STOCHASTIC HEAT EQUATION WITH GENERAL ROUGH NOISE

YAOZHONG HU AND XIONG WANG

Abstract. We study the well-posedness of a nonlinear one dimensional stochastic heat equation driven by Gaussian noise: \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sigma(u) W \), where \( W \) is white in time and fractional in space with Hurst parameter \( H \in \left( \frac{1}{4}, \frac{1}{2} \right) \). In a recent paper [12] by Hu, Huang, Lê, Nualart and Tindel a technical and unusual condition of \( \sigma(0) = 0 \) was assumed which is critical in their approach. The main effort of this paper is to remove this condition. The idea is to work on a weighted space \( L^{p}_{-\lambda, \beta} \) for some power decay weight \( \lambda(x) = c_H (1 + |x|^2)^{H - 1} \). In addition, when \( \sigma(u) = 1 \) we obtain the exact asymptotics of the solution \( u_{\text{add}}(t, x) \) as \( t \) and \( x \) go to infinity. In particular, we find the exact growth of \( \sup_{|x| \leq L} |u_{\text{add}}(t, x)| \) and the sharp growth rate for the Hölder coefficients, namely, \( \sup_{|x| \leq L} \left[ u_{\text{add}}(t, x + h) - u_{\text{add}}(t, x) \right] / |h|^\alpha \)

Abstract. Nous étudions une équation de chaleur stochastique à une dimension spatiale non linéaire entraînée par le bruit gaussien: \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sigma(u) W \), où \( W \) est blanc dans le temps et fractionnaire dans l’espace avec le paramètre Hurst \( H \in \left( \frac{1}{4}, \frac{1}{2} \right) \). Dans un article récent [12] par Hu, Huang, Lê, Nualart et Tindel une condition technique et inhabituelle de \( \sigma(0) = 0 \) a été supposée, ce qui est critique dans leur approche. Le principal effort de ce document est de supprimer cette condition. L’idée est de travailler sur un espace pondéré \( L^{p}_{-\lambda, \beta} \) pour un certain poids de décroissance de puissance \( \lambda(x) = c_H (1 + |x|^2)^{H - 1} \). Lorsque \( \sigma(u) = 1 \) nous obtenons les asymptotiques exactes de la solution \( u_{\text{add}}(t, x) \) as \( t \) et \( x \) vont à l’infini. En particulier, nous trouvons la croissance exacte des \( \sup_{|x| \leq L} |u_{\text{add}}(t, x)| \) et la croissance exacte des coefficients de Hölder, c’est-à-dire, \( \sup_{|x| \leq L} \left[ u_{\text{add}}(t, x + h) - u_{\text{add}}(t, x) \right] / |h|^\alpha \) et \( \sup_{|x| \leq L} \left[ u_{\text{add}}(t, x + h) - u_{\text{add}}(t, x) \right] / |h|^\alpha \).

1. Introduction and main results

In this paper, we consider the following one dimensional (in space variable) nonlinear stochastic heat equation driven by the Gaussian noise which is white in time and fractional in space:

\[
\frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + \sigma(t, x, u(t, x)) W(t, x), \quad t \geq 0, \quad x \in \mathbb{R}, \quad (1.1)
\]

Key words and phrases. Rough Gaussian noise, (nonlinear) stochastic heat equation, additive noise, temporal and spatial asymptotic growth, Hölder constants over unbounded domain, Majorizing measure, weak solution, strong solution, weighted spaces, weighted heat kernel estimates, pathwise uniqueness.

Supported by an NSERC discovery grant and a startup fund from University of Alberta at Edmonton.
where $W(t, x)$ is a centered Gaussian process with covariance given by

$$E[W(t, x)W(s, y)] = \frac{1}{2}(s \wedge t)(|x|^2H + |y|^2H - |x - y|^2H)$$

(1.2)

and where $\frac{1}{4} < H < \frac{1}{2}$ and $W(t, x) = \frac{\partial}{\partial t} W(t, x)$.

There has been a lot of work on stochastic heat equations driven by general Gaussian noises. We refer to [11] for a short survey and for more references. The main feature of this work is that the noise is rough (e.g. $\frac{1}{4} < H < \frac{1}{2}$) in the space variable. We mention three works that are directly related to this specific Gaussian noise structure. The first two are [3] and [13], where the authors study the existence, uniqueness and some properties such as moment bounds of the mild solution when the diffusion coefficient $\sigma$ is affine (i.e. $\sigma(t, x, u) = \sigma(u) = au + b$) in [3] or linear (i.e. $\sigma(u) = au$) in [13]. After these works researchers tried to study (1.1) for general nonlinear $\sigma$. However, the method effective for affine (and linear) equations cannot no longer work. One difficulty for general nonlinear diffusion coefficient $\sigma$ is that we cannot no longer bound $|\sigma(u_1) - \sigma(u_2) - \sigma(v_1) + \sigma(v_2)|$ by a multiple of $|u_1 - u_2 - v_1 + v_2|$ (which is possible only in the affine case). A breakthrough was made in [12]. However, to solve equation (1.1) the authors in [12] have to assume that $\sigma(0) = 0$, which does not even cover the affine case studied in [3]! The main motivation of this paper is to remove the condition $\sigma(0) = 0$ assumed in [12]. To this end we need to understand why this condition is so crucial there.

We first find out that this condition $\sigma(0) = 0$ can ensure the solution lives in the space $\mathcal{Z}_p^0$ (see [12] or (4.4) in Section 4 of this paper with $\lambda(x) = 1$). As we shall see that even in the simplest case $\sigma(u) \equiv 1$ (of the case $\sigma(0) \neq 0$), the solution is no longer in $\mathcal{Z}_p^0$ (see e.g. Theorem 1.1 and Proposition 3.11). Moreover, the initial condition in [12] must be integrable to guarantee the solution belongs to $\mathcal{Z}_p^0$, which means $u_0(x) = 1$ is excluded. Thus, to remove the restriction $\sigma(0) = 0$, we must find another appropriate solution space. Our idea is to introduce a decay weight (as the spatial variable $x$ goes to infinity) to enlarge the solution space $\mathcal{Z}_p^0$ to a weighted space $\mathcal{Z}_p^0(\lambda, T)$ for some suitable power decay function $\lambda(x)$. This weight function will have to be chosen appropriately (not too fast and not too slow. See Section 2 for details).

The introduction of the weight makes all the tools used in [12] collapse. As we can see we shall need a whole set of new understandings of the heat kernel to complete our program. People may wonder whether one can still just use $\mathcal{Z}_p^0$ for our solution space. This question is natural since we work on the whole real line $\mathbb{R}$ for the space variable. A constant function is in $L^\infty(\mathbb{R})$ but not in $L^p(\mathbb{R})$ for any finite $p$. If it happens to be possible to use $\mathcal{Z}_p^0$ (without weight), then many computations in [12] will still be valid and the problem becomes greatly simplified.

To see if this is possible or not we consider the solution $u_{add}(t, x)$ to the equation with additive noise, which is the solution to (1.1) with $\sigma(u) = 1$ and with initial condition $u_0(x) = 0$. This is the simplest case that $\sigma(0) \neq 0$. To find out if $u_{add}(t, x)$ is in $\mathcal{Z}_p^0$ or not (or to see if the introduction of decay weight $\lambda$ is necessary or not), we shall find the sharp bound of the solution $u_{add}(t, x)$ as $x$ goes to infinity. In other words, we shall find the exact explosion rate of $\sup_{|x| \leq L} |u_{add}(t, x)|$ as $L$ goes to infinity. This problem has a great value of its own. To study the supremum of a family of random variables, there are two powerful tools: one is to use the independence and the other one is to use the martingale inequalities. However, $u_{add}(t, x)$ is not a martingale with respect to the spatial variable $x$ (nor
it is a martingale with respect to the time variable $t$ and since the noise $\tilde{W}$ is not independent in the spatial variable either, the application of independence may be much more involved (We refer, however, to [4, 5, 6, 8] for some successful applications of the independence in the stochastic heat equation (1.1)). In this work, we shall use instead the idea of majorizing measure to obtain sharp growth of $\sup_{|x| \leq L} |u_{\text{add}}(t, x)|$ and $\sup_{0 \leq t \leq T, |x| \leq L} |u_{\text{add}}(t, x)|$, as $L$ and $T$ go to infinity, both in terms of expectation and almost surely. More precisely, we have

**Theorem 1.1.** Let the Gaussian field $u_{\text{add}}(t, x)$ be the solution to (1.1) with $\sigma(t, x, u) = 1$ and $u_0(x) = 0$. Then, we have the following statements.

1. There are two positive constants $c_H$ and $C_H$, independent of $T$ and $L$, such that

$$c_H \Psi(T, L) \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T, -L \leq x \leq L} u_{\text{add}}(t, x) \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T, -L \leq x \leq L} |u_{\text{add}}(t, x)| \right] \leq C_H \Psi(T, L),$$

where $\Psi_0(T, L) := 1 + \sqrt{\log_2 \left( L/\sqrt{T} \right)}$, $L \geq \sqrt{T}$ and

$$\Psi(T, L) := \begin{cases} T^{\frac{H}{2}} \Psi_0(T, L) & \text{if } L \geq \sqrt{T}, \\ T^{\frac{H}{2}} & \text{if } L < \sqrt{T}. \end{cases}$$

2. There are two strictly positive random constants $c_H$ and $C_H$, independent of $T$ and $L$, such that almost surely

$$c_H T^{\frac{H}{2}} \Psi_0(T, L) \leq \sup_{(t, x) \in \mathcal{Y}(T, L)} u_{\text{add}}(t, x) \leq \sup_{(t, x) \in \mathcal{Y}(T, L)} |u_{\text{add}}(t, x)| \leq C_H T^{\frac{H}{2}} \Psi_0(T, L),$$

where $\mathcal{Y}(T, L) = \{(t, x) \in [0, T] \times [-L, L] : L \geq \sqrt{T} \}$.

Let us point out that Theorem 1.1 is an extension of Theorem 1.2 of [6] and Theorem 2.3 of [8] to spatial rough noise.

It is well-known that the solution to equation (1.1), if exists, is usually Hölder continuous on any bounded domain. But usually it is not Hölder continuous on the whole space. An interesting question to ask is how the Hölder coefficient depends on the size of the domain. Since the additive solution $u_{\text{add}}(t, x)$ is a Gaussian random field we will be able to obtain sharp dependence on the size of the domain of the Hölder coefficient. In the following theorem we state our result on the Hölder continuity in spatial variable over unbounded domain.

**Theorem 1.2.** Let $u_{\text{add}}(t, x)$ be the solution to (1.1) with $\sigma(t, x, u) = 1$ and $u_0(x) = 0$ and denote

$$\Delta_h u_{\text{add}}(t, x) := u_{\text{add}}(t, x + h) - u_{\text{add}}(t, x).$$

Let $0 < \theta < H$ be given. Then, there are positive constants $c, c_H$ and $C_{H, \theta}$ such that the following inequalities hold true:

$$c_H |h|^{\frac{H}{2}} \Psi_0(t, L) \leq \mathbb{E} \left[ \sup_{-L \leq x \leq L} \Delta_h u_{\text{add}}(t, x) \right] \leq \mathbb{E} \left[ \sup_{-L \leq x \leq L} |\Delta_h u_{\text{add}}(t, x)| \right] \leq C_{H, \theta} t^{\frac{H}{2} - \theta} |h|^\theta \Psi_0(t, L).$$

3
for all $L \geq \sqrt{t} > 0$ and $0 < |h| \leq c(\sqrt{t} \land 1)$. Moreover, there are two (strictly) positive random constants $c_H$ and $C_{H,0}$

$$c_H |h|^2 \Psi_0(t, L) \leq \sup_{-L \leq x \leq L} |\Delta_h u_{add}(t, x)| \leq C_{H,0} t^{\frac{u_d}{2}} |h|^2 \Psi_0(t, L)$$  \hspace{1cm} (1.7)

for all $L \geq \sqrt{t} > 0$ and $0 < |h| \leq c(\sqrt{t} \land 1)$.

Next, we study the Hölder continuity in time over the unbounded domain. We state the following.

**Theorem 1.3.** Let $u_{add}(t, x)$ be the solution to (1.1) with $\sigma(t, x, u) = 1$ and $u_0(x) = 0$ and denote

$$\Delta_\tau u_{add}(t, x) := u_{add}(t + \tau, x) - u_{add}(t, x).$$

Let $0 < \theta < H/2$. Then, there are positive constants $c$, $c_H$ and $C_{H,0}$ such that

$$c_H \tau^{\frac{u_d}{2}} \Psi_0(t, L) \leq \mathbb{E} \left[ \sup_{-L \leq x \leq L} |\Delta_\tau u_{add}(t, x)| \right] \leq C_{H,0} t^{\frac{u_d}{2} - \theta} \Psi_0(t, L)$$  \hspace{1cm} (1.8)

for all $L \geq \sqrt{t} > 0$ and $0 < \tau \leq c(t \land 1)$. We also have the almost sure version of the above result.

$$c_H \tau^{\frac{u_d}{2}} \Psi_0(t, L) \leq \sup_{-L \leq x \leq L} |\Delta_\tau u_{add}(t, x)| \leq C_{H,0} t^{\frac{u_d}{2} - \theta} \Psi_0(t, L)$$  \hspace{1cm} (1.9)

for all $L \geq \sqrt{t} > 0$ and $0 < \tau \leq c(t \land 1)$. Now, $c_H$ and $C_{H,0}$ are random.

The above Theorems 1.1-1.3 are proved in Section 3. Now let us return to the equation (1.1). To make things precise we give here the definitions of strong and weak solutions.

**Definition 1.4.** Let $\{u(t, x), t \geq 0, x \in \mathbb{R}\}$ be a real-valued adapted stochastic process such that for all $t \in [0, T]$ and $x \in \mathbb{R}$ the process $\{G_{t-s}(x-y)\sigma(s, y, u(s, y))1_{[0, t]}(s)\}$ is integrable with respect to $W$ (see Definition 2.4), where $G_t(x) := \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right)$ is the heat kernel on $\mathbb{R}$ associated with the Laplacian operator $\Delta$.

(i) We say that $u(t, x)$ is a strong (mild) solution to (1.1) if for all $t \in [0, T]$ and $x \in \mathbb{R}$ we have

$$u(t, x) = G_t \ast u_0(x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(s, y, u(s, y))W(ds, dy)$$  \hspace{1cm} (1.10)

almost surely, where the stochastic integral is understood in the sense of Definition 2.4.

(ii) We say (1.1) has a weak solution if there exists a probability space with a filtration $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}_t)$, a Gaussian random field $\tilde{W}$ identical to $W$ in law, and an adapted stochastic process $\{u(t, x), t \geq 0, x \in \mathbb{R}\}$ on this probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}_t)$ such that $u(t, x)$ is a mild solution to (1.1) with respect to $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}_t)$ and $\tilde{W}$.
Before stating our theorem on the existence of a weak solution, we make the following assumption.

(H1) \( \sigma(t, x, u) \) is jointly continuous over \([0, T] \times \mathbb{R}^2\) and is at most of linear growth in \( u \) uniformly in \( t \) and \( x \). This means

\[
\sup_{t \in [0,T], x \in \mathbb{R}} |\sigma(t, x, u)| \leq C(|u| + 1) \tag{1.11}
\]

for some positive constant \( C \). We also assume that it is uniformly Lipschitzian in \( u \), namely, \( \forall u, v \in \mathbb{R} \)

\[
\sup_{t \in [0,T], x \in \mathbb{R}} |\sigma(t, x, u) - \sigma(t, x, v)| \leq C|u - v|, \tag{1.12}
\]

for some constant \( C > 0 \).

We can now state our main theorem of the paper.

**Theorem 1.5.** Let \( \lambda(x) = c_H (1 + |x|^2)^{H-1} \) satisfy \( \int_{\mathbb{R}} \lambda(x)dx = 1 \). Assume that \( \sigma(t, x, u) \) satisfies hypothesis (H1) and that the initial data \( u_0 \) belongs to \( Z^p_{0,0} \) for some \( p > \frac{6}{4H-1} \) (see Section 4.1 for the definition of \( Z^p_{0,0} \)). Then, there exists a weak solution to (1.1) with sample paths in \( C([0,T] \times \mathbb{R}) \) almost surely. In addition, for any \( \gamma < H - \frac{2}{p} \), the process \( u(\cdot, t) \) is almost surely Hölder continuous on any compact sets in \([0,T] \times \mathbb{R}\) of Hölder exponent \( \gamma/2 \) with respect to the time variable \( t \) and of Hölder exponent \( \gamma \) with respect to the spatial variable \( x \).

From Theorem 1.1 we see that when \( \sigma(0) \neq 0 \) we expect that the solution will not be in the space \( Z^p_T \). We enlarge it to the weighted space \( Z^p_{\lambda,T} \) in the above theorem. As we said earlier that the introduction of the weight \( \lambda \) makes the computations in [12] no longer applicable. For example, now we need to control, roughly speaking, a certain norm of \( \lambda(\cdot) \int_{\mathbb{R}} G_{s-a}(\cdot - y)u(s, y)dW(s, y) \) and its fractional derivative (with respect to spatial variable) by the similar norm of \( \lambda(\cdot)u(\cdot, \cdot) \) and its fractional derivative. This would require us to study the delicate properties of \( \lambda(x)G_t(x - y)\lambda^{-1}(y) \). Thus, we need some very subtle and very sharp bounds on the heat kernel \( G_t(x - y) \) with respect to the weight function \( \lambda(x) \), which are of interest in their own. This is done in Section 2. After these preparations, we shall show the above theorem in Section 4. Although the techniques of [12] are no longer effective in our new situation we still follow the same spirit there.

It is always interesting to have existence and uniqueness of the strong solution. As we said earlier, due to the roughness of the noise we need to handle, as in [12], the square increment \( |\sigma(u_1) - \sigma(u_2) - \sigma(v_1) + \sigma(v_2)| \). It seems too complicated for the weighted space. So, to show the existence and uniqueness of strong solution we assume that the derivative of the diffusion coefficient in (1.1) possesses a decay itself as \( x \to \infty \). More precisely, we make the following assumptions.

(H2) Assume that \( \sigma(t, x, u) \in C^{0,1,1}([0, T] \times \mathbb{R}^2) \) satisfies the following conditions:

\[
|\sigma'(t, x, u)| \text{ and } |\sigma''(t, x, u)| \text{ are uniformly bounded}, \text{i.e. there is a constant } C > 0 \text{ such that}
\]

\[
\sup_{t \in [0,T], x \in \mathbb{R}, u \in \mathbb{R}} |\sigma'(t, x, u)| \leq C; \tag{1.13}
\]

\[
\sup_{t \in [0,T], x \in \mathbb{R}, u \in \mathbb{R}} |\sigma''(t, x, u)| \leq C. \tag{1.14}
\]
Moreover, we assume that for some \( p > \frac{6}{4H - 1} \),
\[
\sup_{t \in [0, T], x \in \mathbb{R}} \lambda^{-\frac{1}{p}} (x) |\sigma_u(t, x, u_1) - \sigma_u(t, x, u_2)| \leq C|u_1 - u_2|.
\] (1.15)

**Theorem 1.6.** Let \( \sigma \) satisfy the above hypothesis (\( H_2 \)) and assume that for some \( p > \frac{6}{4H - 1}, u_0 \in \mathbb{Z}^p \). Then (1.1) has a unique strong solution with sample paths in \( C([0, T] \times \mathbb{R}) \) almost surely. Moreover, the process \( u(\cdot, \cdot) \) is uniformly H"older continuous almost surely on any compact subset of \( [0, T] \times \mathbb{R} \) with the same temporal and spatial H"older exponents as those in Theorem 1.5.

This theorem will be proved in Section 5. Let us point out that if \( \sigma (u) \) is affine, then it satisfies the assumption (\( H_2 \)).

## 2. Auxiliary Lemmas

In this section, we shall obtain some estimates about the heat kernel \( G_t(x) = \frac{1}{4\pi t} e^{-\frac{|x-y|^2}{4t}} \) associated with the Laplacian \( \Delta \) combined with the decay weight \( \lambda(x) \). These estimates are the key ingredients to establish our results.

### 2.1. Covariance structure

We start by recalling some notations used in [12]. Denote by \( \mathcal{D} = \mathcal{D} (\mathbb{R}) \) the space of smooth functions on \( \mathbb{R} \) with compact support, and by \( \mathcal{D}' \) the dual of \( \mathcal{D} \) with respect to the \( L^2(\mathbb{R}, dx) \). The Fourier transform of a function \( f \in \mathcal{D} \) is defined as
\[
\hat{f}(\xi) = \mathcal{F} f(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx,
\]
and the inverse Fourier transform is then given by \( \mathcal{F}^{-1}g(x) = \frac{1}{2\pi} \mathcal{F}g(-x) \).

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space and let \( H \in (\frac{1}{4}, \frac{1}{2}) \) be given and fixed. Our noise \( \tilde{W} \) is a zero-mean Gaussian family \( \{ W(\phi), \phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}) \} \) with covariance structure given by
\[
\mathbb{E} [ W(\phi) W(\psi) ] = c_{1,H} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F} \phi(s, \xi) \mathcal{F} \psi(s, \xi) |\xi|^{1-2H} ds d\xi,
\] (2.1)
where \( c_{1,H} \) is given below by (2.7) and \( \mathcal{F} \phi(s, \xi) \) is the Fourier transform with respect to the spatial variable \( x \) of the function \( \phi(s, x) \). Let \( \mathcal{F}_t \) be the filtration generated by \( \tilde{W} \). This means
\[
\mathcal{F}_t = \sigma \{ W(\phi(x) 1_{[0, t]} (s)) : r \in [0, t], \phi(x) \in \mathcal{D}(\mathbb{R}) \}.
\]
Equation (2.1) defines a Hilbert scalar product on \( \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}) \). To express this product without the use of Fourier transform, we recall the Marchaud fractional derivative \( D^\beta \) of order \( \beta \in (0, 1) \). For a function \( \phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \), the Marchaud fractional derivative \( D^\beta \) is defined as:
\[
D^\beta \phi(t, x) = \lim_{\epsilon \downarrow 0} D^\beta_{-\epsilon,x} \phi(t, x) = \lim_{\epsilon \downarrow 0} \frac{\beta}{\Gamma(1 - \beta)} \int_{\epsilon}^{\infty} \frac{\phi(t, x) - \phi(t, x + y)}{y^{1+\beta}} dy.
\] (2.2)
We also define the Riemann-Liouville fractional integral of order \( \beta \) of a function \( \phi \) by
\[
\int^\beta \phi(t, x) = \frac{1}{\Gamma(\beta)} \int_x^\infty \phi(t, y)(y - x)^{\beta-1} dy.
\]
Set
\[ S = \{ \phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \mid \exists \psi \in L^2(\mathbb{R}_+ \times \mathbb{R}) \text{ s.t. } \phi(t, x) = I^{\frac{1}{2} - H}_x \psi(t, x) \}. \quad (2.3) \]
With these notations we can express the Hilbert space obtained by completing \( D(\mathbb{R}_+ \times \mathbb{R}) \) with respect to the scalar product given by (2.1) in the following proposition (see e.g. [21] for a proof).

**Proposition 2.1.** The function space \( S \) is a Hilbert space equipped with the scalar product
\[ \langle \phi, \psi \rangle_S = c_{1, H} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F} \phi(s, \xi) \mathcal{F} \psi(s, \xi) |\xi|^{1-2H} d\xi ds \]
\[ = c_{2, H} \int_{\mathbb{R}_+ \times \mathbb{R}} D^{\frac{1}{2} - H} \phi(t, x) D^{\frac{1}{2} - H} \psi(t, x) dx dt \]
\[ = c_{3, H} \int_{\mathbb{R}_+} \int_{\mathbb{R}^2} |\phi(t, x + y) - \phi(t, x)| |\psi(t, x + y) - \psi(t, x)| y^{2H-2} dx dy dt, \]
(2.6)
where
\[ c_{1, H} = \frac{1}{2\pi} \Gamma(2H + 1) \sin(\pi H) ; \]
\[ c_{2, H} = \left[ \Gamma \left( H + \frac{1}{2} \right) \right]^2 \left( \int_0^\infty \left[ (1 + t)^{H-\frac{1}{2}} - t^{H-\frac{1}{2}} \right] dt + \frac{1}{2H} \right)^{-1} ; \]
\[ c_{3, H} = \sqrt{H(\frac{1}{2} - H)} c_{2, H}^{1/2}. \]

The space \( D(\mathbb{R}_+ \times \mathbb{R}) \) is dense in \( S \).

