The elliptic stochastic quantization of some two dimensional Euclidean QFTs

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Abstract. We study a class of elliptic SPDEs with additive Gaussian noise on \( \mathbb{R}^2 \times M \), with \( M \) a \( d \)-dimensional manifold equipped with a positive Radon measure, and a real-valued non linearity given by the derivative of a smooth potential \( V \), convex at infinity and growing at most exponentially. For quite general coefficients and a suitable regularity of the noise we obtain, via the dimensional reduction principle discussed in our previous paper [8], the identity between the law of the solution to the SPDE evaluated at the origin with a Gibbs type measure on an abstract Wiener space over \( M \). The results are then applied to the elliptic stochastic quantization equation for the scalar field with polynomial interaction over \( \mathbb{T}^2 \), and with exponential interaction over \( \mathbb{R}^2 \) (known also as Höegh-Krohn or Liouville model in the literature). In particular for the exponential interaction case, the existence and uniqueness properties of solutions to the elliptic equation over \( \mathbb{R}^2+2 \) is derived as well as the dimensional reduction for the values of the “charge parameter” \( \sigma = \frac{\alpha}{\pi} \sqrt{\pi} < \sqrt{\frac{4}{4-\sqrt{3}}} \pi \simeq \sqrt{4.23\pi} \), for which the model has an Euclidean invariant, reflection positive probability measure (hence also permitting to get the corresponding relativistic invariant model on the two dimensional Minkowski space).

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1. Introduction

The Euclidean approach to quantization of relativistic non-linear wave equations requires the construction of certain probability measures supported on distributions satisfying a set of quite strong requirements among which invariance under the full group of rigid motions of the Euclidean space and reflection positivity. Such constructions have succeeded in the case where the dimension \( d \) of the Euclidean space is equal to 1, 2, 3 (that is also the dimension of the space-time of the original relativistic equation). The non-linearity is quite general if \( d = 1 \), whereas it is restricted to the derivatives of polynomials of even degree respectively suitable superpositions of trigonometric or exponential functions for \( d = 2 \), or a cubic monomial function for \( d = 3 \) (the \( \varphi^4_3 \) model). In these cases a number of interesting physical and mathematical properties of the quantized relativistic fields models have been put in evidence.

In recent years new methods coming from the study of singular stochastic partial differential equations (SSPDEs) have been developed and have opened the possibility of new constructions of such Euclidean measures (and other similar measures associated with problems coming from other areas of applications, like statistical mechanics, hydrodynamics, wave propagation in random media ...). In these approaches the relevant Euclidean invariant (probability) measures are obtained as invariant measures for certain parabolic semilinear SPDEs called stochastic quantization equations (SQEs).

Typically the measures to be constructed have the heuristic form

\[
\mu(d\varphi) = Z^{-1} e^{-S(\varphi)} D\varphi,
\]

where \( \varphi = \varphi(x) \in \mathbb{R}, x \in \mathbb{R}^d \), \( D\varphi = \Pi_{x \in \mathbb{R}^d} d\varphi(x) \) is a “flat measure”, \( Z \) is a “normalization constant”, \( S(\varphi) = S_0(\varphi) + \lambda S_{int}(\varphi) \) with \( S_0(\varphi) = \frac{1}{2} \left( \int_{\mathbb{R}^d} |\nabla \varphi(x)|^2 dx + m^2 \int_{\mathbb{R}^d} |\varphi(x)|^2 dx \right) \), and where \( \lambda \geq 0, m \geq 0 \) are parameters,

\[
S_{int}(\varphi) = \int_{\mathbb{R}^d} V(\varphi(x)) dx,
\]
The associated stochastic quantization equation (first introduced in [68]) is of the form
\[ dX_\tau = (\Delta - m^2)X_\tau \, dt - V'(X_\tau) \, dt + dW_\tau \]
where \( \tau \) is an additional “computer time”, \( \Delta \) is the Laplacian in \( \mathbb{R}^d \), \( X_\tau = X_\tau(x), \, x \in \mathbb{R}^d \), is a random variable taking values, for fixed \( \tau \), in the space of “(generalized) functions” of \( x \), \( dW_\tau(x) \) is a Gaussian white noise in \( \tau \) and \( x \), and \( V' \) in the derivative of \( V \).

Following a terminology used especially when \( V \) is a polynomial \( P \), we call the above SPDE the \( V(\phi)_d \)-stochastic quantization equation (SQE). Various detailed results have been obtained on \( P(\phi)_2 \) SQEs, see the introductions of [20] and [46, 47] for references. In particular in [36] there were obtained a strong solution and a unique ergodic measure in the case where \( \mathbb{R}^2 \) is replaced by the 2-torus \( T^2 \).

As for the case of the \( \varphi^4 \) SQE, a breakthrough for local in time solutions of the SQE on the 3-torus \( T^3 \) came from [49], see also [34, 48]. More recently still for the case of the 3-torus \( T^3 \) a construction of invariant solutions has been given in [20]. This can be looked upon as a construction of the \( \varphi^4 \)-measure on \( T^3 \), by methods different from the previous ones provided by mathematical physics approaches, see references in [44]. The methods in [20] have been simplified and extended to the case of \( \mathbb{R}^3 \) in [46], yielding a full alternative construction of the measures which qualify to be called \( \varphi^4 \)-Euclidean measures. A previous approach following [49] is in [64]. A number of open problems has been mentioned in [20, 46], the main one concerns the uniqueness of the invariant measures.

Other approaches are however possible if one understands stochastic quantization in the broader sense of associating to a given heuristic “Euclidean measure” a stochastic process having this very measure as suitable marginal. A first approach consists in constructing a jump type process via infinite dimensional quasi-regular jump-type Dirichlet forms [5, 7, 22, 23] (for such Dirichlet forms see also [6, 14]), that has by construction the given Euclidean measure as invariant measure. Another approach obtains the invariant measure of an SQE in dimension \( d \) by looking at solutions of a suitable associated elliptic SPDE in dimension \( d + 2 \) and restricting the solution to the \( d \) dimensional Euclidean space obtained sending to zero the two additional variables. This latter procedure is known in the physical literature as Parisi-Sourlas dimensional reduction and has been implemented by using algebraic methods of supersymmetry (see [67]). For the case of a regularized non-linear term in the original SQE and regularized noise it has been studied from a mathematical point of view in [58].

Elliptic stochastic PDEs related to stochastic quantization of the \( \varphi^4 \) and \( \varphi^3 \) models respectively have been discussed both on the torus and in all space in [47]. A systematic mathematical investigation of the mechanism of dimensional reduction has been initiated in [8], for the \( V(\phi)_0 \) model, for both the cases where \( V \) is a convex function and also the case where \( V \) is non-convex, where an interesting phenomenon of non-uniqueness of solutions has been put in evidence.

In fact in [8] an explicit formula for the law of the solution of a class of elliptic SPDE in \( \mathbb{R}^2 \) taken at the origin has been obtained, by means of a rigorous version of dimensional reduction. Besides proving an instance of elliptic stochastic quantization for scalar fields and for the case of underlying Euclidean dimension \( d = 0 \), it also provided a realization of the relation between a supersymmetric quantum field model and an elliptic SPDE in 2 dimensions. For other approaches to the relationship between elliptic SPDEs and stochastic quantization see [7, 28].

The present paper has two principal aims. The first one is to prove the dimensional reduction principle for elliptic SPDEs with non singular noise for equations in \( d + 2 \) dimensions with \( d > 0 \). The second is to extend the result to elliptic singular SPDE (i.e. with space-time Gaussian white noise) having applications in stochastic quantization program and analyzing the corresponding dimensional reduction with respect to removal of spatial cut-offs. We restrict our attention to singular SPDEs with \( d = 2 \) and analyze polynomial and exponential interactions. In particular, in the case of potentials of the form \( \exp(\sigma\phi)_2 \), for suitable real \( \sigma \) we are able to complete the dimensional reduction picture removing all the spatial cutoffs (i.e. in both the “fictitious” and “real” spatial variables).

Before passing to a more detailed description of our results, let us stress the importance of the \( \exp(\sigma\phi)_2 \) model (and related ones) in relativistic and Euclidean quantum field theory. The \( \exp(\sigma\phi)_2 \) model over \( \mathbb{R}^2 \) has been first introduced by Høegh-Krohn [53] (see also [11] and [77], p. 178 and 307-313) and constructed for an interaction of the form
\[ U_g(\phi) := \int_{\mathbb{R}^2} e^{\sigma g(z)} \, g(z) \, dz \, d\nu(\sigma), \]
for any positive space cut-off function \( g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) and any positive finite measure \( \nu \) with support in \((-\sqrt{4\pi}, \sqrt{4\pi})\), in the sense that the interacting measure

\[
\mu_g(d\varphi) := \frac{e^{-U_g(\varphi)} \mu_0(d\varphi)}{\int e^{-U_g(\varphi)} \mu_0(d\varphi)},
\]

for \( m > 0 \), is absolutely continuous with respect to Nelson’s free field Gaussian measure \( \mu_0 \) with mean zero and covariance \((m^2 - \Delta)^{-1}, z \in \mathbb{R}^2\) (realized as a probability measure on, for example, \( S^1(\mathbb{R}^2) \)). See [13, 53] and [77] p. 178 for further properties of the measure \( \mu_g \). This model has been discussed in the infinite volume limit \( g \to 1 \), that is unique, in [11], under the more restrictive assumption that \( \text{supp}(\nu) \subset \left(-\frac{4}{\sqrt{4\pi}}, \frac{4}{\sqrt{4\pi}}\right) \) (see also [77] pp. 307-313 and [40]).

In fact in [11] the authors prove that the measure of the \( \exp(\sigma \varphi)_2 \) model for \( |\sigma| < \sqrt{4\pi} \) (or more generally of suitable superpositions \( \int \text{Clh}(\varphi) \text{d} \nu(\sigma) \), where \( \text{supp}(\nu) \subset \left(-\frac{4}{\sqrt{4\pi}}, \frac{4}{\sqrt{4\pi}}\right) \)) satisfies all axioms of Euclidean quantum field theory and leads to relativistic quantum fields over two-dimensional Minkowski space-time satisfying all Wightman axioms with interaction and a positive, finite mass gap at the lower end of the spectrum of the corresponding Hamiltonian.

Further properties of such models were discussed in, e.g., [3, 4, 40, 43, 52, 77, 81] and [7, 12, 14, 21, 60]. The case \(|\sigma| < \sqrt{8\pi}\) was discussed by S. Kusuoka in [59] and the relation with independent work on multiplicative chaos by J. P. Kahane [57] (see also below) was pointed out by H. Sato. In [11, 13] and also [77] estimates in \( L^p \) (for the interaction in a bounded region) were given for \(|\sigma| < \sqrt{\frac{4\pi}{p-1}}\). The Gaussian character of the model for \(|\sigma|\) sufficiently large or for any \( \sigma \) if the Euclidean space dimension \( d \) satisfies \( d \geq 3 \) has been pointed out in [13] and [2]. The relevance of the \( \exp(\sigma \varphi)_2 \) model for Polyakov string theory has been discussed in [15–17] (|\sigma| < \sqrt{4\pi} corresponding to the embedding dimension \( D < 13 \)) and rediscovered more recently in connection with topics like Liouville model, quantum gravity and multiplicative chaos. Let us mention in this connection [25, 38, 69] see also [30, 37, 45, 76] (for the literature for other approaches to Liouville type models not directly connected with probabilistic methods see, e.g., [26]).

Finally a connection of the \( \exp(\sigma \varphi)_2 \) model with the irreducibility of a unitary representation of the group of mappings of a manifold into a compact Lie group has been pointed out and studied in [24], see also the recent work [1].

The \( \exp(\sigma \varphi)_2 \) model is also closely connected with the \( \sin(\sigma \varphi)_2 \) model ("Sine-Gordon equation") as discussed in [10, 12, 41]. In the latter reference all Wightman axioms are proved again in the region \(|\sigma| < \sqrt{4\pi}\).

Both in the \( \exp(\sigma \varphi)_2 \) model and the \( \sin(\sigma \varphi)_2 \) model a replacement of the \( \exp \) (respectively \( \sin \)) function by a Wick ordered version is needed (see [9]) but also suffices for the existence and non-triviality of the model for \(|\sigma| \) up to \( \sqrt{4\pi} \). For larger values of \(|\sigma|\), in the case of the \( \sin(\sigma \varphi)_2 \) model, up to \( \sqrt{8\pi}\), further renormalization by counter-terms is required (see [29]).

The study of the stochastic quantization equation associated with the latter class of models has been initiated in [9] (where strong solutions have been discussed and the necessity of renormalization has been pointed out). In the case where \( \mathbb{R}^2 \) (or \( \mathbb{T}^2 \)) is replaced by \( \mathbb{R} \) (respectively \( \mathbb{T} \)), i.e. for the model \( \exp(\sigma \varphi)_1 \), a deeper analysis is possible and has been pursued in [19], where the existence of solutions and the strong uniqueness of the invariant measure are proven (for the corresponding stochastic quantization of \( P(\varphi)_1 \) see [56]).

The case of the SQE in \( d = 2 \) and with a regularized noise on the torus (and corresponding changes in the coefficients of the stochastic quantization equation needed to keep the same invariant measure) has been discussed in [62] (see also [18] for further developments), where existence of solutions was proven using essentially the properties of the \( \exp(\sigma \varphi)_2 \) model established in [11, 40] and methods of [36]. Uniqueness problems are also discussed in [18] in conjunction with an approach using Dirichlet forms, for all \( \sigma^2 < 4\pi \), in a setting similar to [62] (see the introduction of [18] for further references).

Hairer and Shen [51] introduced powerful methods to handle the dynamical \( \sin(\sigma \varphi)_2 \) model on \( \mathbb{T}^2 \), and via regularity structures local existence is shown up to \( \sigma^2 < \frac{16}{3}\pi \). More recently these results have been extended in [35] to cover all the subcritical regime \( \sigma^2 < 8\pi \).

Concerning the exponential interaction, the work of Garban [42] appeared recently in which the author studies the SQE on the torus \( \mathbb{T}^2 \) and on the sphere \( S^2 \). After subtracting the solution to the linear equation Garban obtains an SPDE driven by a multi-fractal and intermitted multiplicative chaos. When \(|\sigma| < \sqrt{4 - 2\sqrt{3}} \sqrt{\pi} \) \( (\sqrt{4\pi}) \) (a regime correspondingly called Da Prato-Debussche phase) he shows the existence of a strong solution and the convergence of the solution to the equation with regularized noise to the singular one both on the torus \( \mathbb{T}^2 \) and on the sphere \( S^2 \) (see Theorem 1.7 and
Theorem 1.9 in [42]). The method used is based on the Besov regularity of the Gaussian multiplicative chaos (related to the theory of [57]). The paper [42] also refers to other interesting relations to quantum gravity, conformal fields theory, string theory, multiplicative chaos and exciting new results on random measures. A result on existence and uniqueness for solutions of SQE (without the proof of the convergence of the solution to the equation with regularized noise to the singular one) is also proved (Theorem 1.11 in [42]) for $|\sigma| < (4 - 2\sqrt{2}) \sqrt{\pi} (< \sqrt{4\pi})$. Moreover a comparison with the SQE for the Sine-Gordon model is provided (cfr. Section 7 in [42]).

Let us briefly review and comment the main results obtained in the present paper. In Section 2 we study the case of the following elliptic SPDE (equation (2) in Section 2)

$$(-\Delta_x + m^2 + \Sigma)(\phi)(x,z) + \mathcal{A}^2[f g \partial V(\phi)](x,z) = \xi^A(x,z)$$

on $\mathbb{R}^2 \times M$, with $M$ a $d$-dimensional manifold with or without boundary, equipped with a positive Radon measure $dz$ and a real-valued non-linearity given by the derivative $\partial V$ of a positive smooth function $V$ defined on the real line growing at most exponentially at infinity. We use the shorthand $fgV'(\phi)$ to mean the function $fg\partial V(\phi) : \mathbb{R}^2 \times M \rightarrow \mathbb{R}$ such that $(x,z) \mapsto f(x)g(z)\partial V(\phi(x,z))$ (where $x \in \mathbb{R}^2$ and $z \in M$). Additional assumptions are as follows. The function $V$ is taken to be convex or “quasi convex” (see Hypothesis QC below). $\Sigma$ and $\mathcal{A}$ are two positive self-adjoint operators acting on $L^2(M)$, $\mathcal{A}$ is bounded, $\Sigma$ is not-necessarily bounded but has a dense domain in $L^2(M)$, $\mathcal{A}$ and $\Sigma$ commute and both have completely discrete spectrum with some additional hypotheses on the eigenfunctions and the eigenvalues of $\mathcal{A}$. The noise $\xi^A$ is supposed to be Gaussian with mean zero and covariance corresponding to a regularization given by $\mathcal{A}$. Finally $f$ is a $C^2$ real-valued cut-off function on $\mathbb{R}^2$, decaying exponentially at infinity and $g$ is a smooth real-valued cut-off with compact support on $M$.

We introduce finite dimensional projections relative of (2) and prove the existence of a weak solution $\nu$ satisfying a form of the dimensional reduction principle described in our previous paper [8], extended to the $d$ dimensional case with the presence of a non-trivial operator $\Sigma$. See Theorem 8 for a precise statement.

Next we prove extensions of these results in two directions: in Section 2.2 we consider the case $M = \mathbb{R}^d$, $\Sigma = -\Delta_x$ (cfr. Theorem 16), in Section 2.3 we consider the problem of the removal of the cutoffs $f, g$ in the case where $V$ is a convex function (cfr. Theorem 18).

The first part of the present paper contains the first example of rigorous dimensional reduction for elliptic SPDE in $d + 2$ dimension with $d > 0$. Indeed, the only paper which to our knowledge study dimensional reduction from the rigorous point of view before our work was [58] whose main result is a theorem about the dimensional reduction of an integral in a space of functions in $d + 2$ variables to an integral with respect to a Gibbs type measure in a space of functions in $d$ variables. In particular the authors do not attack the problem of the relation with the elliptic SPDE. Furthermore in [58] only polynomial potentials $V$ are considered and the regularization of the noise is chosen to be compactly supported in Fourier space. Our Hypothesis QC on the potential $V$ and Hypotheses H4 and H4.1 for the regularization of the noise considered here are quite more general. Let us stress that the results obtained under these more general hypotheses use in essential way the SPDE formulation of dimensional reduction.

In Section 3 we extend our results to singular SPDEs. In particular the elliptic stochastic quantization of the exponential interaction is proven, where $V$ is a suitable renormalized exponential function $\exp(\alpha \phi - \infty)$, for $|\alpha| < \alpha_{\text{max}}$, where

$$\alpha_{\text{max}} = 4\sqrt{8 - 4\sqrt{3}\pi},$$

(1)

(corresponding to the reduced index $|\sigma| < \frac{\alpha_{\text{max}}}{2\sqrt{\pi}} = \sqrt{4(8 - 4\sqrt{3})\pi}$, and the white noise is unregularized. Existence and uniqueness is first proven for a suitable regularized model using the results of the previous sections (cfr. Theorem 21). Subsequently the removal of the regularization in the noise is achieved in the Besov space $B^{s,p}_{p,\ell}(\mathbb{R}^d)$, where $-1 < s < 0$ and $1 < p \leq 2$ are suitable constants depending only on $\alpha$. Using the existence, uniqueness and convergence results for the elliptic SPDEs over $\mathbb{R}^d$ proven in Section 2.3, the existence and uniqueness results for the elliptic stochastic quantization equation for exponential interaction follows in Theorem 25. The dimensional reduction result is then obtained (cfr. Theorem 35) with uniqueness for all $|\alpha| < \alpha_{\text{max}}$. Our last result in this section, Theorem 37, shows the convergence and uniqueness of the equation when the spatial cut-off $g$ is removed. As an easy corollary one obtains the full Euclidean invariance of the law of the solutions.

Finally in Section 4 we prove the existence for singular elliptic SPDE with Wick power non linearity in $d + 2 = 4$ dimension. Furthermore we prove a dimensional reduction principle for some weak solutions to SPDEs of this form providing the first example of elliptic stochastic quantization for the polynomial interaction model $P(\varphi)_2$. Let us stress
that for polynomial interactions the problem of uniqueness of weak solutions to the elliptic SPDE is open and this prevents a more detailed analysis of the dimensional reduction.

The results contained in this second part of the paper gives the first example of dimensional reduction for singular elliptic SPDE as originally formulated by Parisi and Sourlas in [67]. In particular with respect to [47], where the existence for singular elliptic SPDEs with polynomial type non-linearity is proven for both $d + 2 = 4$ and, with cubic type non-linearity, $d + 2 = 5$ dimension, in Section 4 we prove, at least in the $d + 2 = 4$ dimensional case, that they can be used as stochastic quantization equation for quantum field theory. However many problems still remain open, for the polynomial models, with respect to the removal of the spatial cutoffs.

Let us stress that, in Section 3, we prove existence and uniqueness of solutions for elliptic SPDE and the relation with the Liouville measure in the regime $|\sigma| < \frac{\alpha_{\text{min}}}{4\sqrt{\pi}}$. This result is achieved using in an essential way two properties of the exponential model: the fact that the Wick exponential of a distribution is a positive measure (this fact is already exploited in [42]) and the multifractality of Wick exponentials (see Lemma 24 for a precise formulation of this property).

In particular we prove the existence and uniqueness for the elliptic SPDE with Wick exponential non-linearity using the space $B_{p,p,\ell}^s(\mathbb{R}^4)$, instead of $B_{\infty,\infty,\ell}^s(\mathbb{R}^4)$, and in addition using only the Da Prato-Debbussche trick. This is possible since for any exponent $|\sigma| < \frac{\alpha_{\text{min}}}{4\sqrt{\pi}} = \sqrt{4(8 - 4\sqrt{3})\pi}$ the noise lives inside $B_{p,p,\ell}^s(\mathbb{R}^4)$ for some $-1 < s < 0$ and $1 < p \leq 2$, instead for $|\sigma| \geq (1 - 2\sqrt{2})\sqrt{\pi} = \left(\sqrt{\frac{4\pi}{\pi}} < \frac{\alpha_{\text{min}}}{2\sqrt{\pi}}\right)$, and, besides that, the Wick power of the noise is in $B_{\infty,\infty,\ell}^s(\mathbb{R}^4)$ with $s < -2$. The other novelty is the fact that we are able to solve the equation in the full space and we are then easily able to prove the Euclidean invariance of the law.

The other best result, to our knowledge and before our paper, concerning stochastic quantization of exponential interaction, is the paper [42], which studied the stochastic quantization in the parabolic setting for the charge parameter $|\sigma| < (4 - 2\sqrt{2})\sqrt{\pi} = \left(\sqrt{\frac{4\pi}{\pi}} \right)$. We think that the analytic methods, developed here in Section 3.2, joined with the probabilistic results, typical of the parabolic setting obtained in [42], can also be applied to the parabolic case for the regime $|\sigma| < \frac{\alpha_{\text{min}}}{2\sqrt{\pi}}$.

After the submission of this article for publication we became aware of [54, 55] that address the stochastic quantization of the exponential model in the parabolic setting on the two dimensional torus $T^2$ for $\sigma^2 < 4\pi$ and $\sigma^2 < 8\pi$ respectively. Let us mention also the papers, which have appeared after the submission of the present article, [66] where the stochastic quantization of the exponential model is addressed in the parabolic and hyperbolic setting on the two dimensional torus $\mathbb{T}^2$ and for $\sigma^2 < 4\pi$, and [65] where the parabolic stochastic quantization of the model related to Liouville quantum gravity on a general compact Riemannian manifold is treated in the $L^2$ regime. Our paper seems to be the only one studying the elliptic case, or more generally the exponential model on the whole $\mathbb{R}^2$.

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2. Dimensional reduction with regularized noise

2.1. Discrete spectrum

Consider the following elliptic SPDE

$$(-\Delta_x + m^2 + \mathcal{L})(\phi)(x, z) + \mathcal{A}^2[fg\partial V(\phi)](x, z) = \xi^A(x, z)$$

(2)

where $x \in \mathbb{R}^2$ and $z \in M$ (where $M$ is a $d$ dimensional manifold with or without boundary, equipped with a Radon positive measure $dz$) and $V$ is a smooth function on $\mathbb{R}$ growing at most exponentially at infinity, $\partial V$ its gradient, $\phi : \mathbb{R}^2 \times M \to \mathbb{R}$ a scalar random field, and the function $fg\partial V(\phi)$ is defined as follows

$$fg\partial V(\phi) : \mathbb{R}^2 \times M \to \mathbb{R}
\begin{align*}
(x, z) &\mapsto f(x)g(z)\partial V(\phi(x, z))
\end{align*}$$

(3)
We now list the hypotheses and assumptions on the various elements of equation (2) and of its generalization to the vector case, where \( V \) is replaced by \( V_n \) being defined on \( \mathbb{R}^n \) (when \( \mathbb{R}^n = \mathbb{R} \) hereafter instead of writing \( V_1 \) we simply write \( V \)).

**Hypothesis C.** The potential \( V_n : \mathbb{R}^n \to \mathbb{R} \) is a positive smooth function such that
\[
y \in \mathbb{R}^n \mapsto V_n(y),
\]
is strictly convex, and it and its first and second partial derivatives grow at most exponentially at infinity.

**Hypothesis QC.** The potential \( V_n : \mathbb{R}^n \to \mathbb{R} \) is a positive smooth function, such that it and its first and second partial derivatives grow at most exponentially at infinity and additionally such that there exists a function \( \mathcal{H} : \mathbb{R} \to \mathbb{R} \) with exponential growth at infinity such that we have
\[
-\langle \hat{n}, \partial V_n(y + r\hat{n}) \rangle \leq \mathcal{H}(y), \quad \hat{n} \in \mathbb{S}^{n-1}, y \in \mathbb{R}^n \text{ and } r \in \mathbb{R}^+,
\]
with \( \mathbb{S}^{n-1} \) is the \( n-1 \) dimensional sphere.

**Hypothesis Hf.** The non-negative function \( f : \mathbb{R}^2 \to \mathbb{R} \), is invariant with respect to rotations (i.e. there exists \( \tilde{f} : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( f(x) = \tilde{f}(|x|^2) \)), it has at least \( C^2 \) smoothness and in addition satisfies \( f'(x) = \tilde{f}'(|x|^2) \), it decays exponentially at infinity and fulfills \( \Delta(f) \leq b^2 \tilde{f} \) for \( b^2 \ll m^2 \) (some examples of such functions are given in [58]).

