Crossing estimates from metric graph and discrete GFF

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Abstract

We compare level-set percolation for Gaussian free fields (GFFs) defined on a rectangular subset of \( \delta \mathbb{Z}^2 \) to level-set percolation for GFFs defined on the corresponding metric graph as the mesh size \( \delta \) goes to 0. In particular, we look at the probability that there is a path that crosses the rectangle in the horizontal direction on which the field is positive. We show this probability is strictly larger in the discrete graph. In the metric graph case, we show that for appropriate boundary conditions the probability that there exists a closed pivotal edge for the horizontal crossing event decays logarithmically in \( \delta \). In the discrete graph case, we compute the limit of the probability of a horizontal crossing for appropriate boundary conditions.

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1 Introduction

In this article, we will prove several results on level-set percolation of Gaussian free fields on the square lattice \( \mathbb{Z}^2 \). The thrust of our results is that the probability that there exists a crossing of the domain in which the field is defined differs depending on whether we consider a discrete Gaussian free field or a metric graph Gaussian free field. Additionally, in the metric graph case we prove a bound on the probability that there exists a closed pivotal edge for the crossing. We begin with some basic definitions before stating our main results.

We call two vertices \( u, v \in \mathbb{Z}^2 \) adjacent, and write \( u \sim v \), if \(|u - v| = 1\), where \(|\cdot|\) denotes the standard Euclidean norm. Throughout, we will let \( V \subset \mathbb{Z}^2 \) be a proper subset of \( \mathbb{Z}^2 \). For such a set, we let \( \partial V = \{v \in V : \exists u \in V^c, u \sim v\} \) and \( V^o = V \setminus \partial V \). We will denote by \( \{S_t : t \geq 0\} \) the continuous-time simple random walk on \( \mathbb{Z}^2 \) with expected holding time 1 at each vertex, by \( \mathbb{P}_v \) the law of \( S \) where \( S_0 = v \), and by \( \mathbb{E}_v \) the expectation with respect to \( \mathbb{P}_v \). For \( V \neq \mathbb{Z}^2 \) and \( f : \partial V \to \mathbb{R} \) we call a Gaussian process \( \{\phi(v) : v \in V\} \) a discrete Gaussian free field (discrete GFF) on \( V \) with boundary condition \( f \) if its mean is the harmonic extension of \( f \) to \( V \) and its covariance is the Green’s function on \( V \). That is, letting \( \zeta = \inf\{t \geq 0 : S_t \in \partial V\} \),

\[
\mathbb{E}[\phi(v)] = \mathbb{E}_v[f(S_\zeta)], \quad v \in V \tag{1.1}
\]

\[
\text{Cov}[\phi(u)\phi(v)] = G(u,v) = \mathbb{E}_u \left[ \int_0^\zeta \mathbb{1}_{\{S_t = v\}} dt \right], \quad u, v \in V. \tag{1.2}
\]

We note that \( \phi(v) = f(v) \) for \( v \in \partial V \) (since \( \text{Var}[\phi(v)] = G(v,v) = 0 \)), and that the distribution of \( \phi \) is uniquely determined by (1.1) and (1.2). We note that the choice that \( S \) has expected holding time 1 at

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each vertex is made for consistency with [LL10] and corresponds to having conductance equal to 1/4 on each edge of the graph.

Next, we extend these definitions to metric graphs on $\mathbb{Z}^2$. To simplify notation, we will identify each subset $V$ of the lattice with the graph with vertex set $V$ and where two vertices $u, v \in V$ are connected by an edge if $u \sim v$. We denote by $E$ the edge set of this graph. To each $e \in E$ we associate a different compact interval $I_e$ of length $2 = 1/2C(e)$ (where $C(e)$ denotes the conductance of edge $e$) and identify the endpoints of this interval with the two vertices adjacent to $e$. The metric graph $\tilde{V}$ associated to $V$ is then defined to be $\tilde{V} = \cup_{e \in E} I_e$. With these definitions, it was shown in [Lu16] that the metric graph Gaussian free field (metric graph GFF) on $\tilde{V}$ with boundary condition $f$, denoted by $\{\tilde{\phi}(v) : v \in \tilde{V}\}$, can be constructed by extending $\phi$ to $\tilde{V}$ in the following manner: for adjacent vertices $u$ and $v$, the value of $\tilde{\phi}$ on the edge $e(u, v)$, conditioned on $\phi(u)$ and $\phi(v)$, is given by an independent bridge of length 2 of a Brownian motion with variance 2 at time 1 with boundary values $\phi(u)$ and $\phi(v)$.

With these definitions in place, we specify the crossing events that we will study in this paper. It will be convenient to think of $\mathbb{Z}^2$ as a subset of the complex plane $\mathbb{C}$. That is, taking $\mathbb{Z}^2$ as a subset of $\mathbb{C}$ in the obvious way, we identify the interval $I_e$ corresponding to an edge $e = e(u, v)$ in the discrete graph with the line segment between $u$ and $v$ (this requires re-scaling $I_e$). For $L > 0$, we let $R_L$ be the rectangle

$$R_L = \{z : 0 < \text{Re}(z) < L, 0 < \text{Im}(z) < 1\}.$$ 

Given such a rectangle, we let $(a, b, c, d)$ be its corners, listed in counter-clockwise order with $b = 0$. For $\delta > 0$, we let $V_\delta = R_L \cap \delta \mathbb{Z}^2$ and $(a_\delta, b_\delta, c_\delta, d_\delta)$ be the corners of $V_\delta$, listed in counter-clockwise order so that $b_\delta$ is closest to the origin (we always take $\delta \leq (L \wedge 1)/3$ so that the corners are distinct and $V_\delta^o$ is non-empty). For two vertices $u, v \in \partial V_\delta$, we let $[u, v)$ be the counter-clockwise arc from $u$ to $v$ in $\partial V_\delta$ which contains $u$ but not $v$. See Figure 1 for an illustration.

We let $\phi_\delta$ be a discrete GFF on $V_\delta$. We say $\phi_\delta$ has zero boundary condition if $\phi_\delta(v) = 0$ for $v \in \partial V_\delta$, and that it has alternating boundary condition if there exists $\theta > 0$ such that

$$\phi_\delta(v) = \begin{cases} 
\theta, & v \in [a_\delta, b_\delta) \cup [c_\delta, d_\delta), \\
-\theta, & v \in [b_\delta, c_\delta) \cup [d_\delta, a_\delta). 
\end{cases} \quad (1.3)$$

In either case, we let $\tilde{\phi}_\delta$ be the corresponding Gaussian free field on the metric graph $\tilde{V}_\delta$ such that $\tilde{\phi}_\delta(v) = \phi_\delta(v)$ for $v \in V_\delta$. Our goal is to study the probability that $\phi_\delta$ (resp. $\tilde{\phi}_\delta$) gives a positive horizontal crossing of $V_\delta$ (resp. $\tilde{V}_\delta$). That is, we want to study the event that there exists a path in $V_\delta$ (resp. $\tilde{V}_\delta$)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{$R_L$ and $V_\delta$ with $[a_\delta, b_\delta)$ and $[c_\delta, d_\delta)$ in red and $[b_\delta, c_\delta)$ and $[d_\delta, a_\delta)$ in blue.}
\end{figure}
from $[a_\delta, b_\delta)$ to $[c_\delta, d_\delta)$ such that $\phi_\delta$ (resp. $\tilde{\phi}_\delta$) is non-negative on this path. We denote this event by

$$\left\{ [a_\delta, b_\delta) \xleftrightarrow{\phi_\delta \geq 0} [c_\delta, d_\delta) \right\}$$

in the discrete case and similarly in the metric graph case. If $\tilde{\phi}_\delta$ has zero boundary condition, the probability of a positive crossing as defined above is equal to zero. For $\phi_\delta$ this probability is one since we can just take a path on $\partial V_\delta$. Therefore, in this case we take $[a_\delta', b_\delta') = \{ u \in V_\delta^\circ : \exists v \in [a_\delta, b_\delta), u \sim v \}$ and similarly for $[c_\delta', d_\delta')$, and consider the event

$$\left\{ [a_\delta', b_\delta') \xleftrightarrow{\phi_\delta \geq 0} [c_\delta', d_\delta') \right\},$$

and similarly for $\tilde{\phi}_\delta$. It is clear that with probability 1, $\phi_\delta(v) \neq 0$ for all $v \in V_\delta^\circ$, and $\tilde{\phi}_\delta$ has no local minima or maxima at 0. We will assume that these conditions hold throughout.

In the zero-boundary case, we show that the probability that there is a positive crossing in the metric graph decays to 0 as $\delta \to 0$ (and provide a bound on the rate of decay), while the probability that there is a positive crossing in the discrete graph remains bounded away from zero.

**Theorem 1.1.** Let $L > 0$ be a constant, $\phi_\delta$ be a zero-boundary discrete GFF on $V_\delta$, and $\tilde{\phi}_\delta$ be the corresponding metric graph GFF on $\hat{V}_\delta$. There exist constants $c = c(L) > 0$ and $\delta_0 = \delta_0(L) > 0$ such that for $\delta \leq \delta_0$,

$$\mathbb{P}\left( [a_\delta', b_\delta') \xleftrightarrow{\tilde{\phi}_\delta \geq 0} [c_\delta', d_\delta') \right) \leq \frac{c}{\sqrt{|\log(\delta)|}}. \quad (1.4)$$

In contrast there exists $\eta_1 = \eta_1(L) > 0$ such that for $\delta \leq \delta_0$,

$$\eta_1 \leq \mathbb{P}\left( [a_\delta', b_\delta') \xleftrightarrow{\phi_\delta \geq 0} [c_\delta', d_\delta') \right) \leq 1 - \eta_1. \quad (1.5)$$

For alternating boundary conditions, we show that the probability that there is a positive crossing in the metric graph remains bounded away from zero as $\delta \to 0$, but is also strictly smaller than the probability that there is a positive crossing in the discrete graph.

**Theorem 1.2.** Let $L, \theta > 0$ be positive constants, $\phi_\delta$ be a discrete GFF on $V_\delta$ with alternating boundary condition (1.3), and $\tilde{\phi}_\delta$ be the corresponding metric graph GFF on $\hat{V}_\delta$. There exist constants $\eta_2 = \eta_2(L, \theta) > 0$, $\eta_3 = \eta_3(L, \theta) > 0$, and $\delta_0 = \delta_0(L) > 0$ such that for $\delta \leq \delta_0$,

$$\eta_2 \leq \mathbb{P}\left( [a_\delta, b_\delta) \xleftrightarrow{\tilde{\phi}_\delta \geq 0} [c_\delta, d_\delta) \right) \leq 1 - \eta_2, \quad (1.6)$$

$$\eta_3 \leq \mathbb{P}\left( [a_\delta, b_\delta) \xleftrightarrow{\phi_\delta \geq 0} [c_\delta, d_\delta) \right) \leq 1 - \eta_3. \quad (1.7)$$

Additionally, there exists $\eta_4 = \eta_4(L, \theta) > 0$ such that for $\delta \leq \delta_0$,

$$\mathbb{P}\left( [a_\delta, b_\delta) \xleftrightarrow{\phi_\delta \geq 0} [c_\delta, d_\delta) \right) - \mathbb{P}\left( [a_\delta, b_\delta) \xleftrightarrow{\tilde{\phi}_\delta \geq 0} [c_\delta, d_\delta) \right) \geq \eta_4. \quad (1.8)$$

It is then natural to ask what are the limits of the crossing probabilities in (1.5), (1.6) and (1.7). We will give an explicit answer to the one in (1.7); for the other cases, see the discussion in Section 6. Throughout the rest of the paper, we let $2\lambda$ be the height gap in [SS09]. If $\theta = \lambda$ then the level line in discrete GFF with alternating boundary condition (1.3) converges to the SLE$_4$($-2; -2$) process, see Section 6. Consequently, we have the following.
Theorem 1.3. When θ = λ, the crossing probability in (1.7) has the following limit:
\[
\lim_{\delta \to 0} \mathbb{P} \left( [a_\delta, b_\delta] \xleftarrow{\phi_\delta} [c_\delta, d_\delta] \right) = \frac{(\varphi(b) - \varphi(a))(\varphi(d) - \varphi(c))}{(\varphi(c) - \varphi(a))(\varphi(d) - \varphi(b))},
\]
where \( \varphi \) is any conformal map from \( \mathbb{R} \) onto the upper-half plane \( \mathbb{H} \) with \( \varphi(a) < \varphi(b) < \varphi(c) < \varphi(d) \).

For the metric graph GFF, we can also prove a bound on the probability that there exists a closed pivotal edge for the event that there is a positive crossing. To specify what we mean, we begin by defining an edge percolation model. Let \( E_\delta \) be the edge set of the nearest-neighbor graph on \( V_\delta \) and \( \omega : E_\delta \to \{0, 1\} \) be such that \( \omega(e) = 1 \) if \( \phi_\delta(u) \geq 0 \) for all \( u \in I_e \) and \( \omega(e) = 0 \) otherwise. We say that \( e \) is open if \( \omega(e) = 1 \) and that \( e \) is closed otherwise. For an edge \( e \in E_\delta \), we let \( \omega^e \) and \( \omega_e \) be defined by
\[
\omega^e(f) = \begin{cases} 1, & f = e, \\ \omega(f), & f \neq e. \end{cases} \quad \omega_e(f) = \begin{cases} 0, & f = e, \\ \omega(f), & f \neq e. \end{cases}
\]
We write
\[\left\{(a_\delta, b_\delta) \xleftarrow{\omega^e} [c_\delta, d_\delta]\right\}\]
for the event that there is a path of open edges in \( \omega \) from \( [a_\delta, b_\delta] \) to \( [c_\delta, d_\delta] \) (note that this is exactly the event that there is a positive horizontal crossing in \( V_\delta \)). We can then define
\[
\{e \text{ is pivotal}\} = \left\{(a_\delta, b_\delta) \xleftarrow{\omega_e} [c_\delta, d_\delta]\right\} \setminus \left\{(a_\delta, b_\delta) \xleftarrow{\omega} [c_\delta, d_\delta]\right\},
\]
\[
\{e \text{ is closed pivotal}\} = \{e \text{ is pivotal}\} \cap \{\omega(e) = 0\},
\]
\[
\{\text{there exists a closed pivotal edge}\} = \cup_{e \in E_\delta} \{e \text{ is closed pivotal}\}.
\]
With these definitions, we obtain the following result

Theorem 1.4. Let \( L, \theta > 0 \) be positive constants and \( \tilde{\phi}_\delta \) be the metric graph GFF on \( \tilde{V}_\delta \) with alternating boundary condition (1.3). There exist constants \( c = c(L, \theta) > 0 \) and \( \delta_0 = \delta_0(L) > 0 \) such that for \( \delta \leq \delta_0 \),
\[
\mathbb{P}(\text{there exists a closed pivotal edge}) \leq \frac{c}{\sqrt{\log(\delta)}}.
\]

The rest of the paper is organized as follows. In Section 2 we introduce the main tools used in the proofs. In Section 3 we prove the results in Theorems 1.1 and 1.2 pertaining to the metric graph GFF \( \tilde{\phi}_\delta \), and in Section 4 we prove the results in Theorems 1.1 and 1.2 pertaining to the discrete GFF \( \phi_\delta \). Assuming the conclusions proved there, we are able to conclude the proof of Theorem 1.1 and 1.2 here. Finally, we will discuss the limits of the probabilities in (1.5), (1.6) and (1.7), and prove Theorem 1.3 in Section 6.

Proof of Theorem 1.1. The bound (1.4) is proved in Section 3.1. The bound (1.5) is proved in Section 4.2.

Proof of Theorem 1.2. Note that it suffices to prove the lower bound in (1.6) to prove (1.6) and (1.7). This follows from the following considerations. First, since \( \phi_\delta \) and \( \tilde{\phi}_\delta \) coincide on \( V_\delta \), the existence of a positive crossing in the metric graph implies the existence of such a crossing in the discrete graph. That is,
\[
\left\{(a_\delta, b_\delta) \xleftarrow{\phi_\delta} [c_\delta, d_\delta]\right\} \subseteq \left\{(a_\delta, b_\delta) \xleftarrow{\tilde{\phi}_\delta} [c_\delta, d_\delta]\right\}.
\]
Therefore, the lower bound in (1.6) implies the lower bound in (1.7). Next, by symmetry, the lower bound in (1.6) implies that
\[ P\left(\left[ b_\delta, c_\delta \right] \overset{\#}{\subseteq} L^{-1} \left[ d_\delta, a_\delta \right] \right) \geq \eta_2(L^{-1}, \theta). \]

The same bound then holds for the probability that there is a strictly negative vertical crossing in the discrete graph (here we use the fact that \( \phi_\delta \) is almost surely not equal to zero on \( V_\delta \)). Finally, since the existence of a strictly negative vertical crossing in the discrete graph implies that there is no positive horizontal crossing (either in the metric graph or the discrete graph), the upper bounds in (1.6) and (1.7) follow. Thus, both (1.6) and (1.7) follow from the lower bound in (1.6) which is proved in Section 3.2.

Finally, the bound in (1.8) is proved in Section 4.3.

Relation to previous works

• Relation of (1.4) and (1.6) to [ALS20] where the authors prove the convergence of first passage set in metric graph GFF to the analog in continuum GFF in Hausdorff metric.
  - From [ALS20], the level loops of zero-boundary metric graph GFF converge to the so-called conformal loop ensemble CLE with \( \kappa = 4 \). This implies that the left hand-side of (1.4) converges to zero. In contrast, our proof in Section 3.1 not only gives the convergence to zero, but also give a quantitative upper bound on the convergence rate.
  - The estimate (1.6) is indeed a consequence of conclusions in [LW18] and [ALS20], see also discussion in [LW21, Section 5.3]. But our proof in Section 3.2 is distinct and we use exploration martingale.

• Relation of (1.5), (1.7), and (1.9) to [SS09] where the authors prove the convergence of level lines in discrete GFF to SLE(4) process and its variants. Their work involves two kinds of convergence topologies: when the limiting curve of the level line does not touch the boundary, the convergence topology is local uniform topology on curves; whereas, when the limiting curve of the level line touches the boundary, the convergence topology is local uniform topology on the driving function. Although the conclusion in [SS09] does not cover our setup directly, their approach applies to our setting in Theorem 1.3 with proper modification, see Section 6.2. The convergence on curves implies the convergence of crossing probability. However, the convergence on driving function is not strong enough to conclude the convergence of crossing probability. In particular, Eq. (1.5) does not follow from [SS09]. Our proof for (1.5) and (1.7) in Section 4 uses exploration martingale and entropic repulsion method, and does not use any SLE theory.

• Our main results (1.8) and Theorem 1.4 do not appear anywhere before to our knowledge.

Future directions

• To derive the limit of crossing probabilities in (1.5), (1.6), and (1.7) is an interesting question. To our knowledge, the limit of the crossing probability in (1.5) is unknown, see the discussion in the end of Section 6.2. The limit of the crossing probability in (1.5) is known for \( \theta = \lambda \), as in (1.9), but it is unknown for \( \theta \neq \lambda \). The limit of the crossing probability in (1.6) is known for \( \theta = 2\lambda \), see (6.10), but it is unknown for \( \theta \neq 2\lambda \).

• We provide an estimate on the existence of pivotal edge in metric graph GFF. It is more important to understand the pivotal vertices in discrete GFF and its connection to continuum GFF.
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2 Preliminaries

In this section we collect some technical tools and introduce some notation that will be used in the proofs of the main results.

Notation. For a real vector $x$ (in any dimension), we denote by $|x|$ the Euclidean norm of $x$, by $|x|_{\ell_1}$ its $\ell_1$-norm, and by $|x|_{\ell_\infty}$ its $\ell_\infty$-norm. For a finite set $A$, we will also use $|A|$ to denote the cardinality of $A$. The meaning will be clear from context. We use $A^c$ to denote the complement of the set (or event) $A$.

Throughout the proofs we let $c,c'$ be positive constants which only depend on $L$ and $\theta$ as in the main theorems, and whose value may change each time they appear.

2.1 Discrete GFF and metric graph GFF

We begin with an alternate construction of the metric graph GFF which more clearly shows it is a natural analog to the discrete GFF. Namely, one can construct $\tilde{\phi}$ by first defining a Brownian motion $\{\tilde{B}_t : t \geq 0\}$ on $\tilde{\mathbb{Z}}^2$ as in [Lu16, Section 2]: $\tilde{B}$ behaves like a standard Brownian motion in the interior of the edges, while on the vertices it chooses to do excursions on each incoming edge uniformly at random. For a subset $V \subset \mathbb{Z}^2$, we let $\tilde{\zeta} = \inf\{t \geq 0 : \tilde{B}_t \in \partial V\}$, and $\{G(u,v) : u,v \in V\}$ be the density of the 0-potential of $\{\tilde{B}_t : 0 \leq t < \tilde{\zeta}\}$ (with respect to the Lebesgue measure on $\tilde{V}$), where $u$ and $v$ are now arbitrary points in $\tilde{V}$ (not necessarily vertices). It is shown in [Lu16] that the trace of $\tilde{B}$ on $V$ (when parametrized by its local time at the vertices) is exactly the continuous-time simple random walk on $V$, and therefore the definition of $G$ here coincides with the one in (1.2) for $u,v \in V$, justifying the abuse of notation. Due to this relation between the metric graph Brownian motion and the simple random walk, we will, by a slight abuse of notation, let $\mathbb{P}_v$ be the law of $\tilde{B}_t$ where $\tilde{B}_0 = v$ and $\mathbb{E}_v$ be the expectation with respect to $\mathbb{P}_v$. It was also shown in [Lu16] that the value of $G$ on $\tilde{V} \setminus V$ can be obtained by interpolation from the value on $V$. For two pairs of adjacent vertices $(u_1,v_1)$ and $(u_2,v_2)$ in $V$, and two points $w_1$ and $w_2$ on the corresponding edges, taking the convention that either the edges are distinct or $(u_1,v_1) = (u_2,v_2)$ and letting $r_1 = |w_1 - u_1|$ and $r_2 = |w_2 - u_2|$, we have by [Lu16, Equation (2.1)] that

$$G(w_1,w_2) = (1 - r_1)(1 - r_2)G(u_1,u_2) + r_1r_2G(v_1,v_2) + (1 - r_1)r_2G(u_1,v_2) + r_1(1 - r_2)G(v_1,u_2) + 4(r_1 \wedge r_2 - r_1r_2)1_{(u_1,v_1) = (u_2,v_2)}.$$  \hspace{1cm} (2.1)

We note that the factor of 4 in front of the last term in (2.1) comes from multiplying the corresponding factor of 2 in [Lu16, Equation (2.1)] by $\rho(\{u_1,v_1\}) = \rho(\{u_2,v_2\}) = 2$.

