A FAMILY OF FRACTIONAL DIFFUSION EQUATIONS
DERIVED FROM STOCHASTIC HARMONIC CHAINS WITH
LONG-RANGE INTERACTIONS

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Abstract. We consider one-dimensional infinite chains of harmonic oscillators
with stochastic perturbations and long-range interactions which have polynomial
decay rate \(|x|^{-\theta}, x \to \infty, \theta > 1\), where \(x \in \mathbb{Z}\) is the interaction range. We prove that
if \(2 < \theta \leq 3\), then the time evolution of the macroscopic thermal energy distribution
is superdiffusive and governed by a fractional diffusion equation with exponent
\(\frac{3}{7} - \theta\), while if \(\theta > 3\), then the exponent is \(\frac{3}{4}\). The threshold is \(\theta = 3\) because the
derivative of the dispersion relation diverges as \(k \to 0\) when \(\theta \leq 3\).

1. Introduction

1.1. Background: Exponential decay model. Chains of oscillators are classical
microscopic systems which are used widely to understand the macroscopic behavior
of energy. A typical example is the Fermi-Pasta-Ulam-Tsingou-chain (FPUT-chain),
which is named after the seminal study by Enrico Fermi, John Pasta, Stanislaw
Ulam, and Mary Tsingou in 50’s. In late 90’s, anomalous heat transport in one-
dimensional unpinned FPUT-Chains is numerically observed [18], and then many
groups started to investigate this phenomenon in various ways, see the reviews [6,
17, 19]. Since the mathematical analysis of such nonlinear deterministic systems is
out of reach of current techniques, the problem has been studied in models where
the nonlinealities are replaced by random exchange of momenta which conserve total
energy and total momentum, see the review [17, Chapter 5]. The model is defined
as follows. Denote by \((p_x(t), q_x(t)) \in \mathbb{R} \times \mathbb{R}\) the momentum and the position of the
particle labeled by \(x \in \mathbb{Z}\) at time \(t \geq 0\). Their stochastic dynamics is given by the
following stochastic differential equation:

\[
\begin{align*}
  dq_x(t) &= p_x(t)dt \\
  dp_x(t) &= \left( - (\alpha * q)_x(t) - \frac{\gamma}{2} (\beta * p)_x(t) \right) dt + \sqrt{\gamma} \sum_{z=-1,0,1} \left( Y_{x+z} p_{x+z}(t) \right) dw_{x+z},
\end{align*}
\]

where \(\ast\) is the discrete convolution on \(\mathbb{Z}\) defined in Section 2 and \(\alpha : \mathbb{Z} \to \mathbb{R}\) is the
interaction potential which satisfies

(a.1) \(\alpha_x \leq 0\) for all \(x \in \mathbb{Z} \setminus \{0\}\), \(\alpha_x \neq 0\) for some \(x \in \mathbb{Z}\).

(a.2) \(\alpha_x = \alpha_{-x}\) for all \(x \in \mathbb{Z}\).

(a.3) There exists some positive constant \(C > 0\) such that \(|\alpha_x| \leq C e^{-|x|}\) for all
\(x \in \mathbb{Z}\).

(a.4) \(\tilde{\alpha}(k) > 0\) for all \(k \neq 0\), \(\tilde{\alpha}(0) = 0\), \(\tilde{\alpha}''(0) > 0\).

Here \(\tilde{\alpha}\) is the discrete Fourier transform defined as

\[
\tilde{\alpha}(k) := \sum_{x \in \mathbb{Z}} \alpha_x e^{-2\pi i x k}, \quad k \in \mathbb{T},
\]
where $T$ is the one-dimensional torus of size one. In addition, the parameter $\gamma > 0$ is the strength of the noise, the vector fields $Y_x$, $x \in \mathbb{Z}$ are defined as

$$Y_x := (p_x - p_{x+1})\partial_{p_{x+1}} + (p_{x+1} - p_{x})\partial_{p_x} + (p_{x-1} - p_{x})\partial_{p_{x+1}},$$

the family of stochastic processes $\{w_x(t); x \in \mathbb{Z}, t \geq 0\}$ are i.i.d. one-dimensional standard Brownian motions, and $\beta_x$, $x \in \mathbb{Z}$ is defined as

$$\beta_x := \begin{cases} 6, & x = 0, \\ -2, & x = \pm 1, \\ -1, & x = \pm 2, \\ 0, & \text{otherwise}. \end{cases}$$

Denote by $r_x(t), e_x(t)$ the inter-particle distance and the energy of particle labeled by $x \in \mathbb{Z}$ defined as

$$r_x(t) := q_x(t) - q_{x-1}(t),$$
$$e_x(t) := \frac{1}{2}|p_x(t)|^2 - \frac{1}{4}\sum_{x' \in \mathbb{Z}, x' \neq x} \alpha_{x-x'}|q_x(t) - q_{x'}(t)|^2.$$

The divergence of the thermal conductivity for this model is proved in [3]. In [13], the authors decompose the energy into the sum of thermal energy and phononic energy, and show that the stochastic perturbation decouple the phononic energy from the thermal energy in the sense that they converge at different time scalings. The phononic energy converges at ballistic scaling. Actually, the following convergence of the scaled empirical measure of $\left(p_x(t), r_x(t), e_x(t)\right)$ holds:

$$\lim_{\epsilon \to 0} \epsilon \sum_{x \in \mathbb{Z}} \frac{p_x(\frac{\epsilon}{2})}{r_x(\frac{\epsilon}{2})} = \int_{\mathbb{R}} dy \left( \bar{p}(y, t) \frac{\bar{r}(y, t)}{\bar{e}_{ph}(y, t)} \right) = J(y)$$

for any test function $J$ where $(\bar{p}, \bar{r}, \bar{e}_{ph})$ is the solution of the following linear wave equation:

$$\begin{cases} \partial_t \bar{r}(y, t) = \partial_y \bar{p}(y, t), \\ \partial_t \bar{p}(y, t) = \frac{\bar{\alpha}(0)}{8\pi^2} \partial_y \bar{r}(y, t), \\ \partial_t \bar{e}_{ph}(y, t) = -\frac{\bar{\alpha}(0)}{8\pi^2} \partial_y (\bar{p} \bar{r})(y, t). \end{cases}$$

Notice that the stochastic perturbation does not explicitly affect the time evolution of the phononic energy. On the other hand, the Boltzmann-type equation is obtained as the time evolution law of the microscopic thermal energy distribution:

$$\partial_t W_\epsilon(y, k, t) + \frac{\epsilon \omega'(k)}{2\pi} \partial_y W_\epsilon(y, k, t) = \gamma \left( \mathcal{L}W_\epsilon(y, \cdot , t) \right) (k) + o(\epsilon),$$

$$\mathcal{L}f(k) = \int_T dk R(k, k') \left( f(k') - f(k) \right),$$

where the local spectral density of energy $W_\epsilon$ depends on the position $y \in \mathbb{R}$ along the chain, the wave number $k \in \mathbb{T} = [-\frac{1}{2}, \frac{1}{2}]$, and time $t \geq 0$. In addition, $\omega(k) = \sqrt{\bar{\alpha}(k)}$ is the dispersion relation, $\mathcal{L}$ is a scattering operator on $\mathbb{T}$ and $R(k, k')$ is the scattering kernel. In [4, 11], under the weak noise assumption $\gamma = \epsilon \gamma_0$, the authors show that the scaled solution of the Boltzmann equation converges to a solution of a fractional
diffusion equation with the exponent \( \frac{3}{4} \) by a two-step procedure. As a first step, by taking a kinetic limit with time scale \( \frac{t}{\epsilon} \), the so-called Boltzmann equation is derived:

\[
\partial_t W(y, k, t) + \frac{\omega'(k)}{2\pi} \partial_y W(y, k, t) = \gamma_0 \left( \mathcal{L} W(y, \cdot, t) \right)(k).
\]

As a second step, they consider a limit of the rescaled energy distribution \( \{ W_N(y, k, Nt) \}_{N \in \mathbb{N}} \) defined as the solution of

\[
\partial_t W_N(y, k, t) + \frac{\omega'(k)}{2N^{\frac{3}{4}}} \partial_y W_N(y, k, t) = \gamma_0 \left( \mathcal{L} W_N(y, \cdot, t) \right)(k).
\]

Thanks to the scattering effect \( \mathcal{L} \), the resulting limit \( u(y, t) := \lim_{N \to \infty} W_N(y, k, Nt) \)

\[
\text{homogenized on } T \text{ and satisfies } \partial_t u(y, t) = -c_{\alpha, \gamma_0}(-\Delta)^{\frac{3}{4}} u(y, t), \quad c_{\alpha, \gamma_0} > 0.
\]

More recently, in [12] the authors proved a direct limit to the fractional diffusion from the microscopic model with the stronger noise \( \gamma = \epsilon^s \gamma_0, 0 \leq s < 1 \), and the time evolution of the direct limit \( \{ W(y, t); y \in \mathbb{R}, t \geq 0 \} \) is governed by

\[
\partial_t W(y, t) = -C_{\alpha, \gamma_0}(-\Delta)^{\frac{3}{4}} W(y, t), \quad C_{\alpha, \gamma_0} > 0.
\]

Note that they did not prove whether \( c_{\alpha, \gamma_0} = C_{\alpha, \gamma_0} \) or not. In general, the two-step limit for an anharmonic chain does not coincide with the direct limit of that [21, 22].

Since the above scaling limit results agree with numerical simulations and theoretical prediction by H. Spohn [22] about FPUT-Chains, the stochastic harmonic chain is considered to be a good approximation of some nonlinear chain. In this way, though the exponential decay interaction potential may have infinite range, the macroscopic behavior of energy is essentially the same as that of the nearest neighbor model \( (\alpha_0 = 2, \alpha_{+1} = -1, \alpha_z = 0, |z| \geq 2) \) and the effect of microscopic long-range interaction on the time evolution of macroscopic energy distribution remains unclear. Hence a natural generalization is to study a model which has slower decay rate.

1.2. Polynomial decay model. In the present study, we consider stochastically perturbed harmonic chains which have polynomial decay rate interaction potentials, that is,

\[
\alpha_x := -|x|^{-\theta}, \quad x \in \mathbb{Z} \setminus \{0\} \quad \alpha_0 := 2 \sum_{x \in \mathbb{N}} |x|^{-\theta} \quad \theta > 1.
\]

Notice that our interaction potentials do not satisfy the condition (3.3). When \( \theta \leq 3 \), (3.4) is not satisfied because \( \bar{\omega}'(k) \) is not a continuous function on \( T \). Our stochastic perturbation is the same as that of exponential decay models. Following the idea of [12], we show the direct limit in Theorem 1, and also by using the strategy of [4, 11] we show the two-step limit under the weak noise assumption in Theorem 2 and 3. The time evolution of the macroscopic thermal energy is governed by a fractional diffusion equation and the exponent of the fractional diffusion changes according to the value of \( \theta \). The equations become

\[
\begin{align*}
\partial_t W(y, t) &= \begin{cases} 
-(2\pi)^{-\frac{6}{\theta}} C_{\theta, \gamma_0}(-\Delta)^{\frac{3}{4}} W(y, t) & 2 \leq \theta \leq 3, \\
-(2\pi)^{\frac{3}{\theta}} C_{\theta, \gamma_0}(-\Delta)^{\frac{3}{4}} W(y, t) & \theta > 3,
\end{cases} 
\text{(Thm 1)} \\
\partial_t u(y, t) &= \begin{cases} 
-(2\pi)^{-\frac{6}{\theta}} c_{\theta, \gamma_0}(-\Delta)^{\frac{3}{4}} u(y, t) & 2 \leq \theta \leq 3, \\
-(2\pi)^{-\frac{3}{\theta}} c_{\theta, \gamma_0}(-\Delta)^{\frac{3}{4}} u(y, t) & \theta > 3,
\end{cases} 
\text{(Thm 2, 3)}
\end{align*}
\]
where $C_{\theta,\gamma_0}, c_{\theta,\gamma_0}$ are positive constants which depend on $\theta > 2, \gamma > 0$. We also show that the superdiffusion coefficient obtained by the direct limit coincides with that obtained by the two-step limit, $c_{\theta,\gamma_0} = C_{\theta,\gamma_0} > 0$. Applying our calculation to the exponential decay model, one can obtain $c_{\alpha,\gamma_0} = C'_{\alpha,\gamma_0}$.

In particular, if $\theta > 3$ then the exponent is the same as that of the exponential decay model. The threshold is $\theta = 3$ because the derivative of the dispersion relation diverges as $k \to 0$ when $\theta \leq 3$:

$$\omega'(k) = \frac{\alpha'(k)}{\sqrt{\alpha(k)}} \sim \begin{cases} |k|^{-\frac{3}{a}} & 2 \theta < 3, \\ \sqrt{-\log(|k|)} & \theta = 3, \\ 1 & \theta > 3, \end{cases} \quad k \to 0.$$  

Roughly speaking, if $\omega'(k) \sim |k|^a$ and the mean value of the scattering kernel $R(k) = \int k' R(k,k') \sim k^b$ as $k \to 0$, then one will get $\frac{b+1}{2(b-a)}$ - fractional diffusion by the scaling limit of the thermal energy distribution when $\frac{b+1}{2(b-a)} \geq 1$ or $a > b \geq 0$. There are several choices of dispersion relation and stochastic perturbation, and an exponent of fractional diffusion depends on one’s choice. Here is a table about asymptotic exponents of feature values of some well-studied models, and one can see that our formal discussion agrees with prior researches, see [5, 12, 14, 15]. Superdiffusion of energy only occurs in the unpinned model of [12] and our models.

<table>
<thead>
<tr>
<th>Model</th>
<th>Potential</th>
<th>Noise</th>
<th>$a$</th>
<th>$b$</th>
<th>$(b+1)/2(b-a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[5]</td>
<td>exp.UP</td>
<td>NLMCN</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>[12]</td>
<td>exp.UP</td>
<td>LMCN</td>
<td>0</td>
<td>2</td>
<td>3/4</td>
</tr>
<tr>
<td>[12]</td>
<td>exp.P</td>
<td>LMCN</td>
<td>1</td>
<td>2</td>
<td>3/2</td>
</tr>
<tr>
<td>[14]</td>
<td>NA</td>
<td>LMCN</td>
<td>1</td>
<td>2</td>
<td>3/2</td>
</tr>
<tr>
<td>[15]</td>
<td>exp.UP</td>
<td>NLMCN</td>
<td>0</td>
<td>0</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Our Model</th>
<th>Potential</th>
<th>Noise</th>
<th>$a$</th>
<th>$b$</th>
<th>$(b+1)/2(b-a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 \theta &lt; 3$</td>
<td>LMCN</td>
<td>$-(3-\theta)/2$</td>
<td>2</td>
<td>3/4</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$\theta = 3$</td>
<td>LMCN</td>
<td>0 with log-corr.</td>
<td>2</td>
<td>3/4</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$\theta &gt; 3$</td>
<td>LMCN</td>
<td>0</td>
<td>2</td>
<td>3/4</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

exp.UP = exp.decay-unpinned, exp.P = exp.decay-pinned, NA = non-acoustic, LMCN = locally momentum conservative noise, NLMCN = non-LMCN.

The effect of the logarithmic correction at $\theta = 3$ does not appear in the exponent of the fractional diffusion, but the space-time scaling is different between the case $\theta > 3$ and $\theta = 3$, see (4.6). The threshold of parameters $a, b$ is obtained by convergence or divergence of the following integral:

$$\lim_{\epsilon \to 0} \int_\Sigma \frac{dk}{R(k)} \bigg( \omega'(k) \bigg)^2 = \begin{cases} \infty & b+1 \in (0, 1], \\ < \infty & b+1 \in (b-a) > 1 \text{ or } a > b. \end{cases}$$  

This integral is expected to be proportional to thermal conductivity and diffusive coefficient and appears in the final part of the proof of Theorem 1, see (5.50). If $\frac{b+1}{2(b-a)} = 1$, then the time scaling should be diffusive with log-correction, and the expected macroscopic behavior is normal diffusion. Note that if there is no stochastic
noise and \( \theta > 2 \), then the behavior of the thermal energy is purely ballistic. Actually, from (4.14) with \( \gamma_0 = 0 \) we see that the local spectral density \( W(y, k, t) \) satisfies the following linear transport equation

\[
\partial_t W(y, k, t) + \frac{\omega'(k)}{2\pi} \partial_y W(y, k, t) = 0,
\]

in some weak sense. We also note that in [7] the same linear transport equation is derived from harmonic chains with local interactions \((\langle |\alpha_z|\rangle = 0 \text{ for sufficiently large } z \in \mathbb{Z})\), and then in [8] the authors extend the analysis of [7] to a wilder class of harmonic chains with exponential decay interactions.

In [24], the authors consider a finite model with periodic boundary condition and they obtain the relationship between the decay rate of the polynomial interaction \( \delta > 0 \) and the decay speed of the current correlation function \( C(t) \sim t^{-\delta}, t \to \infty. \) For the unpinned model, they show that if \( 2 < \delta < 3 \) then \( \beta(\delta) = \frac{\theta - 2}{2} \) and if \( \delta > 3 \) then \( \beta(\delta) = \frac{3}{2} \), which is the same exponent as that of short-range model, so their result agrees with ours.

In a forthcoming article [23], when \( 1 < \theta \leq 3 \) we show the convergence of the scaled empirical measure of \((p_x(t), l_x(t), e_x(t))\) to the limit \((\bar{p}(y, t), \bar{l}(y, t), \bar{e}(y, t))\) at superballistic scaling, while when \( \theta > 3 \) we show the convergence at ballistic scaling, where \( l_x, x \in \mathbb{Z} \) is a new dual variable for \( p_x, x \in \mathbb{Z} \) and is defined via its inverse Fourier transform

\[
\hat{\bar{l}}(k) := -\sqrt{-1} \text{sgn}(k)\omega(k)\hat{q}(k) \quad k \in \mathbb{T},
\]

\[
\text{sgn}(k) := \begin{cases} 
1 & 0 < k < \frac{1}{2} \\
0 & k = 0 \\
-1 & -\frac{1}{2} \leq k < 0.
\end{cases}
\]

We call \( l_x \) generalized tension at \( x \in \mathbb{Z} \), because if \( \theta > 3 \) then \( r_x(t) \) coincides with \( l_x(t) \) macroscopically, that is, the scaled empirical measures of \( r_x(t), l_x(t) \) converge at ballistic scaling and the densities of the limits \( \bar{r}(y, t), \bar{l}(y, t) \) coincide up to a nonzero constant multiple. On the other hand, if \( 1 < \theta \leq 3 \) then \( r_x(t) \) does not coincide with \( l_x(t) \) macroscopically at any time scaling. Especially, the phononic energy converges at superballistic scaling when \( 1 < \theta \leq 3 \), and \( \bar{p}(y, t) \) satisfies the superballistic wave equation:

\[
\partial_t^2 \bar{p}(y, t) = \begin{cases} 
-\frac{C(\theta)}{(2\pi)^{\theta-1}} (-\Delta)^{\frac{\theta-1}{2}} \bar{p}(y, t) & 1 < \theta \leq 3, \\
\frac{C(\theta)}{(2\pi)^2} \Delta \bar{p}(y, t) & \theta > 3,
\end{cases}
\]

where \( C(\theta) > 0 \) is defined in (4.11) in this paper.

Note that we study infinite systems with long-range interactions and non-equilibrium initial distribution of finite total energy, and thus it is appropriate for us to define the dynamics through wave functions \( \{\psi(k, t); k \in \mathbb{T}, t \geq 0\} \). If we start from the wave function, we need some argument to define the energy at \( x \) because the inter-particle distance is “not” a macroscopic variable, see Section 3 and [23]. In this paper, we assume thermal-type condition (4.1), which means that the total phononic energy is macroscopically equal to 0, to derive the scaling limit of the thermal energy. On the
other hand, in [23] we assume the so-called phononic-type condition to derive the hydrodynamic limit for \((p_x(t), l_x(t), e_x(t))\). The above initial conditions guarantee that our systems are in \(L^2\) at any time in some sense, and such \(L^2\) bound enable us to derive the scaling limit.

There are numerical results about anharmonic chains with long-range interactions [1],[2],[10] and they exhibit anomalous heat conduction. In [2], it was observed that the thermal conductivity has non-monotonic dependence with \(\theta\) taking a maximum \(\theta = 2\). However, it is outside of the scope of the current study to consider the thermal energy behavior in the case \(\theta \leq 2\) because \(\tilde{\alpha}'(k)\) is not a continuous function and our proof relies on the asymptotic behavior of \(\tilde{\alpha}'(k)\) as \(k \to 0\). Therefore it remains an important open problem.

Our paper is organized as follows: In Section 2 we prepare some notations. In Section 3 we introduce our model. In Section 4 we state our main results, Theorem 1, 2 and 3. Proofs of Theorem 1, 2 and 3 are given in Section 5, 6, 7 respectively.

2. Notations

Let \(\mathbb{R}\) be the real line, \(\mathbb{R}_0 := \mathbb{R} \setminus \{0\}\) and \(\mathbb{R}_{\geq 0} := [0,\infty)\). Let \(\mathbb{Z}\) be the set of all integers and \(\mathbb{Z}_{\geq 0}\) be the set of all non-negative integers. Let \(T\) be the one-dimensional torus and \(T_0 := T \setminus \{0\}\). We often identify \(T \cong [-\frac{1}{2}, \frac{1}{2}]\).

For \(f, g : \mathbb{Z} \to \mathbb{R}, h \in \ell^2(\mathbb{Z})\) we define \(f * g : \mathbb{Z} \to \mathbb{R}\) and \(\tilde{h} \in \ell^2(T)\) as
\[
(f * g)_x := \sum_{x' \in \mathbb{Z}} f_{x-x'} g_{x'},
\]
\[
\tilde{h}(k) := \sum_{x \in \mathbb{Z}} e^{-2\pi \sqrt{-1} k x} h_x.
\]

For \(J : \mathbb{R} \to \mathbb{C}\) such that \(J(y)\) is rapidly decreasing in \(y \in \mathbb{R}\) we define \(\tilde{J} : \mathbb{R} \to \mathbb{C}\) as
\[
\tilde{J}(p) := \int_{\mathbb{R}} dy \ e^{-2\pi \sqrt{-1} py} J(y).
\]

Denote by \(S(\mathbb{R} \times T)\) the space of smooth functions \(J : \mathbb{R} \times T \to \mathbb{C}\) satisfying
\[
\sup_{y \in \mathbb{R}, k \in T} (1 + y^2)^{n_1} |\partial_y^{n_2} \partial_k^{n_3} J(y, k)| < \infty
\]
for any \(n_i \in \mathbb{Z}_{\geq 0}, i = 1, 2, 3\). We introduce a norm \(|| \cdot ||\) on \(S(\mathbb{R} \times T)\) defined as
\[
||J|| := \int_{\mathbb{R}} dp \sup_k |\tilde{J}(p, k)|
\]
for all \(J \in S(\mathbb{R} \times T)\). The topology of \(S(\mathbb{R} \times T)\) is defined via this norm. We regard \(S(\mathbb{R})\), the Schwartz space on \(\mathbb{R}\), as a subspace of \(S(\mathbb{R} \times T)\):
\[
S(\mathbb{R}) := \{J \in S(\mathbb{R} \times T) ; J(y, k) = J(y)\}.
\]
By \(S(\mathbb{R} \times T)', S(\mathbb{R})'\) we denote the dual spaces of \(S(\mathbb{R} \times T), S(\mathbb{R})\) respectively.

For two functions \(f, g\) defined on common domain \(A\), we write \(f \leq g\) or \(g \geq f\) if there exists some positive constant \(C > 0\) such that \(f(a) \leq C g(a)\) for any \(a \in A\). If \(f \leq g\) and \(g\) is a positive constant, then we also write \(\sup_{a \in A} f(a) \leq 1\).

For two functions \(f, g\) defined on common open subset \(A \subset \mathbb{R}\) and a number \(y_0 \in \overline{A}\), where \(\overline{A}\) is the closure of \(A\), we write \(f(y) = O(g(y))\) as \(y \to y_0\) or \(f(y) = o(g(y))\) as \(y \to y_0\).
\[ y \to y_0 \text{ if} \]
\[ 0 < \lim_{y \to y_0} \left| \frac{f(y)}{g(y)} \right| < \infty. \]

In addition, we write \( f(y) = o(g(y)) \) as \( y \to y_0 \) if
\[ \lim_{y \to y_0} \left| \frac{f(y)}{g(y)} \right| = 0. \]

We introduce some functions. Denote by \( \text{sgn}(y), y \in \mathbb{R} \) the sign function defined as
\[ \text{sgn}(y) := \begin{cases} 
1 & y > 0, \\
0 & y = 0, \\
-1 & y < 0. 
\end{cases} \]

The cosecant function is denote by \( \csc(\pi y) := \left( \sin(\pi y) \right)^{-1}, y \in \mathbb{R} \setminus \mathbb{Z} \). Let \( \text{Re}(z) \) and \( \text{Im}(z) \) denote the real part and the imaginary part of \( z \in \mathbb{C} \) respectively. The gamma function is denote by \( \Gamma(z) := \int_0^\infty dy \, y^{z-1} e^{-y}, \text{Re}(z) > 0, z \in \mathbb{C} \).

3. The dynamics

In this section we define harmonic chains with noise and long-range interactions. Since we analyze the system with finite total energy, it is appropriate for us to define the dynamics through the wave functions \( \{ \psi(k,t); k \in \mathbb{T}, t \geq 0 \} \) as \( L^2(\mathbb{T}) \) solution of the stochastic differential equation (3.10). Then we can reconstruct the classical variables \( \{ p_x(t), q_x(t); x \in \mathbb{Z}, t \geq 0 \} \) from \( \{ \psi(k,t); k \in \mathbb{T}, t \geq 0 \} \) and define the energy \( \{ e_x(t); x \in \mathbb{Z}, t \geq 0 \} \). However, it may be difficult to understand the physical meaning of the important functions such as \( \theta(k) \) and \( R(k) \) from (3.10). To clarify the meaning of the feature values, we first give a formal description of the dynamics in terms of \( \{ p_x(t), q_x(t); x \in \mathbb{Z}, t \geq 0 \} \) in Section 3.1 and 3.3, and introduce the wave function \( \{ \psi(k,t); k \in \mathbb{T}, t \geq 0 \} \) in Section 3.2. Since we do not specify the initial condition \( \{ p_x(0), q_x(0); t \geq 0 \} \) until the last half of Section 3.3, the above construction of the dynamics is formal.

3.1. Deterministic Dynamics. First we consider harmonic chains with long-range interaction without noise. The configuration space is \( (\mathbb{R} \times \mathbb{R})^\mathbb{Z} \) and a configuration is denoted by \( \{(p_x, q_x) \in \mathbb{R} \times \mathbb{R}; x \in \mathbb{Z}\} \). The formal Hamiltonian of the system is given by
\[ H(p, q) := \sum_{x \in \mathbb{Z}} \left( \frac{1}{2} |p_x|^2 - \frac{1}{4} \sum_{x' \in \mathbb{Z}, x \neq x'} \alpha_{x-x'} |q_x - q_{x'}|^2 \right) \]
\[ = \sum_{x \in \mathbb{Z}} e_x \]
where \( e_x := \frac{1}{2} |p_x|^2 - \frac{1}{4} \sum_{x' \in \mathbb{Z}, x \neq x'} \alpha_{x-x'} |q_x - q_{x'}|^2 \) is called the energy at \( x \), the interaction potential \( \alpha: \mathbb{Z} \to \mathbb{R} \) is defined as
\[ \alpha := \begin{cases} 
-|x|^{-\theta}, & x \neq 0, \\
2 \sum_{x \in \mathbb{N}} |x|^{-\theta} & \text{for } x = 0 \end{cases} \quad (3.1) \]
and $\theta > 1$ is a positive constant. The time evolution law of $\{p_x(t), q_x(t); x \in \mathbb{Z}, t \geq 0\}$ is given by the following differential equations

$$
\begin{aligned}
\frac{dq_x(t)}{dt} &= \frac{\partial H}{\partial p_x}(p(t), q(t)) dt = p_x(t) dt \\
\frac{dp_x(t)}{dt} &= -\frac{\partial H}{\partial q_x}(p(t), q(t)) dt = -(\alpha \ast q)_x(t) dt.
\end{aligned}
$$

(3.2)

We introduce an operator $A$ defined as

$$
A = \sum_{x \in \mathbb{Z}} \left( p_x \partial_{q_x} - \sum_{x' \in \mathbb{Z}} \alpha_{x-x'} q_{x'} \partial_{p_x} \right).
$$

Then our deterministic dynamics satisfies $\frac{d}{dt} f(p, q) = (Af)(p, q)$ for any smooth local function $f$.

