ON THE CRITICAL BRANCHING RANDOM WALK III: THE CRITICAL DIMENSION

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ABSTRACT. In this paper, we study the critical branching random walk in the critical dimension, four. We provide the asymptotics of the probability of visiting a fixed finite set and the range of the critical branching random walk conditioned on the total number of offspring. We also prove that conditioned on visiting a finite set, the first visiting point converges in distribution, when the starting point tends to infinity.

1. INTRODUCTION

In this paper, we continue our study on critical branching random walks. A branching random walk is a random process that consists of a finite number of particles performing independent random walks. At every time step, each particle gives birth to a random number of new particles and then dies (due to a prefixed probability measure on \( \mathbb{N} \), called the offspring distribution); the new particles then begin independent random walks (due to another prefixed probability measure on \( \mathbb{Z}^d \), called the jump distribution) starting from the location of their parents. We consider only the critical branching random walk, where ‘critical’ means that the expected number of children of each particle (i.e., the mean of the offspring distribution) is one.

In the previous paper \([9]\), we introduce branching capacity and use it to develop a potential theory for the critical branching random walk in \( \mathbb{Z}^d \), \( d \geq 5 \), in parallel with the classical one for ordinary random walks. One key result is that, when \( d \geq 5 \), fix a finite subset \( K \subseteq \mathbb{Z}^d \) and write \( S_x \) for the branching random walk starting from \( x \), then,

\[
\lim_{x \to \infty} \|x\|^{d-2} \cdot P(S_x \text{ visits } K) = a_d B\text{Cap}(K);
\]

\[
\lim_{x \to \infty} P(S_x(\tau_K) = a|S_x \text{ visits } K) = E\text{S}_K(a)/B\text{Cap}(K),
\]

\[
\lim_{x \to \infty} P(S_x(\xi_K) = a|S_x \text{ visits } K) = E\text{S}_K(a)/B\text{Cap}(K),
\]

where \( \tau_K \) and \( \xi_K \) are the first and the last visiting vertices (according to the depth-first search on the corresponding family tree) respectively, 

\[
a_d = \frac{\Gamma((d-2)/2)}{2^{(d-2)/2} \pi^{(d-3)/2} \sqrt{\det Q}},
\]

\[
\|x\| = \sqrt{x \cdot Q^{-1} x}/\sqrt{d}
\]

with \( Q \) being the covariance matrix of the jump distribution, and \( E\text{S}_K(a), E\text{S}_K(a) \) and \( B\text{Cap}(K) \) are introduced in that paper.

In this paper we concentrate on the case \( d = 4 \) and provide the asymptotics of the probability of visiting a finite subset by a branching random walk and some related results. Before stating our results, let us specify the assumptions that will be in force throughout this paper. We always assume:

- the offspring distribution \( \mu \) is a given distribution on \( \mathbb{N} \) with mean one and finite variance \( \sigma^2 > 0 \);
• the jump distribution $\theta$ is a given distribution on $\mathbb{Z}^4$ with mean zero, not supported on a strict subgroup of $\mathbb{Z}^4$ and has finite exponential moments, i.e., for some $\lambda > 0$,

$$
\sum_{z \in \mathbb{Z}^4} \theta(z) \cdot \exp(\lambda|z|) < \infty.
$$

Our first result is

**Theorem 1.1.** For any nonempty finite subset $K$ of $\mathbb{Z}^4$, we have

$$
\lim_{x \to \infty} (\|x\|^2 \log \|x\|) \cdot P(S_x \text{ visits } K) = \frac{1}{2\sigma^2}. \quad (1.2)
$$

Note that, compared with the analogous result (1.1) in higher dimensions, we have a logarithm correction in the limit and the right hand side is independent of $K$. Similar to the case in higher dimensions, we also show that conditioned on visiting, the first visiting point converges in distribution:

**Theorem 1.2.** Assume further that $\theta$ has finite range. Then, for any nonempty finite subset $K$ of $\mathbb{Z}^4$ and $a \in K$, we have

$$
\lim_{x \to \infty} P(S_x(\tau_K) = a | S_x \text{ visits } K) = \frac{\sigma^2}{4\pi^2 \sqrt{\det Q}} \mathcal{E}_K(a), \quad (1.3)
$$

where $\tau_K$ is the first visiting time (vertex), and $\mathcal{E}_K(a)$ is defined in (4.3).

In higher dimensions $d \geq 5$, conditioned on visiting, the ‘entering measure’ also converges in distribution (see Theorem 9.1 in [9]). However, this is false in the critical dimension $d = 4$. It turns out that conditioned on visiting, the total number of ‘entering points’ will blow up, as the starting point goes to infinity.

Furthermore, we give the asymptotics of the range $R_n$ of a critical branching random walk conditioned on the total size of offspring being $n$, answering a question raised by Le Gall and Lin [5].

**Theorem 1.3.** Assume further that $\theta$ is symmetric. Then, we have

$$
\log \frac{n}{R_n} \xrightarrow{n \to \infty} \frac{16\pi^2 \sqrt{\det Q}}{\sigma^2} \quad \text{in probability.} \quad (1.4)
$$

Note that Theorem 1.3 is independent of Theorem 1.2 and we do not need the assumption that $\theta$ is of finite range. The case when $\mu$ is the critical geometric distribution was proved by Le Gall and Lin [3]:

$$
\log \frac{n}{R_n} \xrightarrow{n \to \infty} \frac{8\pi^2 \sqrt{\det Q}}{\sigma^2} \quad \text{in } L^2.
$$

Now let us turn to a brief description of our methods. Thanks to the formula in [9] (see Proposition 2.4), the visiting probability by a critical branching random walk can be reduced to the visiting probability by a killed random walk. Unlike the case of higher dimensions, the killing rate in the critical dimension is big enough to change the global behavior of the Green function (see (4.1)). This creates serious difficulties. The key step is to establish the following asymptotics (see Proposition 3.2). The probability that the adjusted version of an infinite branching random walk, called a reversed snake, does not return to a fixed set before reaching distance $n$, behaves like $c/\log n$ for some constant $c$. This is analogous to the well-known result for the probability that a simple random walk does not come back to its starting point before reaching distance $n$, but the proof is much
involved in our setting. With the help of this estimate, we can find the asymptotics of visiting probability and the asymptotics of the range of the (infinite) reversed snake (through an invariant shift introduced in [4]). Then, we can use an absolute continuity argument to transfer the asymptotics from the case of an infinite snake to the case of a finite snake with given size, i.e., a branching random walk conditioned on the total number of progeny.

This paper is organized as follows. Section 2 presents our main notation and give some preliminary knowledge and useful results from the previous papers. Section 3, 4 and 5, respectively, are devoted to Theorem 1.1, Theorem 1.2 and Theorem 1.3

2. Preliminaries

We begin with some notations. For a set $A$, we write either $|A|$ or $\#A$ for its cardinality. We write $K \subset \subset \mathbb{Z}^4$ to express that $K$ is a finite nonempty subset of $\mathbb{Z}^4$. For $x \in \mathbb{Z}^4$, we denote by $|x|$ the Euclidean norm of $x$. We will mainly use the norm $\| \cdot \|$ corresponding to the jump distribution $\theta$, i.e., $\|x\| = \sqrt{x \cdot Q^{-1} x}/2$, where $Q$ is the covariance matrix of $\theta$. For convenience, we set $|0| = \|0\| = 1.5$ to avoid dividing by zero. We denote by $\text{diam}(K) = \sup\{|a-b| : a, b \in K\}$ the diameter of $K$, and by $\text{Rad}(K) = \sup\{|a| : a \in K\}$ the radius of $K$ with respect to 0. We write $C(r)$ for the ball $\{z \in \mathbb{Z}^4 : \|z\| \leq r\}$. For any path $\gamma : \{0, \ldots, k\} \rightarrow \mathbb{Z}^4$, we let $|\gamma|$ stand for $k$, the length, i.e., the number of edges of $\gamma$, and $\hat{\gamma}$ for $\gamma(k)$, the endpoint of $\gamma$. Occasionally, we use a sequence of vertices to express a path. For example, we may write $(\gamma(0), \gamma(1), \ldots, \gamma(k))$ for the path $\gamma$. For any $B \subset \mathbb{Z}^4$, we write $\gamma \subseteq B$ to indicate that all vertices of $\gamma$ except for the starting point and the endpoint, lie inside $B$, i.e., $\gamma(i) \in B$ for any $1 \leq i \leq |\gamma| - 1$. If the endpoint of a path $\gamma_1 : \{0, \ldots, |\gamma_1|\} \rightarrow \mathbb{Z}^4$ coincides with the starting point of another path $\gamma_2 : \{0, \ldots, |\gamma_2|\} \rightarrow \mathbb{Z}^4$, then we can define the composite of $\gamma_1$ and $\gamma_2$ by concatenating $\gamma_1$ and $\gamma_2$:

$$\gamma_1 \circ \gamma_2 : \{0, \ldots, |\gamma_1| + |\gamma_2|\} \rightarrow \mathbb{Z}^4,$$

$$\gamma_1 \circ \gamma_2(i) = \begin{cases} \gamma_1(i), & \text{for } i \leq |\gamma_1|; \\ \gamma_2(i - |\gamma_1|), & \text{for } i \geq |\gamma_1|. \end{cases}$$

We now state our convention regarding constants. Throughout the text, we use $C$ and $c$ to denote positive constants that depend only on dimension $d$, the offspring distribution $\mu$ and the jump distribution $\theta$, which may change from place to place. The dependence of constants on additional parameters will be made clear or stated explicitly. For example, $C(\lambda)$ stands for a positive constant depending on $d, \mu, \theta$ and $\lambda$. For functions $f(x)$ and $g(x)$, we write $f \sim g$ if $\lim_{x \to \infty} (f(x)/g(x)) = 1$. We write $f \preceq g$, respectively $f \succeq g$, if there exist a constant $C > 0$ such that, $f \leq Cg$, respectively $f \geq Cg$. We use $f \asymp g$ to express that $f \preceq g$ and $f \succeq g$. We write $f \ll g$ if $\lim_{x \to \infty} (f(x)/g(x)) = 0$.