The Gaussian space \( S \) is the same as the homogeneous Sobolev space \( \mathcal{H}_0^{\beta} \) for \( \beta = \frac{1}{2} - H \in (0, \frac{1}{2}) \) in harmonic analysis ([2]). The Gaussian family \( W = \{ W(\phi), \phi \in D(\mathbb{R}_+ \times \mathbb{R}) \} \) can be extended to an isonormal Gaussian process \( W = \{ W(\phi), \phi \in S \} \) indexed by the Hilbert space \( S \). It is easy to see that \( \phi(t, x) = \chi_{\{0,t\} \times \{0,x\}}, t \in \mathbb{R}_+, \) and \( x \in \mathbb{R}, \) in \( S \) (we set \( \chi_{\{0,t\} \times \{0,x\}} = -\chi_{\{0,t\} \times \{x,0\}} \) if \( x \) is negative). We denote \( W(t, x) = W(\chi_{\{0,t\} \times \{0,x\}}) \).

2.2. **Stochastic integration.** We first define stochastic integral for elementary integrands and then extend it to general ones.

**Definition 2.2.** An elementary process \( g \) is a process of the following form
\[ g(t, x) = \sum_{i=1}^n \sum_{j=1}^m X_{i,j} 1_{(a_i, b_i]}(t) 1_{(h_j, l_j]}(x), \]
where \( n \) and \( m \) are finite positive integers, \( -\infty < a_1 < b_1 < \cdots < a_n < b_n < \infty, \) \( h_j < l_j \) and \( X_{i,j} \) are \( \mathcal{F}_t \)-measurable random variables for \( i = 1, \ldots, n. \) The stochastic integral of such an elementary process with respect to \( W \) is defined as
\[ \int_{\mathbb{R}_+} \int_{\mathbb{R}} g(t, x) W( dt, dx ) = \sum_{i=1}^n \sum_{j=1}^m X_{i,j} W(1_{(a_i, b_i]} \otimes 1_{(h_j, l_j]}) \]
\[ = \sum_{i=1}^n \sum_{j=1}^m X_{i,j} [ W(b_i, l_j) - W(a_i, l_j) - W(b_i, h_j) + W(a_i, h_j) ]. \]

(2.10)
Proposition 2.3. Let $\Lambda_H$ be the space of predictable processes $g$ defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \mathcal{F}$ and $\mathbb{E}[\|g\|_3^2] < \infty$. Then, the space of elementary processes defined in Definition 2.2 is dense in $\Lambda_H$.

Definition 2.4. For $g \in \Lambda_H$, the stochastic integral $\int_{\mathbb{R}_+ \times \mathbb{R}} g(t, x) W(dt, dx)$ is defined as the $L^2(\Omega)$ limit of stochastic integrals of the elementary processes approximating $g(t, x)$ in $\Lambda_H$, and we have the following isometry equality

$$
\mathbb{E} \left[ \left( \int_{\mathbb{R}_+ \times \mathbb{R}} g(t, x) W(dt, dx) \right)^2 \right] = \mathbb{E} \left[ \|g\|^2_3 \right]. \tag{2.11}
$$

2.3. Auxiliary Lemmas. We shall find a solution to equation (1.1) in the space $\mathcal{L}_x^p$. To deal with weight $\lambda$ we need a few technical results concerning the interaction between the weight $\lambda(x)$ and the Green’s function $G_t(x - y)$.

Lemma 2.5. For any $\lambda \in \mathbb{R}$, $\lambda(x) = \frac{1}{1 + |x|^2 \lambda}$ and $T > 0$, we have

$$
\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} \frac{1}{\lambda(x)} \int_{\mathbb{R}} G_t(x - y) \lambda(y) dy < \infty. \tag{2.12}
$$

Remark 2.6. To avoid using too many notations we use the symbol $\lambda$ for a real number and the function induced. Apparently, there will be no confusion.

Proof. Let us rewrite (2.12) as

$$
\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} G_t(y) \frac{\lambda(y + x)}{\lambda(x)} \lambda_{\lambda}(y) dy \leq \sup_{0 \leq t \leq T} \int_{\mathbb{R}} G_t(y) \sup_{x \in \mathbb{R}} \frac{\lambda(y + x)}{\lambda(x)} \lambda_{\lambda}(y) dy.
$$

We discuss the cases $\lambda \geq 0$ and $\lambda < 0$ separately. When $\lambda \geq 0$, we have

$$
\sup_{x \in \mathbb{R}} \frac{\lambda(y + x)}{\lambda(x)} \leq C_{\lambda} \sup_{x \in \mathbb{R}} \left( \frac{1 + |x|}{1 + |x + y|} \right)^{2\lambda} \leq C_{\lambda}(1 + |y|)^{2\lambda}.
$$

On the other hand when $\lambda < 0$ we have

$$
\sup_{x \in \mathbb{R}} \left( \frac{1 + |x + y|}{1 + |x|^2} \right)^{-\lambda} \leq C_{\lambda} \sup_{x \in \mathbb{R}} \left( \frac{1 + |x + y|}{1 + |x|} \right)^{-2\lambda} \leq C_{\lambda}(1 + |y|)^{-2\lambda}.
$$

In both cases we see

$$
\sup_{0 \leq t \leq T} \int_{\mathbb{R}} \lambda_{\lambda}(y) \sup_{x \in \mathbb{R}} \frac{\lambda(y + x)}{\lambda(x)} dy \leq C_{\lambda} \sup_{t \in [0, T]} \int_{\mathbb{R}} \lambda_{\lambda}(y)(1 + |y|)^{2\lambda} dy < \infty.
$$

This finishes the proof. $\square$

Lemma 2.7. Denote $J(x) := \int_0^\infty e^{-\eta^2} \eta^\beta \cos(x \eta) d\eta$, where $\beta > -1$. We have

$$
|J(x)| \leq C_{\beta} \left( \frac{1}{|x|^{\beta + 1}} \right)^{1}. \tag{2.13}
$$

Proof. Notice that this is to estimate the decay rate of the Fourier transform of $e^{-\eta^2} \eta^\beta$ when $|x|$ is large. Since $J(-x) = J(x)$ and since we are only concerned with the large $x$ behaviour we may assume $x \geq 1$. We split the integral $J(x)$ into two parts:

$$
J(x) = \int_0^{s(x)} e^{-\eta^2} \eta^\beta \cos(x \eta) d\eta + \int_{s(x)}^\infty e^{-\eta^2} \eta^\beta \cos(x \eta) d\eta := J_1(x) + J_2(x),
$$

where $s(x) > 0$ is a function to be determined shortly.
First, it is easy to see
\[ |J_1(x)| \leq \int_0^{s(x)} \eta^\beta d\eta \leq C_{\beta}[s(x)]^{\beta+1}. \]

For \( J_2(x) \), an integration by parts implies
\[
|J_2(x)| = \left| \int_{s(x)}^{\infty} e^{-\eta^2} \eta^\beta \cos(x\eta) d\eta \right|
= \left| \frac{1}{x} \int_{s(x)}^{\infty} e^{-\eta^2} \eta^\beta \sin(x\eta) d\eta \right|
\leq C_{\beta} \frac{[s(x)]^\beta}{x} + C_{\beta} \frac{e^{-\eta^2}}{x} \int_{s(x)}^{\infty} \eta^{\beta-1} e^{-\eta^2} \sin(x\eta) d\eta
+ C_{\beta} \left| \int_{s(x)}^{\infty} \eta^{\beta+1} e^{-\eta^2} \sin(x\eta) d\eta \right|.
\]

Let \( k = \lceil \beta \rceil \) denote the least integer greater than or equal to \( \beta \). Continuing the above application of integration by parts another \( k \) times yields
\[ |J_2(x)| \leq \frac{C_{\beta}}{x^{k+1}} + C_{\beta} \sum_{j=0}^{k} \frac{[s(x)]^{\beta-j} + [s(x)]^{\beta+j}}{x^j+1}. \]

Combining the estimates of \( J_1(x) \) and \( J_2(x) \) we have
\[ |J(x)| \leq C_{\beta}[s(x)]^{\beta+1} + \frac{C_{\beta}}{x^{k+1}} + C_{\beta} \sum_{j=0}^{k} \frac{[s(x)]^{\beta-j} + [s(x)]^{\beta+j}}{x^j+1}. \]

The lemmas follows with the choice of \( s(x) = \frac{1}{x} \). ∎

Let us associate two increments related to the Green function \( G_t(x) \), given as follows. The first one is a first order difference:
\[ D_t(x, h) := G_t(x + h) - G_t(x). \quad (2.14) \]

Denote \( D(x, h) = \sqrt{\pi} D_{1/4}(x, h) = e^{-(x+h)^2} - e^{-x^2} \). The second one is a second order difference:
\[ \Box_t(x, y, h) := G_t(x + y + h) - G_t(x + y) - G_t(x + h) + G_t(x). \quad (2.15) \]

As above, we denote \( \Box(x, y, h) = \sqrt{\pi} \Box_{1/4}(x, y, h) \):
\[ \Box(x, y, h) = e^{-(x+y+h)^2} - e^{-(x+h)^2} - e^{-(x+y)^2} + e^{-x^2}. \quad (2.16) \]

For these two increments, we have the following identities which are needed later.

**Lemma 2.8.** For any \( \alpha, \beta \in (0, 1) \), we have
\[
\int_{\mathbb{R}^2} |D_t(x, h)|^2 |h|^{-1-2\beta} dh dx = \frac{C_{\beta}}{t^{\frac{7}{2}+\beta}} \quad (2.17)
\]
and
\[
\int_{\mathbb{R}^3} |\Box_t(x, y, h)|^2 |h|^{-1-2\alpha} |y|^{-1-2\beta} dy dh dx = \frac{C_{\alpha}\beta}{t^{\frac{7}{2}+\alpha+\beta}} \quad (2.18)
\]
Proof. With change of variables, it suffices to show
\begin{align}
\int_{\mathbb{R}^2} |D(x, h)|^2 |h|^{-1-2\beta} dh dx < \infty; \\
\int_{\mathbb{R}^3} |\Box(x, y, h)|^2 |h|^{-1-2\alpha} |y|^{-1-2\beta} dy dh dx < \infty.
\end{align}
(2.19)

The above two inequalities will be derived from Plancherel’s identity. The Fourier transforms with respect to the variable $x$ of $D(x, h)$ and $\Box(x, y, h)$ are, respectively,
\[ \hat{D}(\xi, h) = \mathcal{F}[D(\cdot, h)](\xi) = \sqrt{\pi} e^{-\frac{\xi^2}{4}} [e^{ih\xi} - 1] \]
and
\[ \hat{\Box}(\xi, y, h) = \mathcal{F}[\Box(\cdot, y, h)](\xi) = \sqrt{\pi} e^{-\frac{\xi^2}{4}} [e^{iy\xi} - 1][e^{ih\xi} - 1]. \]

Thus, we have
\[ \int_{\mathbb{R}} |D(x, h)|^2 dx = \int_{\mathbb{R}} |\hat{D}(\xi, h)|^2 d\xi = 4\pi \int_{\mathbb{R}} e^{-\frac{\xi^2}{4}} [1 - \cos(h\xi)] d\xi \]
and
\[ \int_{\mathbb{R}} |\Box(x, y, h)|^2 dx = \int_{\mathbb{R}} |\hat{\Box}(\xi, y, h)|^2 d\xi = 4\pi \int_{\mathbb{R}} e^{-\frac{\xi^2}{4}} [1 - \cos(h\xi)][1 - \cos(y\xi)] d\xi. \]

By Fubini’s theorem
\begin{align}
\int_{\mathbb{R}^2} |D(x, h)|^2 |h|^{-1-2\beta} dh dx &= C \int_{\mathbb{R}} e^{-\frac{\xi^2}{4}} d\xi \int_{\mathbb{R}} [1 - \cos(h\xi)] |h|^{-1-2\beta} dh \\
&= C \int_{\mathbb{R}} e^{-\frac{\xi^2}{4}} \xi^{2\beta} d\xi \int_{\mathbb{R}} [1 - \cos(h\xi)] |h|^{-1-2\beta} dh < \infty
\end{align}
(2.20)

since \( \int_0^\infty \frac{1 - \cos(\theta)}{\theta^\beta} d\theta \) is finite for all \( \theta \in (1, 3) \) which requires \( \alpha, \beta \in (0, 1) \). This proves the first inequality in (2.19). Same argument shows the second inequality in (2.19) under the condition of the lemma. \( \square \)

Remark 2.9. In the rest of our paper, we shall use the lemma for \( \alpha = \beta = \frac{1}{2} - H \in (0, 1/4) \).

Lemma 2.10. For $D(x, h)$ and $D_t(x, h)$ defined in (2.14), we have
\[ F(x) := \int_{\mathbb{R}} |D(x, h)|^2 |h|^{2H-2} dh \leq C_H (1 \wedge |x|^{2H-2}) , \]
(2.21)

and when $t > 0$
\[ F_t(x) := \int_{\mathbb{R}} |D_t(x, h)|^2 |h|^{2H-2} dh \leq C_H \left( t^{H - \frac{1}{2}} \wedge \frac{|x|^{2H-2}}{\sqrt{t}} \right) , \]
(2.22)
where $0 < H < \frac{1}{2}$.

Proof. The assertion (2.22) is an easy consequence of (2.21) by change of variables so we only need to provide a proof for (2.21).

Recall that the Fourier transform of $D(x, h)$ (as a function of $x$) is
\[ \hat{D}(\eta, h) = \mathcal{F}[D(\cdot, h)](\eta) = \sqrt{\pi} e^{-\frac{\eta^2}{4}} [e^{ih\eta} - 1]. \]

By the inverse Fourier transformation $D(x, h)$ can also be written as
\[ D(x, h) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{D}(\eta, h)e^{ix\eta} d\eta = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-\frac{\eta^2}{4}} [e^{ih\eta} - 1] e^{ix\eta} d\eta. \]
Therefore, we can write
\[
F(x) = C_H \pi^2 \int_{\mathbb{R}^2} e^{-\frac{x_1^2 + x_2^2}{4}} \int_{\mathbb{R}} [e^{i\eta_1} - 1][e^{i\eta_2} - 1]|h|^{2H-2} \, dh \, e^{ix(y_1 - y_2)} \, d\eta_1 \, d\eta_2 \\
= C_H \int_{\mathbb{R}^2} e^{-\frac{x_1^2 + x_2^2}{4}} H(\eta_1, \eta_2) e^{ix(y_1 - y_2)} \, d\eta_1 \, d\eta_2 ,
\]
where similar to (2.20), we have
\[
H(\eta_1, \eta_2) = C_H \int_{\mathbb{R}} [e^{i\eta_1} - 1][e^{i\eta_2} - 1]|h|^{2H-2} \, dh \\
= C_H (|\eta_1|^{1-2H} + |\eta_2|^{1-2H} - |\eta_1 - \eta_2|^{1-2H}) .
\]
It is easy to see that \(\sup_{x \in \mathbb{R}} |F(x)| \leq C < \infty\). Now, we want to get the desired decay estimate when \(x\) goes to infinity. We have
\[
F(x) \leq C_H \left[ \int_{\mathbb{R}^2} e^{-\frac{x_1^2 + x_2^2}{4}} |\eta_2|^{1-2H} e^{ix(y_1 - y_2)} \, d\eta_1 \, d\eta_2 \right] \\
+ C_H \left[ \int_{\mathbb{R}^2} e^{-\frac{x_1^2 + x_2^2}{4}} |\eta_1 - \eta_2|^{1-2H} e^{ix(y_1 - y_2)} \, d\eta_1 \, d\eta_2 \right] \\
\leq C_H e^{-x^2} \left[ \int_{\mathbb{R}} \left[ |\eta|^2 \cos(x\eta) \right] \, d\eta \right] \\
+ C_H \left[ \int_{\mathbb{R}} \left[ e^{-\frac{|\eta|^2}{2}} |\eta|^{1-2H} \cos(x\eta) \right] \, d\eta \right] \\
\leq C_H e^{-x^2} \int_{\mathbb{R}^2} e^{-\frac{|\eta|^2}{4}} |\eta|^2 \cos(x\eta) \, d\eta + C_H \int_{\mathbb{R}^2} e^{-\frac{|\eta|^2}{4}} |\eta|^1 \cos(x\eta) \, d\eta \]
\]
since
\[
\int_{\mathbb{R}} e^{-\frac{|\eta|^2}{4}} \, d\eta = C e^{-\frac{|x|^2}{4}} .
\]
Now the inequality (2.21) follows from Lemma 2.7.

**Lemma 2.11.** Recall that \(\Box_t(y, h)\) and \(\Box(x, y, h)\) are defined by (2.15) and (2.16). We have
\[
F(x) := \int_{\mathbb{R}^2} |\Box(x, y, h)|^2 |h|^{2H-2} |y|^{2H-2} \, dy \, dh \leq C_H \left( 1 \wedge |x|^{2H-2} \right) .
\]
Moreover, for any \(t > 0\) we have
\[
F_t(x) := \int_{\mathbb{R}^2} |\Box_t(x, y, h)|^2 |h|^{2H-2} |y|^{2H-2} \, dy \, dh \leq C_H \left( t^{2H-2} \wedge \frac{|x|^{2H-2}}{t^{1-H}} \right) .
\]
**Proof.** As in the proof of Lemma 2.10 we only need to prove (2.24) and last inequality can be derived from (2.24) by a change of variable.

The proof of (2.24) is similar to that of Lemma 2.10. Recall the Fourier transform of \(\Box(x, y, h)\) as a function of \(x\):
\[
\Box(\eta, y, h) = \mathcal{F}[\Box(\cdot, y, h)](\eta) = \sqrt{\pi} e^{-\frac{\eta^2}{4}} [e^{iy\eta} - 1]\, [e^{ih\eta} - 1] .
\]
This means
\[
\Box(x, y, h) = \sqrt{\pi} \int_{\mathbb{R}} e^{-\frac{\eta^2}{4}} [e^{iy\eta} - 1]\, [e^{ih\eta} - 1] \, e^{i\eta x} \, d\eta .
\]
Thus, we have
\[
F(x) = \int_{\mathbb{R}^4} e^{-\frac{r^2}{4} + x^2} [e^{iy\eta_1} - 1][e^{iy\eta_2} - 1] \\ \frac{e^{iy\eta_2} - 1}{|h|^{2H-2}} |y|^{2H-2} e^{ix(\eta_1 - \eta_2)} dy dh d\eta_1 d\eta_2 \\
= 2\pi^2 \int_{\mathbb{R}^2} e^{-\frac{r^2}{4} + \frac{9}{4}} H^2(\eta_1, \eta_2) e^{ix(\eta_1 - \eta_2)} d\eta_1 d\eta_2 ,
\]
where \(H(\eta_1, \eta_2)\) is defined in (2.23) or
\[
H^2(\eta_1, \eta_2) = C_H^2 \left( |\eta_1|^{2-4H} + |\eta_2|^{2-4H} + |\eta_1|^{1-2H}|\eta_2|^{1-2H} + |\eta_1 - \eta_2|^{2-4H} - |\eta_1|^{1-2H}|\eta_1 - \eta_2|^{1-2H} - |\eta_2|^{1-2H}|\eta_1 - \eta_2|^{1-2H} \right) .
\]
It is easy to see that \(\sup_{x \in \mathbb{R}} |F(x)| \leq C < \infty\). Now we want to show the desired decay rate as \(x \to \infty\). By the symmetry \(F(-x) = F(x)\), we can and will assume \(x \geq 1\). The argument in the proof of Lemma 2.10 can be used to obtain the desired bound for each of the above terms except the terms \(|\eta_1 - \eta_2|^{2-4H}\) and \(|\eta_1|^{1-2H}|\eta_1 - \eta_2|^{1-2H}\), which can be handled analogously.

For term \(|\eta_1 - \eta_2|^{2-4H}\), letting \(\xi_1 = \eta_1 - \eta_2\) and \(\xi_2 = \eta_1 + \eta_2\) implies
\[
\int_{\mathbb{R}^2} e^{-\frac{\xi_1^2 + \xi_2^2}{2}} |\eta_1 - \eta_2|^{2-4H} e^{ix(\eta_1 - \eta_2)} d\eta_1 d\eta_2 \\
= C \int_{\mathbb{R}^2} e^{-\frac{\xi_1^2 + \xi_2^2}{8}} |\xi_1|^{2-4H} e^{ix\xi_1} d\xi_1 d\xi_2 = C \int_{\mathbb{R}^2} e^{-\frac{\xi_2^2}{8}} |\xi_2|^{2-4H} \cos(x\xi) d\xi.
\]
Then using Lemma 2.7, we see that this term is bounded by \(1 \wedge |x|^{4H - 3} \lesssim 1 \wedge |x|^{2H - 2}\) for \(\frac{1}{4} < H < \frac{1}{2}\).