### 3.2. Wave function

In this subsection we define the wave function of the deterministic Hamiltonian dynamics. From (3.2), we have the time evolution of $\{\overline{p}(k, t), \overline{q}(k, t); k \in \mathbb{T}, t \geq 0\}$

$$
\frac{d}{dt} \begin{pmatrix}
\overline{q}(k, t) \\
\overline{p}(k, t)
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-\overline{\alpha}(k) & 0
\end{pmatrix} \begin{pmatrix}
\overline{q}(k, t) \\
\overline{p}(k, t)
\end{pmatrix},
$$

(3.3)

The eigenvalues of the matrix appeared in the right hand side of (3.3) are $\pm \sqrt{-1} \omega(k)$, $\omega(k) = \sqrt{\overline{\alpha}(k)}$, $k \in \mathbb{T}$ and corresponding eigenvectors $\overline{\psi}(k, t), \overline{\psi}^*(k, t), k \in \mathbb{T}, t \geq 0$ can be written as

$$
\overline{\psi}(k, t) = \omega(k) \overline{q}(k, t) + \sqrt{-1} \overline{p}(k, t),
$$

$$
\overline{\psi}^*(k, t) = \omega(k) \overline{q}(-k, t) - \sqrt{-1} \overline{p}(-k, t),
$$

(3.4)

and the time evolution of $\overline{\psi}(k, t), \overline{\psi}^*(k, t)$ are given by

$$
\frac{d}{dt} \overline{\psi}(k, t) = -\sqrt{-1} \omega(k) \overline{\psi}(k, t) dt,
$$

$$
\frac{d}{dt} \overline{\psi}^*(k, t) = \sqrt{-1} \omega(k) \overline{\psi}^*(k, t) dt.
$$

Here we normalize eigenvectors to satisfy $\int_{\mathbb{T}} dk |\overline{\psi}(k, t)|^2 = 2H(p, q)$. We call $\{\overline{\psi}(k, t); k \in \mathbb{T}, t \geq 0\}$ the wave function of the (deterministic) dynamics.

**Remark 3.1.** From (3.4), we can represent $\{\overline{p}(k, t), \overline{q}(k, t); k \in \mathbb{T}, t \geq 0\}$ by using the wave function:

$$
\begin{aligned}
\overline{q}(k, t) &= \frac{1}{2\omega(k)} \left( \overline{\psi}(k, t) + \overline{\psi}^*(-k, t) \right), \\
\overline{p}(k, t) &= -\frac{\sqrt{1}}{2} \left( \overline{\psi}(k, t) - \overline{\psi}^*(-k, t) \right).
\end{aligned}
$$

(3.5)

Actually, as we will see in Section 3.3, we define $\{\overline{\psi}(k, t); k \in \mathbb{T}, t \geq 0\}$ first as the dynamics on $L^2(\mathbb{T})$ and then we define $\{\overline{p}(k, t), \overline{q}(k, t); k \in \mathbb{T}, t \geq 0\}$ by the equations (3.5). Note that from (B.1) in Appendix B we see that the asymptotic behavior of $\omega(k)$ is as follows:

$$
\omega(k) \sim \begin{cases} 
|k|^{\frac{2\theta}{2}} & 2 < \theta < 3, \\
|k|^{\sqrt{-\log(|k|)}}, & \theta = 3, \\
|k| & \theta > 3,
\end{cases}
$$

for $k \to 0$. 


Thus, the variable $\tilde{q}(k, t)$ is not necessarily defined as an element of $L^2(T)$ in general.

3.3. Stochastic noise and rigorous definition of the dynamics. Next we add to this deterministic dynamics (3.2) a stochastic noise which conserves $p_{x-1} + p_x + p_{x+1}$ and $p_{x-1}^2 + p_{x}^2 + p_{x+1}^2$, $x \in \mathbb{Z}$. The corresponding stochastic differential equations are written as

$$
\begin{aligned}
\{dq_x(t) = p_x(t) dt \\
\{dp_x(t) = \left(- (\alpha * q)_x(t) - \frac{\gamma}{2} (\beta * p)_x(t) \right) dt + \sqrt{\gamma} \sum_{z=-1,0,1} \left(Y_{x+z} p_x(t) \right) dw_{x+z},
\end{aligned}
$$

(3.6)

where $\gamma > 0$ is the strength of the noise and $Y_x, x \in \mathbb{Z}$ are vector fields defined as

$$
Y_x := (p_x - p_{x+1}) \partial_{p_{x-1}} + (p_{x+1} - p_x) \partial_{p_x} + (p_{x-1} - p_x) \partial_{p_{x+1}},
$$

$\{w_x(t); x \in \mathbb{Z}, t \geq 0 \}$ are i.i.d. one-dimensional standard Brownian motions, and

$$
\beta_x := \begin{cases} 
6, & x = 0, \\
-2, & x = \pm 1, \\
-1, & x = \pm 2, \\
0, & \text{otherwise.}
\end{cases}
$$

In other words, $\{p_x(t), q_x(t); t \geq 0 \}$ is a Markov process generated by $A + \gamma S$, where $S := \sum_{x \in \mathbb{Z}} (Y_x)^2$. Note that $A + \gamma S$ conserves the total energy and the total momentum.

From (3.6), the time evolution of $\{\bar{p}(k, t), \tilde{q}(k, t); k \in \mathbb{T}, t \geq 0 \}$ is given by

$$
\begin{aligned}
\{d\bar{p}(k, t) = \bar{p}(k, t) dt \\
\{d\tilde{q}(k, t) = \left(- \tilde{\alpha}(k) \tilde{q}(k, t) - 2\gamma R(k) \bar{p}(k, t) \right) dt \\
+ 2\sqrt{-1} \int_{\mathbb{T}} r(k, k') \bar{p}(k - k', t) B(\text{dk}', dt),
\end{aligned}
$$

(3.7)

where

$$
R(k) := \frac{\tilde{\beta}(k)}{4} = 2 \sin^4(\pi k) + \frac{3}{2} \sin^2(2\pi k),
$$

(3.8)

$$
r(k, k') := 2 \sin^2(\pi k) \sin \left(2\pi(k - k') \right) + \sin \left(2\pi k \right) \sin \left(2\pi (k - k') \right)
$$

(3.9)

and the process $B(\text{dk}, dt)$ is a cylindrical Wiener noise on $L^2(T)$ defined as

$$
B(\text{dk}, dt) := \sum_{x \in \mathbb{Z}} e^{2\pi \sqrt{-1} k x} w_x(dt).
$$

Then we also define $\{\bar{\tilde{q}}(k, t); k \in \mathbb{T}, t \geq 0 \}$ by (3.4) for this stochastic system. From (3.7), the time evolution of $\{\bar{\tilde{q}}(k, t); k \in \mathbb{T}, t \geq 0 \}$ can be written as

$$
\begin{aligned}
\{d\bar{\tilde{q}}(k, t) = \left[ - \sqrt{-1} \omega(k) \bar{\tilde{q}}(k, t) - \gamma R(k) \left( \bar{\tilde{q}}(k, t) - \bar{\tilde{q}}^*( -k, t) \right) \right] dt \\
+ \sqrt{-1} \sqrt{\gamma} \int_{\mathbb{T}} r(k, k') \left( \bar{\tilde{q}}(k - k', t) - \bar{\tilde{q}}^*(k' - k, t) \right) B(\text{dk}', dt).
\end{aligned}
$$

(3.10)
Remark 3.2. Another well-studied stochastic noise, which is local and conserves the total energy and the total momentum, is the jump-type momentum exchange noise defined as follows. We introduce an operator \( S' \) defined as
\[
(S' f)(p) := \sum_{x \in \mathbb{Z}} \left( f(p^{x,x+1}) - f(p) \right)
\]
for any bounded local function \( f \) where \( p^{x,x+1} \) is defined as
\[
p^{x,x+1} := \begin{cases} 
p_{x+1} & z = x, \\
p_x & z = x + 1, \\
p_z & \text{otherwise.}
\end{cases}
\]
Then one can consider the Markov process \( \{p_x(t), q_x(t) ; t \geq 0\} \) generated by \( A + \gamma S' \) or the corresponding stochastic differential equation for \( \{\tilde{\psi}(k,t) ; k \in \mathbb{T}, t \geq 0\} \). For instance, in the review [17, Chapter 5] the jump-type noise is adopted. Note that one can obtain the same scaling limit as that of our model because the asymptotic behavior of the mean value of the scattering kernel for the jump-type noise \( R_S(k) \) is also \( k^2 \) as \( k \to 0 \). See also [13, Remark 3.3] at the end of [13, Section 3.3] for exponential-decay models with jump-type noises.

Now we present the rigorous definition of the dynamics. As we have mentioned at the beginning of this section, we define \( \{\tilde{\psi}(k,t) \in \mathbb{L}^2(\mathbb{T}) ; k \in \mathbb{T}, t \geq 0\} \) as the solution of (3.10) with initial distribution \( \mu_0 \) where \( \mu_0 \) is an arbitrary probability measure on \( \mathbb{L}^2(\mathbb{T}) \). For \( \theta > 1 \), we can show the existence and uniqueness of the solution by using a classical fixed point theorem, see Appendix A for the sketch of the proof. Once we define the dynamics \( \{\tilde{\psi}(k,t) \in \mathbb{L}^2(\mathbb{T})\} \), then we can also define \( \{\tilde{p}(k,t), \tilde{q}(k,t) ; k \in \mathbb{T}, t \geq 0\} \) by (3.5). Since \( \tilde{q}(k) \) may not be in \( \mathbb{L}^2(\mathbb{T}) \), we cannot define \( q_x \) as the Fourier coefficient of \( \tilde{q}(k) \). But still it is sufficient to define \( \sum_{x'} \alpha_{x-x'}(q_x - q_{x'})^2 \) and so \( e_x(t) \) by the following argument: suppose that \( \{q_x, x \in \mathbb{Z}\} \) is an \( \ell^2(\mathbb{Z}) \) element, then the Fourier transform of \( \sum_{x'} \alpha_{x-x'}|q_x - q_{x'}|^2 \) is equal to
\[
\frac{1}{4} \int_{\mathbb{T}} dk' F(k-k', k') \left( \tilde{\psi}(k') + \tilde{\psi}(-k')^* \right) \left( \tilde{\psi}(k-k') + \tilde{\psi}(-k'-k')^* \right).
\]
Therefore when we start from the wave functions to define the dynamics, we define \( \sum_{x'} \alpha_{x-x'}|q_x - q_{x'}|^2 \) as the Fourier coefficient of the above integration:
\[
\sum_{x'} \alpha_{x-x'}|q_x - q_{x'}|^2 := \frac{1}{4} \int_{\mathbb{T}} dk' e^{2\pi i (k+k')x} F(k,k') \times \left( \tilde{\psi}(k) + \tilde{\psi}(-k)^* \right) \left( \tilde{\psi}(k') + \tilde{\psi}(-k')^* \right). \tag{3.11}
\]
Then we can define the energy of particle \( x \) in the usual way:
\[
e_x(t) := \frac{1}{2} |p_x(t)|^2 - \frac{1}{4} \sum_{x' \in \mathbb{Z}, x' \neq x} \alpha_{x-x'}|q_x(t) - q_{x'}(t)|^2.
\]

4. Main Result : Superdiffusive behavior of the energy

In this section we state our main results about the scaling limit of the energy. First we introduce the Wigner distribution, which is a good substitute of the empirical measure of the energy. We show that space-time-noise-scaled Wigner distribution
converges to a solution of the fractional diffusion equation, see Theorem 1, Theorem 2 and Theorem 3. Since the strength of the noise means the strength of some nonlinear effect, if the noise is weaker, then the scaling of the time should be slower (cf. Theorem 1). Critical scaling is space : time : noise = $\epsilon : \epsilon : \epsilon$, $\epsilon \to 0$. (cf. Theorem 2, Theorem 3)

4.1. **Wigner distribution (local spectral density).** Let $0 < \epsilon < 1$ be a scale parameter and $\{\mu_\epsilon\}_{0<\epsilon<1}$ be a family of probability measures on $L^2(\mathbb{T})$. We define $\{\hat{\psi}(k,t) = \hat{\psi}_\epsilon(k,t) \in L^2(\mathbb{T}); k \in \mathbb{T}, t \geq 0\}_{0<\epsilon<1}$ as the solution of (3.10) with initial condition $\mu_\epsilon$ and $\gamma := \epsilon^s \gamma_0$, $\gamma_0 > 0$, $0 \leq s \leq 1$. The exponent $0 \leq s \leq 1$ represents the strength of the noise. If $s = 1$ (resp. $0 \leq s < 1$), then we say that the noise is weak (resp. strong). Denote by $E_{\mu_\epsilon}$ the expectation with respect to $\mu_\epsilon$, and denote by $E_{\epsilon}$ the expectation with respect to the distribution of $\{\hat{\psi}(k,t); k \in \mathbb{T}, t \geq 0\}$ which starts from $\mu_\epsilon$. We assume the following energy bound condition, called thermal type condition (cf.[13]):

$$\sup_{0<\epsilon<1} \int_{\mathbb{T}} dk \epsilon^2 |E_{\epsilon}[\hat{\psi}(k)]|^2 \leq K_1$$

(4.1)

where $K_1$ is a positive constant. Note that this assumption and the energy conservation law

$$E_{\epsilon}[|\hat{\psi}(k,t)|^2] = E_{\epsilon}[|\hat{\psi}(k)|^2]$$

(4.2)

imply

$$\sup_{0<\epsilon<1} \int_{\mathbb{T}} dk \epsilon E_{\epsilon}[|\hat{\psi}(k,t)|^2] = \sup_{0<\epsilon<1} \int_{\mathbb{T}} dk \epsilon E_{\epsilon}[|\hat{\psi}(k)|^2] \leq \sqrt{K_1},$$

(4.3)

for any $J \in \mathbb{S}(\mathbb{R}^2), t \geq 0$. We can easily show (4.2) by substituting $p = 0$ for both sides of (5.6) and by taking the integral with respect to $k \in \mathbb{T}$.

Now we define the **Wigner distribution** $W_{\epsilon,s}(t) \in \mathbb{S}(\mathbb{R} \times \mathbb{T})$, which is the local spectral density of the energy as we will see later in Remark 4.1, as follows:

$$\langle W_{\epsilon,s}(t), J \rangle := \frac{\epsilon}{2} \sum_{x,x' \in \mathbb{Z}} E_{\epsilon} \left[ \hat{\psi}_x^*(t) \frac{t}{f_{\theta,s}(\epsilon)} \hat{\psi}_{x'}(t) \frac{t}{f_{\theta,s}(\epsilon)} \right] \int_{\mathbb{T}} dk \epsilon^2 e^{2\pi i (x - x') k} f(t(x + x')^2, k)^*$$

(4.4)

$$= \frac{\epsilon}{2} \int_{\mathbb{R} \times \mathbb{T}} dp dk E_{\epsilon} \left[ \hat{\psi}(k - \frac{ep}{2}) \frac{t}{f_{\theta,s}(\epsilon)} \hat{\psi}(k + \frac{ep}{2}) \frac{t}{f_{\theta,s}(\epsilon)} \right] \mathcal{J}(p,k)^*,$$

(4.5)

for $t \geq 0$, $J \in \mathbb{S}(\mathbb{R} \times \mathbb{T})$. The ratio of space-time scaling $f_{\theta,s}(\epsilon)$ is given by

$$f_{\theta,s}(\epsilon) :=
\begin{cases} 
\epsilon^{\frac{6-s(\theta-1)}{3-\theta}} & 1 < \theta < 3, 0 \leq s \leq 1, \\
\epsilon^{s|\epsilon|^3} & \theta = 3, 0 \leq s < 1, \\
\epsilon & \theta = 3, s = 1, \\
\epsilon^{\frac{3-s}{2}} & \theta > 3, 0 \leq s \leq 1,
\end{cases}$$

(4.6)

where $h_s(\cdot)$ is the inverse function of $y \mapsto \left( \frac{y^4}{-\log(y)} \right)^{\frac{1}{(1-s)}}$ on $[0, 1]$. 

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Remark 4.1. Actually, \( W_{e,+}(t), t \geq 0 \) is well-defined on a wider class of test functions than \( S(\mathbb{R} \times \mathbb{T}) \). If \( J(y, k) = J(k), (y, k) \in \mathbb{R} \times \mathbb{T} \) and \( J(k), k \in \mathbb{T} \) is bounded, then we can define \( \langle W_{e,+}(t), J \rangle \) by (4.4) and we have

\[
\langle W_{e,+}(t), J \rangle = \frac{\epsilon}{2} \int_{\mathbb{T}} dk \mathbb{E}_0 \left[ \left| \hat{\psi}(k, \frac{t}{\hat{f}_{\theta,s}(\epsilon)}) \right|^2 \right] J(k).
\]

(4.7)

From (4.7), we see that the Wigner distribution contains information of the spectral distribution as the local spectral density of the energy. In addition, if \( J \in S(\mathbb{R} \times \mathbb{T}) \) and \( J(y, k) = J(y), (y, k) \in \mathbb{R} \times \mathbb{T} \), then we have

\[
\langle W_{e,+}(t), J \rangle = \frac{\epsilon}{2} \sum_{x \in \mathbb{Z}} \mathbb{E}_0 \left[ \left| \psi_x \left( \frac{t}{\hat{f}_{\theta,s}(\epsilon)} \right) \right|^2 \right] J(\epsilon x).
\]

(4.8)

From (4.8), we see that the Wigner distribution is the empirical measure of \( \{ \left| \psi_x \left( \frac{t}{\hat{f}_{\theta,s}(\epsilon)} \right) \right|^2; x \in \mathbb{Z} \} \). According to (4.7) and (4.8), we can think of the Wigner distribution as the local spectral density of the energy.

Next Proposition ensures that the limit of the Wigner distribution is the macroscopic distribution of the energy:

**Proposition 4.1.** Assume (4.1). Then for any \( t \geq 0, J \in S(\mathbb{R}) \), we have

\[
\lim_{\epsilon \to 0} \left| \langle W_{e,+}(t), J \rangle - \epsilon \sum_{x \in \mathbb{Z}} \mathbb{E}_0 \left[ \left| \psi_x \left( \frac{t}{\hat{f}_{\theta,s}(\epsilon)} \right) \right|^2 \right] J(\epsilon x) \right| = 0.
\]

We postpone the proof of this Proposition to Appendix C.

4.2. Superdiffusive behavior of the energy : direct limit. Now we state one of our main results.

**Theorem 1.** Suppose that \( \theta > 2, 0 \leq s < 1 \), (4.1) and there exists some \( W_0 \in L^1(\mathbb{R}) \) such that

\[
\lim_{\epsilon \to 0} \langle W_{e,+}(0), J \rangle = \int_{\mathbb{R}} dy W_0(y) J(y),
\]

(4.9)

for \( J(y, k) \equiv J(y) \in S(\mathbb{R} \times \mathbb{T}) \). Then for \( J \in C^0_{\infty}(\mathbb{R} \times \mathbb{R}_{\geq 0}) \), we have

\[
\lim_{\epsilon \to 0} \int_{0}^{\infty} dt \langle W_{e,+}(t), J(c,t) \rangle = \int_{\mathbb{R}} dy \int_{0}^{\infty} dt W(y,t) J(y,t),
\]

where \( W(y,t) \) is given via its Fourier transform

\[
\hat{W}(p,t) = \begin{cases} 
\exp \left( -C_{\theta,s} \langle p \rangle \frac{\epsilon}{\sqrt{t}} \right) \hat{W}_0(p) & 2 < \theta \leq 3, \\
\exp \left( -C_{\theta,s} \langle p \rangle \frac{\epsilon}{\sqrt{t}} \right) \hat{W}_0(p) & \theta > 3
\end{cases}
\]

for any \( p \in \mathbb{R} \), moreover \( C_{\theta,s} \) is a positive constant defined as

\[
C_{\theta,s} := \begin{cases} 
\frac{24\pi^2 \csc \left( \frac{(4-\theta)\pi}{2\pi} \right)}{2^{7-\theta}} \left( \frac{\theta - 1}{24\pi^2} \right)^{\frac{\theta}{2}} \frac{2}{\gamma_0^{\frac{\theta-1}{2}}} C(\theta)^{\frac{\theta}{2}} & 2 < \theta \leq 3, \\
\frac{\sqrt{6\gamma_0^{-\frac{1}{2}} C(\theta)^{\frac{3}{2}}}}{12} & \theta > 3,
\end{cases}
\]

(4.10)
and C(\theta) > 0 is defined as
\[
C(\theta) := \begin{cases} 
(2\pi)^{\theta-1} \int_0^\infty dy \frac{2 - 2\cos(y)}{|y|^\theta} & 1 < \theta < 3, \\
(2\pi)^2 & \theta = 3, \\
(2\pi)^2 \sum_{x \geq 1} |x|^{2-\theta} & \theta > 3.
\end{cases}
\]

Remark 4.3. From Theorem 1 and Proposition 4.1, we have the uniform boundness of the Wigner distribution:
\[
\frac{|\{W_{\epsilon,\nu}(t), J]\}}{|J|} \leq \frac{\sqrt{K_1}}{2} \quad 0 < \epsilon < 1, t \geq 0, J \in \mathcal{S}(\mathbb{R} \times \mathbb{T}).
\]

Thanks to (4.12), we obtain the \( \ast \) - weakly sequential compactness of the Wigner distribution in \( \mathcal{L}^\infty([0,T] \times \mathcal{S}) \) for any \( T > 0 \), that is, there exists some subsequence \( \{W_{\epsilon(n)}(t), \nu\}\} \) and an element \( \{W(t), \nu\} \in \mathcal{S}'(\mathbb{R} \times \mathbb{T}) \) such that
\[
\lim_{n \to \infty} \left| \int_0^T dt \langle W_{\epsilon(n)}(t), J \rangle f(t) - \int_0^T dt \langle W(t), \nu \rangle f(t) \right| = 0
\]
for any \( J \in \mathcal{S}(\mathbb{R} \times \mathbb{T}) \), \( f \in \mathcal{L}^1([0,T]) \). Therefore, if we verify the uniqueness of limits of convergent subsequences, then we can conclude the proof of Theorem 1. Note that any limit of a convergent subsequence \( \{W(t), \nu\} \) is positive a.e. \( t \) and we can extend \( \{W(t), \nu\} \) to a finite positive measure \( \mu(t)(dy) \) on \( \mathbb{R} \), see Section 6.1.2.

From Theorem 1 and Proposition 4.1 we conclude that when \( 2 < \theta \leq 3 \), the time evolution of the macroscopic energy distribution is governed by \( \frac{4}{7-\theta} \) fractional diffusion equation.

Remark 4.3. We can also consider pinned models and for these models the interaction potential \( \alpha \) is defined as follows:
\[
\alpha_x := -|x|^{-\theta}, \quad x \in \mathbb{Z} \setminus \{0\} \quad \alpha_0 := \nu + 2 \sum_{x \in \mathbb{N}} |x|^{-\theta} \quad \nu > 0, \theta > 1.
\]

The definition of the wave function and the Wigner distribution are the same as those of unpinned chains, but the time scaling \( f_{\theta,\alpha}(\epsilon) = f_{\theta,\nu,\alpha}(\epsilon) \) is changed as
\[
f_{\theta,\nu,\alpha}(\epsilon) := \begin{cases} 
\epsilon^{\frac{3-(\theta-1)}{4-\theta}} & 2 < \theta < \frac{5}{2}, \ 0 \leq s < 1, \\
\epsilon^{2-s} \log (\epsilon^{-1}) & \theta = \frac{5}{2}, \ 0 \leq s < 1, \\
\epsilon^{2-s} & \theta > \frac{5}{2}, \ 0 \leq s < 1.
\end{cases}
\]

Assume that \( (\theta, s) \in (2, \frac{5}{2}] \times [0,1) \cup (\frac{5}{2}, \infty) \times (0,1) \). Then by using the strategy of Section 5, we have the direct limit for the Wigner distribution as follows:
\[
\lim_{\epsilon \to 0} \int_0^\infty dt \langle W_{\epsilon,\nu}(t), J(\cdot,t) \rangle = \int_\mathbb{R} dy \int_0^\infty dt \ W(y,t)J(y,t),
\]

\( \square \)
where $W(y,t)$ is given via its Fourier transform

$$
\tilde{W}(p,t) = \begin{cases} 
\exp\left(-C_{\theta,\nu,\gamma_0}|p|^4 t\right)\tilde{W}_0(p) & 2 < \theta \leq \frac{5}{2}, \\
\exp\left(-C_{\theta,\nu,\gamma_0}|p|^2 t\right)\tilde{W}_0(p) & \theta > \frac{5}{2}, 
\end{cases}
$$

for any $p \in \mathbb{R}$, moreover $C_{\theta,\nu,\gamma_0}$ is a positive constant defined as

$$
C_{\theta,\nu,\gamma_0} := \begin{cases} 
\frac{12\pi^3 \csc\left(\frac{3(\theta-2)p}{2\theta-4} + \frac{\pi}{4}\right)}{4 - \theta} & 2 < \theta \leq \frac{5}{2}, \\
\gamma_0^{-1} \int_{\mathbb{T}} dk \frac{(\omega'(k))^2}{2R(k)} & \theta > \frac{5}{2}.
\end{cases}
$$

Although the chain is pinned, we see that if $2 < \theta \leq \frac{5}{2}$ then the macroscopic energy behavior is anomalous, and this result also agrees with the result of [24]. Since we want to avoid complicated division into cases, in this paper we do not give the proof for pinned chains.

Note that we can also consider the cases $(\theta,s) \in (\frac{5}{2},\infty) \times \{0\}$, and with some additional computations we can prove that the macroscopic energy behavior is normal, see the remark at the end of Section 5.2.

**Remark 4.4.** It is impossible for us to consider the case $1 < \theta \leq 2$ by using the strategy of [12] because their proof relies only on the asymptotic behavior of $\omega'(k)$ at $k = 0$. In the case $1 < \theta \leq 2$, $\tilde{\alpha}(k)$ and $\omega(k)$ are not continuous functions on $\mathbb{T}_0$.

4.3. Derivation of the Boltzmann equation: micro to meso. If $s = 1$ we need the two-step scaling limit, the first one is from the microscopic scale to the mesoscopic scale, the second one is from the mesoscopic scale to the macroscopic scale, to derive the fractional diffusion. Note that the following Theorem is also true for $\gamma_0 = 0$.

**Theorem 2.** Suppose that $\theta > 2$, $s = 1$, $\gamma_0 \geq 0$ and there exists a finite measure $\mu_0$ on $\mathbb{R} \times \mathbb{T}$ such that

$$
\lim_{\epsilon \to 0} \{W_{\epsilon,\gamma}(0), J\} = \int_{\mathbb{R} \times \mathbb{T}} d\mu_0 J(y,k)
$$

for any $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$. Then we have the following results:

1. For any $t \geq 0$, there exists $W(t) \in \mathcal{S}(\mathbb{R} \times \mathbb{T})^\prime$ such that

$$
\lim_{\epsilon \to 0} \{W(t), J\} - \{W_{\epsilon,\gamma}(t), J\} = 0,
$$

for any $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$.

2. $W(t)$ can be extended to a finite measure $\mu(t)$ on $\mathbb{R} \times \mathbb{T}$.

3. $\mu(t)$ is the unique solution of the following measure-valued Boltzmann equation:

$$
\begin{align*}
\partial_t \int_{\mathbb{R} \times \mathbb{T}} d\mu(t) J(y,k) &= \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{T}} d\mu(t) \omega'(k) \partial_y J(y,k) + \gamma_0 \int_{\mathbb{R} \times \mathbb{T}} d\mu(t) \left( \mathcal{L} J(y,\cdot) \right)(k), \\
\int_{\mathbb{R} \times \mathbb{T}} d\mu(0) J(y,k) &= \int_{\mathbb{R} \times \mathbb{T}} d\mu_0 J(y,k) \quad \mu(t)(dy,\{0\}) = 0,
\end{align*}
$$

(4.14)
for any $J \in \mathcal{S}(\mathbb{R} \times T_0)$, where $L$ is a scattering operator on $T$ defined as

\[
\mathcal{L}J(k) := 2 \int_T dk' R(k, k')(J(k') - J(k)), \quad J \in \mathcal{L}^1(T),
\]

(4.15)

\[
R(k, k') := \frac{1}{2}(r(k, k + k')^2 + r(k, k - k')^2)
\]

\[
= 4\left(\sin^4(\pi k) \sin^2(2\pi k') + \sin^4(\pi k') \sin^2(2\pi k)\right).
\]

(4.16)

Note that the scattering kernel $R(k, k')$ satisfies

\[
\int_T dk' R(k, k') = R(k)
\]

(4.17)

for any $k \in T$ where $R(k)$ is defined in (3.8). We usually expect that the limit of the local spectral density homogenize on $T$ due to the scattering effect. However, $\mu(t)$ does not homogenize on $T$ in the critical case $s = 1$. This implies that there should exist a longer time scaling on which the homogenization on $T$ occurs and the time evolution of the energy on $\mathbb{R}$ is non-trivial, and leads us to the problem of the scaling limit for the rescaled solution of (4.14).