**Hypothesis Hg.** The non-negative function \( g : M \to \mathbb{R} \) is smooth and with compact support.

**Hypothesis \( L \).** The operator \( \mathcal{L} \) is closed (possibly unbounded) and defined on the vector space \( D_\mathcal{L} \subset L^2(M) \) (where \( L^2(M) \) is the space of measurable \( L^2 \) functions defined on \( M \) with respect to the measure \( dz \) on \( M \), which is dense in \( L^2(M) \). The range of \( \mathcal{L} \) is a subset of \( L^2(M) \). The operator \( \mathcal{L} \) is positive and self-adjoint and has a completely discrete spectrum. Furthermore we suppose that there exists at least an orthonormal basis \( H_1, \ldots, H_k, \ldots \) in \( L^2(M) \) composed by eigenfunctions of \( \mathcal{L} \) which are \( C^0_\text{loc}(M) \) functions (where \( C^0_\text{loc}(M) \) is the space of continuous functions \( h \) defined on \( M \) such that \( \|h(z)\|_{\text{loc}} \| < +\infty \) where \( \nu \) is a positive continuous function uniformly bounded from above). Finally we denote by \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \) the eigenvalues of \( \mathcal{L} \) corresponding to the eigenfunctions \( H_1, H_2, \ldots, H_k, \ldots \)

**Hypothesis \( A \).** The operator \( A : L^2(M) \to L^2(M) \) is a linear, injective, continuous, positive self-adjoint operator commuting with \( \mathcal{L} \). If we denote by \( \sigma_1, \ldots, \sigma_k, \ldots \) the eigenvalues of \( A \) corresponding to the basis \( H_1, \ldots, H_k, \ldots \) we assume that
\[
\sum_{k=1}^{\infty} \sigma_k < +\infty \quad \sum_{k=1}^{\infty} \sigma_k \|H_k\|^2_{L^2,M} < +\infty,
\]
where \( \| \cdot \|_{L^2,M} \) denotes the \( L^2 \)-norm in the weighted space \( C^0_\text{loc}(M) \).

**Hypothesis \( \xi \).** The noise \( \xi^A \) is Gaussian with mean zero and a covariance corresponding to a regularization provided by \( A \), namely such that if \( h_1, h_2 \in C^0_\text{loc}(\mathbb{R}^2 \times M) \) we have
\[
\mathbb{E}[\langle h_1, \xi_A^A \rangle \langle h_2, \xi_A^A \rangle] = \int_{\mathbb{R}^2 \times M} (I \otimes A)(h_1)(x,z) (I \otimes A)(h_2)(x,z) dz dx.
\]

Hereafter when \( V \) is a scalar function (i.e. \( V : \mathbb{R} \to \mathbb{R} \)) we shall use the notation \( V' := \partial V \).

**Remark 1.** When \( V_1 = V : \mathbb{R} \to \mathbb{R} \), Hypothesis QC can be reformulated as
\[
\forall V'(y + r) \leq \mathcal{H}(y),
\]
where \( r \geq 0 \).

In the rest of this Subsection 2.1 we will always assume Hypothesis Hf, Hg, H\( L \), H\( A \) and we specify the use of Hypothesis QC or C case by case. Moreover, hereafter we systematically use the following abuse of notation: we denote by \( \mathcal{L} \) and \( A \) both the operators with the same name defined on the space \( L^2(M) \) and also the operators \( I \otimes \mathcal{L} \) and \( I \otimes A \) defined on \( L^2(\mathbb{R}^2 \times M) \).

We describe now the setting for equation (2). We assume that \( \xi^A \) is defined on an abstract Wiener space (in the sense of L. Gross) \( (\mathcal{W}, \mathcal{H}, \mu^A) \) where the Cameron-Martin space is
\[
\mathcal{H} = L^2(\mathbb{R}^2) \otimes_M A(L^2(M)),
\]
(here $\otimes_H$ is the natural tensor product of Hilbert spaces) equipped with the following scalar product

$$\langle h_1, h_2 \rangle = \int_{\mathbb{R}^2 \times M} A^{-1}(h_1)(x, z) A^{-1}(h_2)(x, z) dx dz, \quad h_1, h_2 \in \mathcal{H}.$$ 

The Wiener space $\mathcal{W}$ is given by

$$\mathcal{W} = (-\Delta_x + 1)(C^0_\ell(\mathbb{R}^2) \cap W_{\ell}^{1-p}(\mathbb{R}^2)) \otimes_{e} A^{1/2}(L^2(M)),$$

where $\ell \geq 0$ and $p > 1$, $C^0_\ell(\mathbb{R}^2)$ is the space of continuous functions on $\mathbb{R}^2$ such that

$$\|k\|_{\infty, \ell} = \sup_{x \in \mathbb{R}^2} (|k(x)|(1 + |x|^2)^{-\ell/2}),$$

$W_{\ell}^{1-p}(\mathbb{R}^2)$ denotes the Sobolev space of regularity $1-$ and weight

$$r_{\ell}(x) = (1 + |x|^2)^{-\ell/2},$$

$\otimes_{e}$ denotes the injective tensor product whose norm is given by

$$\|w\|_\mathcal{W} = \sup \left\{ \left\| \sum_{i=1}^{n} \langle b, k_i \rangle_{\mathcal{O}_\ell} \langle c, h_i \rangle_{A^{1/2}(L^2(M))} \right\|, b \in B_{\mathcal{O}_\ell}^*, c \in B_{A^{1/2}(L^2(M))}^* \right\}$$

$$= \sup \left\{ \left\| \sum_{i=1}^{n} \langle b, k_i \rangle_{\mathcal{O}_\ell} h_i \right\|_{A^{1/2}(L^2(M))}, b \in B_{\mathcal{O}_\ell}^* \right\}$$

$$= \sup \left\{ \left\| \sum_{i=1}^{n} \langle c, h_i \rangle_{A^{1/2}(L^2(M))} k_i \right\|_{\mathcal{O}_\ell}, c \in B_{A^{1/2}(L^2(M))}^* \right\}$$

where we suppose that $w = \sum_{i=1}^{\infty} k_i \otimes h_i \in \mathcal{W}$, $\mathcal{O}_\ell = (-\Delta_x + 1)(C^0_\ell(\mathbb{R}^2) \cap W_{\ell}^{1-p}(\mathbb{R}^2))$, $(\cdot, \cdot)_{\mathcal{O}_\ell}$ and $(\cdot, \cdot)_{A^{1/2}(L^2(M))}$ are the natural duality of the Banach spaces $\mathcal{O}_\ell$ and $A^{1/2}(L^2(M))$ respectively and $B_{\mathcal{O}_\ell}^*$ and $B_{A^{1/2}(L^2(M))}^*$ are the unit balls of $\mathcal{O}_\ell^*$ and $(A^{1/2}(L^2(M)))^*$ respectively (i.e., the natural dual Banach spaces of $\mathcal{O}_\ell$ and $A^{1/2}(L^2(M))$; see [72] for more details). In particular if $w = \sum_{i=1}^{\infty} k_i \otimes h_i$ (where the series is supposed to converge in $\mathcal{W}$ with respect to its natural strong topology) we have

$$\|w\|_\mathcal{W} \leq \sup_{\tau \in B_{A^{-1/2}(L^2(M))}} \left\| \left( \sum_{i=1}^{\infty} \left( \int_{M} \tau(z) h_i(z) dz \right)^2 \right)^{1/2} \right\|_{B_{\mathcal{O}_\ell}^*},$$

where $B_{A^{-1/2}(L^2(M))}$ is the unit ball of $A^{-1/2}(L^2(M))$ (here we exploit the fact that $A^{-1/2}(L^2(M))$ is the dual space of $A^{1/2}(L^2(M))$ with respect to the standard scalar product of $L^2(M)$), we use Cauchy-Schwarz inequality and the fact that $A^{-1/2}(L^2(M))$ is the dual of $A^{1/2}(L^2(M))$ with respect to the scalar product of $L^2(M)$.

The measure $\mu^A$ is the centered Gaussian measure with Cameron-Martin space $\mathcal{H}$.

In order to prove that $(\mathcal{W}, \mathcal{H}, \mu^A)$ is actually an abstract Wiener space it is sufficient to prove that $\mathbb{E}[\|\xi^A\|_\mathcal{W}] < +\infty$ where $\| \cdot \|_\mathcal{W}$ denotes the norm of $\mathcal{W}$. We have that

$$\xi^A(x, z) = \sum_{k=1}^{\infty} \sigma_k \xi^k(x) H_k(z),$$

where $x \in \mathbb{R}^2$, $z \in M$, $\xi^k$ are a sequence of independent Gaussian white noises defined on $\mathbb{R}^2$ and $\sigma_k$ are defined as in Hypothesis H.A.

Using inequality (5) we have that

$$\|\xi^A\|_\mathcal{W} \leq \sup_{\tau \in B_{A^{-1/2}(L^2(M))}} \left( \sum_{k=1}^{\infty} \sigma_k \left( \int_{M} \tau(z) H_k(z) dz \right)^2 \right)^{1/2}$$

where $\tau \in B_{A^{-1/2}(L^2(M))}$.
where in the second inequality we use the fact that \( \sqrt{\sum_{k=1}^{\infty} \sigma_k \|(-\Delta_x + 1)^{-1}(\xi_k)\|_{C_0^p(\mathbb{R}^2) \cap W_0^{1,p}(\mathbb{R}^2)}} \leq \sqrt{\sum_{k=1}^{\infty} \sigma_k \|(-\Delta_x + 1)^{-1}(\xi_k)\|_{C_0^p(\mathbb{R}^2) \cap W_0^{1,p}(\mathbb{R}^2)}} \) (7)

where in the second inequality we use the fact that \( \sqrt{\sum_{k=1}^{\infty} \sigma_k \|(-\Delta_x + 1)^{-1}(\xi_k)\|_{C_0^p(\mathbb{R}^2) \cap W_0^{1,p}(\mathbb{R}^2)}} \leq \sqrt{\sum_{k=1}^{\infty} \sigma_k \|(-\Delta_x + 1)^{-1}(\xi_k)\|_{C_0^p(\mathbb{R}^2) \cap W_0^{1,p}(\mathbb{R}^2)}} \) whenever \( \tau \in B_{A^{-1/2}(L^2(M))} \).

On the other hand, by Hypothesis H.A, we obtain

\[
\mathbb{E} \left[ \sum_{k=1}^{\infty} \sigma_k \|(-\Delta_x + 1)^{-1}(\xi_k)\|_{C_0^p(\mathbb{R}^2) \cap W_0^{1,p}(\mathbb{R}^2)}^2 \right] \leq \sum_{k=1}^{\infty} \sigma_k < +\infty,
\]

(where \( \lesssim \) stands for \( \leq \) modulo a multiplicative positive constant), and thus \( \mathbb{E}[\|\xi^A\|_{W}] < +\infty \). This proves that \((W, \mathcal{H}, \mu^A)\) is an abstract Wiener space when \( \xi^A : W \to W \), defined as \( \xi^A(w) = w \), has exactly \( \mu^A \) as probability distribution.

For later use, it is important to note that \( W \) is continuously embedded in

\[ W' = (-\Delta_x + 1)(C_0^p(\mathbb{R}^2) \cap W_0^{1,p}(\mathbb{R}^2)) \otimes_\varepsilon C_{0w}^0(M), \]

since \( A^{1/2}(L^2(M)) \) is continuously embedded in \( C_{0w}^0(M) \) by Hypothesis H.A. Furthermore by definition of injective tensor product we have \( W' \subset C_{r_L(z)C_0^0}(\mathbb{R}^2 \times M) \), with \( r_L(x) \) is defined in (4), indeed

\[
\|w\|_{r_L(z)C_0^0} = \sup_{x_0 \in \mathbb{R}^2, z_0 \in M} |\langle r_L(x_0)w(z_0) \delta_{x_0} \otimes \delta_{z_0}, w \rangle| \leq \sup \left\{ \|b \otimes w\|, b \in B_{C_0^0}, c \in B_{C_0^0}(M) \right\} = \|w\|_{W'},
\]

where \( \delta_{x_0} \) and \( \delta_{z_0} \) are Dirac deltas in \( x_0 \in \mathbb{R}^2 \) and \( z_0 \in M \) respectively (see [72] Section 3.2 for the details of the proof).

Equation (2) can be written as a problem defined on the abstract Wiener space \((W, \mathcal{H}, \mu^A)\). In particular we define the map \( U : W \to \mathcal{H} \) by

\[ U(w) = A^2[f(x)g(z)V'(\mathcal{I}w)], \]

where \( w \in W, x \in \mathbb{R}^2, z \in M \), \( f \) and \( g \) as in equation (2), and \( \mathcal{I} : W \to (-\Delta_x + 1)^{-1}(W) \) is the linear operator given by \( \mathcal{I} = (-\Delta_x + m^2 + \mathcal{L})^{-1} \). We define the map \( T : W \to W \) as \( T(w) = w + U(w) \).

**Definition 2.** It is clear that if \( S \) is any measurable map \( S : W \to W \) satisfying \( T \circ S = \text{id}_W \mu^A \)-almost surely we have that \( \phi(x, z, w) = \mathcal{I}(S(w))(x, z) \) is a (strong) solution to equation (2). Furthermore if \( \nu \) is a probability law on \( W \) such that \( T_*(\nu) = \mu^A \) then a random variable \( \phi \) taking values on \( \mathcal{I}(W) \) and having the same law as \( \tilde{\nu} = T_*(\nu) \) is a (weak) solution to equation (2).

**Remark 3.** Let \( \tilde{T} : (-\Delta + 1)^{-1}(W) \to (-\Delta + 1)^{-1}(W) \) be the map defined as \( \tilde{T}(\tilde{w}) = \tilde{w} + \tilde{U}(\tilde{w}) \) where \( \tilde{w} \in (-\Delta + 1)^{-1}(W) \)

\[
\tilde{U}(\tilde{w}) = \mathcal{I}(A^2[f(x)g(z)V'(\mathcal{I}w)]).
\]

Then we have that \( \phi(x, z, w) \) is a weak solution to equation if and only if \( \tilde{T}_*(\nu_\phi) = \mathcal{I}(\mu^A) \) (where is the law of \( \phi \)). Furthermore we have that if \( S \) is a strong solution to equation (2) in the sense of Definition 2 if and only if

\[
\tilde{T}(\mathcal{I}(S((-\Delta + m^2 + \mathcal{L})\tilde{w}))) = \tilde{w}
\]

for \( \mathcal{I}_*(\mu^A) \)-almost every \( \tilde{w} \in (-\Delta + 1)^{-1}(W) \) (we implicitly use the fact that \((-\Delta + m^2 + \mathcal{L})(\tilde{w}) \in W \) for \( \mathcal{I}(\mu^A) \)-almost every \( \tilde{w} \in (-\Delta + 1)^{-1}(W) \)).
In order to solve equation (2) (or equivalently equation (9)) we need to introduce an approximation. Let $P_n$ be the orthogonal projection in $L^2(M)$ onto the finite dimensional subspace generated by $H_1, \ldots, H_n$. The restrictions $P_n|_{A[L^2(M)]}$ and $P_n|_{A^1/V[L^2(M)]}$ are the orthogonal projections on the subspace generated by $H_1, \ldots, H_n$ too. This implies also that $I \otimes P_n$ (in the following denoted also simply by $P_n$) is a continuous linear operator on $\mathcal{W}$.

Let $\phi_n$ be the solution to the following approximated equation

$$(-\Delta x + \mathcal{L} + m^2)(\phi_n)(x, z) + P_n A^2 f g V'(P_n(\phi_n)))(x, z) = \xi^A(x, z), \quad \text{(10)}$$

where $fgV'(P_n(\phi_n))$ is defined by equation (3), and let $U_n(w) = P_n(U(P_n(w)))$ and $T_n(w) = w + U_n(w)$ be the maps, analogous to $U$ and $T$, related to equation (10). Using these objects we can define weak and strong solutions to equation (10) as in Definition 2.

Let us now study equation (10). We introduce the following function $V_n : \mathbb{R}^n \to \mathbb{R}$

$$V_n(y) = \int_M g(z) V\left( \sum_{k=1}^{n} y^k H_k(z) \right) dz.$$ 

Then, since $P_n$ commutes with $\mathcal{L}$ and $\mathcal{A}$, being the projection on a subset of common eigenfunctions of both $\mathcal{L}$ and $\mathcal{A}$, we have that $\phi_n$ is of the form

$$\phi_n(x, z) = I\xi^A(x, z) + \tilde{\phi}_n(x, z) = I\xi^A(x, z) + \sum_{k=1}^{n} \tilde{\psi}_k^A(x) H_k(z),$$

where $\tilde{\psi}_k^A(x)$ solves the set of equations

$$0 = (\Delta x + \mathcal{L} + m^2) \tilde{\psi}_k^A(x) + \sum_{j=1}^{n} \tilde{\psi}_j^A(x) H_j(z) = (\Delta x + \mathcal{L} + m^2) \tilde{\psi}_k^A(x) + \sum_{j=1}^{n} \tilde{\psi}_j^A(x) H_j(z) \quad \text{(11)}$$

with $\mathcal{L} = (-\Delta x + \mathcal{L} + m^2)^{-1}, \xi^A(x) = (\sigma_k \mathcal{L} \xi^A(x))_{k=1\ldots n}, \partial_y^k$ is the partial derivatives with respect the $k$-th variables $y^k = \tilde{\psi}_k^A$, and we used the fact that $\mathcal{A}$ is self-adjoint in $L^2(M)$. If we denote by $\mathcal{A}_n$ the $n \times n$ diagonal matrix such that $\mathcal{A}_n^T = \sigma_j \delta^{ij}, i, j = 1, \ldots, n$, and by $\mathcal{L}_n$ the $n \times n$ matrix such that $\mathcal{L}_n^T = \lambda_i \delta^{ij}$ we can write equation (11) in the following way

$$(-\Delta x + \mathcal{L} + \xi^A(x)) \tilde{\psi}_n(x) + \mathcal{A}_n^2 \cdot \partial V_n(\psi_n) = \mathcal{A}_n \cdot \xi_n,$$ \quad \text{(12)}$$

where $\tilde{\psi}_n = (\tilde{\psi}_n^1, \ldots, \tilde{\psi}_n^n)$ and $\xi_n = (\xi_k(x))_{k=1\ldots n}$. Equation (12) can be reformulated defined on the abstract Wiener space $(\mathcal{W}_n, \mathcal{H}_n, \mu_n)$, where $\mathcal{H}_n$ is the Cameron-Martin space

$$\mathcal{H}_n := L^2(\mathbb{R}^n; \mathbb{R}^n),$$

with the scalar product and norm given by $\langle h, g \rangle = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \int_{\mathbb{R}^2} h^i(x)g^i(x)dx$; the Banach space $\mathcal{W}_n$ (in which $\mathcal{H}_n$ is densely embedded) is defined as

$$\mathcal{W}_n := W_{\text{loc}}^{-1,p}(\mathbb{R}^n) \cap (1 - \Delta x)(C^0(\mathbb{R}^2; \mathbb{R}^n))$$

(\text{where $p \geq 1$ is large enough,}) and $\mu_n$ is the law of the noise $\xi_n = (\sigma_1 \xi_1, \ldots, \sigma_n \xi_n)$. On $\mathcal{W}_n$ we can introduce the corresponding maps $T_n(\tilde{\psi}_n(x)) = f(x) \mathcal{A}_n^2 \cdot \partial V_n(\mathcal{I}_n(\tilde{\psi}_n(x)))$ and $\tilde{T}_n(\tilde{\psi}_n) = \tilde{\psi}_n + \mathcal{H}_n(\tilde{\psi}_n)$ where $\mathcal{I}_n$ is the matrix valued operator defined as $\mathcal{I}_n^T = \delta^{ij} \mathcal{L}_n$, and $\tilde{\psi}_n \in \mathcal{W}_n$. Using $\tilde{T}_n$ we can define the concept of strong and weak solutions to equation (12).

Remark 4. It is important to note that the relationship between $\tilde{\psi}_n$ and the strong solution $\tilde{S}_n$ (defined using the map $\tilde{T}_n$) is given by $\tilde{S}_n(\tilde{\psi}_n(x)) = \mathcal{I}_n(\tilde{S}_n(\tilde{\psi}_n(x)))$, $x \in \mathbb{R}$. Furthermore if $\tilde{n}$ is a weak solution to equation (12) (i.e. such that $\tilde{T}_n(\tilde{n}(\tilde{\psi}_n)) = \tilde{n}$) we have that $\tilde{n}$ is the law of $(-\Delta x + \mathcal{L} + \xi_n)(\tilde{\psi}_n)$. 

In the following we shall denote by $\hat{\kappa}^n$ the probability law on $\mathbb{R}^n$ such that
\[
\frac{d\hat{\kappa}^n}{dy} = \frac{\exp\left(-4\pi \left(\frac{1}{2}\left(\sigma_n^2 + A_n^{-1} \cdot y\right)^2 + V_n(y)\right)\right)}{Z_{\hat{\kappa}^n}},
\]
where $Z_{\hat{\kappa}^n}$ is a suitable constant and $dy$ is the Lebesgue measure on $\mathbb{R}^n$. Finally we define $\hat{\Upsilon}_{f,n}$ as the random variable defined on $\hat{\mathcal{W}}_n$ such that
\[
\hat{\Upsilon}_{f,n}(\hat{\omega}_n) := \exp\left(4 \int_{\mathbb{R}^2} f'(x) V_n(\mathcal{I}_{\mathcal{L}}(\hat{\omega}_n)(x))dx\right)
\]
where $\hat{\omega}_n \in \hat{\mathcal{W}}_n$. In the following we introduce the function
\[
V_n^\mathcal{A}(y) = V_n(\sigma_1 y^1, \ldots, \sigma_n y^n) = \int_{\mathcal{M}} g(z)V\left(A \left(\sum_{k=1}^n y^k H_k(z)\right)\right) dz.
\]

**Theorem 5.** If $V_n^\mathcal{A}$ satisfies Hypothesis QC then there exists at least a weak solution $\hat{v}_n$ to equation (11) such that for any bounded measurable function $F : \mathbb{R}^n \to \mathbb{R}$ we have
\[
\int_{\hat{\mathcal{W}}_n} F(\mathcal{I}_{\mathcal{L}}(\hat{\omega}_n(0))) \hat{\Upsilon}_{f,n}(\hat{\omega}_n)d\hat{\nu}_n(\hat{\omega}_n) = Z_{f,n} \int_{\mathbb{R}^n} F(y)d\hat{\kappa}^n \tag{13}
\]
where
\[
Z_{f,n} := \int_{\hat{\mathcal{W}}_n} \hat{\Upsilon}_{f,n}(\hat{\omega}_n)d\hat{\nu}_n(\hat{\omega}_n). \tag{14}
\]

**Proof.** The proof is given in [8] Theorem 1 for the case $\Sigma_n = 0$ and $A_n = I_n$. The case of generic $A_n$ can be obtained with a change of variables as it is described in Remark 5 of [8]. The case with $\Sigma_n$ a generic positive diagonal matrix is a trivial extension. \qed

It is simple to prove that if the potential $V$ satisfies Hypothesis QC then the potential $V_n^\mathcal{A}$ satisfies Hypothesis QC too, indeed the following proposition holds.

**Proposition 6.** If $V : \mathbb{R} \to \mathbb{R}$ satisfies Hypothesis QC then $V_n^\mathcal{A}$ satisfies Hypothesis QC. Finally if $V$ is strictly convex (i.e. it satisfies Hypothesis C) also $V_n^\mathcal{A}$ and $V_n$ are convex.

**Proof.** First of all we note that
\[
\partial_y V_n^\mathcal{A}(y) = \int_{\mathcal{M}} g(z)V'\left(A \left(\sum_{j=1}^n y^j H_j(z)\right)\right) A(H_k)(z) dz, \quad y \in \mathbb{R}^n.
\]
This means that if $\mathbf{n} \in \mathbb{S}^{n-1}$ we have
\[
-\mathbf{n} \cdot \partial V_n^\mathcal{A}(y + t\mathbf{n}) = -\sum_{k=1}^n n^k \partial_{y^k} (V_n^\mathcal{A})(y + t\mathbf{n})
\]
\[
= -\sum_{k=1}^n \int_{\mathcal{M}} n^k g(z)V'\left(A \sum_{j=1}^n y^j H_j(z) + tA \sum_{j=1}^n n^j H_j(z)\right) A(H_k)(z) dz
\]
\[
\lesssim \int_{\mathcal{M}} g(z)\mathfrak{d} \left(A \sum_{j=1}^n y^j H_j(z)\right) dz
\]
\[
\lesssim \mathfrak{d} \left(\|y\|_\infty \sum_{k=1}^n \sigma_k \|H_k\|_\infty \right)
\]
where we use that \( g \) has compact support, the fact that \( V \) satisfies Hypothesis QC and the fact that \( \tilde{y} \) is increasing. Since
\[
\sum_{k=1}^{+\infty} \sigma_k \left\| H_k \mathbb{I}_{g(z) \neq 0} \right\|_{\infty} \leq \left\| \frac{1}{\nu_0} \mathbb{I}_{g(z) \neq 0} \right\|_{\infty} \sum_{k=1}^{+\infty} \sigma_k \| H_k \|_{\infty, \nu_0} < +\infty,
\]
where \( \nu_0 : M \to \mathbb{R}_+ \) is the weight function in Hypothesis HA, and thus by Hypothesis HA, the thesis is proved.

Using the weak solution \( \tilde{\nu}_n \) to equation (12), given by Theorem 5, we are able to construct a weak solution to equation (10) satisfying the dimensional reduction principle.

