A-LINKED COUPLING FOR LANGEVIN DIFFUSIONS

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Abstract. We ask when a measurable map driven by a Markov process is Markovian and how it forms a Λ-linked coupling. This question leads us to develop (1) a general construction of intertwining dual via Liggett duality, and (2) a coupled realization of diffusion processes in a form of stochastic flow. We are able to show criteria and examples of construction for Langevin diffusions of higher dimension. In particular, all the examples use a constructive framework by means of Skorohod equations. It includes Pitman’s theorem for one-dimensional Brownian motion as a special case.

1. Introduction

This study was inspired by the recent development for intertwining duals by Fill and Lyzinski [5] and Miclo [14, 15]. In this introduction we consider a diffusion operator

\[ \mathcal{A}f = -\beta(x) \frac{df}{dx} + \frac{1}{2} \frac{d^2f}{dx^2} \]

of one-dimensional Brownian motion with drift coefficient \(-\beta(x)\), and called it (overdamped) Langevin diffusion. Then we illustrate the connection between Liggett and intertwining dual, and demonstrate the construction of Λ-linked coupling by Skorohod equations. In Example 1.1 we show how the Λ-linked coupling can be related back to the work of Rogers and Pitman [20].

1.1. Intertwining dual. The Langevin diffusion (1.1) has a transition density function \(p(t, x, y)\), and it is “time-reversible” with respect to an invariant function

\[ \nu(x) = \exp \left( -2 \int_0^x \beta(\xi)d\xi \right) \]

satisfying \(\nu(x)p(t, x, y) = \nu(y)p(t, y, x)\). Here we introduce a state space \(D^*\) dual to \(D = \mathbb{R}\) by

\[ D^* = \{(z, y) \in \mathbb{R}^2 : z < y\}, \]

and a duality function \(\Gamma\) on \(D^* \times D\) by

\[ \Gamma((z, y), x) = \begin{cases} 1 & \text{if } z < x \leq y; \\ 0 & \text{otherwise.} \end{cases} \]

We present a diffusion operator \(B\) on \(D^*\) by

\[ Bf = \left( \beta(y) \frac{\partial}{\partial y} + \beta(z) \frac{\partial}{\partial z} \right) f + \frac{1}{2} \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right)^2 f \]
with boundary condition that \( f(z, y) \) tends to zero as \((z, y)\) approaches the boundary \( \partial D^* = \{(x, x) \in \mathbb{R}^2 : x \in \mathbb{R}\} \). Then we find a harmonic function

\[
h(z, y) = \int_{-\infty}^{\infty} \Gamma((z, y), x) \nu(x) dx = \int_{z}^{y} \nu(x) dx
\]
on \( D^* \), for which \( Bh = 0 \) holds. This enables us to construct an operator \( B^* \) by the Doob \( h \)-transform

\[
B^* f = \frac{1}{h} B[h f] = B f + \frac{1}{h} \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right) h \times \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right) f
\]

We define a Markov kernel density \( \lambda \) from \( D^* \) to \( D \) by

\[
\lambda((z, y), x) = \frac{1}{h(z, y)} \Gamma((z, y), x) \nu(x)
\]
and the corresponding Markov kernel \( \Lambda \) by

\[
\Lambda f(z, y) = \int_{-\infty}^{\infty} \lambda((z, y), x) f(x) dx
\]
for any bounded measurable function \( f \) on \( D \). Then \( B^* \) satisfies \( \Lambda A f = B^* \Lambda f \), and it is called an intertwining dual to \( A \). In the setting of Markov chains Diaconis and Fill [4] observed that an intertwining dual can be viewed as a Doob \( h \)-transform of the Siegmund dual of the time-reversed Markov chain, and Fill and Lyzinski [5] demonstrated the analogous result for diffusions on \([0, 1]\). The above construction of intertwining dual coincides with the one obtained by Miclo [14]. In Chapter 3 we examine the notion of intertwining duality in terms of Markov semigroups, and present a general construction of intertwining dual (Proposition 3.3).

1.2. \( \Lambda \)-linked coupling. We apply an imputation procedure for missing data \((\omega(t))_{t \geq 0}\) of Brownian motion. Provided a sample path \((\xi(t))_{t \geq 0}\) we reconstruct (or “impute” as commonly known in statistics) \( \omega(t) \) by

\[-\omega(t) = \xi(t) - \xi(0) - \int_{0}^{t} \beta(\xi(v)) dv\]
as if \( \xi(t) \) were governed by

\[\xi(t) = \xi(0) + \int_{0}^{t} \beta(\xi(v)) dv - \omega(t).\]
Assuming \( \xi(0) \leq y \), there exists a unique pair \((L, Y)\) of sample paths satisfying

\[
Y(t) = y + \int_{0}^{t} \beta(Y(v)) dv + \omega(t) + L(t);
\]
\[
L(t) = \int_{0}^{t} I(\xi(v) = Y(v)) dL(v),
\]
where \( I(\xi(v) = Y(v)) = 1 \) or \( 0 \) accordingly as \( \xi(v) = Y(v) \) or not. The equations (1.8)–(1.9) of \( \check{S}k\) orohod type were first introduced by Saisho and Tanemura [23]. We define

\[
\Theta_0(y, \xi) = \begin{cases} -\omega & \text{if } \xi(0) > y; \\ \omega + L & \text{if } \xi(0) \leq y, \end{cases}
\]
which maps from a sample path \((\xi(t))_{t \geq 0}\) to a sample path \((\Theta_0(y, \xi)(t))_{t \geq 0}\).
By \((W(t))_{t \geq 0}\) we denote a Brownian motion starting from \(W(0) = 0\). Fix some \(T > 0\), and construct a sample path \((\hat{X}(s))_{0 \leq s \leq T}\) of the Langevin diffusion (1.1) moving backward in time by

\[
\hat{X}(s) = x - \int_s^T \beta(\hat{X}(u))du + W(T - s), \quad 0 \leq s \leq T,
\]

starting from \(\hat{X}(T) = x\). Then the law of \((\Theta_0(y, \hat{X})(t))_{0 \leq t \leq T}\) becomes that of a Brownian motion; this property is presented as Lemma 7.9. Consequently the diffusion operator (1.4) is associated with a solution \((Y(t), Z(t))_{0 \leq t \leq T}\) to the equations

\[
\begin{align*}
Y(t) &= y + \int_0^t \beta(Y(v))dv + \Theta_0(y, \hat{X})(t), \\
Z(t) &= z + \int_0^t \beta(Z(v))dv - \Theta_0(y, \hat{X})(t),
\end{align*}
\]

starting from \((Z(0), Y(0)) = (z, y) \in D^*\) until the absorbing time \(\zeta = \inf\{0 \leq t \leq T : Z(t) = Y(t)\}\). In Proposition 6.5 and Example 6.6 we establish

\[
(1.11) \quad \Gamma((z, y), X(0)) = \Gamma((Z(T), Y(T)), x),
\]

where we posit \(\Gamma((Z(T), Y(T)), x) = 0\) if \(\zeta \leq T\). Liggett [13] proposed the duality relation (1.11), notably generalizing a notion of duality by Siegmund [24]. Thus, we call (1.4) a Liggett dual to (1.1).

Suppose that an initial state \(X(0)\) is distributed as the Markov kernel density \(\lambda((z, y), \cdot)\) in (1.7), and that a sample path \((X(t))_{t \geq 0}\) of the Langevin diffusion (1.1) is generated by

\[
(1.12) \quad X(t) = X(0) - \int_0^t \beta(X(v))dv + W(t), \quad t \geq 0.
\]

Then we demonstrate in Section 8.1 that the intertwining dual (1.6) is associated with a solution \((Y(t), Z(t))_{t \geq 0}\) to the equations

\[
(1.13) \quad \begin{cases}
Y(t) &= y + \int_0^t \beta(Y(v))dv + \Theta_0(y, X)(t), \\
Z(t) &= z + \int_0^t \beta(Z(v))dv - \Theta_0(y, X)(t),
\end{cases}
\]

starting from \((z, y) \in D^*\). The resulting sample path \(((Z(t), Y(t)), X(t))_{t \geq 0}\) is a Markov process on \(D^* \times D\), and it is called a \(\Lambda\)-linked coupling. The notion of \(\Lambda\)-linked coupling was originally proposed by Diaconis and Fill [4] in the context of Markov chains.

**Example 1.1.** Let \(\beta(x) \equiv \mu\) be a constant drift. The process (1.13) can start from \((Z(0), Y(0)) = (y, y)\), and never hits the boundary \(\partial D^*\) again; see further discussion in Section 8.1. We generate \((X(t))_{t \geq 0}\) by (1.12) starting from \(X(0) = y\), and impute \(\omega(t) = -W(t) + 2\mu t\). Then \(\Theta_0(y, X) = \omega + L\) is explicitly obtained by

\[
L(t) = 2 \max_{0 \leq v \leq t} [W(v) - 2\mu v],
\]

which satisfies the equations (1.8)–(1.9) of Skorohod type. Consider the pair \(U(t) = [Y(t) + Z(t)]/2\) and \(V(t) = [Y(t) - Z(t)]/2\), which is governed by the diffusion operator

\[
\mu \frac{\partial f}{\partial u} + 2\mu \coth(2\mu v) \frac{\partial f}{\partial v} + \frac{1}{2} \frac{\partial^2 f}{\partial v^2}.
\]

In particular we can observe that

\[
V(t) = L(t) - [W(t) - 2\mu t].
\]
This construction of $V(t)$ was obtained by Pitman [19] for $\mu = 0$, and extended by Rogers and Pitman [20]. In light of Lemma 3.2 the regular conditional probability distribution $P(X(t) \in \cdot | (Z(t), Y(t)) = (y + \mu t - v, y + \mu t + v))$ has the density $\lambda((y + \mu t - v, y + \mu t + v), \cdot)$. The equivalent observation by [20] is that $P(\{W(t) - 2\mu t\} \in \cdot | V(t) = v)$ has the density on $(-v, v)$ proportional to $\nu(x) = e^{-4\mu x}$.

1.3. Langevin diffusions of higher dimension. The main objective of this paper is to introduce a construction of Liggett dual and $\Lambda$-linked coupling for an $n$-dimensional Langevin diffusion (Propositions 6.5 and 7.10). In Section 2 we briefly review a stochastic process by means of differential operator, Markov semigroup, and stochastic differential equation (SDE). Kent [9] observed that a symmetric diffusion can be designed for arbitrary invariant function $\nu$ on $\mathbb{R}^n$. In Definition 2.1 we present a $\nu$-symmetric Langevin diffusion when we know an invariant function $\nu$ of interest.

In Section 3 we consider the $\nu$-symmetric Langevin diffusion, and investigate an intertwining dual. In Section 3.1 the $\Lambda$-linked coupling $((Z^*(t), Y^*(t)), X(t))_{t \geq 0}$ is defined in a general term. Intertwining duality has been studied in relation with the question regarding when a function $\phi(X(t))$ of a Markov process $X(t)$ is Markovian; see [18] for a brief review of the literature. General criteria such as Theorem 2 of [20] have been used for the result of Example 1.1. In this paper we ask when a process $(Z^*(t), Y^*(t))$ is Markovian, provided that $X(t)$ is Markovian. Our criteria (Proposition 3.4) for the existence of $\Lambda$-linked coupling is successfully applied for examples of higher dimension in Section 5.1, 6.2, and 8.2, when $Y^*(t)$ takes a value of hypographic closed subset. In Section 5.1 we give two conditions (a) and (b) to be assumed for classes of hypographical closed subsets, and present three different examples of Langevin diffusion where these conditions can be verified. In general they are hard to verify, or simply false for numerous choices of invariant function. This is undoubtedly a serious issue with the constructive framework of $\Lambda$-linked coupling proposed in this paper.

In Section 4 we examine the Euler approximation of a strong solution $(X(t))_{0 \leq t \leq T}$ to the Langevin diffusion and its time-reversed $(\hat{X}(s))_{0 \leq s \leq T}$, which provides a necessary ingredient for various algorithmic constructions. In Section 5 we introduce a stochastic process $(\partial Y^*(t))_{0 \leq t \leq T}$ of hypographical surface as an upper bound for $(X(t))_{0 \leq t \leq T}$, and present a coupled construction of the time-reversed $(\hat{X}(s))_{0 \leq s \leq T}$ in Algorithm 5.4. In Section 6 we examine SDE of Skorohod type (Section 6.1) and propose a construction of Liggett dual (Section 6.2). In Section 7 we present another algorithmic construction of $(X(t))_{0 \leq t \leq T}$ and $(Y^*(t))_{0 \leq t \leq T}$ forward in time (Algorithm 7.1), and devise a $\Lambda$-linked dual (Section 7.3). The exploration of different examples culminates in Section 8. The diffusion operator of intertwining dual is derived, and their connection to a three-dimensional Bessel process is presented for each case of the examples.

2. Langevin diffusions

We introduce a diffusion operator $\mathcal{A}$ on $\mathbb{R}^n$ by

$$
\mathcal{A} f = \sum_{i=1}^{n} \left[ \frac{1}{2} \frac{\partial^2}{\partial x_i^2} f(x) - \beta_i(x) \frac{\partial}{\partial x_i} f(x) \right],
$$

(2.1)
where the subscript $x$ in $A_x$ indicates a variable $x$ to differentiate. By

$$A_x^If = \sum_{i=1}^n \left[ \frac{1}{2} \frac{\partial^2}{\partial y_i^2} f(y) + \frac{\partial}{\partial y_i} (\beta_i(y)f(y)) \right]$$

we denote the adjoint operator of $A$ in $L^2(\mathbb{R}^n)$. In what follows we assume that the drift coefficients $\beta_i(x)$’s are smooth enough (differentiability and Hölder continuity for their derivatives) so that a fundamental solution exists; see [6, 8, 25] for sufficient conditions for existence and uniqueness. Thus, the differential operator $A$ uniquely determines a positive and conservative [i.e., $\int p(t, x, y) \, dy = 1$] transition density function $p(t, x, y)$. It satisfies the parabolic equations

$$\frac{\partial}{\partial t} p(t, x, y) = A_x p(t, x, y);$$

$$\frac{\partial}{\partial t} p(t, x, y) = A_y^i p(t, x, y),$$

which are respectively referred as Kolmogorov backward and forward equation.

Let $\mathbb{R}_+$ be the half line $[0, \infty)$, and let $\Omega = C(\mathbb{R}_+, \mathbb{R}^n)$ be the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^n$. In terms of SDE the distribution determined by the fundamental solution to (2.1) corresponds to the law of a solution to

$$dX(t) = -\beta(X(t)) \, dt + dW(t),$$

where $\beta(x) = [\beta_1(x), \dots, \beta_n(x)]^T$ is the column vector of drift coefficients and $W(t)$ is an $n$-dimensional Brownian motion starting from $W(0) = 0$. That is, a sample path $(X(t))_{t \geq 0}$ starting at $X(0) = x$ is governed by the probability measure $\mathbb{P}_x$ on $\Omega$ which satisfies

$$\mathbb{P}_x(X(t_i) \in dx_i, i = 1, \dots, N) = \prod_{i=1}^N p(t_i - t_{i-1}, x_{i-1}, x_i) \, dx_i$$

with $x_0 = x$ and $t_0 = 0 < t_1 < \cdots < t_N$. Here the event “$X(t_i) \in dx_i, i = 1, \dots, N$” is the measurable set $\{X \in \Omega : X(t_i) \in dx_i, i = 1, \dots, N\}$, and $X$ is identified with an element of $\Omega$. By $\mathbb{E}_{\mathbb{P}_x}[F(X)]$ we denote the expectation with respect to the probability measure $\mathbb{P}_x$ for any measurable function $F$ on $\Omega$.

**Definition 2.1.** A strictly positive function $\nu$ on $\mathbb{R}^n$ is called invariant if it satisfies

$$\nu(y) = \int \nu(x) p(t, x, y) \, dx$$

for any $t > 0$ and $y \in \mathbb{R}^n$. Then

$$\tilde{\nu}(t, x, y) = \frac{\nu(y)}{\nu(x)} p(t, y, x)$$

is the time-reversed transition density with respect to $\nu$, and it satisfies

$$\frac{\partial}{\partial t} \tilde{\nu}(t, x, y) = \frac{1}{\nu(x)} A_y^i [\nu(x) \tilde{\nu}(t, x, y)].$$

The transition density function $p$ is called $\nu$-symmetric if $p = \tilde{\nu}$.