We call a process $\{\tilde{\phi}(v) : v \in \tilde{V}\}$ a Gaussian free field on $\tilde{V}$ if it is a continuous Gaussian process such that there exists a function $f : \partial V \to \mathbb{R}$ for which

$$\mathbb{E}[\tilde{\phi}(v)] = \mathbb{E}_v[f(\tilde{B}_t)], \quad \text{Cov}[\tilde{\phi}(u)\tilde{\phi}(v)] = G(u,v).$$

We note that this definition extends to any compact, connected subset $K \subset \tilde{\mathbb{Z}}^2$.

We remark on a consequence of the construction above. For any finite subset $A \subset \tilde{\mathbb{Z}}^2$, we can define an electric network with vertex set $W = \mathbb{Z}^2 \cup A$, where any two vertices $u,v \in W$ are connected with an edge of conductance $(4|u-v|^{-1}$ if there is a continuous path in $\mathbb{Z}^2$ from $u$ to $v$ which does not contain any other points in $W$ (note this path is always contained in $I_v$ for some edge in standard nearest-neighbor graph). Then, if $V \subset W$ is a connected (proper) subset and $\tilde{\phi}$ is a metric graph GFF on $\tilde{V}$, $\{\tilde{\phi}(v) : v \in V\}$ is a discrete GFF on (the electric network with vertex set) $V$. 
2.2 Excursion sets and first passage sets

Next, we introduce certain random subsets of $V_\delta$ and $\tilde{V}_\delta$ that are related to the existence of positive crossings. Let $E_\delta^{\geq 0}$ (resp. $\tilde{E}_\delta^{\geq 0}$) be the excursion set of $\phi_\delta$ (resp. $\tilde{\phi}_\delta$) above 0. That is,

$$E_\delta^{\geq 0} = \{ v \in V_\delta : \phi_\delta(v) \geq 0 \},$$

and similarly for $\tilde{E}_\delta^{\geq 0}$. The excursion set below zero, $E_\delta^{\leq 0}$ (resp. $\tilde{E}_\delta^{\leq 0}$), is defined similarly. Next, we introduce the first passage set of $\phi_\delta$ (resp. $\tilde{\phi}_\delta$) above 0. This set, which we denote by $A_{\delta,0}$ (resp. $\tilde{A}_{\delta,0}$) is the union of all connected components of $E_\delta^{\geq 0}$ that intersect the boundary. That is,

$$A_{\delta,0} = \{ v \in V_\delta : \exists \text{ nearest-neighbor path } \gamma \text{ from } v \text{ to } \partial V_\delta \text{ such that } \phi_\delta \geq 0 \text{ on } \gamma \}.$$

We will denote by $A_{\delta,0}^l$ the union of all connected components of $A_{\delta,0}$ intersecting $[a_\delta, b_\delta]$. That is, the left part of the first passage set. Similarly, we denote by $A_{\delta,0}^r$ the union of all connected components of $A_{\delta,0}$ intersecting $[c_\delta, d_\delta]$ (the right part of the first passage set). Note that there is a positive horizontal crossing of $R_L$ when $A_{\delta,0}^l = A_{\delta,0}^r$. We denote by $\forall_{\delta,0}$ the first passage set below 0, by $\forall_{\delta,0}^b$ the union of all connected components of $\forall_{\delta,0}$ intersecting $[b_\delta, c_\delta]$ (the bottom part of the first passage set), and by $\forall_{\delta,0}^r$ the union of all connected components of $\forall_{\delta,0}$ intersecting $[d_\delta, a_\delta]$ (the top part of the first passage set). The metric graph first passage sets $A_{\delta,0}^l$, $A_{\delta,0}^r$, $\tilde{A}_{\delta,0}^l$, and so on are defined similarly.

2.3 Exploration martingales

In this section we introduce a family of martingales which form the basis of the proofs of our results. We begin with some basic definitions and some fundamental results. For a deterministic subset $W \subset \tilde{V}_\delta$, we let $F_W$ be the sigma algebra generated by $\{ \hat{\phi}_\delta(w) : w \in W \}$.

**Definition 2.1** (Optional set). Let $K$ be a random compact subset of $\tilde{V}_\delta$ that almost surely has finitely many connected components. We say $K$ is an optional set for $\hat{\phi}_\delta$ if for every deterministic, open subset $W$ of $\tilde{V}_\delta$, $\{ K \subset W \} \in F_W$. Similarly, we say $K$ is optional for $\tilde{\phi}_\delta$ if the following holds. For any deterministic open subset $W$ of $\tilde{V}_\delta$, $\{ K \subset W \} \in F_W \cap F_\tilde{V}_\delta$.

We note that in the definition above, we use the subspace topology on $\tilde{V}_\delta$ considered as a subspace of $\mathbb{Z}^2$.

For an optional set $K$, we define its $\sigma$-field $F_K$ by

$$F_K = \left\{ A \in F_{\tilde{V}_\delta} : A \cap \{ K \subset W \} \in F_W, \text{ for all deterministic, open } W \subset \tilde{V}_\delta \right\}.$$ 

Sometimes, it will be useful to consider the field $\text{sign}(\hat{\phi}) = \{ \text{sign}(\hat{\phi}(v)) : v \in \tilde{V}_\delta \}$, where sign is the function

$$\text{sign}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

For a deterministic subset $W \subset \tilde{V}_\delta$, we let $G_W$ be the $\sigma$-field generated by $\{ \text{sign}(\hat{\phi}(w)) : w \in W \}$. We extend this notation to optional sets in the obvious way.

The strong Markov property of the metric graph GFF, stated below, will allow us to perform detailed analysis of the exploration martingales and is thus fundamental to our proofs.

**Theorem 2.2** (Strong Markov property, [Lu16]). Let $K$ be optional for $\hat{\phi}_\delta$. Given $F_K$, the process $\{ \hat{\phi}_\delta(v) : v \in \tilde{V}_\delta \setminus K \}$ is a metric graph GFF on $\tilde{V}_\delta \setminus K$ with boundary condition given by the restriction of $\hat{\phi}_\delta$ to $\partial K \cup \partial \tilde{V}_\delta$. 

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It is straightforward that the strong Markov property holds for the discrete GFF as well. We introduce some notation for harmonic extension. Assume the same setup as in Theorem 2.2, and let \( \hat{B} \) be a Brownian motion on \( \hat{V}_\delta \). Set \( \hat{\zeta} = \inf\{t \geq 0 : \hat{B}_t \in \partial \hat{V}_\delta\} \) as in Section 2.1 and \( \tau = \inf\{t \geq 0 : \hat{B}_t \in K\} \) be the hitting time of \( K \). We define, for \( v \in \hat{V}_\delta \) and \( w \in K \cup \partial \hat{V}_\delta \),

\[
Hm(v, w; K) = \mathbb{P}_v \left( \hat{B}_{\tau \wedge \hat{\zeta}} = w \right).
\]

Note that the harmonic measure \( Hm(v, \cdot; K) \) is always supported on a finite set of points since \( |\partial K| < \infty \) for compact \( K \) with finitely many connected components. With this notation, in Theorem 2.2, we have in fact, for \( u, v \in \hat{V}_\delta \),

\[
\mathbb{E}[\hat{\phi}_\delta(v) \mid F_K] = \sum_{w \in \hat{V}_\delta} Hm(v, w; K) \hat{\phi}_\delta(w),
\]

\[
\text{Cov}[\hat{\phi}_\delta(u), \hat{\phi}_\delta(v) \mid F_K] = G(u, v) - \sum_{w \in K} Hm(u, w; K)G(w, v).
\]

We define

\[
Hm(v, K) = \sum_{w \in K} Hm(v, w; K) = \mathbb{P}_v (\tau \leq \hat{\zeta}).
\]

For \( U \subset \hat{V}_\delta \) a finite subset, we define

\[
Hm(U, K) = \sum_{u \in U} Hm(u, K).
\]

We can now introduce the exploration martingales. For a finite subset \( U \subset \hat{V}_\delta \), we define the “observable” \( X_U \) by

\[
X_U = \sum_{v \in U} \hat{\phi}_\delta(v).
\]

For \( I_0 \) a deterministic, compact subset of \( \hat{V}_\delta \) (with finitely many connected components), we will let \( M \) be the Doob martingale for \( X_U \) as we explore \( E^\geq_\delta \) or \( E^\leq_\delta \) from \( I_0 \). We will specify whether the exploration happens on the metric graph or discrete graph whenever we use an exploration martingale.

To make this precise, we specify what we mean by exploring the excursion set from \( I_0 \). We begin with the exploration on \( E^\geq_\delta \) as it is simpler to explain. Let \( \hat{D}^\geq_\delta \) be the metric graph distance on \( \hat{E}^\geq_\delta \). We let the set explored by time \( t \geq 0 \), which we denote by \( \hat{I}_t \), be equal to the ball of radius \( t \) around \( I_0 \) with respect to \( \hat{D}^\geq_\delta \). We use the convention that for \( u, v \in \hat{V}_\delta \) with \( u \notin \hat{E}^\geq_\delta \) and \( u \neq v \), \( \hat{D}^\geq_\delta(u, v) = \infty \) and \( \hat{D}^\geq_\delta(u, u) = 0 \). It is easy to show that \( \hat{I}_t \) is an optional set for each \( t \geq 0 \) [DW20].

The exploration corresponding to \( E^\leq_\delta \) is similar in spirit but slightly more cumbersome to describe. In this case we take \( V_0 \subset \hat{V}_\delta \) and \( I_0 \) the metric graph on \( V_0 \). We begin the exploration by setting \( A_0 = V_0 \cap \hat{E}^\geq_\delta \), and \( B_0 = V_0 \cap \hat{E}^\leq_\delta \). For \( k \geq 1 \) an integer, we let

\[
A_k = \{ v \in (V_0 \setminus V_{k-1}) \cap \hat{E}^\geq_\delta : \exists u \in A_{k-1}, u \sim v \},
\]

\[
B_k = \{ v \in (V_0 \setminus V_{k-1}) \cap \hat{E}^\leq_\delta : \exists u \in A_{k-1}, u \sim v \},
\]

\[
V_k = V_{k-1} \cup A_k \cup B_k,
\]

and \( \hat{I}_k \) be the metric graph on \( V_k \). In words, at each time step we explore all unexplored vertices that are adjacent to an explored vertex on which the field is non-negative. Note that \( \hat{I}_k \) is an optional set, and in fact \( \hat{I}_k \in \mathcal{G}_{\hat{V}_{k-1}} \). To extend this definition to real \( t \), we proceed by interpolation. That is, if we let \( E_k = \{ (u, v) : u \in \hat{I}_{k-1}, v \in A_k \cup B_k, u \sim v \} \) be the edges between \( \hat{I}_{k-1} \) and \( \hat{I}_k \setminus \hat{I}_{k-1} \) then for \( k - 1 < t < k \) we let

\[
\hat{I}_t = \hat{I}_{k-1} \cup \left( \bigcup_{(u, v) \in E_k} [u, (k-t)u + (t-(k-1))v] \right).
\]
Note that as before $\mathcal{I}_t \in \mathcal{G}_{V_{k-1}}$ and so $\mathcal{I}_t$ is an optional set for any $t$.

Explorations on $E_t^{\leq 0}$ and $\tilde{E}_t^{\leq 0}$ are defined in the same way. As alluded to above, we take

$$M_t = \mathbb{E}[X_U \mid \mathcal{F}_{\mathcal{I}_t}].$$

It is straightforward to check that $M$ is a continuous martingale [DW20]. We now turn to the quadratic variation of the exploration martingale.

It is straightforward to check that for a Doob martingale, the quadratic variation is equal to the decrease in the conditional variance [DW20]. That is,

$$\langle M \rangle_t = \text{Var}[X_U \mid \mathcal{F}_{\mathcal{I}_0}] - \text{Var}[X_U \mid \mathcal{F}_{\mathcal{I}_t}].$$

We denote by $G_t$ the Green’s function on $\tilde{V}_\delta \setminus \mathcal{I}_t$, and write

$$H_m(u, v) = H_m(u, v; \mathcal{I}_t).$$ (2.2)

We get from Theorem 2.2 that

$$\text{Var}[X_U \mid \mathcal{F}_{\mathcal{I}_t}] = \sum_{u, u' \in U} G_t(u, u') = \sum_{u, u' \in U} \left[ G(u, u') - \sum_{v \in \mathcal{I}_t} H_m(u, v)G(v, u') \right].$$ (2.3)

This gives

$$\langle M \rangle_t = \text{Var}[X_U \mid \mathcal{F}_{\mathcal{I}_0}] - \text{Var}[X_U \mid \mathcal{F}_{\mathcal{I}_t}]$$

$$= \sum_{u, u' \in U} \sum_{v \in \mathcal{I}_t} H_m(u, v)G(v, u') - \sum_{v' \in \mathcal{I}_0} H_m(u, v')G(v', u').$$

Since $\mathcal{I}_0 \subseteq \mathcal{I}_t$, we have by the Markov property of $\tilde{B}$ that $H_m(u, v') = \sum_{v \in \mathcal{I}_t} H_m(u, v) H_m(v, v')$ so the above becomes

$$\langle M \rangle_t = \sum_{u, u' \in U} \sum_{v \in \mathcal{I}_t} H_m(u, v) \left[ G(v, u') - \sum_{v' \in \mathcal{I}_0} H_m(v, v')G(v', u') \right],$$

$$= \sum_{u, u' \in U} \sum_{v \in \mathcal{I}_t} H_m(u, v)G_0(v, u'),$$

$$= \sum_{v \in \mathcal{I}_t} H_m(U, v)G_0(v, U),$$ (2.4)

where $G_0(v, U) = \sum_{u \in U} G_0(v, u)$.

2.4 Brownian motion tools

In this section, we recall two facts about continuous martingales and Brownian motion that will be useful throughout the paper. The first is [RY99, Theorem 1.7 in Chapter V], stated below, which is a version of the Dubins-Schwarz theorem for martingales of bounded quadratic variation.

**Theorem 2.3.** Let be $M$ a continuous martingale, $T_t = \inf\{s : \langle M \rangle_s > t\}$, and $W$ be the following process

$$W_t = \begin{cases} M_{T_t} - M_0 & t < \langle M \rangle_\infty, \\ M_\infty - M_0 & t \geq \langle M \rangle_\infty. \end{cases}$$

Then $W$ is a Brownian motion stopped at $\langle M \rangle_\infty$. 9
When applying this theorem, we will generally denote by $B$ a Brownian motion which satisfies $B_t = M_{T_t} - M_0$ for $t < \langle M \rangle_\infty$ but is not stopped at $\langle M \rangle_\infty$, so that $W_t = B_{t/\langle M \rangle_\infty}$. Suppose $M$ is the exploration martingale in Section 2.3. By Theorem 2.3, the process $\{M_{T_t} - M_0 : t \geq 0\}$ is independent of $\mathcal{F}_{T_0}$, so we will generally take $B$ to be independent of $\mathcal{F}_{T_0}$ as well.

The second result gives the distribution of the hitting time of a line by Brownian motion.

**Proposition 2.4.** Let $B$ be a standard one-dimensional Brownian motion, $\Phi$ be the distribution function of the standard normal distribution, and $\bar{\Phi}(x) = 1 - \Phi(x)$. For $m \in \mathbb{R}$ and $b > 0$, let $\tau = \inf\{t \geq 0 : B_t \leq mt - b\}$. Then for $T > 0$,

$$\mathbb{P}(\tau \leq T) = \bar{\Phi}\left(\frac{b}{\sqrt{T}} - m\sqrt{T}\right) + e^{2bm}\bar{\Phi}\left(\frac{b}{\sqrt{T}} + m\sqrt{T}\right).$$

**Proof.** The density of $\tau$ is given in [BS02, Equation (2.0.2) in Part II]. Taking an integral then gives the desired result. \(\square\)

We note two facts that follow directly from Proposition 2.4 and will be used repeatedly throughout the paper. First, letting $m = 0$ we obtain that for any deterministic $t \geq 0$, sup$\{B_s : 0 \leq s \leq t\}$ and $-\inf\{B_s : 0 \leq s \leq t\}$ have the same distribution as $|B_t|$. Second, for $m < 0$ and $b > 0$ we have by letting $T \to \infty$ that $\mathbb{P}(\tau < \infty) = e^{2bm} < 1$.

### 2.5 Random walk estimates

We conclude this section with some results about Green’s functions and harmonic measures which will be used later. The first result follows from [LL10, Theorem 4.4.4, Proposition 4.6.2]. It will be useful in comparing the Green’s function in different domains. Below and throughout the paper, a subset $V \subset \mathbb{Z}^2$ is simply connected if for any loop in $V$, all the vertices in the interior of the loop (i.e. separated from infinity by the loop) are contained in $V$.

**Lemma 2.5.** There exists a universal constant $C > 0$ such that the following holds. For $V \subset \mathbb{Z}^2$ simply connected, let $G$ be the Green’s function on $V$. For $v, u \in V^o$, let $\Delta = \text{dist}(v, \partial V)$ be the Euclidean distance between $v$ and $\partial V$. Then

$$G(v, u) \leq \frac{2}{\pi} \log\left(\frac{\Delta}{|u - v| + 1} + 1\right) + C.$$

We note that in the statement of the lemma, $u$ and $v$ need not be distinct vertices.

**Proof.** Without loss of generality, we assume $V$ is finite. Following the notation in [LL10], we let $a$ be the potential kernel of the simple random walk in two dimensions. By [LL10, Proposition 4.6.2], we have

$$G(v, u) = \sum_{w \in \partial V} \text{Hm}(v, w; V)a(w, u) - a(v, u). \quad (2.5)$$

By [LL10, Theorem 4.4.4], there exists a universal constant $C > 0$ such that for all $x, y \in \mathbb{Z}^2$,

$$|a(x, y) - \frac{2}{\pi} \log(|x - y| + 1)| \leq C. \quad (2.6)$$

It follows (by adjusting the value of $C$) that

$$G(v, u) \leq \frac{2}{\pi} \sum_{w \in \partial V} \text{Hm}(v, w; V) \log\left(\frac{|w - u| + 1}{|v - u| + 1}\right) + C. \quad (2.7)$$

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For $n \geq 0$, let $A_n = \{ w \in \mathbb{Z}^2 : |w-v| \in [3^n \Delta, 3^{n+1} \Delta) \}$ and $D_n = \partial V \cap A_n$. To conclude, we need to bound the harmonic measure on $D_n$. By the invariance principle, there is a constant $p > 0$ such that for each $n \geq 0$, the following event has probability at least $p$: a random walk (not killed on $\partial V$) started at $v$ completes a loop in $A_n$ before exiting the open ball of radius $3^{n+1} \Delta$ centered at $v$ (i.e., before exiting $A_n$ through its outer boundary). Since $V$ is simply connected and $\text{dist}(v, \partial V) = \Delta$, any such loop must hit $\partial V$. It follows that

$$\text{Hm}(v, D_n; \partial V) \leq (1-p)^n.$$  

Combining this with (2.7) and the fact that $|w-u| \leq |w-v| + |v-u|$ gives

$$G(v, u) \leq \frac{2}{\pi} \sum_{n=0}^{\infty} \text{Hm}(v, D_n; V) \log \left( \frac{3^{n+1} \Delta}{|v-u|+1} + 1 \right) + C.$$  

Next, we provide two estimates on harmonic measures for a random walk started near the boundary of a box. The first provides an upper bound on the probability that a random walk started near the left side of a box doesn’t exit the box through the left side. Recall that $S$ is a simple random walk on $\mathbb{Z}^2$, $\mathbb{P}_v$ is the law of $S$ started at $v$, and that we take $\mathbb{Z}^2 \subset \mathbb{C}$ in the obvious way.

**Lemma 2.6.** There exists a constant $c > 0$ such that the following holds. For $N, M \geq 2$, $R = [0, N] \times [-M, M]$, $V = R \cap \mathbb{Z}^2$, and $\zeta = \min\{n \geq 1 : S_n \in \partial V\}$. We have

$$\mathbb{P}_1(\Re(S_\zeta) \neq 0) \leq \frac{c}{N \wedge M}.$$  

**Proof.** Let $\zeta' = \min\{n \geq 1 : \Re(S_n) = 0\}$ and $K = N \wedge M$. We have by [LL10, Theorem 8.1.2] that

$$\mathbb{P}_1(|\Im(S_{\zeta'})| > K) \leq \frac{c}{K}.$$  

Let $D = \partial V \setminus \{z : \Re(z) = 0\}$ and note that there exists a universal constant $c' > 0$ such that

$$\mathbb{P}_v(|\Im(S_{\zeta'})| > K) \geq c', \quad \forall v \in D.$$  

Finally, the conclusion follows by noting

$$\mathbb{P}_1(|\Im(S_{\zeta'})| > K) = \sum_{v \in D} \mathbb{P}_1(S_\zeta = v) \mathbb{P}_v(|\Im(S_{\zeta'})| > K) \geq c' \mathbb{P}_1(\Re(S_\zeta) \neq 0).$$

The second result provides a lower bound on the probability that a random walk started near the left side of a box will exit the box through the right side.