4.4. Derivation of fractional diffusion equation : meso to macro. In this subsection we construct a solution of (4.14) probabilistically. Let $\{K_n \in \mathcal{T}; n \in \mathbb{Z}_{\geq 0}\}$ be a Markov chain whose transition probability $P(k, dk')$ is given by

\[
P(k, dk') := \frac{R(k, k')}{R(k)} \cdot dk'.
\]

The invariant probability measure $\pi$ for $\{K_n \in \mathcal{T}; n \in \mathbb{N}\}$ is written as

\[
\pi(\cdot) := \frac{2R(k)}{3} \cdot dk.
\]

Suppose that $\{\tau_n; n \in \mathbb{N}\}$ be an i.i.d. sequence of random variables such that $\tau_1$ is exponentially distributed with intensity 1, and $\{K_n \in \mathcal{T}; n \in \mathbb{Z}_{\geq 0}\}$ and $\{\tau_n; n \in \mathbb{N}\}$ are independent. Set $t_n := \sum_{m=1}^{n} \frac{1}{2\pi R(K_m-1)} \cdot \tau_m$, $n \geq 1$, $t_0 := 0$. Now we define a continuous-time Markov process $\{K(t) \in \mathcal{T}; t \geq 0\}$ as $K(t) := K_n$ if $t_n \leq t < t_{n+1}$ for some $n \in \mathbb{Z}_{\geq 0}$. By simple computations we see that $\{K(t) \in \mathcal{T}; t \geq 0\}$ is the continuous random walk generated by $2\pi R(k)$. Now, we can construct a solution of (4.14). Suppose that $u_0 : \mathbb{R} \times \mathcal{T} \to \mathbb{R}$ is a function such that $u(\cdot, k) \in C^0_T(\mathbb{R})$ for all $k \in \mathcal{T}$ and $u(y, \cdot) \in C(\mathbb{R})$ for all $y \in \mathbb{R}$. Denote by $\mathbb{E}_k$ the expectation of the dynamics with initial condition $K_0 = k, k \in \mathcal{T}$. Then we define a function $u : \mathbb{R} \times \mathcal{T} \to \mathbb{R}$ as

\[
u(y, k, t) := \mathbb{E}_k \left[u_0(y + Z(t), K(t))\right],
\]

where

\[
Z(t) := \int_0^t ds \omega(K(s)).
\]

The measure $u(y, k, t) dy dt$ is the unique solution of (4.14) with initial condition $u(y, k, 0) dy dk = u_0(y, k) dy dk$. By applying [11, Theorem 2.8], we obtain the scaling limit of $u(y, k, t)$.

**Theorem 3.**

1. As $N \to \infty$, the finite-dimensional distributions of the scaled process $\{\frac{1}{N(t)} Z(Nt); t \geq 0\}$ converge weakly to those of the Lévy process generated by $-2\pi^\frac{6}{\gamma} C_{\theta, \gamma_0}(-\Delta)^{\frac{3}{\gamma}}$
if $2 < \theta \leq 3$ and by $-(2\pi)^{-\frac{\theta}{2}} C_{\theta,\gamma_0} (-\Delta)^{\frac{3}{2}}$ if $\theta > 3$, where $C_{\theta,\gamma_0}$ is given by (4.10) and the ratio of the space-time scaling $N(\theta)$ is defined as

$$N(\theta) := \begin{cases} N^{\frac{\theta}{2}} & 2 < \theta < 3, \\ \log(N) \frac{1}{2} N^{\frac{\theta}{3}} & \theta = 3, \\ N^{\frac{\theta}{2}} & \theta > 3. \end{cases}$$

(2) Suppose that $u_0 \in C^\infty_0(\mathbb{R} \times \mathbb{T})$. Define $u_N(y, k, t)$ as

$$u_N(y, k, t) := E_k \left[ u_0 \left( \frac{1}{N(\theta)} (y + Z(Nt)), K(Nt) \right) \right].$$

Then for any $(y, t) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$, we have

$$\lim_{N \to \infty} \int_{\mathbb{T}} dk \left| u_N(N(\theta)y, k, t) - \bar{u}(y, t) \right|^2 = 0,$$

where $\bar{u}$ is the solution of the following partial differential equation:

$$\begin{cases} \partial_t \bar{u}(y, t) = -(2\pi)^{-\frac{\theta}{2}} C_{\theta,\gamma_0} (-\Delta)^{\frac{3}{2}} u_0(y, t), & 2 < \theta \leq 3, \\ \bar{u}(y, 0) = \int_{\mathbb{T}} dk u_0(y, k). \\ \partial_t \bar{u}(y, t) = -(2\pi)^{-\frac{3}{2}} C_{\theta,\gamma_0} (-\Delta)^{\frac{3}{2}} u_0(y, t), & \theta > 3, \\ \bar{u}(y, 0) = \int_{\mathbb{T}} dk u_0(y, k). \end{cases}$$

We point out that the coefficient of the fractional diffusion obtained by the two-step scaling limit is equal to that obtained by the direct scaling limit (4.10). From Theorem 3 (2), we see that the time evolution of the macroscopic energy distribution $u(y, t)$ is governed by the fractional diffusion equation.

**5. PROOF OF THEOREM 1**

In this section we prove Theorem 1 by following the scheme from [12, Section 8 - 11] and [13, Section 5]. Since the steps are similar, we may refer to the original results of [12]. However, as mentioned in Introduction, if $2 < \theta \leq 3$ then asymptotic behavior of $\bar{\alpha}(k), k \to 0$ is different from that of the exponential decay model, thus we have to make some modifications. Actually, in many situations the asymptotic behaviors of $\omega(k) = \sqrt{\bar{\alpha}(k)}$ and $\omega'(k)$ as $k \to 0$, have an important role to prove Theorem 1. Especially, the time scaling $f_{\theta,\gamma}(\epsilon)$, defined in (4.6), depends on the order of $|\omega'(k)|$ as $k \to 0$. In Appendix B we compute the asymptotic behaviors of $\bar{\alpha}(k)$ and $\bar{\alpha}'(k)$ as $k \to 0$, and thus from (B.1) and (B.2) we obtain the following:

$$\omega(k) \sim \begin{cases} |k|^{\frac{\theta}{2}} & 2 < \theta < 3, \\ \left| k \sqrt{-\log(|k|)} \right| & \theta = 3, \\ |k| & \theta > 3, \end{cases} \quad k \to 0,$$

$$\omega'(k) \sim \begin{cases} |k|^{-\frac{\theta}{2}} & 2 < \theta < 3, \\ \sqrt{-\log(|k|)} & \theta = 3, \\ 1 & \theta > 3, \end{cases} \quad k \to 0.$$
Since if $2 < \theta \leq 3$ then $|\omega'(k)|$ diverges to infinity as $k \to 0$, we have to be careful when we replace the function $\omega(k + \frac{p}{t}) - \omega(k - \frac{p}{t})$, $p \in \mathbb{R}$, which will appear in the time evolution equation of the Wigner distribution, by $p\omega'(k)$ in some sense. For instance, such problem happens in the proof of Proposition 5.3.

5.1. Overview of the proof. In this subsection we outline the very intuitive, and not rigorous proof of Theorem 1. As we have mentioned in the remark below Theorem 1, all we have to do is to show the uniqueness of the limit of any convergent subsequence. To prove this, we fix a convergent subsequence and for notational simplicity we also denote this subsequence by $\{W_{\epsilon,+,}\}_{\epsilon}$.

First we introduce the function $\widetilde{W}_{\epsilon,+(p,k,t)}, (p,k,t) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_{>0}$, which is the Fourier transform of $\{W_{\epsilon,+,}\}$ in $\mathcal{S}(\mathbb{R} \times \mathbb{T})'$ with respect to the space variable, that is, $\widetilde{W}_{\epsilon,+(p,k,t)}$ satisfies

$$
\langle W_{\epsilon,+(t)}, J \rangle = \int_{\mathbb{R} \times \mathbb{T}} dpdk \widetilde{W}_{\epsilon,+(p,k,t)} \overline{\mathcal{J}(p,k)}
$$

for any $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$. Then we calculate the time evolution of $\widetilde{W}_{\epsilon,+(p,k,t)}$. Roughly we obtain

$$
\frac{d}{dt} \widetilde{W}_{\epsilon,+(p,k,t)} = -\frac{\sqrt{-1}\epsilon \omega'(k)}{f_{\theta,s}(\epsilon)} \widetilde{W}_{\epsilon,+(p,k,t)} + \frac{\gamma}{f_{\theta,s}(\epsilon)} (\mathcal{L} \widetilde{W}_{\epsilon,+(p, \cdot, t)})(k) + o_{\epsilon}(1),
$$

where $\mathcal{L}$ is the scattering operator on $\mathbb{T}$ defined in (4.15). Since this differential equation includes the scattering term on $\mathbb{T}$, the limit of the Wigner distribution homogenizes in $k \in \mathbb{T}$, that is, we will obtain $W_{\epsilon,+(y,k,t)} \to W(y,t)$ as $\epsilon \to 0$ in some sense. Then by taking the Laplace transform of both sides of the above equation, we have

$$
\tilde{w}_{\epsilon,+(p,k,\lambda)} - \tilde{w}_{\epsilon,+(p,k,0)} = -\frac{\sqrt{-1}\epsilon \omega'(k)}{f_{\theta,s}(\epsilon)} \tilde{w}_{\epsilon,+(p,k,\lambda)} + \frac{\gamma}{f_{\theta,s}(\epsilon)} (\mathcal{L} \tilde{w}_{\epsilon,+(p, \cdot, \lambda)})(k) + o_{\epsilon}(1),
$$

where $\tilde{w}_{\epsilon,+(p,k,\lambda)} := \int_{0}^{\infty} dt \ e^{-\lambda t} \tilde{W}_{\epsilon,+(p,k,t)}$, $\lambda > 0$. From the above equation and (4.9), we obtain

$$
\left( \int_{\mathbb{T}} dk \frac{2\gamma R(k)}{f_{\theta,s}(\epsilon)} (1 - \frac{2\gamma R(k)}{f_{\theta,s}(\epsilon)\lambda + 2\gamma R(k) + \sqrt{-1}\epsilon \omega'(k)}) \right) \int_{\mathbb{T}} dk \frac{2}{3} R(k) \tilde{w}_{\epsilon,+(p,k,\lambda)} = \left( \int_{\mathbb{T}} dk \frac{2\gamma R(k)}{f_{\theta,s}(\epsilon)\lambda + 2\gamma R(k) + \sqrt{-1}\epsilon \omega'(k)} \right) \tilde{W}_{0}(p) + o_{\epsilon}(1).
$$

In Section 5.4 and 5.5 we will prove that

$$
\int_{\mathbb{T}} dk \frac{2\gamma R(k)}{f_{\theta,s}(\epsilon)} (1 - \frac{2\gamma R(k)}{f_{\theta,s}(\epsilon)\lambda + 2\gamma R(k) + \sqrt{-1}\epsilon \omega'(k)}) \rightarrow \begin{cases} 
\lambda + C_{\theta,\gamma_{0}}|p|^{\frac{6}{3}} & 2 < \theta \leq 3, \\
\lambda + C_{\theta,\gamma_{0}}|p|^\frac{2}{3} & \theta > 3,
\end{cases}
$$

$$
\int_{\mathbb{T}} dk \frac{2\gamma R(k)}{f_{\theta,s}(\epsilon)\lambda + 2\gamma R(k) + \sqrt{-1}\epsilon \omega'(k)} \rightarrow 1,
$$

and

$$
\int_{\mathbb{T}} dk \frac{2}{3} R(k) \tilde{w}_{\epsilon,+(p,k,\lambda)} \rightarrow w(p,\lambda) = \int_{\mathbb{R}} dy \ e^{-2\pi \sqrt{-1}yp} \int_{0}^{\infty} dt \ e^{-\lambda t} W(y,t),
$$
as \( \epsilon \to 0 \), and thus we have
\[
\begin{cases}
(\lambda + C_{\theta, \gamma_0} |p|^2) w(p, \lambda) + \bar{W}_0(p) = 0 & 2 < \theta \leq 3, \\
(\lambda + C_{\theta, \gamma_0} |p|^3) w(p, \lambda) + \bar{W}_0(p) = 0 & \theta > 3,
\end{cases}
\]
for almost every \( p \in \mathbb{R} \) and any \( \lambda > 0 \). By using the one-to-one correspondence of Laplace-Fourier transform we can show that \( W(y, t) \) is the unique solution of the fractional diffusion equation.

The real situation is more complicated, but the correct proof follows the above intuitive story.

5.2. Time evolution of the Wigner distribution. Define \( \bar{W}_{\epsilon, \pm}(p, k, t) : \mathbb{R} \times \mathbb{T} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{C} \) as
\[
\bar{W}_{\epsilon, \pm}(p, k, t) = \frac{\epsilon}{2} \mathbb{E} \left[ \bar{\psi}(k - \frac{cp}{2}, \frac{t}{f_{\theta,s}(\epsilon)}) \right] \bar{\psi}(k + \frac{cp}{2}, \frac{t}{f_{\theta,s}(\epsilon)})].
\]
(5.2)

From (4.5) and (5.2) one can see that \( \bar{W}_{\epsilon, \pm}(p, k, t) \) satisfies (5.1). To get closed equations governing the dynamics of \( \bar{W}_{\epsilon, \pm} \), we also introduce the Fourier transforms of the anti-Wigner distribution \( \bar{Y}_{\epsilon} \) and the \( * \)-Wigner distribution \( \bar{W}_{\epsilon, -} \). Define \( \bar{W}_{\epsilon, -}(p, k, t), \bar{Y}_{\epsilon, \pm}(p, k, t) : \mathbb{R} \times \mathbb{T} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{C}, \ t = \pm \) as
\[
\bar{W}_{\epsilon, -}(p, k, t) := \bar{W}_{\epsilon, \pm}(p, -k, t),
\]
(5.3)
\[
\bar{Y}_{\epsilon, \pm}(p, k, t) := \frac{\epsilon}{2} \mathbb{E} \left[ \bar{\psi}(-k + \frac{cp}{2}, \frac{t}{f_{\theta,s}(\epsilon)}) \right] \bar{\psi}(k + \frac{cp}{2}, \frac{t}{f_{\theta,s}(\epsilon)})].
\]
(5.4)
\[
\bar{Y}_{\epsilon, -}(p, k, t) := \bar{Y}_{\epsilon, \pm}(t)(-p, k, t)^*.
\]
(5.5)

We may abbreviate the variables \( (p, k, t) \) or some of them for simplicity. However, since the notation of \( R \) will be used several times, we always indicate the variable of \( R \) to avoid confusion. From (3.10) and Itô's formula, we see that the time evolution of \( (\bar{W}_{\epsilon, \pm}, \bar{W}_{\epsilon, -}, \bar{Y}_{\epsilon, \pm}, \bar{Y}_{\epsilon, -}) \) is governed by the following differential equations (see also [12, Section 8]):
\[
\frac{d}{dt} \bar{W}_{\epsilon, \pm} = -\frac{\sqrt{-1} \epsilon (\delta_\epsilon \omega)}{f_{\theta,s}(\epsilon)} \bar{W}_{\epsilon, \pm} + \frac{\gamma}{f_{\theta,s}(\epsilon)} \mathcal{L}_{\epsilon p} \bar{W}_{\epsilon, \pm} - \frac{\gamma}{2 f_{\theta,s}(\epsilon)} \sum_{i=\pm} \mathcal{L}_{\epsilon p}^+ \bar{Y}_{\epsilon, -},
\]
(5.6)
\[
\frac{d}{dt} \bar{W}_{\epsilon, -} = -\frac{\sqrt{-1} \epsilon (\delta_\epsilon \omega)}{f_{\theta,s}(\epsilon)} \bar{W}_{\epsilon, -} + \frac{\gamma}{f_{\theta,s}(\epsilon)} \mathcal{L}_{\epsilon p} \bar{W}_{\epsilon, -} - \frac{2 \gamma}{f_{\theta,s}(\epsilon)} \sum_{i=\pm} \mathcal{L}_{\epsilon p}^+ \bar{Y}_{\epsilon, -},
\]
(5.7)
\[
\frac{d}{dt} \bar{Y}_{\epsilon, \pm} = -\frac{\sqrt{-1} \epsilon \bar{\omega}}{f_{\theta,s}(\epsilon)} \bar{Y}_{\epsilon, \pm} + \frac{\gamma}{f_{\theta,s}(\epsilon)} \mathcal{L}_{\epsilon p} \bar{Y}_{\epsilon, \pm} - \frac{\gamma}{f_{\theta,s}(\epsilon)} \mathcal{R}_{\epsilon p} (\bar{Y}_{\epsilon, +} - \bar{Y}_{\epsilon, -})
\]
\[ - \frac{\gamma}{2 f_{\theta,s}(\epsilon)} \sum_{i=\pm} \mathcal{L}_{\epsilon p}^+ \bar{W}_{\epsilon, -},
\]
(5.8)
\[
\frac{d}{dt} \bar{Y}_{\epsilon, -} = -\frac{\sqrt{-1} \epsilon \bar{\omega}}{f_{\theta,s}(\epsilon)} \bar{Y}_{\epsilon, -} + \frac{\gamma}{f_{\theta,s}(\epsilon)} \mathcal{L}_{\epsilon p} \bar{Y}_{\epsilon, -} + \frac{\gamma}{f_{\theta,s}(\epsilon)} \mathcal{R}_{\epsilon p} (\bar{Y}_{\epsilon, +} - \bar{Y}_{\epsilon, -})
\]
\[ - \frac{\gamma}{2 f_{\theta,s}(\epsilon)} \sum_{i=\pm} \mathcal{L}_{\epsilon p}^+ \bar{W}_{\epsilon, -},
\]
(5.9)
where $\delta_\epsilon \omega, \bar{\omega}_\epsilon$ are functions on $\mathbb{R} \times \mathbb{T}$ defined as
\begin{equation}
\delta_\epsilon \omega(p, k) := \frac{1}{\epsilon} \left( \omega(k + \frac{cp}{2}) - \omega(k - \frac{cp}{2}) \right),
\end{equation}
\begin{equation}
\bar{\omega}_\epsilon(p, k) := \frac{1}{2} \left[ \omega(k + \frac{cp}{2}) + \omega(k - \frac{cp}{2}) \right].
\end{equation}

In addition, $\mathcal{R}_p, \mathcal{L}_p, \mathcal{L}_p^\pm, p \in \mathbb{R}$ are operators on $L^2(\mathbb{T})$ defined as
\begin{equation}
\mathcal{R}_p f(k) := \int_{\mathbb{T}} dk' R(k, k', p) f(k'),
\end{equation}
\begin{equation}
R(k, k', p) := \frac{1}{2} \sum_{i=\pm} r(k + \frac{p}{2}, k + ik') r(k - \frac{p}{2}, k + ik'),
\end{equation}
and
\begin{equation}
\mathcal{L}_p f(k) := 2\mathcal{R}_p f(k) - \left( R(k + \frac{p}{2}) + R(k - \frac{p}{2}) \right) f(k),
\end{equation}
\begin{equation}
\mathcal{L}_p^\pm f(k) := 2\mathcal{R}_p f(k) - 2R(k \pm \frac{p}{2}) f(k)
\end{equation}
for any $f \in L^2(\mathbb{T})$ where $r(k, k')$ is defined in (3.9). We note that $R(k, k', p)$ satisfies $R(k, k', 0) = R(k, k')$ for any $k, k' \in \mathbb{T}$ where $R(k, k')$ is defined in (4.16), and thus from (4.17) we see that the operators $\mathcal{L}_p, \mathcal{L}_p^\pm$ are related to $\mathcal{L}$ through $\mathcal{L}_0 = \mathcal{L}_p^\pm = \mathcal{L}$ where $\mathcal{L}$ is defined in (4.15). Since the above differential equations are linear and the coefficients are bounded, $(\tilde{W}_{\epsilon, +}, \tilde{W}_{\epsilon, -}, \tilde{Y}_{\epsilon, +}, \tilde{Y}_{\epsilon, -})(p, t) \in (L^2(\mathbb{T}))^4$ for any $(p, t) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$.

Now we simplify the righthand side of (5.6) in several steps. In Appendix B, we show the expansion of $R(k, k', p)$ with respect to $ep$ up to the second order, we have
\begin{equation}
\frac{d}{dt} \tilde{W}_{\epsilon, +} = -\left( \text{sgn } \epsilon \right) \sqrt{-1} \epsilon (\delta_\epsilon \omega) \tilde{W}_{\epsilon, +} + \frac{\gamma}{f_{\theta, s}(\epsilon)} \mathcal{L} \left( \tilde{W}_{\epsilon, +} - \frac{1}{2} \tilde{Y}_{\epsilon, +} + \tilde{Y}_{\epsilon, -} \right)
- \left( \text{sgn } \epsilon \right) \gamma \epsilon R(k)(ep) \left( \tilde{Y}_{\epsilon, +} - \tilde{Y}_{\epsilon, -} \right) + \frac{\gamma}{2 f_{\theta, s}(\epsilon)} r^{(1, \epsilon)}_{\epsilon},
\end{equation}
\begin{equation}
\frac{d}{dt} \tilde{Y}_{\epsilon, +} = -\left( \text{sgn } \epsilon \right) \sqrt{-1} \omega_\epsilon \tilde{Y}_{\epsilon, +} + \frac{\gamma}{f_{\theta, s}(\epsilon)} \mathcal{L} \left( \tilde{Y}_{\epsilon, +} - \frac{1}{2} \tilde{W}_{\epsilon, +} + \tilde{W}_{\epsilon, -} \right)
- \left( \text{sgn } \epsilon \right) \left( \frac{\gamma}{f_{\theta, s}(\epsilon)} \mathcal{R}_0 \left( \tilde{Y}_{\epsilon, +} - \tilde{Y}_{\epsilon, -} \right) + \frac{\gamma \epsilon R(k)(ep)}{2 f_{\theta, s}(\epsilon)} \left( \tilde{W}_{\epsilon, +} - \tilde{W}_{\epsilon, -} \right) \right) + \frac{\gamma}{2 f_{\theta, s}(\epsilon)} r^{(2, \epsilon)}_{\epsilon},
\end{equation}
where $\text{sgn } \epsilon$ is the sign of $\epsilon = \pm$ and $r^{(i, \epsilon)}(p, k, t), i = 1, 2, \epsilon = \pm$ are remainder terms and these satisfy
\begin{equation}
||r^{(i, \epsilon)}(p, \cdot, t)||_{L^2(\mathbb{T})} \lesssim \sum_{i = \pm} ||\tilde{W}_{\epsilon, i}(p, \cdot, t)||_{L^2(\mathbb{T})} + ||\tilde{Y}_{\epsilon, i}(p, \cdot, t)||_{L^2(\mathbb{T})}.
\end{equation}
for any $0 < \epsilon < 1, (p, t) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$. The operator $\mathcal{L}$ is defined in (4.15). Define
\begin{equation}
\tilde{U}_{\epsilon, +}(p, k, t) := \frac{1}{2} \tilde{Y}_{\epsilon, +} + \tilde{Y}_{\epsilon, -})(p, k, t),
\end{equation}
\begin{equation}
\tilde{U}_{\epsilon, -}(p, k, t) := \frac{1}{2} \tilde{Y}_{\epsilon, +} - \tilde{Y}_{\epsilon, -})(p, k, t).
\end{equation}
From (5.13), (5.14), (5.16) and (5.17), we have

\[
\frac{d}{dt} \tilde{W}_{\epsilon,t} = -(\text{sgn} \, \epsilon) \sqrt{-1} \epsilon (\delta_{\epsilon,\omega}) \frac{\sqrt{-1} \gamma (k)}{f_{j,\omega}(\epsilon)} \tilde{W}_{\epsilon,t} + \gamma \tilde{L} (\tilde{W}_{\epsilon,t} - \tilde{U}_{\epsilon,t}) \\
\quad - (\text{sgn} \, \epsilon) \sqrt{-1} \gamma R(k) \epsilon p \tilde{U}_{\epsilon,t} + \frac{\gamma \epsilon^2 p^2}{2f_{j,\omega}(\epsilon)} (r_{\epsilon}^{(2,+) - r_{\epsilon}^{(2,-)})},
\]

(5.18)

\[
\frac{d}{dt} \tilde{U}_{\epsilon,t} = - \frac{2\omega_{\epsilon}}{f_{j,\omega}(\epsilon)} \tilde{U}_{\epsilon,t} - \frac{2\gamma R(k)}{f_{j,\omega}(\epsilon)} \tilde{U}_{\epsilon,t} + \frac{\sqrt{-1} \gamma R(k) \epsilon}{2f_{j,\omega}(\epsilon)} (\tilde{W}_{\epsilon,t} - \tilde{W}_{\epsilon,t}) \\
\quad - \frac{\sqrt{-1} \gamma p^2}{2f_{j,\omega}(\epsilon)} (r_{\epsilon}^{(2,+) - r_{\epsilon}^{(2,-)})},
\]

(5.19)

(5.20)

Define \( E(f) = E(f, f), f \in \mathbb{L}^2(\mathbb{T}) \) where \( E(\cdot, \cdot) : \mathbb{L}^2(\mathbb{T}) \times \mathbb{L}^2(\mathbb{T}) \to \mathbb{C} \) is the Dirichlet form corresponding to the generator \( \tilde{L} \):

\[
E(\phi_1, \phi_2) = \int_{\mathbb{T}} dk (-\tilde{L} f_1)(k) f_2(k)^* \\
\quad = \int_{\mathbb{T}} dk d\theta R(k, k') (f_1(k) - f_1(k')) (f_2(k) - f_2(k'))
\]

(5.21)

for any \( f_1, f_2 \in \mathbb{L}^2(\mathbb{T}) \). By using (5.18) – (5.20), we have the following \( \mathbb{L}^2(\mathbb{T}) \) estimate of the Wigner distribution. The proof of Proposition 5.1 follows the proof of [12, Proposition 9.1].

**Proposition 5.1.** There exists some positive constant \( C_1 \) such that

\[
E_{2,\epsilon}(p, T) \leq E_{2,\epsilon}(p, 0) \exp \left( C_1 p^2 \frac{\gamma^2}{f_{j,\omega}(\epsilon)} T \right),
\]

holds for any \( \theta > 2, 0 \leq s \leq 1, 0 < \epsilon < 1, p \in \mathbb{R}, T > 0 \) where

\[
E_{2,\epsilon}(p, t) := \frac{\sqrt{\mathbb{L}^2(\mathbb{T})}}{2} \left\| \tilde{W}_{\epsilon,t} + (p, \cdot, t) \right\|^2_{\mathbb{L}^2(\mathbb{T})} + \sum_{i=1}^{\infty} \left\| \tilde{U}_{\epsilon,t} + (p, \cdot, t) \right\|^2_{\mathbb{L}^2(\mathbb{T})}.
\]

**Proof.** By multiplying both sides of (5.18), (5.19), (5.20) respectively by \( \tilde{W}_{\epsilon,t}^+, \tilde{U}_{\epsilon,t}^+, \epsilon = \pm \) and (5.18)*, (5.19)*, (5.20)* respectively by \( \tilde{W}_{\epsilon,t}^+, \tilde{U}_{\epsilon,t}^+, \epsilon = \pm \), adding them sideways, and taking the integral with respect to \( k \in \mathbb{T} \) and \( t \in [0, T] \), we have

\[
E_{2,\epsilon}(p, T) + \frac{2\gamma}{f_{j,\omega}(\epsilon)} \int_0^T dt \mathbb{E} \left( (\tilde{W}_{\epsilon,t} + (p, \cdot, t)) - (\tilde{U}_{\epsilon,t} + (p, \cdot, t)) \right) \\
\quad + \frac{4\gamma}{f_{j,\omega}(\epsilon)} \int_0^T dt \int_{\mathbb{T}} dk R(k) (\tilde{U}_{\epsilon,t}(p, k, t))^2 \\
\quad + \frac{2\gamma p}{f_{j,\omega}(\epsilon)} \int_0^T dt \int_{\mathbb{T}} dk R(k) \mathbb{I}m(\tilde{W}_{\epsilon,t}^+ \tilde{U}_{\epsilon,t})(p, k, t) \\
= E_{2,\epsilon}(p, 0) + \frac{\gamma^2 p^2}{f_{j,\omega}(\epsilon)} \int_0^T dt \int_{\mathbb{T}} dk R_\epsilon(p, k, t),
\]
where $R_ε(p,k,t)$ is a remainder term which satisfies
\[ |R_ε(p,·,t)|_{L^1(T)} \leq B_2(ε,t), \]
for any $(p,t) ∈ \mathbb{R} × \mathbb{R}_{≥0}$. By using Young’s inequality, we have
\[ ε \int_0^T dt \int_T dk R'(k) \text{Im}(\tilde{W}_{ε,+}^*(p,k,t)) ⩾ -2 \int_0^T dt \int_T dk \frac{1}{R(k)} \left| \tilde{W}_{ε,+(p,k,t)} \right|^2. \]

Since \( \frac{(R'(k))^2}{R(k)} \) is uniformly bounded, we have
\[ E_{2,ε}(p,T) - E_{2,ε}(p,0) ⩽ \frac{C_1 ε^2 p^2}{f_{θ,s}(ε)} \int_0^T dt E_{2,ε}(p,k,t). \]

By using Gronwall’s inequality and (4.1), we conclude the proof of this Proposition. □

Remark 5.1. This Remark is a continuation of Remark 4.3. As we shall emphasize in the next subsection, we often use the limit (5.24) in the proof of Theorem 1. For instance, the term \[ \frac{C_1(ε^2 p^2)}{f_{θ,s}(ε)} \] appears in the scaling factors of the remainder terms because we express the scaled scattering kernel \( R(k,k',ε)p \) and its mean value \( R(k±\frac{ΔT}{2}) \), defined in (5.12) and (3.8) respectively, with respect to \( εp \) up to the second order. But if the chain is pinned and \((θ,s) ∈ \left(\frac{T}{2},∞\right) × \{0\} \) then the time scaling \( f_{θ,s}(ε) \) is \( ε^2 \), and thus we have to expand the functions \( R(k,k',ε)p, R(k±\frac{ΔT}{2}) \) with respect to \( εp \) up to the third order to replace the scaling factors of the remainder terms with \( \frac{C_1(ε^2 p^2)}{f_{θ,s}(ε)} \). In this case, the strategy of the proof is almost the same as that of [12, Theorem 5.1].