Kent showed (in Section 4 of [9]) that $p$ is $\nu$-symmetric if and only if the operators $A$ and $A^T$ satisfy $A_x f(x) = \frac{1}{\nu(x) A_y^i [\nu(x) f(x)]}$, which is equivalently characterized by $\frac{1}{2} \frac{\partial}{\partial x_i} \nu = -\beta_i \nu$ for $i = 1, \dots, n$. Let $\gamma$ be a real-valued function on $\mathbb{R}^n$, and let $\nu(x) = \exp(-2\gamma(x))$. Then (2.1) is $\nu$-symmetric if the drift coefficient $\beta_i$ satisfies
\[ \beta_i = \frac{\partial}{\partial x_i} \gamma \] for each \( i = 1, \ldots, n \). Here the scalar function \(-\gamma(x)\) is regarded as a potential energy, and \( \beta(x) \) is the gradient \( \nabla \gamma(x) \) (Section 4 of [9]).

**Lemma 2.2.** Let \( \nu \) be fixed, and let \( \Omega_T = C([0, T], \mathbb{R}^n) \). Suppose that the probability measure \( \mathbb{P}_x \) on \( \Omega_T \) is determined by \( \nu \)-symmetric Langevin diffusion (2.2), and that a function \( F \) on \( \Omega_T \) is integrable in either side of (2.4). Then we have

\[ \int \nu(x) E_\nu [F(X)] dx = \int \nu(y) E_\nu [F(X(T - \cdot))] dy \] \hspace{1cm} \text{(2.4)}

**Proof.** It suffices to show (2.4) for \( F(X) = \prod_{i=0}^N I_{E_i}(X(t_i)) \) with Borel subsets \( E_i \)'s of \( \mathbb{R} \) and \( 0 = t_0 < \cdots < t_N = T \) (cf. Section II-38 of [21]), where \( I_{E_i}(x) \) denotes the indicator function on \( E_i \). Then the left-hand side of (2.4) can be expressed as

\[ \int_{E_0} \cdots \int_{E_N} \nu(x_0) dx_0 \prod_{i=1}^N p(t_i - t_{i-1}, x_{i-1}, x_i) dx_i \]

and the right-hand side becomes

\[ \int_{E_N} \cdots \int_{E_0} \nu(x_N) dx_N \prod_{i=N}^1 p(t_i - t_{i-1}, x_{i}, x_{i-1}) dx_{i-1} \]

By repeatedly applying the \( \nu \)-symmetry of \( p \) we can verify that they are equal. \( \square \)

3. Duality and Stochastic Flow

In the rest of this paper we set \( D = \mathbb{R}^n \), and consider a semigroup \( P_t \) on \( D \) for \( \nu \)-symmetric Langevin diffusion (Definition 2.1). In this section we introduce a “dual” state space \( D^* \), and assume that \( D^* \) is open relative to its extension \( \bar{D}^* = D^* \cup \partial D^* \) with collection \( \partial D^* \) of “coffin” states, and that \( D^* \) is a Polish space.

3.1. Intertwining and Liggett dual. A Markov kernel density \( \lambda(x^*, x) \) from \( D^* \) to \( D \) is called a link. In particular, \( \lambda(x^*, \cdot) \) is a probability density on \( D \) [i.e., \( \int \lambda(x^*, x) dx = 1 \)]. In what follows we assume that \( E = \{(x^*, x) \in D^* \times D : \lambda(x^*, x) > 0 \} \) is Polish; thus, readily guaranteeing the existence of regular conditional distribution such as (3.2).

**Definition 3.1.** Let \( Q_t^* \) be a Markov semigroup on \( D^* \), and let \( V_t \) be a Markov semigroup on \( E \). The respective transition functions are assumed to exist, and also denoted by \( Q_t^* \) and \( V_t \). Then \( V_t \) is said to be \( \Lambda \)-linked between \( P_t \) and \( Q_t^* \) if (a) \( V_t f = P_t f \) for \( f(x^*, x) = f(x) \) on \( E \), and (b)

\[ \int \lambda(x^*, x) V_t g(x^*, x) dx = \int Q_t^*(x^*, dy^*) \int \lambda(y^*, y) g(y^*, y) dy \]

for any \( x^* \in D^* \) and for any bounded measurable function \( g \) on \( E \).

By \( \Lambda \) we denote the map

\[ \Lambda[f](x^*) = \int \lambda(x^*, x) f(x) dx \]

from bounded measurable functions \( f \) on \( D \) to bounded measurable functions \( \Lambda[f] \) on \( D^* \). When we choose \( f(x^*, x) = f(x) \) on \( E \), by Definition 3.1 we can observe
that
\[ \Lambda[P_tf](x^*) = \int \lambda(x^*, x)V_t f(x^*, x) \, dx \]
\[ = \int Q_t^*(x^*, dy^*) \int \lambda(y^*, y)f(y) \, dy = Q_t^*\Lambda f(x^*); \]
thus, \( P_t \) and \( Q_t^* \) are "\( \Lambda \)-linked." \( Q_t^* \) is commonly called an intertwining dual of \( P_t \) with respect to \( \Lambda \) when \( \Lambda P_t^*=Q_t^*\Lambda \) holds. The corresponding infinitesimal operator \( \mathcal{B}^* \) of \( Q_t^* \), if it exists, is an intertwining dual of the generator \( \mathcal{A} \) of \( P_t \) if \( \Lambda \mathcal{A} = \mathcal{B}^* \Lambda \).

When the distribution of sample path \((X^*(t), X(t))_{t \geq 0}\) on \( E \) is generated by the Markov semigroup \( V_t \) of Definition 3.1, the marginal distribution of \((X^*(t))_{t \geq 0}\) may not be Markovian. However, we obtain the result similar to Theorem 2 of [20].

**Lemma 3.2.** For any \( x_0^* \in D^* \) we can construct a probability measure \( \nu_{\Lambda(x_0^*)} \) on \( C(\mathbb{R}_+, E) \) satisfying for \( 0 = t_0 \leq t_1 < \cdots < t_N \),
\[ \nu_{\Lambda(x_0^*)}(X(t)_n) \in dx^*_n \times dx_i, i = 1, \ldots, N) \]
\[ = \int \lambda(x_0^*, x_0) \, dx_0 \times \prod_{i=1}^{N} V_{t_i-t_{i-1}}((x_{i-1}^*, x_i), dx_i^* \times dx_i). \]

Then \((X^*(t))_{t \geq 0}\) is Markovian with initial state \( X^*(0) = x_0^* \), and governed by the Markov semigroup \( Q_t^* \).

**Proof.** Similarly to the proof of Theorem 2 of [20] we can verify for \( 0 = t_0 \leq t_1 < \cdots < t_N \),
\[ \nu_{\Lambda(x_0^*)}(X(t)_n) \in dx^*_n, i = 1, \ldots, N) \]
\[ = \prod_{i=1}^{N} Q_{t_i-t_{i-1}}(x_i^*, dx_i) \]
by applying (3.1) recursively. \( \square \)

We call the probability measure \( \nu_{\Lambda(x_0^*)} \) of Lemma 3.2 a \( \Lambda \)-linked coupling. It satisfies
\[ \nu_{\Lambda(x_0^*)}(X(t)_n) \in dy^* \times dy) = Q_t^*(x^*, dy^*)\lambda(y^*, y) \, dy, \]
which allows us to derive a regular conditional probability
\[ \nu_{\Lambda(x_0^*)}(X(t) \in dy | X^*(t) = y^*) = \lambda(y^*, y)dy \]
for \( t > 0 \).

Let \( Q_t \) be a sub-Markov semigroup on \( D^* \), and let \( \Gamma(x^*, x) \) be a bounded non-negative measurable function on \( D^* \times D \). The corresponding transition function is assumed to exist, and also denoted by \( Q_t \). Then \( Q_t \) is said to be a Liggett dual of \( P_t \) with respect to \( \Gamma \) if
\[ \int p(t, x, y)\Gamma(x^*, y) \, dy = \int Q_t(x^*, dy^*)\Gamma(y^*, x) \]
holds for \((x^*, x) \in D^* \times D \).

**Proposition 3.3.** Suppose that \( Q_t \) is a Liggett dual of \( P_t \) with respect to \( \Gamma \), and that
\[ h(x^*) = \int \Gamma(x^*, x)\nu(x) \, dx \]
is finite and strictly positive on $D^*$. Then the Markov semigroup

$$Q_t^* f(x^*) = \frac{1}{h(x^*)} Q_t [hf](x^*)$$

is an intertwining dual of $P_t$ with respect to the link

$$\lambda(x^*, x) = \frac{\Gamma(x^*, x)}{h(x^*)} \nu(x)$$

By applying the Liggett duality and the $\nu$-symmetry of $P_t$, we can observe that

$$Q_t h(x^*) = \int Q_t(x^*, dy^*) \int \Gamma(y^*, x) \nu(x) \, dx = \int \Gamma(x^*, y) \nu(y) p(t, x, y) \, dx = h(x^*);$$

thus, $h$ is harmonic for $Q_t$. Given the harmonic function $h$, the semigroup $Q_t^*$ of Proposition 3.3 is known as the Doob $h$-transform, and it is clearly conservative.

**Proof.** We obtain

$$\Lambda[P_t f](x^*) = \int \frac{\Gamma(x^*, x)}{h(x^*)} \nu(x) \, dx \int p(t, x, y) f(y) \, dy$$

$$= \frac{1}{h(x^*)} \int \nu(y) f(y) \, dy \int \Gamma(x^*, x) p(t, y, x) \, dx$$

$$= \frac{1}{h(x^*)} \int Q_t(x^*, dy^*) h(y^*) \int \frac{\Gamma(y^*, y)}{h(y^*)} \nu(y) f(y) \, dy$$

$$= \frac{1}{h(x^*)} Q_t^* [h(Af)](x^*) = Q_t^* [Af](x^*).$$

Hence, $P_t$ and $Q_t^*$ are $\Lambda$-linked. □

The Liggett dual $Q_t$ of Proposition 3.3 may not be conservative, but it can be extended to a Markov semigroup over $D^* = D^* \cup \partial D^*$. We can generate a Markov process $X^*$ on $D^*$ by $Q_t$ with exit boundary $\partial D^*$. If $Q_t$ is conservative [i.e., $\int Q_t(x, dy) = 1$], no Markov process started in $D^*$ reaches the coffin state. By setting the terminal time $\zeta = \inf\{t \geq 0 : X^*(t) \in \partial D^*\}$ accompanied with $X^*$, we can view it as a Markov process $X^*(t)$ over $D^*$ defined for the duration $[0, \zeta]$. For a duality function $\Gamma$ it is understood customarily that $\Gamma(X^*(t), x) = 0$ if $t \geq \zeta$, or equivalently that $\Gamma$ is extended over $D^* \times D$ by setting $\Gamma(x^*, x) = 0$ for all $x^* \in \partial D^*$.

3.2. **Stochastic flow.** For each $T > 0$ we write $\Omega_T = C([0, T], \mathbb{R}^n)$ as in Lemma 2.2. A family $(\Psi_{s,t,T})_{0 \leq s \leq t \leq T}$ of measurable maps $\Psi_{s,t,T}$ from $D^* \times \Omega_T$ to $D^*$ is called a *cocycle dynamical system* if (a) $\Psi_{s,t,T}(x^*, \xi) = \Psi_{u,t,T}(\Psi_{s,u,T}(x^*, \xi), \xi)$ (cocycle property) and $\Psi_{s,t,T}(x^*, \xi) = x^*$ for all $(x^*, \xi) \in D^* \times \Omega_T$ and for all $0 \leq s \leq u \leq t \leq T$, and (b) $\Psi_{s,t,T}(x^*, \xi) \in \partial D^*$ for all $(x^*, \xi) \in \partial D^* \times \Omega_T$. The cocycle dynamical system $(\Psi_{s,t,T})_{0 \leq s \leq t \leq T}$ is driven by a “noise” $\xi \in \Omega_T$, and it is called a **stochastic flow** if for each $T > 0$ there exists a probability measure $P_T$ on $\Omega_T$ such that (c) $\Psi_{t_1, \lambda, t_2}(x^*, \xi), i = 1, \ldots, n$, are independent under $P_T$ for all $0 \leq t_0 < \cdots < t_n \leq T$ and all $x^*_1, \ldots, x^*_n \in D^*$, and (d) the law $P_T(\{\xi \in \Omega_T : \Psi_{s,t,T}(x^*, \xi) \in dy^*\})$ is uniquely determined by $x^*$ and $(t - s)$ regardless of $T$. Thus, the stochastic flow $(\Psi_{s,t,T})_{0 \leq s \leq t \leq T}$ defines a Markov semigroup...
$U_t f(x^*) = \mathbb{E}_{P_T}[f(\Psi_{0,T}(x^*, \xi))].$ Although the map $\Psi_{s,t,S}$ may not be exactly the same as $\Psi_{s,t}^\ast$ and $P_T$ may not be an extension of $P_S$ for $S < T$, we simply write $\Psi_{s,t}$ for $\Psi_{s,t,T}$ when a fixed value $T > 0$ is indicated.

Let $\mathcal{W}_T$ be the Wiener measure for an $n$-dimensional Brownian motion on $[0,T]$. Similarly $\mathcal{P}_{x,T}$ we denote the probability measure (2.3) restricted on $\Omega_T$. Furthermore, we introduce the probability measure $\mathcal{P}_{x,T}$ reversed in time by setting $\mathcal{P}_{x,T}(B) = \mathbb{P}_{x,T}(\{\xi(T - \cdot) : \xi \in B\})$ for any measurable subset $B$ of $\Omega_T$. In the next proposition we consider the Liggett dual $\mathcal{Q}_t$ of $P_t$ with respect to $\Gamma$ and the intertwining dual $\mathcal{Q}^\ast_t$ of Proposition 3.3. We also assume that $E = \{(x^*, x) \in D^\ast \times D : \Gamma(x^*, x) > 0\}$ is a Polish space.

**Proposition 3.4.** Let $(\Xi_{s,t}^\ast)_{0 \leq s \leq t \leq T}$ be a stochastic flow for $Q_t$ under $\mathcal{W}_T$, and let $(\Theta_{s,t})_{0 \leq s \leq t \leq T}$ be a family of measurable maps $\Theta_{s,t}$ from $D^\ast \times \Omega_T$ to $\Omega_T$. Assume that (a) the law of $\Theta_{s,t}(x^*, \xi)$ under $\mathcal{P}_{x,T}$ is the Wiener measure $\mathcal{W}_T$ for all $(x^*, x) \in D \times D^\ast$, and (b) the map

$$\Psi_{s,t}^\ast(x^*, \xi) = \Xi_{s,t}^\ast(x^*, \Theta_{s,t}(x^*, \xi))$$

forms a stochastic flow $(\Psi_{s,t}^\ast)_{0 \leq s \leq t \leq T}$ for $Q_t$ under $\mathcal{P}_{x,T}$, and satisfies

$$\Gamma(x^*, \xi(s)) = \Gamma(\Psi_{s,t}^\ast(x^*, \xi), \xi(t))$$

for any $(x^*, \xi) \in D^\ast \times \Omega_T$. Then the semigroup

$$V_t g(x^*, x) = \mathbb{E}_{P_t}[g(\Psi_{0,T}(x^*, X(T)))], \quad (x^*, x) \in E,$

is $\Lambda$-linked between $P_t$ and $Q_t^\ast$.

