We now recall the Helffer-Sjöstrand formula (see, e.g., [11]). For our \( f \) as above, define the almost analytic extension of \( f \) by,
\[
\tilde{f}(x + iy) = (f(x) + iyf'(x))\chi(y).
\]
Then, for \( \lambda \in \mathbb{R} \), we have
\[
f(\lambda) = \frac{1}{\pi} \int_{\mathbb{R}^2} \overline{\tilde{\tau}_z\tilde{f}(z)} \frac{\lambda - z}{dxdy}
\]
(A.30)
where we use the notation,
\[
z = x + iy, \quad \overline{\tilde{\tau}_z} := \frac{\partial_x + i\partial_y}{2}.
\]
(A.31)
Using this formula and the fact that \( \text{tr} f(A) \) is real for self-adjoint \( A \) we have,
\[
\left| \frac{1}{N} \text{tr} f(M) - \frac{1}{N} \text{tr} f(H) \right| \leq \left| \int \int yf''(x)\chi'(y)\text{Im}[S(x + iy)]dxdy \right| \\
+ \left| \int \int |f(x)||\chi'(y)||\text{Im}[S(x + iy)]|dxdy \right| \\
+ \left| \int \int |yf'(x)\chi'(y)\text{Re}[S(x + iy)]|dxdy \right|.
\]
(A.32)
(A.33)
(A.34)

Lemma A.5. Let \( \varepsilon_1, \varepsilon > 0 \) and \( \delta f > 0 \) be arbitrary. There is an event such that the following holds with overwhelming probability. Suppose that \( f \) is a smooth function so that \( f = 1 \) on \([a, b]\) and \( f = 0 \) outside of \([a - N^{\delta f - 1}, b + N^{\delta f - 1}]\). Assume that,
\[
a \geq 2 - \kappa, \quad |a - b| \leq \frac{N^{\delta f + \varepsilon_1}}{N}.
\]
(A.25)
where \( \kappa = N^{-1/2} \). Assume that \( \|f^{(k)}\|_{L^\infty} \leq C(N^{1-\delta f})^k \) for \( k = 1, 2 \). Assume
\[
0 < \delta f < \frac{1}{20}, \quad 0 < \varepsilon_1 < \frac{1}{20}
\]
(A.26)
Then,
\[
|\text{tr} f(M) - \text{tr} f(H)| \leq N^{\varepsilon - \delta f}.
\]
(A.27)
Proof. We work on the event that the estimate of Lemma A.2 holds with \( \varepsilon > 0 \) and \( \delta = \varepsilon/10 \). We can assume that \( 100\varepsilon < \min\{\varepsilon_1, \delta_f\} \). Fix,
\[
\eta_1 = \frac{N^{\delta_1}}{N}, \quad \varepsilon < \delta_1 < \frac{1}{5}
\]
(A.28)
and let \( \chi \) be a cut-off function so that \( \chi(x) = 1 \) for \( |x| \leq \eta_1 \) and \( \chi(x) = 0 \) for \( |x| > 2\eta_1 \) and \( |\chi^{(k)}| \leq C(\eta_1)^{-k} \) for \( k = 1, 2 \). Below we will make the choice \( \delta_1 = \delta_f + \varepsilon_1 + 10^{-1} \). Denote
\[
S(z) = m_M(z) - m_H(z).
\]
(A.29)
We now recall the Helffer-Sjöstrand formula (see, e.g., [11]). For our \( f \) as above, define the almost analytic extension of \( f \) by,
\[
\tilde{f}(x + iy) = (f(x) + iyf'(x))\chi(y).
\]
Using Lemma A.2 and the assumed $L^\infty$ bounds for $f'$ and $\chi'$, we see that the last term is bounded above by
\[
\int \int |y f'(x) \chi'(y) \text{Re} S(x + iy)|\,dx\,dy \leq C \frac{N^\varepsilon}{N} \left( \frac{1}{N \eta_1} + \sqrt{\kappa + \eta_1} \right)
\leq C N^{\varepsilon - 1 - \delta_1} (N^{-\delta_1} + N^{-1/4}),
\] (A.35)
where we used $\eta_1 < N^{-1/2}$ as well as $\text{Im}[m_{ac}(E + iy)] \leq C \sqrt{|E| - 2| + \eta}$ (see, e.g., Section 3 of [7]). Similarly, the second last term is bounded above by
\[
\int \int |f(x)| \chi'(y) |\text{Im} S(x + iy)|\,dx\,dy \leq C\|f\|_{L^1} \frac{1}{N \eta_1} \left( \frac{1}{N \eta_1} + \sqrt{\kappa + \eta_1} \right)
\leq \frac{C N^{\delta_1 + \varepsilon_1}}{N} (N^{-2\delta_1} + N^{-1/4})
\] (A.36)

For the first term, we first estimate the contribution of $y \leq \eta_2 := \frac{N^\varepsilon}{N}$. From the fact that $y \rightarrow y \text{Im}[m_M(x + iy)]$ is increasing (and the same for $m_H$) and the local semi-circle law, we find the estimate
\[
|y \text{Im}[S(x + iy)]| \leq 2 \frac{N^\varepsilon}{N} + C \eta_2 \text{Im}[m_{ac}]
\] (A.37)
which gives
\[
\left| \int \int_{|y| < \eta_2} y f''(x) \chi(y) \text{Im}[S(x + iy)]\,dx\,dy \right| \leq C \frac{N^\varepsilon}{N^{\delta_1}} \left( \frac{N^\varepsilon}{N} + C \eta_2 \sqrt{\eta_2 + \kappa} \right)
\leq C \frac{N^{2\varepsilon}}{N} N^{-\delta_1}
\] (A.38)