A weak solution \( \nu_n \) to equation (10) is of the form \( \nu_n = \tilde{\nu}_n \otimes \mu_n^A \), where \( \tilde{\nu}_n \) is the law of
\[
(-\Delta_x + m^2 + \Sigma) P_n(\phi_n)(x, z) = \sum_{k=1}^{n} (-\Delta_x + m^2 + \lambda_k)(\psi_n^k)(x) H_k(z), \quad x \in \mathbb{R}^2, z \in M,
\]
on the subspace \( \text{Im}(P_n) \subset \mathcal{W} \) and \( \mu_n^A \) is the law of
\[
Q_n(\phi_n) = (-\Delta_x + m^2 + \Sigma)(I - P_n)(\phi_n)(x, z) = \sum_{k=n+1}^{+\infty} \sigma_k \xi_n^k(x) H_k(z),
\]
which is the law of the Gaussian field on \( \text{Im}(Q_n) \subset \mathcal{W} \). Using the basis \( H_1, \ldots, H_n \) we can identify the law \( \tilde{\nu}_n \) on \( \text{Im}(P_n) \) with a probability measure \( \tilde{\nu}_n \) on \( \mathcal{W}_n \). In this way it is evident that if the law \( \tilde{\nu}_n \) satisfies the dimensional reduction principle on \( \mathcal{W}_n \), then the probability law \( \nu_n = \tilde{\nu}_n \otimes \mu_n^A \) satisfies the dimensional reduction principle on \( \mathcal{W} \), since it is the tensor product of two probability laws satisfying the dimensional reduction principle.

More precisely, we consider the following abstract Wiener space \( (\mathbb{W}, \mathbb{H}, \mu^A, \xi) \) where
\[
\mathbb{W} = \mathcal{A}^{1/2}(L^2(M)),
\]
\[
\mathbb{H} = (\Sigma + m^2)^{-1/2}(\mathcal{A}(L^2(M))),(\text{with } \mathcal{A} \text{ and } \Sigma \text{ as in (2)})\text{ and } \mathbb{W} \text{ is equipped with its natural norm while on } \mathbb{H} \text{ we define the following scalar product}
\]
\[
\langle h_1, h_2 \rangle_{\mathbb{H}} = 4\pi \int_M (\Sigma + m^2)^{1/2}(\mathcal{A}^{-1}(h_1))(z)(\Sigma + m^2)^{1/2}(\mathcal{A}^{-1}(h_2))(z)dz.
\]

Thus \( \mu^A, \xi \) is the centered Gaussian measure on \( \mathbb{W} \) with Cameron-Martin space given by \( \mathbb{H} \). We now denote by \( \kappa_n \) the measure on \( \mathbb{W} \) such that
\[
\frac{d\kappa_n}{d\mu^A, \xi}(\omega) = \exp \left(-4\pi \int_M g(z)V(P_n(\omega)(z))dz\right) / Z_{\kappa_n}
\]
where \( \omega \in \mathbb{W} \) and \( Z_{\kappa_n} \) is a normalization constant.

We introduce also the following random variables
\[
\Upsilon_{f,n}(w) := \exp \left(4 \int_{\mathbb{R}^2 \times M} f'(x)g(z)V(P_n(Iw)(x, z))dx \right)
\]
\[
\Upsilon_f(w) := \exp \left(4 \int_{\mathbb{R}^2 \times M} f'(x)g(z)V((Iw)(x, z))dx \right)
\]
where \( w \in \mathcal{W} \). If we use the previous identification of \( \text{Im}(P_n) \) with \( \mathcal{W}_n \), we have that
\[
\tilde{\Upsilon}_{f,n}(P_n(w)) = \Upsilon_{f,n}(w).
\]
Using the previous observations the next proposition (expressing the dimensional reduction principle) trivially follows.
Proposition 7. If $V$ satisfies Hypothesis QC then there exists a weak solution $\nu_n$ to equation (10) such that for any bounded measurable function $F : \mathbb{W} \to \mathbb{R}$ we have that

$$\int_{\mathbb{W}} F(\mathcal{I}w(0,\cdot))\Upsilon_{f,n}(w)d\nu_n(w) = Z_{f,n} \int_{\mathbb{W}} F(\omega)d\kappa_n(\omega),$$

with $Z_{f,n}$ as in (14).

The rest of this section concerns the generalization of Proposition 7 to solutions to equation (2). In particular we denote by $\kappa$ the probability measure on $\mathbb{W}$ such that

$$\frac{d\kappa}{d\mu_{\mathbb{W}}}(\omega) = \exp\left(-4\pi \int_M g(z)V(\omega(z))dz\right)/Z_\kappa,$$

where $\omega \in \mathbb{W}$.

Theorem 8. Suppose that $V$ satisfies Hypothesis QC then there exists at least a weak solution $\nu$ to equation (2) such that for any bounded Borel measurable function $F : \mathbb{W} \to \mathbb{R}$ we have

$$\int_{\mathbb{W}} F(\mathcal{I}w(0,\cdot))\Upsilon_f(w)d\nu(w) = Z_f \int_{\mathbb{W}} F(\omega)d\kappa(\omega),$$

where $\kappa$ is defined as in (20), $\Upsilon_f$ is given by equation (18), and

$$Z_f = \int_{\mathbb{W}} \Upsilon_f(w)d\nu(w).$$

Remark 9. Since $\kappa_n$ converges weakly to $\kappa$, the right hand side of equation (19) converges to the right hand side of equation (21), as $n \to +\infty$.

In order to prove Theorem 8 we prove the following lemmas. In the space of functions $C^0(\mathbb{R}^2, \mathbb{R}^n)$ we denote by $\|\cdot\|_{C^0_{\ell,k}}$ the following norm

$$\|h\|_{C^0_{\ell,k}} = \sup_{x \in \mathbb{R}^2} \left(\sum_{j=1}^n \sigma_j^2(h^j(x))^2 r_\ell(x)^2\right),$$

where $h \in C^0(\mathbb{R}^2, \mathbb{R}^n)$ and $k, \ell \in \mathbb{R}$ ($r_\ell$ is the weight we introduced in (4)).

Lemma 10. Let $\tilde{\psi}_n$ be a solution to the equation (11) and let $\ell, \ell' \in \mathbb{R}$, then we have

$$\|\tilde{\psi}_n\|_{C^0_{\ell'}}^{-1} \lesssim \|\exp(\alpha(\mathcal{I}^\ell, \mathcal{I}^\ell')r_{\ell'}^{-1}(x))\mathbb{I}_{g(z) \neq 0} f(x)r_\ell(x)\|_\infty,$$

where $\mathbb{I}_{\ell} = \|P_n\mathcal{I}^\ell(x, z)\|_{C^0_{\ell,M}}$, $\alpha, \alpha'$ and the implicit constants do not depend on $n$.

Proof. We write $\tilde{\psi}_n = \left(\frac{\tilde{\psi}_n^k}{\sigma_k}\right)_{k=1}^n$. The equation for $\tilde{\psi}_n$ reads

$$(-\Delta x + m^2 + \lambda_k)(\tilde{\psi}_n^k)(x) + \sigma_k f(x) \int_M g(z)H_k(z)\mathcal{I}^\ell(x, z)dz = 0,$$

where $\tilde{\psi}_n(x, z)$ is related to $\tilde{\psi}_n$ by

$$\tilde{\psi}_n(x, z) = \sum_{k=1}^n \sigma_k \tilde{\psi}_n^k(x)H_k(z).$$
Putting $\tilde{\Psi}^2_n(x) = (1 + |x|^2)^{-\ell} \sum_{k=1}^n (\tilde{\psi}_k^2(x))^2$, writing $r_{\ell,\theta}(x) = (1 + |x|^2)^{-\ell}$, for some $\theta > 0$, and denoting by $\bar{x}$ the maximum of $\tilde{\Psi}_n$, we have

$$
m^2 \tilde{\Psi}^2_n(\bar{x}) \leq -\frac{1}{2} \Delta (\tilde{\Psi}^2_n)(\bar{x}) + m^2 \tilde{\Psi}^2_n(\bar{x}) + r_{\ell,\theta}(\bar{x}) \tilde{\Psi}^2_n(\bar{x}) \cdot \Sigma_n(\tilde{\psi}_n(\bar{x}))$$

$$\leq -r_{\ell,\theta} \tilde{\psi}_n \cdot \Delta \tilde{\psi}_n - r_{\ell,\theta} \sum_{k=1}^n |\nabla \tilde{\psi}_n|^2 - \left(\frac{-2|\nabla r_{\ell,\theta}|^2 + r_{\ell,\theta} \Delta r_{\ell,\theta}}{2r_{\ell,\theta}^2}\right) \tilde{\Psi}^2_n(\bar{x})$$

$$+ m^2 \tilde{\Psi}^2_n(\bar{x}) + r_{\ell,\theta} \tilde{\psi}_n \cdot \Sigma_n(\tilde{\psi}_n(\bar{x}))$$

$$\leq -f(\bar{x}) r_{\ell,\theta}(\bar{x}) \int_M g(z) \tilde{\phi}_n(x, z) V'(P_n M \xi^A(x, z) + \tilde{\phi}_n(x, z)) dz$$

$$- \left(\frac{-2|\nabla r_{\ell,\theta}|^2 + r_{\ell,\theta} \Delta r_{\ell,\theta}}{2r_{\ell,\theta}^2}\right) \tilde{\Psi}^2_n(\bar{x}),$$

where we used that

$$\nabla \left[ \nabla \left( \sum_{k=1}^n (\tilde{\psi}_k(\bar{x}))^2 \right) \right] = \nabla r_{\ell,\theta}(\bar{x}) \sum_{k=1}^n (\tilde{\psi}_k(\bar{x}))^2$$

since $\bar{x}$ is a stationary point for $\tilde{\Psi}_n$. From equation (24) if, for any fixed $\ell$, we choose a $\theta$ in such a way that

$$\left| \frac{-2|\nabla r_{\ell,\theta}|^2 + r_{\ell,\theta} \Delta r_{\ell,\theta}}{2r_{\ell,\theta}^2} \right| < m^2,$$

we obtain

$$\tilde{\Psi}^2_n(\bar{x}) \lesssim \left\| \exp(\alpha \Xi_{\ell,\theta}(x)) w^{-1}(z) \mathbb{1}_{g(z) \neq 0} f(x) (r_{\ell,\theta}(x))^{\frac{1}{2}} \right\|_{\infty} \cdot \left\| (r_{\ell,\theta}(x))^{\frac{1}{2}} \tilde{\phi}_n(x, z) \mathbb{1}_{g(z) \neq 0} \right\|_{\infty},$$

where all the constants do not depend on $n$. On the other hand

$$(r_{\ell,\theta}(x))^{\frac{1}{2}} |\tilde{\phi}_n(x, z)| \leq (r_{\ell,\theta}(x))^{\frac{1}{2}} \sum_{k=1}^n |\tilde{\psi}_k(\bar{x})| \cdot |H_k(z)|$$

$$\lesssim \left\| \frac{1}{w(z)} \mathbb{1}_{g(z) \neq 0} \mathbb{1}_{g(z) \neq 0} \tilde{\Psi}_n(\bar{x}) \sqrt{\sum_{k=1}^\infty \sigma_k \|H_k(z)w(z)\|^2_{L^2}} \right\|_{\infty},$$

where $w(z)$ is the weight function in Hypothesis H.4, for all $x \in \mathbb{R}^2$ and $z \in \text{supp}(g)$. Using Hypothesis H.4 we deduce from this that

$$\tilde{\Psi}_n(\bar{x}) \lesssim \left\| \exp(\alpha \Xi_{\ell,\theta}(x)) w^{-1}(z) \mathbb{1}_{g(z) \neq 0} f(x) (r_{\ell,\theta}(x))^{\frac{1}{2}} \right\|_{\infty}.$$

Since $\|\tilde{\psi}_n\|_{L^2_{\ell-1}} \lesssim_{\theta} \Psi_n(\bar{x})$ inequality (22) is proved. Inequality (23) follows directly from inequality (22) and equation (11).

Corollary 11. Under the hypotheses and notations of Lemma 10, if $\ell < -1$, we have that there exists an increasing continuous function $K_0 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\|(-\Delta_x + m^2 + \mathcal{L})(\tilde{\phi}_n)\|_{\mathcal{H}} \leq K_0(\Xi_{\ell,\theta}),$$

with $\Xi_{\ell,\theta}$ as in Lemma 10.

Proof. The proof consists simply in noting that

$$\|(-\Delta_x + m^2 + \mathcal{L})(\tilde{\phi}_n)\|_{\mathcal{H}} \lesssim \|(-\Delta_x + m^2 + \mathcal{L}_n)(\tilde{\psi}_n)\|_{L^2_{\ell-1}}.$$
indeed
\[
\|(-\Delta_x + m^2 + 2)(\tilde{\phi}_n)\|_{H^n}^2 = \int_{\mathbb{R}^2 \times M} \left|(-\Delta_x + m^2 + 2)(A^{-1}(\tilde{\phi}_n)(x,z))\right|^2 \, dx \, dz
\]
\[
= \sum_{k=1}^n \frac{1}{\lambda_k^2} \int_{\mathbb{R}^2} \left|(-\Delta_x + m^2 + \lambda_k)(\tilde{\psi}_n)(x)\right|^2 \, dx
\]
\[
\leq \left(\|(-\Delta_x + m^2 + \xi_n)(\tilde{\psi}_n)\|_{H^1}^2\right) \int_{\mathbb{R}^2} (1 + |x|^2)^\ell \, dx
\]
which is finite whenever \(\ell < -1\).

\begin{proof}
We write
\begin{equation}
\tag{2}
\end{equation}
\end{proof}

\begin{remark}
Since we can identify \(\xi^A\) with the identity map \(w \mapsto \xi^A(w) = w\) on \(W\), we can identify \(\Xi_{n,\ell}\) with a random variable defined as \(w \mapsto \Xi_{n,\ell}(w) = \|P_n(\xi^A(w))(x,z)\|_{C^0_\mu(\mathbb{R}^2) \odot C^0_p(M)}\).

An important consequence of Corollary 11 is the following lemma.

\begin{lemma}
Let \(K \subset W\) be a compact set, then \(\mathcal{R} = \bigcup_{n \in \mathbb{N}} T_n^{-1}(K)\) is precompact in \(W\).
\end{lemma}

\begin{proof}
We write \(T_n(\hat{\phi}, w) = \hat{\phi} + U_n(\hat{\phi} + w).\) If \(w_{n,y} \in W\) is solution to the equation \(T_n(w_{n,y}) = y\) (where \(y \in W\)) then \(\tilde{\phi}_{n,y} = w_{n,y} - y\) will be a solution to the equation \(T_n(\tilde{\phi}_{n,y}, y) = 0\). This will imply that \(\tilde{\phi}_{n,y} = \mathcal{I}(\tilde{\phi}_{n,y})\) is a solution to equation (11) for the realization of the white noise \(\xi^A(y)\). By Corollary 11 and Remark 12, we have that \(\|\tilde{\phi}_{n,y}\|_H \leq K_0(\Xi_{n,\ell}(y))\). On the other hand we have the estimate
\[
\sup_{n \in \mathbb{N}} \|\tilde{\phi}_{n,y}\|_H \leq \sup_{n \in \mathbb{N}, y \in K} K_0(\Xi_{n,\ell}(y)) \leq \sup_{y \in K} K_0(\|\xi^A(y)\|_W) = C_K < +\infty.
\]

Letting \(\bar{K} = \{h \in H, \|h\|_H \leq C_K\}\) we have that \(\bar{K}\) is compact in \(W\) since the inclusion map \(i : H \rightarrow W\) is compact. This fact implies that \(\mathcal{R} \subset K + \bar{K}\) is precompact since it is contained in the sum of two compact sets.
\end{proof}

\begin{lemma}
The family \((\nu_n)_n\) of measures such that \(T_n(\nu) = \mu^A\) is tight.
\end{lemma}

\begin{proof}
Let \(\tilde{K}\) be a compact set such that \(\mu^A(\tilde{K}) \geq 1 - \epsilon\) for a fixed \(0 < \epsilon < 1\), then, by Lemma 13, \(\mathcal{R} := \bigcup_{n \in \mathbb{N}} T_n^{-1}(\tilde{K})\) is a compact set in \(W\). This implies
\[
\nu_n(\mathcal{R}) \geq \nu_n(\bigcup_{i} T_n^{-1}(\tilde{K})) \geq \nu_n(T_n^{-1}(\tilde{K})) \geq \mu^A(\tilde{K}) \geq 1 - \epsilon,
\]
for any \(n \in \mathbb{N}\). Thus the sequence \(\nu_n\) is tight and the lemma is proven.
\end{proof}

\begin{lemma}
Suppose that \((\nu_n)_n\) is a sequence of weak solutions to equation (10) weakly converging to \(\nu\), then \(\nu\) is a solution to (2), i.e. \(T_n(\nu) = \mu^A\).
\end{lemma}

\begin{proof}
Proving the claim is equivalent to prove that for any function \(h : W \rightarrow \mathbb{R}\) which is bounded with continuous and bounded Fréchet derivative we have \(\int h \circ T \, d\nu = \int h \, d\mu^A\). In order to prove this we note that
\[
\|h \circ T(w) - h \circ T_n(w)\|_W \leq \|h\|_{C^1(W)} : \|U(w) - P_nU(P_n(w))\|_W.
\]

On the other hand we have
\[
\|U(w) - P_nU(P_n(w))\|_W \leq \sum_{k=1}^n \sigma_k \cdot \|U(w)\|_H + \sup_{y \in [w,P_n(w)]} \|\nabla U(y)\|_{L(W)} \|P_n(w) - w\|_W.
\]

Finally we observe that
\[
\|P_n(w) - w\|_W = \|(I - P_n)(T_n(w))\|_W.
\]
Let, for any fixed \( \epsilon > 0, K_\epsilon \subset \mathbb{W} \) be a compact set such that \( \nu_n(\mathbb{W}\backslash K_\epsilon), \nu(\mathbb{W}\backslash K_\epsilon) < \epsilon \), then we have

\[
\left| \int_{\mathbb{W}} (h \circ T - h \circ T_n) d\nu_n \right| \leq \int_{\mathbb{W}}|h|d\mu_A
\]

If we are able to prove that the reduction principle (whose existence is proven by Proposition 7) converges weakly, as \( n \to +\infty \), we have that the right hand side of (27) converges to 0 as \( n \to +\infty \). Since \( \nabla U(w) \) is bounded with respect to \( w \), this implies that the right hand side of equation (26), converges to 0 as \( n \to +\infty \). Exploiting this fact, the fact that \( \nu_n \) weakly converges to \( \nu \) and equation (25) we obtain that \( \int_{\mathbb{W}} h \circ T d\nu - \int_{\mathbb{W}} h d\mu_A \leq \epsilon \|h\|_{C^0(\mathbb{W})} \). Since \( \epsilon > 0 \) is arbitrary the lemma is proved.

**Proof of Theorem 8** Thanks to Lemma 15 we have that the sequence of weak solutions \( \nu_n \), satisfying the dimensional reduction principle (whose existence is proven by Proposition 7) converges weakly, as \( n \to +\infty \), to a solution \( \nu \) of the equation (2). If we are able to prove that \( \mathbb{Y}_{f,n} d\nu_n \to \mathbb{Y}_f d\nu \) weakly, then the theorem would follow.

First of all we note that \( \mathbb{Y}_{f,n}(w) = \mathbb{Y}_f(P_n(w)) \) and that \( \mathbb{Y}_f \) is a bounded Fréchet \( C^1(\mathbb{W}) \) function. We want to prove that \( \nabla \mathbb{Y}_f \) is bounded when \( w \) is in a bounded subset of \( \mathbb{W} \). We have that, for any \( h \in \mathbb{W} \),

\[
\nabla \mathbb{Y}_f(w)[h] = -\mathbb{Y}_f(w) \cdot \int_{\mathbb{R}^2 \times M} f'(x)g(z)V'(\mathbb{I}(w)(x,z))\mathbb{I}(h)(x,z)dx dz.
\]

In particular we have that

\[
\|\nabla \mathbb{Y}_f(w)\|_{\mathbb{W}^*} \leq \int_{\mathbb{R}^2 \times M} |f'(x)g(z)|\exp(\alpha\mathbb{I}(w))\|\mathbb{C}^0(\mathbb{R}^2) \otimes \mathbb{C}_w^0(M)\|_{\mathbb{R}^2}(z)dx dz,
\]

for a suitable positive constant \( \alpha \). On the other hand the linear map \( \mathbb{I} : \mathbb{W} \to \mathbb{C}^0(\mathbb{R}^2) \otimes \mathbb{C}^0_w(M) \) is continuous, which implies that if \( B \) is a bounded subset of \( \mathbb{W} \) given by \( \sup_{w \in B} \|\mathbb{I}(w)\|_{\mathbb{C}^0(\mathbb{R}^2) \otimes \mathbb{C}_w^0(M)} \), \( B \subset \mathbb{W} \). This observation proves that

\[
\sup_{w \in B} \|\nabla \mathbb{Y}_f(w)\|_{\mathbb{W}^*} < +\infty \text{ for any bounded set } B \subset \mathbb{W}.
\]

Let for any given \( \epsilon > 0, K_\epsilon \subset \mathbb{W} \) be a compact set such that \( \nu_n(\mathbb{W}\backslash K_\epsilon), \nu(\mathbb{W}\backslash K_\epsilon) < \epsilon \), then there exists a ball \( B_\epsilon \subset \mathbb{W} \) such that \( K_\epsilon \cup (\cup_{n \in \mathbb{N}} P_n(K_\epsilon)) \subset B_\epsilon \). Let \( F \) be a continuous and bounded function on \( \mathbb{W} \), then

\[
\left| \int_{\mathbb{W}} F(w)\mathbb{Y}_{f,n} d\nu_n - \int_{\mathbb{W}} F(w)\mathbb{Y}_f d\nu \right| \leq \left| \int_{\mathbb{W}} F(w)(\mathbb{Y}_f(P_n(w)) - \mathbb{Y}_f(w))d\nu_n \right|
\]
Theorem 16 does not follow directly from Theorem 8. The rest of this subsection will focus on the proof of Theorem 16.

In this section we consider the following equation

\[ (-\Delta_x - \Delta_z + m^2)(\phi)(x, z) + A^2[f g V'(\phi)](x, z) = \xi^A(x, z), \tag{28} \]

where \( x \in \mathbb{R}^2 \) and \( z \in M \) (see equation (3) for the notation \( f g V'(\phi) := f g \partial V(\phi) \)). We consider the following hypotheses on \( \mathcal{L} \) and \( \mathcal{A} \) (in addition to the previous Hypotheses \( H_f \) and \( H_g \) on the cut-offs):

**Hypothesis H\( \mathcal{L} \)1** \( \mathcal{L} = -\Delta_z \) and \( M = \mathbb{R}^d \) (for any fixed integer \( d \geq 0 \)).

**Hypothesis H\( \mathcal{A} \)1** \( \mathcal{A}(h(z')) = a h'(z') \), for \( z' \in \mathbb{R}^2 \), where \( a(z) \) is a \( C^{1+\delta/2+}(\mathbb{R}^2) \) Hölder continuous function with compact support.

**Hypothesis H\( \xi \)1** The noise \( \xi^A \) is such that if \( h_1, h_2 \in C^\infty_0(\mathbb{R}^2 \times \mathbb{R}^d) \) we have

\[ \mathbb{E}[(h_1, \xi^A)(h_2, \xi^A)] = \int_{\mathbb{R}^2 \times \mathbb{R}^d} (I \otimes \mathcal{A})(h_1)(x, z)(I \otimes \mathcal{A})(h_2)(x, z) \, dx \, dz. \]

In this subsection we assume Hypotheses H\( \mathcal{L} \)1, H\( \mathcal{A} \)1 and H\( \xi \)1 instead of the corresponding assumptions H\( \mathcal{L} \), H\( \mathcal{A} \) and H\( \xi \) respectively we had before. The assumptions H\( f \) and H\( g \) on the space cut-offs however remain.

We define the abstract Wiener space \( (\mathcal{W}, \mathcal{H}, \mu^A) \) relative to equation (28) as follows

\[ \mathcal{H} = L^2(\mathbb{R}^2) \otimes_H \mathcal{A}(L^2(\mathbb{R}^d)) \]

\[ \mathcal{W} = (-\Delta_x + 1)(C^0_0(\mathbb{R}^2) \cap W^{1-\ell}(\mathbb{R}^2)) \otimes_C C^0_{\ell}(\mathbb{R}^d) \]

where we suppose \( \ell, \ell' > 0 \). The maps \( T \) and \( U \), and so the concept of weak and strong solution to equation (28), are defined in the same way as in the previous section. We shall prove the following theorem.

**Theorem 16.** Suppose that \( V \) satisfies Hypothesis QC (in Section 2.1), then there exists a weak solution \( \nu \) to equation (28) such that for any bounded Borel measurable function \( F : \mathcal{W} \rightarrow \mathbb{R} \)

\[ \int_{\mathcal{W}} F(Iw(0)) \mathcal{Y}_f(w) \, d\nu(w) = Z_f \int_{\mathcal{W}} F(\omega) \, d\mathcal{A}(\omega), \]

as in (21).

It is clear that the conditions H\( \mathcal{L} \)1, H\( \mathcal{A} \)1 on \( \mathcal{L}, M, \mathcal{A} \) are incompatible with the previous Hypotheses H\( \mathcal{L} \) and H\( \mathcal{A} \). Thus Theorem 16 does not follow directly from Theorem 8. The rest of this subsection will focus on the proof of Theorem 16.

In order to achieve the proof we first replace \( M \) by the manifolds \( M_R = \mathbb{T}_R^d \), i.e. \( M_R \) is a torus of radius \( R > 0 \). If \( h \) is a real-valued function on \( M = \mathbb{R}^d \) with sufficiently rapid decay at infinity we can easily define a function \( h_R \) on \( M_R \) in the following way

\[ h_R(z_R) = \sum_{j \in \mathbb{Z}^d} h(z_R + Rj) \]
where \( z_R \in [-R/2, R/2]^d \). There is also a standard way of defining a function \( \tilde{h}_R \) on \( M_R \) given one on \( M \), that is the following
\[
\tilde{h}_R(z_R) = h(z_R).
\]
If the function \( h \) has support contained in \([-R/2, R/2]^d\) then \( \tilde{h}_R = h_R \).

On the other hand if we consider a function \( k : M_R \to \mathbb{R} \) we can associate with it a function \( k^p \) defined on all of \( M \) in the following way
\[
k^p(z) = k \left( z - R \left| \frac{z + R}{2} \right| \right),
\]
\( z \in M = \mathbb{R}^d \). In general we have that \( (k^p)_R = k \), and so if \( k \) has compact support contained in \([-R/2, R/2]^d\) we have \( k = (k^p)_R \). Furthermore if \( b \) is a function with compact support on \( \mathbb{R}^d \) and \( k \) is a function on \( M_R \) we have \( b_R k^p = b^k \).