**Proof.** Obviously we have $V_t g = P_t f$ if $g(x^*, x) = f(x)$ on $E$. In order to show (3.1), we apply (3.7), and find for each $(x^*, x) \in E$

$$\int \lambda(x^*, x) \mathbb{E}_{P_t}[g(\Psi_{0,T}(x^*, X(T)))] \, dx$$

$$= \frac{1}{\mathbb{h}(x^*)} \int \nu(x) \mathbb{E}_{P_t}[\Gamma(\Psi_{0,T}(x^*, X(T)), X(T)) g(\Psi_{0,T}(x^*, X), X(T))] \, dx$$

We can use Lemma 2.2 for the above integration, and observe that the law of $\Psi_{0,T}(x^*, X(T - \cdot))$ is determined by $Q_T(x^*, \cdot)$. Then we obtain

$$\frac{1}{\mathbb{h}(x^*)} \int \nu(y) \mathbb{E}_{P_t}[\Gamma(\Psi_{0,T}(x^*, X(T - \cdot)), X(0)) g(\Psi_{0,T}(x^*, X(T - \cdot)), X(0))] \, dy$$

$$= \frac{1}{\mathbb{h}(x^*)} \int \nu(y) \, dy \int \Gamma(y^*, y) g(y^*, y) Q_T(x^*, dy^*)$$

$$= \int \frac{\mathbb{h}(y^*)}{\mathbb{h}(x^*)} Q_T(x^*, dy^*) \int \frac{\nu(y)}{\mathbb{h}(y^*)} \Gamma(y^*, y) g(y^*, y) \, dy$$

$$= \int Q_T^\ast(x^*, dy^*) \lambda(y^*, y) g(y^*, y) \, dy,$

which completes the proof. \qed

**Remark 3.5.** By $P_{\lambda(x^*, \cdot)}$ we denote the law of a solution $X$ to (2.2) when an initial value $X(0)$ is distributed as $\lambda(x^*, \cdot)$. By Lemma 3.2 the stochastic flow $\Psi_{s,t}^\ast$ of Proposition 3.4 determines $Q_t^\ast f(x^*) = \mathbb{E}_{P_{\lambda(x^*, \cdot)}}[f(\Psi_{0,T}(x^*, X))], and it is appropriately called a $\Lambda$-linked dual. In all the examples of Section 8, $x^* \in \partial D^\ast$ becomes
an entrance state for the respective intertwining dual, and a specific construction of \( \Psi_{s,t}^*(x^*, X) \) is obtained.

A stochastic flow \((\varphi_{s,t})_{0 \leq s \leq t \leq T}\) of maps \(\varphi_{s,t}\) from \(D \times \Omega_T\) to \(D\) is called a solution to (2.2) if it satisfies \(P_{t-s}f(x) = E_{\omega_T}[f(\varphi_{s,t}(x, \omega))]\) for all \(x \in D\) and \(0 \leq s \leq t \leq T < \infty\). Suppose that \((\Xi_{s,t}^*)_{0 \leq s \leq t \leq T}\) is a stochastic flow for \(Q_t\) under \(\mathbb{W}_T\) as in Proposition 3.4. Then we call \(\Xi_{s,t}^*\) a Liggett dual with respect to \(\Gamma\) if for each \(T > 0\) and \(x^* \in D^*\) there exists a solution \((\psi_{s,t,x^*}, \mathcal{T})_{0 \leq s \leq t \leq T}\) for (2.2) such that

\[
\Gamma(x^*, \psi_{0,T,x^*}, \mathcal{T}(x, \omega)) = \Gamma(\Xi_{0,T}^*(x^*, \omega), x)
\]

for all \(x \in D\). Clearly (3.8) implies (3.3). In Section 6.2 we present a stochastic flow \(\Xi_{s,t}\) of Proposition 3.4, and demonstrate that there is such a solution \(\psi_{s,t,x^*}, \mathcal{T}\) that (3.8) holds. In Section 7.3 we introduce a construction of \(\Theta_{s,t}\) of Proposition 3.4, and examine the cocycle property and the independent increment of (3.6). We note that the construction \(\Theta_{s,t}\) of Section 7.3 does not depend on \(t\), and we simply write \(\Theta_{s}\) for \(\Theta_{s,t}\) in such constructions.

**Example 3.6.** In the case of \(\beta(x) \equiv \mu = 0\) in Example 1.1 we can consider a dual state space \(D^* = (0, \infty)\) with duality function \(\Gamma(y, x) = 1\) if \(y < x \leq y\); otherwise, \(\Gamma(y, x) = 0\). We choose the entrance state \(y = 0\) for the \(\Lambda\)-linked dual \(\Psi^*\), and set \(X(t)\) starting at \(X(0) = 0\) as a Brownian motion \(W(t)\). In Example 7.11 we show that

\[
\Theta_0(0, X)(t) = X(t) - 2\max\{X(v) : 0 \leq v \leq t\}
\]

and

\[
\Psi_{0,t}^*(0, X) = -\Theta_0(0, X)(t)
\]

A two-dimensional version of Example 3.6 is constructed on a dual state space \(D^* = \{y \in \mathbb{R}^2 : y_1 < y_2\}\) with duality function \(\Gamma(y, x) = 1\) if \(x_1 + x_2 = y_1 + y_2\) and \(y_1 - y_2 < x_2 - x_1 \leq y_2 - y_1\); otherwise, \(\Gamma(y, x) = 0\). Then we obtain a \(\Lambda\)-linked dual

\[
P^*_{s,t}(0, W) = ((W_1 \otimes W_2)(t), (W_2 \otimes W_1)(t))
\]

where \((W_1 \otimes W_2)(t) = \min_{0 \leq s \leq t}(W_1(v) - W_2(v)) + W_2(t)\) and \((W_2 \otimes W_1)(t) = \max_{0 \leq s \leq t}(W_2(v) - W_1(v)) + W_1(t)\). It is also known as a Brownian motion in a Weyl chamber \(D^*\). An \(n\)-dimensional Brownian motion in \(D^* = \{y \in \mathbb{R}^n : y_1 < y_2 < \cdots < y_n\}\) can be obtained explicitly (see, e.g., [17, 2]), but a general construction of \(\Lambda\)-linked dual \(\Psi_{s,t}^*(y, W)\) is not known. The constructive approach of Section 7.3 is not immediately applicable for this type of generalization of Pitman’s theorem.

### 4. Stochastic flow of strong solution

Let \(W(t)\) be an \(n\)-dimensional Brownian motion, and let \(\Phi_{s,t}\) be a stochastic flow for \(P_t\) under the Wiener measure \(\mathbb{W}\). Then we call \(\Phi_{s,t}\) a stochastic flow of strong solution if \(X(t) = \Phi_{0,t}(x, W)\) is a solution to (2.2) starting at \(X(0) = x\) and uniquely determined regardless of \(T > 0\) (cf. Theorem IV-1.1 of [7]). Such a stochastic flow \(\Phi_{s,t}\) exists if \(\beta\) is locally Lipschitz continuous (cf. Theorem IV-3.1 of [7]). In this paper we assume that the drift coefficient \(\beta\) is smooth with bounded first derivatives; thus, it has a Lipschitz constant \(K_{\beta}\).
4.1. **Euler approximation of Langevin diffusions.** Here we fix $T > 0$, and develop an approximation $X_N(s)$ for an $n$-dimensional Langevin diffusion (2.2). Set a uniform increment $0 = s_0 < \cdots < s_N = T$, and define an approximate Markov transition map $\phi_{s,t}$ by

\[
\phi_{s,t}(z, W) = z - \beta(z)(t-s) + W(t) - W(s).
\]

Starting at $X_N(0) = x_N$, we can recursively construct

\[
X_N(s) = \phi_{s_{k-1}, s}(X_N(s_{k-1}), W)
\]

for $s_{k-1} < s \leq s_k$, $k = 1, \ldots, N$. By using $a \wedge b = \min\{a, b\}$ and $[c]_+ = \max\{c, 0\}$, we can formulate (4.2) as

\[
X_N(s) = x_N - \sum_{k=1}^{N} \beta(X_N(s_{k-1}))[(s - s_{k-1}) \wedge (s_k - s_{k-1})] + W(s) - W(0).
\]

When $x_N$ is convergent, the approximation of (4.2) or (4.3) is known to converge, and called an explicit Euler method for numerical solutions of SDE’s; see, e.g., Kloeden and Platen [10].

For $f \in C([0, T], \mathbb{R}^n)$ we define the modulus of continuity (cf. Chapter 2 of [3]) by

\[
\Delta_s f = \sup\{\|f(s + u) - f(s)\| : 0 \leq s + u \leq T, u \leq \delta\}, \quad 0 < \delta \leq T.
\]

We also set

\[
\|f\|_s = \sup\{\|f(u)\| : 0 \leq u \leq s\}, \quad 0 \leq s \leq T,
\]

and write $|f|_s$ instead of $\|f\|_s$ when $f$ is a scalar function. The lemma below shows uniform boundedness and equicontinuity of the approximation $X_N$.

**Lemma 4.1.** For any $\delta > 0$ we have

\[
\|X_N\|_T \leq e^{K_T}(\|X_N\| + \|\beta(0)\|T + 2\|W\|_T);
\]

\[
\Delta_s X_N \leq (K_\beta\|X_N\|_T + \|\beta(0)\|)\delta + \Delta_s W.
\]

**Proof.** Since $\|\beta(X_N(s))\|$ is bounded by $K_\beta\|X_N\|_T + \|\beta(0)\|$ by (4.3) we obtain an upper bound for $\Delta_s X_N$. Observe for $k = 0, \ldots, N$ that

\[
\|X_N(s_k)\| \leq \|x_N\| + \|\beta(0)\|T + 2\|W\|_T + K_\beta \sum_{i=0}^{k-1} \|X_N(s_i)\|(s_{i+1} - s_i).
\]

Then the upper bound for $\|X_N\|_T$ is an immediate consequence of the following version of discrete Gronwall’s inequality: If $x_k \leq \alpha + \sum_{i=0}^{k-1} \gamma_i x_i$ for $k = 0, \ldots, N$ with nonnegative $\alpha$ and $\gamma_i$’s then we have $x_k \leq \alpha \exp\left(\sum_{i=0}^{k-1} \gamma_i\right)$ for $k = 0, \ldots, N$.

Assuming that $x_N$ converges to $X(0)$, a subsequence of $\{X_N\}$ converges uniformly by Ascoli-Arzelà theorem. The limiting process $X$ implies the existence of a stochastic flow of strong solution

\[
\Phi_{0,a}(X(0), W) = X(0) - \int_0^a \beta(X(u)) \, du + W(s) - W(0).
\]

Since the solution must be unique, the whole sequence $\{X_N\}$ must converge uniformly to $X$, and the inequalities of Lemma 4.1 hold for $X$. 

4.2. Inverse stochastic flow. The map $\Phi_{s,t}(\cdot, \omega)$ of (4.4) is diffeomorphic from $D = \mathbb{R}^n$ to itself for each $0 \leq s \leq t$ and $\omega \in \Omega = C(\mathbb{R}_+, \mathbb{R}^n)$ (cf. Theorem V-13.8 of [22]). By $\Phi_{s,t}^{-1}(\cdot, \omega)$ we denote the inverse map of $\Phi_{s,t}(\cdot, \omega)$ for each $\omega \in \Omega$. We view $Y(s) = \Phi_{T-t,T-s}(Y(t), W(T-\cdot))$, $0 \leq s \leq t \leq T$, as a sample path starting from $Y(t)$ and moving backward to $Y(s)$. This allows us to formulate a stochastic flow $Y(t) = \Phi_{T-t,T-s}^{-1}(Y(s), W(T-\cdot))$ of strong solution for

$$
(4.5)
$$

$$
dY(t) = \beta(Y(t)) \, dt + dW(t)
$$

starting from $Y(s)$.

We set the time increment $t_j = T - s_{N-j}$, $j = 0, \ldots, N$. Here we also assume a sufficiently small increment (i.e., a sufficiently large $N$) so that $\phi_{T-t_j, T-t_{j-1}}(\cdot, \omega)$ is diffeomorphic. Starting from $Y_N(0) = x_N$, we can construct an approximation $Y_N(t)$ to (4.5) recursively by

$$
(4.6)
$$

$$
Y_N(t) = \phi_{T-t_j, T-t} \left( \phi_{T-t_j, T-t_{j-1}}^{-1}(Y_N(t_{j-1}), W(T-\cdot)), W(T-\cdot) \right)
$$

for $t_{j-1} < t \leq t_j$, $j = 1, \ldots, N$. It is equivalently formulated by

$$
(4.7)
$$

$$
Y_N(t) = x_N + \sum_{j=1}^{N} \beta(Y_N(t_j)) \left[ (t - t_{j-1}) \wedge (t_j - t_{j-1}) \right] + W(t) - W(0),
$$

and called an implicit Euler scheme. Provided that $x_N$ converges to $Y(0)$, the uniform convergence of $Y_N$ is an immediate consequence to the following lemma.

Lemma 4.2. Let $\delta > 0$ be arbitrarily fixed. For a sufficiently large $N$ we have

$$
\max_{0 \leq j \leq N} \|Y_N(t_j)\| \leq 2e^{2K_\delta T} (\|x_N\| + \|\beta(0)\| T + 2\|W\| T);
$$

$$
\|Y_N\|_T \leq \|x_N\| + 2\|W\| T + \left( \|\beta(0)\| + K_\delta \max_{0 \leq j \leq N} \|Y_N(t_j)\| \right) T;
$$

$$
\Delta \delta Y_N \leq \left( \|\beta(0)\| + K_\delta \max_{0 \leq j \leq N} \|Y_N(t_j)\| \right) \delta + \Delta \delta W;
$$

Proof. By (4.7) we obtain for $j = 1, \ldots, N$,

$$
\|Y_N(t_j)\| \leq \|x_N\| + \|\beta(0)\| T + 2\|W\| T + K_\delta \sum_{i=1}^{j} \|Y_N(t_i)\| (t_i - t_{i-1})
$$

For a sufficiently large $N$ we can find that $[1 - K_\delta (t_j - t_{j-1})] \geq 1/2$ for every $j$, and that

$$
\|Y_N(t_j)\| \leq 2 \left[ \|x_N\| + \|\beta(0)\| T + 2\|W\| T + K_\delta \sum_{i=1}^{j-1} \|Y_N(t_i)\| (t_i - t_{i-1}) \right]
$$

The rest of the proof is completed similarly to that of Lemma 4.1. □

4.3. Stochastic flow of inverse image. By $F$ and $F_0$ we denote respectively the space of closed subsets and of nonempty closed subsets in $D = \mathbb{R}^n$. Equipped with Fell topology the space $F_0$ is a locally compact Polish space, and it is also characterized by Painlevé-Kuratowski convergence (cf. Appendix B of [16]). Let $\{x_N^j\}$ be a sequence in $F_0$. The lower limit, denoted by $\liminf x_N^j$, consists of all the points $x$ such that $x_N \to x$ with $x_N \in x_N^j$. The upper limit, denoted by $\limsup x_N^j$, consists of all the limiting points $x$ of some subsequence $\{x_{N_k}\}$ with $x_{N_k} \in x_N^j$. The sequence $\{x_N^j\}$ converges to $x^*$ in the Painlevé-Kuratowski sense, denoted by
\[ x^* = \text{rkm-lim } x_N^*, \text{ if } x^* = \liminf x_N^* = \limsup x_N^*. \]

An \( \mathbb{F}_0 \)-valued function \( F(t) \) is lower semicontinuous at \( t = t_0 \) if \( \liminf F(t_N) \supseteq F(t_0) \) for every sequence \( \{ t_N \} \) converging to \( t_0 \), and it is upper semicontinuous at \( t = t_0 \) if \( \limsup F(t_N) \subseteq F(t_0) \) for each of such sequences (cf. Appendix D of [16]). Then \( F(t) \) is continuous if it is lower and upper semicontinuous.

Here we consider a stochastic flow \( (\psi_s)_{0 \leq s \leq t \leq T} \) from \( D \times \Omega_T \) to \( D \), construct a stochastic flow \( \phi^{-1}_{T-t} \) for each of such sequences (cf. Appendix D of [16]), and introduce a sufficient condition for upper semicontinuity.

**Lemma 4.4.** Let \( Y^*(t) = \psi^{-1}_{T-t,T}(x^*, \omega(T-\cdot)) \) be a path of inverse image starting from \( Y(0) = x^* \in \mathbb{F}_0 \), and let \( \zeta(x^*, \omega) = \sup\{0 \leq t \leq T : Y^*(t) \neq \emptyset \} \) be a map from \( \mathbb{F}_0 \times \Omega_T \) to \( [0, T] \). If \( \psi_s(x, \omega) \) is continuous on \( (s, x) \in [0, t] \times D \) then \( Y^*(t) \) is \( \mathbb{F}_0 \)-valued and upper semicontinuous for \( 0 \leq t < \zeta(x^*, \omega) \) or for \( 0 \leq t \leq T \) if \( \zeta(x^*, \omega) = T \).

**Proof.** Since \( \psi_{T-t,T}(x, \omega(T-\cdot)) \) is continuous in \( x \), \( Y^*(t) \) takes values on \( \mathbb{F} \). In order to show the upper semicontinuity, we consider a sequence \( \{ (t_N, z_N) \} \) satisfying \( \psi_{T-t_N,T}(z_N, \omega(T-\cdot)) \in x^* \), and observe \( z_0 \in Y^*(t_0) \) if it converges to \( (t_0, z_0) \).

In the case of stochastic flow (4.4) we obtain a continuous stochastic flow of inverse image.

