Uniform spanning forests on biased Euclidean lattices

Z. Shi, V. Sidoravicius, H. Song, L. Wang, K. Xiang

Abstract

The uniform spanning forest measure (USF) on a locally finite, infinite connected graph with conductance $c$ is defined as a weak limit of uniform spanning tree measure on finite subgraphs. Depending on the underlying graph and conductances, the corresponding USF is not necessarily concentrated on the set of spanning trees. Pemantle [20] showed that on \( \mathbb{Z}^d \), equipped with the unit conductance \( c = 1 \), USF is concentrated on spanning trees if and only if \( d \leq 4 \). In this work we study the USF associated with conductances \( c(e) = \lambda^{-|e|} \), where \( |e| \) is the graph distance of the edge \( e \) from the origin, and \( \lambda \in (0, 1) \) is a fixed parameter. Our main result states that in this case USF consists of finitely many trees if and only if \( d = 2 \) or 3. More precisely, we prove that the uniform spanning forest has \( 2^d \) trees if \( d = 2 \) or 3, and infinitely many trees if \( d > 4 \). Our method relies on the analysis of the spectral radius and the speed of the \( \lambda \)-biased random walk on \( \mathbb{Z}^d \).

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1 Introduction and main results

A spanning tree in a finite connected graph \( G \) is a subgraph that is a tree containing every vertex of \( G \). A uniform spanning tree in \( G \) is a subgraph chosen uniformly among all spanning trees. In [20], following a suggestion of R. Lyons, Pemantle investigated the uniform spanning forest on \( \mathbb{Z}^d \) (with \( d \geq 1 \)), as the weak limit of uniform spanning trees in larger and larger finite boxes; see also Häggström [10]. Pemantle showed that the limit exists, that it does not depend on the sequence of boxes, and that every connected component of the USF is an infinite tree. This weak limit is a forest, not necessarily composed of a single tree: loosely speaking, as the size of the boxes increases to infinity, the spanning tree can possibly “split into several pieces”. More precisely, Pemantle proved the following remarkable result.

Theorem A (Pemantle [20]). The uniform spanning forest on \( \mathbb{Z}^d \) has almost surely a single tree if \( d \leq 4 \), and infinitely many trees if \( d \geq 5 \).
For refinements of Theorem A, see Benjamini, Kesten, Peres and Schramm [6], Hutchcroft and Peres [11]. For an account of general properties of uniform spanning trees on infinite graphs, see Benjamini, Lyons, Peres and Schramm [7], as well as Chapter 10 of Lyons and Peres [18].

The aim of this paper is to make appear a phase transition at another dimension for uniform spanning forests on \( \mathbb{Z}^d \). In order to see such a new phase transition, the infinite graph \( \mathbb{Z}^d \) needs to be biased. Let \( \lambda \in (0, 1) \). Each edge \( e \) of the \( \lambda \)-biased lattice \( \mathbb{Z}^d \) is given the conductance \( \lambda^{-|e|} \), where \( |e| \) denotes the distance between \( e \) and the origin \( 0 \in \mathbb{Z}^d \). [If \( x \) and \( y \) are the extremities of \( e \), then \( |e| := \min\{|x|, |y|\} \), with “\( |·| \)” standing for the \( \ell^1 \)-norm.] Following Benjamini, Lyons, Peres and Schramm [7], a uniform spanning tree in a finite connected subgraph \( G \) of \( \lambda \)-biased \( \mathbb{Z}^d \) is defined to be a subgraph chosen among all spanning trees proportional to their conductances. [The conductance of a spanning tree is the product of the conductances of all the edges in the subtree.] We consider uniform spanning trees in \( \lambda \)-biased \( \mathbb{Z}^d \) in finite boxes and are interested in the weak limit when the size of the boxes increases to infinity. There are two ways to consider such weak limits, with either free or wired boundary conditions. In both situations, the existence of weak limit was known (Benjamini, Lyons, Peres and Schramm [7]), yielding the free uniform spanning forest measure (denoted by \( \text{FSF} \)) and the wired uniform spanning forest measure (\( \text{WSF} \)), respectively. We prove in Section 4 that the two measures coincide (except in the simple case \( d = 1 \)), and are referred to as the uniform spanning forests (\( \text{USF} \)) for simplicity. [The identification between free and wired uniform spanning forest measures on the unbiased lattice \( \mathbb{Z}^d \) studied in Theorem A was already proved by Pemantle [20].] For more discussions on relations between \( \text{FSF} \) and \( \text{WSF} \) on general infinite graphs, see Chapter 10 of Lyons and Peres [18].

The main result of the paper is as follows.

**Theorem 1.1.** Let \( \lambda \in (0, 1) \). Almost surely, the number of trees in the uniform spanning forest on the \( \lambda \)-biased lattice \( \mathbb{Z}^d \) is \( 2^d \) if \( d = 2 \) or \( 3 \), and is infinite if \( d \geq 4 \).

Quite recently, Dibene [8] investigated another type of \( \text{USF} \) on \( \mathbb{Z}^d \) with an assignment of conductances \( c(e) = e^{\lambda \max\{x_1, y_1\}} \), where \( x = (x_1, \ldots, x_d) \) and \( (y_1, \ldots, y_d) \) are the extremities of \( e \). The underlying random walk has a uniform drift towards a fixed direction, while in our case it has a drift away from the origin. This gives rise to very
different properties of the uniform spanning forest. For example, the USF in [8] has a.s. one tree if \( d = 1 \) or 2, and infinity many trees if \( d \geq 3 \).

The proof of Theorem 1.1 relies on the study of biased random walks, in particular, on their intersection properties. The rest of the paper is organized as follows. In Section 2, we introduce the notion of biased random walks, and study the spectral radius and the speed. The number of intersections of two independent biased random walks is studied in Section 3. Finally, Theorem 1.1 is proved in Section 4.

For positive functions \( f \) and \( g \) on \( \mathbb{N} \), we write \( f \asymp g \) if there is a constant \( c > 0 \) (possibly depending on \( \lambda \) and \( d \)) such that \( c^{-1}g(n) \leq f(n) \leq cg(n) \) for all \( n \in \mathbb{N} \), and write \( f \sim g \) if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1 \).

2 Spectral radius and speed of biased walks on \( \mathbb{Z}^d \)

This section is devoted to the study of the spectral radius and the speed of biased walks on \( \mathbb{Z}^d \). We first introduce the notion of biased random walks on general infinite graphs.

Let \( G := (V(G), E(G)) \) be a locally finite, connected infinite graph and fix a vertex \( o \) in \( G \) as root. For \( x \in V(G) \), let \( |x| \) be the graph distance of \( x \) from \( o \). We define, for \( n \geq 0 \),

\[
B_G(n) := \{ x \in V(G) : |x| \leq n \}, \quad \partial B_G(n) := \{ x \in V(G) : |x| = n \}.
\]

Let \( \lambda > 0 \). The \( \lambda \)-biased random walk \( (X_n, n \geq 0) \), or \( \text{RW}_\lambda \), is a random walk on \( (G, o) \) with transition probabilities: for \( y \) adjacent to \( x \),

\[
p_\lambda(x, y) = \begin{cases} 
\frac{1}{d_0} & \text{if } x = o, \\
\frac{\lambda}{d_x + (\lambda - 1)d_x^{-}} & \text{if } x \neq o \text{ and } |y| = |x| - 1, \\
\frac{1}{d_x + (\lambda - 1)d_x^{-}} & \text{otherwise}.
\end{cases}
\]

(2.1)

Here \( d_x \) is the degree of vertex \( x \), and \( d_x^{-} \) (resp. \( d_x^{0} \)) is the number of edges connecting \( x \) to \( \partial B_G(|x| - 1) \) (resp. \( \partial B_G(|x|) \)). Note that \( d_x^{-} \geq 1 \) if \( x \neq o \), and \( d_0^{-} = d_0^{0} = 0 \). When \( \lambda = 1 \), \( \text{RW}_\lambda \) is the usual simple random walk on \( G \). For general properties of biased random walks on graphs we refer to [18] and [22].