2.1. Finite and infinite trees. We are interested in rooted ordered trees (plane trees), in particular, Galton-Watson trees and their companions. Recall that $\mu = (\mu(i))_{i \in \mathbb{N}}$ is a critical distribution with finite variance $\sigma^2 > 0$. We exclude the trivial case where $\mu(1) = 1$. Throughout this paper, $\mu$ will be fixed. Define another probability measure $\bar{\mu}$ on $\mathbb{N}$ by setting $\bar{\mu}(i) = \sum_{j=i+1}^{\infty} \mu(j)$. Since $\mu$ has mean 1, $\bar{\mu}$ is indeed a probability measure. The mean of $\bar{\mu}$ is $\sigma^2/2$. A Galton-Watson process with distribution $\mu$ is a process starting with one initial particle, with each particle independently having a random number of children due to $\mu$. The Galton-Watson
tree is just the family tree of the Galton-Watson process, rooted at the initial particle. We simply write \( \mu \)-GW tree for the Galton-Watson tree with offspring distribution \( \mu \). If we change the law of the number of children for the root, using \( \tilde{\mu} \) instead of \( \mu \) (for other particles, still use \( \mu \)), the new tree is called an adjoint \( \mu \)-GW tree. The infinite \( \mu \)-GW tree is constructed as follows: start with a semi-infinite line of vertices, called the spine, and graft to the left of each vertex in the spine an independent adjoint \( \mu \)-GW tree, called a bush. The infinite \( \mu \)-GW tree is rooted at the first vertex of the spine. By ‘left’, we mean that every vertex in the spine except for the root is considered the youngest child (the latest in the depth-first search order) of its parent. The invariant \( \mu \)-GW tree is defined similarly to the infinite \( \mu \)-GW tree except that we graft to the root (the first vertex of the spine) an independent \( \mu \)-GW tree instead of an independent adjoint \( \mu \)-GW tree. We also need to introduce the so-called \( \mu \)-GW tree conditioned on survival. Start with a semi-infinite path, called the spine, rooted at the starting point. For each vertex in the spine, with probability \( \mu(i + j + 1) \) (for any \( i, j \in \mathbb{N} \)), it has a total number of \( i + j + 1 \) children, with exactly \( i \) children that are older than the child corresponding to the next vertex in the spine, and exactly \( j \) children that are younger. For any vertex not in the spine, it has a random number of children due to \( \mu \). The numbers of children for different vertices are independent. The random tree generated in this way is just the \( \mu \)-GW tree conditioned on survival. Each tree is ordered using the classical order according to the depth-first search starting from the root. Note that the subtree of the \( \mu \)-GW tree conditioned on survival, generated by the spine vertices and all vertices ‘on the left’ of the spine, i.e., the descendants of those vertices which are elder siblings of the spine vertices, has the same distribution as the infinite \( \mu \)-GW tree.

2.2. Tree-indexed random walk. Now we introduce the random walk in \( \mathbb{Z}^4 \) with jump distribution \( \theta \), indexed by a random plane tree \( T \). First, choose some \( a \in \mathbb{Z}^4 \) as the starting point. Conditionally on \( T \), we assign independently to each edge of \( T \) a random variable in \( \mathbb{Z}^4 \) according to \( \theta \). Then, we can uniquely define a function \( S_T : T \to \mathbb{Z}^4 \) such that, for every vertex \( v \in T \) (we also use \( T \) for the set of all vertices of the tree \( T \)), \( S_T(v) - a \) is the sum of the variables of all edges belonging to the unique simple path from the root \( o \) to the vertex \( u \) (hence \( S_T(o) = a \)). A plane tree \( T \) together with this random function \( S_T \) is called \( T \)-indexed random walk starting from \( a \). When \( T \) is a \( \mu \)-GW tree, an adjoint \( \mu \)-GW tree, and an infinite \( \mu \)-GW tree respectively, we simply call the tree-indexed random walk a **snake**, an **adjoint snake** and an **infinite snake**, respectively. We write \( S_x, S'_x \) and \( S_x^\infty \) for a snake, an adjoint snake, and an infinite snake, respectively, starting from \( x \in \mathbb{Z}^4 \).

Note that a snake is just the branching random walk with offspring distribution \( \mu \) and jump distribution \( \theta \). We also need to introduce the **reversed infinite snake** starting from \( x, S_x^\infty \), which is constructed in the same way as \( S_x^\infty \) except that the variables assigned to the edges in the spine are now due to not \( \theta \) but rather to the reverse distribution \( \theta^- \) of \( \theta \) (i.e., \( \theta^-(x) = \theta(-x) \) for \( x \in \mathbb{Z}^4 \)). Similarly, the **invariant snake** starting from \( x, S_x^l \), is constructed by using the invariant \( \mu \)-GW tree as the random tree \( T \) and using \( \theta^- \) for all edges in the spine of \( T \) and \( \theta \) for all other edges. For an infinite snake (a reversed infinite snake and an invariant snake), the random walk indexed by its spine, called its backbone, is just a random walk with jump distribution \( \theta^- \) (\( \theta^- \)). Note that all snakes here certainly depend on
\[ \theta \text{ jump distribution } \]

\[ \text{a given function. In other words, the random walk with killing rate } k \text{ probabilities } p, \text{i.e., jumps to a 'cemetery' state in the notation.} \]

Suppose that when the random walk is currently at position \( x \in \mathbb{Z}^4 \), then it is killed, i.e., jumps to a 'cemetery' state, with killing function \( k(x) \) (and jump distribution \( \theta \)) is a Markov chain \( \{X_n : n \geq 0\} \) on \( \mathbb{Z}^4 \cup \{\diamond\} \) with transition probabilities \( p(\cdot, \cdot) \) given by the following: for \( x, y \in \mathbb{Z}^4 \),

\[ p(x, o) = k(x), \quad p(o, o) = 1, \quad p(x, y) = (1 - k(x)) \theta(y - x). \]

For any path \( \gamma : \{0, \ldots, n\} \to \mathbb{Z}^4 \) with length \( n \), its probability weight \( b(\gamma) \) is defined to be the probability that the path consisting of the first \( n \) steps for the random walk with killing starting from \( \gamma(0) \) is \( \gamma \). Equivalently,

\[ b(\gamma) = \prod_{i=0}^{\gamma[-1]} (1 - k(\gamma(i))) \theta(\gamma(i + 1) - \gamma(i)) = s(\gamma) \prod_{i=0}^{\gamma[-1]} (1 - k(\gamma(i))), \]

(2.1)

where \( s(\gamma) = \prod_{i=0}^{\gamma[-1]} \theta(\gamma(i + 1) - \gamma(i)) \) is the probability weight of \( \gamma \) corresponding to the random walk with jump distribution \( \theta \). In this paper, unless otherwise specified, the killing function is always of the form \( k(x) = k_A(x) = r_A(x) \) for some finite subset \( A \subset \mathbb{Z}^4 \). Note that \( r_A(x) = 1 \) when \( x \in A \). Therefore, when \( \gamma \) intersects \( A \) (except for the endpoint), \( b(\gamma) = 0 \). Now we can define the corresponding Green function for \( x, y \in \mathbb{Z}^4 \) by

\[ G_A(x, y) = \sum_{n=0}^{\infty} P(S_x^k(n) = y) = \sum_{\gamma:x\rightarrow y} b(\gamma), \]

where \( S_x^k = (S_x^k(n))_{n \in \mathbb{N}} \) is the random walk (with jump distribution \( \theta \)) starting from \( x \), with killing function \( k_A \), and the last sum is over all paths from \( x \) to \( y \).

For \( x \in \mathbb{Z}^4, K \subset \mathbb{Z}^4 \), we write \( G_A(x, K) \) for \( \sum_{y \in K} G_A(x, y) \).

For any \( B \subset \mathbb{Z}^4 \) and \( x, y \in \mathbb{Z}^4 \), define the harmonic measure by

\[ H_A^B(x, y) = \sum_{\gamma:x\rightarrow y, \gamma \subseteq B} b(\gamma). \]

When the killing function \( k \equiv 0 \) (or equivalently \( A = \emptyset \)), the random walk with this killing is just the ordinary random walk without killing and in this case we write \( H^B(x, y) \) instead.

We will repeatedly use the following first-visit lemma. The idea is to decompose a path according to the first or the last visit of a set.

**Lemma 2.1.** For any \( A \subset \subset \mathbb{Z}^4, B \subset \mathbb{Z}^4 \) and \( a \in B, b \notin B \), we have

\[ G_A(a, b) = \sum_{z \in B} H_A^B(a, z) G_A(z, b) = \sum_{z \in B} G_A(a, z) H_A^B(z, b); \]

\[ G_A(b, a) = \sum_{z \in B} H_A^B(b, z) G_A(z, a) = \sum_{z \in B} G_A(b, z) H_A^B(z, a). \]
Since our jump distribution $\theta$ may be unbounded, we need the following overshoot lemma:

**Lemma 2.2.** For any $r, s > 1$ and $p \in \mathbb{N}^+$, there exists some $C(p) > 0$, such that, for any $A \subset \subset \mathbb{Z}^4$ and $a \in B = C(r)$, we have

$$
\sum_{y \in (C(r+s))^c} \mathcal{H}_A^B(a, y) \leq C(p) \frac{r^2}{s^p}, \quad \sum_{y \in (C(r+s))^c} \mathcal{H}_A^B(y, a) \leq C(p) \frac{r^2}{s^p}. \tag{2.2}
$$

**Proof.** We only need to consider the case when $k \equiv 0$. By the Markov property, one finds that

$$
\sum_{y \in (C(r+s))^c} \mathcal{H}_A^B(a, y) \leq \sum_{z \in C(r)} G^B(a, z) \sum_{w \in (C(r+s))^c} \theta(z, w)
\leq \left( \sum_{z \in C(r)} g(a, z) \right) \cdot C(p) \frac{r^2}{s^p},
$$

where in the last step we use the fact that $\theta$ has finite exponential moments and hence

$$
\sum_{w \in (C(r+s))^c} \theta(z, w) \leq \theta(C^c(s)) \leq \frac{C(p)}{s^p}, \quad \text{for } z \in C(r).
$$

One can show the other inequality in a similar way. \hfill \square

### 2.4. Some facts about random walk and the Green function.

For $x \in \mathbb{Z}^4$, we write $S_x = (S_x(n))_{n \in \mathbb{N}}$ for the random walk with jump distribution $\theta$ starting from $S_x(0) = x$. Recall that the norm $\| \cdot \|$ corresponding to $\theta$ for every $x \in \mathbb{Z}^4$ is defined to be $\|x\| = \sqrt{x \cdot Q^{-1} x}$, where $Q$ is the covariance matrix of $\theta$. Note that $\|x\| \asymp |x|$. The Green function $g(x, y)$ is defined to be

$$
g(x, y) = \sum_{n=0}^{\infty} P(S_x(n) = y) = \sum_{\gamma : x \rightarrow y} s(\gamma).
$$

Since $\mu$ has mean one, we obtain that $g(x, y)$ is also equal to the expected number of visits to $y$ by a snake starting from $x$. We write $g(x)$ for $g(0, x)$.

Our assumptions about the jump distribution $\theta$ guarantee the standard estimate for the Green function (see, e.g., Theorem 4.3.5 in [3]):

$$
g(x) \sim a_4 \|x\|^{-2}, \tag{2.3}
$$

where $a_4 = 1/(8\pi^2\sqrt{\det Q})$ with $Q$ being the covariance matrix of $\theta$.

Additionally by the local central limit theorem (see e.g., Theorem 2.3.11 in [3]), one can get the following lemma.