**Lemma 4.4.** The stochastic flow \( \Phi_{s,t}(x, W) \) is continuous on \( (s, x) \in [0, t] \times D \).

**Proof.** Consider a limiting sequence \( \{ (s_N, x_N) \} \) to \( (s_0, x_0) \) in \([0, t] \times D \). For each \( (s, x) = (s_N, x_N) \) a solution \( X_N(t) \) to

\[ X(t) = x - \int_s^t \beta(X(u)) \, du + W(t) - W(s), \quad 0 \leq t \leq T, \]

is uniquely determined, and the sequence \( \{ X_N(t) \} \) is uniformly bounded and equicontinuous. Thus, we can find a subsequence \( \{ X_{N_k}(t) \} \) which converges to \( X_0(t) \) uniformly. Since \( X_0(t) \) must satisfy (4.8) with \((s, x) = (s_0, x_0)\), the uniqueness of solution implies that the whole sequence \( \{ X_N(t) \} \) must converge to \( X_0(t) \). Consequently \( X_N(t) = \Phi_{s_N,t}(x_N, W) \) converges to \( X_0(t) = \Phi_{s_0,t}(x_0, W) \).

**Proposition 4.5.** Starting from \( Y^*(0) = x^* \in \mathbb{F}_0 \), \( Y^*(t) = \Phi^{-1}_{T-t,T}(x^*, W(T-\cdot)) \) is \( \mathbb{F}_0 \)-valued and continuous for \( 0 \leq t \leq T \).

**Proof.** By Lemma 4.3 and Lemma 4.4 it suffices to show the lower semicontinuity. The proof is naturally related to a selection of continuous process \( X(t) \in Y^*(t) \) (cf. Chapter 9 of [1]). Consider a limiting sequence \( \{ t_N \} \) to \( t_0 \), and choose \( y \in Y^*(t_0) \). Then we can set \( x = \Phi_{T-t_0,T}(y, W(T-\cdot)) \), and find that \( X(t_N) = \Phi^{-1}_{T-t_N,T}(x, W(T-\cdot)) \in Y^*(t_N) \) converges to \( y \).

An approximation \( Y_N^*(t) \) to the path \( Y^*(t) \) of Proposition 4.5 starts with \( Y_N^*(0) = x^* \) and is recursively updated by

\[ Y_N^*(t) = \phi_{T-t_j,T-t_{j-1}} \left( \phi^{-1}_{T-t_{j-1},T-t_{j-2}} \left( \cdots \phi^{-1}_{T-t_1,T-t_0} \left(Y_N^*(t_0), W(T-\cdot) \right) \right) \right) \]

for \( t_j < t \leq t_j, j = 1, \ldots, N \). It can be viewed as the collection of approximations of (4.6) starting from \( Y_N(0) \in x^* \). Then we obtain \( Y^*(t) = \text{rkm-lim}_{N \to \infty} Y_N^*(t) \).
5. Stochastic Flow of Hypographical Surface

By \(X_i\) or \(X_{i,N}\) we denote the \(i\)-th coordinate of \(X\) or \(X_N\), and by \(X_{(i)}\) or \(X_{(-i),N}\) we denote the \((n-1)\)-dimensional vector of \(X\) or \(X_N\), by deleting the \(i\)-th coordinate. A nonempty closed subset \(x^*\) of \(\mathbb{R}^n\) is said to be hypographic at the direction of \(i\)-th coordinate if there exists a unique upper semicontinuous function \(h\) from \(\mathbb{R}^{n-1}\) to \(\mathbb{R}\) satisfying \(x^* = \{(x_i; x_{(i)}): x_i \leq h(x_{(i)}), x_{(i)} \in \mathbb{R}^{n-1}\}\). Note that \(\mathbb{R}^0\) is viewed as the subspace \(\{0\}\) of zero dimension in \(\mathbb{R}\). The boundary \(\partial x^*\) of hypographic closed set uniquely determines the hypographical surface

\[
\partial x^* = \{x \in \mathbb{R}^n: x_i = h(x_{(i)})\}
\]

if \(h\) is continuous. Thus, we denote by \(\partial x^*(\cdot)\) the corresponding function \(h(\cdot)\) in (5.1), though it is a slight abuse of notation. We call a hypographic closed subset \(x^*\) “Lipschitz-continuous” if \(\partial x^*(\cdot)\) is Lipschitz-continuous.

5.1. Examples of hypographical surface. In the rest of this paper we consider the stochastic flow (4.4), and assume the existence of a subclass \(F_1\) of Lipschitz-continuous hypographic closed sets at the direction of first coordinate such that (a) the stochastic flow \((\Phi_{T-t,T}^{-1})_{0 \leq t \leq T}\) of inverse image maps from \(\mathbb{F}_1 \times \Omega_T\) to \(\mathbb{F}_1\), and (b) a Lipschitz constant exists universally for all \(x^* \in F_1\). Provided an \(\mathbb{F}_1\)-valued stochastic flow \(Y^*(t) = \Phi_{T-t,T}^{-1}(Y^*(s), W(T - t))\), we denote by \(\partial Y^*(t, \cdot)\) the corresponding function of hypographical surface at each \(t\), and call it stochastic flow of hypographical surface.

Example 5.1. A trivial example of hypographical surfaces is given by

\[
F_1 = \{x^*: \text{hypographic with constant } \partial x^*(\cdot)\}
\]

if \(n = 1\), or \(X_1(t)\) and \(X_{(1)}(t)\) are independent in (2.2). Suppose that an invariant function \(\nu^*\) is formed by \(\nu(x) = \nu_1(x_1)v_2(x_{(1)})\). Then the drift coefficient \(\beta_1(x)\) is a function of \(x_1\), and the hypographical surface \(\partial Y^*(t, \cdot) \equiv Y_1(t)\) is determined by \(dY_1(t) = \beta_1(Y_1(t))dt + dW_1(t)\). Therefore, it is reduced to an example of one-dimensional Langevin diffusion.

In general neither the initial condition of \(Y^*(0) = x^* \in F_1\) nor the Lipschitz-continuity for \(\beta\) in (2.2) can ensure that a stochastic flow \(Y^*(t) = \Phi_{T-t,T}^{-1}(x^*, W(T - t))\) remains on a subclass \(F_1\) of hypographic closed sets.

Example 5.2. Let \(\nu(x) = e^{-2x_1x_2}\) be an invariant function on \(\mathbb{R}^2\). Then we can find the drift coefficient \(\beta(x) = [x_2, x_1]^T\), and set

\[
F_1 = \{x^*: \text{hypographic with constant } \partial x^*(x_2) = (\tanh \theta)x_2 + \eta \text{ for } \theta, \eta \in \mathbb{R}\}.
\]

A direction \(U(t) = [\sinh(\theta + t), \cosh(\theta + t)]\) of the line \(\partial Y^*(t)\) satisfies \(dU = \beta(U)\) if \(U(0) = [\sinh \theta, \cosh \theta]\). Thus, given an initial direction \(U(0)\) and a point \(Y(0)\) on the line \(\partial Y^*(0)\), the hypographical line \(\partial Y^*(t, x_2) = (U(t)/U_2(t))(x_2 - Y_2(t)) + Y_1(t)\) is determined by \(dU(t) = \beta(U(t))dt\) and \(dY(t) = \beta(Y(t))dt + dW(t)\).

In the next example we view \(D = \mathbb{R}^n\) as a parameter space, and consider a linear regression \(a, x = \sum_{i=1}^n a_i x_i\) with vector \(a = [a_1, \ldots, a_n]^T\) of explanatory variables. By \(S(\theta) = 1/(1 + e^{-\theta})\) we denote the logistic sigmoid function. Then we can generate a binary output \(b = 0\) or \(1\) according to the probability \(S((2b - 1)(a, x))\), and call it a Bernoulli-logistic regression model. In the neural network terminology this is a unit perceptron with input \(a\) and weight vector \(x\). Provided a training
data set \( \{(a^{(1)}, b^{(1)}), \ldots, (a^{(N)}, b^{(N)})\} \) consisting of \( N \) input-output pairs, we can construct the likelihood function

\[
\nu(x) = \prod_{j=1}^{N} S((2b^{(j)} - 1)(a^{(j)}, x))
\]

\[
= \exp \left( \sum_{j=1}^{N} b^{(j)} (a^{(j)}, x) \right) \times \prod_{j=1}^{N} S(- (a^{(j)}, x))
\]

by applying \( S((a, x))bS(- (a, x))^{1-b} = \exp(b(a, x))S(- (a, x)) \).

**Example 5.3.** We can consider the above likelihood function as an invariant function, and obtain the drift coefficient

\[
\beta(x) = \frac{1}{2} \sum_{j=1}^{N} a^{(j)} \left[ S((a^{(j)}, x)) - b^{(j)} \right].
\]

Suppose that the input vectors \( a^{(1)}, \ldots, a^{(N)} \) span an \((n-1)\)-dimensional subspace \( H \), and that a unit normal vector \( d \) to the subspace \( H \) has a positive component \( d_1 = \cos \theta_0 \) to the first coordinate. Furthermore, it is assumed that

\[
(5.2) \quad \sup_{x \in H, ||x||=1} \min_{j=1, \ldots, N} (2b^{(j)} - 1)(a^{(j)}, x) < 0
\]

so that the surface integral \( \int_{H} \nu(x) d\sigma_H < \infty \); thus, in the Bayesian viewpoint the function \( \nu(x) \) on the subspace \( H \) is proportional to the posterior density function \( \nu_H(x) \) given the flat prior. Here we can choose \( \theta_0 < \theta \leq \pi/2 \), and introduce a subclass \( F_1 \) of hypographic closed subsets \( x^* \) satisfying \( ||d, x - y|| \leq ||x - y|| \cos \theta \) for any \( x, y \in \partial x^* \). A Lipschitz constant for \( \partial x^* \) in \( F_1 \) is bounded by \( \cot(\theta - \theta_0) \). Particularly we have

\[
(5.3) \quad F_1 = \{ x^* : \text{hypographic with } \partial x^* = H + z \text{ with some } z \in \mathbb{R}^n \}
\]

when \( \theta = \pi/2 \).

In Example 5.3 we can construct a sample path \( Y^*(t) = \Phi_{t,T}^{-1}(Y^*(0), W(T - \cdot)) \)

starting from \( Y^*(0) = x^* \in F_1 \). Consider two distinct paths \( X(t) = \Phi_{t,T}^{-1}(X(0), W(T - \cdot)) \)

and \( Y(t) = \Phi_{t,T}^{-1}(Y(0), W(T - \cdot)) \) on \( \partial Y^*(t) \). Then we can introduce the difference \( z(t) = Y(t) - X(t) = \alpha(t)d + h(t) \) in the coordinate system with the vector \( d \) and the subspace \( H \) by setting \( \alpha(t) = \langle d, z(t) \rangle \) and \( h(t) = z(t) - \alpha(t)d \). Observe that \( z(t) \) is a solution to the differential equation

\[
\frac{dz}{dt} = \frac{1}{2} \sum_{j=1}^{N} a^{(j)} \left[ S((a^{(j)}, z + X(t))) - S((a^{(j)}, X(t))) \right],
\]

and therefore, that \( \alpha(t) \equiv \alpha(0) \) and \( ||h(t)|| \) is increasing. Thus, we obtain \( ||z(t)|| \cos \theta \), which implies that \( Y^*(t) \in F_1 \).

**5.2. A coupled construction by approximation.** Here we consider the transition map \( \phi_{s,t} \) of Euler approximation in (4.1), and assume that the inverse image \( \phi_{s,t}^{-1}(\cdot, \omega) \) maps from \( F_1 \) to itself for every \( \omega \in \Omega_T \). By \( \phi_{i,s,t} \) we denote the \( i \)-th coordinate of \( \phi_{s,t} \), and by \( \phi_{(-i),s,t} \) the \((n-1)\)-dimensional vector of \( \phi_{s,t} \) by deleting the \( i \)-th coordinate. Since the map \( \phi_{i,s,t}(\cdot, W) \) and \( \phi_{(-i),s,t}(\cdot, W) \) are determined
respectively by $W_s$ and $W_{(-1)}$, we can write $\phi_{t,s,t}(\cdot, W_s)$ and $\phi_{(-1),s,t}(\cdot, W_{(-1)})$. By $W'$ we denote the $n$-dimensional Brownian motion

\begin{equation}
W'(s) = [-W_1(s), W_{(-1)}(s)]
\end{equation}

by changing the sign to the path $W_1(s)$ of the first coordinate.

We reverse the time $t$ with $s = T - t$ in Proposition 4.5, and approximate $Y^*(s) = \Phi_{s,T}(y^*, W')$ recursively by

\begin{equation}
Y^*_N(s) = \phi_{s_{k-1},s}(\phi_{s_{k-1},s_k}^{-1}(Y^*_N(s_k), W'), W')
\end{equation}

backward for $s_{k-1} \leq s < s_k$, $k = N, \ldots, 1$. It is started from $s_N = T$ and $Y^*_N(T) = y^* \in \mathbb{F}_1$, and driven by the Brownian motion (5.4) until $s_0 = 0$. Then we can introduce Algorithm 5.4 which generates $Y_N(s)$ and couples it with $Y^*_N(s)$.

**Algorithm 5.4.** Set the initial value $Y_N(0) = x_N$ and $\sigma_N(0) = 0$. Provided $Y_N(s_{k-1})$ and $\sigma_N(s_{k-1})$, we can construct $Y_N(s)$ and $\sigma_N(s)$ recursively forward for $s_{k-1} < s \leq s_k$, $k = 1, \ldots, N$, in the following steps: (i) Update $Y_{(-1),N}(s)$ for $s_{k-1} < s \leq s_k$ by

\begin{equation}
Y_{(-1),N}(s) = \phi_{-1,s_{k-1},s}(Y_N(s_{k-1}), W_{(-1)}).
\end{equation}

(ii) Update $\sigma_N(s)$ for $s_{k-1} < s \leq s_k$ by

\begin{equation}
\sigma_N(s) = \sigma_N(s_{k-1}) + 2(W_1(s) - W_1(s_{k-1}))
\end{equation}

if $Y_N(s_{k-1})$ and $Y_{(-1),N}(s_k)$ satisfy

\begin{equation}
Y_1,N(s_{k-1}) - \beta_1(Y_N(s_{k-1}))(s_k - s_{k-1}) + |W_1(s_k) - W_1(s_{k-1})|
\end{equation}

\begin{equation}
\geq \partial Y^*_N(s_k, Y_{(-1),N}(s_k));
\end{equation}

otherwise, set $\sigma_N(s) \equiv \sigma_N(s_{k-1})$. (iii) Complete the update of $Y_N(s)$ by setting

\begin{equation}
Y_{1,N}(s) = \phi_{1,s_{k-1},s}(Y_N(s_{k-1}), W_1 - \sigma_N)
\end{equation}

for $s_{k-1} < s \leq s_k$.

**Remark 5.5.** Algorithm 5.4 produces $Y_N(s)$ by the Euler approximation (4.2) with the noise $W_N(s) = [W_1 - \sigma_N, W_{(-1)}](s)$, $0 \leq s \leq T$, and couple it with $Y^*_N(s)$ in such a way to maintain

\begin{equation}
Y_N(s_k) \in Y^*_N(s_k)
\end{equation}

for all $k = 1, \ldots, N$ if it starts with

\begin{equation}
Y_N(0) = x_N \in Y^*_N(0).
\end{equation}

In the case of (5.11) we find $\sigma_N(s_k)$ updated by (5.7) only if $W_1(s_k) - W_1(s_{k-1}) > 0$; otherwise, the left-hand side of (5.8) is equal to

\begin{equation}
Y_{1,N}(s_{k-1}) - \beta_1(Y_N(s_{k-1}))(s_k - s_{k-1}) - (W_1(s_k) - W_1(s_{k-1}))
\end{equation}

and bounded by $\partial Y^*_N(s_k, Y_{(-1),N}(s_k))$ since (5.10) holds at $k-1$. Hence, we observe that

\begin{equation}
\partial Y^*_N(s_k, Y_{(-1),N}(s_k)) - 2(W_1(s_k) - W_1(s_{k-1})) < Y_{1,N}(s_k)
\end{equation}

whenever (5.7) is applied over the interval $(s_{k-1}, s_k]$.

**Lemma 5.6.** If $W(s)$, $0 \leq s \leq T$, is a Brownian motion under $\mathbb{W}_T$ then so is $W_N(s) = [W_1 - \sigma_N, W_{(-1)}](s)$, $0 \leq s \leq T$. 