Let \( p_\lambda^{(n)}(x, y) = \mathbb{P}_x(X_n = y) \) be the \( n \)-step transition probability of \( X_n \), where \( \mathbb{P}_x \) is the law of \( \text{RW}_\lambda \) starting at \( x \). The spectral radius \( \rho(\lambda) \) of \( \text{RW}_\lambda \) is defined to be the
reciprocal of the convergence radius for the Green function

\[ G_\lambda(x, y|z) := \sum_{n=0}^{\infty} p^{(n)}_\lambda(x, y) z^n. \]

Clearly, \( \rho(\lambda) \) does not depend on the choices of \( x \) and \( y \), and can be expressed as

\[ \rho(\lambda) := \limsup_{n \to \infty} [p^{(n)}_\lambda(o, o)]^{1/n}. \]

Define the speed \( S(\lambda) \) of \( \text{RW}_\lambda \) by

\[ S(\lambda) := \lim_{n \to \infty} \frac{|X_n|}{n}, \]

provided the limit exists almost surely.

There are many deep and important questions related to how the spectral radius and the speed depend on the bias parameter \( \lambda \). Lyons, Pemantle and Peres [17] asked whether the speed of \( \text{RW}_\lambda \) on the supercritical Galton–Watson tree without leaves is strictly decreasing. This has been confirmed for \( \lambda \) lying in some regions (cf. [4, 2, 1, 23]), but still remains open for general values of \( \lambda \). For the supercritical Galton–Watson tree with leaves, the speed is expected ([3, Section 3]) to be unimodal in \( \lambda \) (due to presence of traps). On lamplighter graph \( \mathbb{Z} \rtimes \sum_{x \in \mathbb{Z}} \mathbb{Z}_2 \), the speed of \( \text{RW}_\lambda \) is positive if and only if \( 1 < \lambda < (1 + \sqrt{5})/2 \); see [16].

When \( G = \mathbb{Z}^d \), the situation is relatively simple.

**Theorem 2.1.** Let \( \lambda \in (0, 1) \), and let \( (X_n) \) be \( \text{RW}_\lambda \) on \( \mathbb{Z}^d \). Then

\[ p^{(2n)}_\lambda(o, o) \asymp \left( \frac{2\sqrt{\lambda}}{1+\lambda} \right)^{2n} \frac{1}{n^{3d/2}}. \]

In particular, the spectral radius equals \( \rho(\lambda) = \frac{2\sqrt{\lambda}}{1+\lambda} < 1 \), and is strictly increasing in \( \lambda \).

The implicit constant in (2.2) depends on \( \lambda \) and \( d \), and it blows up when \( \lambda \uparrow 1 \). The proof of Theorem 2.1 relies on the following lemma, which is motivated by [21, Exercise 1.7]. Let \( (Y_n) \) be the simple random walk on \( \mathbb{Z} \) standing at 0. Then we have

\[ \mathbb{P}(Y_i > 0, 1 \leq i \leq 2n-1, Y_{2n} = 0) = 4^{-n} C_{n-1}, \]

where \( C_\ell = \frac{1}{\ell+1} \binom{2\ell}{\ell} \) is the \( \ell \)-th Catalan number.
Lemma 2.2. Let $Z_n$ be the number of $1 \leq i \leq 2n$ such that $Y_i = 0$. Then there is a positive constant $c$ such that for $n \in \mathbb{N}$ and $1 \leq k \leq n$,

$$
\mathbb{P}(Y_{2n} = 0, Z_n = k) \leq \frac{ck^{5/2}}{n^{3/2}}.
$$

Proof. The lemma holds trivially if $k \geq \frac{n}{2}$. Assume now $1 \leq k < \frac{n}{2}$. By splitting the paths of the simple random walk into excursions, we see from (2.3) that

$$
\mathbb{P}(Y_{2n} = 0, Z_n = k) = 4^{-n} \sum_{n_1 + \cdots + n_k = n, n_i \geq 1, 1 \leq i \leq k} (2C_{n_1-1})(2C_{n_2-1}) \cdots (2C_{n_k-1})
$$

$$
= 2^{k-2n} \sum_{n_1 + \cdots + n_k = n, n_i \geq 1, 1 \leq i \leq k} C_{n_1-1}C_{n_2-1} \cdots C_{n_k-1}.
$$

Since $n_i \geq \frac{n}{k}$ for some $i$, we have that

$$
\mathbb{P}(Y_{2n} = 0, Z_n = k) \leq k \cdot 2^{k-2n} \sum_{n_1 + \cdots + n_k = n, n_i \geq 1, 2 \leq i \leq k, n_i \geq n/k} C_{n_1-1}C_{n_2-1} \cdots C_{n_k-1}
$$

$$
= k \cdot 2^{k-2n} \sum_{n_2 + \cdots + n_k \leq n -(n/k), n_i \geq 1, 2 \leq i \leq k} C_{n_2-1} \cdots C_{n_k-1}.
$$

Recall that ([9]) $C_\ell < \frac{4^\ell}{(\ell+1)(\pi \ell)^{1/2}}$ for all $\ell$. As a consequence, we have $C_{n_2-1} \cdots C_{n_k-1} < \frac{4^{n_2-1} \cdots n_k-1}{(n_2-1) \cdots n_k-1)^{1/2}}$, which is bounded by $\frac{4^{n_2-1} \cdots n_k-1}{(\pi/2)^{1/2}}$ if $n_2 + \cdots + n_k \leq n - n/k$. Accordingly,

$$
\mathbb{P}(Y_{2n} = 0, Z_n = k) \leq \frac{k \cdot 2^{k-2n}}{n \left(\pi \left(\frac{n}{k} - 1\right)\right)^{1/2}} \sum_{n_2 + \cdots + n_k \leq n -(n/k), n_i \geq 1, 2 \leq i \leq k} \frac{C_{n_2-1} \cdots C_{n_k-1}}{4^{n_2} \cdots 4^{n_k}}
$$

$$
\leq \frac{k \cdot 2^{k-2n}}{n \left(\pi \left(\frac{n}{k} - 1\right)\right)^{1/2}} \left(\sum_{m=1}^\infty \frac{C_{m-1}}{4^m}\right)^{k-1}.
$$

Recall that the generating function of $C_\ell$ is

$$
\sum_{\ell=0}^\infty C_\ell x^\ell = \frac{1 - \sqrt{1 - x}}{2x}, \quad x \in \left[-\frac{1}{4}, \frac{1}{4}\right],
$$

from which it follows that $\sum_{\ell=0}^\infty \frac{C_\ell}{4^\ell} = \frac{1}{2}$. Hence

$$
\mathbb{P}(Y_{2n} = 0, Z_n = k) \leq \frac{k \cdot 2^{k-2n}}{n \left(\pi \left(\frac{n}{k} - 1\right)\right)^{1/2}} \frac{1}{2^{k-1}} = \frac{1}{2} \frac{2^{-1}k}{n \left(\pi \left(\frac{n}{k} - 1\right)\right)^{1/2}}.
$$
Since \(0 < k < \frac{n}{2}\), we have \((\pi(\frac{n}{k}) - 1)^{1/2} = (\frac{\pi n}{k})^{1/2}(1 - \frac{k}{n})^{1/2} \geq (\frac{\pi n}{2k})^{1/2}\), so that
\[
\Pr(Y_{2n} = 0, Z_n = k) \leq \frac{k^{5/2}}{(2\pi)^{n/2}} n^{3/2}
\] as desired.