**Lemma 2.7.** For any $a > 0$ there exists a constant $c_a > 0$ such that the following holds. For $N, M \geq 2$ with $M \geq aN$, $R = [0, N] \times [-M, M]$, $V = R \cap \mathbb{Z}^2$, and $\zeta = \min\{n \geq 1 : S_n \in \partial V\}$, we have

$$\mathbb{P}_1(\Re(S_\zeta) = N) \geq \frac{c_a}{N}.$$  

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Proof. Let $M' = M/2$, $V' = (-\infty, N] \times [-M', M'] \cap \mathbb{Z}^2$, and $\zeta' = \min\{n \geq 1 : S_n \in \partial V'\}$. For an integer $k$, let $v_k = 1 + ik$. By the invariance principle, there exists a constant $c_a > 0$ such that

$$\mathbb{P}_{v_k}(\text{Re}(S_{\zeta'}) = N) \geq c_a, \quad |k| \leq M'/2.$$ 

Let $D = \{v_k : |k| \leq M' - 1\}$, $\tau = \min\{n \geq 1 : S_n \in D\} \wedge \zeta'$, and $G$ be the Green’s function on $V'$. By a last exit decomposition, we have for $v \in D$

$$\mathbb{P}_v(\text{Re}(S_{\zeta'}) = N) = \sum_{v' \in D} G(v, v') \mathbb{P}_{v'}(\text{Re}(S_\tau) = N).$$

By [DL18, Lemma 1], $\max\{G(v, D) : v \in D\} \leq c'_a N$. Therefore, combining the last two displays and summing over $v \in D$ gives

$$c_a N \leq \sum_{v \in D} \mathbb{P}_v(\text{Re}(S_{\zeta'}) = N) \leq c'_a N \sum_{v \in D} \mathbb{P}_v(\text{Re}(S_\tau) = N).$$

Finally, we note that for any $v \in D$, $\mathbb{P}_v(\text{Re}(S_\tau) = N) \leq \mathbb{P}_1(\text{Re}(S_\zeta) = N)$, so the conclusion follows.  

2.6 Overview of the method

Before moving to the proofs, we give an overview of how we use exploration martingales to study excursion sets of GFFs. For concreteness, we will assume we are exploring the excursion set above 0.

By (2.4), the total quadratic variation of the martingale is increasing as a function of the explored set. In particular, if we choose $U$ such that $G_0(v, U)$ is uniformly bounded then $\langle M \rangle_\infty$ is bounded above by (a constant times) $\text{Hm}(U, \mathcal{I}_\infty)$. We can achieve this, for example, by making $U$ a collection of vertices adjacent to the boundary. Moreover, if we choose $U$ such that it separates $\mathcal{I}_0$ from a macroscopic portion of the boundary, then $\langle M \rangle_\infty$ will be bounded below by (a constant times) $\text{Hm}(U, \mathcal{I}_\infty)$ when $\mathcal{I}_\infty$ is large enough (i.e. when a non-negligible portion of the harmonic measure comes from vertices in the bulk of $R_L$).

Next, we illustrate how to obtain bounds on the distribution of $\langle M \rangle_\infty$ (and thus on the distribution of $\text{Hm}(U, \mathcal{I}_\infty)$). We assume for simplicity that $\phi$ is equal to $\theta > 0$ on $I_0$ and equal to 0 on $\partial V_0 \setminus I_0$. In particular, we assume $M_0 = \theta \text{Hm}(U, \mathcal{I}_0)$. If we are exploring $E_{\geq 0}$, then $M_t \geq 0$ because the exploration stops whenever it hits a point where $\phi$ is 0. Therefore, it follows from Theorem 2.3 that $\langle M \rangle_\infty$ is dominated by the time it take a Brownian motion to hit $-\theta \text{Hm}(U, \mathcal{I}_0)$ which is typically of order $\text{Hm}(U, \mathcal{I}_0)^2$. However, if we are exploring $E_{\geq 0}$, then (assuming $\mathcal{I}_t$ never hits $U$) when the exploration stops $\phi$ is strictly negative on the outer boundary of $\mathcal{I}_\infty$. In fact, we will show in Section 4 that with high probability (given $\mathcal{I}_\infty \cap U = \emptyset$), $M_\infty \leq -c \text{Hm}(U, \mathcal{I}_\infty)$ for some $c > 0$. Since by the discussion above $\langle M \rangle_t \leq \text{Hm}(U, \mathcal{I}_t)$, if we assume that $\text{Hm}(U, \mathcal{I}_0) \geq c$ it follows that the event $\mathcal{I}_\infty \cap U = \emptyset$ is dominated by the event that a Brownian motion hits a function of the form $f(t) = -c(1 \wedge t)$ in finite time. This event has probability less than 1 by Proposition 2.4 so (loosely speaking) we have that the exploration will reach $U$ with positive probability. Since all we assumed was that $U$ has macroscopic length and is adjacent to the boundary it follows that the exploration from $\mathcal{I}_0$ will hit any portion of the boundary with positive probability.

This informal analysis illustrates the difference in behavior between the metric graph and discrete graph GFFs explorations. Essentially, it comes from the fact that for the discrete graph exploration to die out, we must have strictly negative values of the field along the boundary of the explored set. It also illustrates the idea that we want to choose $U$ to be close to the boundary and have macroscopic length.
3 Estimates of crossing probabilities in the metric graph

3.1 The zero-boundary case

In this section we prove (1.4). The proof consists of analyzing the exploration martingale $M$, introduced in Section 2.3, corresponding to an exploration on $\bar{E}_\delta^\geq$ with $I_0 = [c_\delta', d_\delta']$ and $U = \{ u \in \partial V_\delta^0 : \text{Re}(u) \leq L/4 \}$. Note that $U$ contains the portion of the boundary $\partial V_\delta^0$ that intersects a macroscopic ball around $a_\delta$ and $b_\delta$. This property will be needed in the proof (i.e. the proof would not work with $U = [a_\delta', b_\delta']$).

Thus, the proof will be complete once we prove (3.2).

We let $I_0^- = \{ v \in I_0 : \tilde{\phi}_\delta(v) < 0 \}$. Since $\tilde{\phi}_\delta$ is non-negative on $I \setminus I_0^-$, we have

$$ M_t = \sum_{v \in I_0^-} \text{Hm}_t(U, v) \tilde{\phi}_\delta(v). $$

Noting that $\text{Hm}_t(U, v)$ is decreasing in $t$ for $v \in I_0^-$ (since $I_0$ is increasing), we conclude

$$ M_t - M_0 \geq - \sum_{v \in I_0} \text{Hm}_0(U, v) (\tilde{\phi}_\delta(v) \vee 0). \quad (3.1) $$

Next, we turn to bounding the quadratic variation. In particular, we claim that there exists $c = c(L) > 0$ such that

$$ \{ [a_\delta', b_\delta'] \overset{\tilde{\phi}_\delta \geq 0}{\xleftarrow{\sim}} [c_\delta', d_\delta'] \} \subseteq \{ (M)_\infty \geq c |\log(\delta)| \}. \quad (3.2) $$

Before proving (3.2), we show how it can be used to conclude the proof of (1.4). Suppose $(B_s)_{s \geq 0}$ is a standard one-dimensional Brownian motion. Combining Theorem 2.3 with (3.1) and (3.2), we have

$$ P \left( [a_\delta', b_\delta'] \overset{\tilde{\phi}_\delta \geq 0}{\xleftarrow{\sim}} [c_\delta', d_\delta'] \right) \leq P \left( \inf_{0 \leq s \leq |\log(\delta)|} B_s \geq - \sum_{v \in I_0} \text{Hm}_0(U, v) (\tilde{\phi}_\delta(v) \vee 0) \right) $$

$$ \leq \frac{1}{\sqrt{c |\log(\delta)|}} \sum_{v \in I_0} \text{Hm}_0(U, v) E[\tilde{\phi}_\delta(v) \vee 0] $$

$$ \leq \frac{c'}{\sqrt{|\log(\delta)|}} \text{Hm}(U, I_0), $$

where in the second inequality we used the fact that, for $t$ and $a > 0$, we have by Proposition 2.4 that $P(\inf_{0 \leq s \leq t} B_s \geq -a) \leq a/\sqrt{t}$; and in the third inequality we used the fact that $G(v, v) \leq 4$ for $v \in I_0$ (since $v$ is adjacent to $\partial V_\delta$). It remains to bound $\text{Hm}(U, I_0)$. Since $U, I_0 \subset V_\delta$, we can consider a discrete time simple random walk $S$ (on $\delta \mathbb{Z}^2$), killed on $\partial V_\delta$ (instead of the metric graph Brownian motion $\tilde{B}$). For a vertex $u \in U$, we have $\text{Hm}(u, I_0) \leq c \delta$ by Lemma 2.6. Since $|U| \leq c/\delta$, we conclude

$$ \text{Hm}(U, I_0) \leq c. $$

Thus, the proof will be complete once we prove (3.2).

Proof of (3.2). We begin by noting that on $\{ [a_\delta', b_\delta'] \overset{\tilde{\phi}_\delta \geq 0}{\xleftarrow{\sim}} [c_\delta', d_\delta'] \}$ there exists a vertex $u^* \in [a_\delta', b_\delta']$ such that the set $I_\infty$ contains a nearest neighbor path $\gamma \subset V_\delta^0$ connecting $u^*$ to the complement of the box of radius $L/4$ around $u^*$. Therefore, on $\{ [a_\delta', b_\delta'] \overset{\tilde{\phi}_\delta \geq 0}{\xleftarrow{\sim}} [c_\delta', d_\delta'] \}$ the following holds almost surely

$$ \text{Var}[X_U | F_{I_\infty}] \leq \max_{\gamma} \text{Var}[X_U | F_{\gamma \cup I_0}], $$

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where the maximum is taking over all deterministic paths $\gamma$ from a vertex in $[a'_0, b'_0]$ to the outside of the box of radius $L/4$ centered at that vertex. Therefore, it suffices to show that for any such deterministic vertex $u^*$ and path $\gamma$, we have
\[
\text{Var}[X_U \mid F_{I_0}] - \text{Var}[X_U \mid F_{\gamma \cup I_0}] \geq c \log(\delta)].
\]
Recall that $G_0$ denotes the Green’s function on $\tilde{V}_0 \setminus I_0$ and that
\[
\text{Var}[X_U \mid F_{I_0}] - \text{Var}[X_U \mid F_{\gamma \cup I_0}] = \sum_{v \in \gamma} H_{G_0}(U, v; \gamma)G_0(v, U).
\]
Let $S$ be a discrete-time simple random walk (on $\delta \mathbb{Z}^2$) killed on $\partial V_0 \cup I_0$ and $T_U$ be the hitting time of $U$ by $S$. It is easy to see that
\[
G_0(v, U) \geq \mathbb{P}_v(T_U < \infty) \geq c, \quad \forall v \in \gamma,
\]
for some universal constant $c > 0$ (note that this wouldn’t be the case if $U = [a'_0, b'_0]$ and $v \in [c'_0, d'_0]$).

Therefore, it remains to show
\[
H_{G_0}(U, \gamma) \geq c |\log(\delta)|. \tag{3.3}
\]
To this end, we partition $U$ and $\gamma$ as follows. For $n \geq 0$ an integer, we let $Q_n$ be the box of radius $r_n = 2^{-n-2}L$ centered at $u^*$, $A_n = Q_n \setminus Q_{n+1}$, $U_n = U \cap A_n$, and $\gamma_n = \gamma \cap A_n$. We note that $N = \max\{n \geq 1 : r_n \geq 1000\delta\}$ satisfies $N \geq c |\log(\delta)|$. Finally, we claim there exists a universal constant $c > 0$ such that
\[
H_{G_0}(U_n; \gamma_n; \gamma) \geq c, \quad 0 \leq n \leq N.
\]
For the proof, let $u_1 \in U_n$ be such that $2r_n/3 - \delta < |u_1 - u^*|_{\ell\infty} \leq 2r_n/3$, $u_2 \in U_n$ be such that $5r_n/6 \leq |u_2 - u^*|_{\ell\infty} < 5r_n/6 + \delta$, and $Q_{n,2}$ be a box of radius $r_n/12$ centered at $u_2$. With these choices, the distances between $u_1$ and $Q_{n,2}$, and between $\{u_1\} \cup Q_{n,2}$ and $A_n^c$ are of the same order as $r_n$. Therefore, if we let $S$ be a simple random walk (not killed at $\partial V_0$) and $E$ be the event that $S$ hits $Q_{n,2} \cap U$ and then completes a loop around $u^*$ before exiting $A_n$, we have $\mathbb{P}_{u_1}(E) \geq c$. Note that such a walk necessarily contains a path from $U_n$ to $\gamma_n$ which does not hit $\partial V_0$. See Figure 2 for an illustration.

Therefore, letting $T$ be the hitting time of $U_n \cup \gamma_n$ by $S$, and $\zeta$ be the hitting time of $A_n^c$, we have by a last exit decomposition
\[
\mathbb{P}_{u_1}(E) \leq \sum_{u \in U} G_n^*(u_1, u)\mathbb{P}_u(S_T \wedge \zeta \in \gamma_n),
\]
where $G_n^*(u_1, u)$ is the expected number of visits a random walk started at $u_1$ makes to $u$ after hitting $Q_{n,2} \cap U_n$, and before exiting $A_n$. It is immediate that $\mathbb{P}_u(S_T \wedge \zeta \in \gamma_n) \leq H_{G_0}(u, \gamma_n; \gamma)$, and we have that
\[
G_n^*(u_1, u) \leq G_n(u_1, u) \wedge \max\{G_n(u', u) : u' \in Q_{n,2} \cap U_n\},
\]
where $G_n$ is the Green’s function on $A_n \cap \delta \mathbb{Z}^2$. It follows from Lemma 2.5 that $G_n^*$ is uniformly bounded. That is, there exists a universal constant $c$ such that
\[
G_n^*(u_1, u) \leq c, \quad u \in U_n.
\]
Finally, this implies that
\[
H_{G_0}(U_n; \gamma_n; \gamma) \geq \sum_{u \in U_n} \mathbb{P}_u(S_T \wedge \zeta \in \gamma_n) \geq c \mathbb{P}_{u_1}(E) \geq c.
\]
This concludes the proof. \qed
3.2 The alternating boundary case

In this section we prove the lower bound in (1.6). The proof consists of two main claims, both of which are proved using an exploration martingale. Recall that $\tilde{A}_{\delta,0}$ (resp. $\tilde{V}_{\delta,0}$) is the first passage set of $\tilde{\phi}_{\delta}$ above zero (resp. below zero). First, letting $\Pi = \{ z : |\text{Im}(z) - 1/2| \leq 1/4 \}$, we claim that there exists a constant $c_1 = c_1(L, \theta) > 0$ such that

$$\mathbb{P}\left( \tilde{V}_{\delta,0} \cap \Pi = \emptyset \right) \geq c_1. \quad (3.4)$$

Note that (3.4) is not enough to prove that there exists a positive crossing with positive probability. This is due to the fact that an edge with a zero of $\tilde{\phi}_{\delta}$ is closed for both a positive and negative crossing which means we cannot use planar duality. Therefore, to conclude the proof we need to show that conditional on $\{ \tilde{V}_{\delta,0} \cap \Pi = \emptyset \}$, the probability that there is a positive crossing is bounded uniformly away from zero. That is, letting $\mathbb{P}^+$ denote the law of $\tilde{\phi}_{\delta}$ given $F_{\tilde{V}_{\delta,0}}$ (and $E^+$ be the expectation with respect to $\mathbb{P}^+$), there exists a constant $c_2 = c_2(L, \theta) > 0$ such that

$$\mathbb{P}^+\left( [a_\delta, b_\delta] \overset{\tilde{A}_{\delta,0}}{\longleftrightarrow} [c_\delta, d_\delta] \right) \geq c_2 \quad \text{a.s. on } \{ \tilde{V}_{\delta,0} \cap \Pi = \emptyset \}. \quad (3.5)$$

Assuming (3.4) and (3.5), we obtain the lower bound in (1.6):

$$\mathbb{P}\left( [a_\delta, b_\delta] \overset{\tilde{A}_{\delta,0}}{\longleftrightarrow} [c_\delta, d_\delta] \right) \geq \mathbb{P}^+\left( [a_\delta, b_\delta] \overset{\tilde{A}_{\delta,0}}{\longleftrightarrow} [c_\delta, d_\delta], \tilde{V}_{\delta,0} \cap \Pi = \emptyset \right) \geq c_1 c_2.$$

Thus it remains to show (3.4) and (3.5). We note that (3.5) can be obtained from Formula (18) in [LW18], which implies that

$$\mathbb{P}^+\left( [a_\delta, b_\delta] \overset{\tilde{A}_{\delta,0}}{\longleftrightarrow} [c_\delta, d_\delta] \right) = 1 - \exp\left( -2C_{V_{\delta}}^{\text{eff}} \left[ a_\delta, b_\delta, [c_\delta, d_\delta] \right] \right),$$

Figure 2: The event $\mathcal{E}$. $U$ in red, $\gamma$ in blue, and the random walk trace in black.
where $C_{\tilde{V}_0 \backslash \tilde{\nu}_{0,0}}^{\text{eff}}([a_{\delta}, b_{\delta}), [c_{\delta}, d_{\delta})]$ denotes the effective conductance between $[a_{\delta}, b_{\delta})$ and $[c_{\delta}, d_{\delta})$ in $\tilde{V}_0 \backslash \tilde{\nu}_{0,0}$ (see [LW18] for details of the definition of the effective conductance). We provide a proof using the exploration martingale method in order to keep the proof of (1.6) self-contained.

**Proof of (3.4).** For this proof, we consider an exploration martingale $M_t$, introduced in Section 2.3, corresponding to an exploration of $\tilde{E}^{\leq 0}_{\delta}$ from $I_0 = [b_{\delta}, c_{\delta}) \cup [d_{\delta}, a_{\delta})$ with observable corresponding to

$$U = \left\{ u : u \in [a_{\delta}', b_{\delta}') \cup [c_{\delta}', d_{\delta}'), \left| \text{Im}(u) - \frac{1}{2} \right| \leq \frac{3}{8} \right\}.$$ 

The following processes will be useful in the analysis

$$\pi_t = \text{Hm}_t(U, [a_{\delta}, b_{\delta}) \cup [c_{\delta}, d_{\delta})), \quad \mu_t = \text{Hm}_t(U, I_t).$$

Note that we have $M_0 = \theta(\pi_0 - \mu_0)$ and $M_t \leq \theta \pi_t$. Since $\pi_t$ is decreasing,

$$M_t - M_0 \leq \theta \mu_0.$$

Note that by Lemma 2.6 there exists $c > 0$ such that $\text{Hm}(u, I_0) \leq c\delta$ for all $u \in U$, which implies $\mu_0 \leq c$. Let $T = \tilde{D}_0^{\leq 0}(I_0, \Pi)$ be the time that the exploration reaches $\Pi$. On $\{\tilde{\nu}_{0, \Pi} \cap \Pi \neq \emptyset\}$, the set $I_T$ contains a nearest neighbor path $\gamma \subset V_0^1$ crossing one of the strips making up the region

$$\Pi' = \left\{ z : \frac{1}{4} \leq \left| \text{Im}(z) - \frac{1}{2} \right| \leq \frac{3}{8} \right\}.$$

See Figure 3 for an illustration. Therefore, on $\{\tilde{\nu}_{0, \Pi} \cap \Pi \neq \emptyset\}$ the following holds almost surely

$$\text{Var}[X_U | F_{\Pi' \cup I_0}] \leq \max_\gamma \text{Var}[X_U | F_{\gamma \cup I_0}],$$

where the maximum is taking over all deterministic paths $\gamma$ crossing one of the strips that make up $\Pi'$. We claim that for there exists a constant $c > 0$ such that for any such path

$$\text{Var}(X_U) - \text{Var}(X_U | F_{\gamma}) \geq c. \quad (3.6)$$
Assuming this bound for now, we have
\[ \langle M \rangle_T \geq \text{Var}(X_U) - \max_{\gamma} \text{Var}(X_U \mid F_{\gamma}) \geq c, \]
almost surely on \( \{ \tilde{V}_{0,\delta} \cap \Pi \neq \emptyset \} \) (here we used the fact that \( F_{I_0} \) is the trivial \( \sigma \)-field since \( I_0 \subset \partial V_\delta \)). Applying Theorem 2.3 and Proposition 2.4 gives
\[ \mathbb{P}(\tilde{V}_{0,\delta} \cap \Pi \neq \emptyset) \leq \mathbb{P}\left( \sup_{0 \leq t \leq \epsilon} B_t \leq c' \theta \right) \leq 1 - c''. \]

Turning to the proof of (3.6), we have by the invariance principle that \( G(v, U) \geq c \) for any \( v \in \gamma \). Therefore
\[ \text{Var}(X_U) - \text{Var}(X_U \mid F_{\gamma}) = \sum_{v \in \gamma} H_m(U, v; \gamma)G(v, U) \geq c \text{Hm}(U, \gamma). \]
Finally, by Lemma 2.7 there exists a constant \( c > 0 \) such that \( \text{Hm}(u, \gamma) \geq c\delta \) for all \( u \in U \cap \Pi \). Since \( |U \cap \Pi| \geq c\delta^{-1} \), it follows that \( \text{Hm}(U, \gamma) \geq c \). This concludes the proof. \( \Box \)

**Proof of (3.5).** Let \( \tilde{V}_\delta^+ \) be the connected component of \( \tilde{V}_\delta \setminus \tilde{V}_{0,\delta} \) containing \([a_\delta, b_\delta) \cup [c_\delta, d_\delta)\), \( V_\delta^+ = \tilde{V}_\delta^+ \cap \delta \mathbb{Z}^2 \) be the vertices in \( \tilde{V}_\delta^+ \), and \( U = \{c_\delta, d'_\delta) \cap V_\delta^+ \). For any \( u \in U \), let \( \text{adj}(u) \) be the (unique) vertex in \( \{c_\delta, d_\delta) \) adjacent to \( u \). For \( 0 < \epsilon \leq 1 \), let \( u_\epsilon = \epsilon u + (1 - \epsilon) \text{adj}(u) \) and
\[ U_\epsilon = \{u_\epsilon : u \in U\}. \]
That is, \( u_\epsilon \) is the point on the edge connecting \( u \) to \( \partial V_\delta \) that is distance \( \epsilon \) away from \( \partial V_\delta \). Finally, we let \( M_\epsilon \) be the exploration martingale corresponding to an exploration on \( \tilde{E}_\delta^{>0} \) from \( I_0 = [a_\delta, b_\delta) \) with observable \( X_{U_\epsilon} \). As in the proof of (3.4), we will use the processes
\[ \pi_{\epsilon,t} = \text{Hm}^+_{\epsilon,t}(U_\epsilon, \{c_\delta, d_\delta\}), \quad \mu_{\epsilon,t} = \text{Hm}^+_{\epsilon,t}(U_\epsilon, I_t), \]
where \( \text{Hm}^+_{\epsilon,t} \) is the harmonic measure on \( I_t \cup \partial \tilde{V}_\delta^+ \). That is, letting \( \tilde{\zeta}^+ \) be the hitting time of \( \partial \tilde{V}_\delta^+ \) and \( \tau_t \) be the hitting time of \( I_t \), \( \text{Hm}^+_{\epsilon,t}(u, v) = \mathbb{P}_u(\tilde{\zeta}^+ \wedge \tau_t = v) \). Note that \( \text{Hm}^+_{0,t} \) is simply the harmonic measure on \( \partial \tilde{V}_\delta^+ \) so we will write \( \text{Hm}^+ \) in this case instead. Note that \( M_{\epsilon,t} \geq \theta \pi_{\epsilon,t} \) and \( M_{\epsilon,0} = \theta(\mu_{\epsilon,0} + \pi_{\epsilon,0}) \), so we obtain
\[ M_{\epsilon,t} - M_{\epsilon,0} \geq -\theta(\mu_{\epsilon,0} + \pi_{\epsilon,0} - \pi_{\epsilon,t}), \]
with equality if and only if \( M_{\epsilon,t} = M_{\epsilon,\infty} \) and \( I_t \cap U_\epsilon = \emptyset \) (i.e. the exploration has stopped by time \( t \) before hitting \( U_\epsilon \)). In particular,
\[ \left\{ M_{\epsilon,t} - M_{\epsilon,0} > -\theta(\mu_{\epsilon,0} + \pi_{\epsilon,0} - \pi_{\epsilon,t}), 0 \leq t \leq \tilde{D}_\delta^{>0}([a_\delta, b_\delta), U_\epsilon) \right\} \subseteq \{[a_\delta, b_\delta) \not\rightarrow U_\epsilon\}. \] (3.7)
To conclude the proof, we need to lower bound \( \mu_{\epsilon,0} \) and upper bound \( \langle M_\epsilon \rangle_T \).