Proof. We set for each \( k = 0, \ldots, N \)
\[
W(s; k) = \begin{cases} 
[W_1(s) - \sigma_N(s), W_{(-1)}(s)] & \text{if } 0 \leq s \leq s_k; \\
[W_1(s) - \sigma_N(s_k), W_{(-1)}(s)] & \text{if } s_k < s \leq T,
\end{cases}
\]
and observe that \( W(s; 0) = W(s) \) and \( W(s; N) = W_N(s) \). We prove by induction that \( W(s; k), 0 \leq s \leq T, \) is a Brownian motion for each \( k = 1, \ldots, N \).

Suppose that \( W(\cdot; k-1) \) is a Brownian motion under \( \mathbb{W}_T \). Set \( B_1(s) = W_1(s) - W_1(s_{k-1}), \) \( s_{k-1} \leq s \leq s_k \), and define
\[
\rho = \begin{cases} 
-1 & \text{if } (5.8) \text{ holds for } Y_N(s_{k-1}) \text{ and } Y_{(-1),N}(s_k); \\
1 & \text{otherwise.}
\end{cases}
\]
Observe that \( Y_N(s_{k-1}) \) is a function of \( Y_N(0) = x_N \) and \( W(\cdot; k-1) \) on \([0, s_{k-1}]\), and that \( Y_N^T(s_k) \) is a function of \( Y_N^T(T) = y^* \) and \( W(\cdot; k-1) - W(s_k; k-1) \) on \([s_k, T]\). Provided \( Y_N(s_{k-1}) \) and \( Y_N^T(s_k) \), we can determine whether (5.8) holds or not by the length \( |B_1(s_k)| \) and the vector \( W_{(-1)}(s_{k-1}) - W_{(-1)}(s_{k-1}) \). Thus, \( \rho \) is a function of \( x_N, y^*, W_1(\cdot; k-1) \) on \([0, s_{k-1}], W_1(\cdot; k-1) - W_1(s_k; k-1) \) on \([s_k, T], W_{(-1)}(\cdot; k-1) \) on \([0, T]\), and \( B_1(s_k) \). Let \( C \) be a measurable subset of \( \Omega_T \) generated by \( W_1(\cdot; k-1) \) on \([0, s_{k-1}], W_1(\cdot; k-1) - W_1(s_k; k-1) \) on \([s_k, T], \) and \( W_{(-1)}(\cdot; k-1) \) on \([0, T]\). For \( s_{k-1} \leq s < s' \leq s_k \) we can observe that \( W_1(s'; k) - W_1(s; k) = \rho(B_1(s') - B_1(s)) \), and that
\[
\mathbb{W}_T(0 \leq W_1(s; k) - W_1(s_{k-1}; k) \leq z, 0 \leq W_1(s'; k) - W_1(s; k) \leq z', C)
= \mathbb{W}_T(0 \leq B_1(s) - B_1(s_{k-1}) \leq z, 0 \leq B_1(s') - B_1(s) \leq z') \cdot \mathbb{W}_T(C)
\]
for all \( z, z' > 0 \). We can establish a similar observation for further subdivision \( s_{k-1} \leq s < s' < s'' < \cdots \leq s_k \). Thus, \( W(\cdot; k) \) is a Brownian motion under \( \mathbb{W}_T \). \( \square \)

We say that a sequence of stochastic processes is tight or weakly converging if the sequence of their distributions is tight or weakly converging (cf. Chapter 2 of [3]). By Lemma 5.6 we can find that \( Y_N \) of Algorithm 5.4 is equal in distribution to the approximation \( X_N \) by (4.2). If \( X_N \) converges weakly to \( X \), so does \( Y_N \); thus, we obtain the following corollary.

**Corollary 5.7.** If \( W(s) \) is a Brownian motion and \( x_N \) converge to \( x \) in Algorithm 5.4 then \( Y_N \) converges weakly to the probability measure \( \mathbb{P}_x \) of (2.2) on \( \Omega_T \).

### 5.3. Boundedness and equicontinuity of approximation

In Section 5.2 we introduced an approximation \( Y_N^T(s) \) by (5.5) starting from the initial state \( Y_N^T(0) = Y^* \) and updating backward. By \( K_1 \) we denote a Lipschitz constant universally for every hypographical surface \( \partial x^* \) from \( \mathbb{F}_1 \), and assume \( K_1 \geq 1 \).

**Lemma 5.8.** Let \( \{Z_{(-1),N}(s)\} \) be a uniformly bounded and equicontinuous sequence of \( \mathbb{R}^{n-1} \)-valued functions on \([0, T]\). Then \( \{\partial Y_N^T(s, Z_{(-1),N}(s))\} \) is uniformly bounded and equicontinuous on \([0, T]\).

**Proof.** By reversing the time \( t \) with \( s = T - t \) in (4.6) we can construct \( X_N(s) \) recursively by
\[
X_N(s) = \phi_{s_{k-1}, s}(\phi_{s_{k-1}, s_k}^{-1}(X_N(s_k), W'), W')
\]

(5.13)
for $s_{k-1} \leq s < s_k$, $k = N, \ldots, 1$. We can set the initial state $X_N(T) \in \partial y^*$, and observe that $X_N(s) \in \partial Y_N^*(s)$. Furthermore, we obtain

$$|\partial Y_N^*(s, Z_{(-1),N}(s))| \leq |X_{1,N}(s)| + K_1 \left( \|X_{N,(-1)}(s)\| + \|Z_{(-1),N}(s)\| \right)
\leq K_1 \left( \sqrt{2}\|X_N(s)\| + \|Z_{(-1),N}(s)\| \right).$$

Since $X_N(s)$ and $Z_{(-1),N}(s)$ are uniformly bounded on $[0, T]$, so is $\partial Y_N^*(s, Z_{(-1),N}(s))$.

Let $\delta > 0$ be arbitrarily fixed. For equicontinuity we can start from $X_N(s) = [\partial Y_N^*(s, Z_{(-1),N}(s))$, $Z_{(-1),N}(s)]$, and construct $X_N(u)$ backward $s - \delta \leq u \leq s$ by (5.13). Since $X_N(s - \delta) \in \partial Y_N^*(s - \delta)$, we obtain

$$|\partial Y_N^*(s, Z_{(-1),N}(s)) - \partial Y_N^*(s - \delta, Z_{(-1),N}(s - \delta))|
\leq |X_{1,N}(s) - X_{1,N}(s - \delta)|
+ |\partial Y_N^*(s - \delta, X_{(-1),N}(s - \delta)) - \partial Y_N^*(s - \delta, Z_{(-1),N}(s))|
+ |\partial Y_N^*(s - \delta, Z_{(-1),N}(s - \delta)) - \partial Y_N^*(s - \delta, Z_{(-1),N}(s))|
\leq \sqrt{2}K_1||X_N(s) - X_N(s - \delta)|| + K_1||Z_{(-1),N}(s) - Z_{(-1),N}(s - \delta)||
\leq K_1[\sqrt{2}\Delta s X_N + \Delta s Z_{(-1),N}].$$

Therefore, the equicontinuity of $\partial Y_N^*(s, Z_{(-1),N}(s))$ is implied by that of $X_N(s)$ and $Z_{(-1),N}(s)$.

Let $Y_N(s)$ be the approximation constructed by Algorithm 5.4. In the following lemma we consider

$$Z_N(s) = x_N - \sum_{k=1}^{N} \beta(Y_N(s_{k-1}))(s - s_{k-1}) \land (s_k - s_{k-1})_+ + W(s)$$

for $0 \leq s \leq T$. Then we can observe that $Y_{1,N}(s) = Z_{1,N}(s) - \sigma_N(s)$ and $Y_{(-1),N}(s) = Z_{(-1),N}(s)$.

**Lemma 5.9.** If $\{x_N\}$ is bounded then $\{\sigma_N(s)\}$ is uniformly bounded and equicontinuous on $[0, T]$.

**Proof.** If $x_N \notin Y_N^*(0)$ then $\sigma_N(s) = 2W(s)$ is uniformly bounded and equicontinuous on $[0, T]$. Let $\delta > 0$ be arbitrarily fixed. In the proof we assume that $x_N \in Y_N^*(0)$, and that $N$ is sufficiently large so that $s_k - s_{k-1} < \delta$ for $k = 1, \ldots, N$. Let $y_N(k) = \max_{0 \leq i \leq k} ||Y_N(s_i)||$ and $z_N(k) = \max_{0 \leq i \leq k} ||Z_N(s_i)||$ for $k = 0, \ldots, N$. Then we can find

$$z_N(k) \leq x_N + \|\beta(0)\|T + K_\beta \sum_{i=0}^{k-1} y_N(i)(s_{i+1} - s_i) + \|W\|_T.$$  

Suppose that the last change for $\sigma_N(s)$ by (5.7) has been made over the $i$-th interval $(s_{i-1}, s_i]$ before $s_k$; otherwise, set $i = 0$. Together with (5.12) we obtain

$$\sigma_N(s_k) = \sigma_N(s_i) = Z_{1,N}(s_i) - Y_{1,N}(s_i)
\leq |Z_{1,N}(s_i) + |\partial Y_N^*(s_i, Z_{(-1),N}(s_i))| + 2\Delta s W_1$$

Let $X_N(s)$ be the approximation by (5.13) which starts from $X_N(T) \in \partial y^*$. By the boundedness proof of Lemma 5.8 we obtain

$$\sigma_N(s_k) \leq \sqrt{2}K_1[z_N(k) + \|X_N\|_T] + 2\Delta s W.$$
Thus, we can apply $y_N(k) \leq z_N(k) + \sigma_N(s_k)$ to (5.15), and show by the discrete Gronwall’s inequality that $y_N(k)$ and $z_N(k)$ are bounded universally regardless of $N$. Hence, we can conclude that $Z_N(s)$ is uniformly bounded and equicontinuous, and that $\sigma_N(s)$ is uniformly bounded.

For the equicontinuity of $\sigma_N(s)$ we present the upper bound for $|\sigma_N(s) - \sigma_N(s - \delta)|$. If no update by (5.7) is completed between $s - \delta$ and $s$ then $|\sigma_N(s) - \sigma_N(s - \delta)| \leq 2\Delta s W$. Otherwise, we can find the first and the last update by (5.7) completed respectively at $s_k$ and $s_{k'}$ on $[s - \delta, s]$. Then we obtain

$$|\sigma_N(s) - \sigma_N(s - \delta)| \leq 8\Delta s W + |Z_{1,N}(s_{k'}) - Z_{1,N}(s_k)| + |\partial Y^*_N(s_{k'}, Z_{(-1),N}(s_{k'})) - \partial Y^*_N(s_k, Z_{(-1),N}(s_k))|.$$  

By the equicontinuity proof of Lemma 5.8 we find $\sigma_N(s)$ equicontinuous. \hfill \Box

In the proof of Lemma 5.9 we can also show that $Y^*_N(s)$ is uniformly bounded and equicontinuous.

6. Skorohod equations

Let $\kappa(s)$ be a real-valued continuous function. Assuming $\kappa(0) \geq 0$, we call

$$\eta(s) = \kappa(s) + \ell(s)$$

a Skorohod equation if $\eta(s)$ is a nonnegative continuous function and $\ell(s)$ is a nondecreasing continuous function with $\ell(0) = 0$, satisfying

$$\ell(s) = \int_0^s I_{\{\eta(u) = 0\}} d\ell(u),$$

where $I_{\{\eta(u) = 0\}}$ is the indicator function of a statement $\{\eta(u) = 0\}$, taking values 1 or 0 accordingly as the statement is true or not. Given $\kappa(s)$, a pair $(\kappa, \ell)$ of functions forms the Skorohod equation, and the nonnegative function $\eta$ of (6.1) is uniquely determined by

$$\ell(s) = -\min_{0 \leq u \leq s} [\kappa(u) \wedge 0].$$

6.1. SDE of Skorohod type. As in the case of Algorithm 5.4 we assume the stochastic flow $Y^*(s) = \Phi_{s,T}^{-1}(y^*, W')$ of inverse image starting from $Y^*(T) = y^* \in F_1$ and evolving backward in time. Provided $Y^*(s), 0 \leq s \leq T$, and $x \in Y^*(0)$, we can introduce an SDE of Skorohod type by

$$Y^T(s) = x - \int_0^s \beta(Y^T(u)) du + [W_1 - L, W_{(-1)}](s);$$

$$L(s) = \int_0^s I_{\{Y^T_1(u) = \partial Y^*(u, Y^T_{(-1)}(u))\}} dL(u),$$

for $0 \leq s \leq T$.

Assuming a solution $(Y^T, L)$ to the SDE (6.4)–(6.5) of Skorohod type, we can set

$$Z_1(s) = Y^T_1(s) + L(s) = x_1 - \int_0^s \beta_1(Y^T(u)) du + W_1(s).$$

Similarly to Section 2 of Saisho and Tanemura [23], we can introduce $\kappa(s) = \partial Y^*(s, Y^T_{(-1)}(s)) - Z_1(s)$ and $\ell(s) = L(s)$, and show that the pair $(\eta, \ell)$ satisfies the Skorohod equations (6.1)–(6.2) with $\eta(s) = \partial Y^*(s, Y^T_{(-1)}(s)) - Y^T_1(s)$. We first
establish the uniqueness of the SDE solution of Skorohod type in Lemma 6.1, and then demonstrate in Proposition 6.2 that the solution \((Y^\dagger, L)\) exists, and that it coincides with the limit of \((Y_N, \sigma_N)\) constructed by Algorithm 5.4.

**Lemma 6.1.** Let \(x, x^\dagger \in Y^*(0)\). Suppose that \((L, Y^\dagger)\) and \((L^1, Y^\dagger)\) are solutions to the SDE (6.4)–(6.5) of Skorohod type starting respectively from \(Y^\dagger(0) = x\) and \(Y^\dagger(0) = x^\dagger\). Then we have
\[
\|Y^\dagger - Y^\dagger\|_T \leq (3 + K_1)\|x - x^\dagger\|e^{(3 + K_1)K_3T}
\]

**Proof.** Accompanying with the respective solutions \((L, Y^\dagger)\) and \((L^1, Y^\dagger)\), we can construct \(Z_1\) and \(Z_1^\dagger\) by (6.6). By the Lipschitz continuity of \(\beta\), we have
\[
|Z_1 - Z_1^\dagger|_s, \|Y^\dagger_{(-1)} - Y^\dagger_{(-1)}\|_s \leq \|x - x^\dagger\| + K_\beta \int_0^s \|Y^\dagger - Y^\dagger\|_u\,du
\]
We can apply (6.3) to obtain an upper bound for \(|L - L^1|_s\) (cf. Lemma 2.1 of [23]). Along with the Lipschitz constant \(K_1\) of hypographical surface we obtain
\[
|L - L^1|_s \leq |\partial Y^*(\cdot, Y^\dagger_{(-1)}(\cdot)) - \partial Y^*(\cdot, Y^\dagger_{(-1)}(\cdot))|_s + |Z_1 - Z_1^\dagger|_s
\]
Together we have
\[
\|Y^\dagger - Y^\dagger\|_s \leq (3 + K_1)\left[\|x - x^\dagger\| + K_\beta \int_0^s \|Y^\dagger - Y^\dagger\|_u\,du\right],
\]
which completes the proof by Gronwall’s inequality. \(\square\)

**Proposition 6.2.** In Algorithm 5.4 if (5.11) converges to \(x \in Y^*(0)\) then \(\sigma_N\) uniformly converges to \(L\) of (6.4)–(6.5).