5.3. Laplace transform of the Wigner distribution. Define
\[ \tilde{ω}_{ε,ε}(p,k,λ) := \int_0^∞ dt e^{-λT} \tilde{w}_{ε,ε}(p,k,t), \]
\[ \tilde{u}_{ε,ε}(p,k,λ) := \int_0^∞ dt e^{-λT} \tilde{u}_{ε,ε}(p,k,t), \]
for \((p,k,λ) ∈ \mathbb{R} × T × (0,∞)\). Thanks to (4.1) and Proposition 5.1, for any \((p,λ) ∈ \mathbb{R} × (0,∞)\), we have \( \tilde{ω}_{ε,ε}(p,·,λ), \tilde{u}_{ε,ε}(p,·,λ) ∈ L^2(T) \) by choosing sufficiently small \( ε > 0 \). Actually, we obtain
\[ \int_T dk |\tilde{ω}_{ε,ε}(p,k,λ)|^2 ⩽ \frac{1}{λ} \int_0^∞ dt e^{-λt} \int_T dk \left| \tilde{W}_{ε,+(p,k,t)} \right|^2 \leq \frac{2K_1}{λ(λ - C_1 \frac{ε^2 p^2}{f_{θ,s}(ε)})}, \]  
(5.22)
\[ \int_T dk |\tilde{u}_{ε,ε}(p,k,λ)|^2 ⩽ \frac{2K_1}{λ(λ - C_1 \frac{ε^2 p^2}{f_{θ,s}(ε)})}. \]  
(5.23)
and from (4.6) we have
\[
\lim_{\epsilon \to 0} \frac{\epsilon^2}{f_{\theta,s}(\epsilon)} = 0, \tag{5.24}
\]
for any \( \theta > 2, 0 \leq s \leq 1 \). Hereafter we will often use the limit (5.24) in the rest of Section 5.

Taking the Laplace transforms of both sides of (5.18) - (5.20), we see that \( \bar{w}, \bar{u} \) satisfies the following equations:

\[
\begin{align*}
\lambda \bar{w}_{\epsilon,t} - \bar{W}^{(0)}_{\epsilon,t} &= -\text{sgn}(\epsilon) \left( \frac{\sqrt{-1\epsilon}(s_0 \omega)}{f_{\theta,s}(\epsilon)} \bar{w}_{\epsilon,t} + \frac{\sqrt{-1\gamma} R'(k) e^p}{f_{\theta,s}(\epsilon)} \bar{u}_{\epsilon,-} \right) \\
&\quad + \frac{\gamma}{f_{\theta,s}(\epsilon)} L(\bar{w}_{\epsilon,t} - \bar{u}_{\epsilon,+}) + \frac{\gamma^2 \epsilon^2 p^2}{f_{\theta,s}(\epsilon)} \rho_1^{(1,t)} , \\
\lambda \bar{u}_{\epsilon,+} - \bar{U}^{(0)}_{\epsilon,+} &= \frac{2\bar{w}_t}{f_{\theta,s}(\epsilon)} \bar{u}_{\epsilon,+} + \frac{\gamma}{f_{\theta,s}(\epsilon)} L\left( \bar{u}_{\epsilon,+} - \frac{1}{2} (\bar{w}_{\epsilon,+} + \bar{w}_{\epsilon,-}) \right) + \frac{\gamma \epsilon^2 p^2}{f_{\theta,s}(\epsilon)} \rho_2^{(2,+)} , \\
\lambda \bar{u}_{\epsilon,-} - \bar{U}^{(0)}_{\epsilon,-} &= -\frac{2\bar{w}_t}{f_{\theta,s}(\epsilon)} \bar{u}_{\epsilon,-} - 2\gamma R(k) \frac{\sqrt{-1\gamma} R'(k) e^p}{2 f_{\theta,s}(\epsilon)} (\bar{w}_{\epsilon,+} - \bar{w}_{\epsilon,-}) \\
&\quad + \frac{\gamma \epsilon^2 p^2}{f_{\theta,s}(\epsilon)} \rho_2^{(2,-)} ,
\end{align*}
\]

where \( \bar{W}^{(0)}_{\epsilon,t} := \bar{W}_{\epsilon,t}(\epsilon,t,0), \bar{U}^{(0)}_{\epsilon,t} := \bar{U}_{\epsilon,t}(\epsilon,t,0), \epsilon = \pm \). \( \rho_{i,t}^{(i,t)}(p,k,\lambda), i = 1, 2, \epsilon = \pm \), are the Laplace transforms of the remainder terms \( \rho_{i,t}^{(i,t)}(p,k,t), i = 1, 2, \epsilon = \pm \) and they satisfy

\[
|\rho_{i,t}^{(i,t)}(p,\cdot,\lambda)|^2_{L^2(\mathbb{T})} \leq |\bar{w}_{\epsilon,+}(p,\cdot,\lambda)|^2_{L^2(\mathbb{T})} + \sum_{i=\pm} |\bar{u}_{\epsilon,i}(p,\cdot,\lambda)|^2_{L^2(\mathbb{T})} . \tag{5.28}
\]

By using the above equations, we can show the following estimate.

**Proposition 5.2.** For any positive constant \( M > 0 \) and any compact interval \( I \subset \mathbb{R}_{>0} \), there exists some positive constant \( C_{M,I} > 0 \) such that

\[
\limsup_{\epsilon \to 0} \frac{\gamma}{f_{\theta,s}(\epsilon)} \sup_{|p| \leq M, \lambda \in I} \left[ \int_{\mathbb{T}} dk \frac{R(k)}{\lambda f_{\theta,s}(\epsilon) + 2\gamma R(k)} |\tilde{w}_t(p,k,\lambda)|^2 |\bar{u}_{\epsilon,+}(p,k,\lambda)|^2 \right] \leq C_{M,I} , \tag{5.29}
\]

for any \( \theta > 2, 0 \leq s \leq 1 \).

**Proof.** First we will show that

\[
\limsup_{\epsilon \to 0} \frac{\gamma}{f_{\theta,s}(\epsilon)} \sup_{|p| \leq M, \lambda \in I} \left[ \int_{\mathbb{T}} dk \frac{R(k)}{\lambda f_{\theta,s}(\epsilon) + 2\gamma R(k)} |\tilde{w}_t(p,k,\lambda)|^2 |\bar{u}_{\epsilon,+}(p,k,\lambda)|^2 \right] \leq C_{M,I} , \tag{5.30}
\]

where \( \mathcal{E} \) is the Dirichlet form defined in (5.21). By multiplying both sides of (5.25), (5.26), (5.27) respectively by \( \tilde{w}_{\epsilon,+} \), \( \tilde{u}_{\epsilon,+}, t = \pm \), taking the real parts of them,
adding them sideways and taking the integral with respect to \(k \in \mathbb{T}\), we obtain

\[
\lambda \left( \left\| \bar{w}_{e,+}(p, \cdot, \lambda) \right\|_{L^2(\mathbb{T})}^2 + \sum_{l=\pm} \left\| \bar{u}_{e,l}(p, \cdot, \lambda) \right\|_{L^2(\mathbb{T})}^2 \right) + \frac{2\gamma}{f_{\theta,s}(\epsilon)} \int_{\mathbb{T}} \kappa k R(k) |\bar{u}_{e,-}(p, k, \lambda)|^2 \\
+ \frac{2\gamma \epsilon p}{f_{\theta,s}(\epsilon)} \int_{\mathbb{T}} \kappa k R(k) \text{Im} \left( \bar{w}_{e,+}^* \bar{u}_{e,-} \right)(p, k, \lambda) + \frac{\gamma}{f_{\theta,s}(\epsilon)} \mathcal{E} \left( \bar{w}_{e,+}(p, \cdot, \lambda) - \bar{u}_{e,+}(p, \cdot, \lambda) \right)
\]

\[
= \int_{\mathbb{T}} \kappa k \text{Re} \left( \bar{W}_{e,+}^{(0)} \bar{w}_{e,+}^* + \sum_{l=\pm} \bar{U}_{e,l}^{(0)} \bar{u}_{e,-} \right)(p, k, \lambda) + \frac{\gamma \epsilon p^2}{f_{\theta,s}(\epsilon)} \int_{\mathbb{T}} \kappa k r_e(p, k, \lambda),
\]

where \(r_e(p, k, \lambda)\) is a remainder term which satisfies

\[
|r_e(p, k, \lambda)|_{L^1(\mathbb{T})} \leq \| \bar{w}_{e,+}(p, \cdot, \lambda) \|_{L^2(\mathbb{T})}^2 + \sum_{l=\pm} \| \bar{u}_{e,l}(p, \cdot, \lambda) \|_{L^2(\mathbb{T})}^2.
\]

By using Young’s inequality, we have

\[
\epsilon p \int_{\mathbb{T}} \kappa k R' \left( \text{Im} \left( \bar{w}_{e,+}^* \bar{u}_{e,-} \right) \right)(p, k, \lambda)
\geq -\frac{1}{2} \int_{\mathbb{T}} \kappa k R(k) |\bar{u}_{e,-}(p, k, \lambda)|^2 - \frac{\epsilon^2 p^2}{2} \int_{\mathbb{T}} \kappa k \left( \frac{R'(k)}{R(k)} \right)^2 |\bar{w}_{e,+}(p, k, \lambda)|^2.
\]

From the above, (4.1), (5.22), (5.23) and the uniform boundness \(0 \leq \| R'(k) \|_{L^2(\mathbb{T})} \leq 1, k \in \mathbb{T}\), we obtain

\[
\frac{\gamma}{f_{\theta,s}(\epsilon)} \int_{\mathbb{T}} \kappa k R(k) |\bar{u}_{e,-}(p, k, \lambda)|^2 + \frac{\gamma}{f_{\theta,s}(\epsilon)} \mathcal{E} \left( \bar{w}_{e,+} - \bar{u}_{e,+} \right)(p, \cdot, \lambda)
\leq \int_{\mathbb{T}} \kappa k \text{Re} \left( \bar{W}_{e,+}^{(0)} \bar{w}_{e,+}^* + \sum_{l=\pm} \bar{U}_{e,l}^{(0)} \bar{u}_{e,-} \right)(p, k, \lambda) + \frac{\gamma \epsilon p^2}{f_{\theta,s}(\epsilon)} \int_{\mathbb{T}} \kappa k r_e(p, k, \lambda)
- \frac{2\gamma \epsilon p}{f_{\theta,s}(\epsilon)} \int_{\mathbb{T}} \kappa k R'(k) \text{Im} \left( \bar{w}_{e,+} \bar{u}_{e,-} \right)(p, k, \lambda) - \frac{\gamma}{f_{\theta,s}(\epsilon)} \int_{\mathbb{T}} \kappa k R(k) |\bar{u}_{e,-}(p, k, \lambda)|^2
\leq \int_{\mathbb{T}} \kappa k \text{Re} \left( \bar{W}_{e,+}^{(0)} \bar{w}_{e,+}^* + \sum_{l=\pm} \bar{U}_{e,l}^{(0)} \bar{u}_{e,-} \right)(p, k, \lambda) + \frac{\gamma \epsilon p^2}{f_{\theta,s}(\epsilon)} \int_{\mathbb{T}} \kappa k r_e(p, k, \lambda)
+ \frac{\gamma \epsilon p^2}{f_{\theta,s}(\epsilon)} \int_{\mathbb{T}} \kappa k \left( \frac{R'(k)}{R(k)} \right)^2 |\bar{w}_{e,+}(p, k, \lambda)|^2
\leq \| \bar{w}_{e,+}(p, \cdot, \lambda) \|_{L^2(\mathbb{T})}^2 \| \bar{W}_{e,+}^{(0)}(p, \cdot) \|_{L^2(\mathbb{T})}^2 + \sum_{l=\pm} \| \bar{u}_{e,l}(p, \cdot, \lambda) \|_{L^2(\mathbb{T})}^2 \| \bar{U}_{e,l}^{(0)}(p, \cdot) \|_{L^2(\mathbb{T})}^2
+ \frac{\gamma \epsilon p^2}{f_{\theta,s}(\epsilon)} \left( \| \bar{w}_{e,+}(p, \cdot, \lambda) \|_{L^2(\mathbb{T})}^2 + \sum_{l=\pm} \| \bar{u}_{e,l}(p, \cdot, \lambda) \|_{L^2(\mathbb{T})}^2 \right)
\leq \left( \frac{1}{\lambda(\lambda - C_1 \epsilon^2 p^2 f_{\theta,s}(\epsilon))} \right)^{\frac{1}{2}} + \frac{\gamma \epsilon p^2}{f_{\theta,s}(\epsilon)} \frac{1}{\lambda(\lambda - C_1 \epsilon^2 p^2 f_{\theta,s}(\epsilon))}
\]

and thus by using (5.24) we get (5.30).

Next we will verify that

\[
\lim_{\epsilon \to 0} \sup_{|p| \leq M} \int_{\mathbb{T}} \kappa k R(k) \left( \frac{\bar{w}_e(p, k)}{\lambda f_{\theta,s}(\epsilon)} + 2\gamma R(k) \right)^2 |\bar{u}_{e,+}(p, k, \lambda)|^2 \leq C_{M,I}.
\]
From (5.27), we have
\[ \frac{\omega_\epsilon(p,k)}{\lambda f_{\theta,s}(\epsilon)} + 2\gamma R(k) \tilde{u}_{\epsilon,+}(p,k,\lambda) = \tilde{u}_{\epsilon,-} + \frac{1}{\lambda f_{\theta,s}(\epsilon)} + 2\gamma R \left( f_{\theta,s}(\epsilon) \tilde{u}_{\epsilon,-} - \frac{\sqrt{-1}\gamma R'\epsilon}{2} (\tilde{w}_{\epsilon,-} - \tilde{w}_{\epsilon,-}) + \gamma \epsilon^2 \gamma R' R_{\epsilon}^{(2,-)} \right). \]

By using Schwarz’s inequality, we obtain
\[ \int_T dk \ R(k) \left( \frac{\omega_\epsilon(p,k)}{\lambda f_{\theta,s}(\epsilon)} + 2\gamma R(k) \right)^2 |\tilde{u}_{\epsilon,+}(p,k,\lambda)|^2 \]
\[ \leq \int_T dk \ R(k) \left[ \left( \frac{f_{\theta,s}(\epsilon)}{\lambda f_{\theta,s}(\epsilon) + 2\gamma R(k)} \right)^2 |\tilde{U}_{\epsilon,-}^{(0)}(p,k)|^2 + \left( \frac{\gamma \epsilon^2 p^2}{\lambda f_{\theta,s}(\epsilon) + 2\gamma R(k)} \right)^2 |R_{\epsilon}^{(2,-)}|^2 \right] \]
\[ + \int_T dk \ R(k) |\tilde{u}_{\epsilon,-}(p,k,\lambda)|^2 + \int_T dk \ R(k) \left( \frac{\gamma R'(k) \epsilon p}{\lambda f_{\theta,s}(\epsilon) + 2\gamma R(k)} \right)^2 |\tilde{u}_{\epsilon,-}(p,k,\lambda)|^2. \]

Since \((\frac{1}{\lambda f_{\theta,s}(\epsilon) + 2\gamma R(k)})^2 \leq \frac{1}{\lambda f_{\theta,s}(\epsilon) + 2\gamma R(k)}\), we have
\[ \int_T dk \ R(k) \left( \frac{f_{\theta,s}(\epsilon)}{\gamma f_{\theta,s}(\epsilon) + \gamma^4} |\tilde{U}_{\epsilon,-}^{(0)}(p,k)|^2 + \left( \frac{\gamma \epsilon^2 p^2}{\gamma f_{\theta,s}(\epsilon) + \gamma^4} \right)^2 |R_{\epsilon}^{(2,-)}|^2 \right] \]
\[ \leq \frac{f_{\theta,s}(\epsilon)}{\gamma} + \frac{\gamma^4}{f_{\theta,s}(\epsilon)} \]
for any \(\theta > 2, 0 \leq s \leq 1, |p| \leq M, \lambda \in I\). By using (5.30) and the uniform boundness \(0 \leq (R'(k))^2 R(k) \leq 1, k \in T\), we have
\[ \int_T dk \ R(k) |\tilde{u}_{\epsilon,-}(p,k,\lambda)|^2 + \int_T dk \ R(k) \left( \frac{\gamma R'(k) \epsilon p}{\gamma f_{\theta,s}(\epsilon) + \gamma^4} \right)^2 |\tilde{u}_{\epsilon,+}(p,k,\lambda)|^2 \]
\[ \leq \frac{f_{\theta,s}(\epsilon)}{\gamma} + \frac{\gamma^4}{f_{\theta,s}(\epsilon)} \]
for any \(\theta > 2, 0 \leq s \leq 1, |p| \leq M, \lambda \in I\). Therefore from the above and (5.24) we obtain (5.31).

Finally we will show that
\[ \lim_{\epsilon \to 0} \frac{\gamma}{f_{\theta,s}(\epsilon)} \sup_{|p| \leq M, \lambda \in I} \mathcal{E} \left( \tilde{u}_{\epsilon,+}(p,\cdot,\lambda) \right) \leq C_{M,I}. \] (5.32)

If we get (5.32), then from (5.30), (5.31) and (5.32) we obtain the estimate (5.29). Since
\[ \mathcal{E} \left( \tilde{u}_{\epsilon,+}(p,\cdot,\lambda) \right) \leq \int_T R(k) |\tilde{u}_{\epsilon,+}(p,k,\lambda)|^2, \]
for any \((p, \lambda) \in \mathbb{R} \times (0, \infty)\), it is sufficient to show that
\[ \lim_{\epsilon \to 0} \frac{\gamma}{f_{\theta,s}(\epsilon)} \sup_{|p| \leq M, \lambda \in I} \int_T R(k) |\tilde{u}_{\epsilon,+}(p,k,\lambda)|^2 \leq C_{M,I}. \] (5.33)

To prove (5.33), we divide the domain of the integration into two parts. Since \(R(k) \leq \frac{f_{\theta,s}(\epsilon)}{\gamma} \) on \(|k| \leq \left( \frac{f_{\theta,s}(\epsilon)}{\gamma} \right)^{-\frac{1}{2}} \), we have
\[ \int_{|k| \leq \left( \frac{f_{\theta,s}(\epsilon)}{\gamma} \right)^{-\frac{1}{2}}} dk \ R(k) |\tilde{u}_{\epsilon,+}(p,k,\lambda)|^2 \leq \frac{f_{\theta,s}(\epsilon)}{\gamma}, \]
for any $\theta > 2, 0 \leq s \leq 1, |p| \leq M, \lambda \in I$. On the other hand, from (3.8), (5.11) and (B.1) we have
\[
\left( \frac{\tilde{\omega}_s(p, k)}{\lambda f_{\theta,s}(\epsilon) + 2\gamma R(k)} \right)^2 = \gamma^{-2} \left( \frac{\tilde{\omega}_s(p, k)}{k^2} \right)^2 \left( \frac{k^2}{\lambda f_{\theta,s}(\epsilon) + 2\gamma R(k)} \right)^2 \leq \begin{cases} 
\gamma^{-2} |k|^{-3(5-\theta)} & 2 < \theta \leq 3, \\
\gamma^{-2} |k|^{-2} & \theta > 3 
\end{cases}
\]
on $\{k > (\frac{f_{\theta,s}(\epsilon)}{\gamma})^{\frac{1}{2}}\}$ for any $|p| \leq M, \lambda \in I$. Especially, we can take a positive constant $c_M$ which only depends on $M > 0$ such that
\[
\left( \frac{\tilde{\omega}_s(p, k)}{\lambda f_{\theta,s}(\epsilon) + 2\gamma R(k)} \right)^2 \geq c_M \gamma^{-2}
on \]
on $\{k > (\frac{f_{\theta,s}(\epsilon)}{\gamma})^{\frac{1}{2}}\}$ for any $\theta > 2, 0 \leq s \leq 1, |p| \leq M, \lambda \in I$. By using the above estimate and (5.31) we obtain
\[
\int_{|k| > (\frac{f_{\theta,s}(\epsilon)}{\gamma})^{\frac{1}{2}}} dk \ R(k)|\tilde{u}_{\epsilon,s}(p, k, \lambda)|^2 \\
\leq \int_{|k| > (\frac{f_{\theta,s}(\epsilon)}{\gamma})^{\frac{1}{2}}} dk \ R(k) \left( \frac{\tilde{\omega}_s(p, k)}{\lambda f_{\theta,s}(\epsilon) + 2\gamma R(k)} \right)^2 |\tilde{u}_{\epsilon,s}(p, k, \lambda)|^2 \\
\leq f_{\theta,s}(\epsilon) \gamma
\]
for any $\theta > 2, 0 \leq s \leq 1, |p| \leq M, \lambda \in I$ and thus we have (5.33). \qed

5.4. Homogenization of a limit point of the Wigner distribution. In this subsection, by following the proof of [12, Theorem 10.2], we will show a homogenization result. Define $e_i(k), \ k \in \mathbb{T}, \ i = 1, 2$ as
\[
e_1(k) := \frac{8}{3} \sin^4(\pi k), \quad e_2(k) := 2\sin^2(2\pi k).
\]
The functions $e_i, i = 1, 2$ are normalized to satisfy $\int_{\mathbb{T}} dk \ e_i(k) = 1$, and by using $e_i, i = 1, 2$ the scattering kernel $R(k, k'), k, k' \in \mathbb{T}$ is expressed as
\[
R(k, k') = \frac{3}{4} \sum_{i=1,2} e_i(k)e_{3-i}(k').
\]
Define $w_{\epsilon,i}(p, \lambda), (p, \lambda) \in \mathbb{R} \times (0, \infty), \ i = 1, 2$ as
\[
w_{\epsilon,i}(p, \lambda) := \int_{\mathbb{T}} dk \ e_i(k)\tilde{\omega}_{\epsilon,s}(p, k, \lambda), \quad i = 1, 2.
\]

**Theorem 4.** Suppose the same assumption of Theorem 1. For any constant $M > 0$ and a compact interval $I \subset (0, \infty)$, we have
\[
\lim_{\epsilon \to 0} \sup_{|p| \leq M} \int_{\mathbb{T}} dk \ |\tilde{u}_{\epsilon,s}(p, k, \lambda) - w_{\epsilon,i}(p, \lambda)| = 0, \quad i = 1, 2, \quad (5.36)
\]
\[
\lim_{\epsilon \to 0} \sup_{|p| \leq M} \int_{\mathbb{T}} dk \ |\tilde{u}_{\epsilon,i}(p, k, \lambda)| = 0, \quad i = \pm. \quad (5.37)
\]
Proof. First we show that there is no mass concentration at $k = 0$ macroscopically, that is,

$$\lim_{\epsilon \to 0} \sup_{\lambda \in I} \sup_{|\ell| \leq M} \int_{|k| < \delta} dk \sum_{\ell \in \mathbb{Z}} |\tilde{w}_{\epsilon,i}(p, k, \lambda)| + |\tilde{u}_{\epsilon,i}(p, k, \lambda)| = 0. \quad (5.38)$$

Actually, by Schwarz’s inequality, we have

$$\left( \int_{|k| < \delta} dk \sum_{\ell \in \mathbb{Z}} |\tilde{w}_{\epsilon,i}(p, k, \lambda)| + |\tilde{u}_{\epsilon,i}(p, k, \lambda)| \right)^2 \leq 2\delta \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{T}} dk |\tilde{w}_{\epsilon,i}(p, k, \lambda)|^2 + |\tilde{u}_{\epsilon,i}(p, k, \lambda)|^2$$

for $0 < \delta < \frac{1}{2}$. From (5.22) and (5.23), for any $i = \pm$ we obtain

$$\sup_{\lambda \in I} \sup_{|\ell| \leq M} \int_{\mathbb{T}} dk |\tilde{w}_{\epsilon,i}(p, k, \lambda)|^2 \leq \frac{2K_1}{\lambda(\lambda - C_1 e^{2M^2/\int_{\mathbb{T}} |\epsilon,\pi||f_\theta(x)|})},$$

$$\sup_{\lambda \in I} \sup_{|\ell| \leq M} \int_{\mathbb{T}} dk |\tilde{u}_{\epsilon,i}(p, k, \lambda)|^2 \leq \frac{2K_1}{\lambda(\lambda - C_1 e^{2M^2/\int_{\mathbb{T}} |\epsilon,\pi||f_\theta(x)|})}$$

for sufficiently small $\epsilon > 0$ and the above estimates imply (5.38).

Thanks to (5.38), to verify (5.36) it is sufficient to show

$$\lim_{\epsilon \to 0} \sup_{\lambda \in I} \sup_{|\ell| \leq M} \int_{\mathbb{T}} dk \ e_i(k)|\tilde{w}_{\epsilon,i}(p, k, \lambda) - w_{\epsilon,i}(p, \lambda)| = 0, \quad i = 1, 2. \quad (5.39)$$

From the definition of $w_{\epsilon,i}$, the property $\int_{\mathbb{T}} dk \ e_i(k) = 1$ and Schwarz’s inequality, we have

$$\int_{\mathbb{T}} dk \ e_i(k)|\tilde{w}_{\epsilon,i}(p, k, \lambda) - w_{\epsilon,i}(p, \lambda)|$$

$$\leq \int_{\mathbb{T}^2} dkdk' \ e_i(k) e_i(k') |\tilde{w}_{\epsilon,i}(p, k, \lambda) - \tilde{w}_{\epsilon,i}(p, k', \lambda)|$$

$$\leq \left( \int_{\mathbb{T}^2} dkdk' \ R(k, k') |\tilde{w}_{\epsilon,i}(p, k, \lambda) - \tilde{w}_{\epsilon,i}(p, k', \lambda)| \right)^{\frac{1}{2}} \left( \int_{\mathbb{T}^2} dkdk' \ \frac{e_i^2(k) e_i^2(k')}{R(k, k')} \right)^{\frac{1}{2}}$$

$$= \mathcal{E}\left( \tilde{w}_{\epsilon,i}(p, \cdot, \lambda) \right)^{\frac{1}{2}} \left( \int_{\mathbb{T}^2} dkdk' \ \frac{e_i^2(k) e_i^2(k')}{R(k, k')} \right)^{\frac{1}{2}}.$$

Since $e_i^2(k) e_i^2(k')$, $i = 1, 2$ are uniformly bounded in $k, k' \in \mathbb{T}$, by using Proposition 5.2 we have (5.39).