First, we claim that there exists \( c_1 > 0 \) such that \( \mu_{\epsilon,0} \geq c_1 \epsilon \). Indeed, we have by Lemma 2.7 and the assumption \( \tilde{V}_{0,\delta} \cap \Pi = \emptyset \) that for any \( u \in U \) such that \( |\text{Im}(u) - 1/2| \leq 1/8 \)
\[ \text{Hm}^+(u, [a_\delta, b_\delta)) \geq c_1 \delta. \]
It follows that \( \text{Hm}^+(U, [a_\delta, b_\delta)) \geq c_1 \). Next, by construction a metric graph Brownian motion started at \( u_\epsilon \) will hit \( u \) before \( \text{adj}(u) \) with probability \( \epsilon \). This gives
\[ \mu_{\epsilon,0} = \epsilon \text{Hm}^+(U, [a_\delta, b_\delta)) \geq c_1 \epsilon. \]
Second, for the upper bound on the quadratic variation, we claim that there exists \( c_2 > 0 \) such that

\[
\langle M_\epsilon \rangle_t \leq c_2 \epsilon (\pi_{\epsilon,0} - \pi_{\epsilon,t}), \quad \forall t \leq \tilde{D}_\delta^{\geq 0}([a_\delta, b_\delta), U_\epsilon).
\]

For the proof, note that for such \( t \) we have

\[
\pi_{\epsilon,0} - \pi_{\epsilon,t} = \sum_{v \in \partial I_t} \text{Hm}^+_t(U_\epsilon, v) \text{Hm}^+(v, [c_\delta, d_\delta]),
\]

\[
\langle M_\epsilon \rangle_t = \sum_{v \in \partial I_t} \text{Hm}^+_t(U_\epsilon, v) G^+(v, U_\epsilon),
\]

where \( G^+ \) is the Green’s function of a metric graph Brownian motion killed on \( \partial \tilde{V}_\delta^+ \). To proceed, let \( \tau_{u_\epsilon} \) be the hitting time of \( u_\epsilon \) and recall \( \tilde{\epsilon}^+ \) is the hitting time of \( \partial \tilde{V}_\delta^+ \). We have

\[
G^+(v, U_\epsilon) = \sum_{u \in U} \mathbb{P}_v(\tau_{u_\epsilon} < \tilde{\epsilon}^+) G^+(u_\epsilon, u_\epsilon),
\]

\[
= \epsilon \sum_{u \in U} \mathbb{P}_v(\tau_{u_\epsilon} < \tilde{\epsilon}^+) [\epsilon G^+(u, u) + 4(1 - \epsilon)],
\]

\[
\leq 4 \epsilon \sum_{u \in U} \mathbb{P}_v(\tau_{u_\epsilon} < \tilde{\epsilon}^+),
\]

where in the second equality we used (2.1) to express \( G^+(u_\epsilon, u_\epsilon) \) as a sum of terms involving \( u \) and \( \text{adj}(u) \). Since \( \text{adj}(u) \in \partial \tilde{V}_\delta^+ \), we have \( G^+(u, \text{adj}(u)) = G^+(\text{adj}(u), \text{adj}(u)) = 0 \). In the inequality we used the fact that \( G^+(u, u) \leq 4 \) since \( u \) is adjacent to \( \partial \tilde{V}_\delta^+ \). Finally, assuming from now on that \( \epsilon < 1/2 \), we have

\[
\text{Hm}^+(v, \text{adj}(u)) \geq \frac{1}{2} \mathbb{P}_v(\tau_{u_\epsilon} < \tilde{\epsilon}^+).
\]

It follows that \( \langle M_\epsilon \rangle_t \leq 16 \epsilon (\pi_{\epsilon,0} - \pi_{\epsilon,t}) \). Plugging both bounds into (3.7) we conclude that, conditional on \( \{\tilde{V}_{\delta,0} \cap \Pi = \emptyset\} \),

\[
\left\{ M_{\epsilon,t} - M_{\epsilon,0} > -\theta/c_2 \epsilon (\langle M_\epsilon \rangle_t - c_1 \theta \epsilon), 0 \leq t \leq \tilde{D}_\delta^{\geq 0}([a_\delta, b_\delta), U_\epsilon) \right\} \subseteq \{[a_\delta, b_\delta) \leftarrow U_\epsilon \}.
\]

Applying Theorem 2.3 and Proposition 2.4 (after re-scaling the Brownian motion by \( \epsilon^{-1} \)) we obtain for some \( c > 0 \) (independent of \( \epsilon \)),

\[
\mathbb{P}^+\left([a_\delta, b_\delta) \leftarrow U_\epsilon \right) \geq \mathbb{P}(B_t \geq -\theta(c_2 t + c_1), t \geq 0) \geq c,
\]

where \( B \) is a standard Brownian motion. Letting \( \epsilon \to 0 \) concludes the proof. \( \square \)

4 Estimates of crossing probabilities in the discrete graph

The goal of this section is to prove (1.5) and (1.8). Both proofs are based on the observation that in the discrete graph, when we explore the excursion set \( E_{\delta}^{\geq 0} \) starting from \( \mathcal{I}_0 \), the set \( \partial I_\infty \setminus \partial V_\delta \) is contained in \( E_{\delta}^{< 0} \). We show that in fact, suitably defined averages of the field on this set are bounded away from 0 with high probability. Following the literature on the subject, we call this phenomenon entropic repulsion. It was previously studied in a similar context in [DL18].
Note that since by assumption for any \( \{v \in \mathbb{Z} \} \) is such that dist\( (v, \partial K) \geq 1 \). The reason we write our bound in terms of \( Hm(U,v;D_1) \) instead of \( Hm(U,v;I) \) is that if \( v \in D \setminus D^* \) is such that dist\( (v, \partial K) \) is very small, the variance of \( \phi(v) \) is very small and thus \( \phi(v) \) is close to 0 with high probability.

To this end, we will need some geometric assumptions on the pair \((U,I)\). First, that \( I \setminus D \) contains all points in \( V \) that are separated from \( U \) by \( D \cup \partial K \). Second, that for any \( v \in D^* \), there exists \( w \in (I \setminus D) \cup \partial K \) such that \( v \sim w \). This assumption ensures that given \( \{\phi(v) : v \in I \setminus D\} \), the variance of \( \phi(v) \) is bounded by a universal constant for all \( v \in D \) and the variables \( \{\phi(v) : v \in D\} \) are not too correlated (as every point in \( D \) is close to a point where the value of the field is known). In order to establish a high probability bound on \( Y \), we need to ensure that \( Y \) does not depend too strongly on \( \phi(v) \) for any one \( v \in D^* \). Therefore, we will assume that the harmonic measure on \( D \) does not concentrate on any one point in \( D^* \). In fact, for technical reasons we will need to assume that the harmonic measure on \( D' \) does not concentrate on any one point either. Therefore, we quantify the concentration of the harmonic measures by the following quantity

\[
\xi(U,I) = \frac{1}{Hm(U,D^*;I)} \max \{Hm(U,w;I \setminus D) : w \in D' \}.
\]

Note that since by assumption for any \( v \in D^* \) there exists \( w \in D' \) such that \( v \sim w \) we have

\[
Hm(U,w;I \setminus D) \geq \frac{1}{4} Hm(U,v;I),
\]

and therefore \( Hm(U,v;I) \leq 4\xi(U,I) Hm(U,D^*;I) \) for all \( v \in D^* \).

---

### 4.1 Entropic repulsion for explorations on the discrete graph

In this section we formalize the idea, stated above, that when there is no positive horizontal crossing in the discrete GFF, the values of the field on the outer boundary of \( K_{\delta_0} \) are strictly negative and bounded away from 0. We begin with some setup. For this section, we let \( K \subset \mathbb{Z}^2 \) be compact and simply connected and \( V = K \cap \mathbb{Z}^2 \). For \( U, I \subset V \), we let \( D, D^*, \) and \( D' \) be

\[
D = \{ v \in I : Hm(U,v;I) > 0 \}, \quad D^* = \{ v \in D : \text{dist}(v, \partial K) \geq 1 \}, \quad D' = \{ v \in I : Hm(U,v;I \setminus D) > 0 \},
\]

where \( Hm(\cdot,:I) \) is the harmonic measure on \( I \cup \partial K \) (and likewise for \( I \setminus D \)). That is, \( D \) is the subset of \( \partial I \) that can be reached from \( U \), \( D' \) is the subset of \( \partial I \setminus D \) that can be reached from \( U \) (equivalently from \( D \)), and \( D^* \) is the subset of \( D \) that is distance at least 1 from \( \partial K \). See Figure 4 for an illustration.

As the notation suggests, \( U \) corresponds to the “observable” in the definition of the exploration martingale, while \( I \) corresponds (roughly) to the explored set. Let \( \phi \) be a discrete GFF on \( \partial K \cup V \) with boundary condition \( f : \partial K \to \mathbb{R} \) such that \( |f(v)| \leq \theta \) for all \( v \in \partial K \). Let \( Y \) be given by

\[
Y = \sum_{v \in D} Hm(U,v;I) \phi(v).
\]

![Figure 4: An example of a pair \((U,I)\) with the various parts of \( I \) labeled by color.](image)

We think of \( Y \) as the average value of \( \phi \) on \( D \) as seen from \( U \). We want to show that on the event \( \{\phi(v) < 0 : v \in D\} \), \( Y \leq -c Hm(U,D^*;I) \) with high probability (for some constant \( c > 0 \) independent of \( (U,I) \)). The reason we write our bound in terms of \( Hm(U,D^*;I) \) instead of \( Hm(U,I) \) is that if \( v \in D \setminus D^* \) is such that \( \text{dist}(v, \partial K) \) is very small, the variance of \( \phi(v) \) is very small and thus \( \phi(v) \) is close to 0 with high probability.

To this end, we will need some geometric assumptions on the pair \((U,I)\). First, that \( I \setminus D \) contains all points in \( V \) that are separated from \( U \) by \( D \cup \partial K \). Second, that for any \( v \in D^* \), there exists \( w \in (I \setminus D) \cup \partial K \) such that \( v \sim w \). This assumption ensures that given \( \{\phi(v) : v \in I \setminus D\} \), the variance of \( \phi(v) \) is bounded by a universal constant for all \( v \in D \) and the variables \( \{\phi(v) : v \in D\} \) are not too correlated (as every point in \( D \) is close to a point where the value of the field is known). In order to establish a high probability bound on \( Y \), we need to ensure that \( Y \) does not depend too strongly on \( \phi(v) \) for any one \( v \in D^* \). Therefore, we will assume that the harmonic measure on \( D \) does not concentrate on any one point in \( D^* \). In fact, for technical reasons we will need to assume that the harmonic measure on \( D' \) does not concentrate on any one point either. Therefore, we quantify the concentration of the harmonic measures by the following quantity

\[
\xi(U,I) = \frac{1}{Hm(U,D^*;I)} \max \{Hm(U,w;I \setminus D) : w \in D' \}.
\]

Note that since by assumption for any \( v \in D^* \) there exists \( w \in D' \) such that \( v \sim w \) we have

\[
Hm(U,w;I \setminus D) \geq \frac{1}{4} Hm(U,v;I),
\]

and therefore \( Hm(U,v;I) \leq 4\xi(U,I) Hm(U,D^*;I) \) for all \( v \in D^* \).
Proposition 4.1. Let $\tilde{K}$, $U$, $D$, $I$, and $\phi$ satisfy the conditions above. Let $I^-$ and $I^+$ be disjoint subsets of $I$ such that $D \subset I^-$. Let $\mathcal{E}$ be the following event

$$\mathcal{E} = \{\phi(v) < 0 : v \in I^-\} \cap \{\phi(v) \geq 0 : v \in I^+\}.$$ 

Let $Y$ be as in (4.1). There exists a universal function $r$ satisfying $r(x) \to 0$ as $x \to 0$ and a constant $\Delta = \Delta(\theta) > 0$ such that

$$\mathbb{P}(Y \leq -\Delta \text{Hm}(U, D^*; I) \mid \mathcal{E}) \geq 1 - r(\xi(U, I)),$$

where $\text{Hm}(U, D^*; I) = \sum_{v \in D^*} \text{Hm}(U, v; I)$.

Proof of Proposition 4.1: The proof follows along the same lines as [DL18, Lemma 6], and consists of bounding the conditional mean and variance of $Y$ given $\mathcal{E}$ and the field on $I^+$. We begin with three lemmas that will be used throughout the proof.

Lemma 4.2. Let $A \subset \tilde{K}$ be finite. Let $\{B_1_v\}_{v \in A}$ and $\{B_2_v\}_{v \in A}$ be a sequence of non-empty intervals satisfying

$$\inf B_1_v \leq \inf B_2_v, \quad \sup B_1_v \leq \sup B_2_v, \quad \forall v \in A.$$ 

Let $\tilde{\phi}_1$ and $\tilde{\phi}_2$ be metric graph GFFs on $\tilde{K}$ with boundary conditions $f_1$ and $f_2$ such that $f_1(v) \leq f_2(v)$ for $v \in \partial \tilde{K}$, and let $\phi_1$ and $\phi_2$ be the restrictions of the metric graph GFFs to $A$. Finally, for $j = 1, 2$ let $\mathcal{E}_j$ be the event

$$\mathcal{E}_j = \{\phi_j(v) \in B_j_v, v \in A\}.$$ 

Then the law of $\phi_1$ given $\mathcal{E}_1$ is stochastically smaller than the law of $\phi_2$ given $\mathcal{E}_2$.

Note that in the lemma above, one or both of the endpoints of each interval $B_j,v$ may be infinite.

Proof. The proof is the same as the proof of [DL18, Equation (49)]; we reproduce it here for completeness. Without loss of generality, we assume that $A$ contains $V$ and $\partial \tilde{K}$ (otherwise, we let $B_j,v = \mathbb{R}$ for $v \in V \setminus A$ and $B_j,v = \{f_j(v)\}$ for $v \in \partial \tilde{K} \setminus A$). Let $\nu_1$ be the law of $\phi_1$ given $\mathcal{E}_1$ and $\nu_2$ be the law of $\phi_2$ given $\mathcal{E}_2$. Further, let $\tilde{K}'$ be the closure of an open neighborhood of $\tilde{K}$ such that $\tilde{K}' \cap \mathbb{Z}^2 = V$. Let $\tilde{\phi}'$ be a metric graph GFF on $\tilde{K}'$ with zero boundary condition, say, and $\phi'$ be the restriction of $\tilde{\phi}'$ to $A$. We note that $\mu$, the law of $\phi'$, has density $\mu(dr) = \exp(-H(r))dr$ (where $r$ is an $|A|$-dimensional vector) such that for every $r, r' \in \mathbb{R}^{|A|}$

$$H(r \land r') + H(r \lor r') \leq H(r) + H(r'),$$

where $\land$ and $\lor$ are taken coordinate by coordinate. Additionally, we note $\nu_j$ is simply the law of $\phi'$ conditioned on the event

$$\mathcal{E}_j' = \{\phi'(v) \in B_j,v, v \in A\}.$$ 

For $q > 0$ and $B \subset \mathbb{R}$, we define the function

$$W_B^{(q)}(t) = q \text{dist}(t, B)^4.$$ 

(4.2)

We can then approximate $\nu_1$ and $\nu_2$ by probability measures satisfying the following

$$\nu_j^{(q)}(dr) \propto \exp \left( - \sum_{v \in A} W_B^{(q)}(r_v) \right) \mu(dr),$$
It is clear that for any $t, t' \in \mathbb{R}$ and any $v \in A$ we have

$$W_{B_{1,v}}^{(q)}(t \land t') + W_{B_{2,v}}^{(q)}(t \lor t') \leq W_{B_{1,v}}^{(q)}(t) + W_{B_{2,v}}^{(q)}(t').$$

Therefore, it follows from [Pr74] that $\nu_1^{(q)}$ is stochastically smaller than $\nu_2^{(q)}$ for any $q > 0$. As $q \to \infty$, $\nu_1^{(q)}$ converges weakly to $\nu_j$, so it follows $\nu_1$ is stochastically smaller than $\nu_2$.

Let

$$\mathcal{E}^* = \{ \phi(v) < 0 : v \in D^* \} \cap \{ \phi(v) \geq 0 : v \in I \setminus D \} \cap \{ \phi(v) = 0 : v \in D \setminus D^* \}.$$  

In light of Lemma 4.2 (and the fact that $Y$ is an increasing function of $\phi$) it suffices to prove the conclusion of Proposition 4.1 holds when we replace $E$ with $\mathcal{E}^*$ and the boundary condition $f$ is equal to $\theta$.

In what follows, we let $I \geq 0 = I \setminus D^*$ and $\mathcal{F}^*$ be the $\sigma$-algebra generated by $\mathcal{F}_{I \geq 0}$ and $\mathcal{G}_{D^*}$ (recall that $\mathcal{G}_A$ is the $\sigma$-algebra generated by $\{ \text{sign}(\phi(v)) : v \in A \}$). In particular, $\mathcal{E}^*$ is measurable with respect to $\mathcal{F}^*$.

We are ready to begin the moment analysis. We start with the variance as the argument is easier.

**Lemma 4.3.**

$$\text{Var}[Y \mid \mathcal{F}^*] \leq \text{Var}[Y \mid \mathcal{F}_{I \geq 0}] \quad \text{a.s.}$$

**Proof.** The proof is the same as that of [DL18, Equation (56)]; we reproduce it here for completeness. Let $Y = (\phi(v))_{v \in D}$, $g : I \geq 0 \to [0, \infty)$ be any function on $I \geq 0$ and $s : D^* \to \{-1, 0, 1\}$ be any assignment of signs to the vertices of $D^*$. Let $\mu$ be the law of $Y$ given $\phi|_{I \geq 0} = g$, and $\nu$ be the law of $Y$ given $\phi|_{I \geq 0} = g$ and $\text{sign}(\phi)|_{D^*} = s$. Let $m$ and $\Sigma$ be the mean and variance of $\mu$ and note that $\Sigma$ does not depend on $g$. Recall $\mu(dr) \propto \exp\left(-\frac{1}{2}(r-m)\Sigma^{-1}(r-m)dr\right)$. For $q > 0$, we approximate $\nu$ by a probability measure $\nu^{(q)}$ satisfying the following

$$\nu^{(q)} \propto \exp\left(-q \sum_{v \in D^*} r_q \cdot \text{sign}(r_v) \neq r_v \mu(dr).$$

Since the second derivative of $f_a(t) = t^{4 \cdot \text{sign}(t)} \neq a$ is non-negative, we have that $\nu^{(q)}$ is of the form $\nu^{(q)}(dr) = \exp(-H(r))dr$ where $\inf \text{Hess}(H)(r) \geq \frac{1}{2}\Sigma^{-1}$. Therefore, by the Brascamp-Lieb inequality [BL76], for a random vector $X^{(q)} \sim \nu^{(q)}$, a random vector $X \sim \mu$, and any $l \in \mathbb{R}^{|D|}$, we have $\text{Var}[l \cdot X^{(q)}] \leq \text{Var}[l \cdot X]$. As $q \to \infty$, $\nu^{(q)}$ converges weakly to $\nu$, so we have

$$\text{Var}[l \cdot Y \mid \mathcal{F}^*] \leq \text{Var}[l \cdot Y \mid \mathcal{F}_{I \geq 0}] \quad \text{a.s.}$$

Since $Y$ is of the form $l \cdot Y$, the conclusion follows.