**Lemma 2.3.**

$$
\lim_{n \to \infty} \sup_{x \in \mathbb{Z}^4} \left( \sum_{\gamma : 0 \rightarrow x, |\gamma| \geq n \|x\|^2} s(\gamma)/g(x) \right) = 0. \tag{2.4}
$$
2.5. Some results in the previous paper. We first recall some notations from [9]. Fix a finite subset \( K \subset \subset \mathbb{Z}^d \). For any \( x \in \mathbb{Z}^d \), define the position-dependent distribution \( \mu_x = \mu_{x,K} \) on \( \mathbb{N} \) by

\[
\mu_x(m) = \begin{cases} 
\sum_{l \geq 0} \mu(l + m + 1)/(1 - r_{x,K}(l)), & \text{when } x \notin K; \\
\mu(m), & \text{when } x \in K; 
\end{cases}
\]

(2.5)

where \( r_x \) is the probability that a snake starting from \( x \) conditioned on the initial particle having only one child does not visit \( K \). Write \( N(x) = N_{K}(x) = \sum_{m \geq 0} m \mu_x(m) \) for the mean of \( \mu_x \). Since \( \lim_{x \to \infty} r_{x,K}(x) = 0 \) and \( \lim_{x \to \infty} r_x(x) = 1 \), we deduce that

\[
\lim_{x \to \infty} N(x) = \sum_{m \geq 0} m \sum_{l \geq 0} \mu(m + 1 + l) = \frac{\sigma^2}{2}.
\]

When a snake \( S_x = (T, S_T) \) visits \( K \), we denote by \( \tau_K \), the first vertex in \( \{ v \in T : S_T(v) \in K \} \) due to the depth-first search order. We say \( S_T(\tau_K) \) is the first visiting point. Let \( (v_0, v_1, \ldots, v_k) \) be the unique simple path in \( T \) from the root \( o \) to \( \tau_K \) and say \( S_x \) visits \( K \) via \( (S_T(v_0), \ldots, S_T(v_k)) \). Let \( \tilde{b}_i \) be the number of younger siblings of \( v_i \), for \( i = 1, \ldots, k \).

The key observation is the following proposition.

**Proposition 2.4.** For any \( \gamma = (\gamma(0), \ldots, \gamma(k)) \subseteq K^c \) with \( \gamma(0) = x \) and \( \gamma(k) \in K \), we have

\[
P(S_x \text{ visits } K \text{ via } \gamma) = b(\gamma); \tag{2.6}
\]

\[
p_K(x) = \sum_{\gamma:x \to K} b(\gamma) = G_K(x, K). \tag{2.7}
\]

Furthermore, conditionally on the event that \( S_x \) visits \( K \) via \( \gamma, \tilde{b}_1, \ldots, \tilde{b}_k \) are independent, with the following distribution:

\[
\tilde{b}_i \overset{d}{=} \mu_{\gamma(i-1)}.
\]

The above proposition is a summary of Section 5 in [9]. In particular, (2.6) is stated as Proposition 5.1 in [9], (2.7) is given as (5.5) in [9]. The distribution \( \mu_x \) is introduced in Section 9.1; the last assertion follows from (5.1) in [9](see the paragraph before Proposition 9.2 in [9]).

For \( r_K(x) \), we have (see (8.6) in [9])

\[
r_K(x) \asymp p_K(x). \tag{2.8}
\]

Note that in [9] we consider the supercritical dimensions \( d \geq 5 \). However, for the results above, we do not need the assumption \( d \geq 5 \). They hold for any \( d \in \mathbb{N}^+ \) (and even for more general graphs).

We also need the following result from [8]: when \( ||x|| \geq 2 \),

\[
p_{(0)}(x) \asymp (||x||^2 \log ||x||)^{-1}. \tag{2.9}
\]

3. The visiting probability.

The main goal of this section is to prove Theorem 1.1. In this section and the next, we fix a finite nonempty subset \( K \subset \subset \mathbb{Z}^d \) and therefore the constants appearing in this and the next sections may depend also on \( K \).

The first step is to establish the following estimate of the Green function:
Lemma 3.1. For any $\alpha \in (0,1/2)$, we have
\[
\lim_{x,y \to \infty; \|x\|/(\log \|x\|)^\gamma \leq \|y\| \leq \|x\|/(\log \|x\|)^\alpha} \frac{G_K(x,y)}{g(x,y)} = 1. \tag{3.1}
\]

Remark 3.1. In the supercritical dimensions, we have $G_K(x,y) \sim g(x,y)$ (see Lemma 6.1 in [9]). In the critical dimension, this holds only when $x,y$ are not too far away from each other, relative to their norms. We will give a more precise asymptotic behavior of $G_K$ in the next section.

Proof of Lemma 3.1. We use the same idea in the proof for a similar form in the supercritical dimensions (see Section 6 of [9]). Since $\alpha < 1/2$, we can choose $\beta, \epsilon > 0$ with $\epsilon + 2\alpha + 2\beta < 1$. Without loss of generality, assume $\|y\| \geq \|x\|$. Let $r = \|x\|/\log^\beta \|x\|$ and
\[
\Gamma_1 = \{ \gamma : x \to y \mid |\gamma| \geq (\log \|x\|)^\epsilon \|x-y\|^2 \};
\]
\[
\Gamma_2 = \{ \gamma : x \to y \mid \gamma \text{ visits } C(r) \}.
\]
By Lemma 2.3, we find that $\sum_{\gamma \in \Gamma_1} s(\gamma)/g(x,y)$ tends to 0. On the other hand, by considering the first visiting point, we have (let $B = C(r)$)
\[
\sum_{\gamma \in \Gamma_2} s(\gamma) = \sum_{a \in B} \mathcal{H}^B(x,a)g(a,y) \leq \sum_{a \in B} \mathcal{H}^B(x,a)g(x,y) = P(S_x \text{ visits } B) \cdot g(x,y) \ll g(x,y),
\]
where for the last line the estimate of $P(S_x \text{ visits } B) \asymp (r/\|x\|)^2 \to 0$ is standard. Therefore, we have
\[
g(x,y) \sim \sum_{\gamma : x \to y, \gamma \notin \Gamma_1 \cup \Gamma_2} s(\gamma).
\]
On the other hand,
\[
\frac{G_K(x,y)}{g(x,y)} = \sum_{\gamma \in \Gamma_1} \frac{b(\gamma)}{g(x,y)} + \sum_{\gamma \in \Gamma_2} \frac{b(\gamma)}{g(x,y)} + \sum_{\gamma : x \to y, \gamma \notin \Gamma_1 \cup \Gamma_2} \frac{b(\gamma)}{g(x,y)}
\]
From $b(\gamma) \leq s(\gamma)$ and what has already been proved, we deduce that the first two sums on the right hand side tend to zero. Hence, it suffices to verify
\[
b(\gamma)/s(\gamma) \to 1, \text{ for any } \gamma : x \to y, \notin \Gamma_1 \cup \Gamma_2.
\]
By [2.9] and [2.8], the killing function $k(z) = r_K(z) \asymp p_K(z) \preceq 1/(\|z\|^2 \log \|z\|)$. Hence, for any $\gamma : x \to y, \notin \Gamma_1 \cup \Gamma_2$,
\[
b(\gamma)/s(\gamma) = \prod_{i=0}^{\lfloor |\gamma|-1 \rfloor} (1 - k(\gamma(i))) \geq (1 - c/(r^2 \log r))^{\!|\gamma|} \geq 1 - c|\gamma|/(r^2 \log r)
\]
\[
\geq 1 - c(\log \|x\|^r \|y\|^2/((\|x\|/\log^\beta \|x\|)^2 \log \|x\|))
\]
\[
\geq 1 - c(\log \|x\|^r(\|x\| \log^\alpha \|x\|^2/(\|x\|/\log^\beta \|x\|)^2 \log \|x\|))
\]
\[
\geq 1 - c(\log \|x\|)^{r+2\alpha+2\beta}/\log \|x\| \to 1.
\]
Now the proof is complete. \qed
Write \( N \) for the number of visits of \( K \) by \( S_x \). Note that \( E[N] = g(x, K) \). We would like to estimate \( E(N|S_x \text{ visits } K \text{ via } \gamma) \). Inspired by Proposition 2.4 set
\[
\mathcal{N}(\gamma) = \sum_{i=0}^{\lfloor |\gamma| \rfloor} N(\gamma(i))g(\gamma(i), K); \quad \mathcal{N}^-(\gamma) = \sum_{i=0}^{\lfloor |\gamma| - 1 \rfloor} N(\gamma(i))g(\gamma(i), K),
\]
for any finite path \( \gamma \). By Proposition 2.4 we have
\[
E(N|S_x \text{ visits } K \text{ via } \gamma) = \mathcal{N}(\gamma).
\]
Hence,
\[
\sum_{\gamma:x \to K} b(\gamma)\mathcal{N}(\gamma) = E[N] = g(x, K) \sim a_4|K||x|^{-2}. \tag{3.3}
\]

The main step is to control the sum of the escape probabilities.

**Proposition 3.2.**
\[
\sum_{\gamma:(C(2n)) \not\subseteq C(n) \to K, |\gamma| \leq C(n)} b(\gamma) \sim \frac{4\pi^2 \sqrt{\det Q}}{\sigma^2} \frac{1}{\log n}. \tag{3.4}
\]

**Remark 3.2.** By the overshoot lemma, we also have
\[
\sum_{\gamma:|\gamma| \not\subseteq C(n)} b(\gamma) \sim \frac{4\pi^2 \sqrt{\det Q}}{\sigma^2} \frac{1}{\log n}.
\]

Note that the above sum is equal to \( \sum_{a \in K} \sum_{\gamma:|\gamma| \not\subseteq C(n)} b(\gamma) \) and \( b(\gamma) \) is the probability that a reversed snake escapes to distance \( n \) along \( \gamma \). Therefore, the above sum can be interpreted as the total measure that a reversed snake starting from \( K \) does not return to \( K \) before reaching distance \( n \).

**Proof of Theorem 1.1 under Proposition 3.2**
Let \( n = \|x\|/\log \|x\| \), \( B = C(n) \), \( B_1 = C(2n) \setminus B \) and \( B_2 = C(2n)^c \). By (2.7) and the first-visit lemma, we have
\[
P(S_x \text{ visits } K) = \sum_{\gamma:x \to K} b(\gamma) = \sum_{b \in B^c} G_K(x, b) \sum_{a \in K} \mathcal{H}^B_K(b, a) = \sum_{b \in B_1} G_K(x, b) \sum_{a \in K} \mathcal{H}^B_K(b, a) + \sum_{b \in B_2} G_K(x, b) \sum_{a \in K} \mathcal{H}^B_K(b, a).
\]

We argue that the first term has the desired asymptotics and the second is negligible:
\[
\sum_{b \in B_1} G_K(x, b) \sum_{a \in K} \mathcal{H}^B_K(b, a) \sim a_4\|x\|^{-2} \sum_{b \in B_1} \sum_{a \in K} \mathcal{H}^B_K(b, a) \sim \frac{1}{2\sigma^2 \|x\|^2 \log \|x\|}.
\]

\[
\sum_{b \in B_2} G_K(x, b) \sum_{a \in K} \mathcal{H}^B_K(b, a) \leq \sum_{a \in K} \sum_{b \in B_2} \mathcal{H}^B_K(b, a) \leq |K|n^2/n^5 < 1/\|x\|^2 \log \|x\|.
\]

In order to prove Proposition 3.2 we need two lemmas about random walks. They are adjusted versions of Lemma 17 and Lemma 18 in [4]. Let \((S_j)_{j \in \mathbb{N}}\) be a random walk (with jump distribution \( \theta^- \)) starting from 0, \( \tau_\alpha \) be the first visiting time of \( C(n)^c \) by the random walk and \( h : \mathbb{Z}^4 \to \mathbb{R}^+ \) is a fixed positive function such that \( h(x) \sim a_4\|x\|^{-2} \).