To prove (5.37), it is sufficient to show that

$$\lim_{\epsilon \to 0} \sup_{\lambda \in I} \sup_{|\ell| \leq M} \int_{\mathbb{T}} dk \ R(k)|\tilde{u}_{\epsilon,i}(p, k, \lambda)| = 0, \quad \iota = \pm. \quad (5.40)$$

By using Schwarz’s inequality, Proposition 5.2 and (5.33) we obtain

$$\sup_{\lambda \in I} \sup_{|\ell| \leq M} \int_{\mathbb{T}} dk \ R(k)|\tilde{u}_{\epsilon,i}(p, k, \lambda)| \leq \sup_{\lambda \in I} \sup_{|\ell| \leq M} \left( \int_{\mathbb{T}} dk \ R(k) \right)^{\frac{1}{2}} \left( \int_{\mathbb{T}} dk \ R(k)|\tilde{u}_{\epsilon,i}(p, k, \lambda)|^2 \right)^{\frac{1}{2}}$$

$$\leq \left( \int_{\mathbb{T}} d\theta \ |f_\theta| \right)^{\frac{1}{2}} \left( \int_{\mathbb{T}} d\theta \ |f_\theta| \right)^{\frac{1}{2}}$$

for any $\iota = \pm$ and thus we have (5.40).
5.5. Characterization of a limit point of the Wigner distribution. In this subsection we characterize a limit point of any convergent subsequence of \( \{ \tilde{w}_{\epsilon,\ast}(p, k, \lambda) \}_{\epsilon} \). From Remark 4.2 and (5.36), for any \( J \in \mathbb{S}(\mathbb{R}) \) we have
\[
\int_0^\infty dt \ e^{-\lambda t} \langle W(t), J \rangle = \lim_{\epsilon \to 0} \int_0^\infty dt \ e^{-\lambda t} \langle W_{\epsilon}(t), J \rangle
\]
\[
= \lim_{\epsilon \to 0} \int_{\mathbb{R}} dp \left( \int_{\mathbb{T}} dk \ \tilde{w}_{\epsilon,\ast}(p, k, \lambda) \right) \tilde{J}(p)
\]
\[
= \lim_{\epsilon \to 0} \int_{\mathbb{R}} dp \ w_{\epsilon,i}(p, \lambda) \tilde{J}(p)
\]
\[
= \int_{\mathbb{R}} dp \ w(p, \lambda) \tilde{J}(p),
\]
where \( w(p, \lambda) \) is the Laplace-Fourier transform of \( \langle W(t), \cdot \rangle = \mu(t)(dy) \):
\[
w(p, \lambda) = \int_0^\infty dt \ e^{-\lambda t} \tilde{W}(p, t), \quad \tilde{W}(p, t) = \int_{\mathbb{R}} \mu(t)(dy) \ e^{2\pi i p y}.
\]
In the present subsection we will show the following equation:
\[
\left\{ \begin{array}{l}
\int_{\mathbb{R}} dp \left( \lambda + C_{\theta, \gamma_{\ast}} |p|^{\frac{s}{\gamma}} \right) w(p, \lambda) J(p) = \int_{\mathbb{R}} dp \ \tilde{W}_0(p) J(p) \quad 2 < \theta \leq 3, \\
\int_{\mathbb{R}} dp \left( \lambda + C_{\theta, \gamma_{\ast}} |p|^{\frac{s}{\gamma}} \right) w(p, \lambda) J(p) = \int_{\mathbb{R}} dp \ \tilde{W}_0(p) J(p) \quad \theta > 3,
\end{array} \right.
\]
(5.41)
for any \( J \in \mathbb{S}(\mathbb{R}) \) and \( \lambda > 0 \). By using one-to-one correspondence of the Laplace-Fourier transform, we see that any limit of convergent subsequence \( W(y, t) \) is given via its Fourier transform
\[
\left\{ \begin{array}{l}
\tilde{W}(p, t) = e^{-C_{\theta, \gamma_{\ast}} |p|^{\frac{s}{\gamma}}} \tilde{W}_0(p) \quad 2 < \theta \leq 3, \\
\tilde{W}(p, t) = e^{-C_{\theta, \gamma_{\ast}} |p|^{\frac{s}{\gamma}}} \tilde{W}_0(p) \quad \theta > 3,
\end{array} \right.
\]
for almost every \( (p, t) \in \mathbb{R} \times \mathbb{R}_{\geq 0} \). We have thus established the theorem.

Now we will derive (5.41). First by direct computation we rewrite (5.25), \( \epsilon = + \) as follows:
\[
D_{\epsilon} \tilde{w}_{\epsilon,\ast} = f_{\theta,s}(\epsilon) \tilde{W}_{\epsilon,\ast}^{(0)} + \frac{3\gamma}{2} \sum_{i=1,2} e_i w_{\epsilon,3-i} + q_{\epsilon},
\]
(5.42)
where \( e_i(k), i = 1, 2 \) and \( w_{\epsilon,i}(p, \lambda), i = 1, 2 \) are defined in (5.34) and (5.35) respectively. In addition, \( D_{\epsilon}(p, k, \lambda) \) and \( q_{\epsilon}(p, k, \lambda) \) are defined as
\[
D_{\epsilon}(p, k, \lambda) := f_{\theta,s}(\epsilon) \lambda + 2\gamma R(k) + \sqrt{-1} \epsilon \delta_{\epsilon} \omega(p, k),
\]
\[
q_{\epsilon}(p, k, \lambda) := -\sqrt{-1} \gamma R(k) \epsilon p \bar{u}_{\epsilon,\ast}(p, k, \lambda) - \gamma \left( L \bar{u}_{\epsilon,\ast}(p, \cdot, \lambda) \right)(k) + \gamma \epsilon^2 p^2 r_{\epsilon,1}^{(1,\ast)}(p, k, \lambda).
\]
Multiplying both sides of (5.42) by \( \frac{\gamma}{f_{\theta,s}(\epsilon) D_{\epsilon}} \) and taking the integral with respect to \( k \in \mathbb{T} \), we have
\[
\left( \int_{\mathbb{T}} dk \frac{\gamma}{f_{\theta,s}(\epsilon)} \left( 1 - \frac{3e_i e_{3-i}}{2D_{\epsilon}} \right) \right) w_{\epsilon,i} = \left( \frac{3}{2} \int_{\mathbb{T}} dk \gamma^2 e_i^2 \right) w_{\epsilon,3-i}
\]
\[
= \int_{\mathbb{T}} dk \frac{\gamma e_i \tilde{W}_{\epsilon,\ast}^{(0)}}{D_{\epsilon}} + \int_{\mathbb{T}} dk \frac{\gamma e_i q_{\epsilon}}{f_{\theta,s}(\epsilon) D_{\epsilon}}, \quad i = 1, 2.
\]
Adding sideways the above equations corresponding to both values of \( i \), we have

\[
a_i w_{e,1} - b_i (w_{e,1} - w_{e,2}) = \int_T dk \left( \frac{2\gamma R(k) \overline{\bar{W}}_{e,+}^{(0)}}{D_e} + \int_T dk \frac{2\gamma R(k) q_e}{f_{\theta,s}(\epsilon) D_e} \right), \tag{5.43}
\]

where

\[
a_e(p, \lambda) := \int_T dk \frac{2\gamma R(k)}{f_{\theta,s}(\epsilon)} \left( 1 - \frac{2\gamma R(k)}{D_e} \right), \tag{5.44}
\]

\[
b_e(p, \lambda) := \frac{3}{2} \int_T dk \frac{\gamma e_i}{f_{\theta,s}(\epsilon)} \left( 1 - \frac{2\gamma R(k)}{D_e} \right). \tag{5.45}
\]

In the rest of this subsection, we will show the following Proposition by using the strategy of the proof of [12, Proposition 11.1].

**Proposition 5.3.** Suppose the same assumptions of Theorem 1. For any \( J \in S(\mathbb{R}) \) and \( \lambda > 0 \), we have

\[
\begin{align*}
\lim_{\epsilon \to 0} & \int_{\mathbb{R}} dp \left| a_e(p, \lambda) - \lambda - C_{\theta,70} \| p \|^{\frac{2}{\theta}} \right| \| J(p) \| = 0 \quad 2 < \theta \leq 3, \\
\lim_{\epsilon \to 0} & \int_{\mathbb{R}} dp \left| a_e(p, \lambda) - \lambda - C_{\theta,70} \| p \|^{\frac{3}{\theta}} \right| \| J(p) \| = 0 \quad \theta > 3,
\end{align*}
\]

where \( C_{\theta,70} \) is a positive constant defined in (4.10). In addition, we have

\[
\lim_{\epsilon \to 0} \left| \int_{\mathbb{R} \times T} dpdk \frac{2\gamma R(k) \overline{\bar{W}}_{e,+}^{(0)}(p,k)}{D_e(p,k,\lambda)} J(p) - \int_{\mathbb{R}} dp \overline{\bar{W}}_0(p) J(p) \right| = 0, \tag{5.46}
\]

and

\[
\begin{align*}
\lim_{\epsilon \to 0} & \int_{\mathbb{R}} dp \left| b_e(p, \lambda)(w_{e,1} - w_{e,2})(p, \lambda) \right| \| J(p) \| = 0, \\
\lim_{\epsilon \to 0} & \int_{\mathbb{R}} dp \left| \int_T dk \frac{\gamma R(k) q_e(p,k,\lambda)}{f_{\theta,s}(\epsilon) D_e(p,k,\lambda)} \right| \| J(p) \| = 0. \tag{5.47}
\end{align*}
\]

From (5.36), (5.43) and Proposition 5.3, we obtain (5.41).

### 5.5.1. Proof of (5.46). From (4.9), it is sufficient to show that

\[
\lim_{\epsilon \to 0} \left| \int_{\mathbb{R} \times T} dpdk \left( \frac{2\gamma R(k)}{D_e} - 1 \right) \overline{\bar{W}}_{e,+}^{(0)} J \right| = 0.
\]

By Schwarz’s inequality, we have

\[
\left| \int_{\mathbb{R} \times T} dpdk \left( \frac{2\gamma R(k)}{D_e} - 1 \right) \overline{\bar{W}}_{e,+}^{(0)} J \right|^2 \leq \left( \int_{\mathbb{R} \times T} dpdk \left( \frac{2\gamma R(k)}{D_e} - 1 \right)^2 \| J \| \right) \left( \int_{\mathbb{R} \times T} dpdk \| \overline{\bar{W}}_{e,+}^{(0)} \|^2 \| J \| \right).
\]
From (4.1), \( \int_{\mathbb{R} \times T} dp \text{div} [\bar{V}_{\epsilon,\lambda}^2 |J] \) is bounded from above by \( K_1 \int_{\mathbb{R}} dp |J| \). On the other hand, \( \frac{\partial \bar{R}}{\partial \epsilon} \) is uniformly bounded in \( (p, k) \in \mathbb{R} \times \mathbb{T}, 0 < \epsilon < 1 \):

\[
\left| \frac{2 \gamma R(k)}{D_{\epsilon}} \right| = \frac{2 \gamma R(k) D_{\epsilon}^*}{|D_{\epsilon}|^2} = \frac{2 \gamma R(k) \left( f_{\theta,s}(\epsilon) \lambda + 2 \gamma R(k) - \sqrt{-1} \epsilon \lambda \sigma \omega \right)}{\left( f_{\theta,s}(\epsilon) \lambda + 2 \gamma R(k) \right)^2 + \epsilon^2 (\delta_\omega)^2} \\
\leq \frac{2 \gamma R(k) f_{\theta,s}(\epsilon) \lambda + 2 \gamma R(k)}{\left( f_{\theta,s}(\epsilon) \lambda + 2 \gamma R(k) \right)^2 + \epsilon^2 (\delta_\omega)^2} + \frac{2 \gamma R(k) \epsilon |\delta_\omega|}{\left( f_{\theta,s}(\epsilon) \lambda + 2 \gamma R(k) \right)} \\
\leq \frac{5}{4},
\]

where we apply a fundamental inequality \( \frac{a+b}{2} \geq \sqrt{ab}, a, b \geq 0 \) to \( \frac{1}{|D_{\epsilon}|^2}, \) that is,

\[
\frac{1}{|D_{\epsilon}|^2} \leq \frac{1}{4 f_{\theta,s}(\epsilon) \lambda + 2 \gamma R(k)} \epsilon |\delta_\omega|, \quad (5.49)
\]

for any \( (p, k, \lambda) \in \mathbb{R} \times \mathbb{T} \times (0, \infty) \). In addition, \( \lim_{\epsilon \to 0} \sup_{(p,k) \in \mathbb{R} \times \mathbb{T}_0} \frac{\partial \bar{R}}{\partial \epsilon} = 1 \) for any \( (p, k) \in \mathbb{R} \times \mathbb{T}_0 \). By the dominated convergence theorem, we conclude the proof of (5.46).

5.5.2. Proof of (5.45). From (5.44), we have

\[
a_{\epsilon} = \int_{\mathbb{T}} \frac{2 \gamma R(k)}{f_{\theta,s}(\epsilon)} \left( f_{\theta,s}(\epsilon) \lambda + \sqrt{-1} \epsilon \lambda \sigma \omega \right) \left( f_{\theta,s}(\epsilon) \lambda + 2 \gamma R(k) - \sqrt{-1} \epsilon \lambda \sigma \omega \right) \frac{dk}{|D_{\epsilon}|^2} \\
= \int_{\mathbb{T}} \frac{2 \gamma R(k)}{f_{\theta,s}(\epsilon)} \frac{f_{\theta,s}(\epsilon) \lambda \left( f_{\theta,s}(\epsilon) \lambda + 2 \gamma R(k) \right) + \epsilon^2 (\delta_\omega)^2}{|D_{\epsilon}|^2} + \int_{\mathbb{T}} \frac{2 \gamma R(k) \epsilon^2 (\delta_\omega)^2}{f_{\theta,s}(\epsilon) |D_{\epsilon}|^2} \frac{dk}{|D_{\epsilon}|^2} \\
= \lambda \int_{\mathbb{T}} \frac{2 \gamma R(k) \left( f_{\theta,s}(\epsilon) \lambda + 2 \gamma R(k) \right)}{|D_{\epsilon}|^2} + \int_{\mathbb{T}} \frac{2 \gamma R(k) \epsilon^2 (\delta_\omega)^2}{f_{\theta,s}(\epsilon) |D_{\epsilon}|^2} \frac{dk}{|D_{\epsilon}|^2},
\]

where we use the property that \( k \to \delta_\omega(p,k) \) is an odd function for any \( p \in \mathbb{R} \), and \( k \to R(k), k \to |D_{\epsilon}(p,k,\lambda)|^2 \) are even functions for any \( (p, \lambda) \in \mathbb{R} \times (0, \infty) \). Since

\[
0 \leq \frac{2 \gamma R(k) \left( f_{\theta,s}(\epsilon) \lambda + 2 \gamma R(k) \right)}{|D_{\epsilon}|^2} \leq 1
\]

and

\[
\lim_{\epsilon \to 0} \frac{2 \gamma R(k) \left( f_{\theta,s}(\epsilon) \lambda + 2 \gamma R(k) \right)}{|D_{\epsilon}|^2} = 1
\]
for any \((p, k, \lambda) \in \mathbb{R} \times T_0 \times (0, \infty)\), by using the dominated convergence theorem we have

\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}} dp \left| \lambda \int_T dk \frac{2\gamma R(k) (f_{\theta,s}(\epsilon) \lambda + 2\gamma R(k))}{|D_k|^2} - \lambda |J(p)| \right| = 0
\]

for any \(J \in \mathcal{S}(\mathbb{R})\).

From now on we will show that

\[
\begin{align*}
\lim_{\epsilon \to 0} \int_{\mathbb{R}} dp \left| I_\epsilon(p, \lambda) - C_{\theta, \gamma_0} |p|^{\frac{3}{2}} \right| J(p) &= 0 & 2 < \theta \leq 3, \\
\lim_{\epsilon \to 0} \int_{\mathbb{R}} dp \left| I_\epsilon(p, \lambda) - C_{\theta, \gamma_0} |p|^\frac{3}{2} \right| J(p) &= 0 & \theta > 3,
\end{align*}
\]

(5.50)

for any \(\lambda > 0\) where

\[
I_\epsilon(p, \lambda) := \int_T dk \frac{2\gamma R(k) \epsilon^2 (\delta_\epsilon \omega)^2}{|f_{\theta,s}(\epsilon)| D_k^2}.
\]

Fix a \(p \in \mathbb{R}_0\), where \(\mathbb{R}_0 = \mathbb{R} \setminus \{0\}\). First we change the variable \(k \in T\) as

\[
k \mapsto \begin{cases} 
\gamma_0^{-\frac{3}{2}} |p|^{-\frac{3}{2}} (f_{\theta,s}(\epsilon) \frac{e^s}{e^\theta})^{-\frac{1}{2}} k & 2 < \theta \leq 3, \\
\gamma_0^{-\frac{1}{2}} |p|^{-\frac{1}{2}} (f_{\theta,s}(\epsilon) \frac{e^s}{e^\theta})^{-\frac{1}{2}} k & \theta > 3,
\end{cases}
\]

and then the domain \(T = [-\frac{1}{2}, \frac{1}{2}]\) is also changed as

\[
T \mapsto T_\epsilon := \begin{cases} 
- \frac{1}{2} \gamma_0^{-\frac{3}{2}} |p|^{-\frac{3}{2}} (f_{\theta,s}(\epsilon) \frac{e^s}{e^\theta})^{-\frac{1}{2}} \leq k < \frac{1}{2} \gamma_0^{-\frac{3}{2}} |p|^{-\frac{3}{2}} (f_{\theta,s}(\epsilon) \frac{e^s}{e^\theta})^{-\frac{1}{2}} & 2 < \theta \leq 3, \\
- \frac{1}{2} \gamma_0^{-\frac{1}{2}} |p|^{-\frac{1}{2}} (f_{\theta,s}(\epsilon) \frac{e^s}{e^\theta})^{-\frac{1}{2}} \leq k < \frac{1}{2} \gamma_0^{-\frac{1}{2}} |p|^{-\frac{1}{2}} (f_{\theta,s}(\epsilon) \frac{e^s}{e^\theta})^{-\frac{1}{2}} & \theta > 3.
\end{cases}
\]

Define

\[
k_\epsilon := \begin{cases} 
\gamma_0^{-\frac{3}{2}} |p|^{-\frac{3}{2}} (f_{\theta,s}(\epsilon) \frac{e^s}{e^\theta})^{\frac{1}{2}} k & 2 < \theta \leq 3, \\
\gamma_0^{-\frac{1}{2}} |p|^{-\frac{1}{2}} (f_{\theta,s}(\epsilon) \frac{e^s}{e^\theta})^{\frac{1}{2}} k & \theta > 3,
\end{cases}
\]

for any \(k \in T_\epsilon\). Then we define scaled function \(R_\epsilon(k), (\delta_\epsilon \omega)_\epsilon(p, k)\) as

\[
R_\epsilon(k) := \begin{cases} 
\gamma_0^{-\frac{3}{2}} |p|^{-\frac{3}{2}} (f_{\theta,s}(\epsilon) \frac{e^s}{e^\theta})^{\frac{1}{2}} R(k_\epsilon) & 2 < \theta \leq 3, \\
\gamma_0 |p|^{-\frac{1}{2}} R(k_\epsilon) & \theta > 3,
\end{cases}
\]

\[
(\delta_\epsilon \omega)_\epsilon(p, k) := \begin{cases} 
|p|^{-\frac{1}{2}} \left[ - \log \left( f_{\theta,s}(\epsilon) \frac{e^s}{e^\theta} \right) \right]^{\frac{1}{\theta}} \delta_\epsilon \omega(p, k_\epsilon) & 2 < \theta < 3, \\
|p|^{-\frac{1}{2}} \delta_\epsilon \omega(p, k_\epsilon) & \theta = 3,
\end{cases}
\]

\(\theta > 3\),
for any \( k \in \mathbb{T}_e \). Then we have

\[
I_\epsilon(p, \lambda) = \begin{cases} 
\gamma_0^{-\frac{3-\theta}{2}} |p|^\frac{3}{2} \int_{T_e} dk & 2 \theta \leq 3, \\
\gamma_0^{-\frac{3-\theta}{2}} |p|^\frac{3}{2} \int_{T_e} dk & \theta > 3.
\end{cases}
\]

From Lemma B.1, we see that \( R_\epsilon(k) \) converges to \( 6\pi^2k^2 \) as \( \epsilon \to 0 \) for any \((p, k) \in \mathbb{R}_0 \times \mathbb{R}\). On the other hand, we have

\[
\lim_{\epsilon \to 0} (\delta_\epsilon \omega)_\epsilon(p, k) = \begin{cases} 
\text{sgn}(k) \frac{(\theta - 1)\sqrt{C(\theta)}}{2} |k|^{-\frac{3-\theta}{2}} & 2 \theta < 3, \\
\text{sgn}(k) \sqrt{C(\theta)} & \theta \geq 3,
\end{cases} \tag{5.51}
\]

for any \((p, k) \in \mathbb{R}_0 \times \mathbb{R}_0\), see Appendix D. In addition, the integrand of \( I_\epsilon(p, \lambda) \) is bounded from above by \( 1_{\{|k| \leq 1\}} R_\epsilon + 1_{\{|T, \epsilon| |k| > 1\}} \frac{(\delta_\epsilon \omega)_\epsilon^2 R_\epsilon}{R_\epsilon} \), and from (D.2) we obtain

\[
1_{\{|k| \leq 1\}} R_\epsilon + 1_{\{|T, \epsilon| |k| > 1\}} \frac{(\delta_\epsilon \omega)_\epsilon^2}{R_\epsilon} \leq \begin{cases} 
1_{\{|k| \leq 1\}} |k|^2 + 1_{\{|k| > 1\}} |k|^{-\theta} & 2 \theta < 3, \\
1_{\{|k| \leq 1\}} |k|^2 + 1_{\{|k| > 1\}} \log |k| & \theta = 3, \\
1_{\{|k| \leq 1\}} |k|^2 + 1_{\{|k| > 1\}} |k|^{-2} & \theta > 3.
\end{cases}
\]

Therefore by the dominated convergence theorem, we have

\[
\lim_{\epsilon \to 0} \int_{T_e} dk \frac{2R_\epsilon (\delta_\epsilon \omega)_\epsilon^2}{\left[ \gamma_0^{-\frac{3-\theta}{2}} \left( \epsilon^{2s} f_{\theta, s}(\epsilon) \right)^{\frac{3}{2}} \lambda + 2R_\epsilon \right]^2 + (\delta_\epsilon \omega)_\epsilon^2} = \int_{\mathbb{R}} dk \frac{12\pi^2 \theta (\theta - 1)^2 C(\theta) |k|^2}{576\pi^4 |k|^{1-\theta} + (\theta - 1)^2 C(\theta)}
\]

for \( 2 \theta \leq 3 \), and

\[
\lim_{\epsilon \to 0} \int_{T_e} dk \frac{2R_\epsilon (\delta_\epsilon \omega)_\epsilon^2}{\left[ \gamma_0^{-\frac{3-\theta}{2}} \left( \epsilon^{2s} f_{\theta, s}(\epsilon) \right)^{\frac{3}{2}} \lambda + R_\epsilon \right]^2 + (\delta_\epsilon \omega)_\epsilon^2} = \int_{\mathbb{R}} dk \frac{12\pi^2 C(\theta) |k|^2}{144\pi^4 |k|^4 + C(\theta)}
\]

for \( \theta > 3 \). By using the calculus of residua, we get

\[
\int_{\mathbb{R}} dk \frac{|k|^2}{|k|^{1+1/|k|^{\tau}}} = \frac{\pi \csc (\frac{\pi}{2} \tau + \frac{\pi}{4})}{2}, \quad 0 \leq \tau < 1,
\]
and thus we obtain

\[
\int_{\mathbb{R}} dk \frac{12\pi^2 (\theta - 1)^2 C(\theta) |k|^2}{576\pi^4 |k|^{7-\theta} + (\theta - 1)^2 C(\theta)} \\
= \frac{48\pi^2}{7-\theta} \left( \frac{\theta - 1}{24\pi^2} \right)^{\frac{6}{7-\theta}} \int_{\mathbb{R}} dk \frac{|k|^2}{|k|^4 + 1} \frac{1}{|k|^\frac{4(3-\theta)}{7-\theta}} \\
= \frac{24\pi^3 \csc \left( \frac{3\pi(3-\theta)}{4(7-\theta)} + \frac{\pi}{4} \right)}{7-\theta} \left( \frac{\theta - 1}{24\pi^2} \right)^{\frac{6}{7-\theta}} C(\theta)^{\frac{3}{7-\theta}}
\]

for \(2 < \theta \leq 3\) where in the second equation we change the variable as \(k \rightarrow sgn(k) \left( \frac{(\theta - 1)\sqrt{C(\theta)}}{24\pi^2} \right)^{\frac{2}{7-\theta}} |k|^{\frac{4}{7-\theta}}\), and

\[
\int_{\mathbb{R}} dk \frac{12\pi^2 C(\theta) |k|^2}{144\pi^4 |k|^4 + C(\theta)} = \frac{\sqrt{3}C(\theta)^{\frac{3}{2}}}{6\pi \int_{\mathbb{R}} dk \frac{|k|^2}{|k|^4 + 1}} = \frac{\sqrt{6}C(\theta)^{\frac{3}{2}}}{12}
\]

for \(\theta > 3\). From the above arguments and the dominated convergence theorem, we obtain (5.50).

5.5.3. Proof of (5.47). From the proof of (5.45), we see that \(|b_s(p, \lambda)J(p)|\) is bounded by some integrable function on \(\mathbb{R}\). By using (5.36), we can verify (5.47).

5.5.4. Proof of (5.48). First we divide \(\int_{\mathbb{T}} dk \frac{\gamma R_k}{f_\theta, s(\epsilon) D_\epsilon}\) into three parts:

\[
\int_{\mathbb{T}} dk \frac{\gamma R(k) q_e}{f_\theta, s(\epsilon) D_\epsilon} = \sum_{i=1}^{3} Q^{(i)}_\epsilon,
\]

\[
Q^{(1)}_\epsilon := -\sqrt{-1} \int_{\mathbb{T}} dk \frac{\gamma R(k) R'(k) \epsilon \bar{u}_\epsilon}{f_\theta, s(\epsilon) D_\epsilon}, \quad Q^{(2)}_\epsilon := -\int_{\mathbb{T}} dk \frac{\gamma^2 \epsilon p R(k)(L \bar{u}_\epsilon)}{f_\theta, s(\epsilon) D_\epsilon}, \quad Q^{(3)}_\epsilon := \int_{\mathbb{T}} dk \frac{\gamma^2 \epsilon^2 p^2 R(k)^2 p R^{(1)}_\epsilon}{f_\theta, s(\epsilon) D_\epsilon}.
\]

From the boundness of \(\frac{\gamma R}{\epsilon^2}\) and (5.28), we have

\[
|Q^{(3)}_\epsilon| \leq p^2 \frac{\epsilon^{2+s}}{f_\theta, s(\epsilon)},
\]

and thus we obtain

\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}} dp \left| Q^{(3)}_\epsilon(p, \lambda) J(p) \right| = 0,
\]

for any \(\theta > 2, 0 \leq s < 1\).
Next we consider the limit of $Q^{(1)}_\epsilon(p, \lambda)$. Since $k \to \delta_\epsilon \omega(k)$, $k \to R'(k)$ are odd and $k \to R(k)$, $k \to |D_x(p, k, \lambda)|^2$ are even for any $(p, \lambda) \in \mathbb{R} \times (0, \infty)$, we have

$$\begin{align*}
Q^{(1)}_\epsilon(p, \lambda) &= -\sqrt{-1} \int_T dk \frac{\gamma^2 R(k) R'(k) \epsilon p \hat{u}_{\epsilon,-} \left( f_{\theta,s}(\epsilon) \lambda + 2 \gamma R(k) - \sqrt{-1} \epsilon \delta_\epsilon \omega \right)}{f_{\theta,s}(\epsilon)|D_x|^2} \\
&= - \int_T dk \frac{\gamma^2 \epsilon^2 p R(k) R'(k) \delta_\epsilon \omega}{f_{\theta,s}(\epsilon)|D_x|^2} \hat{u}_{\epsilon, -} \\
&= - \sqrt{-1} \int_T dk \frac{\gamma^2 R(k) R'(k) \epsilon p \hat{u}_{\epsilon,-} \left( f_{\theta,s}(\epsilon) \lambda + 2 \gamma R(k) \right)}{f_{\theta,s}(\epsilon)|D_x|^2} \\
&= - \int_T dk \frac{\gamma^2 \epsilon^2 p R(k) R'(k) \delta_\epsilon \omega}{f_{\theta,s}(\epsilon)|D_x|^2} \hat{u}_{\epsilon, -}.
\end{align*}$$

By using Schwarz’s inequality, Proposition 5.2 and (5.49), we have

$$\begin{align*}
|Q^{(1)}_\epsilon|^2 &\leq \frac{\gamma^2 \epsilon^2|p|^2}{\left(f_{\theta,s}(\epsilon)\right)^2} \left( \int_T dk R(k)|\hat{u}_{\epsilon, -}|^2 \right) \left( \int_T dk R(k) \left| \frac{\gamma R'(k) \epsilon \delta_\epsilon \omega}{|D_x|^2} \right|^2 \right) \\
&\leq \frac{\gamma^2 \epsilon^2|p|^2}{\left(f_{\theta,s}(\epsilon)\right)^2} \left( \int_T dk R(k)|\hat{u}_{\epsilon, -}|^2 \right) \left( \int_T dk R(k) \left| \frac{\gamma R'(k) \epsilon}{f_{\theta,s}(\epsilon) \lambda + 2 \gamma R} \right|^2 \right) \\
&\leq |p|^2 \frac{\epsilon^{2+s}}{f_{\theta,s}(\epsilon)}.
\end{align*}$$

Therefore we obtain

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} dp \, |Q^{(1)}_\epsilon(p, \lambda) J(p)| = 0,$$

for any $\theta > 2, 0 \leq s < 1$.