**Proof of Theorem 2.1.** Let \((S_n) = (S^1_n, \ldots, S^d_n)\) be the simple random walk on \(\mathbb{Z}^d\) starting at \(o\). For \(n \geq 0\), let \(\kappa(n)\) be the number of coordinates of \(S_n\) that are zero. It is easy to see that

\[
  p^{(2n)}_\lambda(o, o) = \left(\frac{2\sqrt{\lambda}}{1 + \lambda}\right)^{2n} E\left[1_{\{S_{2n} = o\}} \prod_{j=0}^{2n-1} \frac{d(1 + \lambda)}{d + \kappa(j) + (d - \kappa(j))\lambda}\right].
\] (2.4)

To prove the theorem, it suffices to show that

\[
E\left[1_{\{S_{2n} = o\}} \prod_{j=0}^{2n-1} \frac{d(1 + \lambda)}{d + \kappa(j) + (d - \kappa(j))\lambda}\right] \approx n^{-3d/2}.
\] (2.5)

For \(1 \leq i \leq d\), let \(Y^i_0, Y^i_1, \ldots\) be the \(i\)-th coordinate of the sequence \(S_0, S_1, \ldots\), with the null moves deleted. In other words, if we denote by \(N_i\) the number of steps among the first \(2n\) steps of the simple random walk \((S_n)\) that are taken in the \(i\)-th coordinate, then \(Y^i_{N_i} = S^i_{2n}\). Conditioned on \(N_i = 2n_i\), \(1 \leq i \leq d\), \((Y^i_k)_{0 \leq k \leq n_i}\) is a one-dimensional simple random walk. By [21, Lemma 1.4], there exist constants \(c_1 > 0\) and \(c_2 > 0\), depending only on \(d\), such that

\[
\sum_{n_1 + \cdots + n_d = n} \Pr(N_i = 2n_i, 1 \leq i \leq d) \leq c_1 \exp(-c_2n).
\]

For \(n_1 + \cdots + n_d = n\) with \(n_i \in \left[\frac{n}{d}, \frac{3n}{d}\right]\) for all \(1 \leq i \leq d\), we have from (2.3) that

\[
E\left[1_{\{S_{2n} = o\}} \prod_{j=0}^{2n-1} \frac{d(1 + \lambda)}{d + \kappa(j) + (d - \kappa(j))\lambda} \mid N_i = 2n_i, 1 \leq i \leq d\right]
\]
\[
\geq \Pr(Y^i_{n_i} = 0, Y^j_{n_j} > 0, 1 \leq j < n_i, 1 \leq i \leq d \mid N_i = 2n_i, 1 \leq i \leq d)
\]
\[
\geq c_3 \prod_{i=1}^{d} n_i^{-3/2}
\]

for some \(c_3 > 0\). Therefore the left-hand side of (2.5) is at least

\[
c_3 \sum_{n_1 + \cdots + n_d = n, n_i \in \left[\frac{n}{d}, \frac{3n}{d}\right], 1 \leq i \leq d} \Pr(N_i = 2n_i, 1 \leq i \leq d) \prod_{j=1}^{d} n_i^{-3/2} \geq c_4 n^{-3d/2},
\]

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which proves the lower bound for \( p^{(2n)}_\lambda(o, o) \).

To show the upper bound for \( p^{(2n)}_\lambda(o, o) \), we note that \( d+k+(d-k)\lambda \geq d(1+\lambda)+(1-\lambda) \) for \( k \geq 1 \). Writing \( \eta := \frac{d(1+\lambda)}{d+\kappa(j)+(d-\kappa(j))\lambda} \in (0, 1) \), we get,

\[
\prod_{j=0}^{2n-1} \frac{d(1+\lambda)}{d+\kappa(j)+(d-\kappa(j))\lambda} \leq \eta^{2n},
\]

where \( Z_n \) is the number of hits (but excluding the initial hit) to the axial hyperplanes of the simple random walk \( (S_n) \) among the first \( 2n \) steps. Denote by \( Z^i_n \) the number of \( 1 \leq j \leq N_i \) such that \( Y^i_j = 0 \). Then \( Z_n \geq Z^1_n + \cdots + Z^d_n \). For \( n_1 + \cdots + n_d = n \) with \( n_i \in \left[ \frac{n}{d}, \frac{3n}{d} \right] \) for all \( 1 \leq i \leq d \), we have

\[
E \left[ \prod_{j=0}^{2n-1} \frac{d(1+\lambda)}{d+\kappa(j)+(d-\kappa(j))\lambda} \mid N_i = 2n_i, 1 \leq i \leq d \right]
\leq E \left[ \prod_{i=1}^d \prod_{k=1}^{n_i} \frac{d(1+\lambda)}{d+\kappa(j)+(d-\kappa(j))\lambda} \mid N_i = 2n_i, 1 \leq i \leq d \right]
\leq c_5 \prod_{i=1}^d \sum_{k=1}^{n_i} \frac{k^{5/2}}{\eta^k} n_i^{-3/2}
\leq c_6 n^{-3d/2},
\]

where we used Lemma 2.2 in the second last inequality. Therefore,

\[
E \left[ \prod_{j=0}^{2n-1} \frac{d(1+\lambda)}{d+\kappa(j)+(d-\kappa(j))\lambda} \right]
\leq c_1 e^{-c_2 n} + \sum_{n_1+\cdots+n_d=n \atop n_i \in \left[ \frac{n}{d}, \frac{3n}{d} \right], 1 \leq i \leq d} \mathbb{P}(N_i = 2n_i, 1 \leq i \leq d)n^{-3d/2}
\leq c_7 n^{-3d/2}.
\]

This proves (2.5) and the theorem follows.

Let \( \mathcal{X} := \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : x_i = 0 \text{ for some } i\} \).

**Lemma 2.3.** Almost surely, \( \text{RW}_\lambda \) with \( \lambda \in (0, 1) \) visits \( \mathcal{X} \) only finitely many times.

**Proof.** In dimension \( d = 2 \), the lemma is a consequence of [13, Proposition 2.1], whose proof relies on properties of Riemann surfaces, and does not seem to be easily extended to higher dimensions.

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Let $\lambda \in (0, 1)$. Let $(X_n)_{n=0}^{\infty} := (X_0, \ldots, X_n)_{n=0}^{\infty}$ be RW with the step distribution $\mu$ with the step distribution $\mu$.

We get (i) by sending $n$ the transition probability from a state in $X \setminus \{Y\}$, starting at $F_i$, i.e.,

$$\sigma_1 := \inf\{n > 0 : Y_n \in \mathbb{Z}_+ \setminus \mathcal{X}\}, \quad \tau_1 := \inf\{n > \sigma_1 : Y_n \in \mathcal{X}\},$$

and recursively for $i \geq 2$,

$$\sigma_i := \inf\{n > \tau_i-1 : Y_n \in \mathbb{Z}_+ \setminus \mathcal{X}\}, \quad \tau_i := \inf\{n > \sigma_i : Y_n \in \mathcal{X}\},$$

with the convention that $\inf \emptyset := \infty$. Let $(\mathcal{F}_n)_{n=0}^{\infty}$ be the filtration generated by $(Y_n)_{n=0}^{\infty}$, i.e., $\mathcal{F}_n := \sigma(Y_1, \ldots, Y_n)$. We claim that

(i) For any $i > 1$, conditioned on $\{\tau_{i-1} < \infty\}$ and $\mathcal{F}_{\tau_{i-1}}$, $\sigma_i < \infty$ a.s.

(ii) There exists a constant $0 < q < 1$ such that for any $i \geq 1$, $\mathbb{P}(\tau_i < \infty | \sigma_i < \infty, \mathcal{F}_{\sigma_i}) \leq q$.

Indeed, conditionally on $\{\tau_{i-1} < \infty\}$ and $\mathcal{F}_{\tau_{i-1}}$, $(Y_{\tau_{i-1}+n})_{n=0}^{\infty}$ is a Markov chain starting at $Y_{\tau_{i-1}}$ with the same transition probability as that of $(Y_n)_{n=0}^{\infty}$. At each step, the transition probability from a state in $\mathcal{X} \setminus \{o\}$ to another state in $\mathcal{X}$ is $\frac{d-k+(d-k)\lambda}{(d-1)(1+\lambda)}$ for some $1 \leq k \leq d-1$, which is at most $\frac{(d-1)(1+\lambda)}{(d-1)(1+\lambda)+2} < 1$. Since the number of visits to $o$ in the first $2n$ steps is at most $n$, we have

$$\mathbb{P}(\sigma_i - \tau_{i-1} > 2n | \tau_{i-1} < \infty, \mathcal{F}_{\tau_{i-1}}) = \mathbb{P}(Y_{\tau_{i-1}+k} \in \mathcal{X} \text{ for } 1 \leq k \leq 2n | \tau_{i-1} < \infty, \mathcal{F}_{\tau_{i-1}}) \leq \left(\frac{(d-1)(1+\lambda)}{(d-1)(1+\lambda)+2}\right)^n.$$

We get (i) by sending $n$ to $\infty$.