Furthermore it is important to note that if \( \Phi_R \) is the Green function associated with the operator \((-\Delta_x - \Delta_z + m^2)^{-1}\) on \( \mathbb{R}^2 \times M_R \) and \( G \) is the Green function of the same operator on \( \mathbb{R}^2 \times M \) we have
\[
G_R = \Phi_R.
\]

We define the operator \( A_R \) on \( L^2(M_R) \) by \( A_R(k) = a_R k \). In this way we can define the abstract Wiener space \((\mathcal{W}_R, \mathcal{H}_R, \mu^{A^p}_R)\) as follows
\[
\mathcal{H}_R = L^2(\mathbb{R}^2) \otimes_R A_R(L^2(M_R)),
\]
\[
\mathcal{W}_R = (\Delta_x + 1)(C_0^0(\mathbb{R}^2) \cap W_1^{1, -p}(\mathbb{R}^2)) \otimes_x A^{1/2}_R(L^2(M_R))
\]
and \( \mu^{A^p}_R \) is the law of the noise \( \xi^A_R \) on \( \mathcal{W}_R \) with covariance \( A_R \).

Using the map \((\cdot)^p\) we can extend the noise \( \xi^A_R \), defined on \( M_R \), to the noise \( \xi^{A^p}_R \) defined on all of \( M \). This means that the law \( \mu^{A^p}_R \) of \( \xi^A_R \) on \( \mathcal{W}_R \), thanks to the map \((\cdot)^p\), induces a Gaussian measure \( \mu^{A^p_R}_R \) on \( \mathcal{W} \), which is the underlying probability measure of the noise \( \xi^{A^p}_R \). It is simple to prove that \( \mu^{A^p_R}_R \) weakly converges to \( \mu^A \) when \( R \to +\infty \) and the support of \( a \) is compact.

Let \( U : \mathcal{W}_R \to \mathcal{H}_R \) be the function defined by
\[
U_R(w_R) := f(x) A_R(g_R(z_R)V(\Phi_R+ w_R)) = f(x) A_R(g_R(z_R)V(\Phi_R+ w_R)).
\]
We set \( T_R(w_R) = w_R + U_R(w_R) \). The map \( U_R \) induces a map \( U^p_R \) on \( \mathcal{W} \) in the following way
\[
U^p_R(w) = f(x) A_R(g_R^p(z_R)V(\Phi^p_R+ w_R^p)),
\]
and \( T^p_R(w) = w + U^p_R(w) \). Let \( \nu_R \) be a probability law on \( \mathcal{W}_R \) such that \( T^p_R(\nu_R) = \mu^{A^p}_R \), then it induces a probability law \( \nu^p_R \) on \( \mathcal{W} \) such that \( T^p_R(\nu^p_R) = \mu^{A^p_R}_R \).

**Lemma 17.** Let \( R_n \in \mathbb{R}_+ \) be a sequence such that \( R_n \to +\infty \), then the sequence \( \nu^p_{R_n} \) is tight on \( \mathcal{W} \) and \( \nu^p_{R_n} \to \nu \), as \( n \to \infty \) and \( R_n \to \infty \), then \( T^p_{R_n}(\nu) = \mu^A \).

**Proof.** We note that the support of \( \nu^p_{R_n} \) is contained in the set of \( R_n \) periodic distributions contained in \( \mathcal{W} \) and furthermore the map \( T^p_{R_n} \) sends \( R_n \) periodic distributions into \( R_n \) periodic distributions.

Using the methods of Lemma 10 and Corollary 11, it is simple to prove that if \( y_n \in \mathcal{W} \) is an \( R_n \) periodic distribution and if \( w_{y_n} \in \mathcal{W} \) is an \( R_n \) periodic distribution such that \( w_{y_n} \in T^p_{R_n}(y_n) \) then
\[
\| w_{y_n} - y_n \|_n \leq K_{f,g}(\tilde{\Xi}_{t,t'}(y_n)),
\]
where
\[
\tilde{\Xi}_{t,t'}(y_n) = \sup_{(x,z) \in \mathbb{R}^2 \times M} |G*y_n(x,z)| r(x)r_{t'}(z),
\]
and \( K_{f,g} \) is a positive continuous function depending only on the functions \( f \) and \( g \). It is simple to prove that the map \( \tilde{\Xi}_{t,t'} \) is continuous on \( \mathcal{W} \). Using the fact that, as \( n \to \infty \) and \( R_n \to \infty \), \( \mu^{A^p_R}_R \) converges to \( \mu^A \) weakly and so the sequence \( \mu^{A^p_R}_R \) is tight, we can use the bound (29) and the same methods of Lemma 14 to prove that \( \nu^p_{R_n} \) is tight.
Suppose that \( \nu_{R_n}^p \) weakly converges to \( \nu \), we want to prove that \( T_*(\nu) = \mu^A \). Let \( F_{R_n} \) be a function of the form 
\[
F_{R_n}(w) = G((h_1, w), \ldots, (h_r, w)),
\]
where \( G: \mathbb{R}^r \to \mathbb{R} \) is a continuous and bounded function and \( h_1, \ldots, h_r \) are smooth functions with support in \((-R/2 + \tau, R/2 - \tau)^n\) and \( \tau = \text{diam}(\text{supp}(\alpha)) \). For this kind of functions we have that \( T_{R_k} F_{R_n} = F_{R_n} \circ T \) for \( k \geq n \). From this observation we get
\[
\int F_{R_n} \circ T \nu = \lim_k \int F_{R_n} \circ T \nu_{R_k}^p = \lim_k \int F_{R_n} \circ T_{R_k}^p \nu_{R_k}^p = \lim_k \int F_{R_n} \nu_{R_k}^p A_{R_k}^p = \int F_{R_n} \nu A^p,
\]
where the limit is taken for \( k \to \infty \) and \( R_k \to \infty \). Since the functions of the form \( F_{R_n} \), for \( n \in \mathbb{N} \), generate the space of all \( \mathcal{W} \) Borel measurable functions the lemma is proved. \( \square \)

**Proof of Theorem 16** By Theorem 8 there exists a probability law \( \nu_{R_n} \) on \( \mathcal{W}_{R_n} \) such that \( T_{R_n,\ast}(\nu_{R_n}) = \mu_{R_n}^{A_{R_n}} \). On the other hand this implies that \( T_{R_n,\ast}^p(\nu_{R_n}) = \mu_{R_n}^{A_{R_n}} \) and so, by Lemma 17, there exists at least one probability measure \( \nu \) on \( \mathcal{W} \) such that \( T_*(\nu) = \mu^A \) and \( \nu_{R_n}^p \to \nu \) weakly, as \( n \to \infty \).

Then, using the notations of the proof of Lemma 17, we have that for any bounded continuous function \( F_{R_n}: \mathbb{W} \to \mathbb{R} \)
\[
\int_{\mathbb{W}} F_{R_n}(G*\nu_{R_k}(0, z)) \Upsilon_f(w) \nu_{R_k}^p(w) = \int_{\mathbb{W}} F_{R_n}(\omega) \nu_{R_k}^p(\omega),
\]
where we used that
\[
\int_{\mathbb{W}} F_{R_n}(G*\nu_{R_k}(0, z)) \Upsilon_f(w) \nu_{R_k}^p(w) = \int_{\mathbb{W}} F_{R_n}(G*\nu_{R_k}(0, z)) \Upsilon_f(w) \nu_{R_k}^p(w),
\]
for \( k \geq n \) and a similar equality for \( \kappa_{R_k} \). Since \( F_{R_n}(G*) \) and \( \Upsilon_f \) are continuous on \( \mathbb{W} \), the left hand side of (30), converges to \( \int_{\mathbb{W}} F_{R_n}(\omega) \Upsilon_f(\omega) \nu(\omega) \) as \( k \to + \infty \).

Furthermore, since
\[
\frac{d\kappa_{R_k}}{d\mu^{A_{R_k}}} = Z_{R_k}^{-1} \exp \left( -4\pi \int_{\mathbb{R}^2} g(z) V(\omega(z)) dz \right)
\]
and since \( \mu_{R_k}^{A_{R_k}} \) weakly converges to \( \mu^A \), we have that \( \kappa_{R_k} \) weakly converges to \( \kappa \), as \( k, R_k \to \infty \). This proves that the right hand side of (30) converges to \( \int_{\mathbb{W}} F_{R_n}(\omega) \nu(\omega) \) as \( k, R_k \to \infty \). Since the functions of the form \( F_{R_n} \), for \( n \in \mathbb{N} \), generate the space of \( \mathbb{W} \) measurable functions, the theorem is proved. \( \square \)

### 2.3. Cut-off removal with convex potential

Hereafter we denote by \( \omega_\beta(x) \) the function
\[
\omega_\beta(x) := \exp(-\beta \sqrt{1 + |x|^2}),
\]
\( \beta > 0, x \in \mathbb{R}^2 \), and introduce the space \( \mathcal{W}_\beta \) in the following way
\[
\mathcal{W}_\beta := (-\Delta + 1) C_0^0(\mathbb{R}^2) \otimes \mathcal{A}^{1/2}(L^2(M))
\]
where \( C_0^0 \) is the space of continuous functions with respect to the weighted \( L^\infty \) norm
\[
\|g\|_{\infty, \exp_\beta} := \sup_{x \in \mathbb{R}^2} |\omega_\beta(x) g(x)|,
\]
and \( M \) as before in Section 2.1.

In this section we want to prove the following theorem.

**Theorem 18.** Suppose that \( V \) is a convex function, and suppose that \( A, \) and \( \Sigma \) satisfy Hypotheses \( \mathcal{H}A, \mathcal{H} \Sigma \) and \( \mathcal{H} \xi \) (or Hypotheses \( \mathcal{H}A_1, \mathcal{H} \Sigma_1 \) and \( \mathcal{H} \xi_1 \)) then there exists a unique strong solution \( \phi(x, z) \) to equation (2) (or to equation (28)) with \( f \equiv 1 \) taking values on \( \mathcal{W}_\beta \) (for a \( \beta \) such that \( 0 < \beta \leq \beta_0 \) and depends only on \( m^2 \)) such that for any \( \mathbb{W} \) measurable bounded function \( F \) we have
\[
\mathbb{E}[F(\phi(0, z))] = \int_{\mathbb{W}} F(\omega) d\kappa(\omega)
\]
Let \( \kappa \) as in equation (20)).

The proof is very similar to those of Theorem 8 and Theorem 16. For this reason we report here only on the main differences. First of all we need a replacement for Proposition 7.

**Proposition 19.** Let \( V \) (and so \( V_n \)) be a convex function, then under assumptions \( \mathcal{H}_\Sigma, \mathcal{H}_A, \mathcal{H}_\Sigma \) there exists a unique strong solution \( \phi_n \) to equation (10) such that for all \( \mathbb{W} \) measurable bounded continuous function \( F \) we have

\[
\mathbb{E}[F(\phi_n(0, z))] = \int_{\mathbb{W}} F(\omega) d\kappa_n(\omega)
\]

where \( \kappa_n \) is given by expression (16).

**Proof.** The proof is given in [8], Theorem 2 in the case \( \mathcal{L}_n = 0 \). The case considered here is a trivial extension.

In order to pass from equation (33) to equation (32) we need a generalization of Lemma 10. We denote by \( \| \cdot \|_{A}^{\mathcal{U}, \beta, \ell} \) respectively \( \| \cdot \|_{A}^{\mathcal{U}, \beta, \ell} \) the following norms

\[
\|h\|_{A}^{\mathcal{U}, \beta, \ell} := \sup_{x \in \mathbb{R}^2} \sqrt{\sum_{j=1}^n \sigma_j^2(h_j(x))^2 \omega_\beta(x)^2},
\]

\[
\|h\|_{A}^{\mathcal{U}, \beta, \ell} := \sup_{x \in \mathbb{U}} \sqrt{\sum_{j=1}^n \sigma_j^2(h_j(x))^2}.
\]

**Lemma 20.** There exists a number \( \beta_0 > 0 \) (depending on \( m^2 \)) such that for any \( 0 < \beta \leq \beta_0 \), and for any open bounded set \( \mathbb{U} \subset \mathbb{R}^2 \) and under Hypotheses \( \mathcal{C}, \mathcal{H}_g, \mathcal{H}_A, \mathcal{H}_\Sigma, \mathcal{H}_\Sigma \) and \( f \equiv 1 \) we have

\[
\| \psi_n \|_{A}^{\mathcal{U}, \beta, -1} \lesssim \| \exp(\alpha E \omega^{-1} \phi_{\beta}(x)) \|_{\infty}
\]

\[
\|(-\Delta + m^2 + \mathcal{L}_n) \psi_n\|_{A}^{\mathcal{U}, -1} \lesssim \| \exp(\alpha' E \omega^{-1} \phi_{\beta}(x)) \psi_n \|_{4}^{\mathcal{U}, \beta,-1} \omega_{\beta}(x) \|_{C^n(\mathbb{U})}
\]

where \( \mathcal{L}_n = \mathcal{U} \times \{ g(z) \neq 0 \} \subset \mathbb{R}^2 \times M \), uniformly in \( n \) (where \( \mathcal{C}_n \) is defined as in Lemma 10).

**Proof.** The proof is verbatim the same as for Lemma 10 where we replace the function \( r_{\ell, \theta}(x) = (1 + \theta|x|^2)^{-\ell} \) by the function \( \omega_\beta \), defined by (31), and we use the fact for \( \beta \) small enough we have

\[
\frac{-2|\nabla \omega_\beta|}{2 \omega_\beta} < m^2.
\]

The inequality (35) implies that, for any bounded open subset \( \mathbb{U} \) of \( \mathbb{R}^2 \), we have

\[
\|(-\Delta + m^2 + \mathcal{L}(\omega_\beta))\|_{\mathcal{H}^\mathcal{U}} \leq K_U(\mathcal{C}_n)
\]

where \( \| \cdot \|_{\mathcal{H}^\mathcal{U}} \) is the natural norm of the Hilbert space \( \mathcal{H}^\mathcal{U} = L^2(\mathcal{U}) \otimes \mathcal{A}(L^2(M)) \), and \( K_U \) is a positive increasing continuous function depending only on \( \mathcal{U} \). Inequality (36) guarantees us enough compactness to generalize Lemma 13, Lemma 14 and Lemma 15 in order to prove the existence of a weak solution to equation (2) satisfying the dimensional reduction principle when \( f \equiv 1 \) and under Hypotheses \( \mathcal{C}, \mathcal{H}_g, \mathcal{H}_A, \mathcal{H}_\Sigma, \mathcal{H}_\Sigma \). We can generalize the described result under Hypotheses \( \mathcal{C}, \mathcal{H}_g, \mathcal{H}_A, \mathcal{H}_\Sigma, \mathcal{H}_\Sigma \) using a similar strategy.

It remains to prove the uniqueness of the solution to equation (2) (or (28)) when \( V \) is convex. This can be done using the methods of Lemma 10 and Lemma 20. We give here only a sketch of the proof.

Let \( \phi_1 \) and \( \phi_2 \) be two strong solutions to equation (2). It is simple to prove that \( \phi_1(x, \cdot) - \phi_2(x, \cdot) \in \mathcal{A}(L^2(M)) \), so the following function is well defined

\[
\Psi_\beta(x)^2 = \omega_{2\beta}(x) \int_M (\mathcal{A}^{-1}(\phi_1(x, z') - \phi_2(x, z')))^2 dz,
\]

where \( \omega_{2\beta} \) is defined as in equation (31). The function \( \Psi_\beta^2 \) belongs to \( C^2(\mathbb{R}^2) \) and goes to zero at infinity. This means that the maximum is realized in at least one point \( \bar{x} \in \mathbb{R}^2 \). Making some computations similar to the ones of Lemma 10 we obtain

\[
m^2 \Psi_\beta(\bar{x})^2 \leq - \int_M g(z)(\phi_1(\bar{x}, z) - \phi_2(\bar{x}, z))(V'(\phi_1(\bar{x}, z)) - V'(\phi_2(\bar{x}, z))) dz
\]

\[
- \left( \frac{-2|\nabla \omega_\beta|^2}{2 \omega_\beta} \right) \Psi_\beta(\bar{x})^2.
\]
On the other hand there exists a point \( \theta_{x,z} \in [\phi_1(x, z), \phi_2(x, z)] \) such that
\[
V'((\phi_1(x, z)) - V'(\phi_2(x, z)) \leq V''(\theta_{x,z})(\phi_1(x, z) - \phi_2(x, z)).
\]
Choosing \( \beta > 0 \) small enough we obtain
\[
\sup_{x \in \mathbb{R}^d} \Psi_\beta(x)^2 \leq - \int_M g(z) V''(\theta_{x,z})(\phi_1(x, z) - \phi_2(x, z))^2 dz \leq 0,
\]
since \( V \) is convex. This implies that \( \Psi_\beta(x) \equiv 0 \) and so that \( \phi_1(x, z) = \phi_2(x, z) \).

3. The exponential interaction on \( \mathbb{R}^2 \)

In this section we want to consider the following elliptic SPDE
\[
(-\Delta_x - \Delta_z + m^2)(\phi) + g(z)\alpha \exp(\alpha \phi - \infty) = \xi \tag{37}
\]
where \( x \in \mathbb{R}^2, z \in M = \mathbb{R}^2, \xi = \xi(x, z) \) is a standard Gaussian white noise on \( \mathbb{R}^4, |\alpha| < 4\sqrt{2}\pi, m > 0, g : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a non-negative smooth function with compact support, and where \( -\infty \) means that the equation should be properly renormalized. In order to give a meaning to the previous equation we formally subtract from the solution \( \phi \) the solution to the linear equation (i.e. equation (37) with \( g = 0 \)) which means that we consider the equation for the unknown \( \bar{\phi} \):
\[
(-\Delta_x - \Delta_z + m^2)(\bar{\phi}) + g(z)\alpha \exp(\alpha \bar{\phi})\eta(x, z) = 0, \tag{38}
\]
where \( \eta(x, z) = \exp(\alpha \bar{\phi}) \) is a renormalized version of the distribution \( \exp(\alpha \bar{\phi}) \). It is well known that \( \bar{\xi} \in B^{\delta} \) for any \( 1 \leq p \leq \infty, \delta > 0 \) and \( \ell > 0 \) (see e.g. [47]).

In the following we shall take \( d = 2 \) and give a rigorous meaning to equation (38) (and so to equation (37)) when the exponent \( |\alpha| < \alpha_{\max} \) (see equation (1) for the definition \( \alpha_{\max} \)) and, when \( \alpha \bar{\phi} \leq 0 \), and prove that there exists only one solution to equation (37).

Furthermore we want to prove that dimensional reduction holds for the unique solution to equation (37), namely that, if we consider the measure \( \kappa_s \) given by
\[
\frac{d\kappa_s}{d\mu}(\omega) = \exp \left( -4\pi \int_{\mathbb{R}^2} g(z) \exp(\alpha \omega)(dz) \right), \tag{39}
\]
where \( \omega \in B^{\delta} \) and \( \mu \) is the law of \( (-\Delta_x + m^2)^{-1/2}(\xi) \) on \( B^{\delta} \), we have
\[
\mathbb{E}[F(\phi(0, \cdot))] = \int F(\omega) d\kappa_s(\omega),
\]
for any bounded measurable function \( F \) defined on \( B^{\delta} \).

In order to prove the existence and uniqueness of the solution \( \bar{\phi} \) to equation (38) and dimensional reduction (39) for the random field \( \phi = \bar{\phi} + \bar{\xi} \), we need to introduce the following two approximate equations
\[
(-\Delta_x - \Delta_z + m^2)(\phi_x) + a^{*2}g(z)\alpha \exp(\alpha \phi_x - C_\varepsilon) = a^{\varepsilon} \bar{\xi} \tag{40}
\]
\[
(-\Delta_x - \Delta_z + m^2)(\phi_x) + a^{*2}g(z)\alpha \exp(\alpha \phi_x)\eta_x(x, z) = 0, \tag{41}
\]
where \( x, z \in \mathbb{R}^2, a \) is a positive function satisfying Hypothesis HA1 (see Section 2.2) and \( C_\varepsilon := \frac{\alpha^2}{\varepsilon} \mathbb{E}[I(a^{\varepsilon} \bar{\xi})], \bar{\phi}_x := \phi_x - I(a^{\varepsilon} \bar{\xi}), \eta_x \) is the positive measure defined as
\[
\eta_x(dx, dz) := \exp(\alpha I(a^{\varepsilon} \bar{\xi})) dz = \exp(\alpha I(a^{\varepsilon} \bar{\xi}) - C_\varepsilon) dz.
\]
The constant \( C_\varepsilon \) is chosen in such a way that \( \eta_x \rightarrow \eta \) in \( B^{\delta} \) for suitable \( s < 0, 1 < p \leq 2 \) and \( \ell > 0 \) (see Section 3.1).

For equations (40) and (41) we have the following fundamental dimensional reduction result.
Theorem 21. Equation (41) admits a unique solution in $C^0_\ell (\mathbb{R}^4)$ for $\ell$ big enough. Furthermore we have that, for any $\epsilon > 0$ and any bounded continuous function $F$

$$
\mathbb{E}[F(\phi_\epsilon(0, z))] = \int_{C^0_\ell (\mathbb{R}^2)} F(\omega) d\mu_\epsilon(\omega), \quad (42)
$$

for all $z \in \mathbb{R}^2$, where

$$
\frac{d\mu_\epsilon}{d\mu_\epsilon} = \exp \left( -4\pi \int_{\mathbb{R}^2} g(z) \exp(\alpha \omega(z) - C_\epsilon) dz \right)
$$

and $\mu_\epsilon$ is the Gaussian measure on $C^0_\ell$ with covariance $\mathbb{E}[\omega(z)\omega(z')] = \alpha^2_r \mathcal{G}_z(z - z')$, where $\mathcal{G}_z$ is the Green function of the operator $(-\Delta_z + m^2)$. Finally the unique solution to equation (41) satisfies $\alpha \phi_\epsilon \leq 0$.

Proof. The proof is an application of Theorem 16 to equation (40) using the fact that $\exp(\alpha y - C_\epsilon)$ satisfies Hypothesis $C$ in Section 2.

The fact that $\alpha \phi_\epsilon \leq 0$ follows from an application of the maximum principle to the function $\hat{\phi}(\hat{\lambda})$ where $\hat{\lambda} = (\lambda, \hat{\xi})$ in $\mathbb{R}^4$ and $\lambda > 0$ small enough.

In order to prove existence, uniqueness and the reduction principle for equation (37) we have now to study the behavior of the regularized noises $\eta_\epsilon$ and their convergence to $\eta$ in the Besov spaces of the form $B^s_{p,p,\ell}$, as $\epsilon \to 0$. Once we have established this convergence, we can give a meaning to equation (38) and we are able to prove that there exists a subsequence of $\epsilon_n \to 0$ such that $\phi_{\epsilon_n} \to \phi$ in probability in $B^s_{p,p,\ell}$, where $\phi$ solves (37). This fact will permit us to prove equality (37).

3.1. Probabilistic analysis

In this subsection we propose an analysis of the regularity of the noises $\eta$ and $\eta_\epsilon$, for $\epsilon > 0$. First of all we note that

$$
\eta = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} (\mathcal{I} \xi)^{\otimes k}, \quad (43)
$$

$$
\eta_\epsilon = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} (\mathcal{I} \xi_\epsilon)^{\otimes k}, \quad (44)
$$

where $\mathcal{I}$ is the integral operator defined in Section 2,

$$(\mathcal{I} \xi)^{\otimes k} = \mathcal{I} \xi \circ \ldots \circ \mathcal{I} \xi, \quad \text{times}$$

the symbol $\circ$ denotes the Wick product, $\xi_\epsilon = \alpha_{\epsilon} \ast \xi$ and $\mathcal{I} \xi_\epsilon$ is defined correspondingly. The previous expressions are well defined as $L^2(\mu)$ convergent series for $|\alpha| < 4\sqrt{2\pi}$. Indeed we have that for any smooth function $g$ exponentially decaying at infinity

$$
\mathbb{E}[|\langle \eta_\epsilon, g \rangle|^2] = \int_{\mathbb{R}^8} g(\hat{\xi}) g(\hat{\xi}') \exp(\alpha^2 \mathcal{G}_\epsilon(\hat{\xi} - \hat{\xi}')) d\hat{\xi} d\hat{\xi}', \quad (45)
$$

where, hereafter, we write $\mathcal{G}_\epsilon = \alpha_{\epsilon}^2 \ast \mathcal{G}$, where $\mathcal{G}$ is the Green function associated with the operator $(-\Delta_z + m^2)^{-2}$, $\hat{\xi} = (\xi, z) \in \mathbb{R}^4$. It is well known (see Proposition A.1 of [7]) that for $\hat{\xi} \in \mathbb{R}^4$ such that $|\hat{\xi}| \leq 1$ there exists a constant $C_1 > 0$ for which the following inequality holds

$$
\mathcal{G}(\hat{\xi}) \leq -\frac{2}{(4\pi)^2} \log_+(|\hat{\xi}|) + C_1. \quad (46)
$$

Furthermore for $|\hat{\xi}| \geq 1$ there exists two constants $C_2, C_3 > 0$ for which we get

$$
\mathcal{G}(\hat{\xi}) \leq C_2 \exp(-C_3|\hat{\xi}|). \quad (47)
$$
An easy consequence of the inequalities (46) and (47) are the following inequalities
\[
\alpha_2^2 \mathcal{G}(\hat{z}) \leq -\frac{2}{(4\pi)^2} \log_+ (|\hat{z}|) + C_4, \tag{48}
\]
when \(4 \text{diam(supp}(a))\epsilon \leq |\hat{z}| \leq 1\) and for some constant \(C_4 > 0\);
\[
\alpha_2^2 \mathcal{G}(\hat{z}) \leq -\frac{2}{(4\pi)^2} \log(\epsilon) + C_5, \tag{49}
\]
when \(|\hat{z}| < 2 \text{diam(supp}(a))\epsilon\) and for some constant \(C_5 > 0\); and finally
\[
\alpha_2^2 \mathcal{G}(\hat{z}) \leq C_6 \exp(-C_7|\hat{z}|) , \tag{50}
\]
when \(|\hat{z}| \geq 1\) and for some constants \(C_6, C_7 > 0\). It is important to note that the constants \(C_4, C_5, C_6, C_7\) are independent of \(\epsilon\). The previous inequalities and eq. (45) imply that \(\mathbb{E}[|\langle \eta, g \rangle|^2] < +\infty\) for any \(\epsilon \geq 0\) and \(|\alpha| < 4\sqrt{2}\pi\) (with \(\eta_0 = \eta\)).