In order to deal with the second term \(|\eta_1|^{1-2H}|\eta_1 - \eta_2|^{1-2H}\), we make the substitution \(\xi = \eta_1\) and \(\eta = \frac{1}{2}(\eta_1 - \eta_2)\) to obtain
\[
J(x) := \int_{\mathbb{R}^2} e^{-\frac{\xi_1^2 + \xi_2^2}{2}} |\eta_1|^{1-2H}|\eta_1 - \eta_2|^{1-2H} e^{ix(\eta_1 - \eta_2)} d\eta_1 d\eta_2 \\
= C \int_{\mathbb{R}^2} \exp \left( -\frac{\xi^2}{2} \right) \exp \left( -\frac{\eta^2}{2} \right) |\xi|^{1-2H} |\eta|^{1-2H} e^{ix\eta} d\xi d\eta .
\]
Denote
\[
E(\eta) := \int_\mathbb{R} \exp \left( -\frac{\xi^2}{2} \right) |\xi|^{1-2H} d\xi .
\]
We need to show a similar inequality to that in Lemma 2.7:
\[
|J(x)| = \left| \int_0^\infty e^{-\frac{\xi^2}{2}} \eta^{1-2H} E(\eta) \cos(2x\eta) d\eta \right| \leq C_H \left( 1 \wedge |x|^{2H - 2} \right) .
\]
First, we observe that \(|E(\eta)| \leq C_H (1 + |\eta|^{1-2H})\) and both \(|E'(\eta)|\) and \(|E''(\eta)|\) can be bounded by a multiple of
\[
\int_\mathbb{R} \exp \left( -\frac{\xi^2}{4} \right) |\xi|^{1-2H} d\xi \leq C_H (1 + |\eta|^{1-2H}) .
\]
We only need to care the case when \(x\) is large. Let us split \(J(x)\) into two parts of which one integrates from 0 to \(s(x)\), denoted by \(J_1(x)\), and the other integrates
from \( s(x) \) to infinity, denoted by \( J_2(x) \), such that \( s(x) \to 0 \) as \( x \) goes to infinity and whose precise form will be given later. For the first part

\[
|J_1(x)| \leq [s(x)]^{1-2H} \int_0^{s(x)} |E(\eta)| d\eta \leq C_H ([s(x)]^{2-2H} + [s(x)]^{3-4H}).
\]

For \( J_2(x) \), an integration by parts yields

\[
|J_2(x)| = \left| \int_{s(x)}^\infty e^{-\frac{\eta^2}{2}} \eta^{1-2H} E(\eta) \cos(2x\eta) d\eta \right|
\]

\[
\leq C_H \left[ [s(x)]^{1-2H} \eta^{1-2H} E(s(x)) \sin(2x\eta) + \int_{s(x)}^\infty \eta^{-2H} e^{-\frac{\eta^2}{2}} \sin(2x\eta) E(\eta) d\eta \right]
\]

\[
+ C_H \left[ \int_{s(x)}^\infty \eta^{2-2H} e^{-\frac{\eta^2}{2}} \sin(2x\eta) E(\eta) d\eta \right] + C_H \left[ \int_{s(x)}^\infty \eta^{1-2H} e^{-\frac{\eta^2}{2}} \sin(2x\eta) E'(\eta) d\eta \right]
\]

\[=: J_{21} + J_{22} + J_{23} + J_{24}. \]

The first term is bounded by

\[
J_{21}(x) \leq C_H \frac{1}{x} [s(x)]^{1-2H}.
\]

As for \( J_{22}(x) \) an integration by parts yields

\[
J_{22}(x) := \frac{1}{x} \left| \int_{s(x)}^\infty \eta^{2-2H} e^{-\frac{\eta^2}{2}} \sin(2x\eta) E(\eta) d\eta \right|
\]

\[
\leq C \left( \frac{E(s(x))}{x^2} [s(x)]^{-2H} + \frac{C_H}{x^2} \int_{s(x)}^\infty \left| \frac{d}{d\eta} \left[ \eta^{2-2H} E(\eta) e^{-\frac{\eta^2}{2}} \right] \right| d\eta \right)
\]

\[
\leq \frac{C_H}{x^2} [s(x)]^{-2H} + \frac{C_H}{x^2} [s(x)]^{1-4H} + \frac{C_H}{x^2}.
\]

In the same way we can bound \( J_{23}(x) \) as follows.

\[
J_{23}(x) := \frac{1}{x} \left| \int_{s(x)}^\infty \eta^{2-2H} e^{-\frac{\eta^2}{2}} \sin(2x\eta) E(\eta) d\eta \right|
\]

\[
\leq C \left( \frac{E(s(x))}{x^2} [s(x)]^{2-2H} + \frac{C_H}{x^2} \int_{s(x)}^\infty \left| \frac{d}{d\eta} \left[ \eta^{2-2H} E(\eta) e^{-\frac{\eta^2}{2}} \right] \right| d\eta \right)
\]

\[
\leq \frac{C_H}{x^2} [s(x)]^{2-2H} + \frac{C_H}{x^2} [s(x)]^{3-4H} + \frac{C_H}{x^2}.
\]

The term \( J_{24}(x) \) satisfies

\[
J_{24}(x) := \frac{1}{x} \left| \int_{s(x)}^\infty \eta^{1-2H} e^{-\frac{\eta^2}{2}} \sin(x\eta) E'(\eta) d\eta \right|
\]

\[
\leq C \left( \frac{E'(s(x))}{x^2} [s(x)]^{1-2H} + \frac{C_H}{x^2} \int_{s(x)}^\infty \left| \frac{d}{d\eta} \left[ \eta^{1-2H} E'(\eta) e^{-\frac{\eta^2}{2}} \right] \right| d\eta \right)
\]

\[
\leq \frac{C_H}{x^2} [s(x)]^{1-2H} + \frac{C_H}{x^2} [s(x)]^{2-4H} + \frac{C_H}{x^2}.
\]

Noticing that \( \frac{1}{4} < H < \frac{1}{2} \), and taking \( s(x) = \frac{1}{x} \) imply our result. \( \square \)
Lemma 2.22. Denote \( \lambda(x) = \frac{1}{(1+|x|^2)^{m/2}} \) and recall \( D_t(x,h) \) defined by (2.14) and \( \square_t(x,y,h) \) defined by (2.15). We have

\[
\int_{\mathbb{R}^2} |D_t(x,h)|^2 |h|^{2H-2} \lambda(z-x) dx dh \leq C_T H t^{H-1} \lambda(z), \tag{2.27}
\]

\[
\int_{\mathbb{R}^3} |\square_t(x,y,h)|^2 |h|^{2H-2} |y|^{2H-2} \lambda(z-x) dx dy dh \leq C_T H t^{2H-\frac{3}{2}} \lambda(z).
\]

Proof. Set

\[ R(x,z) = \frac{\lambda(z-x)}{\lambda(z)} \simeq \left( \frac{1 + |z|}{1 + |x-z|} \right)^{2-2H}, \]

where and throughout the paper for two functions \( f \) and \( g \), notation \( f \asymp g \) means that there exist two positive constants \( c_H \) and \( C_H \) such that \( c_H g \leq f \leq C_H g \). By Lemma 2.8, we have by change of variables \( x \to x \sqrt{t} \), \( h \to h \sqrt{t} \) and \( z \to z \sqrt{t} \)

\[
\int_{\mathbb{R}^2} |D_t(x,h)|^2 |h|^{2H-2} R(x,z) dx dh 
\leq C_H t^{H-1} \int_{\mathbb{R}^2} |D(x,h)|^2 |h|^{2H-2} R(\sqrt{t}x, \sqrt{t}z) dx dh 
\leq C_H t^{H-1} \int_{\mathbb{R}} \left( 1 \wedge |x|^{2H-2} \right) R(\sqrt{t}x, \sqrt{t}z) dx. \tag{2.28}
\]

Similarly, making substitutions \( x \to x \sqrt{t} \), \( y \to y \sqrt{t} \), \( h \to h \sqrt{t} \) and \( z \to z \sqrt{t} \) we can get rid of the \( t \) in \( \square_t \):

\[
\int_{\mathbb{R}^3} |\square_t(x,y,h)|^2 |h|^{2H-2} |y|^{2H-2} R(x,z) dx dy dh 
= C_H t^{2H-\frac{3}{2}} \int_{\mathbb{R}^3} |\square(x,y,h)|^2 |h|^{2H-2} |y|^{2H-2} R(\sqrt{t}x, \sqrt{t}z) dx dy dh 
\leq C_H t^{2H-\frac{3}{2}} \int_{\mathbb{R}} \left( 1 \wedge |x|^{2H-2} \right) R(\sqrt{t}x, \sqrt{t}z) dx. \tag{2.29}
\]

Notice that the above change of variable with respect to \( z \) is not essential because we will take its supremum over \( \mathbb{R} \). But it will be convenient for us to split the intervals. From (2.28) and (2.29) to show our lemma it is sufficient to show

\[
\sup_{t \in [0,T]} \sup_{z \in \mathbb{R}} \int_{\mathbb{R}} \left( 1 \wedge |x|^{2H-2} \right) R(\sqrt{t}x, \sqrt{t}z) dx < \infty. \tag{2.30}
\]

Notice that we assume that \( t \in [0,T] \) is bounded. If \( z \) is bounded, then \( R(\sqrt{t}x, \sqrt{t}z) \) is also bounded. Then, we have

\[
\sup_{t \in [0,T]} \sup_{|z| \leq 2} \int \left( 1 \wedge |x|^{2H-2} \right) R(\sqrt{t}x, \sqrt{t}z) dx \leq C_T H \int_{\mathbb{R}} 1 \wedge |x|^{2H-2} dx < \infty. \tag{2.31}
\]

This means that we only need to consider the case \(|z| \geq 2\). Due to the symmetry \( R(-\sqrt{t}x, -\sqrt{t}z) = R(\sqrt{t}x, \sqrt{t}z) \), we can assume \( z \geq 2 \).

Next we split the integral domain into two parts.

(i) The domain \( x \leq z/2 \) or \( x \geq 2z \). On this domain \( R(\sqrt{t}x, \sqrt{t}z) \) is bounded. Thus

\[
\sup_{t \in [0,T]} \sup_{|z| \geq 2} \int \left( 1 \wedge |x|^{2H-2} \right) R(\sqrt{t}x, \sqrt{t}z) dx \leq C_T \int_{\mathbb{R}} 1 \wedge |x|^{2H-2} dx < \infty. \tag{2.32}
\]
(ii) The domain $z/2 \leq x \leq 2z$. On this domain we have $x \geq z/2 \geq (z+1)/3 \geq 1$ and then

$$1 \land |x|^{2H-2} \leq |x|^{2H-2} \leq \frac{3^{2-2H}}{(1+z)^{2-2H}}.$$ 

Thus,

$$I := \int_{\frac{z}{2} < x < 2z} (1 \land |x|^{2H-2}) R(\sqrt{t}x, \sqrt{t}z) dx \leq C_H \left( \frac{1 + \sqrt{t}z}{1 + z} \right)^{2-2H} \int_{\frac{z}{2}}^{2z} \frac{1}{(1 + \sqrt{t}|x-z|)^{2-2H}} dx.$$ 

By the symmetry of the above integrand we know that the integrals $I^\ast_0$ and $I^\ast_z$ are the same. Hence, we have

$$I \leq C_H \frac{1 + (\sqrt{t}z)^{2-2H}}{1 + z^{2-2H}} \int_{\frac{z}{2}}^{2z} \frac{1}{(1 + \sqrt{t}|x-z|)^{2-2H}} dx = C_H \frac{1 + (\sqrt{t}z)^{2-2H}}{\sqrt{t} \left( 1 + z^{2-2H} \right)} \left[ 1 - (1 + \sqrt{t}z)^{2H-1} \right] \leq C_H T^{\frac{1}{2}-H} \frac{1 + (\sqrt{t}z)^{2-2H}}{\sqrt{t}z \left( 1 + (\sqrt{t}z)^{1-2H} \right)} \left[ 1 - (1 + \sqrt{t}z)^{2H-1} \right].$$

Consider the function

$$f(u) = \frac{1 + u^{2-2H}}{u(1 + u^{2-2H})} [1 - (1 + u)^{2H-1}], \quad u > 0.$$ 

This is a continuous function on $(0, \infty)$. When $u \to 0$ and when $u \to \infty$ we have

$$\lim_{u \to 0^+} f(u) = 1 - 2H, \quad \lim_{u \to \infty} f(u) = 1.$$ 

Thus, $f(u)$ is bounded on $(0, \infty)$ and this in turn proves

$$\sup_{t \in [0,T]} \sup_{z \geq 1} \int_{\frac{z}{2} \leq x \leq 2z} (1 \land |x|^{2H-2}) R(\sqrt{t}x, \sqrt{t}z) dx < \infty. \quad (2.33)$$

Combining (2.31)–(2.32) together with our above symmetry argument we prove (2.30) and hence we complete the proof of the lemma.

\[ \square \]

**Remark 2.13.** From this lemma, we see why we choose the above decay rate for our weight function. If we consider $\lambda(x) = (1 + |x|^2)^{-\lambda}$ with $\lambda > 1 - H$, then for $|z|$ sufficiently large one has

$$\int_{\mathbb{R}} (1 \land |x|^{2H-2}) R(x, z) dx \gtrsim \int_{|x-z| < 1} |x|^{2H-2} R(x, z) dx \gtrsim \frac{(1 + |z|)^{\lambda}}{|z|^{2-2H}},$$

which diverges as $|z| \to \infty$. This elementary fact suggests us that $\lambda$ must be in $(\frac{1}{2}, 1 - H]$, and it is obvious $L^\lambda(\mathbb{R})$ is the largest space when $\lambda = 1 - H$.
3. Additive noise

When the diffusion coefficient \( \sigma(t, x, u) = 1 \) (or a general constant), the noise is additive and the solution to (1.1) can be written explicitly as

\[
    u(t, x) = \int_{\mathbb{R}} G_t(x - y) u_0(y) dy + \int_0^t \int_{\mathbb{R}} G_{t-s}(x - y) W(ds, dy),
\]

where \( G_t(x) = \frac{1}{\sqrt{4\pi t}} \exp \left( - \frac{x^2}{4t} \right) \) is the heat kernel. To focus on the stochastic part we assume \( u_0 = 0 \). Thus, the resulted solution is written as

\[
    u_{add}(t, x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x - y) W(ds, dy).
\]

This solution \( u_{add}(t, x) \) defines a (symmetric) centered Gaussian process. We shall study how it grows as the parameters \( t \) and \( x \) go to infinity. It is expected that \( u_{add}(t, x) \) is Hölder continuous in \( t \) and \( x \). More precisely, for any positive constants \( \gamma < H, T, L \in (0, \infty) \), there is a constant \( C_{T,L,\gamma} \), depending only on \( T, L \) and \( \gamma \), such that

\[
    \sup_{0 \leq s, t \leq T, \|x\|, \|y\| \leq L} |u_{add}(s, x) - u_{add}(t, y)| \leq C_{T,L,\gamma} \left( |t - s|^{\gamma/2} + |x - y|^{\gamma} \right).
\]

We want to consider the Hölder continuity of \( u_{add}(t, x) \) on the whole space \( \mathbb{R} \). Namely, we want to know how the constant \( C_{T,L,\gamma} \) grows as \( T \) and \( L \) go to infinity (for any fixed \( \gamma \)).

### 3.1. Majorizing measure theorem

To find the sharp bound for \( C_{T,L,\gamma} \) we shall utilize Talagrand’s majorizing measure theorem which we recall below.

**Theorem 3.1.** (Majorizing Measure Theorem, see e.g. [23, Theorem 2.4.2]). Let \( T \) be a given set and let \( \{X_t, t \in T\} \) be a centered Gaussian process indexed by \( T \). Denote \( d(t, s) = \mathbb{E}[X_t - X_s]^2 \), the associated natural metric on \( T \). Then

\[
    \mathbb{E} \left[ \sup_{t \in T} X_t \right] \leq \gamma_2(T, d) := \inf_{A} \sup_{t \in T} \sum_{n \geq 0} 2^{n/2} \text{diam} (A_n(t)),
\]

where the infimum is taken over all increasing sequence \( A := \{A_n, n = 1, 2, \cdots\} \) of partitions of \( T \) such that \( \#A_n \leq 2^n \) (\( \#A \) denotes the number of elements in the set \( A \)), \( A_n(t) \) denotes the unique element of \( A_n \) that contains \( t \), and \( \text{diam} (A_n(t)) \) is the diameter (with respect to the natural distance \( d \)) of \( A_n(t) \).

This theorem provides a powerful general principle for the study of the supremum of Gaussian process.

**Remark 3.2.** The natural metric \( d(t, s) \) is actually only a pseudo-metric because \( d(t, s) = 0 \) does not necessarily imply \( t = s \) (e.g. \( X_t \equiv 1 \)). It is also called the canonical metric.

It is more convenient for us to use the following theorem to obtain the lower bound.

**Theorem 3.3.** (Sudakov minoration theorem, see e.g. [23, Lemma 2.4.2]). Let \( \{X_i, i = 1, \cdots, L\} \) be a centered Gaussian family with natural distance \( d \) and assume

\[
    \forall p, q \leq L, \; p \neq q \Rightarrow d(t_p, t_q) \geq \delta.
\]
Then, we have
\[
\mathbb{E}\left( \sup_{1 \leq i \leq L} X_{t_i} \right) \geq \frac{\delta}{C} \sqrt{\log_2(L)},
\]
where \( C \) is a universal constant.

The following “concentration of measure” type theorem allows us to obtain deviation inequalities for the supremum of a Gaussian family.

**Theorem 3.4.** (Borell, see e.g. [1, Theorem 2.1]). Let \( \{X_t, t \in T\} \) be a centered separable Gaussian process on some topological index set \( T \) with almost surely bounded sample paths. Then \( \mathbb{E}\left( \sup_{t \in T} X_t \right) < \infty \), and for all \( \lambda > 0 \)
\[
P\left\{ \left| \sup_{t \in T} X_t - \mathbb{E}\left( \sup_{t \in T} X_t \right) \right| > \lambda \right\} \leq 2 \exp\left( -\frac{\lambda^2}{2\sigma_T^2} \right),
\]
where \( \sigma_T^2 := \sup_{t \in T} \mathbb{E}(X_t^2) \).

We have the following observation which can be deduced immediately from [23, Lemma 2.2.1]. This simple fact tells us \( \mathbb{E}[\sup_{t \in T} |X_t|] = \mathbb{E}[\sup_{t \in T} X_t] \). So, we only need to consider \( \mathbb{E}[\sup_{t \in T} X_t] \).

**Lemma 3.5.** If the process \( \{X_t, t \in T\} \) is symmetric, then we have
\[
\mathbb{E}\left[ \sup_{t \in T} |X_t| \right] \leq 2 \mathbb{E}\left[ \sup_{t \in T} X_t \right] + \inf_{t_0 \in T} \mathbb{E}\left[ |X_{t_0}| \right].
\]

### 3.2. Asymptotics of the Gaussian solution.

For the mild solution \( u_{\text{add}}(t,x) \) to (1.1) with additive noise (e.g. \( \sigma(t,x,u) = 1 \)), defined by (3.2), we shall first obtain the sharp upper and lower bounds for its associated natural metric:

\[
d_1((t,x),(s,y)) = \sqrt{\mathbb{E}[u_{\text{add}}(t,x) - u_{\text{add}}(s,y)]^2},
\]

The following lemma gives a sharp bounds for this induced natural metric for the Gaussian solution \( u_{\text{add}}(t,x) \).

**Lemma 3.6.** Let \( d_1((t,x),(s,y)) \) be the natural metric defined by (3.7). Then, there are positive constants \( c_H, C_H \) such that
\[
c_H(|x - y|^H \wedge (t \land s)^{\frac{H}{2}} + |t - s|^{\frac{H}{2}}) \leq d_1((t,x),(s,y)) \leq C_H(|x - y|^H \wedge (t \land s)^{\frac{H}{2}} + |t - s|^{\frac{H}{2}})
\]
for any \((t,x),(s,y) \in \mathbb{R}_+ \times \mathbb{R}\).

**Remark 3.7.** The above property of the natural metric can also be written as
\[
d_1((t,x),(s,y)) \asymp d_{1,H}((t,x),(s,y)) := |x - y|^H \wedge (t \land s)^{\frac{H}{2}} + |t - s|^{\frac{H}{2}}.
\]

\( d_{1,H}((t,x),(s,y)) \) is no longer a distance but it is very convenient for us to obtain the desired results.

**Proof.** Without loss of generality, let us assume \( t > s \). Plancherel’s identity and the independence of the stochastic integrals over the time intervals \([0,s]\) and \([s,t]\)
give
\[
d_1((t, x), (s, y)) = \mathbb{E}|u_{\text{add}}(t, x) - u_{\text{add}}(s, y)|^2
\]
\[
= \mathbb{E} \left| \int_0^s \int_{\mathbb{R}} \left( G_{t-r}(x-z) - G_{s-r}(y-z) \right) W(dr, dz) \right|^2
\]
\[
+ \mathbb{E} \left| \int_s^t \int_{\mathbb{R}} G_{t-r}(x-z) W(dr, dz) \right|^2
\]
\[
= \int_{\mathbb{R}^+} [1 - \exp(-2s\xi^2)] [1 + \exp(-2(t-s)\xi^2)]
\]
\[
- 2 \exp(-(t-s)\xi^2) \cos(|x-y|\xi) \cdot \xi^{-1-2H} d\xi + 2^{H-1} \kappa_H (t-s)^H,
\]
where \( \kappa_H = H^{-1}\Gamma(1-H) \) is a positive constant. We start to obtain the upper bound of (3.8). The triangular inequality gives
\[
d_1((t, x), (s, y)) \leq d_1((t, x), (s, x)) + d_1((s, x), (s, y)).
\]

Let us deal with the two terms on the right hand side of the above inequality separately. For the first term, Plancherel’s identity (3.10) implies
\[
d_1^2((t, x), (s, x)) = \kappa_H \left[ 2^{H-1}t^H + 2^{H-1}s^H - (t + s)^H \right] + (2^{H-1} + 1)\kappa_H (t-s)^H
\]
\[
\leq C_H (t-s)^H,
\]
because \( 2^{H-1}t^H + 2^{H-1}s^H - (t+s)^H \leq 0 \) when \( t \geq s \). Again from (3.10), the second term on the right hand side of (3.11) is given by
\[
d_1^2((s, x), (s, y)) = \int_0^s \int_{\mathbb{R}} \exp[-2(s-r)\xi^2] \cdot |\xi|^{1-2H} |1 - \cos(\xi|x-y|)| d\xi dr
\]
\[
= C_H |x-y|^{2H} \int_{\mathbb{R}^+} \left[ 1 - \exp\left(-\frac{2s\xi^2}{|x-y|^2}\right) \right] \cdot \xi^{-1-2H} |1 - \cos(\xi)| d\xi,
\]
which can be controlled by \( C_H |x-y|^{2H} \). On the other hand, we have
\[
d_1^2((s, x), (s, y)) = \mathbb{E}[|u_{\text{add}}(s, x) - u_{\text{add}}(s, y)|^2]
\]
\[
\leq 2(\mathbb{E}[|u_{\text{add}}(s, x)|^2] + \mathbb{E}[|u_{\text{add}}(s, y)|^2]) \leq C_H s^H.
\]
Thus, the quantity of \( d_1^2((s, x), (s, y)) \) is bounded by the minimum of \( C_H |x-y|^{2H} \) and \( C_H s^H \). We can summarize the above argument as
\[
d_1((t, x), (s, y)) \leq C_H (|x-y|^H \wedge s^{\frac{H}{2}} + (t-s)^{\frac{H}{2}}),
\]
which is the upper bound part of (3.8).

Now we turn to show the lower bound part of (3.8). From Plancherel’s identity it is sufficient to bound the first summand in (3.10) from below by \( c_H (|x-y|^H \wedge s^{\frac{H}{2}}) \)
for some constant $c_H > 0$. We denote this first summand by $I$:

$$ I := \int_{\mathbb{R}} [1 - \exp(-2s\xi^2)][1 + \exp(-2(t - s)\xi^2) \]
- 2\exp(-(t - s)\xi^2) \cos(|x - y|\xi)|\xi|^{-2H} \, d\xi

= c|x - y|^{2H} \int_{\mathbb{R}^+} \left[ 1 - \exp \left( - \frac{2s\xi^2}{|x - y|^2} \right) \right] \cdot \left[ 1 - \exp \left( - \frac{(t - s)\xi^2}{|x - y|^2} \right) \cos(\xi) \right]^2 \, d\xi. \quad (3.13) $$

To bound it from below, we divide our argument into two cases:

$$ |x - y| > \sqrt{s} \quad \text{and} \quad |x - y| \leq \sqrt{s}. $$

When $|x - y| \leq \sqrt{s}$, we can bound (3.13) from below by

$$ I \geq c_H |x - y|^{2H} \sum_{n=1}^{\infty} \int_{2n\pi + \frac{\pi}{2}}^{2n\pi + \frac{3\pi}{2}} \left[ 1 - \exp \left( - 2\xi^2 \right) \right] \cdot \xi^{-1-2H} \, d\xi

\geq c_H |x - y|^{2H}, \quad (3.14) $$

since $1 - \exp(-2s\xi^2/|x - y|^2) \geq 1 - \exp(-2\xi^2)$ by the assumption and $\cos(\xi)$ is negative on the intervals $\bigcup_{n=1}^{\infty}[2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}]$.