□
Lemma 3.3. For $p \in \mathbb{N}^+$, there exists a constant $C(p)$ (depending also on $h$) such that, for every $n \geq 2$,
\[ E\left(\sum_{j=0}^{\tau_n} h(S_j)\right)^p \leq C(p)(\log n)^p. \] (3.5)

Lemma 3.4. For every $\alpha, p > 0$, there exists a constant $C_\alpha(p)$ (depending also on $h$) such that, for every $n \geq 2$, we have
\[ P\left(\sum_{j=0}^{\tau_n} h(S_j) - 4a_4 \log n \geq \alpha \log n\right) \leq C_\alpha(p)(\log n)^{-p}. \] (3.6)

Let $h(x) = 2N(x)g(x,K)/(\sigma^2|K|)$. Recall that $N(x) = \sum_{k \geq 0} k\mu_x(k) \sim \sigma^2/2$ and $N(\gamma) = \sum \sigma^2|K|h(\gamma(i))/2$. Applying both lemmas, we get: for any $a \in K$,
\[ \sum_{\gamma: C(n)^\gamma \to a, \gamma \subseteq C(n)} s(\gamma)(N(\gamma))^2 \lesssim (\log n)^2, \]
\[ \sum_{\gamma: C(n)^\gamma \to a, \gamma \subseteq C(n), |N(\gamma) - 2a_4|K|\sigma^2 \log n| \geq \alpha \log n} s(\gamma) \leq C_\alpha(\log n)^{-4}. \] (3.7)

By monotonicity and summation, we obtain
\[ \sum_{\gamma: C(n)^\gamma \to K, \gamma \subseteq C(n) \setminus K} s(\gamma)(N(\gamma))^2 \lesssim (\log n)^2, \]
\[ \sum_{\gamma: C(n)^\gamma \to K, \gamma \subseteq C(n) \setminus K, |N(\gamma) - 2a_4|K|\sigma^2 \log n| \geq \alpha \log n} s(\gamma) \leq C_\alpha(\log n)^{-4}. \] (3.8)

Let us make some comments about the proofs. Lemma 18 in [5] states that
\[ P\left(\sum_{j=0}^{n} g(S_j) - 2a_4 \log n \geq \alpha \log n\right) \leq C_\alpha(\log n)^{-3/2}, \]
where $g$ is the Green function. Their argument is to derive an analogous result for Brownian motion and then to transfer this result to the random walk via the strong invariance principle. This argument also works here with small adjustments. Note that it is assumed there that the jump distribution is symmetric (besides having exponential moments). However if one checks the proof there, one can see that the assumption of symmetry is not needed (when assuming that Lemma 17 there without symmetry assumption, or Lemma 3.3 here) and $g(x)$ can be replaced by any $h(x)$ satisfying $h(x) \sim a_4||x||^{-2}$. Moreover, the exponent $3/2$ can be replaced by any positive constant $p$ with minor modifications. Combing this with the fact that for any fixed $\epsilon > 0$, $P(\tau_n \notin [n^{2-\epsilon}, n^{2+\epsilon}]) = o((\log n)^{-p})$, one can get Lemma 3.4.

For Lemma 3.3, we give a different proof.

\[ E\left(\sum_{j=0}^{\tau_n} h(S_j)\right) \leq \sum_{z \in C(n)} h(z) E\left(\sum_{j=0}^{\tau_n} 1_{S_j = z}\right) \leq \sum_{z \in C(n)} |z|^{-2} E\left(\sum_{j=0}^{\tau_n} 1_{S_j = z}\right) \]
\[ = \sum_{z \in C(n)} |z|^{-2} g(0,z) \asymp \sum_{z \in C(n)} |z|^{-4} \asymp \log n. \]
Case II: p=2.

\[ E(\sum_{j=0}^{\tau_n} h(S_j))^2 \leq E(\sum_{z \in \mathcal{C}(n)} h(z) \sum_{j=0}^{\tau_n} 1_{S_j = z})^2 \leq E(\sum_{z \in \mathcal{C}(n)} |z|^{-2} \sum_{j=0}^{\tau_n} 1_{S_j = z})^2 \]

\[ = \sum_{z, w \in \mathcal{C}(n)} |z|^{-2} |w|^{-2} E(\sum_{j=0}^{\tau_n} 1_{S_j = z} \sum_{i=0}^{\tau_n} 1_{S_i = w}). \]

Write \( A_x = \sum_{j=0}^{\tau_n} 1_{S_j = x} \) and \( A = A_z + A_w \). We point out that

\[ E(A_z A_w) \leq (|z|^{-2} + |w|^{-2})|z - w|^{-2}. \tag{3.9} \]

If so, note that

\[ \sum_{z, w \in \mathcal{C}(n)} |z|^{-2} |w|^{-2} (|z|^{-2} + |w|^{-2})|z - w|^{-2} \leq \sum_{z, w \in \mathcal{C}(n)} |z|^{-4} |w|^{-2} |z - w|^{-2} \]

\[ \leq \sum_{w \in \mathcal{C}(n)} (\sum_{z:|z| \leq |w|, |z-w| \geq |w|/2} |z|^{-4} |w|^{-4} + \sum_{z:|z| \leq |w|, |z-w| < |w|/2} |w|^{-6} |z - w|^{-2}) \]

\[ \leq \sum_{w \in \mathcal{C}(n)} ((\log |w|)|w|^{-4} + |w|^{-4}) \asymp (\log n)^2, \]

and then one can get \( E(\sum_{j=0}^{\tau_n} h(S_j))^2 \asymp (\log n)^2 \).

We now only need to show (3.9). Assume \( z \neq w \) (the case \( z = w \) can be addressed similarly with small adjustments). Note that

\[ E(A_z A_w) \leq E(A^2; A_z > 0, A_w > 0) \asymp \sum_{k \geq 2} k P(A \geq k, A_z > 0, A_w > 0). \]

By the Markov property, one can see that

\[ P(A \geq k, A_z > 0, A_w > 0) \leq P(A_z > 0)((k-1)P_z(A_w > 0)c^{k-2}) + P(A_w > 0)((k-1)P_w(A_z > 0)c^{k-2}), \]

where we write \( P_x \) for the law of random walk starting from \( x \) and

\[ c = \sup_{x \neq y \in Z^d} P_x(A_x + A_y > 1) < 1. \]

Hence, we have

\[ \sum_{k \geq 2} k P(A \geq k, A_z > 0, A_w > 0) \leq \sum_{k \geq 2} k(k-1)c^{k-2} \]

\[ \leq P(A_z > 0)P_z(A_w > 0) + P(A_w > 0)P_w(A_z > 0) \]

\[ \asymp (|z|^{-2} + |w|^{-2})|z - w|^{-2}. \]

Case III: \( p \geq 3 \) (Sketch). We need to generalize the method in Case II. First we still have

\[ E(\sum_{j=0}^{\tau_n} h(S_j))^p \leq \sum_{z_1, z_2, \ldots, z_p \in \mathcal{C}(n)} |z_1|^{-2} \ldots |z_p|^{-2} E(A_{z_1} A_{z_2} \ldots A_{z_p}). \]
We now introduce some notations. For any spanning tree \( t \) of the complete graph on the vertex set \([p] = \{0, 1, \ldots, p\}\) and \( z_0, z_1, \ldots, z_p \in \mathbb{Z}^d \), define

\[
f(t; z_0, \ldots, z_p) = \prod_{(i,j) \in E_t} |z_i - z_j|^{-2},
\]

where \( E_t \) is the edge set of \( t \).

It is not difficult to establish the following estimate by induction:

\[
\sum_{z_1, \ldots, z_p \in C(n)} |z_1|^{-2} \cdots |z_p|^{-2} f(t; z_0, \ldots, z_p) \lesssim (\log n)^p,
\]

for any spanning tree \( t \) of the complete graph on the vertex set \([p]\). For example, one may choose a leaf vertex, say, \( p \) on \( t \) and use the following estimate in the induction step:

\[
\sum_{z_p \in C(n)} |z_p|^{-2} |z_p - z|^{-2} \lesssim \log n, \quad \text{for any } z \in C(n).
\]

Hence, it suffices to show: there exists some \( C_p > 0 \), such that, for any \( z_0, \ldots, z_p \in \mathbb{Z}^d \), we have

\[
E_{z_0}(A_{z_1} A_{z_2} \cdots A_{z_p}) \leq C_p \sum f(t; z_0, \ldots, z_p), \tag{3.10}
\]

where the sum is over all spanning trees of the complete graph on the vertex set \([p]\) and \( E_{z_0} \) is the expectation under \( P_{z_0} \), the law of the random walk starting from \( z_0 \). Pick some \( R > 0 \), such that

\[
c = \sup \left\{ P_{z_0}(A_{z_0} + \cdots + A_{z_p} > 1) : z_0, \ldots, z_p \in \mathbb{Z}^d, |z_i - z_j| \geq R, \text{ for } i \neq j \right\} < 1.
\]

Note that if for any \( i, j \in [p] \), either \( z_i = z_j \) or \( |z_i - z_j| \geq R \), then we can modify the argument in Case II. Hence, for such \((z_0, \ldots, z_p)\), (3.10) holds.

For general \((z_0, \ldots, z_p)\), we use the following reduction. For any \((z_0, \ldots, z_p)\), we define an equivalence relation on \([p]\) as follows. Write \( i \equiv j \), if \(|z_i - z_j| < R \). Let \( \equiv \) be the equivalence relation generated by \( \sim \). Note that when \( i \equiv j \), we have \(|z_i - z_j| < pR \approx 1 \) (the constants here are allowed to depend on \( p \) and hence on \( R \)).