Let \(D_k\), with \(k \geq -1\), be the functions related to Littlewood-Paley block (see Appendix A for the definition of this concept). Using Remark 47, we can suppose that there exist some constants \(\gamma_{-1}, \gamma > 0\) and \(0 < \theta_{-1}, \theta < 1\) such that
\[
|D_{-1}(\hat{z})| \lesssim \exp(-\gamma_{-1}|\hat{z}|^{\theta_{-1}}) \quad \text{and} \quad |D_k(\hat{z})| \lesssim 2^k \exp(-\gamma 2^k|\hat{z}|^{\theta}),
\]
where \(k > 0\). Using this decay at infinity, we get
\[
\mathbb{E}[|\langle \eta, D_k(\hat{z} - \cdot) \rangle|_{L^2}^2] = \int_{\mathbb{R}^s} \mathbb{E}[|\langle \eta, D_k \rangle|^2](1 + |\hat{z}|)^{-2}\hat{z} d\hat{z} \lesssim \mathbb{E}[|\langle \eta, D_k \rangle|^2] < +\infty,
\]
whenever \(\alpha < 4\sqrt{2}\pi\), and where we used the invariance in law of \(\eta\) with respect to translations. Since we have \(D_k(x, z) = 2^k D_0(2^k x, 2^k z)\) and using (48) and (50), we obtain
\[
\mathbb{E}[|\langle \eta, D_k \rangle|^2] = \int_{\mathbb{R}^s} D_k(\hat{z}) D_k(\hat{z}') \exp(\alpha^2 \cdot \alpha_2^2 \mathcal{G}(\hat{z} - \hat{z}')) d\hat{z} d\hat{z}'
\]
\[
\lesssim \int_{\mathbb{R}^s} |D_k(\hat{z}) D_k(\hat{z}')||\hat{z} - \hat{z}'|^{-\frac{\alpha^2}{(4\pi)^2}} d\hat{z} d\hat{z}'
\]
\[
+ \left(1 + \epsilon^{-4\frac{\alpha^2}{(4\pi)^2}}\right) \int_{\mathbb{R}^s} |D_k(\hat{z}) D_k(\hat{z}')| d\hat{z} d\hat{z}'
\]
\[
\lesssim 2^{\frac{\alpha^2}{(4\pi)^2} k} \int_{\mathbb{R}^s} \left(1 + |\hat{z} - \hat{z}'|^{-\frac{\alpha^2}{(4\pi)^2}}\right) |D_0(\hat{z}) D_0(\hat{z}')| d\hat{z} d\hat{z}' \leq 2^{\frac{\alpha^2}{(4\pi)^2} k}
\]
where all the constants implied in the symbol \(\lesssim\) are independent of \(\epsilon\). This means that
\[
\mathbb{E}[|\eta|_{L^2}^2] \lesssim \sum_{k=-1}^{+\infty} 2^{\frac{\alpha^2}{(4\pi)^2} k + 2k},
\]
which is finite and uniformly bounded in \(\epsilon\) whenever \(s < -\frac{\alpha^2}{(4\pi)^2}\) and for \(\ell > 0\) large enough. Furthermore, we have that
\[
\mathbb{E}[|\eta - \eta|_{L^2}^2]^{1/2} \lesssim \left(\sum_{k=-1}^{+\infty} 2^{2k} \mathbb{E}[|\langle \eta, D_k \rangle|^2]\right)^{1/2}
\]
\[
\lesssim \sum_{k=-1}^{+\infty} 2^{2k} \sum_{n=0}^{+\infty} \left|\frac{\alpha}{n}\right|^n \mathbb{E}[|\langle I_{\xi^*} - I_{\xi^*}, D_k \rangle|^2]^{1/2}
\]
for \(k \geq \ell\) large enough. On the other hand we get
\[
\sum_{k=-1}^{+\infty} 2^{2k} \sum_{n=0}^{+\infty} \left|\frac{\alpha}{n}\right|^n \mathbb{E}[|\langle I_{\xi^*} - I_{\xi^*}, D_k \rangle|^2]^{1/2} \leq \sum_{k=-1}^{+\infty} 2^{2k} \sum_{n=0}^{+\infty} \left|\frac{\alpha}{n}\right|^n \mathbb{E}[|\langle I_{\xi^*}, D_k \rangle|^2]^{1/2}
\]
\[
+ \sum_{k=-1}^{+\infty} 2^{2k} \sum_{n=0}^{+\infty} \left|\frac{\alpha}{n}\right|^n \mathbb{E}[|\langle I_{\xi^*}, D_k \rangle|^2]^{1/2}
\]
Lemma 22. \( \leq \frac{1}{2} \sum_{k=1}^{+\infty} 2^{sk} \sum_{n=0}^{+\infty} \frac{\beta^{2n} |\alpha|^{2n}}{(n!)^2} E[|\langle I\xi^\epsilon_n, D_k \rangle|^2] \)

For some constant \( C > 0 \) independent of \( \epsilon \), whenever \( \beta > 1 \) is arbitrary small and \( s < -\frac{\beta \alpha^2}{(4\pi)^2} \), where we use an estimate similar to (51) and (52) for the measure \( \exp(\beta \epsilon I\xi^\epsilon) \). Furthermore, since \( \alpha_\epsilon \) is a regular mollifier and by the properties of the Wick product, we obtain

\[
E[|\langle I\xi^\epsilon_n - I\xi^\epsilon_n, D_k \rangle|^2] \to 0,
\]
as \( \epsilon \to 0 \) and for any \( k \geq -1 \) and any \( n \in \mathbb{N} \). The above inequality, together with the Lebesgue dominated convergence theorem, implies that

\[
E[\|\eta - \eta_\epsilon\|_{L^2(\mathbb{R}^4)}^2] \to 0,
\]
when \( \epsilon \to 0 \), for any \( \ell > 0 \) large enough. We have thus proven the following lemma.

Lemma 22. For \( |\alpha| < 4\sqrt{2}\pi \) and \( s < -\frac{\epsilon^2}{(4\pi)^2} \) and \( \ell > 0 \) large enough we have that \( \eta_\epsilon \to \eta \) as \( \epsilon \to 0 \) in \( L^2(\mathcal{W}; B^s_{2,2,\ell}(\mathbb{R}^4), d\mu) \) and thus in probability in \( B^s_{2,2,\ell}(\mathbb{R}^4) \).

We want to use the previous lemma to prove the following theorem.

Theorem 23. For \( |\alpha| < 4\sqrt{2}\pi \), \( 1 < p \leq 2 \), \( s < -\frac{\epsilon^2}{(4\pi)^2} \) and \( \ell > 0 \) large enough we have that \( \eta_\epsilon \to \eta \) as \( \epsilon \to 0 \) in \( L^p(\mathcal{W}; B^s_{p,p,\ell}(\mathbb{R}^4), d\mu) \) and thus in probability in \( B^s_{p,p,\ell}(\mathbb{R}^4) \).

In order to prove Theorem 23 we introduce the following lemma.

Lemma 24. Let \( B_r(\hat{z}) \) be the ball of radius \( r \) and center in \( \hat{z} \in \mathbb{R}^4 \) then for any \( r < R \), and for any \( 1 < p < 2 \)

\[
E \left[ \left( \int_{B_r(\hat{z})} d\eta_\epsilon \right)^p \right] \lesssim r^{-\frac{\alpha^2}{(4\pi)^2} p(p-1) + 4p}
\]

where the constants depend only \( R \) and are uniform on \( \epsilon \to 0 \) and \( |\alpha| < 4\sqrt{2}\pi \).

Proof. The proof can be found in Proposition 2.7 of [70].

Proof of Theorem 23 First of all we have that, for any \( \epsilon > 0 \) and \( k \geq 0 \),

\[
E[\|\eta_\epsilon, D_k(\hat{z} - \cdot)\|_{L^p}^p] = \int_{\mathbb{R}^4} E[\|\langle \eta_\epsilon, D_k \rangle\|^p] (1 + |\hat{z}|)^{-p\ell} d\hat{z} \lesssim E[\|\langle \eta_\epsilon, D_k \rangle\|^p].
\]
Then
\[ E[|\langle \eta_k, D_k \rangle|^p] \leq E \left[ \left( \int_{B_{r_k}(0)} |D_k(\hat{z})| d\eta_k(\hat{z}) \right)^p \right] + E \left[ \left( \int_{B_{r_k}(0)} |D_k(\hat{z})| d\eta_k(\hat{z}) \right)^{2p/2} \right]. \]

If we choose \( r_k = 2^{-k}(\log(2^\beta k))^\alpha \), for some \( \beta, \alpha > 0 \), from Lemma 24 we have, for any \( k \geq 0 \),
\[ E \left[ \left( \int_{B_{r_k}(0)} |D_k(\hat{z})| d\eta_k(\hat{z}) \right)^p \right] \leq 2^k p^2 2^{\alpha^2/2(p(p-1)k-4p)} \log(2^\beta k) u \left( -\frac{\alpha^2}{4\pi\gamma} p(p-1)+4p \right) \]

Furthermore, if we denote by \( \gamma > 0 \) and \( 0 < \theta < 1 \) the real number such that \( |D_k(\hat{z})| \lesssim \exp(-\gamma |\hat{z}|^\theta) \) and choosing \( u = \frac{1}{\theta} \) where \( 0 < \theta' < \theta \), we obtain
\[ E \left[ \left( \int_{B_{r_k}(0)} |D_k(\hat{z})| d\eta_k(\hat{z}) \right)^{2p/2} \right] \leq 2^{k \alpha - \beta k} \int_{|\hat{z}|>r_k, |\hat{z}|>r_k} e^{\gamma \beta k - \gamma 2^k (|\hat{z}|^\alpha + |\hat{z}'|^{\alpha'}) + \alpha^2 \gamma^2 (\hat{z}' - \hat{z})^2} d\hat{z} d\hat{z}' \]
\[ \leq 2^{k \alpha - \beta k} \int_{|\hat{z}|>r_k, |\hat{z}|>r_k} e^{-\gamma 2^k (|\hat{z} - \hat{z}'| + |\hat{z}' - \hat{z}'|) + \alpha^2 \gamma^2 (\hat{z}' - \hat{z})^2} d\hat{z} d\hat{z}' \]
\[ \leq 2^{k \alpha - \beta k} \int_{\hat{z} \in \mathbb{R}^d} e^{-\gamma (|\hat{z}' - \hat{z}|^\alpha + |\hat{z}' - \hat{z}'|^{\alpha'}) + \alpha^2 \gamma^2 (\hat{z}' - \hat{z})^2} d\hat{z} d\hat{z}' \lesssim 2^{k \alpha - \beta k} \]

If we choose \( \beta \) such that \( 8 - \gamma \beta < \frac{\alpha^2}{4\pi\gamma} \) uniformly in \( \epsilon \). This implies that \( E[||\eta_k||_{p,p,\ell}^p] \) is bounded for \( s < -\frac{\alpha^2}{4\pi\gamma} (p-1) \) uniformly in \( \epsilon > 0 \).

Furthermore, by Lemma 22, \( E[||\eta_k||_{p,p,\ell}^p] \) is uniformly bounded. This means that the random variables \( ||\langle \eta_k, D_k \rangle||^p \) (for \( p < 2 \)) are uniformly integrable. On the other hand, by Lemma 22 and since \( D_k \in B_{2,2,\ell}^2 \), \( ||\langle \eta_k, D_k \rangle||^p \) converges to \( ||\langle \eta, D_k \rangle||^p \) in probability, we have that \( E[||\eta_k, D_k||^p] \rightarrow E[||\eta, D_k||^p] \), as \( \epsilon \rightarrow 0 \). Thus by the bound (53) we can use the Lebesgue dominated convergence theorem for computing the limit of \( E[||\eta_k||_{p,p,\ell}^p] \), obtaining
\[ E[||\eta_k||_{p,p,\ell}^p] \rightarrow E[||\eta||_{p,p,\ell}^p], \]

as \( \epsilon \rightarrow 0 \). But, since \( E[||\eta_k||_{p,p,\ell}^p] \) is uniformly bounded and since, by Lemma 22, as \( \epsilon \rightarrow 0 \), \( \eta_k \rightarrow \eta \) in probability in \( B_{2,2,\ell}^2 \), we have that \( \eta_k \) converges weakly to \( \eta \) in \( L^p(\mathbb{W}, B_{p,p,\ell}^s, d\mu) \). Since \( L^p(\mathbb{W}, B_{p,p,\ell}^s, d\mu) \) is a uniformly convex space, being the space of \( L^p \) functions taking values in the uniformly convex space \( B_{p,p,\ell}^s \) (see [33]), by the weak convergence \( \eta_k \rightarrow \eta \) and the convergence of the \( L^p \) norm \( E[||\eta_k||_{p,p,\ell}^p] \rightarrow E[||\eta||_{p,p,\ell}^p] \), we obtain that \( \eta_k \) converges strongly to \( \eta \) in \( B_{p,p,\ell}^s(\mathbb{R}^d) \), as \( \epsilon \rightarrow 0 \). \( \square \)

### 3.2. Analysis of the elliptic SPDE

In this section we want to prove the following theorem.

**Theorem 25.** For any \( |\alpha| < \alpha_{\max} \) (see equation (1)), there are some \( p, s, \delta, \beta \in \mathbb{R} \) such that \( 1 < p \leq 2 \) and \( p < \frac{2(4\pi)^2}{\alpha^2} \), \(-1 < s < -\frac{(p-1)}{(4\pi)^2} \alpha^2 \) and \( 0 < \delta < s+1 \), for which equation (38) admits a unique solution in \( B_{p,p,\ell}^{s+2} \) for \( \ell > 0 \) large enough and such that \( \alpha \phi \leq 0 \). Furthermore we have that there exists a subsequence \( \epsilon_n \rightarrow 0 \) such that \( \phi_{\epsilon_n} \rightarrow \phi \) in \( B_{p,p,\ell}^{s+2-\delta} \) (for some \( \delta' > 0 \) small enough), as \( n \rightarrow \infty \), almost surely.

For the proof we have first to establish several lemmas. Hereafter, for \( t > 0 \), we write \( \mathbb{B}_{p,p,\ell}^t(\mathbb{R}^d) \). The space \( \mathbb{B}_{p,p,\ell}^t \) has a natural norm given by the sum of the norms of \( B_{p,p,\ell}^t(\mathbb{R}^d) \), and \( \mathbb{L}^\infty(\mathbb{R}^d) \). Furthermore we can equip the space \( \mathbb{B}_{p,p,\ell}^t \) with a different notion of convergence of sequences: we say that a sequence \( h_n \in \mathbb{B}_{p,p,\ell}^t \) converges to \( h \in \mathbb{B}_{p,p,\ell}^t \) if \( h_n \) converges to \( h \) strongly in \( B_{p,p,\ell}^t \) and \( \sup_n ||h_n|| \infty < +\infty \) (it is important to note that \( ||h||_\infty \leq \lim inf_{n \rightarrow +\infty} ||h_n||_\infty \)).
Lemma 26. For any $t > 0$ and $1 < p \leq 2$, we have that $\mathbb{B}_{p,p,\ell}^t \subset B_{\frac{2t}{k},\frac{2}{q},\ell}^{(p-1)\gamma}$ (where $1/p + 1/q = 1$ and $0 < \gamma < 1$) and the immersion is continuous with respect to the natural convergence in $\mathbb{B}_{p,p,\ell}^t$. Furthermore $\mathbb{B}_{p,p,\ell}^t$ is a Banach algebra.

Proof. By Proposition 54, the interpolation space $(B_{\infty,\infty,0}^0, B_{p,p,\ell}^t)_{\frac{1}{q},\gamma}$ between $L_{\infty} \subset B_{\infty,\infty,0}^0$ and $B_{p,p,\ell}^t$ is exactly $B_{\frac{2t}{k},\frac{2}{q},\ell}^{(p-1)\gamma}$, where $q = \frac{p-1}{p}$. This means that $\mathbb{B}_{p,p,\ell}^t = L_{\infty} \cap B_{p,p,\ell}^t \subset B_{\frac{2t}{k},\frac{2}{q},\ell}^{(p-1)\gamma}$ continuously embedded in $B_{\frac{2t}{k},\frac{2}{q},\ell}^{(p-1)\gamma}$. The fact that $\mathbb{B}_{p,p,\ell}^t$ is an algebra is proven in Corollary 2.86 of [27].

Lemma 27. Under the hypotheses of Theorem 25 on $\alpha$ there are some $p, s, \delta$ such that $1 < p \leq 2$ and $p < \frac{2(4\pi)^2}{\alpha^2}$, $-1 < s < -\left(\frac{\alpha^2(p-1)}{4\pi^2} \vee (p-1)\right)$ and $0 < \delta < s + 1$ and for which there is a $0 < \gamma < 1$ such that

$$\frac{(2 + s - \delta)(p-1)}{p} + s > 0 \quad (54)$$

$$\frac{s + 1}{1 - \delta} < \gamma < 1 \quad (55)$$

$$\left(1 - \frac{s + 1}{4} p\right) \gamma < p - 1. \quad (56)$$

Proof. Here we introduce the new variables $Z = 1 + s$ and $k = p - 1$. The statement of the lemma can be reformulated as follows: for any values of the quotient $0 < \frac{1 - Z}{k} < 2$ we can find $0 < Z < 1$ and $0 < k < 1$ such that

$$\frac{Z}{1 - \delta} < \frac{k}{(1 - \frac{Z}{4}(k + 1))} \quad (57)$$

$$\frac{(1 + Z - \delta)k}{k + 1} > 1 - Z \quad (58)$$

From the inequality (58) we get that

$$k > \frac{1 - Z}{2Z - \delta}$$

and inequality (57) is equivalent to

$$k > \frac{Z(4 - Z)}{(Z^2 + 4(1 - \delta))}.$$  

This means that the thesis of the lemma holds if and only if $k > G(Z, \delta)$ where

$$G(Z, \delta) = \max \left(\frac{Z(4 - Z)}{(Z^2 + 4(1 - \delta))}, \frac{1 - Z}{2Z - \delta}\right) = \begin{cases} \frac{1 - Z}{2Z - \delta} & \text{for } Z \leq 4 - 2\sqrt{3} \\ \frac{Z(4 - Z)}{(Z^2 + 4(1 - \delta))} & \text{for } Z > 4 - 2\sqrt{3} \end{cases}$$

for $0 < \delta < 1$ and $\frac{1}{2} < Z < 1$. The bounds on the possible $\alpha$ can be obtained as follows

$$\frac{\alpha^2}{(4\pi)^2} = \sup_{s,p} \frac{-s}{(p-1)} = \sup_{k,Z} \frac{1 - Z}{k} = \sup_{\frac{1}{2} < Z < 1, 0 < \delta < 1} \frac{1 - Z}{G(Z, \delta)} = \frac{1 - (4 - 2\sqrt{3})}{G(4 - 2\sqrt{3}, 0)} = 8 - 4\sqrt{3}.$$  

We introduce the space $\mathcal{M}_{p,p,\ell}^s \subset B_{p,p,\ell}^s$ (with $s, p$ and $\ell$ as in Theorem 25) which is the set of Radon $\sigma$-finite measures $\mu$ contained in $B_{p,p,\ell}^s$ such that $|\mu| \in B_{p,p,\ell}^s$, where $|\mu| = \mu_+ + \mu_-$, and where $\mu_+$ and $\mu_-$ are the unique positive measures such that $\mu = \mu_+ - \mu_-$. We can define a notion of convergence on $\mathcal{M}_{p,p,\ell}^s$ in the following way: a sequence $\mu_n \in \mathcal{M}_{p,p,\ell}^s$ converges to $\mu \in \mathcal{M}_{p,p,\ell}^s$ if $\mu_n$ converges strongly in $B_{p,p,\ell}^s$ and $\sup_n \|\mu_n\|_{B_{p,p,\ell}^s} < +\infty$.

We now consider the natural product $\cdot$ between smooth functions which are in $\mathbb{B}_{r,r,\ell'}^t$ and smooth measures in $\mathcal{M}_{p,p,\ell}^s$. 

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Lemma 28. If \( s, p, \delta, \gamma \) satisfy the thesis of Lemma 27, then the product \( \cdot \) can be extended in a unique and associative way from \( \mathbb{B}^{s+2-\delta}_{r,r',\ell} \times \mathcal{M}^s_{p,p',\ell} \) into \( \mathcal{M}^s_{p,p',\ell} \), where

\[
    r = \frac{p}{(1 - \frac{s+1}{4} p) (\gamma + 1)}.
\]

This extension is (weakly) continuous with respect to the natural convergence in \( \mathbb{B}^{s+2-\delta}_{r,r',\ell} \) and \( \mathcal{M}^s_{p,p',\ell} \). Furthermore for any \( h \in \mathbb{B}^{s+2-\delta}_{r,r',\ell} \) and \( \mu \in \mathcal{M}^s_{p,p',\ell} \) we have

\[
    \| h \cdot \mu \|_{\mathcal{M}^s_{p,p',\ell}} \lesssim \| h \|_\infty \cdot \| \mu \|_{\mathcal{M}^s_{p,p',\ell}}.
\]

Proof. First of all we note that, by Proposition 53, the product \( \cdot \) is well defined and (strongly) continuous as bilinear functional from \( \mathbb{B}^{(s+2-\delta)\frac{1}{p} + \frac{1}{r'}}_{r,r',\ell} \times \mathcal{M}^s_{p,p',\ell} \) into \( \mathcal{M}^s_{p,p',\ell} \), where \( r = \frac{p}{((1 - \frac{s+1}{4} p) (\gamma + 1))} \) for some \( \gamma \) such that \( \frac{s+1}{4} < \gamma < 1 \) and \( (1 - \frac{s+1}{4} p) < p - 1 \) (whose existence is proven in Lemma 27).

Indeed if \( \gamma \) satisfies the previous conditions, we have

\[
    (s + 2 - \delta)\frac{(p-1)}{p} + s > (s + 2 - \delta)\frac{(p-1)}{p} + s > 0
\]

Since by hypotheses \( r = \frac{p}{((1 - \frac{s+1}{4} p) (\gamma + 1))} > 1 \) and \( (s+2-\delta)(p-1) + s > 0 \).

The only thing that remains to prove is inequality (59), since the other statements of the lemma easily follow from it. First of all, using the equivalent norm on Besov spaces of Proposition 49, we note that if \( \mu \) is a Radon measure we have

\[
    \| \mu \|_{\mathcal{M}^s_{p,p',\ell}} \sim \left( \int_{\mathbb{R}^4 \times (0,1)} \left( \frac{|\mu| (B_\lambda (\hat{z}))}{\lambda^{s+4}} \right)^p \hat{r}_\ell (\hat{z})^p d\lambda \right)^{1/p}.
\]

Using again Proposition 49, if \( h \) is a continuous bounded function we have that

\[
    \| h \cdot \mu \|_{\mathcal{M}^s_{p,p',\ell}} \lesssim \left( \int_{\mathbb{R}^4 \times (0,1)} \left( \frac{|h \cdot \mu| (B_\lambda (\hat{z}))}{\lambda^{s+4}} \right)^p \hat{r}_\ell (\hat{z})^p d\lambda \right)^{1/p} \lesssim \| h \|_\infty \cdot \| \mu \|_{\mathcal{M}^s_{p,p',\ell}}.
\]

If \( h \) is a generic function on \( \mathbb{B}^{s+2-\delta}_{r,r',\ell} \) there exists a sequence of smooth functions such that, as \( n \to \infty \), \( h_n \to h \) in \( \mathbb{B}^{s+2-\delta}_{r,r',\ell} \) and \( \| h_n \|_\infty \leq \| h \|_\infty \). By the first part of the lemma we have that \( |h_n \cdot \mu| \) converges weakly in \( \mathcal{S}'(\mathbb{R}^4) \) to \( |h \cdot \mu| \) from which we obtain

\[
    \| h \cdot \mu \|_{\mathcal{M}^s_{p,p',\ell}} \leq \liminf\| h_n \cdot \mu \|_{\mathcal{M}^s_{p,p',\ell}} \leq \| \mu \|_{\mathcal{M}^s_{p,p',\ell}} \liminf\| h_n \|_\infty \lesssim \| h \|_\infty \| \mu \|_{\mathcal{M}^s_{p,p',\ell}}.
\]

We introduce the following map

\[
    K_\gamma (\mu, \tilde{\phi}) := -\alpha (-\Delta + m^2)^{-1} (g(z) G(\alpha \tilde{\phi}(\bar{z})) \cdot \mu(\bar{z}))
\]

for any \( |\alpha| < \alpha_{\text{max}} \), where \( G : \mathbb{R} \to \mathbb{R}_+ \) is an increasing smooth bounded function with all bounded derivatives such that \( G(x) = \exp(x) \) for \( x \leq 0 \). It is clear that if \( \alpha \tilde{\phi} \leq 0 \) and if \( \tilde{\phi} \) solves equation (38) then we have

\[
    \tilde{\phi} = K_\gamma (\eta, \tilde{\phi})
\]

Hereafter we denote by \( \mathcal{M}^s_{r,p,p',\ell} \) the space formed by the positive distributions contained in \( \mathcal{M}^s_{r,p,p',\ell} \).

Lemma 29. Let \( G \) be the function defined by \( \varphi \mapsto G(\varphi) \), and let \( p, s, \delta, \gamma \) be as in the thesis of Lemma 27 then, when \( \ell > 0 \) is big enough, we have that \( G \) is a continuous map from \( \mathbb{B}^{s+2-\delta}_{p,p',\ell} \) into \( \mathbb{B}^{s+2-\delta}_{r,r',2\ell} \) where \( r \) is chosen as in Lemma 28.