The case $|x - y| > \sqrt{s}$ is a little bit more involved. Denote

$$ n_0 := \inf \left\{ n \in \mathbb{N}_0 : 2n\pi + \frac{\pi}{2} \geq \frac{-\ln(1 - c^*)}{2s} |x - y| \right\} $$

with the choice $c^* = 1 - \exp(-\pi^2/2)$ such that

$$ \sqrt{\frac{-\ln(1 - c^*)}{2s}} |x - y| \geq \frac{\pi}{2}. $$

It is then easy to see that $n_0$ is a well defined finite positive integer. This way, we have the lower bound for (3.13):

$$ I \geq \sum_{n=n_0}^{\infty} |x - y|^{2H} \int_{2n\pi + \frac{\pi}{2}}^{2n\pi + \frac{3\pi}{2}} \left[ 1 - \exp \left( - \frac{2s\xi^2}{|x - y|^2} \right) \right] \cdot \xi^{-1-2H} \, d\xi

\geq c^* |x - y|^{2H} \sum_{n=n_0}^{\infty} \int_{2n\pi + \frac{\pi}{2}}^{2n\pi + \frac{3\pi}{2}} \xi^{-1-2H} \, d\xi \geq \frac{c^*}{2} |x - y|^{2H} \int_{2n_0\pi + \frac{\pi}{2}}^{\infty} \xi^{-1-2H} \, d\xi,$$

where the last inequality follows from the fact that $\xi^{-1-2H}$ is a decreasing function on $(0, \infty)$. From the definition of $n_0$, it follows

$$ I \geq c_H |x - y|^{2H} \left( \sqrt{\frac{-\ln(1 - c^*)}{2s}} |x - y| + 2\pi \right)^{-2H} \geq c_H s^H \quad (3.15) $$

19
since \( |x - y| > \sqrt{s} \) and consequently

\[
|x - y|^{2H} \left( \sqrt{\frac{-\ln(1 - c^\epsilon)}{2s}} |x - y| + 2\pi \right)^{-2H} = \left( \sqrt{\frac{-\ln(1 - c^\epsilon)}{2s}} + \frac{2\pi}{|x - y|} \right)^{-2H} \geq \left( \sqrt{\frac{-\ln(1 - c^\epsilon)}{2s}} + \frac{2\pi}{\sqrt{s}} \right)^{-2H} = c_H s^H.
\]

Thus, (3.14) together with (3.15) imply

\[
d_1((t, x), (s, y)) \geq c_H (|x - y|^{\frac{H}{2}} \wedge s^\frac{H}{4} + (t - s)^\frac{H}{4}). \tag{3.16}
\]

Combining (3.12) and (3.16), we complete the proof of this lemma. \(\square\)

Now we are ready to prove Theorem 1.1, which gives a sharp bound for

\[
E \left[ \sup_{0 \leq t \leq T, -L \leq x \leq L} |u_{\text{add}}(t, x)| \right].
\]

**Proof of the first part of Theorem 1.1.** To simplify notation we denote

\[ T = [0, T] \quad \text{and} \quad L = [-L, L]. \]

Since \( u_{\text{add}}(t, x) \) is a symmetric and centred Gaussian process Lemma 3.5 states that

\[
E \left[ \sup_{(t, x) \in T \times L} |u_{\text{add}}(t, x)| \right] \simeq E \left[ \sup_{(t, x) \in T \times L} u_{\text{add}}(t, x) \right]. \tag{3.17}
\]

Hence, to show (1.3) it is equivalent to show

\[
c_H \Psi(T, L) \leq E \left[ \sup_{t \in T, x \in L} u_{\text{add}}(t, x) \right] \leq C_H \Psi(T, L), \tag{3.18}
\]

where \( \Psi(T, L) \) is defined by (1.4). We shall prove the upper and lower bound parts of (3.18) separately. Let us first consider the upper bound part in (3.18). We shall use the majorizing measure method (Theorem 3.1) and our bound for the natural distance (Lemma 3.6). Let us separate the proof into the cases \( L > \sqrt{T} \) and \( L \leq \sqrt{T} \). First, we assume \( L > \sqrt{T} \). We choose the admissible sequences \( (A_n) \) as uniform partition of \( T \times L = [0, T] \times [-L, L] \) such that \( \text{card}(A_n) \leq 2^n \). More precisely, we partition \([0, T] \times [-L, L]\) as

\[
\begin{cases}
[0, T] = \bigcup_{j = 0}^{2^{n-1} - 1} \left[ j \cdot 2^{-2^n - 1} T, (j + 1) \cdot 2^{-2^n - 1} T \right), \\
[-L, L] = \bigcup_{k = -2^{n-2}}^{2^{n-2} - 1} \left[ k \cdot 2^{-2^n - 2} L, (k + 1) \cdot 2^{-2^n - 2} L \right].
\end{cases}
\]

Theorem 3.1 states

\[
E \left[ \sup_{(t, x) \in T \times L} u_{\text{add}}(t, x) \right] \leq C \gamma_2(T, d) \leq C \sup_{(t, x) \in T \times L} \sum_{n \geq 0} 2^{n/2} \text{diam} \,(A_n(t, x)). \tag{3.19}
\]

20
Here $A_n(t, x)$ is the element of uniform partition $\mathcal{A}_n$ that contains $(t, x)$, i.e., $A_n(t, x) = \left[ j : 2^{-2^{n-1}}T, (j + 1) : 2^{-2^{n-1}}T \right] \times \left[ k : 2^{-2^{n-2}}L, (k + 1) : 2^{-2^{n-2}}L \right]$ such that $j \cdot 2^{-2^{n-1}}T \leq t < (j + 1) \cdot 2^{-2^{n-1}}T$ and $k \cdot 2^{-2^{n-2}}L \leq x < (k + 1) \cdot 2^{-2^{n-2}}L$. We only need to estimate diameter of each $A_n(t, x)$. Since $(\mathcal{A}_n)$ is a uniform partition, the diameter of $A_n(t, x)$ with respect to $d_{1,H}((t, x), (s, y))$ defined in (3.9) can be estimated as

$$\text{diam} \left( A_n(t, x) \right) \leq C_H \left( T^{\frac{H}{2}} \land \left( 2^{-H2^{n-2}}L^H \right) \right) + C_H 2^{-H2^{n-2}}T^{\frac{H}{2}}.$$  

For $L \geq \sqrt{T}$, we can split it into two cases: $\sqrt{T} \leq L < 2\sqrt{T}$ and $L \geq 2\sqrt{T}$. It is clear that the case $L \geq 2\sqrt{T}$ is more complicated. We consider it at first. Let $N_0$ be the smallest integer such that $2^{-2^{-2}L} \leq \sqrt{T}$, i.e. $\log_2(\log_2(L/\sqrt{T})) + 2 \leq N_0 < \log_2(\log_2(L/\sqrt{T})) + 3$. By (3.19) we have

$$\mathbb{E} \left[ \sup_{(t, x) \in T \times L} u(t, x) \right] \leq C_H \sum_{n=0}^{N_0} 2^{n/2} \text{diam} \left( A_n(t, x) \right) + \sum_{n=N_0+1}^{\infty} 2^{n/2} \text{diam} \left( A_n(t, x) \right)$$  

$$\leq C_H \sum_{n=0}^{N_0} 2^{n/2} \left[ \sum_{n=0}^{N_0} 2^{n/2} \left( \frac{2^{N_0-2}}{2^{n-2}} \right)^H \right] + C_H T^{\frac{H}{2}}$$  

$$\leq C_H T^{\frac{H}{2}} \cdot 2^{N_0/2} + C_H T^{\frac{H}{2}} \leq C_H T^{\frac{H}{2}} \Psi_0(T, L),$$  

where $\Psi_0(T, L) = 1 + \sqrt{\log_2(L/\sqrt{T})}$ and $L \geq 2\sqrt{T}$. The case $\sqrt{T} \leq L < 2\sqrt{T}$ is easy because $\Psi_0(T, L)$ is bounded now. One can prove directly that $\mathbb{E} \left[ \sup_{(t, x) \in T \times L} u(t, x) \right] \leq C_H T^{\frac{H}{2}}$ same as (3.21). This concludes proof of the upper bound in (3.18) when $L \geq \sqrt{T}$.

Now, we prove the upper bound part in (3.18) when $L < \sqrt{T}$. The same uniform partition discussed above is still applicable. We have

$$\mathbb{E} \left[ \sup_{(t, x) \in T \times L} \left| u(t, x) \right| \right] \leq C_H \sum_{n=0}^{\infty} 2^{n/2} \sup_{(t, x) \in T \times L} \text{diam} \left( A_n(t, x) \right)$$  

$$\leq C_H T^{\frac{H}{2}} \sum_{n=0}^{\infty} 2^{n/2} \cdot 2^{-H2^{n-1}} + C_H T^{\frac{H}{2}} \leq C_H T^{\frac{H}{2}},$$  

because

$$\sup_{(t, x) \in T \times L} \text{diam} \left( A_n(t, x) \right) \leq C_H \left( 2^{-2^{n-2}}L^H \right) + \left( 2^{-2^{n-1}}T \right)^{\frac{H}{2}} \leq C_H 2^{-H2^{n-2}}T^{\frac{H}{2}}.$$  

This completes the upper bounds part of (3.18).

We will utilize Theorem 3.3 (Sudakov minoration Theorem) to prove the lower bound in (3.18). We also divide the proof into two cases: $L \geq \sqrt{T}$ and $L < \sqrt{T}$. 

21
First, we consider the case $L \geq \sqrt{T}$. Select $\delta$ in Theorem 3.3 as $c_H T^{\frac{m}{2}}$ with certain relatively small $c_H > 0$. For the sequence $\{u(T, x_i), i = 0, 1, \cdots, N\}$, where $N = [L/\sqrt{T}]$ (\geq 1 by the assumption) and
\[
x_0 = 0, x_{\pm 1} = \pm \sqrt{T}, \cdots, x_{\pm N} = \pm N \sqrt{T},
\]
we have
\[
d_{1, H}((T, x_i), (T, x_j)) \geq c_H T^{\frac{m}{2}} = \delta \quad \text{if } i \neq j.
\]
Sudakov’s minoration theorem implies
\[
\mathbb{E} \left[ \sup_{(t, x) \in T \times L} |u(t, x)| \right] \geq \mathbb{E} \left[ \sup_{i} u(T, x_i) \right] \geq c_H \delta \sqrt{\log_2 (2N + 1)} \geq c_H T^{\frac{m}{2}} \Psi_0(T, L).
\]
(3.22)
The lower bound in (3.18) is established when $L \geq \sqrt{T}$.

Now we prove the lower bound part in (3.18) when $L < \sqrt{T}$. We choose $\delta = c_H T^{\frac{m}{2}}$ as above and we choose $u(T/2, 0), u(T, 0)$ as our comparison set. We have
\[
d_{1, H}((T/2, 0), (T, 0)) \geq c_H (T/2)^{\frac{m}{2}} \geq \delta.
\]
Theorem 3.3 gives
\[
\mathbb{E} \left[ \sup_{(t, x) \in T \times L} u(t, x) \right] \geq \mathbb{E}[u(T/2, 0) \vee u(T, 0)] \geq c_H T^{\frac{m}{2}}.
\]
(3.23)
Thus, the proof of the lower bound part in (3.18) is completed.

Notice that from (3.9), it follows that for any fixed $t \in \mathbb{R}_+$
\[
d_1((t, x), (t, y)) \asymp d_{1, H}(x, y) := t^{\frac{m}{2}} \wedge |x-y|^H,
\]
and for fixed $x \in \mathbb{R}$
\[
d_1((t, x), (s, x)) \asymp d_1(t, s) := |t-s|^{\frac{m}{2}}.
\]
(3.25)
Using a similar argument to that in the proof of inequality (1.3) we have the following corollary.

**Corollary 3.8.** Let the Gaussian field $u_{\text{add}}(t, x)$ be defined by (3.2). There are positive universal constants $c_H$ and $C_H$ such that
\[
\begin{align*}
    c_H T^{\frac{m}{2}} &\sqrt{\log_2 (L)} \leq \mathbb{E} \left[ \sup_{-L \leq x \leq L} |u_{\text{add}}(t, x)| \right] \\
    &\leq \mathbb{E} \left[ \sup_{-L \leq x \leq L} u_{\text{add}}(t, x) \right] \leq C_H T^{\frac{m}{2}} \sqrt{\log_2 (L)};
\end{align*}
\]
(3.26)
\[
    c_H T^{\frac{m}{2}} \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} u_{\text{add}}(t, x) \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} |u_{\text{add}}(t, x)| \right] \leq C_H T^{\frac{m}{2}}.
\]

Next, we shall explain that the almost sure version of Theorem 1.1 is a consequence of (1.3) with the aid of Borell’s inequality (Theorem 3.4).

**Proof of the second part of Theorem 1.1.** First, we shall prove (1.5) for $T = n^{\alpha}$ for some $\alpha$ and for all sufficiently large integer $n$. Denote $L := [-L, L]$, $T^\alpha = [0, n^\alpha]$. 

Let $\varepsilon > 0$ and let $L \geq n^{\frac{1+\alpha}{2}}$ be sufficiently large. We start with the lower bound. Theorem 1.1 gives
\[
\mathbb{E} \left[ \sup_{(t,x) \in \mathbb{T}^n \times \mathbb{L}} u_{\text{add}}(t, x) \right] \geq c_H \left( n^\frac{\alpha}{2} + n^\frac{\alpha}{2} \sqrt{\log_2 \left( \frac{L}{n^{\alpha/2}} \right)} \right)
\]
for some positive number $c_H$. Denote
\[
\lambda_H := \lambda_H(\mathbb{T}^n \times \mathbb{L}) = \frac{1}{2} \mathbb{E} \left[ \sup_{x \in \mathbb{T}^n \times \mathbb{L}} u_{\text{add}}(t, x) \right],
\]
and
\[
\sigma_H^2 := \sigma_H^2(\mathbb{T}^n \times \mathbb{L}) = \sup_{(t,x) \in \mathbb{T}^n \times \mathbb{L}} \mathbb{E} [ |u_{\text{add}}(t, x)|^2 ] = C_H n^\frac{\alpha}{2}.
\]
Then, Borel’s inequality implies
\[
P \left\{ \sup_{(t,x) \in \mathbb{T}^n \times \mathbb{L}} u_{\text{add}}(t, x) < \frac{1}{2} \mathbb{E} \left[ \sup_{(t,x) \in \mathbb{T}^n \times \mathbb{L}} u_{\text{add}}(t, x) \right] \right\} \leq 2 \exp \left( -\frac{\lambda_H^2}{2\sigma_H^2} \right) \leq 2 \exp \left( -c_H \left[ 1 + \log_2 \left( \frac{L}{n^{\alpha/2}} \right) \right] \right) \leq C_H \left( \frac{n^{\alpha}}{n^{\alpha(1+\varepsilon)}} \right)^{\frac{\alpha}{2}} \leq C_H n^{-\alpha-\varepsilon}, \tag{3.27}
\]
where $c_H, C_H > 0$ are some constants independent of $n$. Select real number $\alpha$ sufficiently large such that $\alpha \varepsilon \cdot \frac{\alpha}{2} > 1$ and define the events $F_n$
\[
F_n := \left\{ \sup_{(t,x) \in \mathbb{T}^n \times \mathbb{L}} u_{\text{add}}(t, x) < \frac{1}{2} \mathbb{E} \left[ \sup_{(t,x) \in \mathbb{T}^n \times \mathbb{L}} u_{\text{add}}(t, x) \right] \right\}.
\]
The bound (3.27) means $\sum_{n=1}^{\infty} P(F_n) < \infty$. An application of Borel-Cantelli’s lemma yields that $P(\lim \sup_n F_n) = 0$. This means that
\[
\sup_{(t,x) \in \mathbb{T}^n \times \mathbb{L}} u_{\text{add}}(t, x) \geq \frac{1}{2} \mathbb{E} \left[ \sup_{(t,x) \in \mathbb{T}^n \times \mathbb{L}} u_{\text{add}}(t, x) \right] \geq c_H T^\frac{\alpha}{2} \Psi_0(T, L), \tag{3.28}
\]
almost surely for sufficiently large values of $T = n^{\alpha}$. Then letting $\varepsilon \to 0$ proves lower bound part of (1.5).

The proof of the upper bound in (1.5) can be done in exactly the same manner as that in the proof of the lower bound except now we replace (3.27) by
\[
P \left\{ \sup_{(t,x) \in \mathbb{T}^n \times \mathbb{L}} u_{\text{add}}(t, x) > \frac{3}{2} \mathbb{E} \left[ \sup_{(t,x) \in \mathbb{T}^n \times \mathbb{L}} u_{\text{add}}(t, x) \right] \right\} \leq 2 \exp \left( -\frac{\lambda_H^2}{2\sigma_H^2} \right) \leq 2 \exp \left( -c_H \left[ 1 + \log_2 \left( \frac{L}{n^{\alpha/2}} \right) \right] \right) \leq C_H \left( \frac{n^{\alpha}}{n^{\alpha(1+\varepsilon)}} \right)^{\frac{\alpha}{2}} \leq C_H n^{-\alpha-\varepsilon}, \tag{3.29}
\]
with some positive constant $c_H, C_H$ independent of $n$. Similar to (3.28) we have
\[
\sup_{(t,x) \in \mathbb{T}^n \times \mathbb{L}} u_{\text{add}}(t, x) \leq \frac{3}{2} \mathbb{E} \left[ \sup_{(t,x) \in \mathbb{T}^n \times \mathbb{L}} u_{\text{add}}(t, x) \right] \leq C_H T^\frac{\alpha}{2} \Psi_0(T, L) \tag{3.30}
\]
a almost surely for sufficiently large $T = n^{\alpha}$. And then $\varepsilon \to 0$ implies the upper bound in (1.5).

Finally, we conclude the proof of (1.5) for $\sup_{(t,x)} u_{\text{add}}(t, x)$ by combining (3.28), (3.30) and the property that $\sup_{(t,x) \in \mathbb{T} \times \mathbb{L}} u_{\text{add}}(t, x)$ is an increasing function of $L$. 

and $T$ almost surely. On the other hand, it is easy to see

$$\sup_x |f(x)| \leq \sup_x |f(x)| + \sup_x [-f(x)]$$

since $|f(x)| \leq \sup_x [f(x)] + \sup_x [-f(x)]$ for any function $f(x)$. Since $u_{\text{add}}(t, x)$ is symmetric, we see that $\sup_{t,x} [-u_{\text{add}}(t, x)]$ and $\sup_{t,x} [u_{\text{add}}(t, x)]$ have the same law. Then, we have

$$\sup_{t,x} |u_{\text{add}}(t, x)| \leq 2 \sup_{t,x} [u_{\text{add}}(t, x)].$$  \quad (3.31)

This completes the proof of (1.5).

One can show the following asymptotic (3.32) by combining (3.26) and Borell’s inequality and we omit the details.

**Corollary 3.9.** Let $u_{\text{add}}(t, x)$ be defined by (3.2) and let $T$ satisfy $T \leq L^2$. Then, there are two positive random constants $c_H$ and $C_H$ such that for any fixed $t \in [0, T]$ we have

$$c_H t^{\frac{H}{2}} \sqrt{\log_2(L)} \leq \sup_{-L \leq x \leq L} u_{\text{add}}(t, x) \leq \sup_{-L \leq x \leq L} \left| u_{\text{add}}(t, x) \right| \leq C_H t^{\frac{H}{2}} \sqrt{\log_2(L)} \quad \text{almost surely.}$$  \quad (3.32)

**Remark 3.10.** As in [6, 8], the inequality (3.32) implies that there exist some constants $c, C > 0$ such that

$$ct^{\frac{H}{2}} \leq \liminf_{|x| \to \infty} \frac{u_{\text{add}}(t, x)}{\sqrt{\log_2(|x|)}} \leq \limsup_{|x| \to \infty} \frac{u_{\text{add}}(t, x)}{\sqrt{\log_2(|x|)}} \leq Ct^{\frac{H}{2}},$$  \quad (3.33)

for any $t \in \mathbb{R}_+$ almost surely.

We now turn to show Theorem 1.2.

**Proof of Theorem 1.2.** $\Delta_h u_{\text{add}}(t, x)$ is centered symmetric and stationary Gaussian process. As before, we only need to find appropriate bounds for $\Delta_h u_{\text{add}}(t, x)$. The conclusion with respect to $|\Delta_h u_{\text{add}}(t, x)|$ follows from (3.31). Our strategy to prove Theorem 1.2 for $\Delta_h u_{\text{add}}(t, x)$ is also to apply Talagrand’s majorizing measure theorem and Sudakov’s minoration theorem to the following Gaussian process

$$\Delta_h u_{\text{add}}(t, x) := u_{\text{add}}(t, x + h) - u_{\text{add}}(t, x)$$

$$= \int_0^t \int_\mathbb{R} [G_{t-s}(x+h-z) - G_{t-s}(x-z)]W(ds,dz),$$  \quad (3.34)

with fixed $t > 0$ and fixed $h \neq 0$. Without loss of generality, we assume $h > 0$. The natural metric is given by

$$d_{2,t,h}(x, y) := \left( \mathbb{E}[\Delta_h u_{\text{add}}(t, x) - \Delta_h u_{\text{add}}(t, y)]^2 \right)^{\frac{1}{2}}.$$

We need to obtain good upper and lower bounds of $d_{2,t,h}(x, y)$. Let us first focus on the upper bound. Similar to (3.10) Plancherel’s identity yields

$$d_{2,t,h}^2(x, y) = C_H \int_{\mathbb{R}_+} \left[ 1 - \exp(-2t\xi^2) \right] \left[ 1 - \cos(|x-y|\xi) \right] \left[ 1 - \cos(h\xi) \right] \xi^{-1-2H} d\xi.$$
By the same argument as that in the proof of the upper bound of $d_1((s,x),(s,y))$ in Lemma 3.6 it is easy to see that for any $0 \leq \theta \leq 1$

\[
d_2^2(x,y) \leq C_H \int_{R_+} \left[ 1 - \exp(-2t\xi^2) \right] \left[ 1 - \cos(h\xi) \right] \cdot \xi^{-1-2H} d\xi
\]

\[
\leq C_H t^H \wedge h^{2H} \leq C_H t^{H-\theta} h^{2\theta}.
\]

On the other hand, an application of the elementary inequality $1 - \cos(x) \leq C_0 x^{2\theta}$, where $\theta \in (0, H)$ is as above, and a substitution $\xi \to \xi/|x - y|$ yield

\[
d_2^2(x,y) \leq C_0, H h^{2\theta} |x - y|^{2H-2\theta} \int_{R_+} \left[ 1 - \cos(\xi) \right] \xi^{2\theta-1-2H} d\xi
\]

\[
\leq C_0, H h^{2\theta} |x - y|^{2H-2\theta}.
\]

In conclusion, we have the following bound analogous to upper bound part of (3.9):

\[
d_2^2(x,y) \leq C_{H, \theta} h^\theta (|x - y|^{H-\theta} \wedge t H^{-\theta} - \frac{2H}{\pi} \sum_{n \in Z}.)
\]

(3.35)

for any $\theta \in (0, H)$.