**Proof.** By Lemma 5.9 we can find a uniformly converging subsequence of pairs \((\sigma_N, Y_N)\), and obtain the respective limit \(\sigma(s)\) and \(Y(s) = \Phi_{0,s}(x, [W_1 - \sigma, W_{(-1)}])\), \(0 \leq s \leq T\). By Remark 5.5 in the case of (5.11) we can show that \(\sigma(s)\) is non-decreasing, and that \(Y(s)\) satisfies \(Y_1(s) \leq \partial Y^*(s, Y_{(-1)}(s))\) for \(0 \leq s \leq T\). By Lemma 5.8 \(\partial Y^*_N(s, Y_{(-1)}(s))\) converges uniformly to \(\partial Y^*(s, Y_{(-1)}(s))\). Note in Remark 5.5 that (5.12) holds whenever \(\sigma_N(s_k) - \sigma_N(s_{k-1}) = 2(W_1(s_k) - W_1(s_{k-1})) > 0\). Thus, for arbitrary \(\varepsilon_0 > 0\) we can find a sufficiently large \(N_i\) so that \(0 \leq \partial Y^*(s_k, Y_{(-1)}(s_k)) - Y_1(s_k) < \varepsilon_0\) whenever \(\sigma_N(s_k) - \sigma_N(s_{k-1}) > 0\), and
\[
\sigma_N(s_l) = \sum_{k=1}^l I_{\{0 \leq \partial Y^*(s_k, Y_{(-1)}(s_k)) - Y_1(s_k) < \varepsilon_0\}} \times (\sigma_N(s_k) - \sigma_N(s_{k-1}))
\]
for \(l = 1, \ldots, N_i\). In addition we can choose a sufficiently large \(N_i\) for arbitrary \(\varepsilon_1 > 0\) such that \(|\sigma_N - \sigma|_T < \varepsilon_1/3T\). Hence, we have
\[
\left|\sigma(s_l) - \sum_{k=1}^l I_{\{0 \leq \partial Y^*(s_k, Y_{(-1)}(s_k)) - Y_1(s_k) < \varepsilon_0\}} \times (\sigma(s_k) - \sigma(s_{k-1}))\right| < \varepsilon_1,
\]
which implies that
\[
\sigma(s) = \int_0^s I_{\{0 \leq \partial Y^*(u, Y_{(-1)}(u)) - Y_1(u) < \varepsilon_0\}}\,d\sigma(u).
\]
Since $\varepsilon_0 > 0$ is arbitrary, the pair $(Y, \sigma)$ must be the SDE solution (6.4)–(6.5) of Skorohod type. By the uniqueness of solution the whole sequence $\sigma_N$ must converge to $L$. □

6.2. Liggett dual. Provided $Y_N^*(s)$, $S \leq s \leq T$, by (5.5) terminating at $Y_N^*(T) = y^* \in F_I$, we can construct a sequence $(\sigma_N, Y_N)$ of Algorithm 5.4 starting from $\sigma_N(S) = 0$ and $Y_N(S) = x$ at $s = S$. Correspondingly provided $Y^*(s) = \Phi^{-1}_{*,T}(y^*, W')$, $S \leq s \leq T$, and $x \in Y^*(S)$, we can consider an SDE solution $(L, Y')$ of Skorohod type (6.4)–(6.5) starting from $L(S) = 0$ and $Y'(S) = x$ at $s = S$. If $x \in Y^*(S) \setminus \partial Y^*(S)$ then $x \in Y_N^*(S)$ holds for sufficiently large $N$ and $\sigma_N$ converges to $L$ uniformly on $[S, T]$ by Proposition 6.2. If $x \notin Y^*(S)$ then we find $x \notin Y_N^*(S)$ for sufficiently large $N$ and $\sigma_N(s) = 2(W_1(s) - W_1(S))$, $S \leq s \leq T$. Since $\mathbb{W}_T \left( x \in \Phi^{-1}_{*,T}(\partial y^*, W') \right) = 0$, $\sigma_N$ converges uniformly to

$$
\sigma(s) = \begin{cases} L(s) & \text{if } x \in \Phi^{-1}_{*,T}(y^*, W'); \\ 2(W_1(s) - W_1(S)) & \text{otherwise}, \end{cases} \quad S \leq s \leq T,
$$

almost surely. Consequently we can introduce

$$(6.7) \quad \Theta_{x,y^*,T}(W)(s) = [W_1 - \sigma, W_{(-1)}](s), \quad S \leq s \leq T,$$

and show by Lemma 5.6 that

$$
\Theta_{x,y^*,T}(W)(s) - \Theta_{x,y^*,T}(W)(S), \quad S \leq s \leq T,
$$

is a Brownian motion under $\mathbb{W}_T$.

Since $\Phi_{s,t}(x, \omega)$ is driven by a noise $\omega$ on $[s, t]$, we can define a map $\psi_{s,t,y^*,T}$ from $D \times \Omega_T$ to $D$ by

$$(6.8) \quad \psi_{s,t,y^*,T}(x, \omega) = \Phi_{s,t}(x, \Theta_{x,y^*,T}(\omega))$$

Provided $x \in \Phi^{-1}_{*,T}(y^*, W')$, $\psi_{s,t,y^*,T}(x, W)$, $S \leq s \leq T$, represents the solution $Y'(s)$ to (6.4)–(6.5). Furthermore, we obtain

Lemma 6.3. $(\psi_{s,t,y^*,T})_{0 \leq s \leq t \leq T}$ is a stochastic flow for $P_t$ under $\mathbb{W}_T$.

Proof. Fix $0 \leq s \leq u \leq t \leq T$ and $x \in D$, and set $\xi = \Theta_{x,y^*,T}(\omega)$ and $\tilde{\xi} = \Theta_{\psi_{s,u,y^*,T}(x, \omega), y^*, T}(\omega)$. Then we can observe $\xi(u') - \xi(u) = \tilde{\xi}(u' - u)$, $u \leq u' \leq T$, and obtain the cocycle property

$$
\psi_{u,t,y^*,T}(\psi_{s,u,y^*,T}(x, \omega), \omega) = \Phi_{u,t}(\psi_{s,u,y^*,T}(x, \omega), \tilde{\xi})
$$

Similarly we can fix $x, y \in D$, and consider the processes $\xi = \Theta_{x,y^*,T}(W)$ and $\eta = \Theta_{y,u,y^*,T}(W)$ driven by a Brownian motion $W$ under $\mathbb{W}_T$. Since $\xi(s') - \xi(s)$, $s \leq s' \leq u$, and $\eta(u') - \eta(u)$, $u \leq u' \leq t$, are independent Brownian motions, $\psi_{s,u,y^*,T}(x, W) = \Phi_{s,u}(x, \xi)$ and $\psi_{u,t,y^*,T}(y, W) = \Phi_{u,t}(y, \eta)$ are independent and their laws are determined respectively by $P_{u-s}$ and $P_{t-u}$. □

In terms of continuity of the stochastic flow $\psi_{s,t,y^*,T}$ we obtain the following lemma.

Lemma 6.4. The stochastic flow $\psi_{s,t,y^*,T}(x, W)$ is continuous on $\{(s, x) \in [0, t] \times D : x \in \Phi^{-1}_{*,T}(y^*, W)\}$. 
Consider a limiting sequence \(\{(s_N, x_N)\}\) to \((s_0, x_0)\) in \([0, t] \times D\) such that 
\(x_N \in \Phi_{s_N, t}^{-1}(y^*, W)\). Without loss of generality we can assume that \(b = \sup s_N < t\). Then we must have \(x_0 \in \Phi_{s_0, t}^{-1}(y^*, W)\) by Proposition 4.5. For each pair \((s_N, x_N)\) we can find the SDE solution \(Y_N^t(s)\) of Skorohod type, \(s_N \leq s \leq t\), to (6.4)–(6.5) starting from \(Y_N^t(s_N) = x_N\). By Lemma 5.9 we can find \(\{Y_N^t\}\) uniformly bounded and equicontinuous on \([b, t]\). Similarly to the proof of Lemma 4.4 we can argue that \(Y_N^t\) uniformly converges to the SDE solution \(Y^t\) of Skorohod type to (6.4)–(6.5) starting from \(Y^t(s_0) = x_0\). \(\square\)

It should be noted that \(\psi_{s, t, y^*}^*, T(x, W)\) cannot be continuous on \((s, x)\) for the unrestricted domain of \([0, t] \times D\). When \(x_N \notin \Phi_{s, t}^{-1}(y^*, W)\) converges to \(x_0 \in \Phi_{s, t}^{-1}(y^*, W)\), it is almost likely observed that \(\psi_{s, t, y^*}^*, T(x_0, W) \in y^* \setminus \partial y^*\); thus, the continuity fails.

**Proposition 6.5.** Let \(D^* = \{(z^*, y^*) \in \mathbb{F} \times \mathbb{F}_1 : z^* \subseteq y^*\}\), and let
\[
\Xi_{s,t}^*((z^*, y^*), \omega) = (\psi_{T^{-1}T^-S, y^*, T^-S}^*(z^*, \omega(T - )), \Phi_{T^{-1}T^-S}^{-1}(y^*, \omega'(T - )))
\]
be a map from \(D^* \times \Omega_T\) to \(D^*\) for \(0 \leq s \leq t \leq T\). Then \(\Xi_{s,t}^*\) is a stochastic flow under \(\mathbb{W}_T\), and it is a Liggett dual for (2.2) with respect to
\[
\Gamma((z^*, y^*), x) = \begin{cases} 1 & \text{if } x \in y^* \setminus z^*; \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** Fix \(0 \leq s \leq u \leq t \leq T\) and \((z^*, y^*) \in D^*\), and set \(Z^*(u) = \psi_{T^{-1}u, y^*, T^-s}^*(z^*, \omega(T - ))\), \(Y^*(u) = \Phi_{T^{-1}u, T^-s}^{-1}(y^*, \omega'(T - ))\), and \(Z^*(t) = \psi_{T^{-1}T^-s, y^*, T^-s}^*(z^*, \omega(T - ))\) and \(Y^*(t) = \Phi_{T^{-1}T^-s, y^*, T^-s}^{-1}(y^*, \omega'(T - ))\). Then we can observe that \(\psi_{T^{-1}T^-u, T^-s, y^*, T^-s}^*(Z^*(u), \omega'(T - )) = Z^*(t)\) and therefore, that \(\Xi_{s,t}^*\) is a cocycle dynamical system. Similarly to the proof of Lemma 6.3 we can show that \(\Xi_{s,t}^*\) has independent increments and that the law of \(\Xi_{s,t}^*\) is determined by \((t - s)\) under \(\mathbb{W}_T\); thus, it is a stochastic flow. For each \(x^* = (z^*, y^*) \in D^*\) we set \(\psi_{s, t, x^*, T}(x, \omega)\) to be the solution \(\psi_{s, t, y^*, T}(x, \omega(T - ))\) of Lemma 6.3. Then (3.8) holds for all \(x \in D\). \(\square\)

We can view the Liggett dual \(\Xi_{s,t}^*((z^*, y^*), W)\) of Proposition 6.5 under \(\mathbb{W}_T\) as a stochastic process \((Z^*(t), Y^*(t))_{0 \leq t \leq T}\) on the state space \(D^* = \{(z^*, y^*) \in \mathbb{F} \times \mathbb{F}_1 : z^* \subseteq y^*\}\) until the absorbing time \(\zeta = \inf\{t \geq 0 : Z^*(t) = Y^*(t)\}\). For all the examples below we can choose
\[
D^* = \{(z^*, y^*) \in \mathbb{F} \times \mathbb{F}_1 : z^* \subseteq y^*\},
\]
and form a Liggett dual by
\[
Z^*(t) = \Phi_{T^{-1}T^-s}^{-1}(z^*, W(T - ))\quad \text{and} \quad Y^*(t) = \Phi_{T^{-1}T^-s}^{-1}(y^*, W'(T - ))
\]
until the absorbing time \(\zeta\).

**Example 6.6.** In Example 5.1 a Liggett dual (6.11) is formed by a stochastic process \((Z_1(t), Y_1(t))\) with respect to the duality function \(\Gamma((z_1, y_1), x) = 1\) if \(z_1 < x_1 \leq y_1\); otherwise, \(\Gamma((z_1, y_1), x) = 0\). The SDE of \((Z_1(t), Y_1(t))\) corresponds to the diffusion operator (1.4) on the dual state space (1.2) in Section 1.1.

**Example 6.7.** In Example 5.2 the pair \(\partial Y^*(t, x_2) = (U_1(t)/U_2(t))(x_2 - Y_2(t)) + Y_1(t)\) and \(\partial Z^*(t, x_2) = (U_1(t)/U_2(t))(x_2 - Z_2(t)) + Z_1(t)\) of hypographical line share the common direction determined by \(dU(t) = \beta(U(t))dt\). Thus, the Liggett dual is
formulated by the triplet \((U(t), Z(t), Y(t))\) of \(\mathbb{R}^2\)-valued processes on a dual state space

\[
D^* = \{(u, z, y) \in \mathbb{R}^6 : \langle [u_2, -u_1]^T, y - z \rangle > 0, |u_1| < u_2 \}.
\]

We can set a duality function \(\Gamma((u, z, y), x) = 1\) if \(\langle [u_2, -u_1]^T, y - x \rangle \geq 0\) and \(\langle [u_2, -u_1]^T, x - z \rangle > 0\); otherwise, \(\Gamma((u, z, y), x) = 0\). Here the governing SDE’s correspond to the differential operator

\[
\tag{7.1}
X(t)
\]

tractable.

Here we consider (5.3) with \(\theta = \pi/2\) in Example 5.3. Then the pair \(\partial Y^*(t) = H + Y(t)\) and \(\partial Z^*(t) = H + Z(t)\) of hypographical surfaces is determined by (7.2) and set an initial hyperplane \(\partial Y^*(0) = H_1\) which is not necessarily parallel to \(H\). Then Proposition 6.5 is still applicable but the exact stochastic process \((Z^*(t), Y^*(t))\) of Liggett dual is no longer tractable.

7. **Time-reversed Skorohod equations**

In this section we start with a construction of

\[
\tag{7.3}
L(t) = \int_0^t I_{(X_t(v) = \partial Y^*(v, X_{(-1})(v)))} dL(v),
\]

so that a solution \((L, Y^*)\) satisfies \(X(t) \in Y^*(t)\) for \(0 \leq t \leq T\).

Recall in Section 6.1 that we start from \(\tilde{Y}^*(T) = y^*\) at \(s = T\) and set \(\tilde{Y}^*(s) = \Phi_{s,T}(y^*, \tilde{W}')\) backward until \(s = 0\). Provided \(x \in \tilde{Y}^*(0)\), we can find a solution
The construction of (7.1) is given by \[ \hat{Y}, \hat{L} \] to the SDE (6.4)–(6.5) of Skorohod type. Suppose that we choose \( W'(t) = [\hat{W}_1 - \hat{L}, \hat{W}_{(-1)}(T-t)] \) starting from \( W'(0) = [\hat{W}_1 - \hat{L}, \hat{W}_{(-1)}(T)] \). Then the construction of (7.1) is given by \( X(t) = \hat{Y}(T-t) \), and the pair of \( L(t) = \hat{L}(T) - \hat{L}(T-t) \) and \( Y^*(t) = \hat{Y}^*(T-t), 0 \leq t \leq T \), satisfies (7.2)–(7.3). Thus, we can appropriately call (7.1)–(7.3) time-reversed Skorohod equations.

### 7.1. An approximation of time-reversed solution.
In order to approximate a solution to (7.2)–(7.3), we adopt Algorithm 5.4 and reverse steps in time. Here we start from \( X_N(T) = x \), and formulate recursively
\[
X_N(t) = \phi_{t-t_j, T-t}(X_N(t_j), W'(T-\cdot)) \equiv X_N(t_j) - \beta(X_N(t_j))(t_j - t) + W'(t) - W'(t_j)
\]
backward for \( t_j-1 \leq t < t_j \), \( j = N, \ldots, 1 \). In Algorithm 7.1 we build \( \sigma_N(t) \) forward, and use it to approximate \( Y_N^*(t_j) \) recursively for \( j = 1, \ldots, N \).

**Algorithm 7.1.** Set the initial value \( Y_N^*(t_0) = y^* \) and \( \sigma_N(t_0) = 0 \). At the \( j \)-th step for \( j = 1, \ldots, N \), provided \( Y_N^*(t_j-1) \) and \( \sigma_N(t_{j-1}) \), (i) set for \( t_{j-1} \leq t \leq t_j \)
\[
\sigma_N(t) = \sigma_N(t_{j-1}) - 2(W_1(t) - W_1(t_{j-1}))
\]
if \( X_{1,N}(t_j) - \beta_1(X_N(t_j))(t_j - t_{j-1}) + |W_1(t_j) - W_1(t_{j-1})| \geq \partial Y_N^*(t_{j-1}, X_{(-1),N}(t_{j-1})) \); otherwise, set
\[
\sigma_N(t) \equiv \sigma_N(t_{j-1})
\]
for \( t_{j-1} \leq t < t_j \). (ii) Set
\[
W_N(t) = [W_1 + \sigma_N, W_{(-1)}](t)
\]
for \( t_{j-1} \leq t \leq t_j \), and update
\[
Y_N^*(t_j) = \phi_{t-t_j, T-t}(Y_N^*(t_{j-1}), W_N(T-\cdot))
\]
at \( t = t_j \).