Therefore, we have

\[
f(t; z_0, \ldots, z_p) \asymp f(t; z_{i_0}, \ldots, z_{i_k}),
\]

as long as \( k \equiv i_k \) for every \( k \in [p] \). Note that the constants behind \( \asymp \) depend on \( p, R \) (also \( \theta \), as default) but not on \( z_i \). On the other hand, we can combine equivalent terms by using the inequality of arithmetic and geometric means. For example, when \( z_1 \equiv z_2 \equiv z_3 \), we have

\[
E_{z_0}(A_{z_1} A_{z_2} A_{z_3} A_{z_4} \cdots A_{z_p}) \leq \frac{1}{3} (E_{z_0}(A_{z_1} A_{z_2} A_{z_1} A_{z_4} \cdots A_{z_p}) + \sum_{j \neq i} E_{z_0}(A_{z_2} A_{z_3} A_{z_4} \cdots A_{z_j} A_{z_p}) + E_{z_0}(A_{z_3} A_{z_4} A_{z_4} A_{z_5} \cdots A_{z_p})).
\]

Therefore, we could assume that in (3.10), either \( z_i = z_j \) or \(|z_i - z_j| \geq pR \), for any \( i, j \in [p] \) and finish the reduction. \( \square \)

**Proof of Proposition 3.2.** We first show the following weaker result.

**Lemma 3.5.** When \( n \geq 2 \), we have

\[
\sum_{\gamma : C(2n) \cap C(n) \rightarrow K, \gamma \subseteq C(n)} b(\gamma) \lesssim (\log n)^{-1}. \tag{3.11}
\]
Proof. By (2.9) and (2.6), we have
\[
\sum_{\gamma:x \to K} b(\gamma) \leq (\|x\|^2 \log \|x\|)^{-1}.
\] (3.12)

Pick some \(x \in \mathbb{Z}^d\) such that \(n = \|x\| (\log \|x\|)^{-1/4}\). Let \(B = C(n)\) and \(B_1 = C(2n)\).

By the first-visit lemma, we have
\[
\sum_{\gamma: x \to K} b(\gamma) = \sum_{a \in K} \sum_{z \in B} G_K(x, z) H^K_B(z, a) \geq \sum_{a \in K} \sum_{z \in B \setminus B_1} G_K(x, z) H^K_B(z, a)
\]
\[\geq \sum_{a \in K} \sum_{z \in B \setminus B_1} g(x, z) H^K_B(z, a) \geq \|x\|^2 \sum_{\gamma: \substack{C(2n) \setminus C(n) \to K, \gamma \subseteq C(n)}} b(\gamma).
\] Combining this with (3.11) gives (3.12). \(\square\)

We need to transfer (3.3) to the following form.

Lemma 3.6.
\[
\lim_{n \to \infty} \sum_{\gamma: C(2n) \setminus C(n) \to K, \gamma \subseteq C(n)} N(\gamma) b(\gamma) = |K|.\] (3.13)

Proof. Pick some \(x \in \mathbb{Z}^d\) such that \(n = \|x\| (\log \|x\|)^{-1/4}\). Let \(B = C(n)\) and \(B_1 = C(2n)\).

By decomposing \(\gamma\) at the last visit in \(B\), one can get
\[
\sum_{\gamma: x \to K} b(\gamma) N(\gamma) = \sum_{a \in K} \sum_{z \in B} b(\gamma_1) b(\gamma_2) (N(\gamma_1) + N(\gamma_2)) = \sum_{z \in B} \left( \sum_{\gamma_1: x \to z, \gamma_2: z \to K, \gamma_2 \subseteq B} b(\gamma_1) b(\gamma_2) N(\gamma_1) + \sum_{\gamma_1: x \to z, \gamma_2: z \to K, \gamma_2 \subseteq B} b(\gamma_1) b(\gamma_2) N(\gamma_2) \right)
\]
\[
= \sum_{z \in B} \left( \sum_{\gamma_1: x \to z} b(\gamma_1) \sum_{\gamma_2: z \to K, \gamma_2 \subseteq B} b(\gamma_2) N(\gamma_2) + \sum_{\gamma_1: x \to z} b(\gamma_1) \sum_{\gamma_2: z \to K, \gamma_2 \subseteq B} b(\gamma_2) N(\gamma_2) \right).
\]

We argue that the first term is negligible:
\[
\sum_{z \in B} \sum_{\gamma_1: x \to z} b(\gamma_1) \sum_{\gamma_2: z \to K, \gamma_2 \subseteq B} b(\gamma_2) N(\gamma_1) \ll \|x\|^{-2}.
\] (3.14)

Note that
\[
\sum_{\gamma: x \to z} b(\gamma) N(\gamma) \leq \sum_{w \in \mathbb{Z}^d} N(w) g(w, K) \sum_{\gamma: x \to z} b(\gamma) \sum_{i=0}^{\|\gamma\|} 1_{\gamma(i) = w}
\]
\[
\leq \sum_{w \in \mathbb{Z}^d} |w|^{-2} \sum_{\gamma: x \to w} b(\gamma) \sum_{\gamma: w \to z} b(\gamma) \leq \sum_{w \in \mathbb{Z}^d} |w|^{-2} g(x, w) g(w, z).
\]

In order to estimate the term above, we need the following easy lemma whose proof we postpone

Lemma 3.7. For any \(a, b, c \in \mathbb{Z}^d\), we have
\[
\sum_{z \in \mathbb{Z}^d} |z - a|^{-2} |z - b|^{-2} |z - c|^{-2} \leq \frac{1 \vee \log(M/m)}{M^2},
\] (3.15)

where \(M = \max\{|a - b|, |b - c|, |c - a|\}\) and \(m = \min\{|a - b|, |b - c|, |c - a|\}\).
Similarly, when \( \sum \), we have
\[
\sum_{z \in B_1 \setminus B} b(\gamma_2) \sum_{\gamma_1: x \to z} b(\gamma_1) N^- (\gamma_1) \ll ||x||^{-2}.
\]

Similarly, when \( z \in B_1 \), \( \sum_{\gamma_1: x \to z} b(\gamma_1) N^- (\gamma_1) \ll ||x||^{-2} \). On the other hand, by the overshoot lemma, we have
\[
\sum_{z \in B_1} \sum_{\gamma_2: z \to K, \gamma_2 \leq B} b(\gamma_2) \sum_{\gamma_1: x \to z} b(\gamma_1) N^- (\gamma_1) \ll ||x||^{-2}.
\]

This completes the proof of (3.14). Combining (3.14) with (3.3) gives
\[
\sum_{z \in B^c} \sum_{\gamma_2: z \to K, \gamma_2 \leq B} b(\gamma_2) N (\gamma_2) \sum_{\gamma_1: x \to z} b(\gamma_1) \sim a_4 |K||x||^{-2}.
\]

Now we aim to show
\[
\sum_{z \in B_1 \setminus B} \sum_{\gamma_1: x \to z} b(\gamma_2) N (\gamma_2) \sum_{\gamma_1: x \to z} b(\gamma_1) \ll a_4 |K||x||^{-2}. \tag{3.16}
\]

If so, we have
\[
\sum_{z \in B_1 \setminus B} \sum_{\gamma_1: x \to z} b(\gamma_2) N (\gamma_2) \sum_{\gamma_1: x \to z} b(\gamma_1) \sim a_4 |K||x||^{-2}. \tag{3.17}
\]

By Lemma 3.1, \( \sum_{\gamma_1: x \to z} b(\gamma_1) = G_K (x, z) \sim g(x, z) \sim a_d ||x||^{-2} \). Plugging this into the above estimate gives (3.13).

Since \( \sum_{\gamma_1: x \to z} b(\gamma_1) \leq 1 \) and \( b(\gamma) \leq s(\gamma) \), in order to show (3.16), it suffices to show
\[
\sum_{\gamma: B_1 \to K, \gamma \leq B} s(\gamma) N (\gamma) \ll ||x||^{-2}. \tag{3.18}
\]

By the Cauchy-Schwarz inequality, we get
\[
\sum_{\gamma: B_1 \to K, \gamma \leq B} s(\gamma) N (\gamma) \leq \left( \sum_{\gamma: B_1 \to K, \gamma \leq B} s(\gamma) \right)^{1/2} \left( \sum_{\gamma: B_1 \to K, \gamma \leq B} s(\gamma) (N (\gamma))^2 \right)^{1/2}.
\]

By the overshoot lemma, the first term in the right hand side decays faster than any polynomial of \( n \). On the other hand, due to (3.7), the second term in the right hand side is less than \( \log n \) by a constant multiplier. Combining both gives (3.18) and finishes the proof of (3.13). \( \square \)

Now we are ready to prove Proposition 3.2. Fix any small \( \epsilon > 0 \). Let \( n = ||x||/(\log ||x||)^{1/4} \),
\[
\Gamma = \{ \gamma : C(2n) \setminus C(n) \to K, \gamma \leq C(n) \setminus K \},
\]
\[
\Gamma_1 = \{ \gamma \in \Gamma : |N (\gamma) - 2a_4 \sigma^2 |K| \log n| > \epsilon \log n \} \quad \text{and} \quad \Gamma_2 = \Gamma \setminus \Gamma_1.
\]

By (3.8), we have (when \( ||x|| \) and hence \( n \) are large)
\[
\sum_{\gamma \in \Gamma_1} s(\gamma) \leq (\log n)^{-4}. \tag{3.19}
\]
Hence, we have (when $n$ is large):

$$\sum_{\gamma \in \Gamma} b(\gamma)\mathcal{N}(\gamma) \leq \sum_{\gamma \in \Gamma} s(\gamma)\mathcal{N}(\gamma) \leq \left( \sum_{\gamma \in \Gamma} s(\gamma) \cdot \sum_{\gamma \in \Gamma} s(\gamma)(\mathcal{N}(\gamma))^2 \right)^{1/2} \leq \left( \sum_{\gamma \in \Gamma} s(\gamma) \cdot \sum_{\gamma \in \Gamma} s(\gamma) \right)^{1/2} \leq \left( \sum_{\gamma \in \Gamma} \frac{1}{(\log n)^4 (\log n)^2} \right)^{1/2} = (\log n)^{-1} \ll |K|.$$

Combining this with (3.13) gives:

$$\sum_{\gamma \in \Gamma} b(\gamma)\mathcal{N}(\gamma) \sim |K|.$$

Hence, we have (when $n$ is large):

$$(1 - \epsilon)|K|/ \left( (2a_4 \sigma^2 |K| + \epsilon) \log n \right) \leq \sum_{\gamma \in \Gamma} b(\gamma) \leq (1 + \epsilon)|K|/ \left( (2a_4 \sigma^2 |K| - \epsilon) \log n \right).$$

On the other hand, $\sum_{\gamma \in \Gamma} b(\gamma) \ll (\log n)^{-1}$. Letting $\epsilon \to 0^+$, we get Proposition 3.2.