Now we can follow the same argument ((3.20) in particular) as that in the proof of Theorem 1.1 by invoking Talagrand’s majorizing measure theorem (Theorem 3.1) to prove the upper bound part of (1.6):

\[
E \left[ \sup_{x \in \mathbb{L}} \Delta_h u_{\text{add}}(t, x) \right] \leq C_{H, \theta} |h|^H \sqrt{\frac{2}{\pi} \sum_{n \in Z}.}
\]

if $L \geq \sqrt{t}$. Now we turn to prove the lower bound part of (1.6). To this end, we need the inverse part of (3.35) and we shall use again the Sudakov minoration theorem. Observe that we only need to consider the case when $|x - y| \geq \sqrt{t}$. We claim

\[
d_2^2(x,y) \geq c_H h^{2H} \quad \text{when } |x - y| \geq \sqrt{t} \text{ and } h \leq \sqrt{\frac{4\pi^2}{8 \ln 2}} \wedge 1.
\]

In fact, notice that

\[
1 - \exp \left( -\frac{2t \xi^2}{|x - y|^2} \right) \geq \frac{1}{2} \quad \forall \; \xi \geq \frac{|x - y|}{4h} \text{ and } h \leq \sqrt{\frac{4\pi^2}{8 \ln 2}} \wedge 1.
\]

The simple inequality

\[
1 - \cos(x) \geq x^2/4 \quad \text{if } |x| \leq \pi/2
\]

implies

\[
1 - \cos \left( \frac{h \xi}{|x - y|} \right) \geq \frac{h^2 \xi^2}{4|x - y|^2} \quad \text{if } \xi \leq \frac{|x - y|}{2h}.
\]
Therefore, a substitution $\xi \mapsto \xi/|x-y|$ yields

$$d_{2,t,h}^2(x,y) = c_H |x-y|^{2H} \int_{\mathbb{R}^+} \left[ 1 - \exp \left( - \frac{2t\xi^2}{|x-y|^2} \right) \right] \left[ 1 - \cos \left( \frac{h\xi}{|x-y|} \right) \right] \cdot [1 - \cos(\xi)] \xi^{-1-2H} d\xi \geq c_H |x-y|^{2H} \int_{\mathbb{R}^+} \left[ 1 - \cos \left( \frac{h\xi}{|x-y|} \right) \right] [1 - \cos(\xi)] \cdot \xi^{-1-2H} d\xi \geq c_H h^2 |x-y|^{2H-2} \int_{|x-y| \geq \pi} [1 - \cos(\xi)] \cdot \xi^{-1-2H} d\xi.$$ 

Set

$$k_0 = \inf \left\{ k \in \mathbb{N}_0 : \frac{(2k+1)\pi}{2} \geq \frac{|x-y|\pi}{4h} \right\} ;$$

and

$$k_1 = \sup \left\{ k \in \mathbb{N}_0 : \frac{(2k+3)\pi}{2} \leq \frac{|x-y|\pi}{2h} \right\} .$$

If $h$ is sufficiently small, then

$$\int_{|x-y| \geq \pi} [1 - \cos(\xi)] \cdot \xi^{-1-2H} d\xi = \sum_{k \geq 0} \int_{I_k \cap \left[ \frac{|x-y|\pi}{4h}, \frac{|x-y|\pi}{2h} \right]} [1 - \cos(\xi)] \cdot \xi^{-1-2H} d\xi \geq \frac{1}{2} \int_{\frac{(2k_0+3)\pi}{2}}^{\frac{(2k_0+1)\pi}{2}} \xi^{-1-2H} d\xi = c_H \left( \frac{(2k_0+3)\pi}{2} \right)^{2-2H} - \left( \frac{(2k_0+1)\pi}{2} \right)^{2-2H} \geq c_H \left( \frac{|x-y|}{h} \right)^{2-2H} ,$$

due to the fact that $\xi^{-1-2H}$ is an increasing function. Thus, we have for $|x-y| \geq \sqrt{t}$

$$d_{2,t,h}(x,y) \geq c_H h^H$$

(3.36)

if $h \leq C(\sqrt{t} \wedge 1)$ for some small positive quantity $C$. On the interval $\mathbb{L} = [-L, L]$ for $L$ large enough, let us select $x_j = jL/\sqrt{t}$ for $j = 0, \pm 1, \ldots, \pm [L/\sqrt{t}]$. Similar to (3.22), applying the Sudakov minoration theorem (Theorem 3.3) with $\delta = c_H h^H$ yields

$$\mathbb{E} \left[ \sup_{x \in \mathbb{L}} \Delta_h u_{add}(t,x) \right] \geq \mathbb{E} \left[ \sup_{x_i} \Delta_h u_{add}(t,x) \right] \geq c_H |h|^H \varphi_0(t, L).$$

The proof of (1.7) follows from exactly the same argument as that in the proof of (1.5) by Borel-Cantelli’s lemma. The only difference is that now we have

$$\sigma^2(h) = \sup_{x \in \mathbb{L}^n} \mathbb{E}[|\Delta_h u_{add}(t,x)|^2] \leq C_{H, \theta} |h|^H |h|^{2\theta} ;$$

$$\lambda_L := \frac{1}{2} \mathbb{E} \left[ \sup_{x \in \mathbb{L}^n} \Delta_h u_{add}(t,x) \right] ;$$

$$\exp \left( - \frac{\lambda_L^2}{2\sigma^2(h)} \right) \leq C_{H, \theta} \exp \left( - \left[ \frac{h^2}{T} \right]^{H-\theta} \log_2 \left[ \frac{n^\gamma}{\sqrt{T}} \right] \right) ,$$

26
where $\mathbb{L}^\alpha := [-n^\alpha, n^\alpha]$. We can then complete the proof of the theorem by choosing $\alpha$ appropriately. We omit the details here. \hfill \Box

**Proof of Theorem 1.3.** We will use the same method as that in the proof of Theorem 1.2. The natural metric associated with the time increment of the solution is

$$d_{3,t,t}(x,y) = (\mathbb{E}|\Delta_x u_{\text{add}}(t,x) - \Delta_x u_{\text{add}}(t,y)|^2)^{1/2}.$$  

Using

$$\Delta_x u_{\text{add}}(t,x) = \int_0^{t+\tau} \int_{\mathbb{R}} G_{t+\tau-s}(x-z)W(ds,dz) - \int_0^t \int_{\mathbb{R}} G_{t-s}(x-z)W(ds,dz),$$

and using the isometric property of stochastic integral and Plancherel’s identity one derives

$$d_{3,t,t}^2(x,y) = 2 \int_{\mathbb{R}^+} f(t,\tau,\xi)[1 - \cos(|x - y|\xi)] \cdot \xi^{-1 - 2H} d\xi,$$  \hspace{1cm} (3.37)

where

$$f(t,\tau,\xi) = \left[1 - \exp(-2(t+\tau)\xi^2)\right] + \left[1 - \exp(-2t\xi^2)\right] - 2 \exp(-\tau\xi^2) \left[1 - \exp(-2t\xi^2)\right].$$

Notice that when $x \geq 0$, $1 - e^{-x} \leq C_0 \theta x$ and $1 + e^{-2x} - 2e^{-x} = (1 - e^{-x})^2 \leq C_0^2 x^{2\theta}$ for any $\theta \in (0,1)$. Then, we have

$$f(t,\tau,\xi) \leq C_0(\tau\xi^2)^\theta, \quad \forall \theta \in (0,1).$$

Inserting this bound into (3.37) yields

$$d_{3,t,t}^2(x,y) \leq C\theta^\tau \int_{\mathbb{R}^+} \left[1 - \cos(|x - y|\xi)\right] \cdot \xi^{-1 - 2H + 2\theta} d\xi \leq C_{H,\theta} \tau^{\theta} |x - y|^{2H - 2\theta}$$

for any $0 < \theta < H$.

On the other hand, a substitution $\xi \rightarrow \xi/\sqrt{\tau}$ yields

$$d_{3,t,t}^2(x,y) \leq C \int_{\mathbb{R}^+} \left[1 - \exp(-2\tau\xi^2)\xi^{-1 - 2H} d\xi \right.$$  

$$\left. + \int_{\mathbb{R}^+} \left[1 - \exp(-2\tau\xi^2)\right] \left[1 - \exp(-\tau\xi^2)\right]^2 \xi^{-1 - 2H} d\xi \right]$$

$$\leq C_H \tau^{H} + C_{H,\theta} \tau^{\theta} t^{H-\theta} \leq C_{H,\theta} \tau^{\theta} t^{H-\theta}$$

when $\tau \leq Ct$. Thus, we have

$$d_{3,t,t}(x,y) \leq C_{H,\theta} \tau^{\theta/2} (|x - y|^{H-\theta} \wedge t^{\frac{H-\theta}{2}}),$$  \hspace{1cm} (3.38)

where $0 < \theta < H$, which is the bound needed for us to prove the upper bound part of (1.8).

The Sudakov minoration Theorem 3.3 will still be used to prove the lower bound. We need to obtain an appropriate lower bound of $d_{3,t,t}(x,y)$ for $|x - y| \geq \sqrt{t}$. It is
easy to see
\[ d^2_{3,1,\tau}(x, y) \geq c \int_{\mathbb{R}^+} [1 - \exp(-2\tau \xi^2)][1 - \cos(|x - y|\xi)] \cdot \xi^{-1 - 2H} d\xi \]
\[ \geq c\tau|x - y|^{2H-2} \int_{\mathbb{R}^+} \left(\frac{|x-y|}{\sqrt{\xi}}\right)^{2-2H} [1 - \cos(\xi)] \xi^{-1 - 2H} d\xi. \] (3.39)

Analogously to the obtention of (3.36) we can conclude that the integral in (3.39) is bounded below by a multiple of \( (\frac{|x-y|}{\sqrt{\tau}})^{2-2H} \). Thus, we obtain
\[ d_{3,1,\tau}(x, y) \geq c_H \tau^{H/2} \] (3.40)
if \( \tau \leq C(t \wedge 1) \) for some constant \( C \). This is the bound needed to use Theorem 3.3 to show the lower bound part of (1.8).

Once again, Borell’s inequality (Theorem 3.4) can be combined with Borel-Cantelli’s lemma to show the almost sure asymptotics (1.7), and the proof Theorem 1.3 is completed. \( \Box \)

In [12] (see also next section) to show the existence and uniqueness of the solution to (1.1) (for Hurst parameter \( H \in (1/4, 1/2) \)) it is extensively used the following quantity
\[ N_{\frac{1}{2}-H} u(t, x) = \left( \int_{\mathbb{R}} |u(t, x + h) - u(t, x)|^2 \cdot |h|^{2H-2} dh \right)^{\frac{1}{2}}, \] (3.41)
which plays the role of fractional derivative of \( u \). It is because of the difficulty to appropriately bound this quantity (see [12] or the next section) it is assumed that \( \sigma(0) = 0 \) in [12]. After our work on the bound of the solution \( u_{add}(t, x) \) we want to argue that
\[ \mathbb{E} \left[ \sup_{x \in \mathbb{L}} N_{\frac{1}{2}-H}^2 u_{add}(t, x) \right] \geq c_{t,H} \log_2(L) \quad \text{if } L \text{ is sufficiently large.} \] (3.42)
This fact illustrates that the argument in [12] for the pathwise uniqueness (see Lemma 4.9 in [12] for this argument) is not applicable in the general setting when \( \sigma(0) \neq 0 \). Here is the precise statement of our result, which is also interesting for its own sake.

**Proposition 3.11.** Let \( u_{add}(t, x) \) be defined by (3.2) and let \( N_{\frac{1}{2}-H} u_{add}(t, x) \) be defined by (3.41).

(i) For any fixed \( t > 0 \) and \( L \geq \sqrt{t} \) we have
\[ \mathbb{E} \left[ \sup_{-L \leq x \leq L} N_{\frac{1}{2}-H}^2 u_{add}(t, x) \right] \geq c_{t,H} \log_2(L), \] (3.43)
where \( c_{t,H} \) is a positive constant.

(ii) Moreover, we have almost surely if \( L \geq \sqrt{t} \)
\[ \sup_{-L \leq x \leq L} N_{\frac{1}{2}-H} u_{add}(t, x) \leq C_H t^{2H-\frac{1}{2}} [1 - \log(\sqrt{t} \wedge 1)]\Psi_0(t, L), \] (3.44)
where \( C_H \) is a positive random constant.
Proof. First, we consider the upper bound (3.44). Let $0 < \theta < \frac{1-2H}{2}$. Applying Theorem 1.2 when $|h| \leq \sqrt{t} \wedge 1$ and Theorem 1.1 when $|h| > \sqrt{t} \wedge 1$, respectively, and using the notation $\Delta_h u_{\text{add}}(t,x) := u_{\text{add}}(t,x + h) - u_{\text{add}}(t,x)$ we obtain

\[
\sup_{x \in L} \mathcal{N}_t^{2-H} u_{\text{add}}(t,x) = \sup_{x \in L} \int_R |\Delta_h u_{\text{add}}(t,x)|^2 \cdot |h|^{2H-2} dh
\]

\[
\leq \int_R \left( \sup_{x \in L} |\Delta_h u_{\text{add}}(t,x)| \right)^2 \cdot |h|^{2H-2} dh
\]

\[
\leq \int_{\{|h| \leq \sqrt{t} \wedge 1\}} \left( \sup_{x \in L} |\Delta_h u_{\text{add}}(t,x)| \right)^2 \cdot |h|^{2H-2} dh
\]

\[
+ \int_{\{|h| > \sqrt{t} \wedge 1\}} \left( \sup_{x \in L} |\Delta_h u_{\text{add}}(t,x)| \right)^2 \cdot |h|^{2H-2} dh
\]

\[
\leq C_H \theta t^{H-\theta} \Psi_0(t,L) \int_{\{|h| \leq \sqrt{t} \wedge 1\}} |h|^{2H-2+2\theta} dh + C_H t^H \Psi_0(t,L)
\]

\[
\cdot \left[ \int_{\{|h| > \sqrt{t} \wedge 1\}} |h|^{2H-2} dh + \int_{\{|h| > \sqrt{t} \wedge 1\}} \log_2(|h|/\sqrt{t}) |h|^{2H-2} dh \right],
\]

where we applied an elementary inequality

\[
\log_2 |L + h| \leq \begin{cases} 
\log_2(L) + 1 & \text{when } |h| \leq 1; \\
\log_2(L) + \log_2(|h|) + 1 & \text{when } |h| \geq 1.
\end{cases}
\]

Most of terms of above integrals can be evaluated easily except the one involving with $\log_2(|h|/\sqrt{t})$, which equals to

\[
\int_{\{|h| > \sqrt{t} \wedge 1\}} \left[ \log_2(|h|) - \log_2(\sqrt{t}) \right] |h|^{2H-2} dh
\]

\[
\leq \int_{\{|h| > \sqrt{t} \wedge 1\}} \left[ \log_2(|h|) - \log_2(\sqrt{t} \wedge 1) \right] |h|^{2H-2} dh \leq (\sqrt{t} \wedge 1)^{2H-1} [1 - \log(\sqrt{t} \wedge 1)].
\]

This yields (3.44).

Now we turn to the lower bound (3.43). A simple observation and an application of Jensen’s inequality give

\[
\mathbb{E} \left[ \sup_{x \in L} \mathcal{N}_t^{2-H} u_{\text{add}}(t,x) \right]
\geq c_H \mathbb{E} \left[ \left( \sup_{x \in L} \int_R \Delta_h u_{\text{add}}(t,x) \varrho(h) dh \right)^2 \right] \geq c_H \left( \mathbb{E} \left[ \sup_{x \in L} \int_R \Delta_h u_{\text{add}}(t,x) \varrho(h) dh \right] \right)^2,
\]

(3.45)

where $\varrho(h) = C_H \left[ |h|^{2H-2} 1_{\{|h| \leq 1\}} + |h|^{2H-2} 1_{\{|h| > 1\}} \right]$ such that it is probability density. Denote

\[
u_\varrho(t,x) = \int_R \Delta_h u_{\text{add}}(t,x) \varrho(h) dh = \int_0^t \int_R \left( \int_R D_{t-s}(h, x-z) \varrho(h) dh \right) W(ds, dz),
\]

where $D_t(h,x)$ is defined in (2.14). The above $\nu_\varrho(t,x)$ is a well-defined Gaussian random field since $\varrho(h)$ is integrable for $\frac{1}{4} < H < \frac{1}{2}$. Introduce the induced natural metric

\[
d_{t,x}(x,y) := (\mathbb{E}[\nu_\varrho(t,x) - \nu_\varrho(t,y)]^2)^{\frac{1}{2}}.
\]
We need to bound this distance for $|x - y| \geq 1$. Applying Plancherel’s identity we can find

$$d_{\mathcal{H}}^2(x, y) = c_H \int_{\mathbb{R}^+} \left[ 1 - \exp(-2t\xi^2) \right] [1 - \cos(|x - y|\xi)]$$

$$\cdot \left( \int_{\mathbb{R}^+} [1 - \cos(h\xi)]\varrho(h)dh \right)^2 \cdot \xi^{-1-2H} d\xi.$$  

When $\xi \geq 1$, we have

$$\int_{\mathbb{R}^+} [1 - \cos(h\xi)]\varrho(h)dh \geq \xi^{\frac{1}{2} - 2H} \int_0^\xi [1 - \cos(h)] \cdot h^{2H - \frac{3}{2}} dh \geq c\xi^{\frac{1}{2} - 2H}.$$  

Thus, we conclude that if $|x - y| \geq 1$, then

$$d_{\mathcal{H}}^2(x, y) \geq c_H [1 - \exp(-2t)] \int_1^\infty [1 - \cos(|x - y|\xi)] \cdot \xi^{-6H} d\xi$$

$$\geq c_H [1 - \exp(-2t)]$$ (3.46)

by the same argument as that in proof of lower bound of $\mathbb{E}[\sup_{x \in \mathcal{L}} \Delta_h u_{add}(t, x)]$ in Theorem 1.2. An application of the Sudakov minoration Theorem 3.3 implies the lower bound (3.43).  

4. Weak Existence and Regularity of Solutions

4.1. Basic settings. This section is devoted to prove the existence of a weak solution to (1.1). Let us briefly recall some notations and facts in [12]. Let $(B, \|\cdot\|_B)$ be a Banach space with the norm $\|\cdot\|_B$. Let $\beta \in (0, 1)$ be a fixed number. For any function $f : \mathbb{R} \to B$ denote

$$N^B_\beta f(x) = \left( \int_{\mathbb{R}} \|f(x + h) - f(x)\|_B^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}},$$ (4.1)

if the above quantity is finite. When $B = \mathbb{R}$, we abbreviate the notation $N^R_\beta f$ as $N_\beta f$ (see also (3.41)). As in [12] throughout this paper we are particularly interested in the case $B = L^p(\Omega)$, and in this case we denote $N^B_\beta$ by $N_{\beta, p}$:

$$N_{\beta, p} f(x) = \left( \int_{\mathbb{R}} \|f(x + h) - f(x)\|_{L^p(\Omega)}^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}.$$ (4.2)

The following Burkholder-Davis-Gundy inequality is well-known (see e.g. [12]).

**Proposition 4.1.** Let $W$ be the Gaussian noise defined by the covariance (2.1), and let $f \in \Lambda_H$ be a predictable random field. Then for any $p \geq 2$ we have

$$\left\| \int_0^t \int_{\mathbb{R}} f(s, y)W(ds, dy) \right\|_{L^p(\Omega)} \leq \sqrt{4p} c_H \left( \int_0^t \int_{\mathbb{R}} \left[ N_{\frac{1}{2} - H, p} f(s, y) \right]^2 dy ds \right)^{\frac{1}{2}},$$ (4.3)

where $c_H$ is a constant depending only on $H$ and $N_{\frac{1}{2} - H, p} f(s, y)$ denotes the application of $N_{\frac{1}{2} - H, p}$ with respect to the space variable $y$.

In the work [12], the authors have already proved the existence and uniqueness result in a solution space $\mathcal{Z}_p$ (see [12] or formula (4.4) in next paragraph for the definition of $\mathcal{Z}_p$) under the condition $\sigma(t, x, 0) = 0$. When $\sigma(t, x, 0) \neq 0$ or even in the simplest case $\sigma(t, x, u) = 1$ (as we see from (3.43)) we cannot expect that the
solution is still in $Z^p_{\lambda,T}$. So, the method powerful in [12] is no longer valid to solve (1.1) for general $\sigma(t,x,u)$. Our idea is to add an appropriate weight $\lambda(x)$ to the space $Z^p_{\lambda,T}$ to obtain a weighted space $Z^p_{\lambda,T}$.

Let $\lambda(x) \geq 0$ be a Lebesgue integrable positive function with $\int_{\mathbb{R}} \lambda(x)dx = 1$. Introduce a norm $\| \cdot \|_{Z^p_{\lambda,T}}$ for a random field $v(t,x)$ as follows:

$$
\|v\|_{Z^p_{\lambda,T}} := \sup_{t \in [0,T]} \|v(t,\cdot)\|_{L^p(\Omega \times \mathbb{R})} + \sup_{t \in [0,T]} N^t_{\frac{1}{2}-H,p} v(t),
$$

where $p \geq 2$, $\frac{1}{4} < H < \frac{1}{2}$,

$$
\|v(t,\cdot)\|_{L^p(\Omega \times \mathbb{R})} = \left( \int_{\mathbb{R}} \left( \mathbb{E} |v(t,x)|^p \right) \lambda(x)dx \right)^{\frac{1}{p}},
$$

and

$$
N^t_{\frac{1}{2}-H,p} v(t) = \left( \int_{\mathbb{R}} |v(t,\cdot) - v(t,\cdot + h)|^2_{L^p(\Omega \times \mathbb{R})} |h|^{2H-2}dh \right)^{\frac{1}{2}}.
$$

Then $Z^p_{\lambda,T}$ is the function space consisting of all the random fields $v = v(t,x)$ such that $\|v\|_{Z^p_{\lambda,T}}$ is finite. When the function is independent of $t$, the corresponding space is denoted by $Z^p_{\lambda,0}$.

4.2. Some bounds for stochastic convolutions. To prove the existence of weak solution, we need some delicate estimates of stochastic integral with respect to the weight.

**Proposition 4.2.** Denote the weight function

$$
\lambda(x) = \lambda_H(x) = c_H (1 + |x|^2)^{H-1},
$$

where $c_H$ is a constant such that $\int \lambda(x)dx = 1$, and denote

$$
\Phi(t,x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)v(s,y)W(ds,dy).
$$

We have the following estimates. (In the following $C_{T,p,H,\gamma}$ denotes a constant, depending only on $T$, $p$, $H$ and $\gamma$).

(i) If $p > \frac{6}{7}$, then

$$
\left\| \sup_{t \in [0,T], x \in \mathbb{R}} \lambda^{\frac{1}{2}}(x) \Phi(t,x) \right\|_{L^p(\Omega)} \leq C_{T,p,H} \|v\|_{Z^p_{\lambda,T}}.
$$

(ii) If $p > \frac{6}{4H-1}$, then

$$
\left\| \sup_{t \in [0,T], x \in \mathbb{R}} \lambda^{\frac{1}{2}}(x) N^t_{\frac{1}{2}-H} \Phi(t,x) \right\|_{L^p(\Omega)} \leq C_{T,p,H} \|v\|_{Z^p_{\lambda,T}}.
$$

(iii) If $p > \frac{3}{2}$, and $0 < \gamma < \frac{H}{2} - \frac{3}{2p}$, then

$$
\left\| \sup_{t, t+h \in [0,T]} \lambda^{\frac{1}{2}}(x) \left[ \Phi(t+h,x) - \Phi(t,x) \right] \right\|_{L^p(\Omega)} \leq C_{T,p,H,\gamma} |h|^\gamma \|v\|_{Z^p_{\lambda,T}}.
$$

(iv) If $p > \frac{3}{2}$, and $0 < \gamma < H - \frac{3}{2}$, then

$$
\left\| \sup_{t \in [0,T], x,y \in \mathbb{R}} \frac{\Phi(t,x) - \Phi(t,y)}{\lambda^{\frac{1}{2}}(x) + \lambda^{\frac{1}{2}}(y)} \right\|_{L^p(\Omega)} \leq C_{T,p,H,\gamma} |x - y|^\gamma \|v\|_{Z^p_{\lambda,T}}.
$$
Remark 4.3. The method provided in the following proof depends on the semigroup property of the heat kernel because we need to use the factorization method (e.g. [7]. see also (4.13) below). This means that we can not apply our approach directly to the stochastic wave equation since the wave kernel (the fundamental solution of the wave equation in [3]) lacks the semigroup property.

Proof. For any $\alpha \in (0, 1)$ we set

$$J_\alpha(r, z) := \int_0^r \int_\mathbb{R} (r-s)^{-\alpha}G_{r-s}(z-y)v(s,y)W(ds,dy).$$

(4.12)

A stochastic version of Fubini’s theorem implies

$$\Phi(t, x) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_\mathbb{R} (t-r)^{\alpha-1}G_{t-r}(x-z)J_\alpha(r, z)dzdr.$$  

(4.13)

We are going to show the four different parts of the proposition separately. We divide our proof into six steps. Let us recall $D_t(x, h) := G_t(x+h) - G_t(x)$, and $\Box_t(x, y, h) = D_t(x+y, h) - D_t(x, h)$ defined in (2.14) and (2.15).