We obtain from this

**Corollary 4.4.**

$$\text{Var}[Y \mid \mathcal{F}^*] \leq 16\xi(U, I) \text{Hm}(U, D^*; I)^2 \quad \text{a.s.}$$

**Proof.** Let $G_{\tilde{K} \setminus I \geq 0}$ is the Green's function on $\tilde{K} \setminus (I \geq 0)$. By the Markov property of $\phi$, we have

$$\text{Var}[Y \mid \mathcal{F}_{I \geq 0}] = \sum_{v, v' \in D} \text{Hm}(U, v; I) \text{Hm}(U, v'; I) \text{Cov}(\phi(v), \phi(v') \mid \mathcal{F}_{I \geq 0})$$

$$= \sum_{v, v' \in D} \text{Hm}(U, v; I) \text{Hm}(U, v'; I) G_{\tilde{K} \setminus I \geq 0}(v', v).$$
Note that since $D \setminus D^* \subset I_{20},$ $G_{K \setminus I_{20}}(v', v)$ vanishes unless $v, v' \in D^*$. Further, since by assumption $D^*$ separates $U$ from $I \setminus D$ in $\tilde{K} \setminus I_{20},$ we have

$$
\sum_{v' \in D^*} \text{Hm}(U, v' ; I) G_{\tilde{K} \setminus I_{20}}(v', v) = G_{\tilde{K} \setminus I_{20}}(U, v), \ \forall v \in D^*,
$$

Therefore,

$$\text{Var}[Y \mid I_{20}] = \sum_{v \in D^*} \text{Hm}(U, v; I) G_{\tilde{K} \setminus I_{20}}(U, v).$$

Note that since each $v \in D^*$ has a neighbor in $I_{20},$ $G_{\tilde{K} \setminus I_{20}}(v, D^*) \leq 4$ and therefore

$$G_{\tilde{K} \setminus I_{20}}(U, D^*) \leq 4 \text{Hm}(U, D^*; I).$$

This gives

$$\text{Var}[Y \mid I_{20}] \leq 4 \text{Hm}(U, D^*; I) \sup_{v \in D^*} \text{Hm}(U, v; I) \leq 16 \xi(U, I) \text{Hm}(U, D^*; I)^2.$$

□

To conclude, we need a high-probability (given $\mathcal{E}^*$) bound on $\mathbb{E}[Y \mid F^*]$. We begin with an auxiliary lemma.

**Lemma 4.5.** There exists a universal continuous, increasing function $g : [0, \infty) \to (0, \infty)$ such that for any $w \in \tilde{K}$ and any $\epsilon > 0$ the following holds. Let $\mathcal{E}_w$ be the event

$$\mathcal{E}_w = \{ \hat{\phi}(w) = 0 \} \cap \{ \hat{\phi}(u) > 0 \forall u \in \tilde{K}, \ \text{dist}(u, \partial \tilde{K} \cup \{w\}) \geq \epsilon \}.$$

For all $v \in \tilde{K}$ we have

$$\mathbb{E}[\hat{\phi}(v) \mid \mathcal{E}_w] \leq g(|v - w|) + \theta.$$

**Proof.** We follow the proof of [DL18, Lemma 6]. First, by an argument similar to the proof of [DL18, Equation (48)], there exists a function $R : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $a, b > 0$, there exists a function $h_{a,b}$ that is harmonic on $\tilde{Z}^2 \setminus \{w\}$ and satisfies

$$|h_{a,b}(u) - a \log (|u - w| + 2) - b| \leq R(a), \ \forall u \in \tilde{Z}^2.$$

Let $\tilde{\varphi}_{a,b}$ be a Gaussian free field on $\tilde{K}$ with boundary condition $h_{a,b}$ on $\partial \tilde{K} \cup \{w\}$, and $\tilde{\varphi}_{a,b}$ be the restriction of $\tilde{\varphi}_{a,b}$ to $V$. Let $Q$ be a countable, dense subset of $\tilde{K}$, and $\{Q_n\}_{n=1}^\infty$ be an increasing sequence of finite subsets of $Q$ such that $\bigcup_{n=1}^\infty Q_n = A$. Let $a, b > 0$ be such that $h_{a,b}(u) > 0$ for all $u \in \tilde{Z}^2$. Applying Lemma 4.2 with $A = Q_n, B_{1,v} = B_{2,v} = (0, \infty)$ if $\text{dist}(v, \partial \tilde{K} \cup \{w\}) \geq \epsilon$, and $B_{1,v} = (-\infty, \infty)$ and $B_{2,v} = (0, \infty)$ otherwise and taking the limit as $n \to \infty$ gives

$$\mathbb{E}[\hat{\phi}(v) \mid \mathcal{E}_w] \leq \mathbb{E}[\tilde{\varphi}_{a,b+\theta}(v) \mid \tilde{\varphi}_{a,b+\theta}(u) > 0, \forall u \in \tilde{K}], \ \forall v \in \tilde{K}.$$

Next, we claim there exist universal constants $c_a, c_b > 0$ such that for all $a \geq c_a$ and $b \geq c_b$,

$$\mathbb{P}(\tilde{\varphi}_{a,b}(u) > 0, \forall u \in \tilde{K}) \geq \frac{1}{2}. \quad (4.3)$$
Indeed, it is straightforward to check by a union bound that there exist universal constants \( c'_a, c'_b > 0 \) such that for all \( a \geq c'_a \) and \( b \geq c'_b \)

\[
\mathbb{P}\left( \varphi_{a,b}(u) \geq \frac{h_{a,b}(u)}{2}, \forall u \in V \right) \geq \frac{3}{4}.
\]

By an abuse of notation, we let \( \mathcal{F}_V = \{ \phi_{a,b}(v) : v \in V \} \). Recall from the introduction that given \( \mathcal{F}_V \), for each segment \( e(u, v) \subset \tilde{K} \) with endpoints \( u, v \in V \cup \partial \tilde{K} \cup \{ w \} \) the restriction of \( \tilde{\varphi}_{a,b} \) to \( e(u, v) \) is a Brownian bridge of length \( 2|u - v| \) of a Brownian motion with variance \( 2 \) at time \( 1 \). Thus, applying [BS02, Formula 1.3.8] we see the following holds almost surely on \( \{ \tilde{\varphi}_{a,b}(u) \geq h_{a,b}(u)/2, \forall u \in V \}, \)

\[
\mathbb{P}(\tilde{\varphi}_{a,b}(x) > 0, \forall x \in e(u, v) | \mathcal{F}_V) = 1 - e^{-\frac{1}{2}|u-v|}\varphi_{a,b}(u)\tilde{\varphi}_{a,b}(v) \geq 1 - e^{-\frac{1}{2}h_{a,b}(u)h_{a,b}(v)}.
\]

Therefore, there exist universal constants \( c''_a, c''_b > 0 \) such that for all \( a \geq c''_a \) and \( b \geq c''_b \)

\[
\mathbb{P}\left( \tilde{\varphi}_{a,b}(u) > 0, \forall u \in \tilde{K} | \varphi_{a,b}(u) \geq \frac{h_{a,b}(u)}{2}, \forall u \in V \right) \geq \frac{2}{3}.
\]

Combining this with the second-to-last display and letting \( c_j = c'_j \vee c''_j \) for \( j \in \{ a, b \} \) gives (4.3). Note that \( \text{Var}[\tilde{\varphi}_{a,b}(v)] \leq c \log(|v - w| + 2) \) for some universal constant \( c > 0 \), and by assumption \( \mathbb{E}[\tilde{\varphi}_{a,b}(v)] = h_{a,b}(v) \leq a \log(|v - w| + 2) + b + R(a) \) so it follows that

\[
\mathbb{E}[\tilde{\varphi}_{ca,c\theta}(v) | \tilde{\varphi}_{ca,c\theta}(u) > 0, \forall u \in \tilde{K}] \leq g(|v - w|) + \theta,
\]

for some universal (continuous, increasing) function \( g \) (which can be calculated explicitly in terms of \( c, c_a \), and \( c_b \)).

The following corollary gives the high probability bound on \( \mathbb{E}[Y | \mathcal{F}^*] \).

**Corollary 4.6.** There exists a constant \( \Delta_1 = \Delta_1(\theta) > 0 \) and a universal function \( r_1 \) which satisfies \( r_1(x) \to 0 \) as \( x \to 0 \), such that the following holds

\[
\mathbb{P}(\mathbb{E}[Y | \mathcal{F}^*] \leq -\Delta_1 \text{Hm}(U,D^*; I) | \mathcal{E}) \geq 1 - r_1(\xi(U, I)).
\]

Before proving the corollary, we show how it implies Proposition 4.1. Let \( \Delta = \Delta_1/2 \) and \( X = \mathbb{P}(Y \leq -\Delta \text{Hm}(U, I) | \mathcal{F}^*) \). We have

\[
\mathbb{P}(Y \leq -\Delta \text{Hm}(U, D^*; I) | \mathcal{E}^*) = \mathbb{E}[X | \mathcal{E}^*].
\]

Additionally, we have by Chebyshev’s inequality and Corollary 4.4 that there exists a function \( r_2 \) such that \( r_2(x) \to 0 \) as \( x \to 0 \) and

\[
X \geq 1 - r_2(\xi(U, I)), \text{ a.s. on } \{ \mathbb{E}[Y | \mathcal{F}^*] \leq -\Delta_1 \text{Hm}(U, D^*; I) \}.
\]

It follows that

\[
\mathbb{P}(Y \leq -\Delta \text{Hm}(U, D^*; I) | \mathcal{E}^*) \geq 1 - r_1(\xi(U, I)) - r_2(\xi(U, I)).
\]
Proof of Corollary 4.6. We have

\[ E[Y \mid \mathcal{F}^*] = \sum_{v \in D} \text{Hm}(U, v; I)E[\phi(v) \mid \mathcal{F}^*]. \]

Of course, on \( \mathcal{E}^* \) we have \( E[\phi(v) \mid \mathcal{F}^*] = 0 \) for all \( v \in D \setminus D^* \) almost surely. By Lemma 4.2, for \( v \in D^* \) the following holds almost surely on \( \mathcal{E}^* \)

\[ E[\phi(v) \mid \mathcal{F}^*] \leq E[\phi(v) \mid \mathcal{F}_{I \geq 0}, \mathcal{G}_v] \leq E[\phi(v) \land 0 \mid \mathcal{F}_{I \geq 0}]. \]

Note that given \( \mathcal{F}_{I \geq 0} \), \( \phi(v) \) has a normal distribution with variance at least 1. Therefore, letting \( m_v = E[\phi(v) \mid \mathcal{F}_{I \geq 0}] \), there exist universal constants \( a, b > 0 \) such that

\[ E[\phi(v) \land 0 \mid \mathcal{F}_{I \geq 0}] \leq -ae^{-bm_v^2}. \]

Thus, it suffices to show that there exists a constant \( c = c(\theta) > 0 \) such that the following holds

\[ P \left( \sum_{v \in D^*} \text{Hm}(U, v; I)m_v \leq c \text{Hm}(U, D^*; I) \mid \mathcal{E}^* \right) \geq 1 - r_1(\xi(U, I)). \]  

(4.4)

Indeed, since we assume \( \phi \) has positive boundary condition, \( m_v \geq 0 \) for all \( v \in D^* \) almost surely on \( \mathcal{E}^* \). Therefore, by Markov's inequality \( \sum_{v \in D^*} \text{Hm}(U, v; I)m_v \leq c \text{Hm}(U, D^*; I) \) implies

\[ \sum_{v \in D^*} \text{Hm}(U, v; I)^1_{m_v \leq 2c} \geq \frac{1}{2} \text{Hm}(U, D^*; I), \]

and consequently

\[ E[Y \mid \mathcal{F}^*] \leq -\frac{ae^{-4bc^2}}{2} \text{Hm}(U, D^*; I). \]

Turning to the proof of (4.4), let \( Y' \) be the following random variable

\[ Y' = \sum_{w \in D'} \text{Hm}(U, w; I \setminus D^*)\phi(w) = \sum_{v \in D^*} \sum_{w \in D'} \text{Hm}(U, v; I)\text{Hm}(v, w; I \setminus D^*). \]

(4.5)

By the Markov property of \( \phi \),

\[ m_v = \sum_{w \in \partial K \cup I \geq 0} \text{Hm}(v, w; I \setminus D^*)\phi(w). \]

Since \( \phi \) vanishes on \( D \setminus D^* \) almost surely on \( \mathcal{E}^* \) and we assumed that \( \text{Hm}(U, \cdot; I \setminus D) \) is supported on \( D' \), we have

\[ m_v = \sum_{w \in \partial K \cup D'} \text{Hm}(v, w; I \setminus D^*)\phi(w). \]

Since we assumed \( \phi \) has boundary condition equal to \( \theta \),

\[ m_v = \theta \text{Hm}(v, \partial \tilde{K}; I \setminus D^*) + \sum_{w \in D'} \text{Hm}(v, w; I \setminus D^*)\phi(w), \]

(4.6)

where \( \text{Hm}(v, \partial \tilde{K}; I \setminus D^*) \) is the probability that metric graph Brownian motion started at \( v \) hits \( \partial \tilde{K} \) before hitting \( I \setminus D^* \). By (4.5) and (4.6),

\[ \sum_{v \in D^*} \text{Hm}(U, v; I)m_v - Y' = \theta \sum_{v \in D} \text{Hm}(U, v; I)\text{Hm}(v, \partial \tilde{K}; I \setminus D^*). \]
In particular, we have
\[ \sum_{v \in D^*} \text{Hm}(U, v; I)m_v \leq Y' + \theta \text{Hm}(U, D^*; I). \]

Therefore, to prove (4.4) it suffices to show
\[ \Pr \left( Y' \leq c \text{Hm}(U, D^*; I) \mid \mathcal{E}^t \right) \geq 1 - r_1(\xi(U, I)). \]

We show this by bounding the conditional mean and variance of \( Y' \). First, by Lemma 4.2, we can replace \( \mathcal{E}^t \) with \( \mathcal{E}' = \{ \phi(v) = 0, v \in D \} \cap \{ \phi(v) > 0, v \in I \setminus D \} \). Next, we let \( \mathcal{F}' \) be the \( \sigma \)-algebra generated by \( \mathcal{F}_D \) and \( \mathcal{G}_{I \setminus D} \). Straightforward adaptations of the proofs of Lemma 4.3 and Corollary 4.4 give
\[ \text{Var}[Y' \mid \mathcal{F}'] \leq \text{Var}[Y' \mid \mathcal{F}_D] \leq 4\xi(U, I) \text{Hm}(U, D^*; I)^2. \]

Finally, Lemma 4.5 and Lemma 4.2 (and the fact that every \( w \in D' \) is adjacent to a vertex in \( D \)) imply that the following holds almost surely on \( \mathcal{E}' \)
\[ \mathbb{E}[\phi(w) \mid \mathcal{F}'] \leq g(1) + \theta, \quad \forall w \in D'. \]

The conclusion then follows easily. \( \square \)

### 4.2 Zero boundary case

In this section we prove (1.5). We let \( M \) be an exploration martingale, as introduced in Section 2.3, corresponding to an exploration on \( E^\geq 0 \) from \( I_0 = [c_\delta, d_\delta) \) with \( U = [a_{\delta}', b_{\delta}') \). Here the exploration is on \( E^\geq 0 \), but with the understanding that we do not explore any vertices on \( \partial V_\delta \setminus I_0 \) (equivalently, the exploration is on \( E^\geq 0 \) with the understanding that \( I_1 = [c_\delta', d_\delta') \)). With this setup, we have \( M_0 = 0 \). For \( k \geq 1 \), let \( J_k \) be the set obtained by adding to \( I_k \) all points in \( V_\delta \) separated from \( U \) by \( I_k \cup \partial V_\delta \) (i.e. \( J_k \) is obtained by filling in any holes in \( I_k \)). Let \( D_k = \{ v \in J_k : \text{Hm}(U, v; J_k) > 0 \} \). It is straightforward that \( D_k \subset I_k \). Since a vertex is only explored if it is in \( [c_\delta', d_\delta') \) or it neighbors a vertex in \( I_{k-1} \) with positive value of \( \phi_\delta \), we conclude that for each \( v \in D_k \), there exists \( w \) such that \( v \sim w \) and \( w \in (J_k \setminus D_k) \cup \partial V_\delta \). Therefore, the pair \( (U, J_k) \) satisfies the two geometric assumptions in Section 4.1. The main step of the proof consists of bounding \( \xi_k = \xi(U, J_k) \). In particular, we claim that there exists \( c > 0 \) such that
\[ \sup\{\xi_k : k \geq 1\} \leq \frac{c}{|\log(\delta)|}, \quad \text{a.s.} \quad (4.7) \]

Before proving (4.7), we show how it implies (1.5). To simplify notation, we will write \( \{D^\geq 0(I_0, U) = \infty\} \) for the event that \( [a_{\delta}', b_{\delta}') \) is not connected to \( [c_\delta', d_\delta') \) in \( E^\geq 0 \). Assuming (4.7), Proposition 4.1 implies that there exists \( \Delta > 0 \) such that for any \( \epsilon > 0 \), there exists \( \delta_0 = \delta_0(\epsilon, L) > 0 \) such that
\[ \Pr(M_\infty \leq -\Delta \text{Hm}(U, I_\infty) \mid D^\geq 0(I_0, U) = \infty) \geq 1 - \epsilon, \quad \forall \delta \leq \delta_0. \quad (4.8) \]

To conclude, we need to upper bound the quadratic variation. Recall that \( \text{Hm}_t \) is the harmonic measure on \( I_t \cup \partial V_\delta \). By (2.4)
\[ \langle M \rangle_t = \sum_{v \in I_t} \text{Hm}_t(U, v)G(v, U) \leq 4 \text{Hm}(U, I_t), \quad (4.9) \]

where we have used the fact that a random walk started on \( U \) has probability at least 1/4 of hitting \( \partial V_\delta \) before returning to \( U \), which implies \( G(v, U) \leq 4 \) for all \( v \in \bar{V}_\delta \).

Finally, by Lemma 2.7, \( \text{Hm}(U, I_0) \geq c \). Combining this with (4.9) and applying Theorem 2.3, we see that there exists \( \epsilon = \epsilon(L) > 0 \) such that
\[ \Pr(\exists t \geq 0 \text{ s.t. } M_t \leq -\Delta \text{Hm}(U, I_t)) \leq \Pr \left( \exists t \geq 0 \text{ s.t. } B_t \leq -\Delta \left( c \vee t \frac{1}{4} \right) \right) \leq 1 - 2\epsilon. \]

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Together with (4.8), the last display implies
\[ \mathbb{P}(D^{>0}(I_0, U) = \infty) \leq 1 - \epsilon, \quad \forall \delta \leq \delta_0. \]

**Proof of (4.7).** Let \( D_k = \{ v \in \mathcal{J}_k : \text{Hm}_k(U, v) > 0 \} \), \( \mathcal{J}'_k = \mathcal{J}_k \setminus D_k \), and \( \text{Hm}_k' \) be the harmonic measure on \( \mathcal{J}'_k \cup \partial V_\delta \). Let \( S \) be a random walk on \( \delta \mathbb{Z}^2 \) (not killed on \( \partial V_\delta \)), \( \tau_k = \min\{ n \geq 1 : S_n \in U \cup \mathcal{J}'_k \cup \partial V_\delta \} \), and \( G'_k \) be the Green’s function of the random walk killed on \( \partial V_\delta \cup \mathcal{J}'_k \). We have for \( u \in U \) and \( v \in \partial \mathcal{J}'_k \)
\[
\text{Hm}_k(u, v) = \sum_{u' \in U} G'_k(u, u') \mathbb{P}_{u'}(S_{\tau_k} = v).
\]

By reversibility, \( \mathbb{P}_{u'}(S_{\tau_k} = v) = \mathbb{P}_v(S_{\tau_k} = u') \). Noting also that \( G'_k \) is symmetric, summing over \( u \) gives
\[
\text{Hm}_k(U, v) = \sum_{u' \in U} \mathbb{P}_v(S_{\tau_k} = u') G'_k(u', U) \leq 4 \mathbb{P}_v(S_{\tau_k} \in U),
\]
where we have used the fact that \( G'_k(u, U) \leq G(u, U) \leq 4 \). There exists a universal constant \( p > 0 \) such that for any integer \( n \geq 0 \) a random walk started at \( v \) will complete a loop around \( B(v, 2^n) \) before exiting \( B(v, 2^{n+1}) \) (here \( B(v, r) \) is the open Euclidean ball of radius \( r \) around \( v \)) with probability at least \( p \). Since any loop around \( v \in \mathcal{J}'_k \) that is contained in \( V \) must intersect \( \mathcal{J}'_k \), it follows that there exist \( A, \alpha > 0 \) such that the following holds for all \( k \geq 1 \) and \( v \in \partial \mathcal{J}'_k \),
\[
\mathbb{P}_v(S_{\tau_k} \in U) \leq (1 - p)^{\lfloor \log_2(\text{dist}(v, U)/\delta) \rfloor} \leq A \left( \frac{\delta}{\text{dist}(v, U)} \right)^\alpha.
\]

It remains to lower bound \( \text{Hm}(U, \mathcal{J}_k) \) in terms of \( \text{dist}(\mathcal{J}_k, U) \). We have already noted that there exists \( c > 0 \) such that \( \text{Hm}(U, \mathcal{J}_k) \geq \text{Hm}(U, I_0) \geq c \) for all \( k \). On the other hand, a simple adaptation of the proof of (3.3) shows that there exists a constant \( c' > 0 \) such that whenever \( \text{dist}(\mathcal{J}_k, U) \leq (L \wedge 1)/2 \) we have
\[
\text{Hm}(U, \mathcal{J}_k) \geq c' \log \left( \frac{L \wedge 1}{\text{dist}(U, I_k) + \delta} \right).
\]

Combining the lower bounds on \( \text{Hm}(U, \mathcal{J}_k) \) with the upper bound on \( \text{Hm}_k(U, v) \) gives the desired conclusion. \( \square \)

### 4.3 Alternating boundary case

In this section, we prove (1.8). We recall the inclusion
\[
\left\{ (a_\delta, b_\delta) \xrightarrow{\tilde{\phi}_\delta \geq 0} c_\delta, d_\delta \right\} \subset \left\{ (a_\delta, b_\delta) \xrightarrow{\phi_\delta \geq 0} c_\delta, d_\delta \right\}.
\]

Therefore, it suffices to find an event \( \mathcal{E} \) such that
\[
\mathcal{E} \subset \left\{ (a_\delta, b_\delta) \xrightarrow{\tilde{\phi}_\delta \geq 0} c_\delta, d_\delta \right\} \setminus \left\{ (a_\delta, b_\delta) \xrightarrow{\phi_\delta \geq 0} c_\delta, d_\delta \right\},
\]
and \( \mathbb{P}(\mathcal{E}) \geq c \).