Let $(Z_n)_{n=0}^{\infty}$ be a drifted random walk on $\mathbb{Z}_d$, starting inside the first open orthant, with the step distribution $\mu$ given by $\mu(e_1) = \cdots = \mu(e_d) = \frac{1}{d(1+\lambda)}$ and $\mu(-e_1) = \cdots = \mu(-e_d) = \frac{\lambda}{d(1+\lambda)}$ (where $\{e_1, \ldots, e_d\}$ is the standard basis in $\mathbb{Z}_d$). Let $\tau := \inf\{n \geq 0 : Z_n \in \mathcal{X}\}$. Since the walk has a constant drift whose components are all strictly positive, $\mathbb{P}(\tau < \infty) \leq q < 1$ where $q$ depends on $d$ and $\lambda$.

Conditioned on $\sigma_i < \infty$ and $\mathcal{F}_{\sigma_i}$, $(Y_{\sigma_i+n}, 0 \leq n < \tau_i - \sigma_i)$ has the same distribution as $(Z_n, 0 \leq n < \tau)$. Now (ii) follows readily.

By (i) and (ii), for $i \geq 2$, $\mathbb{P}(\tau_i < \infty) \leq q \mathbb{P}(\tau_{i-1} < \infty)$, hence $\mathbb{P}(\tau_i < \infty) \leq q^i$. The Borel-Cantelli lemma implies that a.s. there are only finitely many $i$’s such that
\[ \tau_i < \infty. \] Let \( m \) be the largest one. The total number of visits to \( X \) of \( (Y_n)_{n=0}^{\infty} \) is \( \sigma_1 + (\sigma_2 - \tau_1) + \cdots + (\sigma_m - \tau_{m-1}) \), which is a.s. finite.

**Theorem 2.4.** Let \( \lambda \in (0, 1) \) and let \( (X_n) \) be RW \( \lambda \) on \( \mathbb{Z}^d \). Then

\[
\lim_{n \to \infty} \frac{1}{n} \left( |X^1_n|, \ldots, |X^d_n| \right) = \frac{1 - \lambda}{1 + \lambda} \left( \frac{1}{d}, \ldots, \frac{1}{d} \right) \quad \text{a.s.}
\]

In particular, the speed \( S(\lambda) = \frac{1 - \lambda}{1 + \lambda} \) of RW \( \lambda \) is positive and strictly decreasing in \( \lambda \in (0, 1) \).

**Proof.** For simplicity, we only prove the theorem for \( d = 2 \). Define functions \( f_1 \) and \( f_2 \) on \( \mathbb{Z}^2 \) by

\[
f_1(x) := \begin{cases} 0, & x_1 = 0, \\ \frac{1 - \lambda}{3 + \lambda}, & x_1 \neq 0, x_2 = 0, \\ \frac{1 - \lambda}{2(1 + \lambda)}, & \text{otherwise,}
\end{cases} \quad f_2(x) := \begin{cases} \frac{1 - \lambda}{3 + \lambda}, & x_1 = 0, x_2 \neq 0, \\ 0, & x_2 = 0, \\ \frac{1 - \lambda}{2(1 + \lambda)}, & \text{otherwise.}
\end{cases}
\]

It is easily seen that \( (|X^1_n| - |X^1_{n-1}| - f_1(X_{n-1}), |X^2_n| - |X^2_{n-1}| - f_2(X_{n-1}))_{n=1}^{\infty} \) is a martingale-difference sequence. By the strong law of large numbers (cf. [18, Theorem 13.1]),

\[
\lim_{n \to \infty} \frac{1}{n} \left( |X^1_n| - \sum_{k=0}^{n-1} f_1(X_k) \right) = \lim_{n \to \infty} \frac{1}{n} \left( |X^2_n| - \sum_{k=0}^{n-1} f_2(X_k) \right) = 0 \quad \text{a.s.}
\]

Since \( |X_n| = |X^1_n| + |X^2_n| \), the theorem follows from Lemma 2.3 and the definitions of \( f_1 \) and \( f_2 \).

### 3 Intersections of two independent random walks

It is known ([7, Theorem 9.4]) that dimension-depending phase transition in the uniform spanning forest has a deep connection with the intersection property of independent random walks on \( \mathbb{Z}^d \). In fact, Theorem A by Pemantle (recalled in the introduction) is closely related to the fact that two independent simple random walks on \( \mathbb{Z}^d \) intersect infinitely often if \( d \leq 4 \) and finitely many times if \( d \geq 5 \) (see for example Lawler [14, 15]).

In this section, we study the intersection property of two independent biased random walks on \( \mathbb{Z}^d \). The main result of the section, Theorem 3.5 below, plays a crucial role in
the forthcoming computation in Section 4 of the number of trees in the uniform spanning forests on \( \lambda \)-biased \( \mathbb{Z}^d \).

We first study a simplified problem, namely, the intersection property of two independent drifted random walks on \( \mathbb{Z}^d \).

### 3.1 Intersections of drifted random walks

Consider the drifted random walk on \( \mathbb{Z}^d \), whose distribution is given by convolutions of step distribution

\[
\mu(e_1) = \cdots = \mu(e_d) = \frac{1}{d(1 + \lambda)}, \quad \mu(-e_1) = \cdots = \mu(-e_d) = \frac{\lambda}{d(1 + \lambda)},
\]

(3.1)

where \( \{e_1, \ldots, e_d\} \) is the standard basis of \( \mathbb{Z}^d \). Before exiting from one of the \( 2^d \) open orthants, the \( \lambda \)-biased random walk and drifted random walk have the same distributions. However, \( \lambda \)-biased random walks exhibit quite different behavior from drifted random walk when they hit some axial hyperplane or the boundary of the orthant.

**Theorem 3.1.** Assume \( \lambda \in (0, 1) \). Let \((Z_n)_{n=0}^{\infty}\) and \((W_n)_{n=0}^{\infty}\) be independent drifted random walks on \( \mathbb{Z}^d \) with the same step distribution \( \mu \) given by (3.1), starting at \( z_0 \) and \( w_0 \) respectively. Then almost surely,

\[
|\{Z_m; m \geq 0\} \cap \{W_n; n \geq 0\}| \text{ is finite for } d \geq 4 \text{ and infinite for } d \leq 3.
\]

Without loss of generality, let us assume \( z_0 = w_0 = 0 \). The expectation of the intersection number for \((Z_m)_{m=0}^{\infty}\) and \((W_n)_{n=0}^{\infty}\) is

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathbb{P}(Z_m = W_n) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{x \in \mathbb{Z}^d} p^{(m)}(o, x) p^{(n)}(o, x),
\]

(3.2)

where \( p^{(n)}(x, y) \) is the \( n \)-step transition probability for \((Z_m)_{m=0}^{\infty}\) from \( x \) to \( y \). By [18, Theorem 10.24], to prove Theorem 3.1, it suffices to prove that the sum on the right-hand side of (3.2) is finite if \( d \geq 4 \), and is infinite if \( d \leq 3 \).

Let \( \mathbf{m} \) and \( \Sigma = (\Sigma_{ij}) \) be respectively the mean and the covariance matrix of \( \mu \). Then

\[
\mathbf{m} = \frac{1}{d(1 + \lambda)} (1, \ldots, 1) \quad \text{and} \quad \Sigma_{ij} = \frac{1}{d} \delta_{ij} - \frac{(1 - \lambda)^2}{d^2 (1 + \lambda)^2} \quad \text{for } 1 \leq i, j \leq n.
\]

By the local limit theorem ([24, Theorem 2]),

\[
p^{(n)}(o, x) = \frac{1}{(2\pi n)^{d/2} (\det \Sigma)^{1/2}} \exp \left( -\frac{(x - n\mathbf{m}) \cdot \Sigma^{-1}(x - n\mathbf{m})}{2n} \right) + o(n^{-d/2}),
\]

(3.3)
where \( n^{d/2}o(n^{-d/2}) \to 0 \) as \( n \to \infty \) uniformly in \( x \in \mathbb{Z}^d \). Since the largest eigenvalue of \( \Sigma \) is \( \frac{1}{d} \), we have

\[
(x - nm) \cdot \Sigma^{-1}(x - nm) \geq d|x - nm|
\]

for \( x \in \mathbb{Z}^d \). The local limit theorem (3.3) immediately implies the following result.