**Proof of Lemma 3.7.** Without loss of generality, assume $m = |a - b|$. Let $B_a = \{ z : |z - a| \leq 3m/4 \}$, $B_b = \{ z : |z - b| \leq 3m/4 \}$ and $B_c = \{ z : |z - c| \leq M/4 \}$. Write $t = (a + b)/2$ and $B = \{ z : |z - t| \leq 2M \}$. Then, we can estimate separately:

$$\sum_{z \in B_a} |z - a|^{-2} |z - b|^{-2} |z - c|^{-2} \asymp \sum_{z \in B_a} \frac{1}{|z - a|^2 m^2 M^2} \asymp \frac{m^2}{m^2 M^2} \leq \frac{1}{M^2};$$

$$\sum_{z \in B_b} |z - a|^{-2} |z - b|^{-2} |z - c|^{-2} \asymp \sum_{z \in B_b} \frac{1}{|z - c|^2 M^2 M^2} \asymp \frac{M^2}{M^4} \leq \frac{1}{M^2};$$

$$\sum_{z \in B} |z - a|^{-2} |z - b|^{-2} |z - c|^{-2} \asymp \sum_{z \in B} \frac{1}{|z - t|^2 |z - t|^2 M^2} \leq \frac{1}{M^2} \log(M/m);$$

$$\sum_{z \in B^c} |z - a|^{-2} |z - b|^{-2} |z - c|^{-2} \asymp \sum_{z \in B^c} \frac{1}{|z - t|^6} \asymp \sum_{n \geq 2M} \frac{n^3}{n^6} \leq \frac{1}{M^2}.$$

Combining all above finishes the proof.

### 4. Convergence of the first visiting point.

We aim to show Theorem 1.2 in this section. For simplicity, we assume in this section that $\theta$ has finite range. Therefore, for any subset $B \subseteq \mathbb{Z}^4$, we can define its outer boundary and inner boundary by:

$$\partial_o B \doteq \{ y \notin B : \exists x \in B, \text{ such that } \theta(x - y) \lor \theta(y - x) > 0 \};$$

$$\partial_i B \doteq \{ y \in B : \exists x \notin B, \text{ such that } \theta(x - y) \lor \theta(y - x) > 0 \}.$$

The first step is to establish the following asymptotical behavior of the Green function:
Lemma 4.1. \[
\lim_{x,y \to \infty, \|x\| \geq \|y\|} \frac{G_K(x, y)}{(\log \|y\|/ \log \|x\|)} g(x, y) = 1. \tag{4.1}
\]

Remark 4.1. It is a bit unsatisfactory that we need to assume \(\|x\| \geq \|y\|\) in the limit. When \(\theta\) is symmetric, this requirement can be removed, for \(G_K(x, y)/(1-k(x)) = G_K(y, x)/(1-k(y))\). Since we do not need the case \(\|x\| \leq \|y\|\) in this paper, we will not address this case here.

Proof. By Lemma 3.1, we can assume \(\|x\| \geq \|y\|\). Let \(n = \|y\|(\log \|y\|)^{1/8}\) and \(B = \mathcal{C}(n)\). As before, we have
\[
p_K(x) = G_K(x, K) = \sum_{z \in \partial B} \mathcal{H}^R_K(x, z) G_K(z, K) = \sum_{z \in \partial B} \mathcal{H}^R_K(x, z) p_K(z).
\]
By Theorem 1.1, \(p_K(x)/p_K(z) \sim n^2 \log n/(\|x\|^2 \log \|x\|)\). Hence,
\[
\sum_{z \in \partial B} \mathcal{H}^R_K(x, z) \sim \frac{n^2 \log n}{\|x\|^2 \log \|x\|}. \tag{4.2}
\]
By Lemma 3.1, we have \(G_K(z, y) \sim g(z, y) \sim a_4 n^{-2}\) for any \(z \in \partial B\). Therefore,
\[
G_K(x, y) = \sum_{z \in \partial B} \mathcal{H}^R_K(x, z) G_K(z, y) \sim \sum_{z \in \partial B} \mathcal{H}^R_K(x, z) a_4 n^{-2}
\]
\[
\sim a_4 \log n/(\|x\|^2 \log \|x\|) \sim a_4 \log \|y\|/(\|x\|^2 \log \|x\|).
\]
This finishes the proof. \(\square\)

Now we give the following asymptotics of the escape probability by a reversed snake.

Lemma 4.2. For any \(x \in \mathbb{Z}^4\), we have
\[
\mathcal{E}_K(x) = \lim_{n \to \infty} \log n \cdot \sum_{z \in \partial \mathcal{C}(n)} \mathcal{H}^{C(n)}_K(z, x) \text{ exists}. \tag{4.3}
\]
Furthermore, when \(\|x\| > \text{Rad}(K)\), \(E_K(x) > 0\).

Remark 4.2. Note that \(\mathcal{H}^{C(n)}_K(z, x) = \mathcal{H}^{C(n)}_K(z, x)\) and \(\sum_{z \in \partial \mathcal{C}(n)} \mathcal{H}^{C(n)}_K(z, x)\) is roughly the probability that a reversed snake starting from \(x\) does not return to \(K\), except for the bush grafted to the root, until the backbone reaches \(\mathcal{C}(n)\). For the random walk in the critical dimension \((d = 2)\), we have the analogous result (e.g. see Section 2.3 in [2]):
\[
E_K(x) = \lim_{n \to \infty} \log n \cdot \sum_{z \in \partial \mathcal{C}(n)} \mathcal{H}^{C(n)} K(z, x) \text{ exists, for any } x \in \mathbb{Z}^2, K \subset \mathbb{Z}^2;
\]
and
\[
\lim_{x \to \infty} P(S_x(\tau_K) = a | S_x \text{ visits } K) = \frac{1}{\pi^2 \sqrt{\det Q}} E_K(a).
\]

Proof. We will first show
\[
\lim_{n \to \infty, y \to \infty, \|y\| \leq n} \frac{\log n}{\|y\|} \sum_{z \in \partial \mathcal{C}(n)} \mathcal{H}^{C(n)}_K(z, y) = 1. \tag{4.4}
\]
Choose some \( x \in \mathbb{Z}^4 \) such that \( \|x\| \geq n \log n \). By the first-visit lemma, we have
\[
G_K(x, y) = \sum_{z \in \partial_\delta C(n)} G_K(x, z) \mathcal{H}_K^{(n)}(z, y).
\] (4.5)

Due to the last Lemma, \( G_K(x, y) \sim a_4 \|x\|^{-2} \cdot \log \|y\| / \log \|x\| \), \( G_K(x, z) \sim a_4 \|x\|^{-2} \cdot \log n / \log \|x\| \). Joining (4.5) with the last two estimates, we obtain (4.4).

Now we are ready to show (4.3). Note that, for any \( n > l > \|x\| \),
\[
\sum_{w \in \partial_\delta C(n)} \mathcal{H}_K^{(n)}(w, x) = \sum_{z \in \partial_\delta C(l)} \mathcal{H}_K^{(l)}(z, x) \sum_{w \in \partial_\delta C(n)} \mathcal{H}_K^{(n)}(w, z).
\] (4.6)

From this, one finds that if (4.3) is correct for \( x \in \partial_\delta C(l) \) then it is also correct for \( x \).

Therefore, we can assume that \( \|x\| \) is sufficiently large and fixed. Write
\[
a(n) = \log n \cdot \sum_{z \in \partial_\delta C(n)} \mathcal{H}_K^{(n)}(z, x).
\]

It suffices to show
\[
\lim_{m,n \to \infty} \frac{a(m)}{a(n)} = 1; \quad \{a(n)\}_{n \geq 2\|x\|} \text{ is uniformly bounded.}
\] (4.7)

Note that \( \sum_{w \in \partial_\delta C(n)} \mathcal{H}_K^{(n)}(w, x) = a(n) / \log n \) and \( \sum_{z \in \partial_\delta C(l)} \mathcal{H}_K^{(n)}(z, x) = a(l) / \log n \). By (4.4),
\[
\sum_{w \in \partial_\delta C(n)} \mathcal{H}_K^{(n)}(w, z) \sim \frac{\log \|z\|}{\log n}.
\]

Plugging these three estimates into (4.6), we get (4.7).

By Lemma 4.1, we may assume that (when \( \|x\| \) is large enough)
\[
\frac{1}{2} \log \|y\| g(z, y) \leq G_K(z, y) \leq 2 \frac{\log \|y\|}{\log \|z\|} g(z, y), \text{ for } \|z\| \geq \|y\| \geq \|x\|.
\] (4.9)

On the other hand, applying the first-visit lemma, we get
\[
G_K(z, x) = \sum_{y \in \partial_\delta C(n)} G_K(z, y) \mathcal{H}_K^{(n)}(y, x).
\] (4.10)

Choose some \( z \in \partial_\delta C(2n) \). Plugging (4.9) into (4.10) and using the standard estimate of \( g \), one finds that
\[
\log n \sum_{y \in \partial_\delta C(n)} \mathcal{H}_K^{(n)}(y, x) \sim \log \|x\|.
\]

The left hand site is \( a(n) \). Hence, we get (4.8) and finish the proof. \( \square \)

**Remark 4.3.** In fact, we have also proved that when \( \|x\| \) is large, i.e., \( \|x\| > \text{Rad}(K), \mathcal{E}_K(x) > 0 \).

**Proof of Theorem 1.2.** Let \( n = \|x\| / \log \|x\| \) and \( B = C(n) \). Then,
\[
P(S_x(\tau_K) = a | S_x \text{ visits } K) = \frac{\sum_{z \in \partial_\delta B} h(z)}{P_K(x)} [1.2] \frac{\sum_{z \in \partial_\delta B} G_K(x, z) \mathcal{H}_K^{(n)}(z, a)}{1/2 \sigma^2 \|x\|^2 \log \|x\|} a_4 \|x\|^{-2} \mathcal{E}_K(a) \log^{-1} n \frac{1}{1/2 \sigma^2 \|x\|^2 \log \|x\|} \sim 2 \sigma^2 a_4 \mathcal{E}_K(a) = \frac{\sigma^2 \mathcal{E}_K(a)}{4 \pi^2 \sqrt{\text{det } Q}}.
\]
5. The range of the branching random walk conditioned on the total size.

The main goal of this section is to establish the asymptotics of the range of the branching random walk conditioned on the total size, i.e., Theorem 1.3. We will use some ideas from [4]. Especially, we need to use the invariant shift on the invariant snake, $S^I$.

For the invariant snake $S^I$, recall that its backbone is just a random walk. We write $\tau_n$ for the hitting time (vertex) of $(C(n))c$ by the backbone. Thanks to Proposition 3.2, we can obtain the following result.

**Proposition 5.1.**

\[
P(S^I_0(v) \neq 0, \forall v \leq \tau_n \text{ not in the spine}) \sim \frac{4\pi^2 \sqrt{\det Q}}{\sigma^2} \frac{1}{\log n};
\]

\[
P(S^I_0(v_i) \neq 0, i = 1, 2, \ldots, n) \sim \frac{16\pi^2 \sqrt{\det Q}}{\sigma^2} \frac{1}{\log n},
\]

where $v_1 < v_2 < v_3 < \ldots$ are all vertices of $S^I_0$ that are not in the spine.