Step 1. The first two steps are to prove part (i). In this step we will obtain the desired growth estimate of $\Phi(t, x)$ in term of $J_\alpha(r, z)$. Applying the bounds of (2.22) and (2.12) to (4.13) we have

$$\sup_{t,x} |\lambda^\theta(x)| \Phi(t, x)| \leq \sup_{t,x} \lambda^\theta(x) \left| \int_0^t \int_\mathbb{R} (t-r)^{\alpha-1}G_{t-r}(x-z)J_\alpha(r, z)dzdr \right|$$

$$\lesssim \sup_{t,x} \lambda^\theta(x) \left( \int_\mathbb{R} |G_{t-r}(x-z)| \lambda^{-\frac{\alpha}{q}}(z)^{q}dz \right)^{\frac{1}{q}} \|J_\alpha(r, \cdot)\|_{L^\theta(\mathbb{R})}dr$$

$$\lesssim \sup_{t,x} \lambda^\theta(x) \left( \int_\mathbb{R} (t-r)^{\alpha-1} \left( \int_\mathbb{R} |G_{t-r}(x-z)| \lambda^{-\frac{\alpha}{q}}(z)^{q}dz \right)^{\frac{1}{q}} \|J_\alpha(r, \cdot)\|_{L^\theta(\mathbb{R})}dr \right.$$

$$\lesssim \sup_{t,x} \lambda^\theta(x) \left( \int_\mathbb{R} (t-r)^{\alpha-1} \cdot (t-r)^{-\frac{\alpha}{q}} \lambda^{-\frac{1}{q}}(x) \cdot \|J_\alpha(r, \cdot)\|_{L^\theta(\mathbb{R})}dr \right.$$}

Setting $\theta = \frac{1}{p}$ and then applying the Hölder inequality we obtain

$$\sup_{t,x} |\lambda^\theta(x)| \Phi(t, x) \lesssim \sup_{t \in [0, T]} \left[ \int_0^t (t-r)^{\alpha-\frac{q}{2}+\frac{1}{2q}} \cdot \|J_\alpha(r, \cdot)\|_{L^\theta(\mathbb{R})}^p dr \right]^{\frac{1}{p}}$$

$$\lesssim \left[ \int_0^T \|J_\alpha(r, \cdot)\|_{L^\theta(\mathbb{R})}^p dr \right]^{\frac{1}{p}}$$

(4.14)

if $q(\alpha - \frac{3}{2} + \frac{1}{2q}) > -1$, i.e. if

$$\alpha > \frac{3}{2p}.$$  

(4.15)

This is possible if $p > 3/2$. Thus, to prove part (i), we only need to show that there exists a constant $C$, independent of $r \in [0, T]$, such that

$$\mathbb{E}[\|J_\alpha(r, \cdot)\|_{L^\theta(\mathbb{R})}^p] \leq C \|v\|_{L^p(\mathbb{R}, x)}.$$

(4.16)
Step 2. We shall prove the above bound (4.16) in this step and to do this let us introduce the following two notations

\[
\mathcal{D}_1(r, z) := \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} |D_{r-s}(y, h)|^2 \|v(s, y + z)\|_{L_p(\Omega)}^2 |h|^{2H-2} \, dh \, dy \right)^{\frac{2}{p}},
\]

and

\[
\mathcal{D}_2(r, z) := \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} |G_{r-s}(y)|^2 \|\Delta_h v(s, y + z)\|_{L_p(\Omega)}^2 |h|^{2H-2} \, dh \, dy \right)^{\frac{2}{p}},
\]

where \(\Delta_h v(t, x) := v(t, x + h) - v(t, x)\). From the definition (4.12) of \(J\) and by Burkholder-Davis-Bundy’s inequality (4.3) stated in Lemma 4.1, we have

\[
\mathbb{E}|J_{o}(r, \cdot)|^{p}_{L^p_{\mathcal{M}}(\mathbb{R})} \lesssim \int_\mathbb{R} \left\{ \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \left[ \mathbb{E}|G_{r-s}(y + h - z)v(s, y + h) - G_{r-s}(y - z)v(s, y)|^p\right]^{2/p} |h|^{2H-2} \, dh \, dy \right\}^{p/2} \lambda(z) \, dz \lesssim \int_\mathbb{R} [\mathcal{D}_1(r, z) + \mathcal{D}_2(r, z)] \lambda(z) \, dz =: D_1 + D_2.
\]

For the first term \(I_1\), thanks to Minkowski’s inequality, we have

\[
D_1 \lesssim \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} |D_{r-s}(y, h)|^2 \cdot \|\Delta_y v(s, \cdot)\|_{L^p_{\mathcal{M}}(\mathbb{R} \times \Omega)}^2 |h|^{2H-2} \, dh \, dy \right)^{\frac{2}{p}} + \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} |D_{r-s}(y, h)|^2 \cdot \|v(s, \cdot)\|_{L^p_{\mathcal{M}}(\mathbb{R} \times \Omega)}^2 |h|^{2H-2} \, dh \, dy \right)^{\frac{2}{p}} \lesssim \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha + \frac{1}{2}} \|\Delta_y v(s, \cdot)\|_{L^p_{\mathcal{M}}(\mathbb{R} \times \Omega)}^2 |y|^{2H-2} \, dy \right)^{\frac{2}{p}} + \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha + H - 1} \|v(s, \cdot)\|_{L^p_{\mathcal{M}}(\mathbb{R} \times \Omega)}^2 |s|^{2H-2} \, ds \right)^{\frac{2}{p}},
\]

(4.17)

where the last inequality follows from inequalities (2.22) and (2.17).

For the second term \(I_2\), we can again use Minkowski’s inequality, Jensen’s inequality with respect to \((r-s)^{1/2} G_{r-s}(y) dy = G_{r-s}(y) dy\) (since when \(p > 2\), the function \(\phi(x) = x^{2/p}, x > 0\), is concave), and then we use (2.12) to obtain

\[
D_2 \lesssim \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} G_{r-s}(y) \left( \int_{\mathbb{R}} \|\Delta_h v(s, y + z)\|_{L^p_{\mathcal{M}}(\Omega)}^p \lambda(z) \, dz \right)^{2/p} |h|^{2H-2} \, dy \, dh \right)^{\frac{2}{p}} \lesssim \left( \int_0^r \int_{\mathbb{R}} (r-s)^{-2\alpha + \frac{1}{2}} \left( \int_{\mathbb{R}} G_{r-s}(y) \|\Delta_h v(s, z)\|_{L^p_{\mathcal{M}}(\Omega)}^p \, dz \cdot \lambda(z - y) \, dy \right)^{2/p} |h|^{2H-2} \, dh \right)^{\frac{2}{p}} \lesssim \left( \int_0^r \int_{\mathbb{R}} (r-s)^{-2\alpha + \frac{1}{2}} \|\Delta_h v(s, \cdot)\|_{L^p_{\mathcal{M}}(\mathbb{R} \times \Omega)}^2 |h|^{2H-2} \, dh \right)^{\frac{2}{p}},
\]

(4.18)

Recall that

\[
\|v\|_{L^p_{\mathcal{M}}(\mathbb{R})} := \sup_{s \in [0, T]} \|v(s, \cdot)\|_{L^p_{\mathcal{M}}(\mathbb{R} \times \Omega)} + \sup_{s \in [0, T]} \mathcal{N}_{-H,p}^2 v(s),
\]

33
where \( N^*_{\frac{1}{2}-H,p} v(t) \) is defined in (4.5). The estimates obtained in (4.17) and (4.18) imply

\[
\mathbb{E}\|J_\alpha(r, \cdot)\|^p_{L^p_\alpha(\mathbb{R})} \lesssim \|v\|^p_{Z^p_{\alpha,T}} \left( \int_0^T \int_\mathbb{R} (r-s)^{-2\alpha - \frac{1}{2}} + (r-s)^{-2\alpha + H - 1} dr \right)^\frac{p}{q}. \tag{4.19}
\]

If we have \(-2\alpha + H - 1 > -1\) and \(-2\alpha - \frac{1}{2} > -1\), i.e. \(\alpha < \frac{H}{2}\), then (4.16) follows.

However, the condition \(\alpha < H/2\) should be combined with (4.15). This gives \(\frac{3}{2p} < \alpha < \frac{H}{2}\) which implies \(p > \frac{2}{H}\). Thus, under the condition of the proposition, the inequality (4.16) holds true. This finishes the proof of (i).

**Step 3.** In this and next steps we prove (ii). The spirit of the proof is similar to that of the proof of (i) but is more involved. In order to obtain the desired decay rate of \( N_{\frac{1}{2}-H}\Phi(t,x) \), we still use the equation (4.13) to express \(\Phi(t,x)\) by \(J\).

\[
\Phi(t, x+h) - \Phi(t, x) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_\mathbb{R} (t-r)^{\alpha-1} D_t \sum_{r} J_n(r, z) dz dr
\]

where \(\Delta_t J_\alpha(t, x+h) := J_\alpha(t, x+h) - J_\alpha(t, x)\).

Invoking Minkowski’s inequality and then Hölder’s inequality with \(\frac{1}{p} + \frac{1}{q} = 1\) we get

\[
\int_\mathbb{R} |\Phi(t, x+h) - \Phi(t, x)|^2 |h|^{2H-2} dh
\]

\[
\lesssim \int_\mathbb{R} \left( \int_0^t \int_\mathbb{R} (t-r)^{\alpha-1} G_t \sum_{r} J_n(r, z) dz dr \right)^2 \cdot |h|^{2H-2} dh
\]

\[
\lesssim \left( \int_0^t \int_\mathbb{R} (t-r)^{\alpha-1} G_t \sum_{r} J_n(r, z) dz dr \right)^2 \cdot \left( \int_\mathbb{R} |\Delta_t J_\alpha(r, z)|^2 |h|^{2H-2} dh \right)^{\frac{q}{2}} dz dr
\]

\[
\lesssim \left( \int_0^t \int_\mathbb{R} (t-r)^{q(\alpha-1)} G_t^{q} \sum_{r} J_n(r, z) \lambda^{-\frac{q}{2}}(z) dz dr \right)^{\frac{q}{2}} \times \left( \int_0^T \int_\mathbb{R} \left[ \int_\mathbb{R} |\Delta_t J_\alpha(r, z)|^2 |h|^{2H-2} dh \right]^{\frac{q}{2}} \lambda(z) dz dr \right)^\frac{q}{2}
\]

\[
\lesssim \lambda(x)^{-\frac{q}{2}} \left( \int_0^t (t-r)^{q(\alpha-\frac{3}{2} + \frac{1}{2q})} dr \right)^{\frac{q}{2}} \times \left( \int_0^T \int_\mathbb{R} \left[ \int_\mathbb{R} |\Delta_t J_\alpha(r, z)|^2 |h|^{2H-2} dh \right]^{\frac{q}{2}} \lambda(z) dz dr \right)^\frac{q}{2},
\]

where in the above last inequality we used \( G_t^{q} \sum_{r} J_n(r, z) = (t-r)^{\frac{1-q}{2}} G_t \sum_{r} J_n(r, z) \) and inequality (2.12). If we take \(\theta = \frac{1}{p}\), and \(q(\alpha - \frac{3}{2} + \frac{1}{2q}) > -1\), i.e.

\[
\alpha > \frac{3}{2p}, \tag{4.20}
\]
then

$$\sup_{t,x} \lambda(x)^\alpha \left( \int_{\mathbb{R}} |\Phi(t, x + h) - \Phi(t, x)|^2 |h|^{2H-2} dh \right)^{\frac{1}{2}}$$

$$\lesssim \left( \int_0^T \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |\Delta h J_\alpha(r, z)|^2 |h|^{2H-2} dh \right]^{\frac{p}{2}} \lambda(z) dz dr \right)^{\frac{1}{p}}.$$

Thus, to prove part (ii) we only need to prove that there exists some constant $C_1$, independent of $r \in [0, T]$, such that

$$\mathcal{I} := \mathbb{E} \left[ \int_{\mathbb{R}} \int_{\mathbb{R}} |\Delta h J_\alpha(r, z)|^2 |h|^{2H-2} dh \right]^{\frac{p}{2}} \lambda(z) dz \leq C_1 \|v\|^p_{L^p_{\lambda, r}}. \quad (4.21)$$

**Step 4.** In this step we show the above inequality (4.21). By the definition (4.12) of $J$ and by an application of Minkowski’s inequality we have

$$\mathcal{I} \lesssim \left( \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \mathbb{E}[|\Delta h J_\alpha(r, z)|^p \lambda(z) dz] \right]^{\frac{p}{2}} |h|^{2H-2} dh \right)^{\frac{p}{2}}$$

$$\lesssim \left( \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \mathbb{E} \left( \int_0^r \int_{\mathbb{R}} (r - s)^{-2\alpha} |D_{r-s}(z - y - l, h) v(s, y + l) - D_{r-s}(z - y, h) v(s, y)|^2 |l|^{2H-2} dl dy ds \right)^{\frac{p}{2}} \lambda(z) dz \right]^{\frac{p}{2}} |h|^{2H-2} dh \right)^{\frac{p}{2}}.$$

We introduce two notations:

$$\mathcal{I}_1(r, z, h) := \mathbb{E} \left( \int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\alpha} |D_{r-s}(z - y - l, h) v(s, y + l) - D_{r-s}(z - y, h) v(s, y)|^2 |l|^{2H-2} dl dy ds \right)^{\frac{p}{2}},$$

and

$$\mathcal{I}_2(r, z, h) := \mathbb{E} \left( \int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\alpha} |D_{r-s}(z - y - l, h) v(s, y + l) - D_{r-s}(z - y, h) v(s, y)|^2 |l|^{2H-2} dl dy ds \right)^{\frac{p}{2}}.$$

Then, we have

$$\mathbb{E} \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |\Delta h J_\alpha(r, z)|^2 |h|^{2H-2} dh \right]^{\frac{p}{2}} \lambda(z) dz$$

$$\lesssim \left( \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \mathcal{I}_1(r, z, h) \lambda(z) dz \right]^{\frac{p}{2}} |h|^{2H-2} dh \right)^{\frac{p}{2}}$$

$$+ \left( \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \mathcal{I}_2(r, z, h) \lambda(z) dz \right]^{\frac{p}{2}} |h|^{2H-2} dh \right)^{\frac{p}{2}} =: I_1^{p/2} + I_2^{p/2}.$$
We shall bound $I_1$ and $I_2$ one by one. For the first term, a change of variable $y \to z - y$ and an application of Minkowski’s inequality yield

$$ I_1 \lesssim \int_R \left( \int_R \left( \int_0^r \int_{R^2} (r - s)^{-2\alpha} |D_{r-s}(y, h)|^2 \right) \|\Delta v(s, y + z)| |l|^{2H-2} dlydz \right)^\frac{p}{2} \lambda(z)dz \|h|^{2H-2} dh $$

$$ \lesssim \int_0^r \left( \int_{R^3} (r - s)^{-2\alpha} |D_{r-s}(y, h)|^2 |l|^{2H-2} |h|^{2H-2} \right) \times \left( \int_{R} \|\Delta v(s, z)| |p\lambda(z - y)dz \right)^\frac{p}{2} dydhds. \quad (4.22) $$

By (2.17) with $\beta = \frac{1}{H} - H$ we see that

$$ \int_{R^2} |D_{r-s}(y, h)|^2 |h|^{2H-2} dhdy \lesssim (r - s)^{H-1}, $$

which is finite. Since $x^{2/p}, x > 0$ is a concave function for $p \geq 2$ we can apply Jensen’s inequality with respect to the probability measure $(r - s)^{1-H} |G_{r-s}(y) - G_{r-s}(y + h)|^2 |h|^{2H-2} dydh$. Thus, we have for $p \geq 2$:

$$ I_1 \lesssim \int_0^r \left( \int_{R} (r - s)^{-2\alpha + H-1} \left( \int_{R^3} (r - s)^{1-H} |D_{r-s}(y, h)|^2 \right) \right) \|h|^{2H-2} \int_{R} \|\Delta v(s, z)| |p\lambda(z - y)dz \right)^\frac{p}{2} \times |l|^{2H-2} dl $$$$ \lesssim \int_0^r \left( \int_{R} (r - s)^{-2\alpha + H-1} \|\Delta v(s, \cdot)\|_{L^p_x(\Omega \times R)}^2 |l|^{2H-2} dl $$

by the first inequality in Lemma 2.12.

In order to bound $I_2(t, x, h)$, we make a change of variable $y \to z - y$ and then split it into two terms. More precisely, we have

$$ I_2(r, z, h) \lesssim I_{21}(r, z, h) + I_{22}(r, z, h) \quad (4.24) $$

$$ := \int_{R} \left( \int_{R^2} (r - s)^{-2\alpha} \|\Box_{r-s}(y, l, h)|^2 |v(s, z)|^2 |l|^{2H-2} dlydz \right)^\frac{p}{2} + \int_{R} \left( \int_{R^2} (r - s)^{-2\alpha} \|\Box_{r-s}(y, l, h)|^2 |\Delta v(s, z)|^2 |l|^{2H-2} dlydz \right)^\frac{p}{2}. $$

Using Minkowski’s inequality, Lemma 2.8, and Lemma 2.11, one can check that

$$ I_{21} := \int_{R} \left( \int_{R} \int_{R^2} (r - s)^{-2\alpha} |D_{r-s}(y, h)|^2 \lambda(z)dz \right)^\frac{p}{2} |h|^{2H-2} dh $$

$$ \lesssim \int_0^r \left( \int_{R^3} (r - s)^{-2\alpha} \|\Box_{r-s}(y, l, h)|^2 |v(s)|\|_{L^p_x(\Omega \times R)}^2 |l|^{2H-2} |h|^{2H-2} dhdydz \right) \quad (4.25) $$

$$ \lesssim \int_0^r (r - s)^{-2\alpha + 2H-2} \|v(s)|\|_{L^p_x(\Omega \times R)}^2 ds, $$

36
and

\[ I_{22} := \int_\mathbb{R} \left( \int_\mathbb{R} J_{22}(r, z, h) \lambda(z)dz \right)^\frac{2}{p} |h|^{2H-2}dh \]

\[ \lesssim \int_0^r \int_\mathbb{R} (r-s)^{-2a} \int_\mathbb{R} \left( \int_\mathbb{R} |\nabla_r \phi(y, l, h)|^2 |t|^{2H-2} |h|^{2H-2} dldh \right) \cdot \|\Delta_y v(s, \cdot)\|_{L^p_h(\Omega \times \mathbb{R})}^2 dyds \]

\[ \lesssim \int_0^r \int_\mathbb{R} (r-s)^{-2a+H-1} \|\Delta_y v(s, \cdot)\|_{L^p_h(\Omega \times \mathbb{R})}^2 |t|^{2H-2} dyds. \]

(4.26)

Recalling the definition of \( \| \cdot \|_{Z^p_{h,T}} \), and combining (4.23), (4.25) and (4.26), we obtain

\[ \mathbb{E} \int_\mathbb{R} \left( \int_\mathbb{R} |\Delta_h J_a(r, z)|^2 |h|^{2H-2}dh \right)^\frac{p}{2} \lambda(z)dz \]

\[ \leq C_2 \|v\|_{Z^p_{h,T}}^p \left( \int_0^r (r-s)^{-2a+2H-\frac{3}{2}} + (r-s)^{-2a+H-1}dr \right)^\frac{p}{2}. \]

(4.27)

Once we have \(-2a + 2H - \frac{3}{2} > -1\) and \(-2a + H - 1 > -1\), i.e. \( \alpha < H - \frac{1}{4} \), we see that (4.21) follows from (4.27). This condition on \( \alpha \) is combined with (4.20) to become \( \frac{3}{2p} < \alpha < H - \frac{1}{4} \). Therefore, we have proved that if \( p > \frac{6}{4H-1} \), then (4.21) holds, finishing the proof of (ii).

**Step 5.** We are going to prove part (iii). We continue to use (4.13). Without loss of generality, we can assume \( h > 0 \) and \( t \in [0, T] \) such that \( t + h \leq T \). We have

\[ \Phi(t+h, x) - \Phi(t, x) \]

\[ = \frac{\sin(\pi\alpha)}{\pi} \left[ \int_0^{t+h} \int_\mathbb{R} (t + h - r)^{\alpha-1} G_{t+h-r}(x-z) J_a(r, z)drdz \right. \]

\[ - \left. \int_t^{t+h} \int_\mathbb{R} (t - r)^{\alpha-1} G_{t-r}(x-z) \times J_a(r, z)drdz \right]. \]

\[ \lesssim \sum_{i=1}^{3} \tilde{J}_i(t, h, x), \]

where

\[ \tilde{J}_1(t, h, x) := \int_0^{t+h} \int_\mathbb{R} [(t + h - r)^{\alpha-1} - (t - r)^{\alpha-1}] G_{t-r}(x-z) J_a(r, z)drdz, \]

\[ \tilde{J}_2(t, h, x) := \int_0^{t+h} \int_\mathbb{R} (t + h - r)^{\alpha-1} [G_{t+h-r}(x-z) - G_{t-r}(x-z)] J_a(r, z)drdz, \]

and

\[ \tilde{J}_3(t, h, x) := \int_t^{t+h} \int_\mathbb{R} (t + h - r)^{\alpha-1} G_{t+h-r}(x-z) J_a(r, z)drdz. \]
As in the proof of (i) and (ii), we insert additional factors of $\lambda^{-\frac{1}{p}}(z) \cdot \lambda^\frac{1}{p}(z)$ and apply Hölder’s inequality in the expression for $\mathcal{J}_1$. Then, $\mathcal{J}_1$ is estimated as follows.

$$
\mathcal{J}_1(t, h, x) \leq \lambda^{-\frac{1}{p}}(x) \int_0^t |(t + h - r)^{\alpha - 1} - (t - r)^{\alpha - 1}| |(t - r)^{-\frac{1}{p}}\|J_\alpha(r, \cdot)\|_{L_\alpha^p(\mathbb{R})}dr
$$

$$
\leq \lambda^{-\frac{1}{p}}(x) \left( \int_0^t |(t + h - r)^{\alpha - 1} - (t - r)^{\alpha - 1}| |(t - r)^{-\frac{1}{p}}\|^{\frac{1}{p}}dr \right)^{\frac{q}{p}}
$$

$$
\times \left( \int_0^T \|J_\alpha(r, \cdot)\|_{L_\alpha^p(\mathbb{R})}^{\frac{1}{p}}dr \right)^{\frac{q}{p}}.
$$

(4.28)

Fix $\gamma \in (0, 1)$. It is easy to see

$$
|(t + h - r)^{\alpha - 1} - (t - r)^{\alpha - 1}| \lesssim |t - r|^{\alpha - 1 - \gamma} h^\gamma.
$$

(4.29)

Thus, we have

$$
\sup_{t, x} \lambda^{1/p}(x)|\mathcal{J}_1(t, h, x)| \lesssim h^\gamma \sup_{t \in [0, T]} \left( \int_0^t (t - r)^{q(\alpha - 1 - \gamma) + \frac{1}{p}}dr \right)^{\frac{1}{q}}
$$

$$
\times \left( \int_0^T \|J_\alpha(r, \cdot)\|_{L_\alpha^p(\mathbb{R})}^{\frac{1}{p}}dr \right)^{\frac{q}{p}}.
$$

In other word, if $\gamma + \frac{3}{2p} < \alpha < \frac{H}{2}$ or equivalently, if $\gamma < \frac{H}{2} - \frac{3}{2p}$, then we have

$$
\mathbb{E}\left|\sup_{t, x} \lambda^\theta(x)|\mathcal{J}_1(t, h, x)|\right|^p \lesssim |h|^p \|v\|^p_{Z_\alpha^{T, T}}.
$$

(4.30)

Let us proceed to bound $\mathcal{J}_2(t, h, x)$. One finds easily

$$
\mathcal{J}_2(t, h, x) \leq \left( \int_0^t \int_{\mathbb{R}} (t + h - r)^{q(\alpha - 1)}|G_{t+h-r}(x - z)|dzdr \right)^{\frac{1}{q}}
$$

$$
-G_{t-r}(x - z)^{\gamma} \lambda^{-\frac{1}{p}}(z)dzdr \right)^{\frac{1}{q}} \left( \int_0^T \|J_\alpha(r, \cdot)\|_{L_\alpha^p(\mathbb{R})}^{\frac{1}{p}}dr \right)^{\frac{q}{p}}.
$$

(4.31)

To bound the above first factor we use the following inequality

$$
\left| \exp \left(-\frac{x^2}{t+h}\right) - \exp \left(-\frac{x^2}{t}\right) \right| \leq C_\gamma h^\gamma t^{-\gamma} \exp \left(-\frac{\gamma x^2}{2(t+h)}\right) \quad \forall \gamma \in (0, 1).
$$

Combining the above inequality with (4.29) (with $\alpha = 1/2$), we have

$$
|G_{t+h-r}(x - z) - G_{t-r}(x - z)| \leq C_\gamma h^\gamma (t-r)^{\gamma} \left[ G_{\frac{\gamma}{2}(t+h-r)}(x - z) + G_{\frac{\gamma}{2}(t-r)}(x - z) \right].
$$

(4.32)
Thus, the first factor in (4.31) is bounded by

\[
\int_0^t \int_{\mathbb{R}} (t + h - r)^q(a - 1)(r - x)^q \lambda^{-\frac{q}{2}}(z) dz dr 
\leq h^{q\gamma} \int_0^t \int_{\mathbb{R}} (t - r)^q(a - 1) + \frac{1}{q^2} G_{\gamma_2(t + h - r)}(x(z)) \lambda^{-\frac{q}{2}}(z) dz dr 
+ h^{q\gamma} \int_0^t \int_{\mathbb{R}} (t - r)^q(a - 1) + \frac{1}{q^2} G_{\gamma_2(t + h - r)}(x(z)) \lambda^{-\frac{q}{2}}(z) dz dr 
\leq h^{q\gamma} \lambda^{-\frac{q}{2}}(x) \int_0^t (t - r)^q(a - 1) + \frac{1}{q^2} dr ,
\]

where the last inequality follows from Lemma 2.5. Hence, if \( \gamma + \frac{3}{2p} < \alpha < \frac{H}{2} \), namely, if \( \gamma < \frac{H}{2} - \frac{3}{2p} \), then we have the following estimation:

\[
\mathbb{E} \left[ \sup_{t,x} \lambda^\theta(x) | \mathcal{J}_2(t, h, x) | \right]^p \lesssim |h|^p \|v\|_{L^p_{\lambda,T}}^p . \tag{4.33}
\]

Now we are going to bound \( \mathcal{J}_3(t, h, x) \). Exactly in the same way as for (4.28), we have

\[
\mathcal{J}_3(t, h, x) \leq \lambda^{-\frac{q}{2}}(x) \left( \int_0^T (t + h - r)^q(a - 1)(t + h - r)^{\frac{1}{q^2}} dr \right)^{\frac{1}{q}} 
\leq \lambda^{-\frac{q}{2}}(x) \left( \int_0^T \| J_\alpha(r, \cdot) \|_{L^p_{\lambda,T}}^p dr \right)^{\frac{1}{p}} 
= C_p \lambda^{-\frac{q}{2}}(x) h^{\alpha - \frac{3}{2p}} \left( \int_0^T \| J_\alpha(r, \cdot) \|_{L^p_{\lambda,T}}^p dr \right)^{\frac{1}{p}} .
\]

If \( \frac{3}{2p} < \alpha < \frac{H}{2} \), which is possible if \( \gamma < \alpha - \frac{3}{2p} < \frac{H}{2} - \frac{3}{2p} \), then

\[
\mathbb{E} \left[ \sup_{t,x} \lambda^\theta(x) | \mathcal{J}_3(t, h, x) | \right]^p \leq C_3 |h|^p a - \frac{3}{2} \|v\|_{L^p_{\lambda,T}} = C_3 |h|^p \|v\|_{L^p_{\lambda,T}}^p . \tag{4.34}
\]

Combining (4.30), (4.33) and (4.34) we prove (4.10).