As usual, we provide a sketch of the proof here, leaving the technical arguments to the end of the section. By (3.4) we can assume that \( \bar{V}_{\delta,0} \cap \Pi = \emptyset \) and work conditional on \( \mathcal{F}_{\bar{V}_{\delta,0}} \). We let \( \mathbb{P}^+ \) denote the law of \( \tilde{\phi}_\delta \) given \( \mathcal{F}_{\bar{V}_{\delta,0}} \). The first step of the proof is to upper bound the probability that \( \tilde{\phi}_\delta \) contains a
positive horizontal crossing. We claim that there exists a constant \( c > 0 \) such that the following holds almost surely on \( \{ \tilde{\mathcal{V}}_{\delta,0} \cap \Pi = \emptyset \} \)

\[
\mathbb{P}^+ \left( \left[ a_\delta, b_\delta \right] \xrightarrow{\tilde{\partial}} [c_\delta, d_\delta] \right) \geq c. \tag{4.10}
\]

Consequently, we assume \( [a_\delta, b_\delta] \) is not connected to \([c_\delta, d_\delta]\) and work conditionally on \( \mathcal{F}_{\tilde{\mathcal{V}}_{\delta,0} \cup \tilde{\mathcal{V}}_{\delta,0}^+} \). We let \( \mathbb{P}^{0,l} \) be the law of \( \tilde{\partial} \delta \) given \( \mathcal{F}_{\tilde{\mathcal{V}}_{\delta,0} \cup \tilde{\mathcal{V}}_{\delta,0}^+} \) and \( \mathbb{E}^{0,l} \) be the expectation with respect to \( \mathbb{P}^{0,l} \). We claim that there exists a further constant \( c' > 0 \) such that the following holds almost surely on \( \{ \tilde{\mathcal{V}}_{\delta,0} \cap \Pi = \emptyset \} \cap \left\{ [a_\delta, b_\delta] \xrightarrow{\tilde{\partial}} [c_\delta, d_\delta] \right\} \)

\[
\mathbb{P}^{0,l} \left( \tilde{\mathcal{A}}^{l}_{\delta,0} \xrightarrow{\tilde{\partial}} [c_\delta, d_\delta] \right) \geq c'. \tag{4.11}
\]

This concludes the proof. We now turn to proving the two technical claims. As in the case of (3.5), (4.10) can be obtained from Formula (18) in [LW18]. However, we provide a proof using exploration martingales to keep the proof of (1.8) self-contained.

### 4.3.1 Proof of (4.10)

As in Section 3.2, we let \( \tilde{V}^{+}_{\delta} \) be the connected component of \( \tilde{V}_{\delta} \setminus \tilde{\mathcal{V}}_{\delta,0} \) containing \([a_\delta, b_\delta]\) and \([c_\delta, d_\delta]\), \( H_{\delta}^+ \) be the harmonic measure on \( \tilde{V}^{+}_{\delta} \), and \( G^+ \) be the Green’s function of a metric graph Brownian motion killed on \( \partial \tilde{V}^{+}_{\delta} \). The proof proceeds by analyzing an exploration martingale \( M \), as introduced in Section 2.3, corresponding to an exploration of \( \tilde{E}_{\delta}^{0} \) from \( \mathcal{I}_0 = [a_\delta, b_\delta] \cup [c_\delta, d_\delta] \) with observable \( X_U \) corresponding to the following set \( U \). For each \( v \in \partial \tilde{\mathcal{V}}_{\delta,0} \), we let \( \eta_v = \text{dist}(v, \partial \mathbb{Z}^2) \). Since \( |\partial \tilde{\mathcal{V}}_{\delta,0}| < \infty \) and \( \partial \tilde{\mathcal{V}}_{\delta,0} \cap V_{\delta} = \emptyset \) almost surely, we have

\[
\eta = \min \{ \eta_v : v \in \partial \tilde{\mathcal{V}}_{\delta,0} \} > 0, \quad \text{a.s.}
\]

Then, we let \( U \) be the following set

\[
U = \left\{ u : u \in \tilde{V}^{+}_{\delta}, \left| \text{Re}(u) - \frac{L}{2} \right| \leq \frac{3L}{8}, \text{dist}(u, \tilde{\mathcal{V}}_{\delta,0}) = \frac{\eta}{2} \right\}.
\]

The following processes will be useful in the analysis

\[
\pi_t = H^{+}_{t}(U, \tilde{\mathcal{V}}_{\delta,0}), \quad \mu_t = H^{+}_{t}(U, \mathcal{I}_t),
\]

where as usual \( H^{+}_{t} \) is the harmonic measure on \( \mathcal{I}_t \cup \partial \tilde{V}^{+}_{\delta} \). We have \( M_t \geq 0 \) for all \( t \) and \( M_0 = \theta \mu_0 \), so in particular \( M_t - M_0 \geq -\theta \mu_0 \). To conclude the proof, we need to upper bound \( \mu_t \) and lower bound the quadratic variation of \( M \). In particular, we claim that there exist constants \( c, c' > 0 \) such that the following holds

\[
\mu_0 \leq c \frac{\eta}{\delta},
\]

\[
\langle M \rangle_{\infty} \geq c' \left( \frac{\eta}{\delta} \right)^2, \quad \text{a.s. on} \left\{ (a_\delta, b_\delta) \xrightarrow{\tilde{\partial}} [c_\delta, d_\delta] \right\}.
\]

Before proving the claim, we note that it implies by an application of Theorem 2.3, Proposition 2.4, and a rescaling of the Brownian motion

\[
\mathbb{P} \left( (a_\delta, b_\delta) \xrightarrow{\tilde{\partial}} [c_\delta, d_\delta] \right) \leq \mathbb{P} \left( \inf_{0 \leq t \leq c' \frac{\eta}{\delta \theta}} B_t \geq -c' \frac{\eta}{\delta} \right) = \mathbb{P} \left( \inf_{0 \leq t \leq c'} B_t \geq -c \theta \right) \leq 1 - c''.
\]

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For the proof, consider the electric network with vertex set \( W = \mathbb{Z}^2 \cup U \cup \partial \tilde{V}^+_\delta \) where two vertices \( u, v \in W \) are connected with an edge of conductance \( C_{u,v} = (\delta^2 |u - v|)^{-1} \) if there exists a path in \( \mathbb{Z}^2 \) connecting \( u \) to \( v \) which does not contain any other points in \( W \). For this section, we will write \( u \sim v \) if \( u \) and \( v \) are connected in this network, and we let \( C_u = \sum_{v \sim u} C_{u,v} \). Note that \( G^+ \) (as defined above) is the Green’s function of the continuous time simple random walk on \( W \) killed on \( \partial \tilde{V}^+_\delta \). To simplify some definitions, we take \( S \) to be a discrete-time simple random walk on \( W \) (not killed on \( \partial \tilde{V}^+_\delta \)).

We now turn to proving the upper bound on \( \mu_0 \). Let \( \tau_0 = \min \{ n \geq 1 : S_n \in U \cup \partial \tilde{V}^+_\delta \} \) be the hitting time of \( U \cup \partial \tilde{V}^+_\delta \). We have by a last exit decomposition

\[
Hm^+(u, I_0) = \sum_{u' \in U} \sum_{v \in I_0} C_{u'} G^+(u, u') \mathbb{P}_{u'}(S_{\tau_0} = v),
\]

where we have used the fact that \( C_{u'} \mathbb{P}_{u'}(S_{\tau_0} = v) = C_v \mathbb{P}_v(S_{\tau_0} = u') \) and \( G^+(u, u') = G^+(u', u) \). Summing over \( u \in U \) gives

\[
\mu_0 = \sum_{u' \in U} \sum_{v \in I_0} C_v \mathbb{P}_v(S_{\tau_0} = u') G^+(u', U) \leq \left[ \max_{u' \in U} G^+(u', U) \right] \sum_{v \in I_0} C_v \mathbb{P}_v(S_{\tau_0} \in U).
\]

It is straightforward to show that \( G^+(u', U) \leq 2\eta/\delta \) for all \( u' \in U \), and it is clear that \( C_v = 1 \) for all \( v \in I_0 \) such that \( \mathbb{P}_v(S_{\tau_0} \in U) > 0 \). Further, we note that \( \mathbb{P}_v(S_{\tau_0} \in U) \leq c\delta \) since it is bounded by the probability that a random walk on \( \delta \mathbb{Z} \) (started from the origin) reaches \( L/8 \) before returning to zero. It follows that \( \mu_0 \leq c\eta/\delta \) as claimed.

Next, we turn to proving the lower bound on the quadratic variation. Begin by noting that if there exists a horizontal crossing, then \( I_\infty \) contains a nearest neighbor path \( \gamma \subset V^+_\delta \) crossing the following strip

\[
\Pi' = \left\{ z : \left| \text{Re}(z) - \frac{L}{2} \right| \leq \frac{L}{4} \right\}.
\]

We claim that for any such \( \gamma \)

\[
\text{Var}^+(X_U) - \text{Var}^+(X_U \mid F_\gamma) = \sum_{v \in \gamma} Hm^+(U, v; \gamma) G^+(v, U) \geq c \left( \frac{\eta}{\delta} \right)^2.
\]

For the proof, note that there exists \( c > 0 \) such that \( Hm^+(v, U) \geq c \) for all \( v \in \gamma \), where \( c \) is independent of \( \gamma \) and \( v \). This follows from the fact that a random walk started in \( \Pi' \) will exit \( V^+_\delta \) before exiting the strip \( \{ z : |\text{Re}(z) - L/4| \leq 3L/8 \} \) with probability bounded uniformly away from zero. Such a walk will necessarily hit \( U \) before \( \partial \tilde{V}^+_\delta \). Next, it is straightforward to show that \( G^+(u, U) \geq G^+(u, u) \geq \eta/\delta \) for all \( u \in U \), so we conclude \( \mu_0 \leq c\eta/\delta \). Consequently

\[
\text{Var}^+(X_U) - \text{Var}^+(X_U \mid F_\gamma) \geq c \frac{\eta}{\delta} Hm^+(U, \gamma).
\]

To lower bound the harmonic measure, the idea is to replace \( \gamma \) with a line. More precisely, we take

\[
\gamma^* = \left\{ w \in V^+_\delta : \text{Im}(w) \in \left[ \frac{1}{2}, \frac{1}{2} + \delta \right], \left| \text{Re}(w) - \frac{L}{2} \right| \leq \frac{L}{4} \right\},
\]

and note that there exists a universal constant \( c > 0 \) such that for \( w \in \gamma^* \), \( Hm^+(w, \gamma) \geq c \). This follows from the fact that a random walk started at \( w \) will hit the line \( \{ z : \text{Re}(z) = L/2 \} \) before \( \partial \tilde{V}^+_\delta \), and then exit \( V^+_\delta \) before exiting \( \Pi' \) with probability bounded uniformly away from zero. Therefore, we have \( Hm^+(U, \gamma) \geq c Hm^+(U, \gamma^*) \). To bound \( Hm^+(U, \gamma^*) \) from below, we proceed by a last-exit decomposition.
Let \( \tau = \min \{ n \geq 1 : S_n \in U \cup \partial \tilde{V}_\delta^+ \cup \gamma^* \} \). By the same argument given in the proof of the upper bound on \( \mu_0 \),
\[
Hm^+(U, \gamma^*) = \sum_{w \in \gamma^*} \sum_{u \in U} C_w \mathbb{P}_w(S_r = u)G^+(u, U) \geq \frac{\eta}{\delta} \sum_{w \in \gamma^*} \mathbb{P}_w(S_r \in U),
\]
where we have used the fact that \( C_w = 1 \) for all \( w \in \gamma^* \) and \( G^+(u, U) \geq \eta/\delta \) for all \( u \in U \). Finally, by Lemma 2.7 we have
\[
\sum_{w \in \gamma^*} \mathbb{P}_w(S_r \in U) \geq c.
\]
This concludes the proof.

### 4.3.2 Proof of (4.11)

Let \( \tilde{V}_0^l \) be the connected component of \( \tilde{V}_\delta \setminus (\tilde{A}_\delta^l \cup \tilde{V}_0) \) containing \( (c_\delta, d_\delta) \), and \( V_0^l = \tilde{V}_0^l \cap \delta \mathbb{Z}^2 \). Let \( W \) be the following set of vertices
\[
W = \{ v \in V_0^l : \text{dist}(v, \tilde{A}_0) \leq \delta \}.
\]
We note that
\[
\{ (c_\delta, d_\delta) \sim_{\delta} \tilde{A}_0 \} = \{ (c_\delta', d_\delta') \sim_{\delta} W \}.
\]
Consequently, we would like to explore \( E_{\delta} \cap V_0^l \) from \( W \), but in order to apply Proposition 4.1 we need to start our exploration “one step” away from the boundary since any vertices that are distance less than one from \( \partial V_0^l \) will not be in \( D^* \). That is, letting \( I_0 \) and \( U \) be given by
\[
I_0 = \{ v \in V_0^l : \delta < \text{dist}(v, \tilde{A}_0) \leq 2\delta \},
\]
\[
U = \{ c_\delta', d_\delta' \cap \{ z : |\text{Im}(z) - 1/2| \leq 1/8 \} \},
\]
we will show that
\[
\mathbb{P}^l \left( U \sim_{\delta} I_0 \right) \geq c. \quad (4.12)
\]
Assuming this result for now, we show how to conclude the proof of (4.11). Recall that \( \mathcal{G}_{V_0^l \setminus W} \) is the \( \sigma \)-field generated by \( \{ \text{sign}(\phi_\delta(v)) : v \in V_0^l \setminus W \} \) and note that we have
\[
\{ U \sim_{\delta} I_0 \} \in \mathcal{G}_{V_0^l \setminus W}.
\]
Note also that if \( U \) is connected to \( I_0 \), there exists a random vertex \( w* \in W \) such that if \( \phi_\delta(w*) > 0 \), then \( U \) is connected to \( W \) (and thus to \( [a_\delta, b_\delta] \)). Note that \( w* \) is measurable with respect to \( \mathcal{G}_{V_0^l \setminus W} \). Therefore, it suffices to show that there exists a universal constant \( c > 0 \) such that for all \( w \in W \)
\[
\mathbb{P}^l \left( \phi_\delta(w) > 0 \mid \mathcal{G}_{V_0^l \setminus W} \right) \geq c \quad \text{a.s.}
\]
It will then follow that
\[
\mathbb{P}^l \left( (c_\delta, d_\delta) \sim_{\delta} \{ a_\delta, b_\delta \} \mid U \sim_{\delta} I_0 \right) \geq \mathbb{P}^l \left( \phi_\delta(w*) > 0 \mid U \sim_{\delta} I_0 \right) \geq c.
\]
By Lemma 4.2, it suffices to show that for \( \mathcal{E} = \{ \phi_\delta(v) < 0, \forall v \in V_0^l \setminus W \} \),
\[
\mathbb{P}^l \left( \phi_\delta(w) > 0 \mid \mathcal{E} \right) \geq c, \quad \forall w \in W.
\]
The proof is as follows. By the Markov property of the GFF, we have for any \( w \in W \),
\[
\phi(\delta(w)) = Y_w + Z_w,
\]
where \( Z_w \) is normal, mean zero, and independent of \( \mathcal{F}_{\delta,0} \) and \( Y_w = \mathbb{E}^{l,0}[\phi(\delta(w)) | \mathcal{F}_{\delta,0}] \). It is straightforward to show that there exist \( c, c' > 0 \) such that
\[
dist(w, \tilde{A}_{\delta,0}) \leq \frac{c}{\delta} \leq \text{Var}^{l,0}[Z_w] \leq \text{Var}^{l,0}[\phi(\delta(w))] \leq \frac{c}{\delta} \text{dist}(w, \tilde{A}_{\delta,0}).
\]
Finally, by Lemma 4.5 there exists \( c'' > 0 \) such that
\[
\mathbb{E}^{l,0}[Y_w | \mathcal{G}_{\delta,0}] \geq -c'' \text{dist}(w, \tilde{A}_{\delta,0}).
\]
Combining these bounds we obtain
\[
\mathbb{P}(\phi(\delta(w)) > 0 | \mathcal{E}) \geq c \text{ as promised. This concludes the proof.}
\]
Proof of (4.12). We assume without loss of generality that \( \text{dist}(U, I_0) \geq 10\delta \). Otherwise the probability of connection is lower bounded by \( 2^{-20} \), say.

As usual, we consider an exploration martingale \( M \) corresponding to an exploration of \( E_{\delta}^{l,0} \) from \( I_0 \) with observable \( X_U \), with \( U \) as above. The goal is to bound both the value of the martingale and its quadratic variation. As usual, we introduce the following processes
\[
\pi_t = \text{Hm}_{l,0}^t(U, [c_\delta, d_\delta]), \quad \mu_t = \text{Hm}_{l,0}^t(U, I_t).
\]
We will use Proposition 4.1 to show that there exists \( c > 0 \) such that for any \( \epsilon \), there exists \( \delta_0 = \delta_0(\epsilon) > 0 \) such that for \( \delta \leq \delta_0 \),
\[
\mathbb{P} \left( M_{\infty} - M_0 \leq -c\mu_0 - \theta(\pi_0 - \pi_{\infty}) | D_{\delta}^{l,0}(I_0, U) = \infty \right) \geq 1 - \epsilon.
\]
(4.13)
Postponing the proof of this claim until the end of the section, we show how it can be used to obtain the desired result. First, we bound the quadratic variation as follows
\[
\langle M \rangle_t = \sum_{v \in I_t} \text{Hm}_{l,0}^t(U, v) G_{l,0}^t(v, U)
\]
\[
\leq 16 \sum_{v \in I_t} \text{Hm}_{l,0}^t(U, v) \text{Hm}_{l,0}^t(v, [c_\delta, d_\delta])
\]
\[
= 16(\pi_0 - \pi_t)
\]
where we used the fact that a random walk started on \( U \) hits \( [c_\delta, d_\delta] \) before returning to \( U \) with probability at least 1/4 to obtain
\[
G_{l,0}^t(v, U) \leq 4 \text{Hm}_{l,0}^t(v, U) \leq 16 \text{Hm}_{l,0}^t(v, [c_\delta, d_\delta]).
\]
Next, we note that by Lemma 2.7 we have for some \( c > 0 \)
\[
\mu_0 \geq \text{Hm}_{l,0}^+(U, [a_\delta, b_\delta]) \geq c.
\]
Thus, for \( \epsilon > 0 \) small enough and \( \delta < \delta_0 \),
\[
\mathbb{P}^{l,0} \left( U \not\xrightarrow{\phi_{\geq \delta}^\geq} I_0 \right) \leq \mathbb{P} \left( \inf \left\{ B_t + \frac{\theta}{16} t : t \geq 0 \right\} \leq -c \right) + \epsilon \leq 1 - c'.
\]
Turning to the proof of (4.13), we begin by upper bounding \( M_\infty \). To this end, we let \( \mathcal{J}_k \) be the set obtained by adding to \( \mathcal{I}_k \) all the vertices separated from \( U \) by \( \mathcal{I}_k \) and \( \xi_k = \xi(U, \mathcal{J}_k) \). As in the zero boundary case, it is straightforward to check that \((U, \mathcal{J}_k)\) satisfy the assumptions of Proposition 4.1 and that there exists \( c > 0 \) such that

\[
\sup\{\xi_k : k \geq 0\} \leq \frac{c}{|\log(\delta)|}.
\]

The proof is the same as that of (4.7) so we omit further details. We can then apply Proposition 4.1 with

\[
I^+ = \{v \in \mathcal{I}_\infty : \text{sign}(\phi_\delta(v)) = 1\}, \quad I^- = \{v \in \mathcal{I}_\infty : \text{sign}(\phi_\delta(v)) = -1\},
\]

and obtain that there exists \( \Delta = \Delta(\theta) > 0 \) such that for any \( \epsilon > 0 \) there exists \( \delta_1 > 0 \) such that for \( \delta \leq \delta_1 \)

\[
P^{t,0}\left(M_\infty \leq -\Delta \mu_\infty + \theta \pi_\infty \mid U \overset{\phi_\delta}{\leftrightarrow} \mathcal{I}_0\right) \geq 1 - \epsilon.
\]

Next, we lower bound \( M_0 \). We have \( \mathbb{E}^{t,0}[M_0] = \theta \text{Hm}^{t,0}(U, [c_\delta, d_\delta]) \geq \theta \pi_0 \). For the variance, we have

\[
\text{Var}^{t,0}[M_0] = \sum_{v \in \mathcal{I}_0} \text{Hm}^{t,0}(U, v) G^{t,0}(v, U) \leq 64 \xi_0 \mu_0^2 \leq \frac{c}{|\log(\delta)|} \mu_0^2,
\]

where we have used the facts

\[
G^{t,0}(U, \mathcal{I}_0) \leq 16 \text{Hm}^{t,0}(U, \mathcal{I}_0), \quad \text{Hm}^{t,0}(U, v) \leq 4 \xi_0 \text{Hm}^{t,0}(U, \mathcal{I}_0), \quad \forall v \in \mathcal{I}_0.
\]

Therefore, we conclude that for any \( \epsilon > 0 \), there exists \( \delta_2 \) such that for \( \delta < \delta_2 \),

\[
P^{t,0}\left(M_0 \geq \frac{\Delta}{2} \mu_0 + \theta \pi_0\right) \geq 1 - \epsilon.
\]

Combining the two bounds, and noting that \( \mu_t \) is increasing we obtain that for any \( \epsilon > 0 \) and \( \delta < \delta_1 \wedge \delta_2 \)

\[
P^{t,0}\left(M_\infty - M_0 \leq -\frac{\Delta}{2} \mu_0 - \theta(\pi_0 - \pi_\infty) \mid U \overset{\phi_\delta}{\leftrightarrow} \mathcal{I}_0\right) \geq 1 - 2\epsilon.
\]

This concludes the proof of (4.13).