**Lemma 3.2.** (i) There exists a constant \( c > 0 \) such that

\[
\sup_{x \in \mathbb{Z}^d} p^{(n)}(0, x) \leq cn^{-d/2}, \quad \forall n \geq 1. \tag{3.4}
\]

(ii) For \( \sigma > 0 \), define

\[
R_{n,\sigma} := \{x \in \mathbb{Z}^d : |x_i - \frac{1 - \lambda}{d(1 + \lambda)}n| \leq \sigma n^{1/2}, \ 1 \leq i \leq d\}.
\]

Then there exists a constant \( c > 0 \), depending on \( \sigma, \lambda \) and \( d \) such that for any \( n \in \mathbb{N} \) with \( R_{n,\sigma} \neq \emptyset \),

\[
p^{(n)}(0, x) \geq cn^{-d/2} \quad \text{for } x \in R_{n,\sigma} \text{ with } n + |x| \text{ being even.} \tag{3.5}
\]

We need another preliminary result.

**Lemma 3.3.** Let \( \varepsilon > 0 \). For any \( n \in \mathbb{N} \), define

\[
Q_n(\varepsilon) := \{x = (x_1, \cdots, x_d) \in \mathbb{Z}^d : |x_i - \frac{1 - \lambda}{d(1 + \lambda)}n| < n^{(1+\varepsilon)/2}, \ 1 \leq i \leq d\}.
\]

Then

\[
\sum_{x \in \mathbb{Z}^d \setminus Q_n(\varepsilon)} p^{(n)}(0, x) \leq 2d \exp(-n^{\varepsilon}/2), \quad \forall n \in \mathbb{N}. \tag{3.6}
\]

**Proof.** By the Azuma–Hoeffding inequality,

\[
P(\max_{1 \leq i \leq d} |Z^i_n - \frac{1 - \lambda}{d(1 + \lambda)}n| \geq t) \leq 2d \exp(-\frac{t^2}{2n}), \quad t > 0,
\]

where \( Z^i_n \) is the \( i \)-th coordinate component of \( Z_n \in \mathbb{Z}^d \). The lemma follows by taking \( t = n^{(1+\varepsilon)/2} \).

Now we are ready to prove Theorem 3.1.
Proof of Theorem 3.1. Case 1: $d \geq 4$.  

Fix a small enough $\varepsilon \in (0, 1)$. Let $n_{\varepsilon} := \max\{n - \frac{2d(1+\lambda)}{1-\lambda} n^{(1+\varepsilon)/2}, 0\}$. Note that if $1 \leq m < n_{\varepsilon}$, then

$$
\frac{1 - \lambda}{d(1+\lambda)} n - n^{(1+\varepsilon)/2} > \frac{1 - \lambda}{d(1+\lambda)} m + m^{(1+\varepsilon)/2}.
$$

This implies $Q_m(\varepsilon) \cap Q_n(\varepsilon) = \emptyset$. In particular,

$$
\sum_{n \in \mathbb{N}} \sum_{1 \leq m < n_{\varepsilon}} \sum_{x \in \mathbb{Z}^d} p(m)(0, x)p(n)(0, x)
\leq \sum_{n \in \mathbb{N}} \sum_{1 \leq m < n_{\varepsilon}} \left( \sum_{x \in \mathbb{Z}^d \setminus Q_n(\varepsilon)} + \sum_{x \in \mathbb{Z}^d \setminus Q_m(\varepsilon)} \right) p(m)(0, x)p(n)(0, x).
$$

By Lemmas 3.2 and 3.3, $\sum_{x \in \mathbb{Z}^d \setminus Q_n(\varepsilon)} p(m)(0, x)p(n)(0, x)$ and $\sum_{x \in \mathbb{Z}^d \setminus Q_m(\varepsilon)} p(m)(0, x)p(n)(0, x)$ are bounded by $2d \exp(-c_2 n^\varepsilon) c_1 m^{-d/2}$ and $2d \exp(-c_2 m^\varepsilon) c_1 n^{-d/2}$, respectively. Hence

$$
\sum_{n \in \mathbb{N}} \sum_{1 \leq m < n_{\varepsilon}} \sum_{x \in \mathbb{Z}^d} p(m)(0, x)p(n)(0, x)
\leq \sum_{n \in \mathbb{N}} \sum_{1 \leq m < n_{\varepsilon}} \left( 2d \exp(-c_2 n^\varepsilon) c_1 m^{-d/2} + 2d \exp(-c_2 m^\varepsilon) c_1 n^{-d/2} \right) < \infty.
$$

On the other hand, by Lemma 3.2,

$$
\sum_{n \in \mathbb{N}} \sum_{n_{\varepsilon} \leq m \leq n} \sum_{x \in \mathbb{Z}^d} p(m)(0, x)p(n)(0, x)
\leq \sum_{n \in \mathbb{N}} \sum_{n_{\varepsilon} \leq m \leq n} \sum_{x \in \mathbb{Z}^d} p(m)(0, x)c_1 n^{-d/2}
= \sum_{n \in \mathbb{N}} \sum_{n_{\varepsilon} \leq m \leq n} c_1 n^{-d/2} \leq \sum_{n \in \mathbb{N}} c_3 n^{-(d-1-\varepsilon)/2} < \infty.
$$

Moreover, by transience of $(Z_n)_{n=0}^\infty$,

$$
\sum_{n \in \mathbb{N}} \sum_{x \in \mathbb{Z}^d} p^{(1)}(0, x)p(n)(0, x) \leq \sum_{n \in \mathbb{N}} \sum_{x \in \mathbb{Z}^d} p^{(1)}(0, x)c_1 n^{-d/2} = \sum_{n \in \mathbb{N}} c_1 n^{-d/2} < \infty.
$$

Assembling these pieces yields $\sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \sum_{x \in \mathbb{Z}^d} p(m)(0, x)p(n)(0, x) < \infty$. A fortiori, we obtain $\sum_{m=0}^\infty \sum_{n=0}^\infty 1_{\{Z_m=W_n\}} < \infty$ a.s., as desired.

Case 2: $d \leq 3$.

By (3.5) in Lemma 3.2, there exist constants $c_4 > 0$ and $c_5 > 0$ such that

$$
\sum_{n=0}^\infty \sum_{m=0}^\infty \sum_{x \in \mathbb{Z}^d} p(n)(0, x)p(m)(0, x) \geq \sum_{n=2}^\infty \sum_{n_{\varepsilon}^{1/2} \leq m \leq n} \sum_{x \in \mathbb{Z}^d} p(n)(0, x)p(m)(0, x).
$$
which is infinity. By [18, Theorem 10.24], \( \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 1 \{ Z_m = W_n \} = \infty \) a.s. 

### 3.2 Intersections of biased random walks

For \( x = (x_1, \ldots, x_d) \in \mathbb{Z}^d \), define \( \phi(x) := (|x_1|, \ldots, |x_d|) \in \mathbb{Z}^d_+ \). We start by studying the number of intersections of the reflecting random walks \( (\phi(X_n))_{n=0}^{\infty} \) and \( (\phi(Y_n))_{n=0}^{\infty} \), where \( (X_n)_{n=0}^{\infty} \) and \( (Y_n)_{n=0}^{\infty} \) are independent RW_\lambda’s on \( \mathbb{Z}^d \). By Theorem 3.1, with positive probability, the number of intersections is infinite if \( d \leq 3 \), and is finite if \( d \geq 4 \). The Liouville property below will ensure that the probability is indeed one.

**Lemma 3.4.** Let \( (X_n)_{n=0}^{\infty} \) be RW_\lambda on \( \mathbb{Z}^d \) with \( \lambda \in (0, 1) \). The Poisson boundary for \( (\phi(X_n))_{n=0}^{\infty} \) is trivial, i.e., all bounded harmonic functions are constants.

**Proof.** When \( d = 2 \), the lemma is a special case of the main result in [12]. Here we provide a proof valid for all \( d \).