Finally we consider the limit of $Q^{(2)}_\epsilon$. Since $f_v \int_T dk \, (\mathcal{L} f) = 0$ for any $f \in L^1(T)$, $k \to \delta_\epsilon \omega(k)$, $k \to R'(k)$ are odd and $k \to R(k)$, $k \to |D_x(p, k, \lambda)|^2$ are
even for any \((p, \lambda) \in \mathbb{R} \times (0, \infty)\), we have

\[
Q_\varepsilon^{(2)} = - \int_T dk \frac{\gamma^2 R(k)(L\bar{u}_{\varepsilon,+})}{f_{\theta,s}(\varepsilon)D_\varepsilon}
= - \int_T dk \frac{\gamma}{2f_{\theta,s}(\varepsilon)}(L\bar{u}_{\varepsilon,+}) + \int_T dk \frac{\gamma(f_{\theta,s}(\varepsilon)\lambda + \sqrt{-1}\varepsilon\delta\omega)}{2f_{\theta,s}(\varepsilon)D_\varepsilon}(L\bar{u}_{\varepsilon,+})
= \int_T dk \frac{\gamma f_{\theta,s}(\varepsilon)\lambda}{2f_{\theta,s}(\varepsilon)D_\varepsilon}(L\bar{u}_{\varepsilon,+})
= \int_T dk \frac{\gamma f_{\theta,s}(\varepsilon)\lambda f_{\theta,s}(\varepsilon) + 2\gamma R(k) + \gamma (\varepsilon\delta\omega)^2}{2f_{\theta,s}(\varepsilon)|D_\varepsilon|^2}(L\bar{u}_{\varepsilon,+})
+ \int_T dk \frac{2\gamma^2(\sqrt{-1}\varepsilon\delta\omega)R(k)}{2f_{\theta,s}(\varepsilon)|D_\varepsilon|^2}(L\bar{u}_{\varepsilon,+})
= \int_T dk \frac{\gamma f_{\theta,s}(\varepsilon)\lambda f_{\theta,s}(\varepsilon) + 2\gamma R(k) + \gamma (\varepsilon\delta\omega)^2}{2f_{\theta,s}(\varepsilon)|D_\varepsilon|^2}(L\bar{u}_{\varepsilon,+})
= \sum_{i=1,2,3} Q_\varepsilon^{(2,i)},
\]

where

\[
Q_\varepsilon^{(2,1)} := \int_T dk \frac{\gamma \lambda f_{\theta,s}(\varepsilon)\lambda + 2\gamma R(k)}{2|D_\varepsilon|^2}(L\bar{u}_{\varepsilon,+}),
Q_\varepsilon^{(2,2)} := \int_T dk \frac{\gamma R(k)(\varepsilon\delta\omega)^2}{2f_{\theta,s}(\varepsilon)|D_\varepsilon|^2}\bar{u}_{\varepsilon,+},
Q_\varepsilon^{(2,3)} := \sum_{i=1,2} |u_{\varepsilon,i}| \int_T dk \frac{3\gamma e_{\varepsilon-i}(\varepsilon\delta\omega)^2}{8f_{\theta,s}(\varepsilon)|D_\varepsilon|^2}
\]

and \(u_{\varepsilon,i}(p, \lambda) := \int_T dk e_i(k)\bar{u}_{\varepsilon,+}(p, k, \lambda), i = 1, 2\). From (5.49) and (5.34), we get

\[
|Q_\varepsilon^{(2,1)}(p, \lambda)| \leq \sum_{i=1,2} |u_{\varepsilon,i}| \left| \int_T dk \frac{3\gamma e_{\varepsilon-i}(\varepsilon\delta\omega)^2}{8|D_\varepsilon|^2}\bar{u}_{\varepsilon,+} \right|
\]

By using (5.37), we have

\[
\lim_{\varepsilon \to 0} \int_\mathbb{R} dp |Q_\varepsilon^{(2,1)}(p, \lambda)J(p)| = 0,
\]
for any $\theta > 2, 0 \leq s < 1$. Next we estimate $Q^{(2,2)}_\epsilon$. Thanks to Schwarz’s inequality, Proposition 5.2 and (5.49) we have
\[
\left| \int_T dk \frac{\gamma R(k)(\epsilon \delta_\omega)^2}{2f_{\theta,s}(\epsilon)} |D_\epsilon|^2 u_{\epsilon,+} \right|^2 \\
\leq \frac{\gamma^2}{4f_{\theta,s}(\epsilon)^2} \left| \int_T dk R(k) |\bar{u}_{\epsilon,+}|^2 \left( \frac{\omega_\epsilon}{\lambda f_{\theta,s}(\epsilon) + \gamma R(k)} \right)^2 \right| \left| \int_T dk \frac{R(k)(\epsilon \delta_\omega)^4 \left( \lambda f_{\theta,s}(\epsilon) + \gamma R(k) \right)^2}{(\omega_\epsilon)^2 |D_\epsilon|^4} \right| \\
\leq \frac{\gamma^2}{4f_{\theta,s}(\epsilon)^2} \left| \int_T dk R(k) |\bar{u}_{\epsilon,+}|^2 \left( \frac{\omega_\epsilon}{\lambda f_{\theta,s}(\epsilon) + \gamma R(k)} \right)^2 \right| \left| \int_T dk \frac{R(k)(\epsilon \delta_\omega)^2}{(\omega_\epsilon)^2} \right| \\
\leq \frac{\epsilon^{2+s}}{f_{\theta,s}(\epsilon)},
\]
and therefore we obtain
\[
\lim_{\epsilon \to 0} \int_R dp |Q^{(2,2)}(p, \lambda) J(p)| = 0,
\]
for any $\theta > 2, 0 \leq s < 1$. Now we consider $Q^{(2,3)}_\epsilon$. From Proposition 5.2, we have
\[
|u_{\epsilon,i}|^2 \leq \left| \int_T dk R|\bar{u}_{\epsilon,+}|^2 \left( \frac{\omega_\epsilon}{\lambda f_{\theta,s}(\epsilon) + \gamma R} \right)^2 \right| \left| \int_T dk \left( \frac{\lambda f_{\theta,s}(\epsilon) + \gamma R}{\omega_\epsilon} \right)^2 e_i^2 \frac{R}{e_i^2} \right| \\
\leq \epsilon^s f_{\theta,s}(\epsilon), \quad i = 1, 2.
\]
Thus by using (5.49) we get
\[
\sum_{i=1,2} |u_{\epsilon,i}| \left| \int_T dk \frac{3\gamma e_{\epsilon,-}(\epsilon \delta_\omega)^2}{8f_{\theta,s}(\epsilon)} |D_\epsilon|^2 \right| \leq \left( \epsilon^s f_{\theta,s}(\epsilon) \right)^\frac{1}{2} \left| \int_T dk \frac{\gamma R(\epsilon \delta_\omega)^2}{f_{\theta,s}(\epsilon)|D_\epsilon|^2} \right| \\
\leq \left( \frac{\epsilon^{2+s}}{f_{\theta,s}(\epsilon)} \right)^\frac{1}{2} \int_T dk |\delta_{\epsilon}\omega|,
\]
and
\[
\lim_{\epsilon \to 0} \int_R dp |Q^{(2,3)}(p, \lambda) J(p)| = 0,
\]
for any $\theta > 2, 0 \leq s < 1$.

6. PROOF OF THEOREM 2

In this section we prove Theorem 2 by following the strategy of the proof of [4, Theorem 5]. Some steps are simplified because we assume the thermal type condition (4.1). For instance, see the proof of Lemma 6.1 and compare it with the computation in [4, Section 4.3.1]. In addition, we prove the uniqueness of solutions of the measure-valued Boltzmann equation (4.14).

6.1. Proof of (1), (2).

6.1.1. * - weakly sequentially compactness of $\{\{W_{\epsilon,+}(\cdot, \cdot)\}_\epsilon\}$. In this subsection we show that $\{\{W_{\epsilon,+}(\cdot, \cdot)\}_\epsilon\}$ is *- weakly sequentially compact in $C([0, T]; S(\mathbb{R} \times T))$ for any $T > 0$. Since $S(\mathbb{R} \times T)$ is separable, we can take a dense countable subset $\{J^{(m)}; m \in \mathbb{N}\} \subset S(\mathbb{R} \times T)$. From (4.12) and (6.1) (which appears in the proof of (3)), we see that $\{\{W_{\epsilon,+}(\cdot, J)\}_\epsilon\}$ is uniformly bounded and equicontinuous for any $J \in S(\mathbb{R} \times T)$. Therefore, by using the diagonal argument we can find a subsequence $\{\{W_{\epsilon(n),+}(\cdot, \cdot)\}_n\} \subset \{\{W_{\epsilon(n),+}(\cdot, \cdot)\}_n\}$ converges in
We follow the proof of [20, Theorem B.4].

On the other hand we have $J \in C(S(\mathbb{R} \times \mathbb{T}))$ for all $m \in \mathbb{N}$. Then by using (4.12) again, we see that $\{W_{\epsilon(n)}(\cdot), J\}_{n \in \mathbb{N}}$ converges in $C([0,T])$ for any $J \in S(\mathbb{R} \times \mathbb{T})$.

6.1.2. Extension to a finite positive measure. Now we verify that the $\star$ - weak limit $\{W(t, \cdot) := \lim_{n \to \infty} W_{\epsilon(n)}(\cdot), \cdot\}$ can be extended to a finite measure $\mu(t)$ on $\mathbb{R} \times \mathbb{T}$. Note that the following discussion does not depend on the time scaling and the noise scaling for the dynamics.

First we will show that $W(\cdot)$ is multiplicatively positive, that is,

$$\langle W(t), |J|^2 \rangle \geq 0$$

for any $t \geq 0$, $J \in S(\mathbb{R} \times \mathbb{T})$. We follow the proof of [20, Theorem B.4]. Fix a $J \in S(\mathbb{R} \times \mathbb{T})$. Since $J$ is smooth,

$$J\left(\frac{\varepsilon}{2}(x + x'), k\right) - J(\varepsilon, k) = \frac{\varepsilon}{2} \int_0^1 dr \ (x' - x) \partial_y J(\varepsilon + r \frac{\varepsilon}{2}(x' - x), k)$$

for any $x, x' \in \mathbb{Z}$. Therefore we have

$$\left| \int_T dk e^{2\pi \sqrt{-1}(x'-x)k} \left( J\left(\frac{\varepsilon}{2}(x + x'), k\right) - J(\varepsilon, k) J\left(\frac{\varepsilon}{2}(x + x'), k\right)^* \right) \right|$$

$$= \left| \frac{\varepsilon}{2} \left( x' - x \right) \int_T dk e^{2\pi \sqrt{-1}(x'-x)k} J\left(\frac{\varepsilon}{2}(x + x'), k\right)^* \int_0^1 dr \partial_y J(\varepsilon + r \frac{\varepsilon}{2}(x' - x), k) \right| .$$

By repeating the integration by parts we have

$$\left| \int_T dk e^{2\pi \sqrt{-1}(x'-x)k} J\left(\frac{\varepsilon}{2}(x + x'), k\right)^* \int_0^1 dr \partial_y J(\varepsilon + r \frac{\varepsilon}{2}(x' - x), k) \right|$$

$$= \left| \int_T dk \left( \frac{1}{2\pi \sqrt{-1}(x'-x)} \right)^3 e^{2\pi \sqrt{-1}(x'-x)k} \partial_k^3 \left[ J\left(\frac{\varepsilon}{2}(x + x'), k\right)^* \int_0^1 dr \partial_y J(\varepsilon + r \frac{\varepsilon}{2}(x' - x), k) \right] \right|$$

$$\leq \frac{1}{8\pi^3 |x - x'|^3} \int_T dk \left| \partial_k^3 \left[ J\left(\frac{\varepsilon}{2}(x + x'), k\right)^* \int_0^1 dr \partial_y J(\varepsilon + r \frac{\varepsilon}{2}(x' - x), k) \right] \right| .$$

Hence, we obtain

$$\left| \int_T dk e^{2\pi \sqrt{-1}(x'-x)k} \left( J\left(\frac{\varepsilon}{2}(x + x'), k\right) - J(\varepsilon, k) J\left(\frac{\varepsilon}{2}(x + x'), k\right)^* \right) \right|$$

$$\leq \frac{\varepsilon}{16\pi^3 |x - x'|^3} \int_T dk \left| \partial_k^3 \left[ J\left(\frac{\varepsilon}{2}(x + x'), k\right)^* \int_0^1 dr \partial_y J(\varepsilon + r \frac{\varepsilon}{2}(x' - x), k) \right] \right|$$

$$\leq \frac{1}{|x - x'|^3} O_J(\varepsilon)$$

for all $x + x' \in \mathbb{Z}$ where $O_J(\varepsilon)$ is the remainder term which satisfies $\lim_{\varepsilon \to 0} |O_J(\varepsilon)| \leq 1$. In the same way, we can show that

$$\left| \int_T dk e^{2\pi \sqrt{-1}(x'-x)k} \left( J(\varepsilon, k) J\left(\frac{\varepsilon}{2}(x + x'), k\right)^* - J(\varepsilon, k) J(\varepsilon, k)^* \right) \right| \leq \frac{1}{|x - x'|^3} O_J(\varepsilon).$$

On the other hand we have

$$\frac{\varepsilon}{2} \sum_{x, x' \in \mathbb{Z}} E_x \left[ \sum_{x, k} \left( \psi_{x'} \left( \frac{t}{\varepsilon} \right) \psi_{x'} \left( \frac{t}{\varepsilon} \right) \right) \int_T dk e^{2\pi \sqrt{-1}(x'-x)k} J(\varepsilon, k) J(\varepsilon, k)^* \right]$$

$$= \frac{\varepsilon}{2} \int_T dk E_x \left[ \sum_{x \in \mathbb{Z}} e^{-2\pi \sqrt{-1}xt} \psi_{x} \left( \frac{t}{\varepsilon} \right) J(\varepsilon, k)^2 \right] \geq 0.$$
Since
\[
\left| \int_{\mathbb{T}} dke^{2\pi \sqrt{-1}(x'-x)k} \left( |J\left( \frac{\epsilon}{2} (x+x'), k\right)|^2 - J(\epsilon x, k)J(\epsilon x', k)^* \right) \right|
\]
\[
\leq \int_{\mathbb{T}} dke^{2\pi \sqrt{-1}(x'-x)k} \left( J\left( \frac{\epsilon}{2} (x+x'), k\right)J\left( \frac{\epsilon}{2} (x+x'), k\right)^* - J(\epsilon x, k)J\left( \frac{\epsilon}{2} (x+x'), k\right)^* \right)
\]
\[
+ \int_{\mathbb{T}} dke^{2\pi \sqrt{-1}(x'-x)k} \left( J(\epsilon x, k)J\left( \frac{\epsilon}{2} (x+x'), k\right)^* - J(\epsilon x, k)J(\epsilon x', k)^* \right),
\]
by combining the above calculations we have
\[
\langle W_{\epsilon,+}(t), |J|^2 \rangle = \frac{\epsilon}{2} \int_{\mathbb{T}} dk \mathbb{E}_\epsilon \left[ \left| \sum_{x\in\mathbb{Z}} e^{-2\pi \sqrt{-1} xk} \psi_\epsilon(x) J(\epsilon x, k) \right|^2 \right] + O(\epsilon),
\]
and thus
\[
\langle W(t), |J|^2 \rangle \geq \liminf_{\epsilon \to 0} \langle W(t)_{\epsilon,+}, |J|^2 \rangle \geq 0,
\]
for any \( t \geq 0 \). Therefore \( W(\cdot) \) is multiplicatively positive.

Next we show that \( W(\cdot) \) is positive, that is,
\[
\langle W(t), J \rangle \geq 0
\]
for any \( t \geq 0, J \in \mathcal{S}(\mathbb{R} \times \mathbb{T}), J \geq 0 \). Since \( \{ J \in \mathcal{S}(\mathbb{R} \times \mathbb{T}); J \in C_0^\infty(\mathbb{R} \times \mathbb{T}), J \geq 0 \} \) is a dense subset of \( \{ J \in \mathcal{S}(\mathbb{R} \times \mathbb{T}); J \geq 0 \} \), it is sufficient to show the positivity on \( C_0^\infty(\mathbb{R} \times \mathbb{T}) \). Fix a positive function \( J \in C_0^\infty(\mathbb{R} \times \mathbb{T}) \). There exists a positive constant \( M > 0 \) such that the support of \( J \) is a subset of \([-M,M] \times \mathbb{T}\). Let \( a(y) \in C_0^\infty(\mathbb{R}) \) be a function such that \( a(y) = 1, y \in [-M,M] \). Define \( J^{(m)}(y, k) = C_0^\infty(\mathbb{R} \times \mathbb{T}), m \in \mathbb{N} \) as

\[
J^{(m)}(y, k) = a(y) \sqrt{J(y, k) + \frac{1}{m}}.
\]

Then the sequence \( \{|J^{(m)}|^2, m \in \mathbb{N}\} \) converges to \( J(y, k) \) in the topology of \( C_0^\infty(\mathbb{R} \times \mathbb{T}) \). Since the embedding of the space \( C_0^\infty(\mathbb{R} \times \mathbb{T}) \) into the space \( \mathcal{S}(\mathbb{R} \times \mathbb{T}) \) is continuous, \( \{|J^{(m)}|^2, m \in \mathbb{N}\} \) also converges to \( J(y, k) \) in the topology of \( \mathcal{S}(\mathbb{R} \times \mathbb{T}) \). By the continuity of \( W(t) \), we have
\[
\langle W(t), J \rangle = \lim_{m \to \infty} \langle W(t), |J^{(m)}|^2 \rangle \geq 0
\]
for any \( t \geq 0 \). Therefore \( W(\cdot) \) is positive.

By the usual method, for example see Lemma 1 in Chapter 2 of [9], we can extend the domain of \( W(\cdot) \) to the space \( C_0(\mathbb{R} \times \mathbb{T}) \). By the Riesz representation theorem there exists a family of positive measures \( \{\mu(t); t \geq 0\} \) such that
\[
\langle W(t), J \rangle = \int_{\mathbb{R} \times \mathbb{T}} \mu(t)(dy, dk) J(y, k),
\]
for all \( t \geq 0 \) and \( J \in C_0(\mathbb{R} \times \mathbb{T}) \).

Finally we show that \( \{\mu(t); t \geq 0\} \) are finite measures. Fix a family of non-negative functions \( J^{(l)}(y, k) = J^{(l)}(y) \in \mathcal{S}, l \in \mathbb{N} \) which satisfies \( J^{(l)}(y) \not< 1, l \to \infty \) for any
\( y \in \mathbb{R} \). From (4.3), (4.8) and the monotone convergence theorem, we have
\[
\sup_{t \geq 0} \mu(t)(\mathbb{R} \times \mathbb{T}) = \lim_{t \to \infty} \int_{\mathbb{R} \times \mathbb{T}} \mu(t)(dy, dk) J^{(t)}(y, k)
= \lim_{t \to \infty} \lim_{\epsilon \to 0} (W_{x, \epsilon}(t), J^{(t)})
= \lim_{t \to \infty} \lim_{\epsilon \to 0} \frac{1}{2} \sum_{x \in \mathbb{Z}} E_x[|\psi(x)|^2] J^{(t)}(\epsilon x)
\leq \sqrt{K_1 / 2}.
\]

6.2. Proof of (3).

6.2.1. Derivation of the Boltzmann equation. In this section we will show that
\[
\partial_t \langle W_{x, \epsilon}(t), J \rangle = \frac{1}{2\pi} \langle W_{x, \epsilon}(t), \omega'(\partial_y J) \rangle + \gamma_0 \langle W_{x, \epsilon}(t), \mathcal{L} J \rangle + o_J(1) \tag{6.1}
\]
for \( J \in \mathcal{S}(\mathbb{R} \times \mathbb{T}) \) where \( o_J(1) \) is the remainder term which satisfies
\[
\sup_{0 < \epsilon < 1} |o_J(1)| \leq 1, \quad \lim_{\epsilon \to 0} |o_J(1)| = 0.
\]

Notice that \( \langle W_{x, \epsilon}(t), \omega'(\partial_y J) \rangle \) is well-defined for any \( J \in \mathcal{S}(\mathbb{R} \times \mathbb{T}) \) because from Schwarz’s inequality and (5.1) with \( p = 0 \) we have
\[
\left| \frac{1}{2\pi} \langle W_{x, \epsilon}(t), \omega'(\partial_y J) \rangle \right|^2 = \left| \int_{\mathbb{R} \times \mathbb{T}} dpdk \sqrt{-1} \omega'(k) \overline{W}_{x, \epsilon}(p, k, t) J(p, k)^* \right|^2 
\leq \left( \int_{\mathbb{R} \times \mathbb{T}} dpdk |\omega'(k)|^2 \overline{W}_{x, \epsilon}(p, k, t) J(p, k) \right) \left( \int_{\mathbb{R} \times \mathbb{T}} dpdk |\overline{W}_{x, \epsilon}(p, k, t) J(p, k)|^2 \right) 
\leq \left( \int_{\mathbb{R} \times \mathbb{T}} dpdk |\omega'(k)|^2 |\overline{J}(p, k)| \right) \left( \int_{\mathbb{R}} dp \sup_{k \in \mathbb{T}} |\overline{J}(p, k)| \int_{\mathbb{T}} dk |\overline{W}_{x, \epsilon}(0, k, t)|^2 \right) 
\leq \int_{\mathbb{R} \times \mathbb{T}} dpdk |\omega'(k)|^2 |\overline{J}(p, k)| < \infty.
\]

To derive (6.1), we need the following lemma.

Lemma 6.1.
\[
\left| \int_{\mathbb{R} \times \mathbb{T}} dpdk \sqrt{-1} \frac{1}{\epsilon} (\delta_x \omega)(p, k) \overline{W}_{x, \epsilon}(p, k, t) J(p, k)^* \right|^2 
\leq \left( \int_{\mathbb{R} \times \mathbb{T}} dpdk |(\delta_x \omega)(p, k) - \omega'(k)| \overline{W}_{x, \epsilon}(p, k, t) J(p, k) \right) \left( \int_{\mathbb{R} \times \mathbb{T}} dpdk |\overline{W}_{x, \epsilon}(p, k, t) J(p, k)|^2 \right) 
\leq \left( \int_{\mathbb{R} \times \mathbb{T}} dpdk |(\delta_x \omega)(p, k) - \omega'(k)|^2 |\overline{J}(p, k)| \right) \left( \int_{\mathbb{R}} dp \sup_{k \in \mathbb{T}} |\overline{J}(p, k)| \int_{\mathbb{T}} dk |\overline{W}_{x, \epsilon}(0, k, t)|^2 \right) 
\leq \int_{\mathbb{R} \times \mathbb{T}} dpdk |(\delta_x \omega)(p, k) - \omega'(k)|^2 |\overline{J}(p, k)|.
\]

Proof. Thanks to Schwarz’s inequality and (5.1) with \( p = 0 \), we have
\[
\left| \int_{\mathbb{R} \times \mathbb{T}} dpdk \sqrt{-1} \frac{1}{\epsilon} (\delta_x \omega)(p, k) \overline{W}_{x, \epsilon}(p, k, t) J(p, k)^* \right|^2 
\leq \left( \int_{\mathbb{R} \times \mathbb{T}} dpdk |(\delta_x \omega)(p, k) - \omega'(k)| \overline{W}_{x, \epsilon}(p, k, t) J(p, k) \right) \left( \int_{\mathbb{R} \times \mathbb{T}} dpdk |\overline{W}_{x, \epsilon}(p, k, t) J(p, k)|^2 \right) 
\leq \left( \int_{\mathbb{R} \times \mathbb{T}} dpdk |(\delta_x \omega)(p, k) - \omega'(k)|^2 |\overline{J}(p, k)| \right) \left( \int_{\mathbb{R}} dp \sup_{k \in \mathbb{T}} |\overline{J}(p, k)| \int_{\mathbb{T}} dk |\overline{W}_{x, \epsilon}(0, k, t)|^2 \right) 
\leq \int_{\mathbb{R} \times \mathbb{T}} dpdk |(\delta_x \omega)(p, k) - \omega'(k)|^2 |\overline{J}(p, k)|.
\]
By using Lemma B.2, (B.3) and an inequality $|x + y|^a \leq |x|^a + |y|^a$, $0 < a < 1$, $x, y \in \mathbb{R}$, we obtain

$$
|\langle \delta, \omega \rangle(p, k)|^2 = \left| \frac{4p}{\omega(p + \frac{k}{2}) \omega(p - \frac{k}{2})} \sum_{x \in \mathbb{Z}} \frac{1}{|x|^{9/2}} \sin(\pi \epsilon px) \sin(2\pi kx) \right|^2
\leq \frac{|p|^2}{|k|^{2/3}} \left( \sum_{x \in \mathbb{Z}} \frac{1}{|x|^{9/2}} \right)^2 \leq \frac{|p|^2}{|k|^{2/3}} \left( |k|^{(3-\theta)} \right) 2 < \theta < 3,
$$

$$
\leq \frac{|p|^2 \log(|k|)^2}{|k|^2} \quad \theta = 3,
$$

$$
\leq \frac{|p|^2}{|k|^2} \quad \theta > 3.
$$

Since

$$
|\langle \delta, \omega \rangle(p, k) - p\omega' (k)|^2 \leq |\langle \delta, \omega \rangle(p, k)|^2 + p^2 |\omega' (k)|^2 \leq \begin{cases} 
|p|^2 |k|^{-(3-\theta)} & 2 < \theta < 3, \\
|p|^2 \log(|k|)^2 & \theta = 3, \\
|p|^2 & \theta > 3,
\end{cases}
$$

and $(\delta, \omega)(p, k)$ converges to $p\omega'(k)$, by the dominated convergence theorem we have

$$
\int_{\mathbb{R} \times \mathcal{T}} dp \, dk \, |\langle \delta, \omega \rangle(p, k) - p\omega' (k)|^2 |J(p, k)| = o_J(1).
$$

From Lemma 6.1, (5.13) and (5.14) we have

$$
\partial_t \{ W_{\epsilon,+, J} \} = \frac{1}{2\pi} \{ W_{\epsilon,+, \omega} (\partial_y J) \} + \gamma_0 \{ W_{\epsilon,+, \mathcal{L}J} \} - \gamma_0 \{ W_{\epsilon,+, \omega_k} \} + o_J(1),
$$

$$
\partial_t \{ Y_{\epsilon,+, J} \} = -\frac{2\sqrt{1}}{\epsilon} \{ Y_{\epsilon,+, \omega_k} \} + \gamma_0 \{ Y_{\epsilon,+, \mathcal{L}J} \} - \gamma_0 \{ W_{\epsilon,+, \omega_k} \} + o_J(1).
$$

From (6.4), for any $T > 0$ we obtain

$$
\lim_{\epsilon \to 0} \int_0^T dt \{ Y_{\epsilon,+, J} \} = 0, \quad \epsilon = +, -.
$$

If $J \in \mathcal{S}(\mathbb{R} \times \mathcal{T})$, then $\mathcal{L}J$, $\mathcal{R}_0J$ are rapidly decreasing in $\mathbb{R}$ and continuous in $\mathcal{T}$. Therefore for any $J \in \mathcal{S}(\mathbb{R} \times \mathcal{T})$ we have

$$
\lim_{\epsilon \to 0} \int_0^T dt \{ Y_{\epsilon,+, J} \} + |\int_0^T dt \{ Y_{\epsilon,+, \mathcal{R}_0J} \}| = 0, \quad \epsilon = +, -.
$$

From (6.3) and (6.5), we can derive (6.1).