**Proof.** By Proposition 3.2 (set $K = \{0\}$) and the overshoot lemma, we have

\[
\sum_{\gamma : (C(n))c \to \gamma \subseteq C(n)} b(\gamma) \sim \frac{4\pi^2 \sqrt{\det Q}}{\sigma^2} \frac{1}{\log n}.
\]

Hence, the first assertion can be obtained if we can show

\[
P(S^I_0(v) \neq 0, \forall v \leq \tau_n \text{ not in the spine}) \sim \sum_{\gamma : (C(n))c \to \gamma \subseteq C(n)} b(\gamma).
\]

Let $p_0 = P(S_0$ does not visit 0 except for the root) and the new killing function $k'(x)$ be the probability that $S'_x$ returns to 0 (except possibly for the starting point). Note that $k'(x) = k_K(x)$ when $x \neq 0$. We write $b_{k'}(\gamma)$ for the probability weight of $\gamma$ with this killing function. Then, we have

\[
P(S^I_0(v) \neq 0, \forall v \leq \tau_n \text{ not in the spine}) \sim p_0 \sum_{\gamma : (C(n))c \to \gamma \subseteq C(n)} b_{k'}(\gamma)
\]

\[
= p_0 \left( \sum_{\gamma : (C(n))c \to \gamma \subseteq C(n) \setminus \{0\}} b_{k'}(\gamma) \right) \left( \sum_{\gamma : 0 \to \gamma \subseteq C(n)} b_{k'}(\gamma) \right).
\]

Note that $\lim_{n \to \infty} \sum_{\gamma : 0 \to \gamma \subseteq C(n)} b_{k'}(\gamma) = \sum_{\gamma : 0 \to \gamma \subseteq C(n)} b_{k'}(\gamma)$ and

\[
\sum_{\gamma : (C(n))c \to \gamma \subseteq C(n)} b(\gamma) = \sum_{\gamma : (C(n))c \to \gamma \subseteq C(n) \setminus \{0\}} b_{k'}(\gamma).
\]

Hence, for the first assertion, it is sufficient to show

\[
p_0 \sum_{\gamma : 0 \to \gamma \subseteq C(n)} b_{k'}(\gamma) = 1. \tag{5.1}
\]

Note that (2.7) is obtained from the viewpoint of 'the first visiting point'. In fact we also have an analogous formula from the viewpoint of 'the last visiting point'
where $v$

**Theorem 5.2.** For $S$ is not in the spine and then removing the vertices that are strict before the parent $(T, T')$ on spatial trees, which appeared in [4]. For any spatial tree $(T, T')$.

As mentioned before, we need to use the invariant shift $\varsigma$. Proof of Theorem 5.2 is the analog of Theorem 9 in [4], which implies that $\rho_{\varsigma}(T) ≥ \tau_{[n/4, + \epsilon]}$ and $P(v_n ≥ \tau_{[n/4, + \epsilon]})$ are $o((\log n)^{-1})$. □

Now we can establish the following result about the range of $S'$:

**Theorem 5.2.** Set $R'_n := \#\{S'_0(o), S'_0(v_1), \ldots, S'_0(v_n)\}$ for every integer $n ≥ 0$, where $v_1, v_2, \ldots$ are the same as in Proposition 5.1. Then, we have

$$\frac{\log n}{n} R'_n \xrightarrow{L^2} \frac{16\pi^2 \sqrt{\det Q}}{\sigma^2} \text{ as } n \to \infty.$$  

**Remark 5.1.** Since the typical number of vertices in the spine that come before $v_n$ is of order $\sqrt{n}$, which is much less than $n/\log n$, one can get

$$\frac{\log n}{n} \#\{S'_0(\bar{v}_0), S'_0(\bar{v}_1), \ldots, S'_0(\bar{v}_n)\} \xrightarrow{L^2} \frac{16\pi^2 \sqrt{\det Q}}{\sigma^2} \text{ as } n \to \infty,$$

where $\bar{v}_0, \bar{v}_1, \ldots$ are all vertices due to the default order in the corresponding plane tree $T$ in $S'_0$.

**Remark 5.2.** Proposition 5.1 is the analog of Proposition 8 in [4], as Theorem 5.2 is the analog of Theorem 9 in [4].

**Proof of Theorem 5.2.** As mentioned before, we need to use the invariant shift $\varsigma$ on spatial trees, which appeared in [4]. For any spatial tree $(T, T')$, set $\varsigma(T, T') = (T', T'_{\varsigma})$. Roughly speaking, one can get $T'$ by ‘rerooting’ $T$ at the first vertex that is not in the spine and then removing the vertices that are strict before the parent of the new root. For $T'_{\varsigma}$, set

$$S'_{\varsigma}(v) = S_T(v) - S_T(o'), \text{ for any } v \in \varsigma(T),$$

where $o'$ is the new root. The key result is that $\varsigma$ is invariant under the law of the invariant snake from the origin. For more details about this shift transformation, see Section 2 in [4].

Now we start our proof. For simplicity, write $\hat{v}_0 = 0(\in \mathbb{Z}^4)$ and $\hat{v}_i = S'_0(v_i)$. First observe that

$$E(R'_n) = E\left(\sum_{i=0}^{n} 1_{\{v_j \neq \hat{v}_i, \forall j \in [i+1, n]\}}\right) = \sum_{i=0}^{n} P(\hat{v}_j \neq \hat{v}_i, \forall j \in [i+1, n]).$$

Using the invariant shift mentioned in the beginning, we have

$$P(\hat{v}_j \neq \hat{v}_i, \forall j \in [i+1, n]) = P(\hat{v}_j \neq \hat{v}_0, \forall j \in [1, n-i]).$$

Therefore by Proposition 5.1 one can see

$$E(R'_n) = \sum_{i=0}^{n} P(\hat{v}_j \neq \hat{v}_0, \forall j \in [1, n-i]) \sim \frac{16\pi^2 \sqrt{\det Q}}{\sigma^2} \frac{n}{\log n}, \quad (5.2)$$
Now we turn to the second moment. Similarly, we have
\[
E((R_n^I)^2) = E\left[ \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 1_{\{\hat{v}_k \neq \hat{v}_l, \forall k \in [i+1,n]; \hat{v}_j \neq \hat{v}_l, \forall l \in [j+1,n]\}} \right]
\]
\[
= 2 \sum_{0 \leq i < j \leq n} P(\hat{v}_k \neq \hat{v}_l, \forall k \in [i+1, n]; \hat{v}_j \neq \hat{v}_l, \forall l \in [j+1, n]) + E(R_n)
\]
\[
= 2 \sum_{0 \leq i < j \leq n} P(\hat{v}_k \neq 0, \forall k \in [1, n-i]; \hat{v}_l \neq \hat{v}_{j-i}, \forall l \in [j-i+1, n-i])
\]
+ \(E(R_n)\),
where the last equality again follows from the invariant shift. For any fixed \(\alpha \in (0, 1/4)\) define
\[
\sigma_n := \sup \{k \geq 0 : v_k \leq u_{\lfloor n^{1-\alpha} \rfloor} \},
\]
where \(u_0 \leq u_1 \leq \ldots\) are the vertices in the spine. By standard arguments, one can show
\[
P(\sigma_n \notin [n^{1-3\alpha}, n^{1-\alpha}]) = o(\log^{-2} n).
\]
Therefore we have
\[
\limsup_{n \to \infty} \left( \frac{\log n}{n} \right)^2 E((R_n^I)^2) = \limsup_{n \to \infty} \left( \frac{\log n}{n} \right)^2 \sum_{0 \leq i < j \leq n} P(\hat{v}_k \neq 0, \forall k \in [1, n-i]; \hat{v}_l \neq \hat{v}_{j-i}, \forall l \in [j-i+1, n-i]; \sigma_n \in [n^{1-3\alpha}, n^{1-\alpha}]).
\]
Obviously, in order to study the limsup in the right hand side, we can reduce the sum to those indices \(i\) and \(j\) with \(j - i > n^{1-\alpha}\). However, when \(i\) and \(j\) are fixed and with \(j - i > n^{1-\alpha}\), we have
\[
P(\hat{v}_k \neq 0, \forall k \in [1, n-i]; \hat{v}_l \neq \hat{v}_{j-i}, \forall l \in [j-i+1, n-i]; \sigma_n \in [n^{1-3\alpha}, n^{1-\alpha}])
\]
\[
\leq P(\hat{v}_k \neq 0, \forall k \in [1, \sigma_n]; \hat{v}_l \neq \hat{v}_{j-i}, \forall l \in [j-i+1, n-i]; \sigma_n \in [n^{1-3\alpha}, n^{1-\alpha}])
\]
\[
= P(\hat{v}_k \neq 0, \forall k \in [1, \sigma_n]; \sigma_n \in [n^{1-3\alpha}, n^{1-\alpha}])P(\hat{v}_l \neq \hat{v}_{j-i}, \forall l \in [j-i+1, n-i])
\]
\[
P(\hat{v}_k \neq 0, \forall k \in [1, \sigma_n]; \sigma_n \in [n^{1-3\alpha}, n^{1-\alpha}])P(\hat{v}_l \neq 0, \forall l \in [1, n-j]).
\]
Note that for the second last line, we use the fact that after conditioning on \(\sigma_n = m(< n^{1-\alpha})\), the event in the second probability is independent of the event in the first, and for the last line, we use the invariant shift. Now,
\[
P(\hat{v}_k \neq 0, \forall k \in [1, \sigma_n]; \sigma_n \in [n^{1-3\alpha}, n^{1-\alpha}]) \leq P(\hat{v}_k \neq 0, \forall k \in [1, n^{1-3\alpha}]),
\]
and then we obtain
\[
\limsup_{n \to \infty} \left( \frac{\log n}{n} \right)^2 E((R_n^I)^2) \leq \limsup_{n \to \infty} \left( \frac{\log n}{n} \right)^2 \sum_{0 \leq i < j \leq n, j-i > n^{1-\alpha}} P(\hat{v}_k \neq 0, \forall k \in [1, n^{1-3\alpha}])P(\hat{v}_l \neq 0, \forall l \in [1, n-j])
\]
\[
= \frac{1}{1 - 3\alpha} \left( \frac{16\pi^2 \sqrt{\det Q}}{\sigma^2} \right)^2.
\]
Let \(\alpha \to 0^+\), we get
\[
\limsup_{n \to \infty} \left( \frac{\log n}{n} \right)^2 E((R_n^I)^2) \leq \left\{ \frac{16\pi^2 \sqrt{\det Q}}{\sigma^2} \right\}^2.
\]
Combining this with (5.2), we finish the proof of Theorem 5.2. \(\square\)
Noting that $S_0^-$ is different from $S_0^+$ only in the subtree grafted to the root, one can also obtain the range of the reversed infinite snake $S^-$:

**Corollary 5.3.** Set $R_n^- := \#\{S_0^-(v_0), S_0^-(v_1), \ldots, S_0^-(v_n)\}$. Then,

$$\log \frac{n}{n} R_n^- \xrightarrow{L^2} \frac{16\pi^2 \sqrt{\det \mathcal{Q}}}{\sigma^2} \quad \text{as } n \to \infty,$$

where $v_0, v_1, \ldots$ are all vertices in the corresponding tree due to the default order in the reversed snake.