Let \( (Z_n)_{n=0}^{\infty} \) be the drifted random walk driven by \( \mu \) specified in Theorem 3.1. As an immediate consequence of the Hewitt–Savage 0-1 law, \( (Z_n) \) has the Liouville property, which is equivalent to the fact that \( \mathbb{P}_x(\{ Z_n \in A \text{ i.o.} \}) \in \{ 0, 1 \} \) for any \( A \subset \mathbb{Z}^d \) and for some (hence for all) \( x \in \mathbb{Z}^d \). Here i.o. stands for infinitely often.

Let

\[ \mathcal{X} := \{ x = (x_1, \ldots, x_d) \in \mathbb{Z}^d_+ : x_i = 0 \text{ for some } 1 \leq i \leq d \}, \]

be the boundary of \( \mathbb{Z}^d_+ \). As in the proof of Lemma (2.3) we define

\[ \sigma_1 := \inf \{ n \geq 0 : \phi(X_n) \in \mathbb{Z}^d_+ \setminus \mathcal{X} \}, \quad \tau_1 := \inf \{ n > \sigma_1 : \phi(X_n) \in \mathcal{X} \}, \]

and recursively for \( i \geq 2, \)

\[ \sigma_i := \inf \{ n > \tau_{i-1} : \phi(X_n) \in \mathbb{Z}^d_+ \setminus \mathcal{X} \}, \quad \tau_i := \inf \{ n > \sigma_i : \phi(X_n) \in \mathcal{X} \}, \]

with the convention that \( \inf \emptyset := \infty \). Let \( (\mathcal{F}_n)_{n=0}^{\infty} \) be the filtration generated by \( (\phi(X_n))_{n=0}^{\infty} \), i.e., \( \mathcal{F}_n := \sigma(\phi(X_1), \ldots, \phi(X_n)) \). Then, starting at \( x \in \mathbb{Z}^d_+ \setminus \mathcal{X} \), \( (\phi(X_n))_{n=0}^{\infty} \)
has the same distribution as \((Z_n)_{n=0}^\infty\), before hitting the boundary \(\mathcal{X}\). Furthermore, for any \(i > 1\), conditioned on \(\{\tau_{i-1} < \infty\}\) and \(\mathcal{F}_{\sigma_i}\), we have that \((\phi(X_{\sigma_i+n}))_{n<\tau_i}\) has the same distribution as \((Z_n, n < \tau^Z)\) with \(Z_0 = \phi(X_{\sigma_i})\), where \(\tau^Z = \inf\{n: Z_n \in \mathcal{X}\}\).

We claim that, for any \(A \subset \mathbb{Z}^d\), if \(P_x(Z_n \in A \text{ i.o.}) = 1\) then \(P_x(\phi(X_n) \in A \text{ i.o.}) = 1\). In fact, we have from the Liouville property of \((Z_n)\) that \(P_y(Z_n \in A \text{ i.o.}) = 1\) for all \(y \in \mathbb{Z}^d\). Therefore, for any \(i \geq 1\),

\[
P_x(\phi(X_{\sigma_i+n}) \in A \text{ i.o., } \tau_i = \infty \mid \tau_{i-1} < \infty, \mathcal{F}_{\sigma_i}) = P_{\phi(X_{\sigma_i})}(Z_n \in A \text{ i.o., } \tau^Z = \infty)
\]

Since \(\{\tau_{i-1} < \infty, \tau_i = \infty\}\), \(i \geq 1\) is a partition of the probability space, we deduce that \(P_x(\phi(X_n) \in A \text{ i.o.}) = 1\).

Similarly, we can prove that \(P_x(Z_n \in A \text{ i.o.}) = 0\) implies \(P_x(\phi(X_n) \in A \text{ i.o.}) = 0\). Thus \(P_x(\phi(X_n) \in A \text{ i.o.}) \in \{0, 1\}\) and therefore \((\phi(X_n))\) is Liouville.

**Theorem 3.5.** Let \((X_n)_{n=0}^\infty\) and \((Y_n)_{n=0}^\infty\) be independent \(\text{RW}_\lambda\)'s on \(\mathbb{Z}^d\) with \(\lambda \in (0, 1)\). Then almost surely the number of intersections of \((\phi(X_n))_{n=0}^\infty\) and \((\phi(Y_n))_{n=0}^\infty\) is infinite if \(d \leq 3\) and is finite if \(d \geq 4\).

**Proof.** By Lemma 3.4 and the proof of [5, Theorem 1.1], the probability that \((\phi(X_n))_{n=0}^\infty\) and \((\phi(Y_n))_{n=0}^\infty\) intersect infinitely often is either 0 or 1.

Before hitting any axial hyperplanes, \((\phi(X_n))_{n=0}^\infty\) and \((\phi(Y_n))_{n=0}^\infty\) has the same joint distribution as that of \((Z_n)_{n=0}^\infty\), \((W_n)_{n=0}^\infty\), where \((Z_n)_{n=0}^\infty\) and \((W_n)_{n=0}^\infty\) are independent drifted random walks on \(\mathbb{Z}^d\) with step distribution \(\mu\) described in Theorem 3.1, and \(Z_0 = \phi(X_0), W_0 = \phi(Y_0)\). Let \(T\) be the first time either \((Z_n)_{n=0}^\infty\) or \((W_n)_{n=0}^\infty\) hits an hyperplane. By Theorem 3.1, on the event \(\{T = \infty\}\), which has positive probability, the number of intersections between \((Z_n)_{n=0}^\infty\) and \((W_n)_{n=0}^\infty\) is infinite if \(d \leq 3\), and is finite if \(d \geq 4\). In view of the aforementioned 0-1 law above prove this lemma.

**4 Uniform spanning forests**

This section is divided into two parts. In the first part, we study free and wired uniform spanning forests on infinite graphs, and identify, in particular, FSF and WSF on biased \(\mathbb{Z}^d\). The second part is devoted to the proof of Theorem 1.1.
4.1 Free and wired uniform spanning forests

Let \( G = (V(G), E(G)) \) be a locally finite, connected infinite graph, rooted at \( o \). To each edge \( e = (x, y) \in E(G) \), we assign a weight or conductance \( c(e) = c(x, y) = c(y, x) \). The weighted graph \((G, c)\) is called an electrical network. Consider a Markov chain on \( G \) with transition probability \( p(x, y) = \frac{c(x, y)}{\sum_z c(x, z)} \), where \( z \sim x \) means that \( z \) and \( x \) are adjacent vertices in \( G \). The chain is referred to as a random walk on \( G \) with conductance \( c \). Biased random walk \( \text{RW}_\lambda \) on \( G \) is a random walk on \( G \) with conductance defined by \( c(e) = c_\lambda(e) := \lambda - |e| \). By Rayleigh’s monotonicity principle (cf. [18], p. 35), there is a critical value \( \lambda_c(G) \in [0, \infty] \) such that \( \text{RW}_\lambda \) is transient for \( \lambda < \lambda_c(G) \) and is recurrent for \( \lambda > \lambda_c(G) \).

For any finite network \((G, c)\), we consider associated spanning trees, i.e., subgraphs that are trees and that include every vertex. We define the uniform spanning tree measure \( \text{UST}_G \) to be the probability measure on spanning trees of \( G \) such that the measure of each tree is proportional to the product of conductances of the edges in the tree.