It should be noted that Algorithm 7.1 does not require
\[
X_N(0) \equiv y^*.
\]
In the setting of Algorithm 5.4 we can view \( Y_N^*(t_j) \) as if \( \hat{Y}_N^*(s_k) = Y_N^*(T - t_j) \) with \( k = N - j \) were generated by (5.5) backward for \( k = N - 1, \ldots, 0 \) with \( \hat{W}'(s) = [W_1 + \sigma_N, W_{(-1)}](T-s) \) which starts from \( \hat{W}'(0) = [W_1 + \sigma_N, W_{(-1)}](T) \), and \( X_N(t_j), t_j \leq t < t_j+1 \), as if \( Y_N^*(s_k) = X_N(T - t_j) \) were updated by (5.6) and (5.9) with \( \hat{W}(s) \) and \( \hat{\sigma}(s) = \sigma_N(T) - \sigma_N(T-s) \).

**Remark 7.2.** The update by (7.5) is determined by \( |W_1(t_j) - W_1(t_{j-1})| \) and by the values of \( X_N(t_j), X_{(-1),N}(t_{j-1}) \), and \( Y^*(t_{j-1}) \), which are generated by \( x, y^*, W_{(-1)}(t) \) on \([0, T]\), \( W_1(t) \) on \([0, t_{j-1}]\), and \( W_1(t) - W_1(t_j) \) on \([t_j, T]\). When (7.7) holds at \( t = 0 \), Algorithm 7.1 maintains \( X_N(t_j) \in Y_N^*(t_j) \) for \( j = 1, \ldots, N \), and find \( \sigma_N(t_j) \) updated by (7.5) only when \( W_1(t_j) - W_1(t_{j-1}) < 0 \) holds.

In light of Remark 7.2 a version of Lemma 5.6 can be similarly established for Algorithm 7.1.

**Lemma 7.3.** If \( W \) is a Brownian motion under \( \mathbb{W}_T \) then so is \( W_N \) of Algorithm 7.1 on \([0, T]\).
In Algorithm 7.1 we define

\[ U_N(t_j) = \partial Y^*_N(t_j, X_{(-1),N}(t_j)), \quad j = 0, \ldots, N. \]

For the next two lemmas we introduce \( Y_N(t_j) \) on the hypographical surface \( \partial Y^*_N(t_j) \) for \( j = 1, \ldots, N \). Initially we set \( Y_N(0) = [U_N(0), X_{(-1).N}(0)] \). If \( \sigma_N(t_j) \) on the \( j \)-th interval \( (t_{j-1}, t_j) \) is updated by (7.5) then we set

\[ Y_N(t_j) = \phi_{T-t_j,T-t_{j-1}}^{-1}([U_N(t_{j-1}), X_{(-1),N}(t_{j-1})], W_N(T - \cdot)) ; \]

otherwise [i.e., \( \sigma_N(t_j) \) is updated by (7.6)], set

\[ Y_N(t_j) = \phi_{T-t_j,T-t_{j-1}}^{-1}(Y_N(t_{j-1}), W_N(T - \cdot)). \]

In what follows we choose a Lipschitz constant \( K_1 \geq 1 \) for the hypographical surfaces \( \partial Y^*_N(t_j, \cdot) \)'s and for all \( N \).

**Lemma 7.4.** Suppose that (7.7) holds, and that the \( j \)-th step of Algorithm 7.1 updates \( \sigma_N \) by (7.5) and the \( k \)-th step is the next update by (7.5) after the \( j \)-th step. Then there exists a constant \( C \) such that

\[ |Y_1,N(t_k-1) - U_N(t_k-1)| - (U_N(t_k-1) - U_N(t_{j-1}))| \leq C(t_{k-1} - t_{j-1}) \]

for every \( N \) and every pair \((j, k)\).

**Proof.** Similarly to (4.7) we can formulate \( Y_N(t_{k-1}) \) by

\[ Y_N(t_{k-1}) = [U_N(t_{j-1}), X_{(-1),N}(t_{j-1})] \]

\[ + \sum_{i=j}^{k-1} \beta(Y_N(t_i))(t_i - t_{i-1}) + W_N(t_{k-1}) - W_N(t_{j-1}). \]

Then the left-hand side of (7.8) is bounded by

\[ |Y_1,N(t_{k-1}) - U_N(t_{k-1})| + \sum_{i=j}^{k-1} |\beta(Y_N(t_i))(t_i - t_{i-1})|. \]

Since \( Y_N(t_{k-1}) \in \partial Y^*_N(t_{k-1}), |Y_1,N(t_{k-1}) - U_N(t_{k-1})| \) is bounded by

\[ K_1 |Y_{(-1),N}(t_{k-1}) - X_{(-1),N}(t_{k-1})| \]

\[ \leq K_1 \left| \sum_{i=j}^{k-1} \beta_{(-1)}(Y_N(t_i))(t_i - t_{i-1}) - \sum_{i=j}^{k-1} \beta_{(-1)}(X_N(t_i))(t_i - t_{i-1}) \right|. \]

Here we assert that \( \|X_N(t_i)\| \) and \( \|Y_N(t_i)\| \) are not far apart. At the \( j \)-th update by (7.5) we can observe that

\[ 0 \leq U_N(t_{j-1}) - X_{1,N}(t_{j-1}) \leq -2(W_1(t_j) - W_1(t_{j-1})). \]

In particular, by Lemma 4.2 we can find a common upper bound for \( \max_{0 \leq i \leq N} \|X_N(t_i)\| \) and \( \max_{0 \leq i \leq N} \|Y_N(t_i)\| \) regardless of \( N \), and therefore, we can choose a constant \( C \) of (7.8) regardless of \( N \) and \((j, k)\).

**Lemma 7.5.** \{\( \sigma_N(t) \)\} is equicontinuous and uniformly bounded on \([0, T]\).

**Proof.** If (7.7) does not hold then \( \sigma_N(t) = -2W_1(t) \). In the rest of proof we assume (7.7), and set \( \delta > 0 \) and \( 0 \leq t < t' \leq T \) such that \( t' - t \leq \delta \). Then we have \( |\sigma_N(t') - \sigma_N(t)| \leq 4\Delta_\delta W_1 \) if there is no complete update by (7.5) over the interval \((t, t']\); otherwise, we find a series of updates by (7.5), say the first one at the \( k_1 \)-step.
to the last one at the $k_t$-step between $t$ and $t'$. By applying (7.8) and (7.9) from Lemma 7.4 we can find $|\sigma_N(t') - \sigma_N(t)|$ bounded by

\[
|W_{1,N}(t_{k_t-1}) - W_{1,N}(t_{k_t-1})| + 7\Delta t\delta W_1 \\
\leq |U_{1,N}(t_{k_t-1}) - U_{1,N}(t_{k_t-1})| + 7\Delta t\delta W_1 + C(t_{k_t-1} - t_{k_t-1}) \\
\leq |X_{1,N}(t_{k_t-1}) - X_{1,N}(t_{k_t-1})| + 11\Delta t\delta W_1 + C(t_{k_t-1} - t_{k_t-1})
\]

Hence, we obtain the equicontinuity by $\Delta t\sigma_N \leq \Delta tX_{1,N} + 11\Delta t\delta W_1 + C\delta$. Since $\sigma_N(0) = 0$, $\{\sigma_N\}$ must be uniformly bounded. \hfill \Box

7.2. **Uniqueness and existence of time-reversed solution.** In order to show the uniqueness of time-reversed solution to (7.2)–(7.3) we consider two strong solutions $X(t)$ and $Y(t)$ of (7.1) with distinct terminating states $X(T)$ and $Y(T)$. In the next lemma we assume a Lipschitz constant $K_1 \geq 1$ of hypographical surface, and the existence of the respective solutions $(L, Z^\ast)$ and $(M, Z)$ to (7.2)–(7.3) provided $X(0), Y(0) \in Z^\ast(0) = Z^\ast(0) = Y^\ast$. Let $U(z,t) = \Phi_{T-t}^{-1}(z, [W_1 - L, W_{-1}](T - t))$ and $V(z,t) = \Phi_{T-t,T}^{-1}(z, [W_1 - M, W_{-1}](T - t))$ be selected paths respectively for (7.2). Then we can set

\[
\gamma(t) = \sup\{|U(z,v) - V(z,v)| : z \in \partial y^\ast, 0 \leq v \leq t\}; \\
\theta(t) = \sup\{|U(z,v) - V(z,v) - [L(v),0] + [M(v),0]| : z \in \partial y^\ast, 0 \leq v \leq t\},
\]

where $[L(t),0]$ and $[M(t),0]$ are the $n$-dimensional vectors of the respective size $L(t)$ and $M(t)$ at the direction of the first coordinate.

**Lemma 7.6.** We have

\[
\gamma(t) \leq \sqrt{2}K_1e^{(1+\sqrt{2}K_1)K\beta t}\|X - Y\|_T; \\
\theta(t) \leq K\beta \int_{\partial y^\ast}^t \gamma(v)dv; \\
|L - M|_{L} \leq \sqrt{2}K_1(\theta(t) + \|X - Y\|_T).
\]

**Proof.** We obtain

\[
\|U(z,t) - V(z,t) - [L(t),0] + [M(t),0]\| = \left\| \int_0^t [\beta(U(z,v)) - \beta(V(z,v))]dv \right\| \\
\leq K\beta \int_0^t \|U(z,v) - V(z,v)\|dv \leq K\beta \int_0^t \gamma(v)dv,
\]

which implies (7.11). Since $\kappa(t) = \partial Y^\ast(t, X_{t-1}(t)) - L(t)$ and $\ell(t) = L(t)$ satisfy a Skorohod equation of (6.1)–(6.2), we can apply (6.3) as in the proof of Lemma 6.1, and show that

\[
|L - M|_{L} \leq \sup_{0 \leq v \leq t} |\partial Y^\ast(v, X_{t-1}(v)) - \partial Z^\ast(v, Y_{t-1}(v))| \\
- L(v) + M(v) - X_1(v) + Y_1(v)
\]
Furthermore, we can find $z \in \partial y^*$ such that $\partial Y^*(v, X_{-(1)}(v)) = U_1(z, v)$ and $X_{-(1)}(v) = U_{-(1)}(z, v)$. Hence we obtain

$$|\partial Y^*(v, X_{-(1)}(v)) - \partial Z^*(v, Y_{-(1)}(v)) - L(v) + M(v) - X_1(v) + Y_1(v)|$$

$$\leq |U_1(z, v) - V_1(z, v) - L(v) + M(v)| + |X_1(v) - Y_1(v)|$$

$$+ K_1(\|V_{-(1)}(z, v) - U_{-(1)}(z, v)\| + \|X_{-(1)}(v) - Y_{-(1)}(v)\|)$$

$$\leq \sqrt{2}K_1(||U(z, v) - V(z, v) - [L(v), 0] + [M(v), 0]|| + \|X(v) - Y(v)\|),$$

which implies (7.12). Together we obtain

$$\|U(z, t) - V(z, t)\| \leq |L - M| + K_1\int_0^t \|U(z, v) - V(z, v)\| dv$$

$$\leq \sqrt{2}K_1(\theta(t) + \|X - Y\|_T + \int_0^t \gamma(v) dv)$$

$$\leq \sqrt{2}K_1\|X - Y\|_T + (1 + \sqrt{2}K_1)K_1\int_0^t \gamma(v) dv,$$

which implies (7.10). □

Lemma 7.6 implies the uniqueness of time-reversed Skorohod solution. Along with Lemma 7.5 we can show Proposition 7.7 exactly in the same manner as in Proposition 6.2; thus, we omit the proof. It should be noted that a result analogous to Lemma 5.8 is needed in the proof, by which the uniform convergence of $\partial Y_N^*(t, X_{-(1)}, N(t))$ is observed for $Y_N^*(t)$ of Algorithm 7.1.

Proposition 7.7. Assuming that $X_N(0) \in y^*$, $\sigma_N$ of Algorithm 7.1 uniformly converges to $L$ of (7.2)–(7.3).

7.3. $\Lambda$-linked dual. Provided $X_N(t), S \leq t \leq T$, by (7.4) terminating at $X_N(T) = x$, we can construct a sequence $(\sigma_N, Y_N^*)$ of Algorithm 7.1 starting from $\sigma_N(S) = 0$ and $Y_N^*(S) = y^*$. Provided (7.1) for $S \leq t \leq T$ terminating at $X(T) = x$, it corresponds to a solution $(L, Y^*)$ of (7.2)–(7.3) starting from $L(S) = 0$ and $Y^*(S) = y^*$. If $X(S) \in y^* \setminus \partial y^*$ then $X_N(S) \in y^* \setminus \partial y^*$ holds for sufficiently large $N$ and $\sigma_N$ converges to $L$ uniformly on $[S, T]$ by Proposition 7.7. If $X(S) \not\in y^*$ then we find $X_N(S) \not\in y^*$ for sufficiently large $N$, and therefore, obtain $\sigma_N(t) = -2(W_1(t) - W_1(S))$ for $S \leq t \leq T$. Since $\mathbb{W}_T(X(S) \in \partial y^*) = 0$, $\sigma_N$ converges uniformly to

$$\sigma(t) = \begin{cases} L(t) & \text{if } X(S) \not\in y^*; \\ -2(W_1(t) - W_1(S)) & \text{otherwise,} \end{cases} \quad S \leq t \leq T$$

almost surely. By Lemma 7.3 we can observe the following corollary.

Corollary 7.8. Let

$$\Theta_{y^*, S, x, T}(W)(t) = \begin{cases} W'(t) & \text{if } 0 \leq t < S; \\ [W_1(t) + \sigma(t) - 2W_1(S), W_{-(1)}(t)] & \text{if } S \leq t \leq T. \end{cases}$$

Then $\Theta_{y^*, S, x, T}(W)(t), 0 \leq t \leq T$, is a Brownian motion under $\mathbb{W}_T$.

Provided $X(0) \in \Omega_T$, we can impute $W(t)$, $0 \leq t \leq T$, by setting

$$(7.13) \quad W'(t) = X(t) - X(0) - \int_0^t \beta(X(v)) dv$$
so that it satisfies $X(t) = \Phi_{0,T-t}(X(T),W'(T-\cdot))$ for $0 \leq t \leq T$. By using the construction $W'$ of (7.13) we can introduce a map $\Theta_{y^*,S,X(T),T}(W)$ of Corollary 7.8. Since $\Theta_{y^*,S,X(T),T}(W)(t)$ is uniquely determined by $S$, $y^*$, and $X(v)$, $0 \leq v \leq t$, we can define the map $\Theta_S$ from $\mathbb{F}_T$ to $\Omega_T$ by

$$\Theta_S(y^*,X) = \Theta_{y^*,S,X(T),T}(W)$$  (7.14)

Lemma 7.9. Provided $X(t) = \Phi_{0,T-t}(x,W'(T-\cdot))$ for $0 \leq t \leq T$, $\Theta_S(y^*,X)(t)$, $0 \leq t \leq T$, is a Brownian motion under $\mathbb{W}_T$.

Proof. Since $\Theta_S(y^*,X) = \Theta_{y^*,S,x,T}(W)$, the claim is merely a restatement of Corollary 7.8.

In Proposition 6.5 we have constructed a Liggett dual $\Xi_{S,t}$ to (2.2) with respect to the duality function (6.9). Together with (7.14) Proposition 3.4 is applicable.

**Proposition 7.10.** The map

$$\Psi_{S,t}((z^*,y^*),X) = \Xi_{S,t}((z^*,y^*),\Theta_S(y^*,X))$$  (7.15)

forms a stochastic flow from $\bar{D}^* \times \Omega_T$ to $\bar{D}^*$, and it is an $\Lambda$-linked dual.

Proof. (a) In Lemma 7.9 we have seen that the law of $X$ is determined by $\bar{P}_{x,T}$ of Proposition 3.4, and that $\Theta_S(y^*,X)(t)$, $0 \leq t \leq T$, is a Brownian motion under $\bar{P}_{x,T}$. (b) The cocycle property holds for (7.15); thus, $\Psi_{S,t}$ forms a stochastic flow.

By using the construction $W'$ of (7.13) we obtain

$$X(t) = \Phi_{T-t,T-S}^{-1}(X(S),W'(T-\cdot))$$

couple it with

$$Y^*(t) = \Phi_{T-t,T-S}^{-1}(y^*,\Theta_{y^*,S,X(T),T}(W)'(T-\cdot));$$

$$Z^*(t) = \psi_{T-t,T-S,y^*,T-S}^{-1}(z^*,\Theta_{y^*,S,X(T),T}(W)'(T-\cdot))$$

so that $(Z^*(t),Y^*(t)) = \Psi_{S,t}((z^*,y^*),X)$ starts from $(Z^*(S),Y^*(S)) = (z^*,y^*)$.