Finally we show that a limit of a convergent subsequence $\mu(t)$ satisfies $\mu(t)\{dy, \{0\}\} = 0$. Define

$$
B^{\lambda, y^*, r}(y) := \exp \left( -\frac{\lambda}{r^2 - |y - y^*|^2} \right)_{B(y^*, r)}(y),
$$

$$
B(y^*, r) = \{ y \in \mathbb{R} : |y - y^*| < r \},
$$

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for any \( \lambda > 0, y^* \in \mathbb{R}, r > 0 \). Note that \( b^{\lambda,y^*,r} \in C_c^\infty(\mathbb{R}), \|b^{\lambda,y^*,r}\|_\infty \leq 1 \) and
\[
\lim_{\lambda \to 0} b^{\lambda,y^*,r}(y) = 1_{B(y^*,r)}(y).
\]
From (5.1) with \( p = 0 \) and Schwarz’s inequality, we have
\[
|\langle W_\epsilon, (t), J(y)b^{\lambda,0,r}(k) \rangle| \leq \|J\| \left( \int_{\mathbb{T}} dk \|b^{\lambda,0,r}(k)\|^2 \right)^{1/2} \leq \|J\| r,
\]
for any \( J \in L^2(\mathbb{R}), \lambda > 0, 0 < r < \frac{1}{2} \). By taking the limit \( \epsilon \to 0 \) and then \( \lambda \to 0 \), we have
\[
\left| \int_{\mathbb{R}} \mu(t)(dy,(-r,r))J(y) \right| \leq r \|J\| \to 0, \quad r \to 0.
\]
Therefore we obtain \( \mu(t)(dy, \{0\}) = 0 \).

6.2.2. Uniqueness of the solution of (4.14). In this subsection we prove the uniqueness of the initial value problem of (4.14). First we reduce the problem to the space-homogeneous case:

Suppose that a family of finite positive measures \( \{\mu(t); t \geq 0\} \) is a solution of the Boltzmann equation (4.14). Then
\[
\tilde{\mu}(t)(dy,dk) := \begin{cases}
\mu(t)(dy + \frac{1}{2\pi}\omega'((k)t,dk) & \text{on } \mathbb{R} \times \mathbb{T}_0, \\
0 & \text{on } \mathbb{R} \times \{0\},
\end{cases}
\]
is a solution of the following space-homogeneous Boltzmann equation
\[
\partial_t \int_{\mathbb{R} \times \mathbb{T}} d\tilde{\mu}(t)J = \int_{\mathbb{R} \times \mathbb{T}} d\tilde{\mu}(t)\mathcal{L}J
\]
\[
\int_{\mathbb{R} \times \mathbb{T}} d\tilde{\mu}(0)J = \int_{\mathbb{R} \times \mathbb{T}} d\tilde{\mu}_0J \quad \tilde{\mu}(t)(dy,\{0\}) = 0,
\]
where
\[
\int d\tilde{\mu}(t)J = \int \mu(t)(dy + \frac{1}{2\pi}\omega'(k)t,dk)J(y,k)
\]
\[
:= \int \mu(t)(dy,dk)J(y - \frac{1}{2\pi}\omega'(k)t,k).
\]
Conversely, if \( \tilde{\mu}(t) \) is a solution of the space-homogeneous Boltzmann equation, then
\[
\mu(t)(dy,dk) := \begin{cases}
\tilde{\mu}(t)(dy - \frac{1}{2\pi}\omega'(k)t,dk) & \text{on } \mathbb{R} \times \mathbb{T}_0, \\
0 & \text{on } \mathbb{R} \times \{0\},
\end{cases}
\]
is a solution of (4.14). Therefore it is sufficient to show the uniqueness of the solution for the space-homogeneous Boltzmann equation.

Suppose that \( J(y,k) = b^{\lambda,y^*,r}(y)G(k) \), where \( b^{\lambda,y^*,r}(y) \) is defined in (6.6) and \( G \in C_c^\infty(\mathbb{T}) \). Let \( \mu(t), \nu(t) \) be solutions of the space-homogeneous Boltzmann equation with a same initial condition. Then
\[
\left| \int d\mu(t)J - \int d\nu(t)J \right|
\]
\[
= \left| \int d\mu(t)b^{\lambda,y^*,r}(y)G(k) - \int d\nu(t)b^{\lambda,y^*,r}(y)G(k) \right|
\]
\[
\leq \int_0^t ds \int d(\mu(s) - \nu(s)) \cdot (b^{\lambda,y^*,r}(y)(\mathcal{L}G)(k)) |.
\]
By taking the limit $\lambda \to 0$, we have

$$
\left| \int_{T} G(k)(\mu(t)(B(y^*, r), dk) - \nu(t)(B(y^*, r), dk)) \right|
\leq \int_{0}^{t} ds \left| \int_{T} (\mu(s)(B(y^*, r), dk) - \nu(s)(B(y^*, r), dk))(\mathcal{L}G) \right|
\leq \int_{0}^{t} ds \| \mu(s)(B(y^*, r), dk) - \nu(s)(B(y^*, r), dk) \|
$$

where $\| \cdot \|$ is the total variation for a bounded signed measure on $T$. Hence,

$$
\| \mu(t)(B(y^*, r), dk) - \nu(t)(B(y^*, r), dk) \|
\leq \int_{0}^{t} ds \| \mu(s)(B(y^*, r), dk) - \nu(s)(B(y^*, r), dk) \|.
$$

Therefore $\mu(t)(B(y^*, r), dk) = \nu(t)(B(y^*, r), dk)$ on $T_0$ for any ball $B(y^*, r) \subset \mathbb{R}$, which implies that $\mu(t) = \nu(t)$ for any $t \geq 0$.

### 7. Proof of Theorem 3

#### 7.1. Proof of (1).

It is sufficient to show that in our case, [11, Condition 2.1, 2.2, 2.3, and (2.12)] are satisfied. Since our scattering operator is same with that of [11], [11, Condition 2.2 and 2.3] are satisfied. In addition, $\int_{T} P(\cdot, dk') \frac{\omega'(k')}{R(k')} \in L^2(\pi)$, [11, (2.12)] is also satisfied.

From now on we will verify that [11, Condition 2.1] is satisfied. Actually, instead of [11, Condition 2.1], it is sufficient to show that

$$
\lim_{N \to \infty} N \pi \left( \frac{\omega'(k)}{2\gamma_0 R(k)} > N(\theta) \lambda \right) = \begin{cases} 
C_\ast(\theta) \gamma_0 \frac{\pi^2}{\pi^2 - \theta} \lambda^{-\frac{\theta}{2}} & 2 < \theta < 3, \\
C_\ast(\theta) \gamma_0 \frac{3}{2} \lambda^{-\frac{3}{2}} & \theta \geq 3,
\end{cases}
$$

for $\lambda > 0$ where $C_\ast(\theta)$ is some positive constant. About the sufficiency, see the proof of [11, Lemma 5.5]. By using a change of variable $k \in T \rightarrow N^{\frac{1}{2}} k \in N^{\frac{1}{2}} T$, we have

$$
N \pi \left( \frac{\omega'(k)}{2\gamma_0 R(k)} > N(\theta) \lambda \right) = \frac{2N}{3} \int \frac{\omega'(k)}{2\gamma_0 R(k)} > N(\theta) \lambda \right) \frac{k}{R(k)}
\leq \frac{2}{3} \int \frac{\omega'(k)}{2\gamma_0 R(N^{\frac{1}{2}} k)} > N(\theta) \lambda \right) \frac{k}{R(N^{\frac{1}{2}} k)}
\leq \begin{cases} 
4\pi^2 \int \frac{N^{\frac{1}{2}}}{24 \gamma_0^2 \pi^2} (|k|^{-\theta} > \lambda, k \geq 0) \frac{k}{R^2} \frac{k^2}{R(N^{\frac{1}{2}} k)} \frac{k^2}{R(N^{\frac{1}{2}} k)} & 2 < \theta \leq 3, \\
4\pi^2 \int \frac{\sqrt{\gamma_0}}{12 \gamma_0 \pi^2} (|k|^{-\theta} > \lambda, k \geq 0) \frac{k^2}{R^2} \frac{k^2}{R(N^{\frac{1}{2}} k)} & \theta > 3,
\end{cases}
$$

$$
\leq \begin{cases} 
C_\ast(\theta) \gamma_0 \frac{\pi^2}{\pi^2 - \theta} |\lambda|^{-\frac{\theta}{2}} & 2 < \theta \leq 3, \\
C_\ast(\theta) \gamma_0 \frac{3}{2} |\lambda|^{-\frac{3}{2}} & \theta \geq 3,
\end{cases}
$$
where \(C_\tau(\theta) := \begin{cases} 
\frac{4\pi^2}{3} \left(\frac{\theta - 1}{24\pi^2}\right)^{\frac{6}{7}} & 2 \leq \theta < 3, \\
\frac{4\pi^2}{3} \left(\frac{\theta}{12\pi^2}\right)^{\frac{2}{7}} & \theta \geq 3.
\end{cases} \)

Therefore, by [11, Theorem 2.8, Theorem 6.1], the finite-dimensional distributions of scaled process \(\frac{1}{N(\theta)}Z(Nt); t \geq 0\) converge weakly to those of a Lévy process whose characteristic function at time 1, denoted by \(\mathbf{1}(y)\), is given by

\[
\mathbf{1}(y) := \begin{cases} 
\exp \left( \frac{9}{7 - \theta} \int d\lambda \left( e^{-\sqrt{\lambda}y} - 1 - \sqrt{\lambda}y \right) C_\tau(\theta) \gamma_0 \left( \frac{6}{7 - \theta} + 1 \right) \Gamma \left( \frac{3}{2} + \frac{6}{7 - \theta} \right) \right) & 2 \leq \theta < 3, \\
\exp \left( \frac{9}{4} \int d\lambda \left( e^{-\sqrt{\lambda}y} - 1 - \sqrt{\lambda}y \right) C_\tau(\theta) \gamma_0 \left( \frac{3}{2} + 1 \right) \Gamma \left( \frac{3}{2} + \frac{6}{7 - \theta} \right) \right) & \theta \geq 3.
\end{cases}
\]

In addition, the generator of the Lévy process is

\[
-\frac{c_{\theta, \gamma_0}}{(2\pi)^{\frac{1}{2}}} (-\Delta)^{\frac{\theta}{2}} - \frac{c_{\theta, \gamma_0}}{(2\pi)^{\frac{1}{2}}} (-\Delta) \frac{\theta}{2} \quad \text{if} \quad 2 \leq \theta < 3,
\]

and

\[
-\frac{c_{\theta, \gamma_0}}{(2\pi)^{\frac{1}{2}}} (-\Delta)^{\frac{\theta}{2}} \quad \text{if} \quad \theta \geq 3,
\]

where \(c_{\theta, \gamma_0}\) is given by

\[
c_{\theta, \gamma_0} := \begin{cases} 
\frac{24\pi^2}{7 - \theta} \gamma_0 \left( \frac{6}{7 - \theta} + 1 \right) \int dy \left( 1 - \cos(y) \right) \left| y \right|^{\frac{6}{7 - \theta} - 1} & 2 \leq \theta < 3, \\
\frac{\sqrt{3}}{12\pi} \gamma_0 \left( \frac{3}{2} + 1 \right) \int dy \left( 1 - \cos(y) \right) \left| y \right|^{\frac{3}{2} - 1} & \theta \geq 3.
\end{cases}
\]

Finally, we show \(c_{\theta, \gamma_0} = C_{\theta, \gamma_0}\), where \(C_{\theta, \gamma_0}\) is given by (4.10). In Appendix E we show

\[
\begin{aligned}
\int_R dy \left( 1 - \cos(y) \right) \left| y \right|^6 & = \frac{7 - \theta}{3} \cos \left( \frac{3\pi}{7 - \theta} \right) \Gamma \left( 1 - \frac{6}{7 - \theta} \right) \quad 2 \leq \theta < 3, \\
\int_R dy \left( 1 - \cos(y) \right) \left| y \right|^{\frac{3}{2} - 1} & = \frac{4}{3} \cos \left( \frac{3\pi}{4} \right) \Gamma \left( 1 - \frac{3}{2} \right) \quad \theta \geq 3.
\end{aligned}
\]

From (7.1) and Euler’s reflection formula \(\Gamma(1 + y)\Gamma(1 - y) = \pi y \csc \pi y, \ y \in \mathbb{R}\), we have

\[
\begin{aligned}
\Gamma \left( \frac{7 - \theta}{3} + 1 \right) \int_R dy \left( 1 - \cos(y) \right) \left| y \right|^{\frac{6}{7 - \theta} - 1} & = \pi \csc \left( \frac{3\pi}{7 - \theta} \right) \Gamma \left( 1 - \frac{6}{7 - \theta} \right) \quad 2 \leq \theta < 3, \\
\Gamma \left( \frac{3}{2} + 1 \right) \int_R dy \left( 1 - \cos(y) \right) \left| y \right|^{\frac{3}{2} - 1} & = \sqrt{2\pi} \quad \theta \geq 3.
\end{aligned}
\]

Hence we obtain \(c_{\theta, \gamma_0} = C_{\theta, \gamma_0}\).

7.2. Proof of (2). From Theorem 3 (1), it suffices to show that

\[
\lim_{N \to \infty} \int_T dk \left| u_N(N(\theta)y, k, Nt) - \mathbb{E}_k \left[ \int_T dk' u_0(y + \frac{1}{N(\theta)} Z(Nt), k') \right] \right|^2 = 0. \quad (7.2)
\]

By using the Fourier transform, we obtain another representation of \(u_N(y, k, t)\).

\[
u_N(N(\theta)y, k, Nt) = \mathbb{E}_k \left[ u_0(y + \frac{1}{N(\theta)} Z(Nt), K(Nt)) \right] = \sum_{x \in \mathbb{Z}} \int_R dp \hat{u}_0(p, x) \mathbb{E}_k \left[ e^{2\pi \sqrt{-1} p Z(Nt)} e^{2\pi \sqrt{-1} x K(Nt)} \right].
\]

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where $Z_N(Nt) := y + \frac{1}{N(t)}Z(Nt)$. Hence we have

$$
\left| u_N(N\theta y, k, Nt) - \mathbb{E}_k \left[ \int_T dk' u_0(y + \frac{1}{N(\theta)}Z(Nt), k') \right] \right|^2
\leq \sum_{x \in \mathbb{Z}} \int \mathbb{R} d \mu_0(p, x) \left\| \mathbb{E}_k \left[ e^{\frac{2\pi}{\sqrt{-1}}pZ_N(Nt)} \left( e^{2\pi\sqrt{-1}xK(Nt)} - \int_T dk' e^{2\pi\sqrt{-1}xk'} \right) \right] \right\|^2.
$$

Thanks to the assumption $u_0 \in C_0^\infty(\mathbb{R} \times T)$ and the dominated convergence theorem, if we get

$$
\lim_{N \to \infty} \int_T dk \left\| \mathbb{E}_k \left[ e^{\frac{2\pi}{\sqrt{-1}}pZ_N(Nt)} \left( e^{2\pi\sqrt{-1}xK(Nt)} - \int_T dk' e^{2\pi\sqrt{-1}xk'} \right) \right] \right\|^2 = 0
$$

(7.3)

for any $p \in \mathbb{R}, x \in \mathbb{Z}$, then we obtain (7.2).

From now on we will show (7.3). We use a trick to consider the convergence of $Z_N(Nt)$ and $K(Nt)$ separately. Let \{m_N\} be a family of increasing positive numbers which satisfies

$$
\lim_{N \to \infty} m_N = \infty, \quad \lim_{N \to \infty} m_N N(\theta)^{-1} = 0.
$$

Then we have

$$
\left\| \mathbb{E}_k \left[ e^{\frac{2\pi}{\sqrt{-1}}pZ_N(Nt)} e^{2\pi\sqrt{-1}xK(Nt)} - e^{\frac{2\pi}{\sqrt{-1}}pZ_N(Nt-m_Nt)} e^{2\pi\sqrt{-1}xK(Nt)} \right] \right\|^2
\leq \mathbb{E}_k \left[ \left| e^{\frac{2\pi}{\sqrt{-1}}p(Z_N(Nt)-Z_N(Nt-m_Nt))} - 1 \right|^2 \right]
\leq p^2 \mathbb{E}_k \left[ \left| Z_N(Nt) - Z_N(Nt-m_Nt) \right|^2 \right],
$$

and

$$
\int_T dk \left\| \mathbb{E}_k \left[ Z_N(Nt) - Z_N(Nt-m_Nt) \right]^2 \right\|
\leq p^2 \frac{m_N^2}{N(\theta)^2} \int_{Nt-m_Nt}^{Nt} ds \int_T dk \left\| \mathbb{E}_k \left[ \omega'(K(s)) \right] \right\|^2
= p^2 \frac{m_N^2 t^2}{N(\theta)^2} \int_T dk \left\| \omega'(k) \right\|^2 \to 0, \quad N \to \infty.
$$

Here we use the fact that the uniform probability measure on $\mathbb{T}$ is the reversible probability measure of \{K(t); t \geq 0\}. Hence we can replace $\mathbb{E}_k \left[ e^{\frac{2\pi}{\sqrt{-1}}pZ_N(Nt)} e^{2\pi\sqrt{-1}xK(Nt)} \right]$ by $\mathbb{E}_k \left[ e^{\frac{2\pi}{\sqrt{-1}}pZ_N(Nt-m_Nt)} e^{2\pi\sqrt{-1}xK(Nt)} \right]$. By the same argument, we can also replace $\mathbb{E}_k \left[ e^{\frac{2\pi}{\sqrt{-1}}pZ_N(Nt)} \int_T dk' e^{2\pi\sqrt{-1}xk'} \right]$ by $\mathbb{E}_k \left[ e^{\frac{2\pi}{\sqrt{-1}}pZ_N(Nt-m_Nt)} \int_T dk' e^{2\pi\sqrt{-1}xk'} \right]$. In addition, by using the Markov property we have

$$
\left\| \mathbb{E}_k \left[ e^{\frac{2\pi}{\sqrt{-1}}pZ_N(Nt-m_Nt)} \left( e^{2\pi\sqrt{-1}xK(Nt)} - \int_T dk' e^{2\pi\sqrt{-1}xk'} \right) \right] \right\|^2
= \left\| \mathbb{E}_k \left[ e^{\frac{2\pi}{\sqrt{-1}}pZ_N(Nt-m_Nt)} \mathbb{E}_K(Nt-m_Nt) \left[ e^{2\pi\sqrt{-1}xK(m_Nt)} - \int_T dk' e^{2\pi\sqrt{-1}xk'} \right] \right] \right\|^2
\leq \mathbb{E}_k \left[ \mathbb{E}_K(Nt-m_Nt) \left[ e^{2\pi\sqrt{-1}xK(m_Nt)} - \int_T dk' e^{2\pi\sqrt{-1}xk'} \right] \right]^2.
$$
Let \( h(k) := e^{2\pi \sqrt{-1} x k} - \int_T dk' e^{2\pi \sqrt{-1} x k'} \) and denote by \( \{ P_t; t \geq 0 \} \) the semigroup generated by \( 2\gamma_0 L \). Since 0 is simple eigenvalue of \( 2\gamma_0 L \) and the uniform probability measure on \( T \) is the reversible probability measure, we have \( \lim_{t \to \infty} \| P^t f \|^2_{L^2(T)} = 0 \) for any \( f \in L^2(T) \).

For any measure on \( T \), let \( h \) be a probability space and \( I \) be a probability measure, we have \( \lim_{N \to \infty} \| P^{mN} h \|^2_{L^2(T)} \to 0 \) by the ergodic theorem. Hence we obtain

\[
\int_T dk \ E_k [ |E_{K(Nt-mNt)} e^{2\pi \sqrt{-1} x K(mNt)} - \int_T dk' e^{2\pi \sqrt{-1} x k'} |^2 ] = \| P^{mN} h \|^2_{L^2(T)} \to 0, \quad N \to \infty.
\]

Summarizing the above, we have

\[
\begin{align*}
&\int_T dk \ E_k [ e^{2\pi \sqrt{-1} p \gamma_0 (Nt)} \{ e^{2\pi \sqrt{-1} x K(Nt)} - \int_T dk' e^{2\pi \sqrt{-1} x k'} \}^2 ] \\
&\leq \int_T dk \ E_k [ e^{2\pi \sqrt{-1} p \gamma_0 (Nt) (Nt-mNt)} \{ e^{2\pi \sqrt{-1} x K(Nt)} - \int_T dk' e^{2\pi \sqrt{-1} x k'} \}^2 ] \\
&\quad + \int_T dk \ E_k [ e^{2\pi \sqrt{-1} p \gamma_0 (Nt-mNt)} \{ e^{2\pi \sqrt{-1} x K(Nt)} - \int_T dk' e^{2\pi \sqrt{-1} x k'} \}^2 ] \\
&\quad + \int_T dk \ E_k [ e^{2\pi \sqrt{-1} p \gamma_0 (Nt-mNt)} \int_T dk' e^{2\pi \sqrt{-1} x k'} - e^{2\pi \sqrt{-1} p \gamma_0 (Nt)} \int_T dk' e^{2\pi \sqrt{-1} x k'} ]^2 \\
&\to 0, \quad N \to \infty.
\end{align*}
\]

We have thus established the theorem.

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**APPENDIX A. EXISTENCE AND UNIQUENESS OF THE SOLUTION OF (3.10)**

Let \((E, F, \mathbb{P})\) be a probability space and \(B\) be a cylindrical Wiener process in \(L^2(T)\) defined on \((E, F, \mathbb{P})\). For any \(T > 0\), we introduce a Banach space \(\mathcal{H}_T\) defined as

\[
\mathcal{H}_T := \{ f : T \times [0,T] \times \Omega \to \mathbb{C} ; \| f \|_{\mathcal{H}} := \left( \sup_{0 \leq t \leq T} \mathbb{E}[ | f(t) |^2_{L^2(T)} ] \right)^{\frac{1}{2}} < \infty \}.
\]

Fix a \( \tilde{\omega} \in L^2(T) \). We define a mapping \( I_T : \mathcal{H}_T \to \mathcal{H}_T \) as

\[
I_T(f)(t) := \tilde{\omega}(k) - \int_0^t ds \sqrt{-1} \omega(k) f(k,s) + \gamma B(k) \left( f(k,s) - f^*(k,-s) \right) + \sqrt{-1} \gamma \int_0^t \int_T B(dk', ds) r(k,k') \left( f(k-k',s) - f^*(k'-k,s) \right).
\]

If \( T > 0 \) is sufficiently small, then \( I \) is contractive and there exists a unique fixed point \( \tilde{\psi} \in \mathcal{H}_T \). The case of general \( T > 0 \) can be handled in the usual way.
Appendix B. Auxiliary Results

Lemma B.1.

\[ R(k, k', p) = 8 \left( \sin^2(\pi k) - \sin^2\left(\frac{\pi p}{2}\right) \right) (\sin^2 \pi k' - \sin^2 \left(\frac{\pi p}{2}\right)) \]
\[ \times \left( \sin^2 \left(\pi (k + k')\right) + \sin^2 \left(\pi (k - k')\right) - 2 \sin^2 (\pi p) \right) \]
\[ =: R(k, k') - \sin^2\left(\frac{\pi p}{2}\right) R_1(k, k') + \sin^4 \left(\frac{\pi p}{2}\right) R_2(k, k') \]
\[ + \sin^2 \left(\frac{\pi p}{2}\right) \sin^2 (\pi p) R_3(k, k') - 16 \sin^4 \left(\frac{\pi p}{2}\right) \sin^2 (\pi p), \]

where \(R(k, k', p)\) is defined in (5.12), and the functions \(R(k, k'), R_i(k, k'), i = 1, 2, 3\) are given by

\[ R(k, k') = 8 \sin^2 (\pi k) \sin^2 (\pi k') \left( (\sin^2 \pi (k + k') + \sin^2 \pi (k - k')) \right) \]
\[ R_1(k, k') = 8 \left( \sin^2 \pi (k) + \sin^2 \pi (k') \right) \left( \sin^2 \pi (k + k') + \sin^2 \pi (k - k') \right) \]
\[ R_2(k, k') = 8 \left( \sin^2 \pi (k + k') + \sin^2 \pi (k - k') \right) \]
\[ R_3(k, k') = 16 \left( \sin^2 \pi (k) + \sin^2 \pi (k') \right). \]

Proof.

\[ \frac{1}{4} r(k - \frac{p}{2}, k - k') r(k + \frac{p}{2}, k - k') \]
\[ = \sin^2 \left(\pi (k + \frac{p}{2})\right) \sin^2 \left(\pi (k - \frac{p}{2})\right) \sin \left(2 \pi (k' + \frac{p}{2})\right) \sin \left(2 \pi (k' - \frac{p}{2})\right) \]
\[ + \sin^2 \left(\pi (k + \frac{p}{2})\right) \sin^2 \left(\pi (k' - \frac{p}{2})\right) \sin \left(2 \pi (k + \frac{p}{2})\right) \sin \left(2 \pi (k - \frac{p}{2})\right) \]
\[ + \sin^2 \left(\pi (k - \frac{p}{2})\right) \sin^2 \left(\pi (k' - \frac{p}{2})\right) \sin \left(2 \pi (k' + \frac{p}{2})\right) \sin \left(2 \pi (k - \frac{p}{2})\right) \]
\[ + \sin^2 \left(\pi (k - \frac{p}{2})\right) \sin^2 \left(\pi (k + \frac{p}{2})\right) \sin \left(2 \pi (k + \frac{p}{2})\right) \sin \left(2 \pi (k' - \frac{p}{2})\right) \]
\[ = \left( \sin^2 \pi (k) - \sin^2 \left(\frac{\pi p}{2}\right) \right) \left( \sin^2 \pi (k') - \sin^2 \left(\frac{\pi p}{2}\right) \right) \]
\[ \times \left\{ 4 \left[ \sin^2 \pi (k) - \sin^2 \left(\frac{\pi p}{2}\right) - \left( \sin^2 \pi (k') + \sin^2 \left(\frac{\pi p}{2}\right) \right) \right] \left( \sin^2 \pi (k) - \sin^2 \left(\frac{\pi p}{2}\right) \right) \right\} \]
\[ + 4 \left[ \sin^2 \pi (k') - \sin^2 \left(\frac{\pi p}{2}\right) - \left( \sin^2 \pi (k) + \sin^2 \left(\frac{\pi p}{2}\right) \right) \right] \left( \sin^2 \pi (k') - \sin^2 \left(\frac{\pi p}{2}\right) \right) \]
\[ + \left( \sin \left(2 \pi k\right) + \sin \left(\pi p\right) \right) \left( \sin \left(2 \pi k\right) + \sin \left(\pi p\right) \right) \]
\[ + \left( \sin \left(2 \pi k'\right) + \sin \left(\pi p\right) \right) \left( \sin \left(2 \pi k'\right) + \sin \left(\pi p\right) \right) \]
\[ = 4 \left( \sin^2 \pi (k) - \sin^2 \left(\frac{\pi p}{2}\right) \right) \left( \sin^2 \pi (k') - \sin^2 \left(\frac{\pi p}{2}\right) \right) \left( \sin^2 \pi (k + k') - \sin^2 \pi p \right). \]
Therefore we have
\[ R(k, k', p) = \frac{1}{2} \sum_{i, k} r(k - \frac{p}{2}, k + i k') r(k + \frac{p}{2}, k + i k') \]
\[ \triangleright 8 \left( \sin^2(\pi k) - \sin^2(\frac{\pi p}{2}) \right) \left( \sin^2(\pi k') - \sin^2(\frac{\pi p}{2}) \right) \]
\[ \times \left( \sin^2(\pi (k + k')) + \sin^2(\pi (k - k')) - 2 \sin^2(\pi p) \right), \]
\[ = 8 \sin^2(\pi k) \sin^2(\pi k') \left( \sin^2(\pi (k + k')) + \sin^2(\pi (k - k')) \right) - 8 \sin^2(\frac{\pi p}{2}) \left( \sin^2(\pi k) + \sin^2(\pi k') \right) \left( \sin^2(\pi (k + k')) + \sin^2(\pi (k - k')) \right) + 8 \sin^4(\frac{\pi p}{2}) \left( \sin^2(\pi k) + \sin^2(\pi k') \right) - 16 \sin^2(\frac{\pi p}{2}) \sin^2(\pi p). \]

\[ \square \]

**Lemma B.2.**

\[
\tilde{\alpha}(k) = \begin{cases} 
(C(\theta)|k|^\theta - 1 + O(k^2), & 2 < \theta < 3, \\
(C(\theta)k^2)\log(|k|) + O(k^2), & \theta = 3, \\
(C(\theta)k^2 + o(k^2), & \theta > 3, 
\end{cases} \tag{B.1}
\]

and

\[
\tilde{\alpha}'(k) = \begin{cases} 
(\theta - 1)C(\theta)\text{sgn}(k)|k|^\theta - 2 + O(k), & 2 < \theta < 3, \\
2C(\theta)k\log(|k|) + O(k), & \theta = 3, \\
2C(\theta)k + o(k), & \theta > 3, 
\end{cases} \tag{B.2}
\]
as \( k \to 0. \)

**Proof.** First we show (B.1). If \( \theta > 3 \) then (B.1) is obvious, so we only consider the case \( 2 < \theta \leq 3 \). From (3.1) we have

\[
\tilde{\alpha}(k) = \sum_{x \neq 0} \left( 1 - e^{-2\pi\sqrt{-1}kx} \right) |x|^{-\theta} = 4 \sum_{x \in \mathbb{N}} \frac{\sin^2(\pi kx)}{|x|^\theta} 
= 4 \int_{x=0}^{\infty} dy \frac{\sin^2(\pi ky)}{|y|^\theta} + 4 \sum_{x \in \mathbb{N}} \left( \int_{x}^{x+1} dy \frac{\sin^2(\pi kx)}{|y|^\theta} - \frac{\sin^2(\pi ky)}{|y|^\theta} \right) 
= 4\pi^{\theta-1}|k|^\theta \left[ \int_{|k|}^{\infty} dy \frac{\sin^2(y)}{|y|^\theta} - \sum_{|k| \leq x \leq |k|+1} \int_{|k|}^{\pi|x|} dy \int_{\pi|x|}^{y} dy' \frac{d}{dy'}(\frac{\sin^2(y')}{|y'|^{\theta}}) \right].
\]

Since

\[
\left| \frac{d}{dy'} \left( \frac{\sin^2(y')}{|y'|^{\theta}} \right) \right| \leq \frac{2 + \theta}{|\pi k x|^{\theta - 1}}, \quad \pi k x \leq y' \leq y \]
\[
|y - \pi k x| \leq \pi |k|, \quad \pi |k| x \leq y \leq \pi |k|(x + 1),
\]
we have

\[
\int_{\pi k x}^{\pi k(x+1)} dy \int_{\pi k x}^{y} dy' \frac{d}{dy'}(\frac{\sin^2(y')}{|y'|^{\theta}}) \leq \int_{\pi k x}^{\pi k(x+1)} dy \frac{y - \pi k x}{|\pi k x|^{\theta - 1}} \leq \frac{|k|^{\theta - 2}}{|x|^{\theta - 1}}.
\]
Therefore we obtain
\[
4\pi^{\theta-1}|k|^{\theta-1} \sum_{x \in \mathbb{N}} \int_{\pi|x|}^{\pi|x(x+1)|} dy \int_{\pi|x|}^{y} dy' \frac{d}{dy'} \left( \frac{\sin^2(y')}{|y'|^{\theta}} \right) = O(k^2).
\]

On the other hand, we get
\[
\begin{align*}
\lim_{k \to 0} \int_{\pi|k|}^{\infty} \sin^2(y) \frac{dy}{|y|^{\theta}} = & \int_{\pi|k|}^{\infty} \sin^2(y) \frac{dy}{|y|^{\theta}} = 2^{\theta-3} \int_{0}^{\infty} 2 - 2 \cos(y) \frac{dy}{|y|^{\theta}} \quad 2 < \theta < 3, \\
\lim_{k \to 0} \frac{1}{\log(|k|)} \int_{\pi|k|}^{\infty} \sin^2(y) \frac{dy}{|y|^{\theta}} = & 1 \quad \theta = 3.
\end{align*}
\]

Hence we have (B.1) when $2 < \theta \leq 3$.