5 (Non)-Existence of closed pivotal edges for metric graph percolation

In this section we prove Theorem 1.4. First, we recall some definitions and notation. Let \( \tilde{\phi}_\delta \) be a metric graph GFF on \( \tilde{V}_\delta \) with alternating boundary condition (1.3). Denote by \( E_\delta \) the edge set of the nearest-neighbor graph on \( V_\delta \). We let \( \omega : E_\delta \to \{0, 1\} \) be such that \( \omega(e) = 1 \) if \( \tilde{\phi}_\delta(u) \geq 0 \) for all \( u \in I_e \) and \( \omega(e) = 0 \) otherwise. In the former case, we say the edge is open, in the latter that it is closed. The notations \( \omega^e \) and \( \omega_c \) are defined in (1.10). An edge \( e \) is pivotal if there is a horizontal crossing in \( \omega^e \) but there is no such crossing in \( \omega_c \). Note that the pivotality of an edge does not depend on the status of the edge itself. We say that \( e \) is a closed pivotal edge (resp. open pivotal edge) if \( e \) is pivotal and it is closed (resp. open). Our goal is to show that the probability that there exists a closed pivotal edge decays to 0 as \( \delta \to 0 \). In particular, we want to show that there exists a constant \( c > 0 \) (depending only on \( L \)) such that the following holds

\[
\mathbb{P}(\text{there exists a closed pivotal edge}) \leq \frac{c}{\sqrt{|\log(\delta)|}}, \quad \text{(5.1)}
\]
By symmetry, it suffices to consider only edges $e$ on the left half of $V_\delta$. That is, if we let
\[ E^l_\delta = \left\{ e \in E : \exists x \in I_e, \Re(x) \leq \frac{L}{2} \right\}, \]
it suffices to show
\[ \mathbb{P} \left( \text{there exists a closed pivotal edge in } E^l_\delta \right) \leq \frac{c}{\sqrt{|\log(\delta)|}}. \]
In fact, we will show that
\[ \mathbb{P} \left( \text{there exists a closed pivotal edge in } E^l_\delta \middle| F_{\tilde{A}^l_{\delta,0}} \right) \leq \frac{c}{\sqrt{|\log(\delta)|}} \text{ a.s.} \]

Note that if there is a positive horizontal crossing in the metric graph (that is, $\tilde{A}^l_{\delta,0} \cap [c_\delta, d_\delta] \neq \emptyset$) then there are no closed pivotal edges and that this event is measurable with respect to $F_{\tilde{A}^l_{\delta,0}}$. Therefore, we assume from now on that this is not the case.

To simplify notation, we let $\mathbb{P}^l$ denote the law of $\tilde{\phi}_\delta$ given $F_{\tilde{A}^l_{\delta,0}}$ and $\mathbb{E}^l$ denote the expectation with respect to $\mathbb{P}^l$. Similarly, we let $\tilde{V}^l_\delta$ be the connected component of $\tilde{V}_\delta \setminus \tilde{A}^l_{\delta,0}$ containing $[c_\delta, d_\delta]$ and $V^l_\delta = \tilde{V}^l_\delta \cap \delta \mathbb{Z}^2$. Note that, given that there is no horizontal crossing in the metric graph, the set of closed pivotal edges consists of all edges with one endpoint in $\tilde{A}^l_{\delta,0}$ and the other in $\tilde{A}^l_{\delta,0}$. Therefore, given $F_{\tilde{A}^l_{\delta,0}}$, the set of pivotal edges is increasing in $\tilde{A}^l_{\delta,0}$ and therefore increasing in $\{ \tilde{\phi}_\delta(v) : v \in \tilde{V}^l_\delta \}$. We will therefore assume from now on that $\tilde{\phi}_\delta$ has zero boundary condition on $[b_\delta, c_\delta] \cup [d_\delta, a_\delta]$.

The proof is essentially the same as that of (4.10) and consists of analyzing an exploration martingale $M$, as introduced in Section 2.3, corresponding to an exploration on $\tilde{E}^l_\delta$ from $I_0 = [c_\delta, d_\delta]$ with observable given by the following set. For each $v \in \partial \tilde{A}^l_{\delta,0} \setminus \partial \tilde{V}_\delta$, we let $\eta_v = \text{dist}(v, V^l_\delta)$. Since $|\partial \tilde{A}^l_{\delta,0}| < \infty$ and $\partial \tilde{A}^l_{\delta,0} \cap \delta \mathbb{Z}^2 \subset \partial \tilde{V}_\delta$ almost surely, we have
\[ \eta = \min \{ \eta_v : v \in \partial \tilde{A}^l_{\delta,0} \setminus \partial \tilde{V}_\delta > 0 \} \text{ a.s.} \]

We then set the observable set $U$ to be
\[ U = \left\{ u \in \tilde{V}^l_\delta : \text{dist}(u, \partial \tilde{V}^l_\delta) = \frac{\eta}{2}, \Re(u) \leq \frac{3L}{4} \right\}. \]
Without loss of generality, we assume that $U$ is non-empty (and indeed that there exists $u \in U$ with $\Re(u) \leq (L/2) + \delta$) since otherwise there can be no pivotal edges in $E^l_\delta$. As usual, we let $H^l_\delta$ be the harmonic measure on $I_t \cup \partial V^l_\delta$ and $\mu_t = H^l_\delta(U, I_t)$. Since $I_0 \subset \partial V^l_\delta$ we write $H^l_\delta$ for $H^l_{\delta,0}$. Note that $M_0 = \theta \mu_0$ and (because we assume $\tilde{\phi}_\delta$ has zero boundary condition on $[b_\delta, c_\delta] \cup [d_\delta, a_\delta]$) $M_t \geq 0$ for all $t$, so $M_t - M_0 \geq -\theta \mu_0$. We claim that there exist constants $c, c' > 0$ such that
\[ \mu_0 \leq c \frac{\eta^2}{\delta}, \]
\[ \langle M \rangle_\infty \geq c' \left( \frac{\eta}{\delta} \right)^2 |\log(\delta)|, \text{ a.s. on } \{ \text{there exists a closed pivotal edge in } E^l_\delta \}. \]
Assuming this claim for now, an application of Theorem 2.3 then gives (after rescaling the Brownian motion)
\[ \mathbb{P}^l(\text{there exists a closed pivotal edge in } E^l_\delta) \leq \mathbb{P} \left( \inf_{0 \leq t \leq c \sqrt{\log(\delta)}} B_t \geq -c \theta \right) \leq \frac{c''}{\sqrt{|\log(\delta)|}}. \]
To prove the claim we introduce the electric network with vertex set $W = \delta \mathbb{Z}^2 \cup U \cup \partial \tilde{V}_\delta^1$ where two vertices $u, v \in W$ are connected with an edge of conductance $C_{u,v} = (4|u - v|/\delta)^{-1}$ if there exists a path in $\delta \mathbb{Z}^2$ connecting $u$ to $v$ which does not contain any other points in $W$. For the rest of this proof, we will write $u \sim v$ if $u$ and $v$ are connected in this network, and we let $C_u = \sum_{v \sim u} C_{u,v}$. We let $G^t$ be the Green’s function corresponding to the continuous time simple random walk on $W$ killed on $\partial \tilde{V}_\delta^1$, but take $S$ to be a discrete-time simple random walk on $W$ (not killed on $\partial \tilde{V}_\delta^1$).

For the upper bound on $\mu_0$, we let $\tau = \min\{n \geq 1 : S_n \in U \cup \partial \tilde{V}_\delta^1\}$. By the arguments given in the proof of (4.10),

$$Hm^1(U, I_0) = \sum_{u \in U} \sum_{v \in I_0} C_{u,v} \mathbb{P}(S_\tau = u) G^t(u, U) \leq 2 \eta \sum_{v \in I_0} \mathbb{P}(S_\tau \in U),$$

where we used the facts that $C_v = 1$ for all $v \in I_0$ such that $\mathbb{P}(S_\tau \in U) > 0$ and $G^t(u, U) \leq 2 \eta / \delta$ for all $u \in U$. Finally, there exists $c > 0$ such that $P_v(S_\tau \in U) \leq c \delta$ since this probability is upper bounded by the probability that a one-dimensional simple random walk hits $L/4$ before returning to zero. Since $|I_0| \geq c' / \delta$ we conclude that $\mu_0 \leq c \eta / \delta$ as promised.

To lower bound the quadratic variation, we note that if there exists a closed pivotal edge in $E_\delta^1$, then $I_\infty$ contains a nearest-neighbor path $\gamma$ in $V_\delta^1$ satisfying the following conditions. First, that it crosses the following strip

$$\Pi' = \left\{ z : \frac{L}{2} + \delta \leq \Re(z) \leq \frac{5L}{8} \right\}.$$ (5.2)

Second, that there exists $v^* \in \gamma$ and $w \in \bar{A}_{\delta,0} \cap \delta \mathbb{Z}^2$ satisfying $\Re(w) \leq L/2 + \delta$ and $|w - v^*| = \delta$. We claim that there exists $c > 0$ such that for any such path

$$\text{Var}^t(X_U) - \text{Var}^t(X_U \mid F_\gamma) \geq c \left( \frac{\eta}{\delta} \right)^2 |\log(\delta)|.$$

Indeed, we have

$$\text{Var}^t(X_U) - \text{Var}^t(X_U \mid F_\gamma) = \sum_{v \in \gamma} Hm(U, v; \gamma \cup \partial \tilde{V}_\delta^1) G^t(v, U),$$

$$\geq c \eta \delta Hm(U, \gamma; \gamma \cup \partial \tilde{V}_\delta^1),$$

where we used the fact that $Hm^t(v, U) \geq c$ for all $v \in \gamma$ since it is lower bounded by the probability that a simple random walk on $\delta \mathbb{Z}$ started at $\delta[5L/8\delta]$ hits $0$ before $\delta[3L/4\delta]$ and the fact that $G^t(u, U) \geq \eta / \delta$ for all $u \in U$. Therefore, we want to show that there exists $c > 0$ such that for any path $\gamma$ that satisfies the conditions above,

$$Hm(U, \gamma; \gamma \cup \partial \tilde{V}_\delta^1) \geq c \frac{\eta}{\delta} |\log(\delta)|.$$

The proof is essentially identical to that of (3.3). For $n \geq 0$ an integer, we let $Q_n$ be the box of radius $r_n = 2^{-n-2}L$ centered at $v^*$, $A_n = Q_n \setminus Q_{n+1}$, $U_n = U \cap A_n$, and $\gamma_n = \gamma \cap A_n$. We note that $N = \max\{n \geq 1 : r_n \geq 1000\delta\}$ satisfies $N \geq c|\log(\delta)|$. Finally, we claim that there exists a universal constant $c > 0$ such that

$$Hm(U_n, \gamma_n; \gamma \cup \partial \tilde{V}_\delta^1) \geq c \frac{\eta}{\delta}, \quad 1 \leq n \leq N.$$

For the proof, we begin as usual with a last-exit decomposition. Take $1 \leq n \leq N$ and let $\tau = \min\{k \geq 1 : S_k \in U \cup \gamma \cup \partial \tilde{V}_\delta^1\}$ (recall $S$ is a random walk on $W$). We have

$$Hm(U_n, \gamma_n; \gamma \cup \partial \tilde{V}_\delta^1) = \sum_{u \in U_n} \sum_{u' \in U} \sum_{v \in \gamma_n} G^t(u, u') C_{u,v} \mathbb{P}_{u'}(S_\tau = v),$$

$$= \sum_{u \in U_n} \sum_{u' \in U} \sum_{v \in \gamma_n} G^t(u', u) C_{v,u} \mathbb{P}_v(S_\tau = u'),$$

$$\geq c \frac{\eta}{\delta} \sum_{v \in \gamma_n} \mathbb{P}_v(S_\tau \in U).$$
Figure 5: An illustration of the level line $\eta_\delta$. Vertices where $\phi$ is positive are in red and vertices where $\phi$ is negative are in blue.

From here, the details of the proof are the same as for the proof of (3.3). Let $v_1 \in \gamma_n$ be such that $2r_n/3 - \delta < |v_1 - v^*|_{\ell_\infty} \leq 2r_n/3$, $v_2 \in \gamma_n$ be such that $5r_n/6 \leq |v_2 - v^*|_{\ell_\infty} < 5r_n/6 + \delta$, and $Q_{n,2}$ be the box of radius $r_n/12$ centered at $v_2$. With these choices, the distance between $v_1$ and $Q_{n,2}$, and between $\{v_1\} \cup Q_{n,2}$ and $A_n^c$ are of the same order as $r_n$. Therefore, letting $E$ be the event that $S$ hits $Q_{n,2} \cap \gamma$ and then hits $U$ before exiting $A_n$, there exists a universal constant $c > 0$ such that $P_{v_1}(E) \geq c$. By a last exit decomposition

$$P_{v_1}(E) \leq \sum_{v \in \gamma} G_{l,v}^{l,*}(v_1,v)P_v(S \tau \in U_n),$$

where $G_{l,v}^{l,*}(v_1,v)$ is the expected number of visits a random walk started at $v_1$ makes to $v$ after hitting $Q_{n,2} \cap \gamma$ and before exiting $A_n$. Finally, it follows from Lemma 2.5 that $G_{l,v}^{l,*}(v_1,\cdot)$ is uniformly bounded. That is, there exists a universal constant $c > 0$ such that

$$G_{l,v}^{l,*}(v_1,v) \leq c, \quad v \in A_n \cap \delta\mathbb{Z}^2.$$

This concludes the proof.

6 Limits of crossing probabilities

In this section, we discuss level lines of discrete GFF and metric graph GFF. Fix the rectangle $R_L = (0, L) \times (0, 1)$ and $V_\delta = R_L \cap \delta\mathbb{Z}^2$. Recall that the four corners of $R_L$ are denoted by $a, b, c, d$ in counter-clockwise order with $b = 0$, and that the four corners of $V_\delta$ are denoted by $a_\delta, b_\delta, c_\delta, d_\delta$ in counter-clockwise order with $b_\delta$ closest to the origin, i.e. $b_\delta = (\delta, \delta) \in \delta\mathbb{Z}^2$.

Consider discrete GFF $\phi_\delta$ on $V_\delta$ with either zero boundary condition or alternating boundary condition (1.3). In either case, we say that the vertices on $[a_\delta, b_\delta)$ have positive value and the vertices on $[b_\delta, c_\delta)$ have negative value. The level line $\eta_\delta$ of $\phi_\delta$ is defined as follows: it starts from $b_\delta^c = (0, 3\delta/2)$, lies on the dual lattice of $\delta\mathbb{Z}^2$ and turns at every dual-vertex in such a way that it has vertices with positive value on its left and negative value on its right. If there is an indetermination when arriving at a dual-vertex, turn left. The level lines stop when they hit the boundary segments $[c_\delta, d_\delta)$ or $[d_\delta, a_\delta)$. See Figure 5 for an illustration. Notice how on the second turn, there is an indetermination as the line could turn right and still have a vertex with positive value on the left and a vertex with negative value on the right.

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We will consider the convergence of \( \eta_\delta \) in this section. We use the following metric on planar curves: suppose \( \eta_1 \) and \( \eta_2 \) are unparameterized continuous curves, then
\[
d(\eta_1, \eta_2) = \inf_{u_1, u_2} \sup_{t \in [0,1]} |\eta_1(u_1(t)) - \eta_2(u_2(t))|,
\]
where the inf is over increasing homeomorphisms \( u_1, u_2 : [0, 1] \to [0, 1] \). Using techniques in [SS09], we have the following convergence of the law on \( \eta_\delta \). Recall that \( 2\lambda \) denotes the height-gap in [SS09].

**Theorem 6.1.** If \( \theta = \lambda \) then the law of \( \eta_\delta \) converges weakly to \( \eta \sim \text{SLE}_4(-2; -2) \) in \( R_L \) from \( b \) to \( d \) with force points \((a; c)\) as \( \delta \to 0 \). As a consequence, (1.9) holds.

To prove Theorem 6.1, we will first introduce SLE process in Section 6.1 and then complete the proof in Section 6.2. We will discuss the limit of crossing probabilities in (1.5) at the end of Section 6.2. We will discuss the limit of crossing probabilities in (1.6) in Section 6.3.

### 6.1 Preliminaries on SLE

We denote by \( \mathbb{H} \) the upper-half plane. We call a compact subset \( K \) of \( \mathbb{H} \) an \( \mathbb{H} \)-hull if \( \mathbb{H} \setminus K \) is simply connected. By Riemann’s mapping theorem, there exists a unique conformal map \( g_K \) from \( \mathbb{H} \setminus K \) onto \( \mathbb{H} \) with the normalization \( \lim_{z \to \infty} |g_K(z) - z| = 0 \). With such normalization, we say that \( g_K \) is normalized at \( \infty \).

We consider the following collections of \( \mathbb{H} \)-hulls. First, consider families of conformal maps \( (g_t, t \geq 0) \) obtained by solving the Loewner equation: for each \( z \in \mathbb{H} \),
\[
\frac{\partial_t g_t(z)}{g_t(z) - W_t} = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,
\]
where \( (W_t, t \geq 0) \) is a real-valued continuous function, which we call the driving function. Second, for each \( z \in \mathbb{H} \), define the swallowing time \( T_z \) to be
\[
\sup \left\{ t \geq 0 : \inf_{s \in [0, t]} |g_s(z) - W_s| > 0 \right\}.
\]
Finally, denote by \( K_t \) the closure of \( \{ z \in \mathbb{H} : T_z \leq t \} \). Then \( g_t \) is the conformal map from \( \mathbb{H} \setminus K_t \) onto \( \mathbb{H} \) normalized at \( \infty \). The collection of \( \mathbb{H} \)-hulls \( (K_t, t \geq 0) \) is called a Loewner chain parameterized by the half-plane capacity.

For \( \kappa \geq 0 \), Schramm Loewner Evolution, denoted by \( \text{SLE}_\kappa \), is the Loewner chain with driving function \( W_t = \sqrt{\kappa} B_t \) where \( (B_t, t \geq 0) \) is a standard one-dimensional Brownian motion. It was proved in [RS05] that \( (K_t, t \geq 0) \) is almost surely generated by a continuous transient curve, i.e. there exists a continuous curve \( \eta \) such that, for each \( t \geq 0 \), the set \( \mathbb{H} \setminus K_t \) is the unbounded connected component of \( \mathbb{H} \setminus \eta([0, t]) \) and \( \lim_{t \to \infty} |\eta(t)| = \infty \). In this section, we focus on \( \kappa \in (0, 4] \) when the curve is simple.

\( \text{SLE}_\kappa(\rho^L; \rho^R) \) is a variant of \( \text{SLE}_\kappa \) process where one keeps track of two extra marked points on the boundary. Let \( y^L \leq 0 \leq y^R \) and \( \rho^L, \rho^R \in \mathbb{R} \). An \( \text{SLE}_\kappa(\rho^L; \rho^R) \) process with force points \((y^L; y^R)\) is the Loewner chain driven by \( W_t \) that solves the following system of SDEs:
\[
dW_t = \sqrt{\kappa} dB_t + \frac{\rho^L dt}{W_t - V_t^L} + \frac{\rho^R dt}{W_t - V_t^R}, \quad dV_t^L = \frac{2 dt}{V_t^L - W_t}, \quad dV_t^R = \frac{2 dt}{V_t^R - W_t},
\]
where \( W_0 = 0, V_0^L = y^L, V_0^R = y^R \). It turns out that such process exists for all time if \( \rho^L, \rho^R > -2 \). If \( \rho^L \leq -2 \) or \( \rho^R \leq -2 \), it exists up to the first time that the process swallows \( y^L \) or \( y^R \). Moreover, the process is generated by continuous curve up to and including the same time.

The above SLE processes are defined in \( \mathbb{H} \), for other simply connected domains we define SLE process via conformal image. Suppose \( \Omega \) is a non-trivial simply connected domain and \( a, b, c, d \) are four boundary
points lying on locally connected components in counterclockwise order. Then \( \text{SLE}_r(\rho^L; \rho^R) \) in \( \Omega \) from \( b \) to \( d \) with force points \((a; c)\) is \( \varphi^{-1}(\eta) \) where \( \eta \) is an \( \text{SLE}_r(\rho^L; \rho^R) \) from \( 0 \) to \( \infty \) with force points \((y^L; y^R)\) and \( \varphi \) is any conformal map from \( \Omega \) onto \( \mathbb{R} \) such that \( \varphi(a) = y^L \leq \varphi(b) = 0 \leq \varphi(c) = y^R < \varphi(d) = \infty \).