An exhaustion of an infinite graph \( G \) is a sequence \( \{V_n\}_{n \geq 1} \) of finite, connected subsets of \( V(G) \) such that \( V_n \subset V_{n+1} \) for all \( n \geq 1 \) and \( \bigcup_n V_n = V(G) \). Given such an exhaustion, we define the network \( G_n \) to be the subgraph of \( G \) induced by \( V_n \) together with the conductances inherited from \( G \). The free uniform spanning forest measure \( \text{FSF}_G \) is defined to be the weak limit of the sequence \( \{\text{UST}_{G_n}\}_{n \geq 1} \) in the sense that

\[
\text{FSF}(S \subset \mathcal{F}) = \lim_{n \to \infty} \text{UST}_{G_n}(S \subset T),
\]

for each finite set \( S \subset E(G) \). For each \( n \), we can also construct a network \( G_n^* \) from \( G \) by gluing (= wiring) every vertex of \( G \setminus G_n \) into a single vertex, denoted by \( \partial_n \), and deleting all the self-loops that are created. The set of edges of \( G_n^* \) is identified with the set of edges of \( G \) having at least one endpoint in \( V_n \). The wired uniform spanning forest measure \( \text{WSF}_G \) is defined to be the weak limit of the sequence \( \{\text{USF}_{G_n^*}\}_{n \geq 1} \) so that

\[
\text{WSF}_G(S \subset \mathcal{F}) = \lim_{n \to \infty} \text{USF}_{G_n^*}(S \subset T),
\]

for each finite set \( S \subset E(G) \). [For the existence of both \( \text{FSF} \) and \( \text{WSF} \), see [18, Chapter 10].] Both measures \( \text{FSF} \) and \( \text{WSF} \) are easily seen to be concentrated on the set of spanning forests on \( G \) with the property that every connected component is infinite. It is also easy to see that \( \text{FSF} \) stochastically dominates \( \text{WSF} \) for any infinite network \( G \).
The number of trees in the wired uniform spanning forest is a.s. a constant; see (4.4) below. In [7], it is asked whether FSF and WSF are mutually singular (also formulated in [18, Question 10.59]) and whether the number of trees in the free uniform spanning forest is a.s. constant (also formulated in [18, Question 10.28]) if FSF $\neq$ WSF. To answer these questions, the first step is to know whether FSF and WSF are identical. When the electric network is not transitive and FSF $\neq$ WSF, it seems interesting to study whether FSF and WSF are singular. A simple situation is when $G$ is a tree, in which case the free uniform spanning forest has one tree (which is the singleton $\{G\}$), whereas the number of trees in the wired uniform spanning forest can be higher if the constant $K$ defined in (4.4) below is at least 2.

Let $\lambda > 0$. Let $c_{\lambda}(\cdot)$ be the conductances associated with RW$_{\lambda}$ on the graph $G$. Write $\text{FSF}_{\lambda}$ and $\text{WSF}_{\lambda}$ for the free and wired uniform spanning forest measures. [When they are identical, we use the notation $\text{USF}_{\lambda}$ instead.]

We give a criterion to determine whether $\text{FSF}_{\lambda} = \text{WSF}_{\lambda}$, compute the number of trees in $\text{USF}_{\lambda}$ on $\mathbb{Z}^d$, and consider the singularity problem when $\text{FSF}_{\lambda} \neq \text{WSF}_{\lambda}$.

On any graph $G$, if $\lambda > \lambda_c(G)$, then RW$_{\lambda}$ is recurrent, so $\text{FSF}_{\lambda} = \text{WSF}_{\lambda}$. The following theorem deals with the case $0 < \lambda < \lambda_c(G)$. Recall ([18, Section 6.5]) that a graph is said to have one end if the deletion of any finite set of vertices leaves exactly one infinite component.

**Theorem 4.1.** Let $G$ be a graph with one end such that

$$\lim_{n \to \infty} \left( \sum_{x \in \partial B_G(n)} (d_x^+ + d_x^0) \right)^{1/n} = 1.$$  \hspace{1cm} (4.1)

Then for $0 < \lambda < \lambda_c(G) = 1$ we have $\text{FSF}_{\lambda} = \text{WSF}_{\lambda}$. In particular, for any $d \geq 2$ and any Cayley graph of additive group $\mathbb{Z}^d$, $\text{FSF}_{\lambda} = \text{WSF}_{\lambda}$ for $\lambda \in (0, 1)$.

The proof of Theorem 4.1 shows that for any graph $G$ with one end and such that

$$\text{gr}_\ast(G) := \limsup_{n \to \infty} \left( \sum_{x \in \partial B_G(n)} (d_x^+ + d_x^0) \right)^{1/n} \in [1, \infty),$$

we have $\text{FSF}_{\lambda} = \text{WSF}_{\lambda}$ for any $0 < \lambda < \frac{1}{\text{gr}_\ast(G)}$.

*For a group acting on a network so that every vertex has an infinite orbit, it is known ([18, Corollary 10.19]) that the action is mixing and ergodic for both FSF and WSF (so the number of trees in the uniform spanning forest is a.s. a constant), and if FSF and WSF are distinct, they are mutually singular. It is unknown whether this remains true without the assumption that each vertex has an infinite orbit.
Proof of Theorem 4.1. For any function \( f : V \to \mathbb{R} \), let \( df \) be the antisymmetric function on oriented edges defined by

\[
df(e) := f(e^-) - f(e^+),
\]

where \( e^- \) and \( e^+ \) are respectively the tail and head of \( e \). Define the space of Dirichlet functions as

\[
D_\lambda := \left\{ f : (df, df)_{c_\lambda} := \sum_{e \in E} |df(e)|^2 c_\lambda(e) < \infty \right\},
\]

where \( E \) is the set of all oriented edges of \( G \). By [7, Theorem 7.3],

\[
\text{FSF}_\lambda = \text{WSF}_\lambda \iff \text{all harmonic functions in } D_\lambda \text{ are constant}.
\]

Clearly \( \lambda_c(G) = 1 \). Let \( \lambda \in (0, 1) \). Let \( f \) be a harmonic function in \( D_\lambda \). We need to prove that \( f \) is a constant.

By the maximum principle, for every \( n \geq 1 \), there are \( v_1(n), v_2(n) \in \partial B_G(n) \) such that \( f \) takes its maximum at \( v_1(n) \) and minimum at \( v_2(n) \) over all vertices in \( B_G(n) \). By the assumption,

\[
(df, df)_{c_\lambda} = \sum_{e \in E} |df(e)|^2 \lambda^{-|e|} < \infty,
\]

where \( |e| \) is, as before, the distance for \( e \) from \( o \). Hence for some constant \( C > 0 \),

\[
\sup_{e \in E} |df(e)|^2 \lambda^{-|e|} \leq C, \quad \forall e \in E.
\]

Combined with (4.1), we see that

\[
\sum_{e \in E} |df(e)| \leq C^{1/2} \sum_{n=0}^{\infty} \lambda^{n/2} \sum_{e \in E, |e| = n} 1 < \infty. \tag{4.2}
\]

Let \( n \geq 1 \). Since \( G \) has one end, \( G \setminus B_G(n) \) is a connected graph, so there is a finite path \( u_0^n u_1^n \cdots u_{k_n}^n \) in \( G \setminus B_G(n) \) such that \( u_0^n = v_1(n+1) \), \( u_{k_n}^n = v_2(n+1) \). As such,

\[
0 \leq f(v_1(n+1)) - f(v_2(n+1)) = \sum_{j=1}^{k_n} [f(u_j^n) - f(u_{j-1}^n)] \leq \sum_{e \in E, |e| \geq n+1} |df(e)|.
\]

By (4.2),

\[
\lim_{n \to \infty} \{ f(v_1(n+1)) - f(v_2(n+1)) \} = 0,
\]

which implies that \( f \) is constant. \qed

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4.2 Proof of Theorem 1.1

Let us consider the uniform spanning forests on the $\lambda$-biased lattice $\mathbb{Z}^d$. Theorem 4.1 says that $\text{FSF}_\lambda = \text{WSF}_\lambda$ on $\mathbb{Z}^d$ ($d \geq 2$) for $\lambda \in (0, 1)$. As such, we denote both of them by $\text{USF}_\lambda$.

It is clear that Theorem 1.1 will be a straightforward consequence of the following theorem.

**Theorem 4.2.** Let $0 < \lambda < 1$.

(i) Almost surely, the number of trees in the uniform spanning forest on the $\lambda$-biased lattice $\mathbb{Z}^d$ is $2^d$ if $d = 2$ or 3, and is infinite if $d \geq 4$. Moreover, when $d \geq 2$, $\text{USF}_\lambda$-a.s. every tree has one end.