Proof. For any \( 0 \leq k < 1, q > 1 \) and \( \ell \) big enough we have that \( G \) is a continuous (actually Lipschitz) map from \( \mathbb{B}^k_{q,q',\ell} \) into \( \mathbb{B}^k_{q,q',\ell} \). We give a sketch of the proof for the weighted Besov spaces following the argument of [71] Section 5.5.
This fact can be proven using the equivalent definition of Besov norm using finite differences (89) for \( N = 0 \) and \( M = 1 \). Indeed we have that

\[
\|G(\varphi)\|_{B_{p,q}^{s+2-\delta}} = \|G(\varphi)\|_{L^p_{\ell}} + \left( \int_{|h|<1} |h|^{-kq} \|G(\varphi(x + h)) - G(\varphi(x))\|_{L^q_{\ell}}^q \frac{dh}{|h|^n} \right)^{\frac{1}{q}} 
\]

\[
\lesssim (\|G\|_{\infty} + \|G'\|_{\infty}) \left( 1 + \|\varphi\|_{L^p_{\ell}} + \left( \int_{|h|<1} |h|^{-kq} \|\Delta_h^\delta \varphi\|_{L^q_{\ell}}^q \frac{dh}{|h|^n} \right)^{\frac{1}{q}} \right) 
\]

\[
\lesssim (\|G\|_{\infty} + \|G'\|_{\infty})(1 + \|\varphi\|_{B_{p,q,\ell}^{s+2-\delta}}). 
\]

This proves that the map \( G \) takes values in \( B_{q,q,\ell}^k \). By using the real interpolation of Sobolev spaces

\[
(L_p^\ell, W^{1,p}_h)_{s,q} = B_{p,q,\ell}^s, 
\]

which is a consequence of the interpolation of unweighted Sobolev spaces (see [31] Theorem 6.2.4) and the isomorphism between weighted and unweighted Besov and Sobolev spaces (see [80] Theorem 6.5), using the Lipschitz continuity of \( G \) in \( L^p_{\ell} \) and in \( W^{1,q}_h \) (see [32] for the proof in the unweighted Sobolev spaces), and exploiting a standard interpolation argument (see [61]), we obtain that \( G \) is also continuous.

Consider \( \varphi \in B^{s+2-\delta}_{p,p,\ell} \), then \( \nabla G(\varphi) = G'(\varphi) \nabla \varphi \). By a Besov embedding theorem we have that \( \varphi \in B^{1-\delta}_{p',p',\ell} \) where \( p' = p \left(1 - \frac{1 + \delta}{p}\right)^{-1} \) for \( \varepsilon > \delta \) small enough. By our arguments above this means that \( G'(\varphi) \in B^{1-\delta}_{p',p',\ell} \), and by Lemma 26, \( G'(\varphi) \in B^{1-\delta}_{p',s+2-\delta} \). This implies that, for \( \gamma > \frac{1}{p'} + \frac{\gamma}{\ell} \), and \( (1 - \frac{1 + \delta}{p}) \gamma < p - 1 \) (we must choose \( \delta < \varepsilon < \frac{1}{p'} - 1 \) here), the product \( G'(\varphi) \nabla \varphi \) is well defined and in \( B^{s+2-\delta}_{r,r,\ell} \). By Remark 51 below, this implies that \( G(\varphi) \in B^{s+2-\delta}_{r,r,\ell} \).

In a similar way, when \( \varphi_n \in B^{s+2-\delta}_{p,p,\ell} \) converges to \( \varphi \in B^{s+2-\delta}_{p,p,\ell} \), it is possible to prove that \( G(\varphi_n) \rightarrow G(\varphi) \) in \( B^{s+2-\delta}_{r,r,\ell} \). Indeed, by Remark 51, we have

\[
\|G(\varphi) - G(\varphi_n)\|_{B^{s+2-\delta}_{r,r,\ell}} \lesssim \|G(\varphi) - G(\varphi_n)\|_{L^{p}_{\ell}} + \|G'(\varphi) \nabla \varphi - G'(\varphi_n) \nabla \varphi_n\|_{B^{s+1-\delta}_{r,\ell}} 
\]

\[
\lesssim \|G(\varphi) - G(\varphi_n)\|_{L^{p}_{\ell}} + \|G'(\varphi)\|_{B^{1-\delta}_{p,s+2-\delta}} \|\nabla \varphi - \nabla \varphi_n\|_{B^{s+1-\delta}_{p,p,\ell}} + 
\]

\[
\sup_n (\|\nabla \varphi_n\|_{B^{s+1-\delta}_{p,p,\ell}}) \|G'(\varphi) - G'(\varphi_n)\|_{B^{1-\delta}_{p,s+2-\delta}} \rightarrow 0 
\]

since \( G' \) is continuous from \( B^{s+2-\delta}_{p,p,\ell} \) into \( B^{1-\delta}_{p',s+2-\delta} \) (as it was proven in the first part of the proof).

**Lemma 30.** For \( p, s, \delta, \gamma \) satisfying the thesis of Lemma 27, for any \( \ell > 0 \) big enough, for any fixed \( \mu \in M^{s}_{p,p,\ell} \) and small enough \( \delta' > 0 \), there exists at least a solution \( \tilde{\varphi} \in B^{s+2-\delta}_{p,s+2-\delta} \) to the equation

\[
\tilde{\varphi} = K_g(\mu, \tilde{\varphi}). 
\]

**Proof.** We want to use Schaefer’s fixed-point theorem (see Theorem 4 Section 9.2 Chapter 9 of [39]) to prove the lemma. In order to do this we have to prove that \( K_g \) is continuous in \( \varphi \), that it maps any bounded set into a compact set and that the set of solutions to the equations

\[
\tilde{\varphi} = \lambda K_g(\mu, \tilde{\varphi}) 
\]

is bounded uniformly for all \( 0 \leq \lambda \leq 1 \).

The continuity of \( K_g \) (in both \( \mu \) and \( \varphi \)) is a consequence of Lemma 28 and Lemma 29. Indeed in the cone \( M^{s}_{p,p,\ell} \) the natural convergence in \( \mathfrak{M}^{s}_{p,p,\ell} \subset M^{s}_{p,p,\ell} \) coincides with the strong convergence in \( B^{s}_{p,p,\ell} \), since if \( \mu \in M^{s}_{p,p,\ell} \) then \( \|\mu\| = \mu \).

Furthermore, by Lemma 29, the map \( \varphi \rightarrow G(\varphi) \) is continuous from \( B^{s+2-\delta}_{p,p,\ell} \) into \( B^{s+2-\delta}_{p,s+2-\delta} \), where \( r = \frac{p}{((1 - \frac{1 + \delta}{p}) \gamma + 1)} \). Thus we can apply Lemma 28, from which we obtain that the map \( \varphi \mapsto G(\varphi) \cdot \mu \) is continuous in \( \varphi \). Finally the linear operator \((-\Delta + m^2)^{-1}\) is continuous from \( B^{s}_{p,p,\ell} \) into \( B^{s+2}_{p,p,\ell} \) and thus compact from \( B^{s}_{p,p,\ell} \) into
The map $K_g$ is compact since the following inequality holds

$$
\|K_g(\mu, \varphi)\|_{B^{s+2-\delta}_{p,p+\ell+\delta'}} \lesssim \|G\|_{\infty} \|\mu\|_{B^{s+2-\delta}_{p,p+\ell}}
$$

(61)

and, by Proposition 52, we have the compact immersion $B^{s+2}_{p,p+\ell} \hookrightarrow B^{s+2-\delta}_{p,p+\ell+\delta'}$.

Finally the uniform boundedness in $\lambda$ follows from inequality (61). This proves the thesis of the lemma. \qed

**Lemma 31.** Under the hypotheses of Theorem 25 the solution to equation (60) is unique in $B^{s+2-\delta}_{p,p+\ell+\delta'}$ for $\delta, \delta' \geq 0$ small enough.

**Proof.** Let $J : \mathbb{R} \to \mathbb{R}$ be a smooth, bounded, strictly increasing function such that $J(0) = 0$ and $J(-x) = -J(x)$, and let $\varphi_1$ and $\varphi_2$ be two solutions to equation (60). By Lemma 26 and Lemma 29, $J(\varphi_1 - \varphi_2) \in B^{(s+2-\delta')2}_{q,q,\ell+\delta'}$ which implies that $\bar{r}_\ell(\lambda z)J(\varphi_1 - \varphi_2) \in (B^{(s+2-\delta')2}_{q,q,\ell+\delta'})$ for $\delta' \geq 0$ small enough, $\ell' > 0$ large enough and any $\lambda > 0$. This means that

$$
\langle \bar{r}_\ell(\lambda z)J(\varphi_1 - \varphi_2), (\Delta z + m^2)(\varphi_1 - \varphi_2 - K_g(\mu, \varphi_1) + K_g(\mu, \varphi_2)) \rangle = 0.
$$

We are going to see that the inequality

$$
\langle \bar{r}_\ell(\lambda z)J(\varphi_1 - \varphi_2), (\Delta z + m^2)(\varphi_1 - \varphi_2) \rangle \geq C \int \bar{r}_\ell(\lambda z)J(\varphi_1 - \varphi_2)(\varphi_1 - \varphi_2) d\lambda z
$$

holds for $\lambda > 0$ small enough and some constant $C > 0$. Indeed let $f_1, f_2$ be two smooth functions then, for $\lambda > 0$ small enough,

$$
\langle \bar{r}_\ell(\lambda z)J(f_1 - f_2), (\Delta z + m^2)(f_1 - f_2) \rangle = \int \bar{r}_\ell(\lambda z)J(f_1 - f_2)(\Delta z + m^2)(f_1 - f_2) d\lambda z
$$

$$
= \int \bar{r}_\ell(\lambda z)\bar{r}_\ell(\lambda z)J(f_1 - f_2)\Delta f_1 - \nabla f_1^2 d\lambda z + \lambda \int \nabla \bar{r}_\ell(\lambda z)J(f_1 - f_2) \cdot (\nabla f_1 - \nabla f_2) d\lambda z
$$

$$
+ m^2 \int \bar{r}_\ell(\lambda z)J(f_1 - f_2)(f_1 - f_2) d\lambda z
$$

$$
\geq -\lambda^2 \int (\Delta \bar{r}_\ell(\lambda z))J^{-1}(f_1 - f_2) d\lambda z + m^2 \int \bar{r}_\ell(\lambda z)J(f_1 - f_2)(f_1 - f_2) d\lambda z
$$

$$
\geq \int (m^2 - \frac{2\lambda \Delta \bar{r}_\ell(\lambda z)}{r_\ell}) \bar{r}_\ell(\lambda z)J(f_1 - f_2)(f_1 - f_2) d\lambda z
$$

$$
\geq C \int \bar{r}_\ell(\lambda z)J(f_1 - f_2)(f_1 - f_2) d\lambda z
$$

where $J^{-1}(t) = \int_0^t J(\tau) d\tau$. For these deductions we used the fact that

$$
\int \nabla \bar{r}_\ell(\lambda z)J(f_1 - f_2) \cdot (\nabla f_1 - \nabla f_2) d\lambda z = \int \nabla \bar{r}_\ell(\lambda z)\nabla J^{-1}(f_1 - f_2) d\lambda z = -\lambda \int \Delta \bar{r}_\ell(\lambda z)J^{-1}(f_1 - f_2) d\lambda z
$$

which is true since $J^{-1}$ is a Lipschitz function such that $J^{-1}(0) = 0$, and thus, by the first part of the proof of Lemma 29, $J^{-1}(f_1 - f_2) \in W_2^{1,p}(\mathbb{R}^4)$. We have, also, exploited the fact that $J^{-1}(t) \leq J(t) t$ since $J$ is increasing and the fact that we can choose $\lambda$ small enough such that the last inequality holds. This proves that

$$
\langle \bar{r}_\ell(\lambda z)J(f_1 - f_2), (\Delta z + m^2)(f_1 - f_2) \rangle \geq C \int \bar{r}_\ell(\lambda z)J(f_1 - f_2)(f_1 - f_2) d\lambda z,
$$

(62)

for $\lambda$ small enough, and some $C > 0$, for smooth functions $f_1, f_2 \in B^{s+2-\delta}_{p,p+\ell+\delta'}$. Since the expressions $\langle \bar{r}_\ell(\lambda z)J(f_1 - f_2), (\Delta z + m^2)(f_1 - f_2) \rangle$ and $\int \bar{r}_\ell(\lambda z)J(f_1 - f_2)(f_1 - f_2) d\lambda z$ are continuous for $f_1, f_2 \in B^{s+2-\delta}_{p,p+\ell+\delta'}$ (with respect to the $B^{s+2-\delta}_{p,p+\ell+\delta'}$ natural norm) we can extend inequality (62) to the case of general functions $f_1, f_2 \in B^{s+2-\delta}_{p,p+\ell+\delta'}$ and so it holds in particular for $f_1 = \varphi_1$ and $f_2 = \varphi_2$. Furthermore we have that $\langle \bar{r}_\ell(\lambda z)J(\varphi_1 - \varphi_2), (\Delta z + m^2)(-K_g(\mu, \varphi_1) + K_g(\mu, \varphi_2)) \rangle \geq 0$. Indeed by Lemma 28 the product is associative and thus we obtain

$$
\langle \bar{r}_\ell(\lambda z)J(\varphi_1 - \varphi_2), (\Delta z + m^2)(-K_g(\mu, \varphi_1) + K_g(\mu, \varphi_2)) \rangle
$$

$$
= \int \bar{r}_\ell(\lambda z)g(z)(\alpha G(\alpha \varphi_1) - \alpha G(\alpha \varphi_2)) d\mu(\varphi_1 - \varphi_2) d\mu(\varphi_1 - \varphi_2) \geq 0,
$$
where we use that $\text{d}\mu(\hat{z})$ is a positive measure and $(\alpha G(\alpha t_1) - \alpha G(\alpha t_2)) \cdot J(t_1 - t_2)$ is positive since both $G$ and $J$ are increasing functions and $J(0) = 0$. Using the previous inequalities we obtain that

$$\int \tilde{r}_\ell(\lambda \hat{z}) J(\hat{\varphi}_1 - \hat{\varphi}_2)(\hat{\varphi}_1 - \hat{\varphi}_2) \text{d}\hat{\varphi} \leq 0$$

which holds only if $\hat{\varphi}_1 - \hat{\varphi}_2 = 0$, since $J$ is a strictly increasing function.

**Remark 32.** Combining Lemma 30 and Lemma 31 we deduce that the map $\mu \mapsto \hat{\varphi}$, associating with the measure $\mu$ the unique solution $\hat{\varphi}$ to equation (60), is continuous with respect to $\mu$. Indeed suppose that $\mu_n \to \mu$ in $\mathcal{M}^s_{+,p,p,\ell}$ (and so in $B^s_{p,p,\ell}$), then by Lemma 28, we have that $\|\hat{\varphi}_n\|_{B^{s+2-\delta}_{p,p,\ell}} \leq \sup \|\mu_n\|_{B^{s}_{p,p,\ell}}$, and so there exists a converging subsequence $\hat{\varphi}_{k_n}$, as $n \to \infty$. On the other hand, since $\mathcal{K}_g$ is continuous in both $\mu$ and $\varphi$, we get that $\hat{\varphi}_{k_n}$ converges to the unique solution $\varphi$ to equation (60) associated with $\mu$. Since the limit does not depend on the subsequence we have that $\varphi_n \to \varphi$ strongly in $B^{s+2-\delta}_{p,p,\ell}$, which proves the continuity of the solution map.

This continuity of the solution map with respect to the measure $\mu$ is similar to the continuity result obtained for solutions of singular SPDEs defined by the methods of paracontrolled calculus or regularity structure theory.

**Proof of Theorem 25** The existence and uniqueness of the solution to equation (38) are proved in Lemma 30 and Lemma 31 considering the equation $\hat{\varphi} = \mathcal{K}(\eta, \hat{\varphi})$. What remains to prove is the convergence of $\hat{\varphi}_{\varepsilon_n}$ to $\hat{\varphi}$, as $\varepsilon_n \to 0$. Let $\varepsilon_n \to 0$ be a sequence of positive numbers such that $\eta_{\varepsilon_n} \to \eta$ almost surely in $\mathcal{M}^s_{+,p,p,\ell}$ and let $w \in W$ be such that $\eta_{\varepsilon_n}(w) \to \eta(w)$ in $\mathcal{M}^s_{+,p,p,\ell}$, as $\varepsilon \to 0$. We note that

$$\bar{\varphi}_{\varepsilon_n}(w) = \mathfrak{a}^{\varepsilon_n}_*((\mathcal{K}_g(\eta_{\varepsilon_n}(w), \hat{\varphi}_{\varepsilon_n}(w)))).$$  

(63)

From the equality (63) and Lemma 28 we obtain that

$$\|\hat{\varphi}_{\varepsilon_n}(w)\|_{B^{s+2-\delta}_{p,p,\ell}} \lesssim \sup \|\eta_{\varepsilon_n}\|_{B^{s}_{p,p,\ell}},$$

uniformly in $n$. This means that there exists a subsequence $\hat{\varphi}_{\varepsilon_k}(w)$ converging to some $\bar{\varphi}$ in $B^{s+2-\delta}_{p,p,\ell}$. On the other hand we have that

$$\varphi = \lim_{n \to +\infty} \hat{\varphi}_{\varepsilon_k}(w) = \lim_{n \to +\infty} \mathfrak{a}^{\varepsilon_n}_*((\mathcal{K}_g(\eta_{\varepsilon_n}(w), \hat{\varphi}_{\varepsilon_n}(w)))) = \mathcal{K}_g(\eta(w), \bar{\varphi}).$$

Since, by Lemma 31, equation (60) has a unique solution we have that all the subsequences $\hat{\varphi}_{\varepsilon_k}(w)$ converge to the same $\bar{\varphi}$ and so $\bar{\varphi} = \hat{\varphi}(w)$ (which is the unique solution to equation (38) evaluated in $w \in W$). Since the previous reasoning holds for almost every $w \in W$ and since $\eta_{\varepsilon_n} \to \eta$ almost surely, we have that $\hat{\varphi}_{\varepsilon_n} \to \hat{\varphi}$ in $B^{s+2-\delta}_{p,p,\ell}$ almost surely, as $\varepsilon_n \to 0$.

### 3.3. Dimensional reduction

In this section we want to prove the reduction principle for equation (37), i.e. that the random field $\phi(0, z) = \mathcal{I}\xi(0, z) + \bar{\phi}(0, z)$ has the law $\kappa_d$ given by expression (39).

Before proving the dimensional reduction for equation (37) we have to prove that the restriction of a solution to a two dimensional hyperplane is a well defined operation.

**Lemma 33.** There exists a version of the functional $x \mapsto \mathcal{I}\xi(x, \cdot)$ which is continuous as a function from $\mathbb{R}^2$ into $C_\ell^{0, -}(\mathbb{R}^2) \subset B_{p,p,\ell+\delta}^0$. Furthermore for any $h \in C_\ell^\infty(\mathbb{R}^2)$ the sequence of random variables $\int_{\mathbb{R}^2} a_* (\mathcal{I}\xi)(0, z) h(z) \text{d}z$ converges to $\langle \mathcal{I}\xi(0, \cdot), h \rangle$ in $L^2(\text{d}\mu)$, as $\varepsilon \to 0$.

**Proof.** In order to prove that $x \mapsto \mathcal{I}\xi(x, \cdot)$ admits a continuous version we prove that for any smooth function $h : \mathbb{R}^2 \to \mathbb{R}$ such that $(-\Delta + m^2)^{-1/2+\varepsilon}(h) \in L^2(\mathbb{R}^2)$, with $\varepsilon > 0$, we have

$$\mathbb{E}[||\mathcal{I}\xi(x, \cdot) - \mathcal{I}\xi(y, \cdot), h||^2] \lesssim |x - y|^c ||(-\Delta + m^2)^{-1/2+\varepsilon}(h)||_{L^2(\mathbb{R}^2)}, \quad x, y \in \mathbb{R}^2,$$

where the constants implied by $\lesssim$ do not depend on $h, x, y$. This result, exploiting hypercontractivity and a version of Kolmogorov continuity criterion for multidimensional random fields, implies the continuity of $\mathcal{I}\xi(x, \cdot)$ with respect to $x \in \mathbb{R}^2$. We note that, for any $x, y \in \mathbb{R}^2$:

$$\mathbb{E}[||\mathcal{I}\xi(x, \cdot) - \mathcal{I}\xi(y, \cdot), h||^2] = 2 \int_{\mathbb{R}^2} \frac{1 - e^{ik_1(\cdot-y)}}{(k_1^2 + k_2^2 + m^2)^{1+2\varepsilon}} \frac{\hat{h}(k_2)^2}{(k_1^2 + k_2^2 + m^2)^{1-2\varepsilon}} \text{d}k_1 \text{d}k_2.$$
Now we observe that \( |1 - e^{i k_1 (x - y)}| \leq 2 |x - y|^{\frac{\varepsilon}{1 - \varepsilon}} |k_1|^{\frac{\varepsilon}{1 - \varepsilon}} \) for any \( p > 1 \). If we choose \( p = \frac{1}{1 - \varepsilon} \), \( 0 < \varepsilon < 1 \), we obtain the claim.

In order to prove that \( \int_{\mathbb{R}^2} a_* (I \xi) (z) h(z) dz \) converges, as \( \varepsilon \to 0 \), to \( \langle I \xi (0, \cdot), h \rangle \) in \( L^2 (d\mu) \), we note that the distribution \( I (\delta_0 (x) \cdot h(z)) \) belongs to \( L^2 (\mathbb{R}^2) \), implying that
\[
\int_{\mathbb{R}^2} a_* (I \xi) (z) h(z) dz = \delta[I (\delta_0 (x) \cdot a_* h(z))] \quad \langle I \xi (0, \cdot), h \rangle = \delta[I (\delta_0 (x) \cdot h(z))],
\]
where \( \delta \) denote the Skorohod integral with respect to the white noise \( \xi \). Since \( I (\delta_0 (x) \cdot a_* h(z)) \) converges to \( I (\delta_0 (x) \cdot h(z)) \) in \( L^2 (\mathbb{R}^2) \) also this claim follows.

**Lemma 34.** The operator \( T_0 : C^0_\infty (\mathbb{R}^4) \to C^0_\infty (\mathbb{R}^4) \) given by \( T_0 (h)(\cdot) = h(0, \cdot) \), for \( h \in C^0_\infty (\mathbb{R}^4) \), can be uniquely extended in a continuous way as an operator from \( B_{p,p,\ell+\delta}^s (\mathbb{R}^4) \) into \( B_{p,p,\ell+\delta}^{s+2-2/p-\delta} (\mathbb{R}^2) \), when the triple \( s, p, \delta \) satisfies the thesis of Lemma 27, \( \ell > 0 \) big enough and \( \delta' > 0 \) small enough.

**Proof.** If \( s, p, \delta \) satisfies the thesis of Lemma 27 then \( s + 2 - \frac{2}{p} - \delta > 0 \). Under this condition the proof can be found in Section 4.4.1 and Section 4.4.2 of [78], see also Section 18.1 of [79].

**Theorem 35.** Let \( \phi \) be the unique solution to equation (37) then, if \( |\alpha| < \alpha_{\text{max}} \) (see equation (1)), there exists \( \ell > 0 \) such that for any measurable and bounded function \( F \) on \( B_{p,p,\ell+\delta'}^0 (\mathbb{R}^2) \), for some \( \delta' > 0 \) small enough, we have
\[
\mathbb{E}[F(\phi(0, \cdot))] = \int_{B_{p,p,\ell+\delta'}^0 (\mathbb{R}^2)} F(\omega)d\kappa_\gamma(\omega).
\] (64)

**Proof.** We prove the equality (64) for the case in which \( F(\omega) = \bar{F}(\omega, f_1, \ldots, \omega, f_n) \) where \( \bar{F} \) is a bounded continuous function, and \( f_1, \ldots, f_n \in C^0_\infty (\mathbb{R}^2) \). Since the functions of the previous form generate all the \( \sigma \)-algebra of Borel measurable functions on \( B_{p,p,\ell+\delta'}^0 \), where \( \delta' > 0 \), proving the theorem for functions of the previous form is equivalent to prove the theorem in general.

Since equation (42) holds, we have only to prove that \( \mathbb{E}[F(\phi_{\epsilon_n}(0, \cdot))] \to \mathbb{E}[F(\phi(0, \cdot))] \), as \( \epsilon_n \to 0 \) and where \( \epsilon_n \) is any subsequence such that \( \bar{\phi}_{\epsilon_n} \to \bar{\phi} \) almost surely (whose existence is proved in Theorem 25) and \( \int_{B_{p,p,\ell+\delta'}^0 (\mathbb{R}^2)} F(\omega)d\kappa_\epsilon(\omega) \to \int_{B_{p,p,\ell+\delta'}^0 (\mathbb{R}^2)} F(\omega)d\kappa_\gamma(\omega) \), as \( \epsilon_n \to 0 \).

The first convergence follows from the fact that \( \phi_{\epsilon_n} = \bar{\phi}_{\epsilon_n} + a_* (I \xi) \). Indeed, by Theorem 25, as \( \epsilon_n \to 0 \),
\[
\langle \bar{\phi}_{\epsilon_n}(0, \cdot), f_i \rangle \to \langle \bar{\phi}_\gamma, T_0^\ast (f_i) \rangle \to \langle \bar{\phi}, T_0^\ast (f_i) \rangle = \langle \bar{\phi}(0, \cdot), f_i \rangle,
\]
for any \( i = 1, \ldots, n, \) almost surely and where the continuity of \( T_0 \), proved in Lemma 34, is used. On the other hand, by Lemma 33, we have \( \langle a_* (I \xi)(0, \cdot), f_i \rangle \to \langle I \xi(0, \cdot), f_i \rangle \) in probability, as \( \epsilon_n \to 0 \), and this implies that \( \langle \phi_{\epsilon_n}(0, \cdot), f_i \rangle \to \langle \phi(0, \cdot), f_i \rangle \) in probability, for any \( i = 1, \ldots, n \).

Since the convergence in probability implies the one in distribution we get \( \mathbb{E}[F(\phi_{\epsilon_n}(0, \cdot))] \to \mathbb{E}[F(\phi(0, \cdot))] \). Finally since \( \kappa_{\epsilon_n} \) converges weakly to \( \kappa_\gamma \), as \( \epsilon_n \to 0 \), the thesis follows.