Now we are ready to prove our main result about the range of the branching random walk conditioned on the total size. This result will follow from Corollary 5.3 by an absolute continuity argument, which is similar to the one in the proof of Theorem 7 in [4]. The idea is as follows. We write $\Xi$ for the law of the $\mu$-GW tree. For every $a \in (0, 1)$, the law under $\Xi^n := \Xi(\cdot|\#T = n)$ of the subtree obtained by keeping only the first $[an]$ vertices of $T$ is absolutely continuous with respect to the law under $\Xi^\infty(\cdot) := \Xi(\cdot|\#T = \infty)$ of the same subtree, with a density that is bounded independently of $n$. Then, a similar property holds for spatial trees, and hence we can use the convergence in Corollary 5.3 for a tree with law $\Xi^\infty$, to get a similar convergence for a tree with law $\Xi^n$.

**Proof of Theorem 1.3.** Let $G$ be the subgroup of $\mathbb{Z}$ generated by the support of $\mu$. In fact, the cardinality of the vertex set of a $\mu$-GW tree is in $1 + G$. For simplicity, we only consider the case $G = \mathbb{Z}$. Minor modifications are needed for the general case. On the other hand, for any sufficiently large integer $n \in 1 + G$, we can define the conditional probability $S^n$ to be $S_0$ conditioned on the total number of vertices being $n$ (since this event is with strictly positive probability).

For a finite plane tree $t$, write $v_t(0), v_t(1), \ldots, v_t(\#t - 1)$ for the vertices of $t$ by the default order. The Lukasiewicz walk (L-walk) of $t$ is the finite sequence $(L(l) = L_t(l), 0 \leq l \leq \#t)$, which can be defined inductively by

$$L(0) = 0, L(l + 1) - L(l) = k_t(v_t(l)) - 1, \quad \text{for every } 0 \leq l \leq \#t,$$

where $k_t(u)$ (for $u \in t$) is the number of children of $u$. Obviously the tree $t$ is determined by its L-walk. A standard result says that under $\Xi$, its L-walk is distributed as a random walk on $\mathbb{Z}$ with jump distribution $\nu$ determined by $\nu(j) = \mu(j + 1)$ for any $j \geq -1$, which starts from 0 and is stopped at the hitting time of $-1$ (in particular, the law of #t coincides with the law of that hitting time). For simplicity, we just let $(Y(k))_{k \geq 0}$ be a random walk on $\mathbb{Z}$ with jump distribution $\nu$. Set

$$\tau := \inf\{k \geq 0 : Y(k) \leq -1\}.$$

We denote by $P_i$ the law of $(Y(k))_{k \geq 0}$ with starting point $Y(0) = i$. A classical application of Kemperman’s formula gives (when $i \geq 0$)

$$P_i[\tau = n] = \frac{i + 1}{n} P_0[Y(n) = -(i + 1)], \quad (5.3)$$

We also need to consider the L-walk for an infinite tree. When $t$ is an infinite tree with only one infinite ray, now the depth-first search sequence $o = v_t(0) < v_t(1) < \cdots < v_t(n) < \ldots$ only examines part of the vertex set of $t$. We define the L-walk of $t$ to be the infinite sequence $(L(i) = L_t(i), i \in \mathbb{N})$:

$$L_t(0) = 0, L_t(l + 1) - L_t(l) = k_t(v_t(l)) - 1, \quad \text{for every } l \in \mathbb{N}.$$
Now, only the ‘left half’ of $t$ (precisely, the subtree generated by $v(0), v(1), \ldots$), not the whole tree $t$, is determined by its L-walk. It is not difficult to verify that when $t$ is a $\mu$-GW tree conditioned on survival, its L-walk is distributed as the random walk on the last paragraph conditioned on $\tau = \infty$, i.e., a Markov chain on $\mathbb{N}$ with transition probability $p(i, j) = \frac{j+1}{i+j} v(j-i)$. Recall that the infinite $\mu$-GW tree is just the ‘left half’ of the $\mu$-GW tree conditioned on survival.

For any $m \in \mathbb{N}^+$ and any finite or infinite tree $t$ with $\#t > m$, denote by $\rho_m(t)$ and $\rho_m^+(t)$, respectively, the subtree generated by the first $m$ vertices of $t$ (i.e., $v(0), \ldots, v(m-1)$) and the one generated by the first $m$ vertices together with all children of the first $m$ vertices. Note that $\rho_m^+(t)$ is uniquely determined by $\rho_m(t)$ and the degrees of the vertices that are ancestors of $V_{m-1}$. In addition, since the parent of a vertex comes before that vertex, we can see that $\rho_m^+(t)$ is uniquely determined by $(L_k(i))_{i \in [0,m]}$. Note that in $\rho_m^+(t)$, there are exactly $L_k(m) + 1$ vertices, which are not in $\rho_m(t)$. We denote by $\partial \rho_m(t)$ the set of these vertices.

Write $T^n$ and $T^\infty$ for the GW-tree conditioned on the total progeny to be $n$ and the one conditioned on survival respectively. We now compare $P[\rho_m^+(T^n) = \rho_m^+(t)]$ to $P[\rho_m^+(T^\infty) = \rho_m^+(t)]$. By considering the L-walks, it is elementary to see that

$$P[\rho_m^+(T^n) = \rho_m^+(t)] = \frac{P(0)[Y(i) = L_k(i), i = 0, 1, \ldots, m; \tau = n]}{P(0)[\tau = n]}.$$  

Using the Markov property at time $m$ and (5.3), one can get

$$P[\rho_m^+(T^n) = \rho_m^+(t)] = \frac{P(0)[Y(i) = L_k(i), i = 0, 1, \ldots, m; \tau = n]}{P(0)[\tau = n]}.$$  

We turn to $T^\infty$. Note that when $\rho_m^+(T^\infty) = \rho_m^+(t)$, there is a unique vertex in $\partial \rho_m(t)$, which is also in the spine of $T^\infty$. On the other hand, for any $u \in \partial \rho_m(t)$, we have

$$P[\rho_m^+(T^\infty) = \rho_m^+(t), \ u \text{ is a vertex in the spine of } T^\infty] = P(0)[Y(i) = L_k(i), i = 0, 1, \ldots, m].$$  

Hence, we get

$$P[\rho_m^+(T^\infty) = \rho_m^+(t)] = P(0)[Y(i) = L_k(i), i = 0, 1, \ldots, m][L_k(m) + 1].$$  

(5.5)

Combining (5.4) and (5.5) leads to

$$\frac{P[\rho_m^+(T^n) = \rho_m^+(t)]}{P[\rho_m^+(T^\infty) = \rho_m^+(t)]} = nP(0)[Y(n-m) = -(L_k(m) + 1)] / (n-m)P(0)[Y(n) = -1].$$

When $n \to \infty$ and $m/n \to a \in (0,1)$, by the local central limit theorem (see, e.g., Theorem 2.3.9 in [2]), we get

$$\frac{P[\rho_m^+(T^n) = \rho_m^+(t)]}{P[\rho_m^+(T^\infty) = \rho_m^+(t)]} = \Gamma_a \left( \frac{L_k(m) + 1}{\sqrt{n}} \right) + o(1),$$

where $\Gamma_a(x) = \exp(-x^2/2\sigma^2(1-a))/(1-a)^{3/2}$. Note that $\Gamma_a$ is bounded above, when $a \in (0,1)$ is fixed.

In other words, $\rho_m^+(T)$ under $\Xi_n$ is absolutely continuous to $\rho_m^+(T)$ under $\Xi^\infty$ and the corresponding constants are uniformly bounded from above (independent
of $n$). Obviously, the above still holds after we add the spatial random mechanism. Therefore, when $\epsilon > 0$ is fixed, we have
\[
\limsup_{n \to \infty} \mathbb{E}_\theta^n \left( 1_{\{ |R_{[an]} - san/\log n| > \epsilon n/\log n \}} \right) 
\leq C(a) \limsup_{n \to \infty} \mathbb{E}_\theta^n \left( 1_{\{ |R_{[an]} - san/\log n| > \epsilon n/\log n \}} \right),
\]
where $s = 16\pi^2 \sqrt{\det \mathcal{Q}}$, $\mathbb{E}_\theta^n$, $\mathbb{E}_\theta^\infty$ are the laws of the corresponding tree-indexed random walks, and $R_{[an]}$ is the range of the subtree $\rho_{[an]}(T)$. Note that under $\mathbb{E}_\theta^n$, $R_{[an]}$ is just the range of $\mathcal{S}_0^n$ for the first $[an]$ vertices. Hence, by Corollary 5.3 (note that $\mathcal{S}_0^n = S_0^n$ since we assume that $\theta$ is symmetric), we obtain that
\[
\lim_{n \to \infty} \mathbb{E}_\theta^n \left( 1_{\{ |R_{[an]} - san/\log n| > \epsilon n/\log n \}} \right) = 0.
\]
Note that $R_n \geq R_{[an]}$ (under $\mathbb{E}_\theta^n$) and $a$ can be chosen arbitrarily close to 1. This finishes the proof of the lower bound.

We still need to show the upper bound. Note that $\rho_{[an]}(T)$ is the subtree lying on the ‘left’ hand side, generated by the first $[an]$ vertices of $T$. Similarly, one can consider the subtree lying on the ‘right’ hand side. Strictly speaking, to get the subtree lying on the ‘right’ hand side, denoted by $\rho_{[an]}^*(T)$, we first reverse the order of children for each vertex in $T$, and then $\rho_{[an]}$ of the same tree $T$ with the new order is just $\rho_{[an]}^*(T)$. Write $R_{[an]}^{n,*}$ for the range of $\rho_{[an]}^*(T)$ corresponding to $\mathcal{S}_0^n$. By symmetry, we also have
\[
\lim_{n \to \infty} \mathbb{E}_\theta^n \left( 1_{\{ |R_{[an]}^{n,*} - san/\log n| > \epsilon n/\log n \}} \right) = 0.
\]
Now fix some $a \in (0, 1)$. Note that $\rho_{[an]}(T)$ and $\rho_{[(1-a)n]}^*(T)$ cover the whole tree $T$ except for a number of vertices. This number is not more than $|\rho_{[an]}(T) \cap \rho_{[(1-a)n]}^*(T)| + 2$. Note that on each generation, there is at most one vertex that is in both $\rho_{[an]}(T)$ and $\rho_{[(1-a)n]}^*(T)$. Hence $|\rho_{[an]}(T) \cap \rho_{[(1-a)n]}^*(T)|$ is not more than the number of generations, which is typically of order $\sqrt{n}$ (under $\mathbb{E}_\theta^n$). Hence, $R_n - (R_{[an]} + R_{[(1-a)n]}^{n,*})$ is less than $n^{0.6}$ with high probability (tending to 1). This finishes the proof of the upper bound. \qed

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References


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