If $X(S) \notin y^*$ then we obtain $\Theta_{y^*,S,X(T),T}(W)(t) = W'(t)$ and $X(t) \notin Y^*(t)$ for $S \leq t \leq T$. Suppose $X(S) \in y^*$. Then $(Y^*(t))_{S \leq t \leq T}$, is a solution to the time-reversed Skorohod equations (7.2)–(7.3) along with $(L(t))_{S \leq t \leq T}$. Thus, we obtain $\Theta_{y^*,S,X(T),T}(W)(t) = |W_1 + L - 2W_1(S), W_{-1}](t)$ and $X(t) \in Y^*(t)$ for $S \leq t \leq T$. Provided $Y^*(s) = Z^*(T-s) = \Phi_{S,t-T-S}^{-1}(y^*,W')$, with $W'(t) = [W_1 + L, W_{-1}](T-s)$ for $0 \leq s \leq T-S$ in the SDE (6.4)–(6.5) of Skorohod type, the pair of $X(s) = X(T-s)$ and $L(s) = L(T) - L(T-s)$ becomes a solution. Consequently we can observe that $\tilde{X}(T-S) = \psi_{T-t,T,S,y^*,T-S}(\tilde{X}(T-t),W')$ for $S \leq t \leq T$, and therefore, that $X(t) \in Z^*(t)$ if and only if $X(S) \in z^*$. Hence, (3.7) holds.

**Example 7.11.** Continuing from Example 3.6, we can now view $D^* = (0,\infty)$ as the subclass $\{(z^*,y^*) : z^* = (-\infty,-y], y^* = (-\infty,y], y > 0\}$ of (6.10). By (6.11) we obtain a Liggett dual $\Xi_{S,t}$ with respect to the duality function of Example 3.6 by

$$\Xi_{S,t}(y,W) = y - W(t) + W(S).$$

Provided $X \in \Omega_T$, we construct (7.14) explicitly by

$$\Theta_S(y,X)(t) = \begin{cases} 
-X(t) + X(0) & \text{if } X(S) > y \text{ or } 0 \leq t < S; \\
X(t) - 2X(S) + X(0) & \text{if } X(S) \leq y \text{ and } S \leq t < \zeta; \\
X(t) - 2X(S) + X(0) - 2 \max_{\zeta \leq v \leq t} (X(v) - X(\zeta)) & \text{otherwise},
\end{cases}$$
where \( \zeta = \inf \{ v \geq S : X(v) = (X(S) + y)/2 \} \). Together we obtain (3.9) and (3.10).

8. Examples of \( \Lambda \)-linked coupling

Recall in the examples of Section 6.2 that the Liggett dual \((Z^*(t), Y^*(t))\) of Proposition 6.5 was formulated by (6.11) on the state space (6.10). In Proposition 7.10 we constructed a \( \Lambda \)-linked coupling \(((Z^*(t), Y^*(t)), X(t))\) of Proposition 3.4 on the state space

\[ E = \{(z^*, y^*), x) \in D^* \times D : x \in y^* \setminus z^* \}. \]

In this section we examine \( \Lambda \)-linked couplings by examples, and demonstrate that the intertwining dual \((Z^*(t), Y^*(t))\) is governed by the Doob \( h \)-transform (3.5) of a Liggett dual.

8.1. One-dimensional example. We complete what we started in Section 1.2 with regard to the construction (1.13) of \( \Lambda \)-linked coupling. Here we generate \( X(t) \) by (1.12) starting from an initial state \( X(0) \) randomly distributed as \( \lambda((z, y), \cdot) \) of (1.7), and impute \( \omega(t) \) from the sample path \( X(t) \). For the \( \Lambda \)-linked coupling \(((Z, Y), X)\), we can find a solution \((L, Y)\) to the SDE (1.8)–(1.9) of Skorohod type, and obtain an intertwining dual (1.13). In the next proposition we show that \((x, x) \in \partial D^* \) becomes an entrance state for the intertwining dual when a \( \Lambda \)-linked coupling \(((Z, Y), X)\) starts from \( Z(0) = Y(0) = X(0) = x \); see Remark 3.5.

**Proposition 8.1.** Let \(((Z, Y), X)\) be a \( \Lambda \)-linked coupling starting from \( Z(0) = Y(0) = X(0) = x \). Fix an arbitrary sequence \((d_N)\) of positive values converging to zero, and construct \( \Lambda \)-linked couplings \(((Z_N, Y_N), X_N)\) starting from \((Z_N(0), Y_N(0)) = (x - d_N, x + d_N)\). Then the law of \((Z_N, Y_N)\) converges weakly to that of \((Z, Y)\), and \((Z, Y)\) is a diffusion process associated with (1.6).

**Proof.** We consider a Polish space \( \mathbb{R}^N \) for a sequence \((X_N(0))\) of initial states, and construct a probability measure \( \mathbb{X} \) on \( \mathbb{R}^N \) satisfying \( \mathbb{X}(X_N(0) \in dx_N) = \lambda((x - d_N, x + d_N), x_N)dx_N \). Then we sample \( W \) from \( \mathbb{W} \) and generate \( X \) and \( X_N \) by (1.12). By applying the Gronwall’s inequality to (4.4) we obtain \( |X_N - X|_{T} = O(d_N) \), and by Lemma 7.6 we find \( |\Theta_0(x + d_N, X_N) - \Theta_0(x, X)|_{T} = O(d_N) \). Consequently we can verify \( |Y_N - Y|_{T} + |Z_N - Z|_{T} = O(d_N) \). This implies the weak convergence (cf. Section 8 of [3]). Since \((Z_N, Y_N)\) is an intertwining dual for each \( N \), it satisfies the Dynkin’s formula

\[
(8.1) \quad E_{\mathbb{X} \otimes \mathbb{W}} \left[ f(Z_N(t), Y_N(t)) - \int_0^t \mathcal{B}^* f(Z_N(t), Y_N(t)) dv \right] - f(x - d_N, x + d_N) = 0.
\]

By letting \( N \to \infty \), we obtain (8.1) for \((Z(t), Y(t))\), which completes the proof. \( \square \)

By Proposition 8.1 we can view \((Z(t), Y(t))\) as an intertwining dual starting from a point \((x, x)\) in \( \partial D^* \). Note that an SDE for \((Z(t), Y(t))\) is formulated by (1.6) using only one-dimensional Brownian motion \( W(t) \). Recall the harmonic function \( h(z, y) \) in (1.5). Then we can apply the Ito chain rule to \( h(Z(t), Y(t)) \), and obtain

\[
(8.2) \quad d(h(Z(t), Y(t))) = \frac{m(Z(t), Y(t))^2}{h(Z(t), Y(t))} dt - m(Z(t), Y(t)) dW(t).
\]
where we set \( m = \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right) h \). Let \( \tau_2 = \inf\{v \geq 0 : R(v) > t\} \) be the time change by an increasing process \( R(t) = \int_0^t m(Z(v), Y(v)) dv \). By the time scale \( \tau_2 \) we find that \( H(t) = h(Z(\tau_2), Y(\tau_2)) \) becomes a three-dimensional Bessel process satisfying \( dH(t) = \frac{dt}{H(t)} - dW(t) \). It starts from \( H(0) = 0 \), and never hits 0 for \( t > 0 \). In this sense \( \partial D^* \) is the collection of entrance states. This connection of intertwining dual to three-dimensional Bessel process was first observed by Miclo [14].

**Example 8.2.** Continue Example 1.1 in the case of \( \beta(x) \equiv \mu = 0 \), and consider a dual state space \( D^* = \{ -y, y : y > 0 \} \) with duality function \( \Gamma([-y, y], x) = 1 \) if \( x \in (-y, y) \); otherwise, \( \Gamma([-y, y], x) = 0 \). Similarly to Proposition 6.5 we obtain a Liggett dual by

\[
\Xi_{0,t}^*([-y, y], W) = \psi_{T-t,T,y,T}^{-1}([-y, y], W(T - \cdot)),
\]

where we use (6.8) by identifying \( y \) with \( y^* = (-\infty, y) \). We have (3.9) for \( X(t) = W(t) \). Starting from the entrance state \{0\}, we can construct the intertwining dual of Proposition 7.10 by

\[
(8.3) \quad \Xi_{0,t}^*(0, \Theta_0(0, W)) = [\Theta_0(0, W)(t), -\Theta_0(0, W)(t)].
\]

Define \( \text{sgn}(x) = 1 \) if \( x > 0 \); otherwise, \( \text{sgn}(x) = -1 \). By means of Tanaka’s SDE Le Jan and Raimond [11, 12] and Miclo [15] introduced a stochastic flow

\[
\psi_{0,t}(x, W) = \begin{cases} 
x + \text{sgn}(x) \int_0^t \text{sgn}(W(v)) dv & \text{if } t < \tau_x; \\
W(t) & \text{if } t \geq \tau_x,
\end{cases}
\]

where \( \tau_x = \inf \left\{ v \geq 0 : |x| + \int_0^t \text{sgn}(W(v)) dv = 0 \right\} \).

Consult [12] for the construction of \( \psi_{s,t} \) for \( 0 < s < t \leq T \). Then we can show that the law of \( \psi_{0,t}(x, W) \) is that of a Brownian motion \( X(t) = x + W(t) \) with initial state \( X(0) = x \), and that

\[
\Xi_{0,t}^*([-y, y], W) = \psi_{T-t,T,y,T}^{-1}([-y, y], W(T - \cdot))
\]

is a Liggett dual to \( X(t) \). It should be noted that an elaborate machinery of Skorohod equation in Chapter 6 is not involved at all. Furthermore, the pair of \( \Xi_{0,t}^* \) and \( \Theta_0(y, W) = W \) provides a \( \Lambda \)-linked dual of Proposition 3.4. Thus, starting from \{0\} we obtain the intertwining dual

\[
(8.4) \quad \psi_{T-t,T}^{-1}(0, W(T - \cdot)) = \left[ -\int_0^t \text{sgn}(W(v)) dv, \int_0^t \text{sgn}(W(v)) dv \right]
\]

whose law is equal to that of (8.3). The intertwining dual and the relationship of (8.4) to Pitman theorem [19] were extensively studied by Miclo [15].

### 8.2. Examples of higher dimensional case

In this section we look at Example 6.7 and 6.8 for further examples of \( \Lambda \)-linked coupling.

**Example 8.3.** In Example 6.7 we consider a dual state space

\[
D^*_\perp = \{ (u, z, y) \in D^* : 0 < u_1 < u_2 \}
\]
so that the harmonic function
\[ h(u, z, y) = \sqrt{2\pi} \int_{(u_2 z_1 - u_1 z_2)/\sqrt{2u_1 w_2}}^{(u_2 z_1 - u_1 z_2)/\sqrt{2u_1 w_2}} e^{\eta^2} d\eta \]
for the Liggett dual (6.12) is finite and strictly positive for every \((u, z, y) \in D^*_t\). Assuming \([|u_2 - u_1|^2, y - z] > 0\), we can construct a \(\Lambda\)-linked coupling \( ((U, Z, Y), X) \) starting from \((U(0), Z(0), Y(0)) = (u, z, y) \). We choose an initial state \(X(0)\) randomly according to the distribution

\[ \lambda((u, z, y), \cdot) = \frac{\Gamma((u, z, y), \cdot)}{\h(u, z, y)} \nu(\cdot) \]

and generate \(X\) by

\[ X(t) = X(0) - \int_0^t \beta(X(v)) dv + W(t). \]

Provided that \(\omega' = [-\omega_1, \omega_2]\) is imputed by

\[ \omega'(t) = W(t) - 2 \int_0^t \beta(X(v)) dv \]

and that \(U\) is obtained as a solution to \(\frac{dU}{dt} = \beta(U)\), there is a solution \((L, Y)\) to

\[ Y(t) = y + \int_0^t \beta(Y(v)) dv + \omega(t) + [L(t), 0]; \]

\[ L(t) = \int_0^t I_{\{|(u_2(v), -u_1(v))|^2, y(v) - X(v)) = 0\}} dL(v) \]

We obtain \(\Theta_0(y, X) = [\omega_1 + L, \omega_2]\) and

\[ Z(t) = z + \int_0^t \beta(Z(v)) dv + \Theta_0'(Y, X)(t) \]

with \(\Theta_0'(y, X) = [-\omega_1 + L, \omega_2]\).

The intertwining dual \((U, Z, Y)\) of Example 8.3 corresponds to the diffusion operator

\[ B^* f = B f + \frac{1}{h} \left( \frac{\partial}{\partial y_1} - \frac{\partial}{\partial z_1} \right) h \times \left( \frac{\partial}{\partial y_1} - \frac{\partial}{\partial z_1} \right) f + \frac{1}{h} \left( \frac{\partial}{\partial y_2} + \frac{\partial}{\partial z_2} \right) h \times \left( \frac{\partial}{\partial y_2} + \frac{\partial}{\partial z_2} \right) f \]

Consider the identical hypographical lines \(\partial y^* = \partial z^* = \{x \in \mathbb{R}^2 : \langle [u_2, -u_1]^T, x - y \rangle = 0\}\) for an initial state. In Example 8.3 we can choose an entrance state \((u, z, y)\) satisfying \(z \in \partial y^*\), and sample \(X(0)\) from the hypographical line \(\partial y^*\) according to the density proportional to \(\nu\). Then we can generate \(((U, Z, Y), X)\) by Example 8.3, and show that \((U, Z, Y)\) is an intertwining dual associated with \(B^*\) (cf. Proposition 8.1). By setting

\[ m = \sqrt{\left( \frac{\partial}{\partial y_1} - \frac{\partial}{\partial z_1} \right) h} + \left( \frac{\partial}{\partial y_2} + \frac{\partial}{\partial z_2} \right) h \]

we can similarly derive an SDE of the from (8.2), and demonstrate that \(h(U(t), Z(t), Y(t))\) never hits 0 for \(t > 0\).
Example 8.4. We continue to assume (5.3) from Example 6.8. Then we can introduce a harmonic function $h(z, y) = \langle d, y - z \rangle$ for the Liggett dual (6.14) on the state space (6.13). Assuming $\langle d, y - z \rangle > 0$, we can construct a $\Lambda$-linked coupling $((Z, Y), X)$ starting from $(Z(0), Y(0)) = (z, y)$ as follows: We sample $X(0)$ randomly according to the density proportional to $\Gamma((z, y), \cdot) \nu(\cdot)$, and generate $X \sim (8.5)$. Provided that $\omega' = [-\omega_1, \omega(-1)]$ is imputed by (8.6), we can find a solution $(L, Y)$ to (8.7) and

$$L(t) = \int_0^t I_{\{\langle d, Y(v) - X(v) \rangle = 0\}} dL(v),$$

and use it to obtain $Z$ by (8.8) with $\Theta_0(y, X) = [-\omega_1 + L, \omega(-1)]$.

The intertwining dual $(Z, Y)$ of Example 8.4 corresponds to the diffusion operator

$$B^* f = Bf + \frac{2d_1}{h(z, y)} \left( \frac{\partial}{\partial y_1} - \frac{\partial}{\partial z_1} \right) f.$$ 

Similarly to Example 8.3 we can set an initial point $(Z(0), Y(0))$ on the boundary (i.e., $\langle d, Y(0) - Z(0) \rangle = 0$), and sample $X(0) \in (H + Y(0))$ randomly so that $X(0) - \langle d, Y(0) \rangle d$ is distributed as the density $\nu_H(x)$ proportional to $\nu(x)$ on $H$. The generated process $(Z, Y)$ is an intertwining dual starting from an entrance state, for which we obtain an SDE of the form (8.2) with $m(Z(t), Y(t)) \equiv 2d_1$.

As indicated after Example 6.8, Proposition 7.10 is still applicable for an initial state $\partial Y^* (0) = H_1$ which is not parallel to $H$, although an intertwining dual $(Z^*, Y^*)$ becomes intractable. In practice, the input vectors $a^{(1)}, \ldots, a^{(N)}$ may span the entire space $\mathbb{R}^n$. In such cases it is no longer guaranteed that $Y^*(t) = \Phi_{T-t, T}^{-1}(Y^*(0), W(T - \cdot), 0 \leq t \leq T$, remains in a class $F_1$ of hypographic closed sets.

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References