Now we want to discuss what happens if we remove the cut-off \( g \) in equation (37) so we consider the equation
\[
(-\Delta + m^2)(\phi) + \alpha \exp(\alpha \phi - \infty) = \xi.
\] (65)

For the meaning of this equation we refer to the discussion at the beginning of the present section. In order to distinguish between the solution to equation (37) and equation (65) we denote by \( \phi_g \) the solution of the former and by \( \phi \) the solution of the latter. We use also the symbols \( \bar{\phi}_g = \bar{\phi}_g - I \xi \) and \( \bar{\phi} = \phi - I \xi \), with \( I = (-\Delta + m^2)^{-1} \).

**Proposition 36.** For any \( |\alpha| < \alpha_{\text{max}} \) there exists a unique solution \( \phi \) to equation (65). Furthermore, for any affine transformation \( \Phi : \mathbb{R}^4 \to \mathbb{R}^4 \) in the Euclidean group of \( \mathbb{R}^4 \), the random field \( \Phi_* (\phi)(\hat{z}) = \phi(\Phi(\hat{z})) \) has the same law as \( \phi(\hat{z}) \).

**Proof.** The existence and uniqueness for the solution to equation (65) can be proven as in Lemma 30 and Lemma 31, since the estimates used to prove those lemmas do not depend on \( g \).

The invariance of the law of the solutions with respect to affine transformations in the Euclidean group of \( \mathbb{R}^4 \) follows from the invariance (in law) of the white noise \( \xi \) and of the left-hand-side of equation (65) with respect to translations and rotations and from the uniqueness of the solution to equation (65).
Theorem 37. Consider $|\alpha| < \alpha_{\text{max}}$ (see equation (1)), if $g_n$ is an (increasing) sequence of cut-offs with compact support such that $g_n \uparrow 1$ then $\bar{\phi}_{g_n} \to \bar{\phi}$ in $B_{p,p,\ell+\delta}^{s+\delta}$ (which means that $\phi_{g_n} \to \phi$ in $B_{p,p,\ell+\delta}^{0}$). This means that the sequence of probability measures $\kappa_{g_n}$ on $B_{p,p,\ell+\delta}^{0}$ converges weakly, as $n \to \infty$, to a probability measure $\kappa$ which is invariant with respect to the natural action of the Euclidean group of $\mathbb{R}^2$ on $B_{p,p,\ell+\delta}^{0}$ and does not depend on the sequence of cut-offs $g_n$.

Proof. The proof is essentially based on the fact that

$$\|\bar{\phi}_g\|_{p,p,\ell} \lesssim \|\eta\|_{B_{p,p,\ell}^{s+\delta}},$$

uniformly in $g$. From the previous inequality and some reasoning similar to the ones used in the proof of Theorem 25 the convergence follows. The properties of the limit measure $\kappa$ follow from the same properties of the solution $\phi$ to equation (65) proved in Proposition 36.

4. Elliptic quantization of the $P(\varphi)_2$ model

In this section we discuss the elliptic stochastic quantization of the $P(\varphi)_2$ model where $P$ is a polynomial of even degree and satisfying Hypothesis QC. In order to avoid technical details we consider only the case $P(\varphi) = \frac{\varphi^n}{2n}$ for $n \in \mathbb{N}$, the general case being then a straightforward generalization. We consider the equation

$$(-\Delta_x - \Delta_z + m^2)\phi + f(x)\phi^{2n-1} = \xi$$

where $x \in \mathbb{R}^2$, $z \in M = T^2$, $\phi$ stands for the Wick product and $\xi$ is a $\mathbb{R}^2 \times T^2$ white noise. Equation (66) can be better understood if we consider the equation for $\phi := \phi - \bar{\xi}$, the usual Da Prato-Debussche trick (introduced in [36]), obtaining the following equation

$$(-\Delta_x - \Delta_z + m^2)\phi + \sum_{k=0}^{2n-1} \frac{2n-1}{k!} f(x) \cdot \bar{\xi}^k \cdot \phi^{2n-1-k} = 0.$$  

Equation (67) is expected to be well defined, since we expect that the solution $\bar{\phi}$ is in $H^1(\mathbb{R}^2 \times T^2) = W^{1,2}(\mathbb{R}^2 \times T^2)$, in $L_{f,1/2n}^{2n}(\mathbb{R}^2 \times T^2)$ (where $L_{f,1/2n}^{2n}$ is the weighted $L^{2n}$ space with respect to the space weight $f^{1/2n}$) and in $B_{p,\ell}^{2-\delta}(\mathbb{R}^2 \times T^2)$, where $p = \frac{2}{2n-1}$. Furthermore it is well known that $\bar{\xi}^k \in C_{\ell}^{-\delta} = B_{\infty,\ell}^{-\delta}(\mathbb{R}^2 \times T^2)$ for any $\ell, \delta > 0$ (see [47]). In general we expect that equation (67), and so also equation (66), for any realization of the noise $\xi$ admits multiple solutions. So we need a notion of weak solution to equation (67).

First of all we consider a fixed probability space $\mathcal{W} = (C_{\ell}^{-\delta}(\mathbb{R}^2 \times T^2))^{2n} \times \mathbb{M}$, where

$$\mathbb{M} = H^{1-\delta_1}(\mathbb{R}^2 \times T^2) \cap L_{f,1/2n+\delta_1}^{2n}(\mathbb{R}^2 \times T^2) \cap B_{p,\ell}^{2-\delta_1}(\mathbb{R}^2 \times T^2),$$

and where $\delta, \delta_1, \delta_2, \delta_3, \delta_4 > 0$ are small enough (we give some more precise conditions in what follows, see inequalities (72), (74), (76), (77) and (78)), that is the space where $(\bar{\xi}, \bar{\xi}^2, \ldots, \bar{\xi}^{2n}, \phi) \in \mathcal{W}$ are defined. We consider a distinguished subspace of $\mathbb{M}$ namely

$$\mathbb{M} = H^{1}(\mathbb{R}^2 \times T^2) \cap L_{f,1/2n}^{2n}(\mathbb{R}^2 \times T^2) \cap \mathbb{M}.$$

Remark 38. Hereafter, we will always use the following additional hypothesis on the cut-off $f$:

Hypothesis $HF1$. The function $f$ satisfies Hypothesis $HF$ and furthermore $f$ satisfies the following properties, for any $x, y \in \mathbb{R}^2$ and $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq 2$

$$0 < f(x)^{\pm 1} < cf(x-y)^{\pm 1} \exp(d|y|)$$

$$|D^\alpha f(x)| \leq cf(x),$$

where $c, e > 0$ and $d \geq 0$ are some constants.
Examples of functions \( f \) satisfying Hypothesis \( Hf1 \) can be found in [58].

The importance of Hypothesis \( Hf1 \) is that an equivalent norm for \( B^s_{p,q,f} (\mathbb{R}^2 \times \mathbb{T}^2) \), i.e. the weighted Besov space with weight \( (f(x))^{\beta} \) (see [74, 75] for the precise definitions), when \( |s| < 2, -1 \leq \beta \leq 1 \) and \( p > 1 \), is given by

\[
\|u\|_{B^s_{p,q,f}} \sim \|f^{\beta} u\|_{B^s_{p,q}}
\]

(see [75] Theorem 4.4). This means that an analogous version of Proposition 52 and Proposition 54 holds also for the space \( B^s_{p,q,f} \).

**Definition 39.** A probability measure \( \nu^\varepsilon \) on \( \mathcal{W}^\varepsilon \) is a weak solution to equation (67) if the projection of \( \nu^\varepsilon \) on \( [C^{-\delta}_\varepsilon (\mathbb{R}^2 \times \mathbb{T}^2)]^{2n} \), \( \delta > 0 \), gives the law of \( (\xi, \mathcal{I}\xi^2, \ldots, \mathcal{I}^2n^{2n-1}) \) (which is the law of a Gaussian noise with covariance \( -\Delta_x - \Delta_z + m^2 \) and its Wick powers) and it is supported on the set of solutions to the equation

\[
(-\Delta_x - \Delta_z + m^2)(\overline{\partial}) + \sum_{k=0}^{2n-1} \left( \frac{2n-1}{k} \right) f(x) \cdot \sigma_k \cdot \overline{\partial}^{2n-1-k} = 0,
\]

(68)

where \( x \in \mathbb{R}^2, z \in \mathbb{T}^2, (\sigma_1, \ldots, \sigma_{2n}, \overline{\partial}) \in \mathcal{W}^\varepsilon \) and \( \sigma_0 = 1 \), defined on \( \mathcal{W}^\varepsilon \).

The weak solution \( \nu \) to equation (66) associated with the weak solution \( \nu^\varepsilon \) to equation (67), is the probability law on \( C^{-\delta}_\varepsilon (\mathbb{R}^2 \times \mathbb{T}^2) + \mathcal{W}, \delta > 0 \), given by the push-forward of \( \nu^\varepsilon \) with respect to the map \( \sigma_1 + \overline{\partial} \) defined on \( \mathcal{W}^\varepsilon \).

We introduce a modified equation on \( \mathcal{W} \) given by

\[
(-\Delta_x - \Delta_z + m^2)(\overline{\partial}) + a^{2s} \sum_{k=0}^{2n-1} \left( \frac{2n-1}{k} \right) f(x) \cdot \sigma_k \cdot \overline{\partial}^{2n-1-k} = 0
\]

(69)

where \( a_\varepsilon \) is some regular enough mollifier on \( \mathbb{T}^2 \) such that the operator \( A_\varepsilon = a_\varepsilon \ast \) satisfies Hypothesis \( H\mathcal{A} \) (for example we can take \( a_\varepsilon \) as the Green function associated with the operator \( -\epsilon \Delta_z + 1 \) for \( k \) large enough), \( \varepsilon > 0 \). Also in this case we consider a special subspace of \( \mathcal{W} \) given by

\[
\mathcal{W}_\varepsilon = A_\varepsilon (H^1(\mathbb{R}^2 \times \mathbb{T}^2)) \cap L^{2n}_{f^{1/2n}}(\mathbb{R}^2 \times \mathbb{T}^2) \cap \mathcal{W}.
\]

It is important to note that \( \mathcal{W}_\varepsilon \subset \mathcal{W} \).

**Lemma 40.** Let \( \overline{\partial} \in \mathcal{W}_\varepsilon \subset \mathcal{W} \) be a solution to equation (69) then \( \overline{\partial} \in B_{p,p}^{2-\delta'} (\mathbb{R}^2 \times \mathbb{T}^2) \) (with \( 0 < \delta < \delta'' < \delta_4 \) as in the definition of \( \mathcal{W}^\varepsilon \)) and

\[
\|\overline{\partial}\|_{H^1} + \|\overline{\partial}\|_{L^{2n}_{f^{1/2n}}} \lesssim \left[ \sum_{k=1}^{2n} \|\sigma_k\|_{C^{-\delta}_\varepsilon} \right]^\beta_1
\]

(70)

\[
\|\overline{\partial}\|_{B^{2-\delta'}_{p,p,f} - \beta} \lesssim \|A_\varepsilon^{-2} \overline{\partial}\|_{B^{2-\delta'}_{p,p,f} - \beta} \lesssim \left[ \|\overline{\partial}\|_{H^1} + \|\overline{\partial}\|_{L^{2n}_{f^{1/2n}}} + \sum_{k=1}^{2n} \|\sigma_k\|_{C^{-\delta}_\varepsilon} \right]^\beta_2
\]

(71)

for any \( 0 \leq \beta < \frac{1}{2n} - (2n-1)\delta_3 \), and where \( \beta_1, \beta_2 \in \mathbb{R}^+ \) and where \( \beta_1, \beta_2 \) and the constants implied by the symbol \( \lesssim \) do not depend on \( \varepsilon, \sigma_k \) and \( \beta \).

**Proof.** Let \( 0 < \gamma < 1 \) be such that

\[
(2n-1) \left( \frac{\gamma}{2n} + \frac{1-\gamma}{2} \right) < 1.
\]

(72)

Then by Remark 38, Proposition 52 and Proposition 54 it is simple to see that \( \overline{\partial} \in B^{(1-\gamma)}_{p,2} (\mathbb{R}^2 \times \mathbb{T}^2) \), where \( p = \left( \frac{1}{2n} + \frac{1-\gamma}{2} \right)^{-1} \), and

\[
\|\overline{\partial}\|_{B^{(1-\gamma)}_{p,2}} \lesssim \|\overline{\partial}\|_{L^{2n}_{f^{1/2n}}} \cdot \|\overline{\partial}\|_{H^1}^{1-\gamma}.
\]
Using the fact that $\hat{\theta} \in A_r(H^1)$ we can multiply both sides of equation (69) by $A^{-2}_r(\hat{\theta})$ and take the integral over $\mathbb{R}^2 \times \mathbb{T}^2$ obtaining

$$
\left\| A^{-1}_r(\hat{\theta}) \right\|_{H^1} \leq - \left\| \hat{\theta} \right\|_{L^{2n}_{j}}^{2n-1} - \sum_{k=1}^{2n-1} \left( \sum_{k=0}^{2n-1} \left( \sum_{k=0}^{2n-1} \int f(x) \cdot \sigma_k(x, z) \cdot \hat{\theta}^{2n-k} \right) dxz \right)
$$

$$
\leq - \left\| \hat{\theta} \right\|_{L^{2n}_{j}}^{2n-1} + C \sum_{k=1}^{2n-1} \| f^{1/2} \sigma_k \|_{L^{\infty}} \left[ \left\| \hat{\theta} \right\|_{L^{2n}_{j}}^{1/\gamma} \right]^{2n-k} \delta
$$

where we choose $\delta < 1 - \gamma$ and for some constant $C > 0$ depending only on $n$. Using the condition (72), the fact that $\| \hat{\theta} \|_{H^1} \leq \| A^{-1}_r(\hat{\theta}) \|_{H^1}$ and Young’s inequality for products we obtain

$$
\left\| \hat{\theta} \right\|_{H^1}^2 + \| \hat{\theta} \|_{L^{2n}_{j}}^{2n-1} \lesssim \sum_{k=1}^{2n-1} \| f^{1/2} \sigma_k \|_{L^{\infty}} \delta.
$$

(73)

for $\epsilon = (1 - (2n - 1) \left( \frac{2}{2n} + \frac{1}{2} \right))$. Let

$$
0 < \beta < \frac{1}{2n} \quad \text{and} \quad \delta < \delta" < (1 - \gamma),
$$

(74)

we have

$$
\left\| (A^{-2}_r \hat{\theta}) f^{-\beta} \right\|_{B_{p,p'}} \lessapprox \left\| (f(x))^{-\beta} \left[ \sum_{k=0}^{2n-1} \left( \sum_{k=0}^{2n-1} \int f(x) \cdot \sigma_k \cdot \hat{\theta}^{2n-k} \right) \right] \right\|_{B_{p,p'}}
$$

$$
\lesssim \sum_{k=0}^{2n-1} \left\| f(x)^{1-\beta} \cdot \sigma_k \cdot \hat{\theta}^{2n-k} \right\|_{B_{p,p'}}
$$

$$
\lesssim \sum_{k=0}^{2n-1} \left\| f^{1/2} \sigma_k \right\|_{L^{\infty}} \left\| \hat{\theta} \right\|_{H^1} \delta
$$

(75)

where $\delta > 0$ is such that $\delta > \delta"$. Inserting now inequality (73) into inequality (75), and using the fact that $\| \hat{\theta} \|_{B_{p,p'}} \lesssim (A^{-2}_r \hat{\theta}) f^{-\beta} \|_{B_{p,p'}}$ we obtain the thesis.

An easy consequence of the previous lemma is the following one.

**Lemma 41.** Let $F_\epsilon : (C_\epsilon) \to \mathcal{P}(W)_{\epsilon}$, $\epsilon > 0$, be the set-valued map associating $(\sigma_1, \ldots, \sigma_{2n-1})$ with the set of solutions to equation (69) in $W_{\epsilon}$. If $K \subset (C_\epsilon)_{2n-1}$ is a compact set then $\bigcup_{\epsilon < 1} F_\epsilon(K)$ is a subset of $W$ and it is compact in $W$.

**Proof.** The proof of the lemma consists only in noticing that a consequence of Proposition 52 is that the inclusion map $i : B^0_{p,p,f-\beta} \cap L^{2n}_{f^{1/2}} \to W$ is compact, when $\beta$ satisfies condition (74) and it is large enough and $\delta" > 0$ is small enough. Indeed using Proposition 52 and Proposition 54, it is simple to prove that $B^0_{p,p,f-\beta}$ is compactly imbedded in $B^0_{p,p}$ when

$$
\delta < \delta" < \delta_4
$$

(76)

since $f^{-\beta} \to +\infty$ as $x \to +\infty$.

Recalling that, by Remark 48 and Proposition 52, $L^p_{f^{1/2}} \subset B^0_{p.p,f^{1/2}}$ when $p \geq 2$, we obtain, using Proposition 54, that $B^0_{p,p,f-\beta} \cap L^{2n}_{f^{1/2}}$ is compactly imbedded in $B^{2-\delta'/\epsilon}_{p,\epsilon,\beta_0,\epsilon,\beta_0}$ where

$$
p_0 = \frac{2n}{2n - 2} + 1
$$

$$
\beta_0 = -\theta \left( \frac{1}{2n} + \beta + \frac{1}{2n} \right)
$$
for any \(0 \leq \theta \leq 1\). Taking \(0 < \theta < \frac{\delta''}{(2n-\delta')(2n-2)}\), \(\beta = 0\) and \(\epsilon\) small enough we have that \(B_{p,p,f-\beta}^{2-\delta''} \cap L^{2n}_{f_{1/2n+\delta_3}}\) is compactly embedded in \(L^{2n-\delta_2}_{f_{1/2n+\delta_3}}\) for any \(\delta_2 < 1\) and \(\delta_3 > 0\).

Furthermore if we take
\[
\frac{1-\delta_1}{2-\delta''} < \theta < \frac{n-1}{2n-2}
\]
\[
\left(\frac{2-\delta''}{1-\delta_1}\right) \frac{1}{2n} - \frac{1}{2n} < \beta < \frac{1}{2n},
\]
which is a non-void set of conditions whenever
\[
\delta < \delta'' < 1
\] (77)

\[
0 < \delta < \delta'' < \delta_1 \frac{2n-2}{n-1},
\] (78)

we obtain that \(B_{p,p,f-\beta}^{(2-\delta'')\theta-\epsilon} \cap L^{2n}_{f_{1/2n}}\) compactly embeds in \(H^{1-\delta_1}\) for \(\epsilon\) small enough.

Using inequalities (70) and fact that the balls of \((B_{p,p,f-\beta}^{2-\delta''} \cap L^{2n}_{f_{1/2n}}\) are closed sets of \(\mathcal{M}\), since the embedding map \(i: B_{p,p,f-\beta}^{2-\delta''} \cap L^{2n}_{f_{1/2n}} \rightarrow \mathcal{M}\) is injective and continuous, \(\bigcup_{\epsilon < 1} F_{\epsilon}(K)\) is contained in \(\mathcal{M}\).

Hereafter we denote by \(\mathcal{P} : \mathcal{W}^\alpha \rightarrow (C^{\epsilon}_0(\mathbb{R}^2 \times \mathcal{T}^2))^{2n}\) the natural projection of the Cartesian product \(\mathcal{W}^\alpha = (C^{\epsilon}_0(\mathbb{R}^2 \times \mathcal{T}^2))^{2n} \times \mathcal{M}\) on the first component.

**Lemma 42.** Let \(\nu^\epsilon\) be a sequence of weak solutions to equation (69) such that \(\nu^\epsilon(((C^{\epsilon}_0(\mathbb{R}^2 \times \mathcal{T}^2))^{2n} \times \mathcal{M})\), 1, and suppose that \(\mathcal{P}^{-1}((\nu^\epsilon))\) is tight. Then \(\nu^\epsilon\) is tight. Furthermore suppose that \(\mathcal{P}_x(\nu^\epsilon)\) weakly converges to the law of \((I_\xi, I_{\xi^0}, \ldots, I_{\xi^{2n-1}})\) as \(\epsilon \rightarrow 0\), then any convergent subsequence \(\nu^\epsilon_n\) converges, as \(\epsilon_n \rightarrow 0\), to a weak solution \(\nu^\epsilon\) to equation (67) such that \(\nu^\epsilon(((C^{\epsilon}_0(\mathbb{R}^2 \times \mathcal{T}^2))^{2n} \times \mathcal{M})\) = 1.

**Proof.** The proof of the tightness of \(\nu^\epsilon\) is similar to the one of Lemma 14, where Lemma 13 is replaced by Lemma 41. The hypothesis that \(\nu^\epsilon(((C^{\epsilon}_0(\mathbb{R}^2 \times \mathcal{T}^2))^{2n} \times \mathcal{M})\) = 1 enters in the proof in the following way: let \(\tilde{F}_x : (C^{\epsilon}_0(\mathbb{R}^2 \times \mathcal{T}^2))^{2n} \rightarrow \mathcal{P}(\mathcal{M})\), \(\epsilon > 0\), be the set-valued map associating \((\sigma_1, \ldots, \sigma_{2n-1})\) with the set of solutions to equation (69) in \(\mathcal{M}\), let \(K \subset (C^{\epsilon}_0(\mathbb{R}^2 \times \mathcal{T}^2))^{2n}\) be a compact such that \(\mu_x(K) > 1 - \epsilon\), where \(\mu_x\) are the probability distributions of \((I_\xi, I_{\xi^0}, \ldots, I_{\xi^{2n-1}})\) which are tight since \(\mu\) converges to \(\mu\), the probability distribution of \((I_\xi, I_{\xi^0}, \ldots, I_{\xi^{2n-1}})\), as \(\epsilon \rightarrow 0\). Furthermore denote \(\hat{R} := \cup_x \tilde{F}_x(K)\), which is compact by Lemma 41, then
\[
\nu^\epsilon(K \times \hat{R}) \geq \nu^\epsilon(K \times \cup_x \tilde{F}_x(K)) \geq \nu^\epsilon(K \times F_x(K)) = \nu^\epsilon(K \times F_x(K)) = \nu^\epsilon(K \times F_x(K)) = \mu_x(K) \geq 1 - \epsilon.
\]
where in the fourth passage we used that \(\nu^\epsilon(K \times F_x(K)) \geq \nu^\epsilon(K \times F_x(K))\) since \(\nu^\epsilon(((C^{\epsilon}_0(\mathbb{R}^2 \times \mathcal{T}^2))^{2n} \times \mathcal{M})\) = 1.

Since \(\mathcal{P}_x(\nu^\epsilon)\) weakly converges, as \(\epsilon \rightarrow 0\), to the law of \((I_\xi, I_{\xi^0}, \ldots, I_{\xi^{2n-1}})\) and so \(\mathcal{P}_x(\nu^\epsilon)\) has the same law as \((I_\xi, I_{\xi^0}, \ldots, I_{\xi^{2n-1}})\), the proof of the fact that \(\nu^\epsilon\) is a weak solution to equation (67) is equivalent to proving that \(\nu^\epsilon\) is such that for any \(C^1\) bounded function \(F\) from \(B_{p,p}^{2-\delta''}\) into \(\mathbb{R}\) with bounded derivative we have
\[
\int F\left((-\Delta_x - \Delta_\theta + m^2)(\bar{\theta}) + \sum_{k=0}^{2n-1} \binom{2n-1}{k} f(x)\sigma_k \bar{\theta}^{2n-1-k}\right) \nu^\epsilon = \int F(E)\nu^\epsilon = F(0).
\] (79)

We know that \(\int F(E)\nu^\epsilon = F(0)\) with
\[
E^\epsilon = (\Delta_x - \Delta_\theta + m^2)(\bar{\theta}) + \alpha^2 (\sum_{k=0}^{2n-1} \binom{2n-1}{k} f(x) \cdot \sigma_k \cdot \bar{\theta}^{2n-1-k}).
\]

On the other hand \(E^\epsilon\) converges to \(E^\epsilon\), as \(\epsilon \rightarrow 0\), uniformly on compact sets (since \(\alpha^2\) strongly converges as an operator to the identity on \(B_{p,p}^{2-\delta''}\)), which implies that \(F \circ E^\epsilon\) converges to \(F \circ E\) uniformly on compact sets, as \(\epsilon \rightarrow 0\).
Finally since the sets of the form $\bigcup_{\epsilon<1} F_\epsilon (K)$ are subsets of $\mathfrak{W}$, they are closed and they have an arbitrary measure we have that any limit $\nu^\epsilon$ is such that $\nu^\epsilon ((C^\infty_\delta (\mathbb{R}^2 \times \mathbb{T}^2))^{2n} \times \mathfrak{W}) = 1$.

This fact and the boundedness of $F$ and of its derivatives implies (using a reasoning similar to the one of Lemma 15, see also Lemma 2 in [8]) that $F \circ E_\epsilon d\nu^\epsilon$ weakly converges to $F \circ E d\nu^\epsilon$, as $\epsilon \to 0$. This implies equation (79) and thus the thesis of the lemma.

**Theorem 43.** There exists at least one weak solution $\nu^\epsilon$ to equation (67) such that, for any continuous bounded function $F : B_{p,p,\ell}(\mathbb{T}^2) \to \mathbb{R}$ (where $\delta > 0$ and $\ell > 0$ as in the definition of $\mathcal{W}^\epsilon$ and $p = \frac{2n}{2n-1}$),

$$
\int_{\mathcal{W}^\epsilon} F(\phi(0,\cdot)) \tilde{Y}_f (\sigma, \theta) d\nu^\epsilon (\sigma, \theta) = Z_f \int_{B_{p,p,\ell}(\mathbb{T}^2)} F(\omega) d\kappa(\omega)
$$

(80)

where $\phi = \sigma_1 + \tilde{\theta}$ (with $\sigma_1$ and $\tilde{\theta}$ as in Definition 39),

$$
\tilde{Y}_f (\sigma, \theta) = \exp \left( \sum_{k=0}^{2n} \binom{2n}{k} \langle \sigma_1 \theta^{2n-k}, f \rangle \right),
$$

$$
Z_f = \int_{\mathcal{W}^\epsilon} \tilde{Y}_f (\sigma, \theta) d\nu^\epsilon (\sigma, \theta)
$$

and $\kappa$ is given by

$$
\frac{d\kappa}{d\mu^{-\Delta_\epsilon}} = \frac{\exp \left( \int_{\mathbb{T}^2} \rho_{2m}(z) d\tilde{z} \right)}{Z_\kappa},
$$

where $\mu^{-\Delta_\epsilon}$ is the Gaussian with covariance given by (15) with $\mathfrak{L} = -\Delta_\epsilon$ and $A = \text{id}$.