**Step 6.** We prove part (iv) of the proposition. As before, we shall again use the representation formula (4.13) and then we apply the Hölder inequality to find

\[
\Phi(t, x) - \Phi(t, y) =\frac{\sin(\pi \alpha)}{\pi} \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} [G_{t-r}(x - z) - G_{t-r}(y - z)] J_\alpha(r, z) dz dr 
\leq \left( \int_0^T \int_{\mathbb{R}} (t - r)^q(a - 1) |G_{t-r}(x - z) - G_{t-r}(y - z)|^q \lambda^{-\frac{q}{2}}(z) dz dr \right)^{\frac{1}{q}} 
\times \left( \int_0^T \int_{\mathbb{R}} |J_\alpha(r, z)|^p \lambda(z) dz dr \right)^{\frac{1}{p}} .
\]

Denote the above first factor by

\[
\mathcal{K}(t, x, y) := \int_0^t \int_{\mathbb{R}} (t - r)^q(a - 1) |G_{t-r}(x - z) - G_{t-r}(y - z)|^q \lambda^{-\frac{q}{2}}(z) dz dr .
\]
YAOZHONG HU AND XIONG WANG

Fix $\gamma \in (0, 1)$. Using Hölder’s inequality we have

$$\mathcal{K}(t, x, y) \lesssim \int_0^t (t-r)^{(\alpha-1)} \left( \int_{\mathbb{R}} |G_{t-r}(x-z) - G_{t-r}(y-z)|^{pq(1-\gamma)} \lambda^{-q}(z) dz \right)^{\frac{1}{p'}} \times \left( \int_{\mathbb{R}} |G_{t-r}(x-z) - G_{t-r}(y-z)|^{q^* q} dz \right)^{\frac{1}{q'}} dr. \quad (4.35)$$

To bound the integral inside the above second bracket, we make the substitutions $\tilde{x} = \frac{x}{\sqrt{r}}$, $\tilde{y} = \frac{y}{\sqrt{r}}$ and $\tilde{z} = \frac{z}{\sqrt{r}}$ to obtain for any $\rho > 0$,

$$\int_{\mathbb{R}} |G_{t-r}(x-z) - G_{t-r}(y-z)|^\rho dz \lesssim (t-r)^{\frac{1-\gamma}{2}} \int_{\mathbb{R}} |\exp(-|\tilde{x} - \tilde{z}|^2) - \exp(-|\tilde{y} - \tilde{z}|^2)|^\rho d\tilde{z} \lesssim (t-r)^{\frac{1-\gamma}{2}} |\tilde{x} - \tilde{y}|^\rho = (t-r)^{\frac{1-\gamma}{2}} |x-y|^\rho. $$

Substituting this bound into (4.35) with $\rho = q^2 r$ we have

$$\mathcal{K}(t, x, y) \lesssim |x-y|^{q^*} \cdot \int_0^t (t-r)^{(\alpha-1) + \frac{1-\gamma(1-q^*)}{p} + \frac{1-\gamma}{2}} \times \left( \int_{\mathbb{R}} \left[ G_{t-r}^{pq(1-\gamma)}(x-z) + G_{t-r}^{pq(1-\gamma)}(y-z) \right] \lambda^{-q}(z) dz \right)^{\frac{1}{p'}} dr \lesssim |x-y|^{q^*} \cdot \left[ \lambda^{-q}(x) + \lambda^{-q}(y) \right] \cdot \int_0^t (t-r)^{(\alpha-\frac{3}{2} + \frac{1}{2} - \gamma) - \frac{q^*}{q'} - \frac{q^*}{q}} dr,$$

where the last inequality follows from Lemma 2.5.

If $q(\alpha - \frac{3}{2} + \frac{1}{2} - \frac{q^*}{2}) > -1$ and $\alpha < \frac{H}{2}$, namely, if $\frac{3}{2r} + \frac{\gamma}{2} < \alpha < \frac{H}{2}$, then with $\theta = \frac{1}{p}$ we have

$$\mathbf{E} \left| \sup_{t \in [0, T]} \left( \lambda^{-\theta}(x) + \lambda^{-\theta}(y) \right)^{-1} |\mathcal{K}(t, x, y)|^{\frac{1}{\theta}} \times \left( \int_0^T \int_{\mathbb{R}} |J_\alpha(r, z)|^p \lambda(z) dz dr \right)^{\frac{1}{p'}} \right| \lesssim |x-y|^{q^*} \cdot \int_0^T \mathbf{E} |J_\alpha(r, z)|^p \lambda(z) dz dr \leq C_4 |x-y|^{q^*} \|v\|^p_{L^p_{\lambda, T}}. \quad (4.36)$$

This proves (4.11). The proof of the proposition is then completed. \hfill \Box

4.3. Weak existence of the solution. In this subsection we show the weak existence of a solution with paths in $C([0, T] \times \mathbb{R})$, the space of all continuous real valued functions on $[0, T] \times \mathbb{R}$, equipped with a metric

$$d_C(u, v) := \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0 \leq t \leq T, |x| \leq n} \left( ||u(t, x) - v(t, x)|| \wedge 1 \right). \quad (4.37)$$

We state a tightness criterion of probability measures on $(C([0, T] \times \mathbb{R}), \mathcal{B}(C([0, T] \times \mathbb{R})))$ that we are going to use (see Section 2.4 in [18] for the case where $[0, T] \times \mathbb{R}$ is replaced by $[0, \infty)$. It is also true for our case as indicated there).
Theorem 4.4. A sequence \( \{P_n\}_{n=1}^{\infty} \) of probability measures on \( (C([0, T] \times \mathbb{R}), \mathcal{B}(C([0, T] \times \mathbb{R})) \) is tight if and only if

1. \( \lim_{T \to \infty} \sup_{n \geq 1} P_n(\{\omega \in C([0, T] \times \mathbb{R}) : |\omega(0, 0)| > \lambda \}) = 0 \),
2. for any \( T > 0 \), \( R > 0 \) and \( \varepsilon > 0 \)

\[
\lim_{\delta \to 0} \sup_{n \geq 1} P_n(\{\omega \in C([0, T] \times \mathbb{R}) : m_{T,R}^{T, R}(\omega, \delta) > \varepsilon \}) = 0
\]

where

\[
m_{T,R}^{T, R}(\omega, \delta) := \max_{0 \leq t, s \leq T, \delta \leq |x|, |y| \leq R} |\omega(t, x) - \omega(s, y)|
\]

is the modulus of continuity on \([0, T] \times [-R, R]\).

We approximate the noise \( W \) with respect to the space variable by the following smoothing of the noise. That is, for \( \varepsilon > 0 \) we define

\[
\frac{\partial}{\partial x} W_\varepsilon(t, x) = \int_{\mathbb{R}} G_\varepsilon(x - y) W(t, dy).
\]  

(4.38)

The noise \( W_\varepsilon \) induces an approximation to mild solution

\[
u_\varepsilon(t, x) = G_t * u_0(x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x - y) \sigma(s, y, u_\varepsilon(s, y)) W_\varepsilon(ds, dy),
\]

(4.39)

where the stochastic integral is understood in the Itô sense. As in [12] due to the regularity in space, the existence and uniqueness of the solution \( u_\varepsilon(t, x) \) to above equation is well-known.

The lemma below asserts that the approximate solution \( u_\varepsilon(t, x) \) is uniformly bounded in the space \( Z^{p}_{\lambda, T} \). More precisely, we have

Lemma 4.5. Let \( H \in \left( \frac{1}{4}, \frac{3}{2} \right) \) and let \( \lambda(x) \) be defined by (4.6). Assume \( \sigma(t, x, u) \) satisfies hypothesis (H1). Assume also that the initial value \( u_0(x) \in Z^{p}_{\lambda, 0} \). Then the approximate solutions \( u_\varepsilon \) satisfy

\[
sup_{\varepsilon > 0} \|u_\varepsilon\|_{Z^{p}_{\lambda, T}} := sup_{\varepsilon > 0} \sup_{t \in [0, T]} \|u_\varepsilon(t, \cdot)\|_{L^{p}_{\lambda}(\Omega \times \mathbb{R})} + sup_{\varepsilon > 0} \sup_{t \in [0, T]} \mathcal{N}_{H, \lambda, \varepsilon} u_\varepsilon(t) < \infty.
\]  

(4.40)

Proof. For notational simplicity we assume \( \sigma(t, x, u) = \sigma(u) \) without loss of generality because of hypothesis (H1). We shall use some similar thoughts to that in [12] but now with special attention to the weight \( \lambda(x) \). To this end, we define the Picard iteration as follows:

\[
u_\varepsilon^0(t, x) = G_t * u_0(x),
\]

and recursively for \( n = 0, 1, 2, \cdots \),

\[
u_\varepsilon^{n+1}(t, x) = G_t * u_0(x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x - y) \sigma(u_\varepsilon^n(s, y)) W_\varepsilon(ds, dy).
\]  

(4.41)

From [13, Lemma 4.12] it follows that for any fixed \( \varepsilon > 0 \) when \( n \) goes to infinity, the sequence \( u_\varepsilon^n(t, x) \) converges to \( u_\varepsilon(t, x) \) a.s. In the following steps 1 and 2, we shall first bound \( \|u_\varepsilon^n\|_{Z^{p}_{\lambda, T}} \) uniformly in \( n \), and \( \varepsilon \). Then, in step 3 we use Fatou’s lemma to show (4.40).

**Step 1.** In this step, we derive a Gronwall-type inequality to bound the \( L^{p}_{\lambda}(\Omega \times \mathbb{R}) \) norm of \( u_\varepsilon^{n+1}(t, x) \) by the \( Z^{p}_{\lambda, T} \) norm of \( u_\varepsilon^n(t, x) \). Rewrite (4.41) as

\[
u_\varepsilon^{n+1}(t, x) = G_t * u_0(x) + \int_0^t \int_{\mathbb{R}} \left[ (G_{t-s}(x - \cdot) \sigma(u_\varepsilon^n(s, \cdot))) * G_\varepsilon \right](y) W(ds, dy).
\]  

(4.41)
In the following, we will continue to use the notations $D_t(x,h)$ and $\square_{t-s}(x,y,h)$ defined in (2.14) and (2.15) previously. Applying the Burkholder-Davis-Gundy inequality (Proposition 4.1) and the isometry equalities (2.4)-(2.6) and then noting $|\sigma(u)| \lesssim |u| + 1$, we have

$$
E\left|u_{\varepsilon}^{n+1}(t,x)\right|^p \leq C_p G_t \ast u_0(x)^p + C_p \mathbb{E}\left(\int_0^t \int_{\mathbb{R}} \mathcal{F}\left[G_{t-s}(x-\cdot) \sigma(u_\varepsilon(s,\cdot))\right](\xi)^2 e^{-\varepsilon|\xi|^2} |\xi|^{1-2H}\,d\xi\,ds\right)^{\frac{p}{2}} \leq C_p (|G_t \ast u_0(x)|^p + \mathcal{D}_1^{n}(t,x) + \mathcal{D}_2^{n}(t,x)) \; ,
$$

(4.42)

where the constant $C_p$ is independent of $\varepsilon$ because $e^{-\varepsilon|\xi|^2} \leq 1$, and where we denote

$$
\mathcal{D}_1^{n}(t,x) := \left(\int_0^t \int_{\mathbb{R}^2} |D_{t-s}(y,h)|^2 \left(1 + \|u_\varepsilon^n(s,x+y)\|_{L^p(\Omega)}^2\right) |h|^{2H-2}\,dy\,ds\right)^{\frac{p}{2}},
$$

and

$$
\mathcal{D}_2^{n}(t,x) := \left(\int_0^t \int_{\mathbb{R}^2} |G_{t-s}(y)|^2 \|\Delta_h u_\varepsilon^n(t,x+y)\|_{L^p(\Omega)}^2 |h|^{2H-2}\,dy\,ds\right)^{\frac{p}{2}} .
$$

This means

$$
\|u_{\varepsilon}^{n+1}(t,\cdot)\|_{L^p_x(\Omega \times \mathbb{R})}^2 = \left(\int_{\mathbb{R}} \mathbb{E}\left|u_\varepsilon^n(t,x)\right|^p \lambda(x)\,dx\right)^{\frac{p}{2}} \leq C_p \left(\|u_0(x)\|_{L^p(\mathbb{R})} + I_1^{n} + I_2^{n}\right) ,
$$

(4.43)

where $I_1^{n}$ and $I_2^{n}$ are defined and bounded as follows.

$$
I_1^{n} := \left(\int_{\mathbb{R}} \mathcal{D}_1^{n}(t,x) \lambda(x)\,dx\right)^{\frac{p}{2}} \leq C_{p,H} \int_0^t (t-s)^{H-1} \left(1 + \|u_\varepsilon^n(s,\cdot)\|_{L^p_x(\Omega \times \mathbb{R})}^2\right)ds ,
$$

(4.44)

and

$$
I_2^{n} := \left(\int_{\mathbb{R}} \mathcal{D}_2^{n}(t,x) \lambda(x)\,dx\right)^{\frac{p}{2}} \leq C_{p,H} \int_0^t \left[\mathcal{N}_{\frac{p}{2}-H,p,u_\varepsilon^n(s)}\right]^2 \frac{1}{\sqrt{t-s}} \,ds .
$$

(4.45)

The above bounds on $I_1^{n}, I_2^{n}$ together with (4.43) yield

$$
\|u_{\varepsilon}^{n+1}(t,\cdot)\|_{L^p_x(\Omega \times \mathbb{R})}^2 \leq C_{p,H} \left(\|u_0\|_{L^p_x(\omega \times \mathbb{R})}^2 + \int_0^t (t-s)^{H-1} \|u_\varepsilon^n(s,\cdot)\|_{L^p_x(\Omega \times \mathbb{R})}^2 ds + \int_0^t (t-s)^{-1/2} \left[\mathcal{N}_{\frac{p}{2}-H,p,u_\varepsilon^n(s)}\right]^2 ds\right) .
$$

(4.46)
SHE with general rough noise

**Step 2.** Next, we obtain a bound for $\mathcal{N}^n_{\frac{3}{2} - H, p} u^{n+1}_\varepsilon(t)$ analogous to (4.46). Similar to (4.42) we have

\[
\mathbb{E}[|u^{n+1}_\varepsilon(t, x) - u^{n+1}_\varepsilon(t, x + h)|^p] \\
\leq C_p |G_t * u_0(x) - G_t * u_0(x + h)|^p \\
+ C_p \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} |D_{t-s}(x - y - z, h)\sigma(u^p_\varepsilon(s, y + z)) \right. \\
- \left. D_{t-s}(x - z, h)\sigma(u^p_\varepsilon(s, z)) |^2 |y|^{2H-2}dzdyds \right) \]

\[
\leq C_p \left( \mathcal{I}_0(t, x, h) + \mathcal{I}^{\varepsilon, n}_1(t, x, h) + \mathcal{I}^{\varepsilon, n}_2(t, x, h) \right),
\]

where

\[
\mathcal{I}_0(t, x, h) := |G_t * u_0(x) - G_t * u_0(x + h)|^p,
\]

\[
\mathcal{I}^{\varepsilon, n}_1(t, x, h) := \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} |D_{t-s}(x - y - z, h)|^2 |\sigma(u^p_\varepsilon(s, y + z)) - \sigma(u^p_\varepsilon(s, z)) |^2 |y|^{2H-2}dzdyds \right) \]

\[
\mathcal{I}^{\varepsilon, n}_2(t, x, h) := \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} |\Box_{t-s}(x - z, y, h)|^2 |\sigma(u^p_\varepsilon(s, z)) |^2 |y|^{2H-2}dzdyds \right)^\frac{p}{q}.
\]

By Minkowski’s inequality we have

\[
\left[ \mathcal{N}^n_{\frac{3}{2} - H, p} u^{n+1}_\varepsilon(t) \right]^2 \leq C_p \sum_{j=0}^2 \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathcal{I}^{\varepsilon, n}_j(t, x, h)\lambda(x)dx \right)^\frac{p}{q} |h|^{2H-2}dh
\]

\[
=: J_0 + J_1 + J_2. \tag{4.47}
\]

Our strategy is to control the above three quantities by using the ideas similar to those when we deal with the terms $\mathcal{I}_1$ and $\mathcal{I}_2$ in the step 4 of the proof of Proposition 4.2 (ii). First, from Lemma 2.5 it follows.

\[
J_0 \leq C_p \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |G_t(x - y)\lambda(x)dx \right)^\frac{p}{q} |\Delta_h u_0(y)|^p dy |h|^{2H-2}dh
\]

\[
\leq C_p \int_{\mathbb{R}} |\Delta_h u_0(y)|^p \lambda(y)dy |h|^{2H-2}dh = C_p \left[ \mathcal{N}^n_{\frac{3}{2} - H, p} u_0 \right]^2. \tag{4.48}
\]

For the term $J_1$, we can use the method similar to that when we obtain (4.22) and (4.23). This is, a change of variable $y \rightarrow z - y$, and applications of Minkowski’s inequality, Jensen’s inequality and Lemma 2.12 give

\[
J_1 \leq C_{p, H} \int_0^t \int_{\mathbb{R}} (t - s)^{H-1} \left( \int_{\mathbb{R}^3} (t - s)^{1-H} |D_{t-s}(z, h)|^2 |h|^{2H-2} \\
\times \mathbb{E} \left[ |\Delta_h u^p_\varepsilon(t, x)|^p \lambda(x - z)dzdh \right] \right)^\frac{2}{p} |y|^{2H-2}dyds \tag{4.49}
\]

\[
\leq C_{p, H} \int_0^t (t - s)^{H-1} \left[ \mathcal{N}^n_{\frac{3}{2} - H, p} u^{n}_\varepsilon(s) \right]^2 ds.
\]

Next, we obtain a bound for $J_2$. Similar to the obtention of (4.24), (4.25) and (4.26) we also make a change of variable $y \rightarrow z - y$, and then split it to two terms.
to obtain
\[ T_2^2(t, x, h) \leq C_p \left( T_1^2(t, x, h) + T_2^2(t, x, h) \right) \]
\[ := C_p \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} |\nabla \cdot (u_s^2(s, x, z))|^2 \left| y \right|^{2H-2} dy dz ds \right)^{\frac{2}{p}} \]
\[ + C_p \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} |\nabla \cdot (u_s^2(s, x + z)) - \nabla \cdot (u_s^2(s, x))|^2 \left| y \right|^{2H-2} dy dz ds \right)^{\frac{2}{p}}. \]

Applying Minkowski’s inequality, the condition \(|\sigma(u)| \lesssim |u| + 1\), and Lemma 2.8 one has
\[ J_{21} := \int_\mathbb{R} \int_\mathbb{R} \left| \mathcal{I}_{21}^n(t, x, h) \lambda(x) dx \right| \left| h \right|^{2H-2} dh \]
\[ \leq C_{p, H} \int_0^t (t - s)^{2H-\frac{2}{p}} \left( 1 + \left\| u_s^n(s, \cdot) \right\|_{L_H^p(\Omega \times \mathbb{R})}^2 \right)^2 ds. \]  (4.50)

Again by Minkowski’s inequality, the Lipschitz condition (1.12) on \( \sigma \), and Lemma 2.11 we obtain
\[ J_{22} := \int_\mathbb{R} \int_\mathbb{R} \left| \mathcal{I}_{22}^n(t, x, h) \lambda(x) dx \right| \left| h \right|^{2H-2} dh \]
\[ \leq C_{p, H} \int_0^t (t - s)^{H-1} \left[ N_{\frac{1}{2}, H, p} u_s^n(s) \right]^2 ds. \]  (4.51)

Using that fact that \( J_3 \leq J_{31} + J_{32} \) and using (4.47)-(4.51) we obtain
\[ \left[ N_{\frac{1}{2}, H, p} u_s^{n+1}(t) \right]^2 \leq C_{p, H} \left[ N_{\frac{1}{2}, H, p} u_0 \right]^2 + C_{p, H} \int_0^t (t - s)^{H-1} \left[ N_{\frac{1}{2}, H, p} u_s^n(s) \right]^2 ds \]
\[ + C_{p, H} \int_0^t (t - s)^{2H-\frac{2}{p}} \left( 1 + \left\| u_s^n(s, \cdot) \right\|_{L_H^p(\Omega \times \mathbb{R})}^2 \right)^2 ds. \]  (4.52)

**Step 3.** Set
\[ \Psi_\varepsilon^n(t) := \left\| u_s^n(t, \cdot) \right\|_{L_H^p(\Omega \times \mathbb{R})}^2 + \left[ N_{\frac{1}{2}, H, p} u_s^n(t) \right]^2. \]

Thus, combining all the estimates (4.48), (4.49), (4.50) and (4.51) yields
\[ \Psi_\varepsilon^{n+1}(t) \leq C_{p, H, T} \left( \left\| u_0 \right\|_{L_H^p(\Omega \times \mathbb{R})}^2 + \left[ N_{\frac{1}{2}, H, p} u_0 \right]^2 + \int_0^t (t - s)^{2H-\frac{2}{p}} \Psi_\varepsilon^n(s) ds \right). \]

Now it is relatively easy to see by fractional Gronwall lemma (e.g. [20, Lemma 1])
\[ \sup_{n \geq 1} \sup_{t \in [0, T]} \Psi_\varepsilon^n(t) \leq C_{T, p, H} < \infty. \]

For any fixed \( \varepsilon > 0 \) since \( u_s^n \) converges to \( u_\varepsilon \) a.s. as \( n \to \infty \), we have by Fatou’s lemma
\[ \left\| u_\varepsilon(t, \cdot) \right\|_{L_H^p(\Omega \times \mathbb{R})} = \left( \int_\Omega \mathbb{E} \left[ \lim_{n \to \infty} \left| u_s^n(t, x, \cdot) \right|^p \lambda(x) dx \right] \right)^{\frac{1}{p}} \leq \lim_{n \to \infty} \left( \int_\Omega \mathbb{E} \left\| u_s^n(t, x, \cdot) \right\|^p \lambda(x) dx \right)^{\frac{1}{p}} \leq \sup_{n \geq 1} \sup_{t \in [0, T]} \Psi_\varepsilon^n(t) < \infty. \]
Thus, we conclude that \( \sup_{\varepsilon > 0} \sup_{t \in [0, T]} \| u_\varepsilon(t, \cdot) \|_{L^p_\lambda(\Omega \times R)} \) is finite. On the other hand, for any \( t, x \) and \( h \) we have \( |u_\varepsilon^n(t + x) - u_\varepsilon^n(t, x)|^2 \to |u_\varepsilon(t, x) + u_\varepsilon(t, x)|^2 \) a.s. So, on the domain |h| \leq 1

\[
\int_{|h| \leq 1} \| u_\varepsilon(t, \cdot + h) - u_\varepsilon(t, \cdot) \|_{L^p_\lambda(\Omega \times R)}^2 |h|^{2H-2} dh
\]

\[
\leq \lim_{n \to \infty} \int_{|h| \leq 1} \| u_\varepsilon^n(t, \cdot + h) - u_\varepsilon^n(t, \cdot) \|_{L^p_\lambda(\Omega \times R)}^2 |h|^{2H-2} dh.
\]

For |h| \geq 1, we simply bound \( \| u_\varepsilon(t, \cdot + h) - u_\varepsilon(t, \cdot) \|_{L^p_\lambda(\Omega \times R)}^2 \) by \( 2 \| u_\varepsilon^n(t, \cdot) \|_{L^p_\lambda(\Omega \times R)}^2 \), which is uniform bounded with respect to \( t, \varepsilon \) and \( n \). When \( H < \frac{1}{2} \), \( \int_{|h| > 1} |h|^{2H-2} < \infty \). Thus, we have that

\[
\sup_{t \in [0, T]} \mathcal{N}^\varepsilon_{\frac{1}{2} - H, p} u_\varepsilon(t) = \sup_{t \in [0, T]} \left( \int_{\Omega} \| u_\varepsilon(t, \cdot + h) - u_\varepsilon(t, \cdot) \|_{L^p_\lambda(\Omega \times R)}^2 |h|^{2H-2} dh \right)^{\frac{1}{2}}
\]

\[
\leq C_H \sup_{n \geq 1} \sup_{t \in [0, T]} \Psi_\varepsilon^n(t) < \infty.
\]

(4.53)

Therefore, \( \sup_{\varepsilon > 0} \sup_{t \in [0, T]} \mathcal{N}^\varepsilon_{\frac{1}{2} - H, p} u_\varepsilon(t) \) is finite.