For $\eta > \eta_2$ we get,
\[
\left| \int \int_{|y| > \eta_2} y f''(x) \chi(y) \text{Im}[S(x + iy)]\,dx\,dy \right| \leq C N^\varepsilon \frac{N}{N^{\delta_1}} \int_{\eta_2 < y < 2\eta_2} \left( \frac{1}{N^2 y} + \frac{1}{N} \sqrt{y + \kappa} \right)
\leq C \left( \frac{N^{2\varepsilon}}{N^{1 + \delta_1}} + N^{\varepsilon - \delta_1} \sqrt{\kappa \eta_1} + N^{\varepsilon - \delta_1} \eta_1^{3/2} \right) \leq C N^{2\varepsilon - 1 - \delta_1}
\] (A.39)

We have proven,
\[
|\text{trf}(M) - \text{trf}(H)| \leq N^{3\varepsilon} \left( N^{-\delta_1} + N^{\delta_1 + \varepsilon_1 - 2\delta_1} + N^{-\delta_1} \right).
\] (A.40)

The claim follows from choosing $\delta_1 = \delta_f + \varepsilon_1 + \frac{1}{10}$.

**Proof of Proposition A.1.** Let now $0 < \delta_f < \frac{1}{20}$ and $\varepsilon > 0$, with $\varepsilon$ sufficiently small. Let us denote by $\lambda_i$ the eigenvalues of $H$ and by $\mu_i$ the eigenvalues of $M$. Applying Lemma A.5 to $f$ with $|a - b| = N^{\delta_f}/N$, and $a > \lambda_1 + N^{\delta_f - 1}$, we see first that
\[
\mu_1 \leq \lambda_1 + N^{\delta_f - 1}.
\] (A.41)

(or else for some $a > \lambda_1 + N^{\delta_f - 1}$ we would have $\text{trf}(M) \geq 1$, contradicting the estimate proven in Lemma A.5 and the fact that $\text{trf}(H) = 0$ for such $f$). Reversing the roles of $\mu_1$ and $\lambda_1$ we then get that
\[
|\mu_1 - \lambda_1| \leq N^{\delta_f - 1}.
\] (A.42)

Let $k_1$ be the smallest index so that
\[
\lambda_{k_1} - \lambda_{k_1 + 1} > 10 \frac{N^{\delta_f}}{N}
\] (A.43)
and let $J_i = [(1, k_i)]$. Similarly, let $k_2 > k_1$ be the smallest index so that
\[
\lambda_{k_2} - \lambda_{k_2 + 1} > 10 \frac{N^{\delta_f}}{N}, \tag{A.44}
\]
and let $J_2 = [(k_1 + 1, k_2)]$, and define $J_i = [(k_{i-1} + 1, k_i)]$ and so on. Let $\ell$ be the smallest integer so that
\[
[(1, N^{1/20})] \subseteq J_1 \cup J_2 \cup \cdots \cup J_\ell. \tag{A.45}
\]
By rigidity we have that for $j - i > N^\varepsilon$,
\[
|\lambda_i - \lambda_j| > c \frac{j - i}{N^{2/3} (j^{1/3})} \approx N^{\delta_f} j - i, \tag{A.46}
\]
if $j \leq N^{1/20}$. Therefore,
\[
|J_i| \leq N^\varepsilon, \text{ for } i \leq \ell. \tag{A.47}
\]
First, using Lemma A.5 we see that there are no eigenvalues $\mu_i$ in the interval
\[
[\lambda_{k_i + 1} + N^{\delta_f - 1}, \lambda_{k_i} - N^{\delta_f - 1}], \tag{A.48}
\]
for $i \leq \ell$, by taking $a > \lambda_{k_i + 1} + N^{\delta_f - 1}$ and $b < \lambda_{k_i} - N^{\delta_f - 1}$, and $|a - b| = N^{\delta_f - 1}$. Next, we apply Lemma A.5 with the choice $b = \lambda_{k_i + 1} + N^{\delta_f - 1}$ and $a = \lambda_{k_i + 1} - N^{\delta_f - 1}$. Note that since $|J_i| \leq N^\varepsilon$, we see that the length of $[a, b]$ in this case is less than $CN^{\varepsilon + \delta_f - 1}$, and so the lemma applies, which gives
\[
\text{trf}(M) = |J_i| + o(1). \tag{A.49}
\]
Since we have already shown that there are no eigenvalues $\mu_i$ in the intervals $[\lambda_{k_i + 1} - 2N^{\delta_f - 1}, \lambda_{k_i + 1} - N^{\delta_f - 1}]$ and $[\lambda_{k_i + 1} + N^{\delta_f - 1}, \lambda_{k_i + 1} + 2N^{\delta_f - 1}]$, it follows that the quantity $\text{trf}(M)$ is precisely the number of eigenvalues in the interval $[\lambda_{k_i + 1} - N^{\delta_f - 1}, \lambda_{k_i + 1} + N^{\delta_f - 1}]$. This must be an integer, and so it equals $|J_i|$. The claim follows. \hfill \Box

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