**Lemma 44.** Let $\nu^\epsilon_\cdot$ be a sequence of weak solutions to equation (69) such that $\mathfrak{P}_\cdot (\nu^\epsilon_\cdot) \sim (a_{+} \ast I \xi, \ldots, (a_{+} \ast I \xi)^{\ast 2n-1})$ (where $\sim$ means “with the same law”) and $\nu^\epsilon_\cdot ((C^\infty_\delta (\mathbb{R}^2 \times \mathbb{T}^2))^{2n} \times \mathfrak{W}) = 1$ then

$$
\int \exp \left( p \sum_{k=0}^{2n} \binom{2n}{k} \langle \sigma_k, f^\epsilon \theta^{2n-k} \rangle \right) d\nu^\epsilon < C_p
$$

for some constant $C_p$ uniform in $\epsilon$ small enough and depending on $p \geq 1$.

**Proof.** We want to use Nelson’s trick to prove the theorem (see Chapter V of [77]). Then we put $E(\sigma_k, \theta) = \sum_{k=0}^{2n} \binom{2n}{k} \langle \sigma_k, f^\epsilon \theta^{2n-k} \rangle$ and introduce the following expression

$$
E_{\epsilon,N} (\sigma_k, \theta) = \begin{cases} 
\int_{\mathbb{R}^2 \times \mathbb{T}^2} \sum_{k=0}^{2n} \binom{2n}{k} \left( a_{+} - \epsilon (\sigma_1)(\tilde{z}) \right)^{\otimes k} f^\epsilon(x) \theta^{2n-k}(\tilde{z}) d\tilde{z}, & \text{for } \epsilon < \frac{1}{N}, \\
\int_{\mathbb{R}^2 \times \mathbb{T}^2} \sum_{k=0}^{2n} \binom{2n}{k} (\sigma_1(\tilde{z}))^{\otimes k} f^\epsilon(x) \theta^{2n-k}(\tilde{z}) d\tilde{z}, & \text{for } \epsilon \geq \frac{1}{N}.
\end{cases}
$$

We have to prove that, for any $p \geq 1$,

$$
\left( \int |E - E_{\epsilon,N}|^p d\nu^\epsilon_\cdot \right)^{1/p} \lesssim (p-1)^u N^{-\alpha}
$$

(81)

for some $u \in \mathbb{N}$ and $\alpha > 0$ and

$$
E_{\epsilon,N} \lesssim -f^\epsilon (x) (\log(N))^{\alpha'},
$$

(82)

for some $\alpha' > 0$, and that both inequalities are uniform in $\epsilon$. Inequality (81) is obvious when $\epsilon \geq \frac{1}{N}$, since then $E - E_{\epsilon,N} = 0$ $\nu^\epsilon_\cdot$-almost surely. Consider the case $\epsilon < \frac{1}{N}$ then

$$
|E - E_{\epsilon,N}| \lesssim \sum_{k=0}^{2n} \int_{\mathbb{R}^2 \times \mathbb{T}^2} f^\epsilon(x) \left( \sigma_k(z) - \left(a_{+} - \epsilon (\sigma_1)(\tilde{z}) \right)^{\otimes k} \theta^{2n-k}(\tilde{z}) \right) d\tilde{z}.
$$
\[
\lesssim \sum_{k=1}^{2n} \left\| f'(x) \right\|_{1/2n} \left( \sigma_k(z) - \left( a_{\frac{1}{n}} - \epsilon (\sigma_1)(\hat{z}) \right)^{\diamond k} \right) \right\|_{\mathcal{C}^{-\delta}} \cdot \left\| f'(x) \right\|_{1-1/2n} \| \bar{\theta} \|_{H_1}^{2n-k},
\]
where \( p = \frac{2n}{2n-1} \). On the other hand, by Lemma 40, we have:
\[
\left\| f'(x) \right\|^{1-1/2n} \| \bar{\theta} \|^{2n-k}_{H_1^p} \lesssim \left( \| \bar{\theta} \|_{L^{2n-1/2n}} + \| \bar{\theta} \|_{H^1} \right)^{2n-k} \lesssim \left( \sum_{k=1}^{2n} \| \sigma_k \|_{\mathcal{C}^{-\delta}} \right)^{\beta_1(2n-1)},
\]
\( \nu_c^\epsilon \)-almost surely and uniformly in \( \epsilon \) (since \( \nu_c^\epsilon (\mathcal{C}^{-\delta}((\mathbb{R}^2 \times \mathbb{T}^2))^{2n} \times \mathbb{M}_e) = 1 \)). Using Hölder’s inequality we obtain, for any \( p \geq 1 \),
\[
\left( \int |E - E_{\epsilon,N}|^p d\nu_c^\epsilon \right)^{\frac{1}{p}} \lesssim \sum_{k=1}^{2n} \left[ \int \left\| f'(x) \right\|_{1/2n} \left( \sigma_k(z) - \left( a_{\frac{1}{n}} - \epsilon (\sigma_1)(\hat{z}) \right)^{\diamond k} \right)^{2p} d\nu_c^\epsilon \right] \times \left( \int \left( \sum_{k=1}^{2n} \| \sigma_k \|_{\mathcal{C}^{-\delta}} \right)^{2\beta_1(2n-1)p} d\nu_c^\epsilon \right)^{\frac{1}{p}},
\]
Since the law of \( \sigma_k \) is a multilinear functional of a Gaussian random field, by hypercontractivity we have that there exists \( m_1, m_2 \in \mathbb{N} \) and \( \alpha > 0 \) such that
\[
\left( \int \left\| f'(x) \right\|_{1/2n} \left( \sigma_k(z) - \left( a_{\frac{1}{n}} - \epsilon (\sigma_1)(\hat{z}) \right)^{\diamond k} \right)^{2p} d\nu_c^\epsilon \lesssim (p-1)^{p_1} N^{-\alpha}
\]
\[
\int \left( \sum_{k=1}^{2n} \| \sigma_k \|_{\mathcal{C}^{-\delta}} \right)^{2\beta_1(2n-1)p} d\nu_c^\epsilon \lesssim (p-1)^{p_2}
\]
uniformly in \( \epsilon \). This proves inequality (81). We note that
\[
\sum_{k=0}^{2n} \binom{2n}{k} \left( a_{\frac{1}{n}} - \epsilon (\sigma_1)(\hat{z}) \right)^{\diamond k} f'(x) \bar{\theta}^{2n-k}(\hat{z}) = f'(x) H_{2n} \left( a_{\frac{1}{n}} - \epsilon (\sigma_1)(\hat{z}) + \bar{\theta}(\hat{z}); c_{N,\epsilon} \right)
\]
where \( H_{2n}(x; \lambda) \) is the Hermite polynomial of degree \( 2n \) of a Gaussian random variable with variance \( \lambda \) and \( c_{N,\epsilon} = \mathbb{E} \left[ \left( a_{\frac{1}{n}} * (I \xi) \right)^2 \right] \sim \log(N) \), as \( N \to +\infty \) for any \( \epsilon > 0 \). It is simple to see that, for any \( x \in \mathbb{R}^2 \) and \( \hat{z} = (x, z) \in \mathbb{R}^2 \times \mathbb{T}^2 \),
\[
f'(x) H_{2n} \left( a_{\frac{1}{n}} - \epsilon (\sigma_1)(\hat{z}) + \bar{\theta}(\hat{z}); c_{N,\epsilon} \right) \lesssim a_n f'(x) \left( a_{\frac{1}{n}} - \epsilon (\sigma_1)(\hat{z}) + \bar{\theta}(\hat{z}) \right)^{2n}
\]
\[
+ f'(x) \sum_{k=1}^{m} a_k \left( a_{\frac{1}{n}} - \epsilon (\sigma_1)(\hat{z}) + \bar{\theta}(\hat{z}) \right)^{2n-2k} c_{N,\epsilon}^k
\]
\[
\lesssim a_n f'(x) \left( a_{\frac{1}{n}} - \epsilon (\sigma_1)(\hat{z}) + \bar{\theta}(\hat{z}) \right)^{2N} - f'(x) c_{N,\epsilon}^n
\]
\[
\lesssim - f'(x) c_{N,\epsilon}^n \lesssim - f'(x) (\log(N))^n,
\]
where we used that \( f' \leq 0 \) and \( a_N, \bar{a}_N > 0 \). This proves inequality (82).

Now we use Lemma V.5 of [77] and the fact that inequality (81) and (82) are uniform in \( \epsilon \), to prove that there exist two constants \( b, \alpha'' > 0 \) depending only on \( f' \) and \( m \) but not on \( \epsilon > 0 \) such that
\[
\nu_c^\epsilon (E(\sigma, \bar{\theta}) > b \log(K)) \leq e^{-K''}.
\]
Inequality (83) implies that \( \exp(E) \in L^p(\nu_c^\epsilon) \) for any \( p \geq 1 \), and we have also a uniform bound on the \( L^p \) norm of \( \exp(E) \) with respect to \( \epsilon > 0 \). This concludes the proof of the lemma. \( \square \)

**Proof of Theorem 43** First we introduce the equation
\[
(-\Delta_x - \Delta_z + m^2)(\phi) + a_{\epsilon,2}^2 (H_{2n}(\phi; c_{\epsilon})) = a_{\epsilon,2} \xi,
\]
(84)
where $c_\epsilon = \mathbb{E}[(\alpha, \epsilon:\xi)^2]$. Equation (84) is of the form (2), and the potential $V_\epsilon(y) = H_{2n}(y; c_\epsilon)$, where $y \in \mathbb{R}$, satisfies Hypothesis QC since it is an even polynomial with positive leading coefficient. This means that, for any $\epsilon > 0$, there exists a weak solution $\nu_\epsilon$ to equation (84), such that

$$
\int_{\mathcal{W}_\epsilon} F(\phi(0, \cdot)) \gamma_{\epsilon}(\phi) d\nu_\epsilon = \int_{\mathcal{W}_\epsilon} F(\omega) d\kappa(\omega)
$$

where $\gamma_{\epsilon}(\phi) = \int_{\mathbb{R}^2 \times \mathbb{T}^2} f'(x) H_{2n}(\phi(\hat{\xi}); c_\epsilon) d\hat{\xi}$. On the other hand if by $\nu_{\epsilon}^I$ we denote the law of $(\alpha_\epsilon, \epsilon:\xi, \ldots, (\alpha_\epsilon, \epsilon:\xi)^{2n}, \phi - \alpha_\epsilon, \epsilon:\xi)$ (where $\phi$ has law $\nu$ and $\alpha_\epsilon, \epsilon:\xi$ is given by equation (84)), we have that $\nu_{\epsilon}^I$ is a solution to equation (69) and

$$
\int_{\mathcal{W}_\epsilon} F(\phi(0, \cdot)) \gamma_{\epsilon}(\phi) d\nu_\epsilon = \int_{\mathcal{W}_\epsilon} F(\phi(0, \cdot)) \tilde{\gamma}_{\epsilon}(\phi) d\nu_{\epsilon}^I(\phi).
$$

Using the Galerkin approximation of Section 2.1 and a method similar to the one of Lemma 42, it is possible to prove that $\nu_{\epsilon}^I((C^0_{-\delta}(g^2 \times \mathbb{T}^2))^{2n} \times \mathbb{T}^n) = 1$.

By Lemma 33 and Lemma 34 we have that the measure $F(\phi(0, \cdot)) d\nu_{\epsilon}^I$ weakly converges to $F(\phi(0, \cdot)) d\nu^I$, as $\epsilon \to 0$. Let $G_{\epsilon} : \mathcal{W}_\epsilon \to \mathbb{R}$, $r \in \mathbb{N}$, be a sequence of continuous functions such that $0 \leq G_\epsilon \leq 1$, $G_\epsilon = 1$ when $\tilde{\gamma}_{\epsilon}(\sigma, \emptyset) \leq r$ and $G_\epsilon = 0$ when $\tilde{\gamma}_{\epsilon}(\sigma, \emptyset) \geq 2r$ (the existence of this kind of functions follows from the fact that $\tilde{\gamma}_{\epsilon}(\sigma, \emptyset)$ is continuous on $\mathcal{W}_\epsilon$). Since $F(\phi(0, \cdot)) d\nu_{\epsilon}^I$ weakly converges to $F(\phi(0, \cdot)) d\nu^I$ then $\int_{\mathcal{W}_\epsilon} G_{\epsilon} F \tilde{\gamma}_{\epsilon} d\nu_{\epsilon}^I \to \int_{\mathcal{W}_\epsilon} G_{\epsilon} F \tilde{\gamma}_{\epsilon} d\nu^I$, as $\epsilon \to 0$. On the other hand, by Lemma 44, for all $\epsilon > 0$ and any $r > 0$,

$$
\left| \int_{\mathcal{W}_\epsilon} (G_{\epsilon} - 1) F \tilde{\gamma}_{\epsilon} d\nu_{\epsilon}^I \right| \leq \left| \int_{\tilde{\gamma}_{\epsilon} \geq r} |F| \tilde{\gamma}_{\epsilon} d\nu_{\epsilon}^I - \int_{\tilde{\gamma}_{\epsilon} \geq r} |F| \tilde{\gamma}_{\epsilon} d\nu^I \right| \leq C_p \frac{|F|_{\infty}}{p-1}
$$

for any $p \geq 1$ (and a similar inequality is true for $\epsilon = 0$). Taking $r \to +\infty$ the thesis follows. \hfill \Box

### Appendix A: Besov spaces

In this appendix we recall some results on weighted Besov spaces used in this paper. We consider only the case of Besov spaces defined on $\mathbb{R}^n$ but all what follows holds also for Besov spaces on $\mathbb{T}^n$ or on $\mathbb{R}^{n_1} \times \mathbb{T}^{n_2}$ (like the ones used in Section 4).

First we recall the definition of Littlewood-Paley blocks: let $\chi, \varphi$ be smooth non-negative functions from $\mathbb{R}^n$ into $\mathbb{R}$ such that

- $\text{supp}(\chi) \subset B_{\frac{1}{2}}(0)$ and $\text{supp}(\varphi) \subset B_{\frac{1}{2}}(0) \setminus B_{\frac{1}{4}}(0)$,
- $\chi \varphi \leq 1$ and $\chi(y) + \sum_{j \geq 0} \varphi(2^{-j}y) = 1$ for any $y \in \mathbb{R}^n$.
- $\text{supp}(\chi) \cap \text{supp}(\varphi(2^{-j} \cdot)) = 0$ for $i \geq 1$,
- $\varphi(2^{-j} \cdot) \cap \text{supp}(\varphi(2^{-i} \cdot)) = 0$ if $|i - j| > 1$,

where by $B_r(x)$ we denote the ball of center $x \in \mathbb{R}^n$ and radius $r > 0$.

We introduce the following notations: $\varphi_{-1} = \chi$, $\varphi_j(\cdot) = \varphi(2^{-j} \cdot)$, $D_j = \varphi_j$ and for any $f \in \mathcal{S}(\mathbb{R}^n)$ we put $\Delta_j(f) = D_j * f$. Furthermore we write, for any $\ell > 0$, $r_\ell(y) = (1 + |y|^2)^{-\ell/2}$, $L^p_l(\mathbb{R}^d)$ is the $L^p$ space with respect to the norm

$$
\|f\|_{p,\ell} = \left( \int_{\mathbb{R}^n} (f(y)r_\ell(y))^p dy \right)^{1/p},
$$

where $p \in [1, +\infty]$.

**Definition 45.** Consider $s \in \mathbb{R}$, $p, q \in [1, +\infty]$ and $\ell \in \mathbb{R}$. If $f \in \mathcal{S}(\mathbb{R}^n)$ we define the norm

$$
\|f\|_{B^s_{p, q, \ell}} = \left( \sum_{j \geq -1} 2^{sq} \|\Delta_j(f)\|_{p, \ell}^q \right)^{1/q}.
$$

The space $B^s_{p, q, \ell}$ is the subset of $\mathcal{S}'(\mathbb{R}^n)$ such that the norm (85) is finite.
Remark 46. It is important to note that $B^s_{2,2}$ is equal to the weighted Sobolev space $W^{s,2}_x = H^s_x$ of $L^2$ distributions, and that for $s > 0$, $s \notin \mathbb{N}$, $B^s_{\infty,\infty}$ is equal to $C^s$, the space of Hölder functions of regularity $s$. Moreover we remark that $B^s_{p,q} = B^s_{p,q,0}$.

Remark 47. It is always possible to choose $\chi$ and $\varphi$ in such a way that there exists some constants $\gamma_{-1}, \gamma > 0$ and $0 < \theta_{-1}, \theta < 1$ such that

$$|D_{-1}(y)| \lesssim \exp(-\gamma_{-1}|y|^{\theta_{-1}}) \quad \text{and} \quad |D_0(y)| \lesssim \exp(-\gamma|y|^{\theta}),$$

(see, i.e., [73] Section 1.2.2 Proposition 1 or [63]).

Remark 48. In this appendix we treat only the case of polynomial weights. More generally all the statements here presented can be extended to the case of regular enough weights $\omega$ satisfying the following conditions

$$0 < \omega(y)^{\pm 1} < c\omega(w - y)^{\pm 1} \exp(d|w|) \quad (86)$$

$$|D^\alpha \omega(y)| \leq c\omega(y) \quad (87)$$

where $c, d, e > 0$ and $y, w \in \mathbb{R}^n$. Here we have decided to present the case with polynomial weights because for finite distributions with respect to weights satisfying only the conditions $(86)$ and $(87)$ the Fourier transform is not defined. This means that we cannot use directly Definition 45 and a more technical discussion is needed. We refer to [74, 75] for the study of weighted Besov spaces with a weight satisfying the conditions $(86)$ and $(87)$.

We denote by $\mathcal{F}_r$, with $r \in \mathbb{N}$, the space of functions $f \in C^r(\mathbb{R}^n)$ with support in $B_1(0)$ and norm $\|f\|_{C^r(\mathbb{R}^n)} \leq 1$. If $f \in \mathcal{F}_r$ we write $f_{y,\gamma}(\cdot) = \lambda^{-\gamma} f\left(\frac{\cdot - y}{\lambda}\right)$, $y \in \mathbb{R}^n$ and $\gamma > 0$.

Proposition 49. For any $s < 0$, $p, q \in [1, +\infty]$ and $\ell \in \mathbb{R}$ an equivalent norm in the space $B^s_{p,q,\ell}$ is given by the following expression

$$\|f\|_{B^s_{p,q,\ell}} \sim \left( \int_0^1 \|\sup_{y \in \mathcal{F}_r} |\langle f, g_{y,\gamma}\rangle|\|_{p,q,\ell}^q \, d\lambda \right)^{1/q}. \quad (88)$$

where $r \in \mathbb{N}$ is the first integer bigger than $-s$ when $s < 0$.

Proof. Theorem 6.15, in [80], proves the equivalence between the norm $(85)$ and the norm of $B^s_{p,q,\ell}$ built using wavelets, while Proposition 2.4 in [50] proves the equivalence between the norm of $B^s_{p,q,\ell}$ built using wavelets and the norm $(88)$. Combining the two results we obtain the thesis. \hfill \square

If $f$ is a measurable function and $N \in \mathbb{N}$ we define

$$\Delta^N_h (f)(y) = \sum_{i=0}^N (-1)^{N-i} \binom{N}{i} f(y + hi)$$

for $x, h \in \mathbb{R}^n$.

Proposition 50. For any $s > 0$, $p, q \geq 1$, $\ell \in \mathbb{R}$, and $N, M \in \mathbb{N}_0$ such that $N < s < M$ an equivalent norm in the space $B^s_{p,q,\ell}$ is given by the following expression

$$\|f\|_{B^s_{p,q,\ell}} \sim \|f\|_{L^p_\ell} + \sum_{\alpha \in \mathbb{N}^n, |\alpha| = N} \left( \int_{|h| < 1} |h|^{-(s-N)q} \|\Delta^N_h M D^\alpha f\|_{L^q_\ell}^q \frac{dh}{|h|^n} \right)^{\frac{1}{q}}. \quad (89)$$

Proof. The proof is given in [80] Theorem 6.9 for the case $N = 0$. The generic case $N > 0$ can be proved applying the techniques of Corollary 2 Section 2.6.1 of [78] (where formula $(89)$ is proven for Besov spaces without weight). \hfill \square

Remark 51. An important consequence of Proposition 50 is that for any $N \in \mathbb{N}$ such that $N < s$ we have that an equivalent norm of $B^s_{p,q,\ell}$ is given by

$$\|f\|_{B^s_{p,q,\ell}} \sim \|f\|_{L^p_\ell} + \sum_{\alpha \in \mathbb{N}^n, |\alpha| = N} \|D^\alpha f\|_{B^{s-N}_{p,q,\ell}}. \quad (90)$$
Proposition 52. Consider $p_1, p_2, q_1, q_2 \in [1, \infty]$, $\ell_1, \ell_2 \in \mathbb{R}$ and $s_1, s_2 \in \mathbb{R}$ such that $s_1 - \frac{p_1}{p_1} > s_2 - \frac{p_2}{p_2}$ and $\ell_1 > \ell_2$ then $B^{s_1}_{p_1, q_1, \ell_1} \subset B^{s_2}_{p_2, q_2, \ell_2}$ and the immersion is compact.

Furthermore for $1 \leq p \leq 2$

$$B^{s}_{p, p, \ell} \subset W^{s,p}_\ell \subset B^{s}_{p, 2, \ell};$$

for $2 \leq p < \infty$

$$B^{s}_{p, 2, \ell} \subset W^{s,p}_\ell \subset B^{s}_{p, p, \ell}$$

and for $p = \infty$ $W^{s,p}_\ell \subset B^{s}_{\infty, \infty}$. Each of above immersions is continuous.

**Proof.** The proof of the first part of the proposition can be found in Theorem 6.7 in [80].

The second part of the proposition is proved in Theorem 6.4.4 and Theorem 6.2.4 of [31] for unweighted spaces. The result for weighted spaces follows from the fact that $f \in B^{s}_{p,q,\ell}$ and $g \in W^{s,p}_\ell$ if and only if $f \cdot r_\ell \in B^{s}_{p,q}$ and $g \cdot r_\ell \in W^{s,p}$ (see [80] Theorem 6.5 point ii).

Proposition 53. Consider $p_1, p_2, q_1, q_2, \ell_1, \ell_2, \ell_3 \in [1, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ and $q_1 = q_3$ and $q_2 = \infty$. Moreover consider $\ell_1, \ell_2, \ell_3 \in \mathbb{R}$ with $\ell_1 + \ell_2 = \ell_3$ and consider $s_1 < 0$ $s_2 \geq 0$ and $s_3 = s_1 + s_2 > 0$. Finally, consider the bilinear functional $\|f \cdot g\|_{\mathcal{S}(\mathbb{R}^n)} = \|f\|_{\mathcal{S}(\mathbb{R}^n)}$ taking values in $\mathcal{S}(\mathbb{R}^n)$. Then there exists a unique continuous extension of $\Pi$ as the map

$$\Pi : B^{s_1}_{p_1, q_1, \ell_1} \times B^{s_2}_{p_2, q_2, \ell_2} \to B^{s_3}_{p_3, q_3, \ell_3}$$

and we have, for any $f, g$ for which the norms are defined:

$$\|\Pi(f, g)\|_{B^{s_3}_{p_3, q_3, \ell_3}} \leq \|f\|_{B^{s_1}_{p_1, q_1, \ell_1}} \|g\|_{B^{s_2}_{p_2, q_2, \ell_2}}.$$  

**Proof.** The proof can be found in [63] in Section 3.3 for Besov spaces with exponential weights. The proof for polynomial weights is similar.

Proposition 54. Consider $p_1, p_2, q_1, q_2 \in [1, \infty]$, $\ell_1, \ell_2, \ell_3 \in \mathbb{R}$ and $s_1, s_2, s_3 \in \mathbb{R}$. For any $\theta \in [0, 1]$ we write $\frac{\theta}{q_1} + \frac{1-\theta}{q_2} = \frac{1}{q_0}$, $\theta \ell_1 + (1-\theta) \ell_2 = \ell_0$ and $\theta s_1 + (1-\theta) s_2 = s_0$. If $f \in B^{s_1}_{p_1, q_1, \ell_1} \cap B^{s_2}_{p_2, q_2, \ell_2}$ then $f \in B^{s_0}_{p_0, q_0, \ell_0}$ and further

$$\|f\|_{B^{s_0}_{p_0, q_0, \ell_0}} \leq \|f\|^\theta_{B^{s_1}_{p_1, q_1, \ell_1}} \|f\|^{1-\theta}_{B^{s_2}_{p_2, q_2, \ell_2}}.$$  

Furthermore if $f \in W^{s_1}_{\ell_1, p_1} \cap W^{s_2}_{\ell_2, p_2}$ then $f \in W^{s_0}_{\ell_0, p_0}$, and furthermore

$$\|f\|_{W^{s_0}_{\ell_0, p_0}} \leq \|f\|^\theta_{W^{s_1}_{\ell_1, p_1}} \|f\|^{1-\theta}_{W^{s_2}_{\ell_2, p_2}}.$$  

**Proof.** The proof is based on the fact that the complex interpolation $(B^{s_1}_{p_1, q_1, \ell_1}, B^{s_2}_{p_2, q_2, \ell_2})_{\theta}$ of the two spaces $B^{s_1}_{p_1, q_1, \ell_1}$, $B^{s_2}_{p_2, q_2, \ell_2}$ is given by $B^{s_0}_{p_0, q_0, \ell_0}$. This interpolation is proved in [31] Theorem 6.4.5 point (6) for unweighted space. For weighted space it follows from the fact that $f \in B^{s_1}_{p_1, q_1, \ell_1}$ if and only if $f \cdot r_\ell \in B^{s_1}_{p_1, q_1}$. A similar reasoning, using the interpolation proved in [31] Theorem 6.4.5 point (7), holds also for the second inequality.

**References**