(ii) If $d = 1$, then $\text{FSF}_\lambda \neq \text{WSF}_\lambda$: the free uniform spanning forest is the singleton of the tree $\mathbb{Z}^1$, whereas the wired uniform spanning forest has two trees and satisfies

$$\text{WSF}_\lambda[\mathcal{F} = \{T_{i-1}^{-}, T_i^+\}] = \frac{1}{2} (1 - \lambda) \lambda^{|i|\land|i-1|}, \quad i \in \mathbb{Z}. \quad (4.3)$$

Here, $\mathcal{F}$ has the distribution $\text{WSF}_\lambda$, $T_{i-1}^{-}$ and $T_i^+$ are subtrees of $\mathbb{Z}^1$ with vertex sets $\{i - 1, i - 2, \ldots\}$ and $\{i, i + 1, \ldots\}$, respectively.

**Proof.** (i) The proof relies on the following general result ([7, Theorem 9.4], see also [19]): Let $G$ be a connected network, and let $\alpha(w_1, \ldots, w_k)$ denote the probability that $k$ independent random walks on the network started at $w_1, \ldots, w_k$ have no pairwise intersections. Then the number of trees in the wired uniform spanning forest is a.s.

$$K = \sup \{k : \exists w_1, \ldots, w_k, \alpha(w_1, \ldots, w_k) > 0\}. \quad (4.4)$$

Moreover, if, with probability 1, two independent random walks intersect finitely often, then the number of trees in the wired uniform spanning forest is a.s. infinite.

We first study the number of trees in the uniform spanning forest.

The case $d \geq 4$ is easy: According to Theorem 3.1, two independent RW’s on $\mathbb{Z}^d$ intersect finitely often a.s., so the number of trees in the uniform spanning forest on $\lambda$-biased $\mathbb{Z}^d$ is a.s. infinite.

Consider now the case $d = 2$ or 3. Let $(X^{(j)}_n)_{n=0}^\infty$, $1 \leq j \leq 2^d$, be independent RW’s on $\mathbb{Z}^d$ starting at $o$. Note that the lower limit

$$\liminf_{n \to \infty} \alpha(X^{(1)}_n, \ldots, X^{(2^d)}_n)$$
is a.s. greater than or equal to the probability that \( (X_n^{(j)})_{n=0}^{\infty}, 1 \leq j \leq 2^d \), eventually
direct into different orthants. The latter probability is strictly positive according to
Theorem 2.1(ii). Consequently, there exist \( \varepsilon_0 > 0 \) and \( n_0 \in \mathbb{N} \) such that
\[
\mathbb{P}\{\alpha(X_{n_0}^{(1)}, \ldots, X_{n_0}^{(2d)}) > \varepsilon_0\} > 0.
\]
A fortiori, there are \( v_1, \ldots, v_{2d} \) such that \( \alpha(v_1, \ldots, v_{2d}) > \varepsilon_0 \). By (4.4), there are at least
\( 2^d \) trees in the uniform spanning forest on \( \lambda \)-biased \( \mathbb{Z}^d \).

To prove that the number of trees is at most \( 2^d \), let us consider \( 2^d + 1 \) independent
RW\( _\lambda \)'s on \( \mathbb{Z}^d \) starting at any initial points. Since there are \( 2^d \) orthants in \( \mathbb{Z}^d \), Lemma
2.3 implies that a.s. there are at least two of them eventually directing into a common
orthant. By Theorem 3.5, these two RW\( _\lambda \)'s intersect i.o. with probability 1. Therefore,
\[
\sup\{k : \exists w_1, \ldots, w_k, \alpha(w_1, \ldots, w_k) > \varepsilon_0\} \leq 2^d.
\]
Therefore the number of trees in the uniform spanning forest is exactly \( 2^d \) by (4.4).

Now fix \( d \geq 2 \) and \( \lambda \in (0, 1) \). Write
\[
|F|_{c_\lambda} = \sum_{e \in F} c_\lambda(e), \ F \subset E\left(\mathbb{Z}^d\right),
\]
\[
|K|_{\pi} = \sum_{x \in K} \pi(x), \ K \subset \mathbb{Z}^d,
\]
\[
\psi(\mathbb{Z}^d, t) = \inf\{|\partial_E K|_{c_\lambda} : t \leq |K|_{\pi} < \infty\}, \ t > 0;
\]
where \( \partial_E K = \{\{x, y\} \in E(\mathbb{Z}^d) : x \in K, y \notin K\} \), and
\( \pi(x) := (d^+_x + d^-_x \lambda) \lambda^{-|x|}, x \in \mathbb{Z}^d \),
is an invariant measure of the walk. Recall from Theorem 2.1 that \( \rho_\lambda = \frac{2\sqrt{\lambda}}{1+\lambda} < 1 \). By
[18, Theorem 6.7],
\[
\inf\left\{\frac{|\partial_E K|_{c_\lambda}}{|K|_{\pi}} : \emptyset \neq K \subset \mathbb{Z}^d \text{ is finite}\right\} \geq 1 - \rho_\lambda > 0.
\]
Thus for any \( t > 0 \), \( \psi(\mathbb{Z}^d, t) \geq (1 - \rho_\lambda)t \). Since
\[
\inf_{x \in \mathbb{Z}^d} (d^+_x + d^-_x \lambda) \lambda^{-|x|} > 2d\lambda > 0,
\]
by [18, Theorem 10.43], USF\( _\lambda \)-a.s. every tree has only one end.

(ii) It remains to prove (4.3). For any \( n \in \mathbb{N} \), let \( G_n = [-n, n] \cap \mathbb{Z}^1 \) be the induced
subgraph of tree \( \mathbb{Z}^1 \), and \( G_n^* \) the graph obtained from \( G_n \) by identifying all vertices of

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$\mathbb{Z}^1 \setminus G_n$ to a single vertex $z_n$ and deleting all the self-loops. Note $G^*_n$ is a simple cycle of length $2(n + 1)$, and $z_n$ is adjacent to $n$ and $-n$. Endow $G^*_n$ with the following edge conductance function $c_\lambda(\cdot)$:

$$c_\lambda(\{i-1, i\}) = \lambda^{-(|i| \wedge |i-1|)}, \quad i \in [-n-1, n] \cap \mathbb{Z}^1,$$

$$c_\lambda(\{z_n, n\}) = c_\lambda(\{z_n, -n\}) = \lambda^{-n}.$$ 

Clearly all spanning trees of $G^*_n$ are of the form $G^*_n \setminus \{e\}$ for some edge $e$ of $G^*_n$. Let

$$\Xi(G^*_n \setminus \{e\}) = \prod_{f \in E(G^*_n) \setminus \{e\}} c_\lambda(f) = \frac{1}{c_\lambda(e)} \prod_{f \in E(G^*_n)} c_\lambda(f).$$

By the definition of WSF, for any $i \in \mathbb{Z}^1$,

$$\text{WSF}_\lambda[\delta = \{T_{i-1}^-, T_i^+\}] = \lim_{n \to \infty} \frac{\Xi(G^*_n \setminus \{i-1, i\})}{\sum_{e \in E(G^*_n)} \Xi(G^*_n \setminus \{e\})} = \lim_{n \to \infty} \frac{c_\lambda(\{i-1, i\})^{-1}}{\sum_{e \in E(G^*_n)} c_\lambda(\{e\})^{-1}}$$

$$= \lim_{n \to \infty} \frac{\lambda^{\lfloor |i| \wedge |i-1| \rfloor}}{2^n \sum_{k=0}^n \lambda^k} = \frac{1}{2} (1 - \lambda) \lambda^{|i| \wedge |i-1|},$$

as desired.

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References


Zhan Shi
LPSM, Sorbonne Université Paris VI
4 place Jussieu, F-75252 Paris Cedex 05
France
E-mail: zhan.shi@upmc.fr

Vladas Sidoravicius
NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai
& Courant Institute of Mathematical Sciences
Shanghai 200062, P. R. China
E-mail: vs1138@nyu.edu

He Song
School of Mathematics and Statistics, Huaiyin Normal University
Huaiyin 223300, P. R. China
Email: tayunzhuiyue@126.com
Longmin Wang  
School of Mathematical Sciences, LPMC, Nankai University  
Tianjin 300071, P. R. China  
E-mail: wanglm@nankai.edu.cn

Kainan Xiang  
School of Mathematics and Computational Science, Xiangtan University  
Xiangtan City 411105, Hunan Province, P. R. China  
E-mails: kainan.xiang@xtu.edu.cn  
                   kainanxiang@gmail.com