Next we show (B.2). When $\theta > 3$, we have
\[
\tilde{a}'(k) = 4\pi \sum_{x \in \mathbb{N}} \frac{\sin(2\pi k x)}{|x|^{\theta-1}}
\]
\[
= 8\pi^2 k \left( \sum_{x \in \mathbb{N}} \frac{1}{|x|^{\theta-2}} \right) + 8\pi^2 k \sum_{x \in \mathbb{Z}} \left( \frac{\sin(2\pi k x)}{2\pi k x} - 1 \right) \frac{1}{|x|^{\theta-2}}
\]
\[
= 2C(\theta) k + 8\pi^2 k \sum_{x \in \mathbb{Z}} \left( \frac{\sin(2\pi k x)}{2\pi k x} - 1 \right) \frac{1}{|x|^{\theta-2}}.
\]

Fix a number $0 < a < \theta - 3$. Since there exists some constant $C_a > 0$ such that $|\frac{\sin(y)}{y} - 1| \leq C_a |y|^a$ for any $y \in \mathbb{R}$, we obtain
\[
\left| \sum_{x \in \mathbb{Z}} \left( \frac{\sin(2\pi k x)}{2\pi k x} - 1 \right) \frac{1}{|x|^{\theta-2-a}} \right| \leq |k|^a \sum_{x \in \mathbb{Z}} \frac{1}{|x|^{\theta-2-a}}.
\]

Therefore we obtain (B.2) when $\theta > 3$. If $2 < \theta \leq 3$, then we have
\[
\tilde{a}'(k) = 4\pi \int_{1}^{\infty} \frac{\sin(2\pi k y)}{|y|^{\theta-1}} dy + 4\pi \sum_{x \in \mathbb{N}} \left( \int_{x}^{x+1} \frac{\sin(2\pi k x)}{|x|^{\theta-1}} - \frac{\sin(2\pi k y)}{|y|^{\theta-1}} \right)
\]
\[
= 2^{\theta} \pi^{\theta-1} \operatorname{sgn}(k) |k|^{\theta-2} \left[ \int_{2\pi|k|}^{\infty} \frac{\sin(y)}{|y|^{\theta-1}} dy + \sum_{x \in \mathbb{N}} \int_{\pi |k| x}^{\pi |k|(x+1)} \frac{\sin(y')}{|y'|^{\theta-1}} dy' \frac{d}{dy'} \left( \frac{\sin(y')}{|y'|^{\theta-1}} \right) \right].
\]

Since
\[
\left| \frac{d}{dy'} \left( \frac{\sin(y')}{|y'|^{\theta-1}} \right) \right| \leq \frac{\theta}{|\pi k x|^{\theta-1}}, \quad \pi |k|x \leq y' \leq y
\]
\[
|y - \pi |k|x| \leq \pi |k|, \quad \pi |k|x \leq y \leq \pi |k|(x + 1),
\]
we have
\[
\int_{\pi k x}^{\pi k(x+1)} dy \int_{\pi k x}^{y} dy' \frac{d}{dy'} \left( \frac{\sin(y')}{|y'|^{\theta-1}} \right) \leq \int_{\pi k x}^{\pi k(x+1)} dy \frac{y - \pi k x}{\pi |k x|^{\theta-1}} \leq \frac{|k|^{3-\theta}}{|x|^{\theta-1}}.
\]

Therefore we obtain
\[
2^{\theta} \pi^{\theta-1} \operatorname{sgn}(k) |k|^{\theta-2} \sum_{x \in \mathbb{N}} \int_{\pi |k| x}^{\pi |k|(x+1)} dy \int_{\pi |k| x}^{y} dy' \frac{d}{dy'} \left( \frac{\sin(y')}{|y'|^{\theta-1}} \right) = O(k).
\]
On the other hand, we get
\[
\lim_{k \to 0} \int_{\pi|k|}^{\infty} dy \frac{\sin(y)}{|y|^\theta} = \int_0^{\pi|k|} \frac{\sin(y)}{|y|^\theta} = 2\theta - 3 \int_0^{\infty} dy \frac{2 - 2\cos(y)}{|y|^\theta} \quad 2 < \theta < 3,
\]
\[
\lim_{k \to 0} \frac{1}{\log(|k|)} \int_{\pi|k|}^{\infty} dy \frac{\sin(y)}{|y|^2} = 1 \quad \theta = 3.
\]
Hence we have (B.2) when \(2 < \theta \leq 3\). Note that from the following inequalities
\[
\left| \frac{\sin(y)}{y} - 1 \right| \leq C_0 |y|^{\alpha} \quad y \in \mathbb{R},
\]
\[
\left| \frac{\sin(2\pi kx)}{|x|^\theta - 1} - \frac{\sin(2\pi ky)}{|y|^\theta - 1} \right| \leq \left| \int_{\pi|k|x}^{\pi|k|y} dy \frac{d}{dy} \left( \frac{\sin(y)}{|y|^\theta - 1} \right) \right| 
\quad x, y \in \mathbb{R}_{\geq 0},
\]
we also obtain
\[
4\pi \sum_{x \in \mathbb{N}} \frac{\sin(2\pi kx)}{|x|^\theta - 1} = \begin{cases} 
O(|k|^{\theta-2}), & 2 < \theta < 3, \\
O(|k|\log(|k|)), & \theta = 3, \\
O(|k|), & \theta > 3.
\end{cases} \tag{B.3}
\]

**APPENDIX C. PROOF OF PROPOSITION 4.1**

For notational simplicity, we omit the variable \(t \geq 0\). From (3.11), we have
\[
\phi_x := \mathbb{E}_t \left[ \frac{1}{2} |\psi|_t^2 - e_x \right] = \frac{1}{16} \int_{\mathbb{T}^2} dk'd \epsilon e^{2\pi \sqrt{-1} (k+k')e} F_1(k,k') \mathbb{E}_t \left[ \left( \hat{\psi}(k) + \hat{\psi}(-k)^* \right) \left( \hat{\psi}(k') + \hat{\psi}(-k')^* \right) \right],
\]
where \(F_1(k,k') := F(k,k') + 2\). Note that \(F_1(k,-k) = 0, k \in \mathbb{T}\). Then by using (4.1) and (5.1) with \(p = 0\), we obtain
\[
\left| \langle W_{e_\epsilon}(t), J - \epsilon \sum_{x \in \mathbb{Z}} e_x \left( \frac{t}{\theta, \epsilon} \right) J(\epsilon x) \rangle \right|
\]
\[
= |\epsilon \sum_{x \in \mathbb{Z}} \phi_x J(\epsilon x)| = |\epsilon \sum_{x \in \mathbb{Z}} \phi_x \int_{\mathbb{R}} dp \ e^{2\pi \sqrt{-1} \epsilon p \epsilon x} J(p)|
\]
\[
\leq \int_{\mathbb{R} \times T} dpm \left| F_1(k,-k-\epsilon p) \right| \mathbb{E}_t \left[ \left| \hat{\psi}(k) + \hat{\psi}(-k)^* \right|^2 \left| \hat{\psi}(-k-\epsilon p) + \hat{\psi}(k+\epsilon p)^* \right|^2 \right] |J(p)|
\]
\[
\leq \int_{\mathbb{R}} dp \left( \int_{\mathbb{T}}dk \left| F_1(k,-k-\epsilon p) \right|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} dp \ e^{2\pi \epsilon p} \left| \hat{\psi}(k) \right|^2 \right)^{\frac{1}{2}} |J(p)|
\]
\[
\leq \int_{\mathbb{R}} dp \left( \int_{\mathbb{T}}dk \left| F_1(k,-k-\epsilon p) \right|^2 \right)^{\frac{1}{2}} |J(p)|.
\]
Hence if we show that \(\left( \int_{\mathbb{T}}dk \left| F_1(k,-k-\epsilon p) \right|^2 \right)^{\frac{1}{2}}\) is bounded from above by some positive constant uniformly in \(0 < \epsilon \ll 1, p \in \mathbb{R}\), then by using the dominated convergence theorem we have
\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}} dp \left( \int_{\mathbb{T}}dk \left| F_1(k,-k-\epsilon p) \right|^2 \right)^{\frac{1}{2}} |J(p)| = 0.
\]
Now we will estimate \(|\alpha(k + k') - \alpha(k) - \alpha(k')|, k' = -k - \epsilon p\). Since \(\sin^2(y_1 + y_2) - \sin^2(y_1) \leq |\sin(y_1)\sin(y_2)|, y_1, y_2 \in \mathbb{R}\), we get

\[
|\alpha(k + k') - \alpha(k) - \alpha(k')| \lesssim \sum_{x \in \mathbb{N}} \frac{|\sin(\pi k x) \sin(\pi k' x)|}{|x|^\theta}
\]

\[
= \int_1^\infty dy \frac{|\sin(\pi k y) \sin(\pi k' y)|}{|y|^\theta} + \sum_{x \in \mathbb{N}} \left( \int_x^{x+1} dy \frac{|\sin(\pi k x) \sin(\pi k' x)|}{|x|^\theta} - |\sin(\pi k y) \sin(\pi k' y)| \right)
\]

\[
\lesssim |k|^{-\frac{\theta-1}{2}} |k'|^{-\frac{\theta-1}{2}} \int_{\sqrt{|kk'|}}^\infty dy \frac{|\sin(\sqrt{\frac{k}{k'} y}) \sin(\sqrt{\frac{k'}{k} y})|}{|y|^\theta} \lesssim |k|^{-\frac{\theta-1}{2}} |k'|^{-\frac{\theta-1}{2}} \lesssim \omega(k) \omega(k'),
\]

and we see that \(F(k, k')\) is uniformly bounded by some positive constant. When \(2 < \theta < 3\), we obtain

\[
|k|^{-\frac{\theta-1}{2}} |k'|^{-\frac{\theta-1}{2}} \int_{\sqrt{|kk'|}}^\infty dy \frac{|\sin(\sqrt{\frac{k}{k'} y}) \sin(\sqrt{\frac{k'}{k} y})|}{|y|^\theta} \lesssim |k|^{-\frac{\theta-1}{2}} |k'|^{-\frac{\theta-1}{2}} \lesssim \omega(k) \omega(k'),
\]

\[
\int_x^{x+1} dy \int_x^y dy' \partial_{y'} \left( \frac{\sin(ky') \sin(k'y')}{|y'|^\theta} \right)
\]

\[
= \int_x^{x+1} dy \int_x^y dy' \log(\frac{\sin(ky') \sin(k'y')}{|y'|^\theta} - \theta \sin(ky') \sin(k'y') |y'|^{\theta+1})
\]

\[
\lesssim \frac{|kk'|}{|x|^\theta} \lesssim \omega(k) \omega(k'),
\]

and thus we have \(|\alpha(k + k') - \alpha(k) - \alpha(k')| \lesssim \omega(k) \omega(k')\). Finally we consider the case \(\theta = 3\). Since

\[
|kk'| \int_{\sqrt{|kk'|}}^\infty dy \frac{|\sin(\sqrt{\frac{k}{k'} y}) \sin(\sqrt{\frac{k'}{k} y})|}{|y|^3} \lesssim |kk'| \log(|kk'|) + |kk'|,
\]

\[
\int_x^{x+1} dy \int_x^y dy' \partial_{y'} \left( \frac{\sin(ky') \sin(k'y')}{|y'|^\theta} \right) \lesssim \frac{|kk'|}{|x|^2},
\]

we obtain

\[
|F_1(k, k')|^2 \lesssim \frac{\sqrt{|\log(|k|)|}}{|\log(|k'|)|} + \frac{\sqrt{|\log(|k'|)|}}{|\log(|k|)|} + \frac{1}{\sqrt{|\log(|k|)|} \log(|k'|)|} + 1^2
\]

\[
\lesssim \frac{|\log(|k|)|}{|\log(|k'|)|} + \frac{|\log(|k'|)|}{|\log(|k|)|} + \frac{1}{|\log(|k|)| \log(|k'|)|} + 1,
\]

\[
\int_\mathbb{T} dk \frac{|\log(|k|)|}{|\log(|k'|)|} = \int_\mathbb{T} dk \frac{|\log(|k'|)|}{|\log(|k|)|} \lesssim \int_\mathbb{T} dk \log(|k'|)| = \int_\mathbb{T} dk \log(|k|)| < \infty,
\]

\[
\int_\mathbb{T} dk \frac{1}{|\log(|k|)| |\log(|k'|)|} \lesssim \int_\mathbb{T} dk \frac{1}{|\log(|k'|)|} = \int_\mathbb{T} dk \frac{1}{|\log(|k|)|} < \infty.
\]
Thus the Proposition is proved.

**APPENDIX D. PROOF OF (5.51)**

From the definition of \((\delta_\epsilon \omega)\), we have

\[
(\delta_\epsilon \omega)(p, k) = \begin{cases} 
4\pi \gamma_0 \frac{-\theta}{\epsilon} |p| \frac{1}{2} \left( \frac{f_{\theta, a}(\epsilon)}{\epsilon^s} \right) \frac{3-\theta}{3} \sin(\pi \epsilon px) \sum_{x \geq 1} \frac{1}{|x|^{\theta-1}} \sin(2\pi k \epsilon x) & 2 < \theta < 3, \\
4\pi \left[ -\log \left( \frac{f_{\theta, a}(\epsilon)}{\epsilon^s} \right) \right] \frac{1}{2} \sin(\pi \epsilon px) \sum_{x \geq 1} \frac{1}{|x|^{\theta-1}} \sin(2\pi k \epsilon x) & \theta = 3, \\
4\pi \omega(k_\epsilon + \frac{\epsilon p}{2}) + \omega(k_\epsilon - \frac{\epsilon p}{2}) \sum_{x \geq 1} \frac{1}{|x|^{\theta-1}} \sin(2\pi k \epsilon x) & \theta > 3.
\end{cases}
\]

If \(2 < \theta < 3\), then we have

\[
(\delta_\epsilon \omega)(p, k) = \frac{1}{2} |x|^{\frac{3-\theta}{2}} \omega(k_\epsilon + \frac{\epsilon p}{2}) + \omega(k_\epsilon - \frac{\epsilon p}{2}) \sum_{x \geq 1} \frac{1}{|x|^{\theta-1}} \sin(2\pi k \epsilon x)
\]

\[
\times \frac{4\pi}{(f_{\theta, a}(\epsilon))^{\frac{2}{\theta-2}}} \frac{2}{\epsilon^s} \sum_{x \geq 1} \frac{1}{|x|^{\theta-1}} \left( \frac{\sin(\pi \epsilon px)}{\pi \epsilon px} - 1 \right) \sin(2\pi k \epsilon x). \tag{D.1}
\]

From Lemma B.2, the first term of (D.1) converges to \(\text{sgn}(k) (\frac{\theta-1}{C(\theta)} |k|)^{\frac{2-\theta}{\theta}}\) for any \((p, k) \in \mathbb{R}_0 \times \mathbb{R}_0\). Now we will estimate the second term of (D.1). Fix a number \(0 < a < \theta - 2\). Since there exists some constant \(C_a > 0\) such that \(|\frac{\sin(\epsilon x)}{x} - 1| \leq C_a |x|^a\) for any \(x \in \mathbb{R}\) and

\[
\lim_{\epsilon \to 0} \epsilon \left( \frac{f_{\theta, a}(\epsilon)}{\epsilon^s} \right)^{\frac{3}{2}} = 0,
\]

for any \(\theta > 2, 0 \leq s \leq 1\), we obtain

\[
\left( \left( \frac{f_{\theta, a}(\epsilon)}{\epsilon^s} \right)^{-\frac{2}{(\theta - 2)}} \sum_{x \geq 1} \frac{1}{|x|^{\theta-1}} \left( \frac{\sin(\pi \epsilon px)}{\pi \epsilon px} - 1 \right) \sin(2\pi k \epsilon x) \right)
\]

\[
\leq C_a \left( \frac{f_{\theta, a}(\epsilon)}{\epsilon^s} \right)^{-\frac{2}{(\theta - 2)}} \epsilon^a \sum_{x \geq 1} \frac{1}{|x|^{\theta-1-a}} |\sin(2\pi k \epsilon x)|
\]

\[
\leq C_{a,\theta} \left( \frac{f_{\theta, a}(\epsilon)}{\epsilon^s} \right)^{-\frac{2}{(\theta - 2)}} \epsilon^a |p|^{\theta-1} |\sin(2\pi k \epsilon x)|
\]

\[
\leq C_{a,\theta,70} \left( \epsilon \left( \frac{f_{\theta, a}(\epsilon)}{\epsilon^s} \right)^{-\frac{3}{2}} \right)^{\frac{1}{a}} |p|^{\frac{(\theta-2)}{2-\theta} + \frac{(\theta-a)}{2-\theta} |k|^{\theta-2-a} \to 0 \quad \epsilon \to 0,
\]

where \(C_{a,\theta}, C_{a,\theta,70}\) are some constants which depend on the variables in their subscripts.
If $\theta = 3$, then we have
\[
(\delta, \omega)_\epsilon(p, k) = \frac{2|k_i|}{2 \omega(k_\epsilon + \frac{cp}{2}) + \omega(k_\epsilon - \frac{cp}{2})} \frac{4\pi}{|k_i|} \sum_{x \geq 1} \frac{1}{|x|^2} \sin(2\pi k_i x) \\
+ \frac{\epsilon}{2} \frac{1}{2} \gamma_0^2 |p|^{-\frac{1}{2}} \frac{2|k_i|}{\omega(k_\epsilon + \frac{cp}{2}) + \omega(k_\epsilon - \frac{cp}{2})} \\
\times \sum_{x \geq 1} \frac{1}{|x|^2} \left( \frac{\sin(\pi px)}{\pi px} - 1 \right) \sin(2\pi k_i x).
\]
From Lemma B.2, the first term converges to $\text{sgn}(k)\sqrt{C(\theta)}$ for any $(p, k) \in \mathbb{R}_0 \times \mathbb{R}_0$. On the other hand, by using the same argument for $2 < \theta < 3$, for a fixed $0 < a < 1$, we obtain
\[
\left( \frac{\theta, \omega(\epsilon)}{e^s} \right)^{-\frac{1}{2}} \log \left( \frac{\theta, \omega(\epsilon)}{e^s} \right)^{-\frac{1}{2}} \sum_{x \geq 1} \frac{1}{|x|^2} \left( \frac{\sin(\pi px)}{\pi px} - 1 \right) \sin(2\pi k_i x) \\
\leq C_{a, \theta} \left( \frac{\theta, \omega(\epsilon)}{e^s} \right)^{-\frac{1}{2}} \log \left( \frac{\theta, \omega(\epsilon)}{e^s} \right)^{-\frac{1}{2}} e^a |p|^{a/2} |k_i|^{1-a} \\
\leq C_{a, \theta, \gamma_0} \left[ \epsilon \left( \frac{\theta, \omega(\epsilon)}{e^s} \right)^{-\frac{1}{2}} \log \left( \frac{\theta, \omega(\epsilon)}{e^s} \right)^{-\frac{1}{2}} - \frac{1}{2} |p|^{\frac{1}{2}} |k_i|^{1-a} \rightarrow 0 \right] \epsilon \rightarrow 0
\]
where we use the limit
\[
\lim_{\epsilon \rightarrow 0} \left[ \epsilon \left( \frac{\theta, \omega(\epsilon)}{e^s} \right)^{-\frac{1}{2}} \log \left( \frac{\theta, \omega(\epsilon)}{e^s} \right)^{-\frac{1}{2}} - \frac{1}{2} |p|^{\frac{1}{2}} |k_i|^{1-a} \right] = 0.
\]
If $\theta > 3$, then we have
\[
(\delta, \omega)_\epsilon(p, k) = \frac{2|k_i|}{2 \omega(k_\epsilon + \frac{cp}{2}) + \omega(k_\epsilon - \frac{cp}{2})} \frac{4\pi}{|k_i|} \sum_{x \geq 1} \frac{1}{|x|^{\theta-1}} \sin(2\pi k_i x) \\
+ \frac{\epsilon}{2} \frac{1}{2} \gamma_0^2 |p|^{-\frac{1}{2}} \frac{2|k_i|}{\omega(k_\epsilon + \frac{cp}{2}) + \omega(k_\epsilon - \frac{cp}{2})} \\
\times \sum_{x \geq 1} \frac{1}{|x|^{\theta-1}} \left( \frac{\sin(\pi px)}{\pi px} - 1 \right) \sin(2\pi k_i x).
\]
From Lemma B.2, the first term converges to $\text{sgn}(k)\sqrt{C(\theta)}$ for any $(p, k) \in \mathbb{R}_0 \times \mathbb{R}_0$. On the other hand, by using the same argument for $2 < \theta < 3$, for a fixed $0 < a < 1$, we obtain
\[
\left( \frac{\theta, \omega(\epsilon)}{e^s} \right)^{-\frac{1}{2}} \sum_{x \geq 1} \frac{1}{|x|^{\theta-1}} \left( \frac{\sin(\pi px)}{\pi px} - 1 \right) \sin(2\pi k_i x) \\
\leq C_{a, \theta} \left( \frac{\theta, \omega(\epsilon)}{e^s} \right)^{-\frac{1}{2}} e^a |p|^{a/2} |k_i|^{1-a} \\
\leq C_{a, \theta, \gamma_0} \left[ \epsilon \left( \frac{\theta, \omega(\epsilon)}{e^s} \right)^{-\frac{1}{2}} - \frac{1}{2} |p|^{\frac{1}{2}} |k_i|^{1-a} \rightarrow 0 \right] \epsilon \rightarrow 0.
\]
In addition, from the above we see that on the set \( T_\epsilon \cap \{|k| > 1\}, |(\delta_\epsilon, \omega)_\epsilon(p, k)| \) is bounded by
\[
|\langle \delta_\epsilon, \omega \rangle_\epsilon(p, k) | \leq \begin{cases} 
 C_{\theta, p, \gamma_0} |k|^{\frac{3-\theta}{2}} & 2 < \theta < 3, \\
 C_{\theta, p, \gamma_0} \log(|k|) & \theta = 3, \\
 C_{\theta, p, \gamma_0} & \theta > 3,
\end{cases}
\]

where \( C_{\theta, p, \gamma_0} \) is some positive constant which depends on \( \theta > 2, p \in \mathbb{R}, \gamma_0 > 0 \) and satisfies
\[
\lim_{p \to 0} C_{\theta, p, \gamma_0} \leq 1, \\
C_{\theta, p, \gamma_0} = O(|p|^b), \quad |p| \to \infty
\]
for some \( b > 0 \).

**APPENDIX E. PROOF OF (7.1)**

First we observe
\[
\begin{align*}
\int_0^\infty dy \left( 1 - \cos(y) \right) |y|^{-\frac{\theta}{2} - 1} &= \frac{7 - \theta}{3} \int_0^\infty dy \left( \sin(y) \right) |y|^{-\frac{\theta}{2} - 1} & 2 < \theta \leq 3, \\
\int_0^\infty dy \left( 1 - \cos(y) \right) |y|^{-\frac{\theta}{2} - 1} &= \frac{4}{3} \int_0^\infty dy \left( \sin(y) \right) |y|^{-\frac{\theta}{2}} & \theta > 3.
\end{align*}
\]

Hence it is sufficient to show that for any \( 1 < a < 2, \)
\[
\int_0^\infty dy \left( \sin(y) \right) |y|^{-a} = \cos \left( \frac{a\pi}{2} \right) \Gamma(1-a).
\]

For any positive constant \( 0 < b < 1 \), we have
\[
\int_0^\infty dy \left( \sin(y) \right) |y|^{-a} = |b|^{-a+1} \int_0^\infty dy \left( \sin(b(y+1)) \right) |y+1|^{-a} = \frac{\sqrt{-1}}{2} |b|^{-a+1} \int_0^\infty dy \left( e^{-\sqrt{-1}b(y+1)} - e^{\sqrt{-1}b(y+1)} \right) |y+1|^{-a}
\]
\[
= \frac{\sqrt{-1}}{2} |b|^{-a+1} \left( e^{-\sqrt{-1}b} \Psi(1, 2 - a; \sqrt{-1}b) - e^{\sqrt{-1}b} \Psi(1, 2 - a; -\sqrt{-1}b) \right) = \frac{\sqrt{-1}}{2} |b|^{-a+1} \left( \Psi(1, 2 - a; \sqrt{-1}b) - \Psi(1, 2 - a; -\sqrt{-1}b) \right) + O(|b|^{2-a}),
\]

where \( \Psi(\cdot, \cdot; \cdot; \cdot) \) is the confluent hypergeometric function of the second kind. For the definition and the property of the confluent hypergeometric function, see [16, Section 9]. From the relationship between the confluent hypergeometric functions \( \Phi(\cdot, \cdot; \cdot; \cdot) \) and \( \Psi(\cdot, \cdot; \cdot; \cdot) \), we have
\[
\Psi(1, 2 - a; \pm \sqrt{-1}b) = \frac{\Gamma(1+a)}{\Gamma(a)} \Phi(1, 2 - a; \pm \sqrt{-1}b) + \frac{\Gamma(1-a)}{\Gamma(a)} (\pm \sqrt{-1}b)^{a-1} \Phi(a, a; \pm \sqrt{-1}b)
\]
\[
= \frac{\Gamma(1+a)}{\Gamma(a)} \Phi(1, 2 - a; \pm \sqrt{-1}b) + \frac{\Gamma(1-a)}{\Gamma(a)} (\pm \sqrt{-1}b)^{a-1} e^{\pm \sqrt{-1}b}.
\]
Since $|\Phi(1, 2 - a; \sqrt{-1}b) - \Phi(1, 2 - a; -\sqrt{-1}b)| = O(|\theta|)$, by taking the limit $b \to 0$ we have

$$\int_0^\infty dy \left( \sin(y) \right) y^{-a} = \frac{1}{2} \left( (\sqrt{-1})^a + (-\sqrt{-1})^a \right) \Gamma(1 - a)$$

$$= \cos\left( \frac{a\pi}{2} \right) \Gamma(1 - a).$$

References


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