In this section, we focus on \( \text{SLE}_4(\rho^L; \rho^R) \) with \( \rho^L = \rho^R = -2 \) and force points \( y^L < 0 < y^R \). Then the above SDEs become

\[
dW_t = 2dB_t + \frac{2dt}{V^L_t - W_t} + \frac{2dt}{V^R_t - W_t}, \quad dV^L_t = \frac{2dt}{V^L_t - W_t}, \quad dV^R_t = \frac{2dt}{V^R_t - W_t}.
\]

Denote by \( \eta \) the continuous curve corresponding to the Loewner chain driven by \( W \). Let \( T \) be the first time that the process swallows \( y^L \) or \( y^R \). We will calculate the probability \( \mathbb{P}(\eta(T) = y^R) \). Set

\[
\hat{W}_t = \frac{2W_t - V^L_t - V^R_t}{V^R_t - V^L_t}.
\]

Then \( T = \inf\{ t : \hat{W}_t \in \{-1, +1\} \} \). Itô’s formula gives

\[
d\hat{W}_t = \frac{4dB_t}{V^R_t - V^L_t}.
\]

In particular, \((\hat{W}_{t\wedge T}; t \geq 0)\) is a bounded martingale. Optional stopping theorem gives \( \mathbb{E}[\hat{W}_T] = \hat{W}_0 \). From there, we obtain

\[
\mathbb{P}(\eta(T) = y^R) = \mathbb{P}(\hat{W}_T = 1) = \frac{-y^L}{y^R - y^L}.
\]

This gives (1.9) assuming the convergence of the level line in Theorem 6.1.

Below, we perform a time change in the system which will be useful in Section 6.2. This is analog of [SS09, Section 4.5]. Define a new time parameter, for \( t \geq 0 \),

\[
s(t) := \log \left( \frac{V^R_t - V^L_t}{y^R - y^L} \right).
\]

Set \( \tilde{W}_s = \hat{W}_t \) when \( s = s(t) \). Then we have

\[
d\tilde{W}_s = q(\tilde{W}_s)d\tilde{B}_s, \quad \text{where } q(x) = \sqrt{2(1 - x^2)},
\]

and \((\tilde{B}_s; s \geq 0)\) is a standard one-dimensional Brownian motion. The process \( \tilde{W} \) starts from \( \tilde{W}_0 = \frac{-y^L - y^R}{y^R - y^L} \), evolves according to (6.3), and stops when it hits \( \{-1, +1\} \) at finite time \( T \). From above, we see how to transform from the process in (6.1) to the one in (6.3). We will show that this is one-to-one transform up to a scaling constant.

**Lemma 6.2.** Suppose \((\tilde{Y}_s)\) is a continuous process starting from \( \tilde{Y}_0 \in (-1, 1) \), evolving according to \( d\tilde{Y}_s = q(\tilde{Y}_s)d\tilde{B}_s \), and stopped when it hits \( \{-1, +1\} \). Define

\[
t(s) = \frac{1}{8} \int_0^s e^{2u} (1 - \tilde{Y}_u^2) du, \quad s(t) = \sup\{ s : t(s) \leq t \}.
\]

Define

\[
Y_t = \frac{1}{2} e^{s(t)} \tilde{Y}_s(t) + \frac{1}{2} \int_0^{s(t)} e^{u} \tilde{Y}_u du - \frac{1}{2} \tilde{Y}_0.
\]

Then \((Y_t)\) is a continuous process starting from 0 and evolving according to

\[
dY_t = 2dB_t + \frac{2dt}{V^L_t - Y_t} + \frac{2dt}{V^R_t - Y_t}, \quad dV^L_t = \frac{2dt}{V^L_t - Y_t}, \quad dV^R_t = \frac{2dt}{V^R_t - Y_t},
\]

where \( V^L_0 = -\frac{1}{2}(1 + \tilde{Y}_0) \) and \( V^R_0 = \frac{1}{2}(1 - \tilde{Y}_0) \).
Proof. Set
\[
B_t = \frac{1}{4} \int_0^{s(t)} e^u q(\tilde{Y}_u) d\tilde{B}_u;
\]
\[
V_t^L = V_0^L - \frac{1}{2} \int_0^{s(t)} e^u (1 - \hat{Y}_u) du, \quad V_t^R = V_0^R + \frac{1}{2} \int_0^{s(t)} e^u (1 + \hat{Y}_u) du.
\]
Then \((B_t)\) is a standard Brownian motion, and \((Y_t, V_t^L, V_t^R)\) solves (6.4).

\[\square\]

6.2 Scaling limits of level lines in discrete GFF

We will derive Theorem 6.1 in this section following the proof in \([SS09]\). Although our setup is different from that in \([SS09]\), similar techniques work. In fact, our setting is much easier to treat, because Theorem 6.1 only considers the level line up to the first hitting time of the boundary and we do not need to treat the process after that time. As the proof in \([SS09]\) is long and technical (and we do not know how to simplify it), we only sketch the proof in our setting.

Recall that \(\phi_\delta\) is discrete GFF on \(V_\delta\) with alternating boundary condition (1.3), and \(\eta_\delta\) is the level line of \(\phi_\delta\) starting from \(b_\delta^0\) and stopped when it hits the boundary segment \([c_\delta, a_\delta]\). Fix a conformal map \(\varphi\) from \(R_t\) onto \(\mathbb{H}\) with \(\varphi(a) < \varphi(b) = 0 < \varphi(c) < \varphi(d) = \infty\). Denote by \(y^L = \varphi(a) < 0\) and by \(y^R = \varphi(c) > 0\). Consider \(\varphi(\eta_\delta)\) in \(\mathbb{H}\) parameterized by the half-plane capacity. Denote by \(W_t^\delta\) its driving function and by \(g_\delta\) the corresponding family of conformal maps. For \(\epsilon > 0\), let \(T_\epsilon^\delta\) be the first time that \(\varphi(\eta_\delta)\) gets within \(\epsilon\)-distance of the points \(\varphi(a_\delta)\) or \(\varphi(c_\delta)\). Recall that \((W_t)\) is the driving function of \(\eta \sim \text{SLE}_4(-2; -2)\) as in (6.1). Let \(T_\epsilon\) be the first time that \(\eta\) gets within \(\epsilon\)-distance of the points \(y^L\) or \(y^R\).

**Lemma 6.3.** For any fixed \(\epsilon, \alpha > 0\) small, there is \(\delta_0 = \delta_0(\epsilon, \alpha) > 0\) such that, if \(\delta \leq \delta_0\), the interface \(\eta_\delta\) can be coupled with \(\eta\) so that
\[
\mathbb{P}\left( \sup_{t \in [0, T_{T_\epsilon} \wedge T_\epsilon]} |W_t^\delta - W_t| \geq \alpha \right) \leq \alpha.
\]

**Proof.** Denote by \(\tilde{W}^\delta\) the coordinate change of \(W^\delta\) as in Section 6.1. Let \(\mathcal{F}_s\) be the \(\sigma\)-field generated by \(\sigma(\tilde{W}^\delta, r \leq s)\). We will first prove the following conclusion which is analog of \([SS09\text{, Proposition 4.2}]\). For any \(\epsilon, \alpha, \beta > 0\) small, there exists \(C > 0\) depending on \(\epsilon\) and there exists \(\delta_0 > 0\) depending on \(\epsilon, \alpha, \beta\) such that the following holds. If \(\delta \leq \delta_0\) and \(s_0, s_1\) are two stopping times for \(\tilde{W}^\delta\) such that almost surely \(s_0 \leq s_1 \leq T_{T_\epsilon}^\delta\), \(s_1 - s_0 \leq \beta^2\), and \(\sup_{s \in [s_0, s_1]} |\tilde{W}_s^\delta - \tilde{W}_{s_0}^\delta| \leq \beta\), then the following two estimates hold with probability at least \(1 - \alpha\):
\[
\mathbb{E}\left[ |\Delta \tilde{W}^\delta | \mathcal{F}_{s_0} \right] \leq C \beta^3, \quad \mathbb{E}\left[ (\Delta \tilde{W}^\delta)^2 - q(\tilde{W}_{s_0}^\delta)^2 \Delta s | \mathcal{F}_{s_0} \right] \leq C \beta^3,
\]
where \(\Delta s = s_1 - s_0\) and \(\Delta \tilde{W}^\delta = \tilde{W}_{s_1}^\delta - \tilde{W}_{s_0}^\delta\). Roughly speaking, (6.5) corresponds to the discrete version of (6.3).

For \(t > 0\), let \(F_t^\delta\) be the function defined on \(V_\delta\) that is \(-\lambda\) on vertices to the right side of \(\eta_\delta[0, t]\), \(+\lambda\) on the vertices to the left side of \(\eta_\delta[0, t]\), equal to the boundary data on \(\partial V_\delta\), and is harmonic at all other vertices in \(V_\delta\). Suppose \(v \in V_\delta\) such that the distance between \(v\) and \(\partial R_L\) is at least \(1/4\). For \(k = 0, 1\), define \(X_k = \mathbb{E}[\phi_\delta(v) | \mathcal{F}_{s_k}]\) and let \(A_k\) be the event
\[
|X_k - F_{s_k}(v)| \geq \beta^5.
\]
By the same argument as in [SS09, Proposition 3.27], one can show that $X_k - F_{s_k}(v) \to 0$ in probability as $\delta \to 0$. Thus, we have $\mathbb{P}(A_k) \leq \frac{1}{4} \alpha \beta^5$ if $\delta$ is small enough. Consequently, we have

\[ |X_0 - F_{s_0}(v)| \leq \beta^5, \quad \text{on } A_0^c; \]
\[ |\mathbb{E}[X_1 - F_{s_1}(v) | F_{s_0}]| \leq \beta^5 + O(1) \mathbb{P}(A_1 | F_{s_0}). \]

Let $\mathcal{A}$ be the event $\mathbb{P}(A_1 | F_{s_0}) \geq \beta^5$, then $\mathbb{P}(\mathcal{A}) \leq \frac{1}{4} \alpha$. Combing the above two estimates and the fact that $\mathbb{E}[X_1 | F_{s_0}] = X_0$, we have

\[ \mathbb{E}[F_{s_1}(v) | F_{s_0}] - F_{s_0}(v) = O(\beta^5), \quad \text{on } A_0^c \cap \mathcal{A}^c. \]

For $k = 0, 1$, let $H_k$ be the (continuous) function that is $-\lambda$ to the right side of $\eta_{\delta}[0, s_k]$, $+\lambda$ to the left side of $\eta_{\delta}[0, s_k]$, equal to the boundary data on $\partial R_L$, and is harmonic in $R_L \setminus \eta_{\delta}[0, s_k]$. Then $H_k(v) - F_{s_k}(v)$ is small if $\delta$ is small enough and the distance between $v$ and $\eta_{\delta}[0, s_k]$ is bounded from below. Precisely, let $B_k$ be the event $|H_k(v) - F_{s_k}(v)| \geq \beta^5$, we have $\mathbb{P}(B_k) \leq \frac{1}{4} \alpha \beta^5$ if $\delta$ is small enough. Let $\mathcal{B}$ be the event $\mathbb{P}(B_1 | F_{s_0}) \geq \beta^5$, then $\mathbb{P}(\mathcal{B}) \leq \frac{1}{4} \alpha$. From the above estimate, we have

\[ \mathbb{E}[H_1(v) | F_{s_0}] - H_0(v) = O(\beta^5), \quad \text{on } A_0^c \cap \mathcal{A}^c \cap B_0^c \cap B^c. \]  

(6.6)

For $k = 0, 1$, denote by

\[ Z_k = \frac{2g_{s_k}^{\delta}(\varphi(v)) - g_{s_k}^{\delta}(y^L) - g_{s_k}^{\delta}(y^R)}{g_{s_k}^{\delta}(y^R) - g_{s_k}^{\delta}(y^L)}. \]

Then we have

\[ H_k(v) = \lambda - \frac{2\lambda}{\pi} \arg(Z_k - 1) + \frac{2\lambda}{\pi} \arg(Z_k - \tilde{W}_{s_k}^{\delta}) - \frac{2\lambda}{\pi} \arg(Z_k + 1). \]

By the calculation in [SS09, Proof of Proposition 4.2], we have

\[ \frac{\pi}{2\lambda} (H_1(v) - H_0(v)) = \frac{y}{x^2 + y^2} \left( \frac{x}{x^2 + y^2} \left( q(\tilde{W}_{s_0}^{\delta})^2 \Delta s - (\Delta \tilde{W}^{\delta})^2 \right) - \Delta \tilde{W}^{\delta} \right) + O(\beta^3), \]

where $x = \Re(Z_0 - \tilde{W}_{s_0}^{\delta})$ and $y = \Im(Z_0)$. Plugging into (6.6), with probability at least $1 - \alpha$, we have

\[ \mathbb{E} \left[ \frac{x}{x^2 + y^2} \left( q(\tilde{W}_{s_0}^{\delta})^2 \Delta s - (\Delta \tilde{W}^{\delta})^2 \right) - \Delta \tilde{W}^{\delta} | F_{s_0} \right] = O(\beta^3). \]

By different choice of $v$, we obtain (6.5).

Recall that $W$ is driving function of SLE$_4(-2; -2)$. Denote by $\tilde{W}$ the coordinate change of $W$ as in Section 6.1. With (6.5) at hand, by arguments in [SS09, Section 4.4], we have the following conclusion. For any fixed $\epsilon, \alpha > 0$ small, there is $\delta_0 > 0$ such that, if $\delta \leq \delta_0$, there is coupling between $\tilde{W}^{\delta}$ and $\tilde{W}$ so that

\[ \mathbb{P} \left( \sup_{s \in [0, T_0^{\delta} \setminus T_1]} |\tilde{W}^{\delta}_s - \tilde{W}_s| \geq \alpha \right) \leq \alpha. \]

Finally, by Lemma 6.2 and arguments in [SS09, Section 4.6], we obtain the conclusion.

Proof of Theorem 6.1. Recall that $\eta_{\delta}$ is the level line of $\phi_{\delta}$, we parameterize $\varphi(\eta_{\delta})$ by the half plane capacity, and we denote by $W^{\delta}_t$ its driving function. Recall that $\eta \sim$ SLE$_4(-2; -2)$ in $\mathbb{H}$ with force points $y^L = \varphi(a) < 0$ and $y^R = \varphi(c) > 0$, we denote by $W_t$ its driving function. From Lemma 6.3, $W^{\delta}$ is close to
\(W\) in local uniform topology. Combing with [SS09, Section 4.7 and Lemma 4.16], \(\varphi(\eta^\theta)\) and \(\eta\) are close in Hausdorff metric. Precisely, for any fixed \(\epsilon, \alpha \geq 0\) small, there is \(\delta_0 \geq 0\) such that, if \(\delta \leq \delta_0\), the interface \(\eta_\delta\) can be coupled with \(\eta\) so that

\[
\mathbb{P}\left(\sup_{t \in [0,T_\delta \wedge T_\epsilon]} d_H(\varphi(\eta_0[0,t],\eta[0,t]) \geq \alpha) \right) \leq \alpha,
\]

where \(d_H\) denotes the Hausdorff metric. For such argument to work, it is important that \((\eta(t), 0 \leq t \leq T)\) hits \(\mathbb{R}\) only at the two end points \(\eta(0) = 0\) and \(\eta(T) \in \{y_L, y_R\}\). Then, using arguments in [SS09, Section 4.8], we arrive at the following conclusion. For any fixed \(\epsilon, \alpha \geq 0\) small, there is \(\delta_0 > 0\) such that, if \(\delta \leq \delta_0\), the interface \(\eta_\delta\) can be coupled with \(\eta\) so that

\[
\mathbb{P}\left(\sup_{t \in [0,T_\delta \wedge T_\epsilon]} |\varphi(\eta_\delta(t)) - \varphi(\eta(t))| \geq \alpha \right) \leq \alpha.
\]

It remains to get the convergence of the whole process. Suppose \(\eta_\epsilon\) is any subsequential limit of \(\eta_\delta\) in Hausdorff metric. From the above argument, we see that \(\varphi(\eta_\epsilon)\) has the same law as \(\eta\) up to \(T_\epsilon\) for any \(\epsilon > 0\). Note that \(\eta\) is continuous up to and including \(T\). We may conclude that \(\varphi(\eta_\epsilon)\) has the same law as \(\eta\) up to \(T\). This gives the convergence of the whole process. Consequently, we obtain the convergence of the crossing probability (combining with (6.2))

\[
\lim_{\delta \to 0} \mathbb{P}\left( (a_\delta, b_\delta) \overset{\varphi(\eta_\delta) \geq 0}{\rightarrow} [c_\delta, d_\delta] \right) = \mathbb{P}(\eta(T) = y^R) = \frac{-y^L}{y^R - y^L}.
\]

We end this section by a discussion on the convergence of discrete GFF level lines in Theorem 6.1 when \(\theta \neq \lambda\). By analyzing level lines of continuous GFF as in [WW17], we believe that the level line \(\eta_\delta\) of \(\phi_\delta\) converges weakly to \(\eta \sim \text{SLE}_4(-2\rho,\rho - 1;\rho - 1, -2\rho)\) in \(R_L\) from \(b\) to \(d\) with force points \((a, b^-; b^+, c)\) where \(\rho = \theta/\lambda\). However, the techniques in [SS09] do not apply to this general setting directly to our knowledge. In particular, the authors in [SS09] derived the convergence of driving function when \(\theta = 0\): they proved that the driving function of \(\eta_\delta\) weakly converges to the driving function of \(\text{SLE}_4(-1; -1)\) in the local uniform topology; however, the convergence in stronger topology is still missing. Assuming the convergence of \(\eta_\delta\) to \(\eta \sim \text{SLE}_4(-1; -1)\) in Hausdorff metric, we may conclude that the crossing probability in (1.5) is convergent:

\[
\lim_{\delta \to 0} \mathbb{P}\left( (a_\delta, b_\delta) \overset{\varphi(\eta_\delta) \geq 0}{\rightarrow} [c_\delta, d_\delta] \right) = \mathbb{P}(\eta \text{ hits } [c, d] \text{ before } [d, a]).
\]

Whereas, to get an explicit formula as in (1.9) for the right-hand side is another open question.

### 6.3 Scaling limits of level lines in metric graph GFF

Recall that \(\tilde{\phi}_\delta\) is metric graph GFF on \(\tilde{V}_\delta\). Suppose the boundary condition is the following: (note that this is different from the one in (1.3))

\[
\tilde{\phi}_\delta(v) = \begin{cases} 
\theta, & v \in [a_\delta, b_\delta) \cup [c_\delta, d_\delta), \\
0, & v \in [b_\delta, c_\delta) \cup [d_\delta, a_\delta). 
\end{cases}
\]

(6.7)

Recall that

\[
\tilde{A}_{\alpha, \delta, 0} = \left\{ v \in \tilde{V}_\delta : \exists \text{ continuous path } \gamma \text{ connecting } v \text{ to } [a_\delta, b_\delta) \text{ such that } \tilde{\phi}_\delta \geq 0 \text{ on } \gamma \right\}.
\]
Define the frontier of $\tilde{A}_{\delta,0}$ as follows: Consider the set of all the points in $\partial \tilde{A}_{\delta,0}$ to $[a_{\delta}, b_{\delta}]$ and in $\tilde{V}_{\delta} \setminus \tilde{A}_{\delta,0}$ to $[b_{\delta}, c_{\delta}]$. This set contains no vertices and the edges it intersects in the dual graph give a path from the point $b_{\delta}$ to the point $(3\delta/2, 1)$ (near $a$) or the point $(L, 3\delta/2)$ (near $c$). We call such path the frontier of $\tilde{A}_{\delta,0}$. Using conclusions in [ALS20, Section 5.2], we may conclude that, when $\theta = 2\lambda$, the frontier of $\tilde{A}_{\delta,0}$ weakly converges to SLE$_4(-2; -2)$ in $R^L$ from $b$ to $d$ with force points $(a; c)$ in Hausdorff metric as $\delta = 2^{-n} \to 0$. Consequently, we have the convergence of the crossing probabilities: (see [ALS20, Remark 5.11])

$$\lim_{\delta = 2^{-n} \to 0} \mathbb{P}\left( [a_{\delta}, b_{\delta}] \xrightarrow{\tilde{\phi}_\delta \geq 0} [c_{\delta}, d_{\delta}] \right) = q(L),$$

where $q$ is the cross ratio of the rectangle. That is, if we let $\varphi$ be any conformal map from $R^L$ onto the upper-half plane $\mathbb{H}$ with $\varphi(a) < \varphi(b) < \varphi(c) < \varphi(d)$, then

$$q(L) = \frac{(\varphi(b) - \varphi(a))(\varphi(d) - \varphi(c))}{(\varphi(c) - \varphi(a))(\varphi(d) - \varphi(b))}. \tag{6.9}$$

We emphasize that the metric graph GFF (6.8) has the boundary condition (6.7) with $\theta = 2\lambda$ which is different from the one in (1.3).

Let us go back to the boundary condition (1.3) and to (1.6) in Theorem 1.2. Suppose we have metric graph GFF with boundary condition (1.3). Using conclusions in [ALS20], we may conclude that the crossing probabilities in (1.6) are convergent. By Theorem 1.2, we know that the limit should be different from the one in the case of discrete GFF. To derive the explicit formula for this limit is an interesting question. We will derive this explicit formula in [LW21] and the answer is the following: Let $\phi_{\delta}$ be the metric graph GFF with boundary condition (1.3) with $\theta = 2\lambda$, then we have

$$\lim_{\delta = 2^{-n} \to 0} \mathbb{P}\left( [a_{\delta}, b_{\delta}] \xrightarrow{\phi_{\delta} \geq 0} [c_{\delta}, d_{\delta}] \right) = q(L)^4, \tag{6.10}$$

where $q(L)$ is the cross ratio of the rectangle defined in (6.9).

References