In conclusion, we have proved \( \sup_{\varepsilon > 0} \| u_\varepsilon \|_{Z^p_{\lambda, T}} := \sup_{\varepsilon > 0} \sup_{t \in [0, T]} \| u_\varepsilon(t, \cdot) \|_{L^p_\lambda(\Omega \times R)} + \sup_{\varepsilon > 0} \sup_{t \in [0, T]} \mathcal{N}^\varepsilon_{\frac{1}{2} - H, p} u_\varepsilon(t) \) is finite. \( \square \)

Recall that \( (C([0, T] \times R), d_C) \) is the metric space with the metric \( d_C \) defined by (4.37).

**Lemma 4.6.** Let \( u_\varepsilon \in Z^p_{\lambda, T} \). If \( u_\varepsilon \to u \) almost surely in \( C([0, T] \times R), d_C \) as \( \varepsilon \to 0 \), then \( u \) is also in \( Z^p_{\lambda, T} \).

**Proof.** Since \( u_\varepsilon \) converges to \( u \) in \( (C([0, T] \times R), d_C) \) almost surely, we have \( u_\varepsilon(t, x) \to u(t, x) \) for each \( (t, x) \in [0, T] \times R \) almost surely. Thus

\[
\| u(t, \cdot) \|_{L^p_\lambda(\Omega \times R)} \leq \lim_{\varepsilon \to 0} \left( \int_{\Omega} \mathbb{E} |u_\varepsilon(t, x)|^p \lambda(x) dx \right)^{\frac{1}{p}} < \infty.
\]

(4.54)

This means that \( \sup_{t \in [0, T]} \| u(t, \cdot) \|_{L^p_\lambda(\Omega \times R)} \) is finite.

On the other hand, for any \( x, h \) we have \( |u_\varepsilon(t, x + h) - u_\varepsilon(t, x)|^2 \to |u(t, x + h) - u(t, x)|^2 \) almost surely. So, on the domain |h| \leq 1 and |h| \geq 1, we can simply apply the same procedure as in the Step 3 of the proof of Lemma 4.5 but replacing \( \lim_{n \to \infty} \lim_{\varepsilon \to 0} \) and bound \( \| u(t, \cdot + h) - u(t, \cdot) \|_{L^p_\lambda(\Omega \times R)}^2 \) by \( 2 \| u(t, \cdot) \|_{L^p_\lambda(\Omega \times R)}^2 \), which is finite. Thus, similar to (4.53) we have

\[
\sup_{t \in [0, T]} \mathcal{N}^\varepsilon_{\frac{1}{2} - H, p} u(t) = \sup_{t \in [0, T]} \left( \int_{\Omega} \| u(t, \cdot + h) - u(t, \cdot) \|_{L^p_\lambda(\Omega \times R)}^2 |h|^{2H-2} dh \right)^{\frac{1}{2}} < \infty.
\]

Together with (4.54), this implies that \( u \in Z^p_{\lambda, T} \). \( \square \)

**Lemma 4.7.** Let \( u_\varepsilon \) be the approximate mild solution defined by (4.39) and assume that \( u_0(x) \) belongs to \( Z^p_{\lambda, 0} \). Then, we have the following statements.
(i) If \( p > \frac{6}{4H-1} \), then
\[
\left| \sup_{t \in [0,T], x \in \mathbb{R}} \lambda_t^\frac{1}{p} x \mathcal{N}_{\frac{1}{2}-H} u_\varepsilon(t, x) \right|_{L^p(\Omega)} \leq C_{T, H} \| u_\varepsilon \|_{Z_{T}^\varepsilon, +1}.
\] (4.55)

(ii) If \( p > \frac{3}{H} \), then
\[
\left| \sup_{t, t+h \in [0,T], x \in \mathbb{R}} \lambda_t^\frac{1}{p} x \lambda_{t+h}^\frac{1}{p} \left( u_\varepsilon(t+h, x) - u_\varepsilon(t, x) \right) \right|_{L^p(\Omega)} \leq C_{T, H} |h| \left( \| u_\varepsilon \|_{Z_{T}^\varepsilon, +1} + 1 \right),
\] (4.56)
for all \( 0 < \gamma < \frac{H}{2} - \frac{3}{2p} \).

(iii) If \( p > \frac{3}{H} \), then
\[
\left| \sup_{t \in [0,T], x, y \in \mathbb{R}} \frac{u_\varepsilon(t, x) - u_\varepsilon(t, y)}{\lambda_t^\frac{1}{p} x + \lambda_{t+h}^\frac{1}{p} y} \right|_{L^p(\Omega)} \leq C_{T, H} |x - y| \left( \| u_\varepsilon \|_{Z_{T}^\varepsilon, +1} + 1 \right),
\] (4.57)
for all \( 0 < \gamma < \frac{H}{2} - \frac{3}{2p} \).

**Proof.** Denote for \( \alpha \in [0, 1] \)
\[
J_\alpha^\varepsilon(r, \xi) = \int_0^r \int_\mathbb{R} (r - s)^{-\alpha} G_{r-s}(\xi - z) \sigma(u_\varepsilon(s, z)) G_\varepsilon(z - y) dz W(ds, dy).
\]
Then, Fubini’s theorem implies
\[
u_\varepsilon(t, x) = G_t * u_0(x) + \frac{\sin(\pi \alpha)}{\pi} \int_0^t \int_\mathbb{R} (t - r)^{-\alpha-1} G_{t-r}(x - \xi) J_\alpha^\varepsilon(r, \xi) d\xi dr.
\]
Applying Proposition 4.2 (ii), (iii), (iv) to \( u_2, u_\varepsilon(t, x) \) yields (4.55)-(4.57) without the constant term 1. However, from the assumption that \( u_0(x) \) belongs to \( Z_{c}^\varepsilon \), we see that left hand sides of (4.55)-(4.57) are finite when \( u_\varepsilon(t, x) \) is replaced by \( u_1(t, x) \). Combining the bounds for \( u_1(t, x) \) and \( u_2, u_\varepsilon(t, x) \) proves the lemma. \( \square \)

**Proof of Theorem 1.5.** We still assume \( \sigma(t, x, u) = \sigma(u) \) to simplify the notations. From Lemma 4.5 and Lemma 4.7 (ii) and (iii) it follows that the two conditions of Theorem 4.4 are satisfied. Hence, the probability measures on the space \( (\mathcal{C}([0, T] \times \mathbb{R}), \mathcal{F}(\mathcal{C}([0, T] \times \mathbb{R}), d\mathcal{C}) \) corresponding to the processes \( \{u_\varepsilon, \varepsilon \in (0, 1] \} \) are tight. Thus, there is a subsequence \( \varepsilon_n \downarrow 0 \) such that \( u_n \to u_\varepsilon \) in probability by Skorohod representation theorem, there is a probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \) carrying the subsequence \( \tilde{u}_n \) and noise \( \tilde{W} \) such that the finite dimensional distributions of \( (\tilde{u}_n, \tilde{W}) \) coincide. Moreover, we have
\[
u_n(t, x) \to \tilde{u}(t, x) \text{ in } (\mathcal{C}([0, T] \times \mathbb{R}), d\mathcal{C}) \quad \tilde{\mathbb{P}}\text{-almost surely}
\] (4.58)
for a certain stochastic process \( \tilde{u} \) as \( j \to \infty \). By Lemma 4.6 we see that \( \tilde{u} \) belongs to space \( \tilde{Z}_{T}^{\varepsilon} \) with respect to the new probability \( \tilde{\mathbb{P}} \). We want to show that \( \tilde{u} \) is a weak solution to (1.1).

Define the filtration \( \tilde{\mathcal{F}}_t \) to be the filtration generated by \( \tilde{W} \). We claim that \( \tilde{u}_n \) satisfies (1.1) with \( W \) replaced by \( \tilde{W} \), namely,
\[
\tilde{u}_n(t, x) = G_t * u_0(x) + \int_0^t \int_\mathbb{R} G_{t-s}(x - \cdot) \sigma(\tilde{u}_n(s, \cdot)) * G_{\varepsilon_j}(y) \tilde{W}(ds, dy).
\] (4.59)
To show the above identity it is sufficient to prove that for any $Z \in L^2(\tilde{\Omega}, \tilde{\mathbb{P}})$ one has
\[
\tilde{\mathbb{E}}[\tilde{u}_{n_j}(t, x)Z] = \mathbb{E}\left[ G_t * u_0(x)Z \right. \\
+ \int_0^t \int_\mathbb{R} G_{t-s}(x-\cdot)\sigma(\tilde{u}_{n_j}(s, \cdot))* G_{\varepsilon_j}(y)\tilde{W}(ds, dy)Z \bigg] ,
\] (4.60)
where $\tilde{\mathbb{E}}$ means the expectation under $\tilde{\mathbb{P}}$.
For any $\phi \in D(\mathbb{R})$, denote
\[
\tilde{W}_t(\phi) = \int_\mathbb{R} \phi(x)\tilde{W}(t, dx) ; \quad W_t(\phi) = \int_\mathbb{R} \phi(x)W(t, dx) .
\]
It is routine to argue that the set
\[
\mathcal{S} := \left\{ f(\tilde{W}_{t_1}(\phi), \ldots, \tilde{W}_{t_n}(\phi)), \ 0 \leq t_1 < \cdots < t_n \leq T \ f \in C_0(\mathbb{R}^n) \right\}
\]
are dense in $L^2(\tilde{\Omega}, \tilde{\mathbb{P}}, \tilde{\mathcal{F}}_T)$. This means that it is sufficient to choose $Z = f(\tilde{W}_{t_1}(\phi), \ldots, \tilde{W}_{t_n}(\phi))$ in (4.60), which is true because we have the following identities:
\[
\tilde{\mathbb{E}}[\tilde{u}_{n_j}(t, x)f(\tilde{W}_{t_1}(\phi), \ldots, \tilde{W}_{t_n}(\phi))] = \mathbb{E}[u_{n_j}(t, x)\tilde{f}(W_{t_1}(\phi), \ldots, W_{t_n}(\phi))];
\]
\[
\tilde{\mathbb{E}}[G_t * u_0(x)f(\tilde{W}_{t_1}(\phi), \ldots, \tilde{W}_{t_n}(\phi)) = \mathbb{E}[G_t * u_0(x)\tilde{f}(W_{t_1}(\phi), \ldots, W_{t_n}(\phi))];
\]
and
\[
\tilde{\mathbb{E}}\left[ \int_0^t \int_\mathbb{R} G_{t-s}(x-\cdot)\sigma(\tilde{u}_{n_j}(s, \cdot))* G_{\varepsilon_j}(y)\tilde{W}(ds, dy)\tilde{f}(\tilde{W}_{t_1}(\phi), \ldots, \tilde{W}_{t_n}(\phi)) \right]
\]
\[
= \mathbb{E}\left[ \int_0^t \int_\mathbb{R} G_{t-s}(x-\cdot)\sigma(u_{n_j}(s, \cdot))* G_{\varepsilon_j}(y)W(ds, dy)\tilde{f}(W_{t_1}(\phi), \ldots, W_{t_n}(\phi)) \right]
\]
due to the fact that the finite dimensional distributions of $(\tilde{u}_{n_j}, \tilde{W})$ coincide with that of $(u_{n_j}, W)$. Therefore, $\tilde{u}_{n_j}(t, x)$ satisfies (4.60), and hence it satisfies (4.59).
From (4.58) and (4.59) it follows that $\tilde{u}$ is a mild solution to (1.1) with $W$ replaced by $\tilde{W}$. Therefore, we have proved the existence of a weak solution to (1.1).
Moreover, for any $\gamma \in (0, H - \frac{3}{p})$ and for any compact set $\mathbf{T} \subseteq [0, T] \times \mathbb{R}$, Lemma 4.7 (parts (ii) and (iii)) implies that there exists constant $C$ such that
\[
\tilde{\mathbb{E}}\left( \sup_{(t, x), (s, y) \in \mathbf{T}} \left| \frac{\tilde{u}(t, x) - \tilde{u}(s, y)}{|t-s|^\frac{\gamma}{2} + |x-y|^\gamma} \right|^p \right) \leq C\|\tilde{u}\|_{L^p_{\mathbf{T}}}^p .
\] (4.61)
This combined with the Kolmogorov lemma implies the desired Hölder continuity.

\[
\square
\]

5. Pathwise Uniqueness and Strong Existence of solutions

In this section we prove the pathwise uniqueness and the existence of strong solution for the equation (1.1). It is well known that once pathwise uniqueness is achieved, together with the existence of weak solution proved in previous section, we can conclude the existence of the unique strong solutions to (1.1) by, for example, the Yamada-Watanabe theorem ([16]). Therefore, we only need to focus on the proof of pathwise uniqueness.
Proof of Theorem 1.6. The proof follows the strategy in the proof of Theorem 4.3 of [12] combined with Proposition 4.2 (part (ii)).

Define the following stopping times

\[ T_k := \inf \left\{ t \in [0, T] : \sup_{0 \leq s \leq t, x \in \mathbb{R}} \lambda^*_H(x) N_{1 - H} u(s, x) \geq k, \right. \]

\[ \left. \quad \text{or} \sup_{0 \leq s \leq t, x \in \mathbb{R}} \lambda^*_H(x) N_{1 - H} v(s, x) \geq k \right\}, \quad k = 1, 2, \ldots \]

Proposition 4.2, part (ii) implies that \( T_k \uparrow T \) almost surely as \( k \to \infty \). We need to find appropriate bounds for the following two quantities:

\[ I_1(t) = \sup_{x \in \mathbb{R}} \mathbb{E} \left[ \mathbf{1}_{\{t < T_k\}} |u(t, x) - v(t, x)|^2 \right] \]

and

\[ I_2(t) = \sup_{x \in \mathbb{R}} \mathbb{E} \left[ \int_{\mathbb{R}} \mathbf{1}_{\{t < T_k\}} |u(t, x) - v(t, x) - u(t, x + h) + v(t, x + h)|^2 |h|^{2H-2} dh \right]. \]

First, it is easy to see

\[ \mathbf{1}_{\{t < T_k\}}(u(t, x) - v(t, x)) \]

\[ = \mathbf{1}_{\{t < T_k\}} \int_{0}^{t} \int_{\mathbb{R}} G_{t-s}(x - y) \mathbf{1}_{\{s \leq T_k\}} [\sigma(s, y, u(s, y)) - \sigma(s, y, v(s, y))] W(ds, dy). \]

Recall \( D_t(x, h) \) defined in (2.14) and denote \( \Delta(t, x, y) = \sigma(t, x, u(t, y)) - \sigma(t, x, v(t, y)) \).

We can decompose

\[ \mathbb{E} \left[ \mathbf{1}_{\{t < T_k\}} |u(t, x) - v(t, x)|^2 \right] \]

\[ \leq \mathbb{E} \left( \int_{0}^{t} \int_{\mathbb{R}^2} \mathbf{1}_{\{s \leq T_k\}} |D_{t-s}(x - y, h)|^2 |\Delta(s, y, y)|^2 |h|^{2H-2} dh dy ds \right) \]

\[ + \mathbb{E} \left( \int_{0}^{t} \int_{\mathbb{R}^2} \mathbf{1}_{\{s \leq T_k\}} G_{t-s}^2(x - y - h) |\Delta(s, y + h, y) - \Delta(s, y, y)|^2 |h|^{2H-2} dh dy ds \right) \]

\[ + \mathbb{E} \left( \int_{0}^{t} \int_{\mathbb{R}^2} \mathbf{1}_{\{s \leq T_k\}} G_{t-s}^2(x - y) |\Delta(s, y, y + h) - \Delta(s, y, y)|^2 |h|^{2H-2} dh dy ds \right) \]

\[ =: J_1 + J_2 + J_3. \] (5.1)

The assumption (1.13) of \( \sigma \) and the equality (2.17) can be used to dominate the above first term \( J_1 \). This is,

\[ J_1 \leq \mathbb{E} \left( \int_{0}^{t} \int_{\mathbb{R}^2} \mathbf{1}_{\{s \leq T_k\}} |D_{t-s}(x - y, h)|^2 |u(s, y) - v(s, y)|^2 |h|^{2H-2} dh dy ds \right) \]

\[ \leq \int_{0}^{t} \int_{\mathbb{R}^2} \mathbf{1}_{\{s \leq T_k\}} |u(s, y) - v(s, y)|^2 \sup_{y \in \mathbb{R}} \mathbb{E} \left[ |u(s, y) - v(s, y)|^2 \right] ds = \int_{0}^{t} (t-s)^{H-1} I_1(s) ds. \]

Using the properties (1.13) of \( \sigma \), we have if \(|h| > 1\)

\[ |\Delta(s, y + h, y) - \Delta(s, y, y)|^2 \lesssim |u(s, y) - v(s, y)|^2 \]

\[ = \left| \int_{u}^{v} [\sigma(s, y + h, \xi) - \sigma(s, y, \xi)] d\xi \right|^2 \lesssim |u(s, y) - v(s, y)|^2, \]

48
and if $|h| \leq 1$ (with the help of additional properties (1.14))

$$
\left[ \Delta(s, y + h, y) - \Delta(s, y, y) \right]^2
= \left| \int_u^v \sigma'_\xi(s, y + h, \xi) - \sigma'_\xi(s, y, \xi) d\xi \right|^2 \lesssim |h|^2 |u(s, y) - v(s, y)|^2.
$$

Thus, the second term $J_2$ in (5.1) is bounded by

$$
J_2 = E \left( \int_0^t \int_\mathbb{R} \int_{|y|>1} 1_{\{s<T_k\}} G_{t-s}^2(x - y - h) |u(s, y) - v(s, y)|^2 |h|^{2H-2} dh dy ds \right)
+ E \left( \int_0^t \int_\mathbb{R} \int_{|y|\leq1} 1_{\{s<T_k\}} G_{t-s}^2(x - y - h) |u(s, y) - v(s, y)|^2 |h|^{2H} dh dy ds \right)
\lesssim \int_0^t I_1(s) \left( \int_\mathbb{R} G_{t-s}^2(x - y) dy \right) ds \lesssim \int_0^t (t-s)^{-\frac{1}{2}} I_1(s) ds.
$$

For the last term $J_3$ in (5.1) we have by (1.13), (1.15)

$$
\left| \Delta(s, y, y + h) - \Delta(s, y, y) \right|^2
= \left| \int_0^1 [u(s, y + h) - v(s, y + h)]^2 \sigma'_\xi(s, y, \theta u(s, y + h) + (1 - \theta)v(s, y + h)) d\theta
- \int_0^1 [u(s, y) - v(s, y)]^2 \sigma'_\xi(s, y, \theta u(s, y) + (1 - \theta)v(s, y)) d\theta \right|^2.
$$

Noticing the additional uniform decay assumption (1.13), we have

$$
\left| \Delta(s, y, y + h) - \Delta(s, y, y) \right|^2
\lesssim |u(s, y + h) - v(s, y + h) - u(s, y) + v(s, y)|^2
+ \lambda^\frac{2}{3}(y)|u(s, y) - v(s, y)|^2 \cdot \left[ |u(s, y + h) - u(s, y)|^2 + |v(s, y + h) - v(s, y)|^2 \right].
$$

Thus, we can dominate the last term in (5.1) by

$$
J_3 \lesssim k \int_0^t (t-s)^{-\frac{1}{2}} [I_1(s) + I_2(s)] ds.
$$

Summarizing the above estimates we have

$$
I_1(t) \lesssim k \int_0^t (t-s)^{H-1} [I_1(s) + I_2(s)] ds.
$$

The similar procedure can be applied to estimate the term $I_2(t)$ to obtain

$$
I_2(t) \lesssim k \int_0^t (t-s)^{2H-\frac{3}{2}} [I_1(s) + I_2(s)] ds.
$$

As a consequence,

$$
I_1(t) + I_2(t) \lesssim k \int_0^t (t-s)^{2H-\frac{3}{2}} [I_1(s) + I_2(s)] ds.
$$

Now Gronwall’s lemma implies $I_1(t) + I_2(t) = 0$ for all $t \in [0, T]$. In particular, we have

$$
E[\mathbf{1}_{\{t<T_k\}} |u(t, x) - v(t, x)|^2] = 0.
$$

Thus, we have $u(t, x) = v(t, x)$ almost surely on $\{t < T_k\}$ for all $k \geq 1$, and the fact $T_k \uparrow \infty$ a.s as $k$ tends to infinity necessarily indicate $u(t, x) = v(t, x)$ a.s. for every $t \in [0, T]$ and $x \in \mathbb{R}$. 49
It is clear that the hypothesis (H2) implies the hypothesis (H1). So the existence of a Hölder continuous modification version of the solution follows from Theorem 1.5. We have then completed the proof of Theorem 1.6. □

Acknowledgements. We are grateful to the anonymous referees for very careful reading and valuable suggestions which significantly improve the paper.

References

SHE with general rough noise

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada
Email address: yaozhong@ualberta.ca

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada
Email address: xiongwang@ualberta.ca